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On the Generation of New Generalized Families and Classes of Lifetime Distributions with Applications

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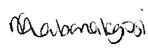
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Abstract

This thesis is concerned with the construction and development of new families of distributions including the Topp-Leone type II exponentiated half logistic-G family of distributions, the Topp-Leone exponentiated half logistic generalized-G family of distributions, the Topp-Leone-Gompertz-G power series class of distributions, the Marshall-Olkin-Topp-Leone-Gompertz-G family of distributions and the gamma odd power generalized Weibull-G family of distributions. Some mathematical and statistical properties of the new families of distributions are explored. The statistical properties studied include the expansion of the density function, hazard rate function and quantile function, moments, probability weighted moments, stochastic ordering, distribution of the order statistics and Rényi entropy. The maximum likelihood technique is employed in the estimation of model parameters and Monte Carlo simulations are conducted to examine the consistency of the estimates considering average bias and root mean square errors. Applications using data sets from various disciplines are presented to demonstrate the importance and the applicability of these new families of distributions.

Keywords

Topp-Leone type II exponentiated half logistic-G; Topp-Leone Exponentiated half logistic generalized-G; Topp-Leone-Gompertz-G power series; Marshall-Olkin-Topp-Leone-Gompertz-G; Gamma odd power generalized Weibull-G; Mathematical and Statistical properties; Maximum likelihood technique; Monte Carlo simulations; Applications; Data sets.

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List of Publications

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- (ii) M. Gabanakgosi, T. Moakofi, B. Oluyede and B. Makubate.(2022). The Gamma Odd Power Generalized Weibull-G Family of Distributions. *Journal of Statistical Modelling: Theory & Applications*. Vol. 2(2), 79-101. http://jsm.yazd.ac.ir/article_2708.html.
- (iii) T. Moakofi, B. Oluyede and M. Gabanakgosi. (2022). The Topp-Leone Odd Burr III-G Family of Distributions: Model, Properties and Applications. *Statistics, Optimization & Information Computing*, Vol. 10(1), 236-262. <http://iapress.org/index.php/soic/article/view/1135>.
- (iv) B. Oluyede and M. Gabanakgosi. (2023). The Type II Exponentiated Half Logistic-Marshall-Olkin-G Family of Distributions with Applications. *Revista Colombiana de Estadística*. Vol. 46(2), 19-53. <https://openurl.ebsco.com/EPDB%3Agcd%3A12%3A17290380/detailv2?sid=ebsco%3Aplink%3Ascholar&id=ebsco%3Agcd%3A169823059&crl=c>.
- (v) G. Warahena-Liyanage, M. Gabanakgosi and B.Oluyede. (2023). Generalized Topp-Leone-G Power Series Class of Distributions. *Electronic Journal of Applied Statistical Analysis*. Vol. (16)(3), 564-583. <http://siba-ese.unisalento.it/index.php/ejasa/article/view/25606>.

2. Accepted Papers

- (i) B. Oluyede and M. Gabanakgosi. (2022). Marshall-Olkin Type II Exponentiated Half Logistic Family of Distributions: Properties and Applications. *Journal of Statistics and Management Systems*.

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- (ii) M. Gabanakgosi, B. Oluyede and G. Warahena-Liyanage. (2022). The Marshall-Olkin-Topp-Leone-Gompertz-G Family of Distributions with Applications. *Statistics, Optimization and Information Computing*.

- (iii) M. Gabanakgosi and B. Oluyede. (2021). The Type II Topp-Leone Exponentiated Half Logistic-G. *Mathematics in Natural Science*.
- (iv) B. Oluyede, M. Gabanakgosi and T. Moakofi. (2022). The Topp-Leone New Power Generalized Weibull-G Family of Distributions with Applications. *Thailand Statistician*.
- (v) W. Sengweni, B. Oluyede and M. Gabanakgosi. (2022). Weibull Topp-Leone-G Power Series Class of Distributions with applications. *Chilean Journal of Statistics*.
- (vi) B. Oluyede, M. Gabanakgosi, D. Wanduku, T. Moakofi and F. Chipepa. (2022). The Topp-Leone Exponentiated Half Logistic-Generalized-G Family of Distributions. *Elsevier Book Chapter*.
- (vii) Morongwa Gabanakgosi and Masego Otladisa. (2023) Marshall-Olkin Exponentiated Half Logistic-Generalized-G family of distributions. *Scientific African*.



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List of Abbreviations

Abbreviation	Definition
ABIAS	Average Bias
AABIAS	Average Absolute Bias
AIC	Akaike Information Criterion
AICC	Consistent Akaike Information Criterion
A^*	Anderson-Darling
BIC	Bayesian Information Criterion
BOL-E	Beta Odd Lindley-Exponential
BGom	Beta-Gompertz
CK	Coefficient of Kurtosis
CS	Coefficient of Skewness
CV	Coefficient of Variation
cdf	Cumulative Distribution Function
EHL-G	Exponentiated Half Logistic-G
Exp-G	Exponentiated-Generalized-G
EGGom	Extended Generalized Gompertz
EMW	Exponentiated Modified Weibull
FIM	Fisher Information Matrix
G-G	Generalized-G
GOR-G	Gamma Odd Rayleigh-G
GONH-G	Gamma Odd Nadarajah Haghighi-G
GOPGW-BXII	The Gamma Odd Power Generalized Weibull-Burr XII
GOPGW-Lx	Gamma Odd Power Generalized Weibull-Lomax
GOPGW-G	Gamma Odd Power Generalized Weibull-G
GW-LLo	Generalized Weibull-Log-Logistic
GWGom	Generalized Weibull Gompertz
hrf	Hazard Rate Function

iid	Independent Identically Distributed
K-M	Kaplan-Meier
K-S	Kolmogorov-Smirnov
KGom	Kumaraswamy Gompertz
KW	Kumaraswamy Weibull
L_n	Log-likelihood
LLoGWP	Log-Logistic-Weibull Poisson
Lx	Lomax
MGF	Moment Generating Function
MLE's	Maximum Likelihood Estimates
MOEGGom	Marshall-Olkin extended generalized Gompertz
MO-G	Marshall-Olkin generator
MO-Gom-W	Marshall-Olkin-Gompertz-Weibull
MO-TL-Gom-BIII	Marshall-Olkin-Topp-Leone-Gompertz-Burr III
MO-TL-Gom-G	Marshall-Olkin-Topp-Leone-Gompertz-G
MO-TL-Gom-LLo	Marshall-Olkin-Topp-Leone-Gompertz-Log-Logistic
MO-TL-Gom-W	Marshall-Olkin-Topp-Leone-Gompertz-Weibull
n	Sample Size
OGE Gom	Odd generalized exponential Gompertz
OLLEW	Odd log-logistic exponentiated Weibull
pdf	Probability Density Function
pmf	Probability Mass Function
PWMs	Probability Weighted Moments
RMSEs	Root Mean Square Errors
SD	Standard Deviation
SS	Sum of Squares
TL	Topp-Leone
TL-EHL-G	Topp-Leone Exponentiated Half Logistic-G
TL-EHL-GG	Topp-Leone exponentiated half logistic generalized-G
TL-EHL-GLLo	Topp-Leone Exponentiated Half Logistic Generalized-Log-Logistic
TL-EHL-GW	Topp-Leone Exponentiated Half Logistic Generalized-Weibull

TL-EHL-GP	Topp-Leone Exponentiated Half Logistic Generalized-Power
TL-Gom-EP	Topp-Leone-Gompertz-Exponential Poisson
TL-Gom-EPS	Topp-Leone-Gompertz-Exponential Power Series
TL-Gom-GPS	Topp-Leone-Gompertz-G Power Series
TL-Gom-LLoP	Topp-Leone-Gompertz-Log-Logistic Poisson
TL-Gom-LLoPS	Topp-Leone-Gompertz-Log-Logistic Power Series
TL-Gom-WP	Topp-Leone-Gompertz-Weibull Poisson
TL-Gom-WPS	Topp-Leone-Gompertz-Weibull Power Series
TLGIKw	Topp-Leone Generalized Inverted Kumaraswamy
TLMOW	Topp-Leone-Marshall-Olkin Weibull
TLGW	Topp-Leone generated Weibull
TL-TIIEHL-G	Topp-Leone type II exponentiated half Logistic-G
TL-TIIEHL-L	Topp-Leone Type II Exponentiated Half Logistic-Lindley
TL-TIIEHL-U	Topp-Leone Type II Exponentiated Half Logistic-Uniform
TL-TIIEHL-W	Topp-Leone Type II Exponentiated Half Logistic-Weibull
TL-TIIHL-G	Topp-Leone type II half Logistic-G
TLWLx	Topp-Leone Weibull-Lomax
TL-WP	Topp-Leone-Weibull Poisson
TTT	Total Time on Test
TIEHL-G	Type I Exponentiated Half Logistic-G
TIHL	Type I Half Logistic
TIIEHL-G	Type II Exponentiated Half Logistic-G
TIIGIE-Lx	Type II General Inverse Exponential Lomax
T-X	Transformed-Transformer
W^*	Cramér-von Mises
WL	Weibull-Lomax
-2logL	-2log-likelihood

1. Introduction and Literature Review

In this chapter, we present a statistical literature review on both previously known and recently constructed distributions. Different types of generators in the literature are also discussed. Lastly, we outline the objectives as well as motivations and justifications for this study.

1.1 Introduction

The development of generalized families of distributions for modelling data in many areas such as economics, biology, insurance, finance and engineering to mention a few is of tremendous practical importance (Lee et al. (2013), Nassar et al. (2018)). This has motivated researchers to study and develop methods of generating new families of distributions from well known distributions in order to get greater flexibility when modelling data in practice.

The development of new lifetime distributions and the generation of new generalized families and classes of distributions in general are of tremendous practical importance in probability and statistics. As implied by the title of this dissertation, the understanding and advancement of lifetime distributions form the center of our investigation. A lifetime distribution provides a way to describe and analyze the time-to-event data. Common examples include the time until a machine fails, the time until a customer makes a purchase, or the time until a person experiences a certain event. These distributions are necessary for understanding the reliability and survival characteristics of systems, products, or processes, and can be applied in various fields, including engineering, finance, biology, and epidemiology.

Recently, scholars have been extending classical and well known distributions in analysing data by using important generators in the literature. Some of those methods of generating new families of distributions include the gamma-G family of distributions by Zografos and Balakrishnan (2009), the Transformed-Transformer method of generating continuous distributions by Alzaatreh et al. (2013), Marshall-Olkin-G family of distributions by Marshall and Olkin (1997), the Topp-Leone generated family of distributions by Al-Shomrani et al. (2016), the TIIEHL-G family of distributions by Al-Mofleh et al. (2020), the type I half-logistic family of distributions by Cordeiro et al. (2016), and the Exp-G class of distributions by Cordeiro et al. (2013), among others. Other useful examples of generators in the literature include the Kumaraswamy-G family of distributions by Cordeiro and de Castro (2011), the beta Odd Lindley-G family of distributions (Chipepa et al. (2019)) and the EHL-G distribution by Cordeiro et al. (2014).

The TL distribution was studied by Nadarajah and Kotz (2003) as a simple bounded J-shaped distribution. The distribution can model failure times data since it has hazard rate function which exhibit bathtub shapes. There are several distributions that has been developed using the Topp-Leone generator, some of those useful distributions are: the odd Weibull inverse Topp-Leone distribution by Almetwally (2022), the Topp-Leone Burr-XII distribution by Reyad and

Othman (2017), the Weibull Topp-Leone generated family by Karamikabir et al. (2020), the Topp-Leone Marshall-Olkin-G family of distributions by Chipepa et al. (2020), Topp-Leone Gompertz distribution by Nzei et al. (2020), Type II exponentiated half logistic Topp-Leone Marshall-Olkin-G family of distributions by Moakofi et al. (2021a) and the generalized Topp-Leone-G power series class of distributions by Warahena Liyanage et al. (2023). Oluyede et al. (2022) developed Topp-Leone Gompertz-G distributions based on the works by Al-Shomrani et al. (2016) and Alizadeh et al. (2017).

The importance of generalizing distributions is that they provide better fits than the existing ones in modeling real-life data. Some examples of generalized distributions via the Marshall-Olkin generator (Marshall and Olkin (1997)), include the Marshall-Olkin-Gompertz-G family of distributions by Chipepa and Oluyede (2021), the Marshall-Olkin extended generalized Gompertz distribution by Lazhar (2017), the Marshall-Olkin Lindley-log-logistic distribution by Moakofi et al. (2021b), the Marshall-Olkin odd exponential half logistic-G family of distributions by Oluyede and Chipepa (2021), the Marshall-Olkin-odd power generalized Weibull-G family of distributions by Chipepa et al. (2022), type II exponentiated half logistic-Topp-Leone-Marshall-Olkin-G distributions by Moakofi et al. (2021a) and the type II exponentiated half logistic-Marshall-Olkin-G family of distributions by Oluyede and Gabanakgosi (2023).

Several recent studies include the work by Oluyede et al. (2018), where they developed the gamma-Weibull-G family of distributions via the gamma-G transformation (Zografos and Balakrishnan (2009)), the generalized modified Weibull distributions using the same generator by Oluyede et al. (2015), Ristic-Balakrishnan-Marshall Olkin-G distributions by Peter et al. (2022), the generalized odd Weibull-G family of distributions by Korkmaz et al. (2018) and a new Weibull-G family by Tahir et al. (2016).

Korkmaz et al. (2018) proposed a wider class of continuous distributions called the generalized odd Weibull-G by using the T-X family (Alzaatreh et al. (2013)) method of generating distributions. Some distributions that utilized this method of generating distributions are, the exponentiated T-X family by Alzaghal et al. (2013), odd power generalized Weibull-G family by Oluyede et al. (2021b), type II exponentiated half logistic-G family by Al-Moffeh et al. (2020) and TIHL family (Cordeiro et al. (2016)), just to name a few.

The power series class of distributions are of substantial importance in the areas of finance, reliability and actuarial sciences. Some of the generalizations that involve power series distributions include the Burr XII power series distribution by Silva and Cordeiro (2015), the generalized exponential power series distribution (Mahmoudi and Jafari (2012)), a compound class of Weibull and power series distributions by Morais and Barreto-Souza (2011), Lindley-Burr XII power series distribution by Makubate et al. (2021), generalized linear failure rate power series distribution by Shams and Alamatsaz (2016), odd Weibull-Topp-Leone-G power series distribution by Oluyede et al. (2021a) and Gompertz-power series distribution by Jafari and Tahmasebi (2016) and the Topp Leone-G power series class of distributions by (Chipepa et al. (2021)). In the

context of survival analysis, modeling lifetime data has become popular. Studies in survival analysis involve examining the lifetimes of both biological and mechanical systems. To model these types of data, several distributions have been introduced in recent years. Examples of those distributions are: the exponential geometric distribution by Adamidis and Loukas (1998), the exponential Poisson distribution by Kuş (2007), the exponential logarithmic distribution by Tahmasbi and Rezaei (2008), Weibull geometric by Barreto-Souza et al. (2011), and Weibull poisson by Lu and Shi (2012), the beta exponential power series distribution by Khojastehbakht et al. (2023), to name a few.

Weibull distribution has many limitations when it comes to applications in real data since it only has monotonically increasing, monotonically decreasing and constant hazard rate function, and this has drawn scholars' attention to develop more generalized Weibull distributions that can give the desired results in applications. A few Weibull distribution generalizations in the literature, are two, three, four and five parameter distributions. The two parameter Weibull-extensions include models by Bebbington et al. (2007) and Zhang and Xie (2011). The three parameter Weibull extensions include models by Marshall and Olkin (1997), Xie et al. (2002) and Nadarajah and Kotz (2005). Some of the four parameter generalizations include the additive Weibull distribution by Xie and Lai (1996), modified Weibull distribution by Sarhan and Zaindin (2009) and the beta-Weibull distribution which was proposed by Famoye et al. (2005). The five parameter modified Weibull distributions include those introduced by Phani (1987), beta modified Weibull distribution by Silva et al. (2010) and Nadarajah et al. (2011) among others. Bagdonavicius and Nikulin (2001) developed an extension of the Weibull distribution called the power generalized Weibull distribution which was later studied by Nikulin and Haghighi (2009). Cordeiro et al. (2015) also studied a new generalized Weibull family of distributions. Other examples of the Weibull generalizations include a three-parameter generalized inverse Weibull distribution with decreasing and unimodal failure rate by Gusmão et al. (2011) and the exponentiated generalized inverse Weibull distribution by Elbatal and Muhammed (2014).

1.2 Some Generators in the Literature

In this section, we explain some of the key generators used in this thesis considering $G(x; \zeta)$ as the baseline continuous distribution, with the survival function $\bar{G}(x; \zeta) = 1 - G(x; \zeta)$ and density function $g(x; \zeta)$, where ζ is the baseline parameter vector.

1.2.1 Type I Exponentiated Half Logistic Generator

The TIEHL-G distribution was developed by Cordeiro et al. (2014) using the gamma generator (Zografos and Balakrishnan (2009)), with the cumulative distribution function (cdf) and probability density function (pdf) given by

$$\begin{aligned} F_{TIEHL-G}(x; \alpha, \lambda, \zeta) &= 1 - \int_0^{-\log[1-G(x;\zeta)]} \frac{2\alpha\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{\alpha-1}}{(1 + e^{-\lambda t})^{\alpha+1}} dt \\ &= [K_G(x; \lambda, \zeta)]^\alpha, \end{aligned} \quad (1.1)$$

and

$$f_{TIEHL-G}(x; \alpha, \lambda, \zeta) = 2\alpha\lambda g(x; \zeta)[1 - G(x; \zeta)]^{\lambda-1} \frac{[1 - [1 - G(x; \zeta)]^\lambda]^{\alpha-1}}{[1 + [1 - G(x; \zeta)]^\lambda]^{\alpha+1}}, \quad (1.2)$$

respectively, for $\alpha, \lambda > 0$ and parameter vector ζ , where $K_G(x; \lambda, \zeta) = \frac{1-[1-G(x;\zeta)]^\lambda}{1+[1-G(x;\zeta)]^\lambda}$, and $\alpha > 0$ and $\lambda > 0$ are shape parameters.

1.2.2 Type II Exponentiated Half Logistic Generator

Based on the gamma generator (Ristić and Balakrishnan (2012)), Al-Mofleh et al. (2020) developed TIEHL-G because it offers increased flexibility in accommodating various types of data, and it has special models that can generate densities with left-skewed, right-skewed, symmetrical, and reversed-J shaped patterns. The TIEHL-G distribution is given by the cdf

$$\begin{aligned} F_{TIEHL-G}(x; a, \lambda, \zeta) &= 1 - \int_0^{-\log G(x;\zeta)} \frac{2a\lambda e^{-\lambda t}(1 + e^{-\lambda t})^{a-1}}{(1 + e^{-\lambda t})^{a+1}} dt \\ &= 1 - [K_G^*(x; \lambda, \zeta)]^a, \end{aligned} \quad (1.3)$$

for $a, \lambda > 0$, where $G(x; \zeta)$ is the baseline cdf with parameter vector ζ . The pdf of the TIEHL-G family of distribution is given by

$$f_{TIEHL-G}(x; a, \lambda, \zeta) = 2a\lambda g(x; \zeta)[G(x; \zeta)]^{\lambda-1} \frac{[1 - [G(x; \zeta)]^\lambda]^{a-1}}{[1 + [G(x; \zeta)]^\lambda]^{a+1}}, \quad (1.4)$$

where $a > 0, \lambda > 0$ are shape parameters, $G(x; \zeta)$ is the baseline cdf with parameter vector ζ , and $K_G^*(x; \lambda, \zeta) = \frac{1-[G(x;\zeta)]^\lambda}{1+[G(x;\zeta)]^\lambda}$.

1.2.3 Topp-Leone Generator

Topp-Leone distribution was first introduced by Topp and Leone (1955), known to exhibit bathtub-shaped hazard rate functions. Based on the Topp-Leone distribution by Nadarajah and Kotz (2003), Al-Shomrani et al. (2016) proposed the Topp-Leone family of distributions with the cdf and pdf given by

$$F_{TL-G}(x; b, \zeta) = [1 - \overline{G}^2(x; \zeta)]^b, \quad (1.5)$$

and

$$f_{TL-G}(x; b, \zeta) = 2b[1 - \overline{G}^2(x; \zeta)]^{b-1} \overline{G}(x; \zeta) g(x; \zeta), \quad (1.6)$$

respectively, for $b > 0$, where $\overline{G}(x; \zeta) = 1 - G(x; \zeta)$ with baseline parameter vector ζ . The type II Topp-Leone-G family was studied by Elgarhy et al. (2018). The cdf

$$F_{TIITL-G}(x; b, \zeta) = 1 - [1 - G(x; \zeta)^2]^b, \quad (1.7)$$

and the corresponding pdf is given as

$$f_{TIITL-G}(x; b, \zeta) = 2b[1 - G(x; \zeta)^2]^{b-1} G(x; \zeta) g(x; \zeta), \quad (1.8)$$

with $b > 0$ as the shape parameter and $G(x; \zeta)$ is the cdf of the baseline distribution which depend on parameter vector ζ .

1.2.4 Transformed-Transformer (T-X) Generator

Let $u(t)$ be the pdf of a random variable $T \in [a, b]$ and $W[G(x)]$ be a function of the cdf of a random variable X , where $W[G(x)]$ satisfies the conditions:

- (i) $W[G(x)] \in [a, b]$
- (ii) $W[G(x)]$ is differentiable and monotonically non-decreasing, and
- (iii) $\lim_{x \rightarrow -\infty} W[G(x)] \rightarrow a$, and $\lim_{x \rightarrow \infty} W[G(x)] \rightarrow b$,

then, Alzaatreh et al. (2013) defined the cdf and pdf of the T-X family of distributions by

$$F(x) = \int_a^{W[G(x)]} u(t) dt, \quad (1.9)$$

and

$$f(x) = \left\{ \frac{d}{dx} W[G(x)] \right\} u \{W[G(x)]\}, \quad (1.10)$$

respectively.

1.2.5 Compounding Method

In the compounding approach of generating new distributions, a continuous and a discrete distribution are merged. A discrete case is expressed as a power series. Power series family of dis-

tributions are binomial, geometric, Poisson and logarithmic distributions (Johnson et al. (1994)).

Let us consider the order statistic X_k as the k^{th} smallest value when the sample observations are arranged in ascending order such that $X_1, X_2, \dots, X_N$ are iid random variables with cdf $G(x; \xi)$, where ξ is a parameter vector. Suppose N is a discrete random variable following a power series distribution and assumed to be truncated at zero, with the pmf given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \dots, \quad (1.11)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n \geq 1}$ a sequence of positive real numbers and let $X_{(1)} = \min(X_1, X_2, \dots, X_N)$, then the cdf of $X_{(1)}|N = n$ is given by

$$F_{X_{(1)}|N=n}(x) = 1 - (\bar{G}(x; \xi))^n, \quad \text{where} \quad \bar{G}(x; \xi) = 1 - G(x; \xi). \quad (1.12)$$

The marginal cdf and pdf of $X_{(1)}$ are

$$F_{X_{(1)}}(x) = 1 - \frac{C(\theta S(x))}{C(\theta)} \quad \text{and} \quad f_{X_{(1)}}(x) = \frac{\theta g(x) C'(\theta S(x))}{C(\theta)}, \quad (1.13)$$

respectively, where $S(x) = S(x; \xi) = 1 - G(x; \xi)$ is the survival function. Similarly, the cdf of $X_{(n)} = \max(X_1, X_2, \dots, X_N)$ is

$$F_{X_{(n)}}(x) = \frac{C(\theta S(x))}{C(\theta)}. \quad (1.14)$$

1.2.6 Marshall-Olkin Generator

The Marshall-Olkin generator (Marshall and Olkin (1997)) is given by the cdf and pdf

$$F_{MO-G}(x; \delta, \xi) = 1 - \frac{\delta \bar{G}(x; \xi)}{1 - \bar{\delta} \bar{G}(x; \xi)}, \quad (1.15)$$

and

$$f_{MO-G}(x; \delta, \xi) = \frac{\delta g(x; \xi)}{[1 - \bar{\delta} \bar{G}(x; \xi)]^2}, \quad (1.16)$$

respectively, where δ is the tilt parameter, $\bar{\delta} = 1 - \delta$ and $G(x; \xi)$ is the baseline cdf which depends on the parameter vector ξ .

1.2.7 Gamma Transformations

For a continuous baseline cdf $G(x; \psi)$, Zografos and Balakrishnan (2009) developed a gamma-generator with the cdf and pdf

$$F(x; \delta, \beta, \psi) = \frac{1}{\Gamma(\delta)\beta^\delta} \int_0^{-\log(\overline{G}(x; \psi))} t^{\delta-1} e^{-\frac{t}{\beta}} dt = \frac{\gamma(\delta, -\beta^{-1} \log(\overline{G}(x; \psi)))}{\Gamma(\delta)}, \quad (1.17)$$

and

$$f(x; \delta, \beta, \psi) = \frac{1}{\Gamma(\delta)\beta^\delta} [-\log(\overline{G}(x; \psi))]^{\delta-1} (1 - G(x; \psi))^{\frac{1}{\beta}-1} g(x; \psi), \quad (1.18)$$

respectively, for $\delta > 0$, ($\beta = 1$) and parameter vector ψ , where $\overline{G}(x; \psi) = 1 - G(x; \psi)$, $\Gamma(\delta) = \int_0^\infty t^{\delta-1} e^{-t} dt$ is the gamma function, and $\gamma(z, \delta) = \int_0^z t^{\delta-1} e^{-t} dt$ is the incomplete gamma function. Similarly, Ristić and Balakrishnan (2012) presented an alternative gamma-generator with the cdf and pdf as

$$F(x; \delta, \beta, \psi) = 1 - \frac{1}{\Gamma(\delta)\beta^\delta} \int_0^{-\log(G(x; \psi))} t^{\delta-1} e^{-\frac{t}{\beta}} dt, \quad (1.19)$$

and

$$f(x; \delta, \beta, \psi) = \frac{1}{\Gamma(\delta)\beta^\delta} [-\log(G(x; \psi))]^{\delta-1} (G(x; \psi))^{\frac{1}{\beta}-1} g(x; \psi), \quad (1.20)$$

respectively.

1.3 Objectives

Main Objective

The central purpose of this work is to propose new generalized families of continuous distributions by extending well known distributions through the use of a combination of several generators in the literature.

1.3.1 Specific Objectives

1. To propose new generalized families of continuous distributions by using the following generators, and a combination of these generators:
 - Type II Exponentiated Half Logistic Generator

- Topp-Leone Generator
 - Compounding Method
 - Marshall-Olkin Generator
 - Gamma Transformations
 - T-X Generator
2. To derive some mathematical and statistical properties of the new generalized families of distributions.
 3. To assess and evaluate the consistency and effectiveness of the maximum likelihood estimates by performing various simulations.
 4. To show the usefulness and the flexibility of the proposed families of distributions through applications to real data sets.

1.3.2 Motivations and Justifications of the Study

Classical distributions have been useful in many statistical applications, they frequently have limits when dealing with complex and diverse datasets. These limitations include inflexibility in capturing various shapes and tail behaviors, failure to model data with heavy tails or skewed distributions. These limits can result in an inadequate representation of the underlying data generating process and inaccurate statistical inference.

To address these challenges we propose new generalized families of distributions with the hope that they will give greater flexibility when fitted to real-life data such as economic data, environmental data, medical data, to mention a few. The new families of distributions have hazard rate functions that are upside-down bathtub, decreasing, increasing, upside-down bathtub followed by bathtub shapes as well as the bathtub shaped and thus, these families of distributions can be important in reliability analysis, and several other areas.

Moreover, we were also motivated by the fact that those newly proposed distributions fit real-world data more accurately than several other generalized distributions with the same number of parameters because their hazard rate functions display monotone and non-monotone shapes. Also, the underlying motivations for developing this new generalized distribution is to achieve skewness for models which are symmetric, to develop special models which exhibit different levels of skewness and kurtosis.

1.3.3 An Overview of the Thesis

The content of this thesis is organized as follows: In Chapter 2, we present the Topp-Leone type II exponentiated half logistic-G family of distributions. Chapter 3 contains the Topp-Leone exponentiated half logistic generalized-G family of distributions, and in Chapter 4, we study the Topp-Leone-Gompertz-G power series class of distributions. In Chapter 5, we present the Marshall-Olkin-Topp-Leone-Gompertz-G family of distributions and Chapter 6 contains the

Gamma odd power generalized Weibull-G family of distributions. Lastly in Chapter 7, we give conclusions and areas of future research.

1.3.4 Chapter Summary

The first chapter of this thesis is briefly summarized in this subsection. Statistical literature review on both previously known and recently constructed distributions are provided in this chapter. Different types of generators in the literature were also discussed in this chapter. Lastly, we give objectives as well as motivations and justifications for this study.



2. The Topp-Leone Type II Exponentiated Half Logistic-G Family of Distributions

In this chapter we extend the work of Al-Mofleh et al. (2020) and Al-Shomrani et al. (2016) by proposing new family of continuous distributions.

2.1 Introduction

The Topp-Leone generator and TIIEHL generator are very useful generators in the literature of statistical distributions. A combination of these two generators was considered to produce a new generalized distribution. We introduce a new family of distributions referred to as the Topp-Leone type II exponentiated half logistic-G distribution. We also present sub-families and some of the special cases of the new family of distributions. Some statistical properties of the new family of distributions are explored. The statistical properties studied include the expansion of the density function, hazard and quantile functions, moments, moment generating functions, probability weighted moments, stochastic ordering, distribution of order statistics and Rényi entropy. Model parameters are estimated using the maximum likelihood technique followed by Monte Carlo simulations. Lastly, the importance of the new family of distributions is examined by means of applications to real data set.

2.2 The Model

In this section, we consider the Topp-Leone family of distributions (Al-Shomrani et al. (2016)) with the cdf and pdf given by equations (1.5) and (1.6). To obtain the new model, we insert the TIIEHL-G family of distributions, equation (1.3) into the Topp-Leone family of distributions given by equation (1.5), that is, equations (1.3) and (1.5) are combined to give a new family of distributions named Topp-Leone type II exponentiated half logistic-G distribution, which has the cdf and pdf

$$F_{TL-TIIEHL-G}(x; \gamma, a, b, \zeta) = \left[1 - \left(\left[\frac{1 - [G(x; \zeta)]^\gamma}{1 + [G(x; \zeta)]^\gamma} \right]^a \right)^2 \right]^b = \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2 \right]^b \quad (2.1)$$

and

$$\begin{aligned} f_{TL-TIIEHL-G}(x; \gamma, a, b, \zeta) &= 4ab\gamma \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2 \right]^{b-1} \\ &\times \frac{(1 - [G(x; \zeta)]^\gamma)^{a-1} [G(x; \zeta)]^{\gamma-1} g(x; \zeta)}{(1 + [G(x; \zeta)]^\gamma)^{a+1}} \\ &\times ([K_G(x; \gamma, \zeta)]^a), \end{aligned} \quad (2.2)$$

respectively, for $a, b, \gamma > 0$ and parameter vector ζ , where $K_G(x; \gamma, \zeta) = \frac{1 - [\bar{G}(x; \zeta)]^\gamma}{1 + [\bar{G}(x; \zeta)]^\gamma}$. The hrf is given by

$$\begin{aligned} h_{TL-TIIEHL-G}(x; \gamma, a, b, \zeta) &= 4ab\gamma \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2 \right]^{b-1} \\ &\times \frac{(1 - [G(x; \zeta)]^\gamma)^{a-1} [G(x; \zeta)]^{\gamma-1} g(x; \zeta)}{(1 + [G(x; \zeta)]^\gamma)^{a+1}} \\ &\times [K_G(x; \gamma, \zeta)]^a \\ &\times \left(1 - \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2 \right]^b \right)^{-1}, \end{aligned} \quad (2.3)$$

for $x, a, b, \gamma > 0$ and baseline vector ζ .

Note that $\bar{G}(x; \zeta)$ is the baseline survival function and it is a fundamental tool in reliability analysis, survival analysis, and related fields, whereas $K_G(x; \gamma, \zeta)$ is part of the type II exponentiated half logistic-G distribution which was developed using the standard half logistic distribution. The range of x in this new family of distribution is from 0 to $+\infty$ and γ is a shape parameter which is greater than zero.

2.2.1 Sub-Families of the TL-TIIEHL-G Distribution

- When $b = 1$, we obtain a distribution with the cdf

$$F(x; \gamma, a, \zeta) = 1 - [K_G(x; \gamma, \zeta)]^{2a},$$

for $\gamma, a > 0$ and parameter vector ζ .

- If we let $a = 1$, we obtain the TL-TIHL-G family of distributions with the cdf

$$F(x; \gamma, b, \zeta) = \left[1 - (K_G(x; \gamma, \zeta))^2 \right]^b,$$

for $\gamma, b > 0$ and parameter vector ζ .

- By letting $a = b = 1$, we get a family of distributions with the cdf

$$F(x; \gamma, \zeta) = 1 - (K_G(x; \gamma, \zeta))^2,$$

for $\gamma > 0$ and parameter vector ζ .

- When $a = b = \gamma = 1$, we get a new family of distributions having the cdf

$$F(x; \zeta) = 1 - \left(\frac{\bar{G}(x; \zeta)}{1 + G(x; \zeta)} \right)^2,$$

for parameter vector ζ .

2.3 Some Statistical Properties

In this section, we present some of the statistical properties of the TL-TIIEHL-G family of distributions. The properties include, expansion of the density, quantile function, moments, generating function, probability weighted moments, stochastic ordering, distribution of order statistics and Rényi entropy.

2.3.1 Linear Representation of Density Function

In this subsection, we express the density function of the TL-TIIEHL-G family of distributions as a linear representation of exponentiated-G (Exp-G) densities by using the following expansions

$$(1+z)^{-(k+1)} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma((k+1)+j)}{\Gamma(k+1)j!} z^j, \quad \text{for } |z| < 1, \quad \text{and}$$

$$(1-z)^b = \sum_{s=0}^{\infty} \binom{b}{s} (-1)^s z^s, \quad \text{for } |z| < 1.$$

Using the generalized binomial expansion

$$\left[1 - ([K_G(x; \gamma, \zeta)]^a)^2\right]^{b-1} = \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i ([K_G(x; \gamma, \zeta)]^a)^{2i},$$

we can write equation (2.2) as

$$\begin{aligned} f(x; \gamma, a, b, \zeta) &= 4ab\gamma \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i ([K_G(x; \gamma, \zeta)]^a)^{2i+1} \\ &\times [K_G(x; \gamma, \zeta)]^{a-1} [G(x; \zeta)]^{\gamma-1} g(x; \zeta) \\ &= 4ab\gamma \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i [K_G(x; \gamma, \zeta)]^{a(2i+2)-1} \\ &\times [G(x; \zeta)]^{\gamma-1} g(x; \zeta) \\ &= 4ab\gamma \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i [1 - [G(x; \zeta)]^\gamma]^{a(2i+2)-1} \\ &\times \frac{g(x; \zeta) [G(x; \zeta)]^{\gamma-1}}{[1 + [G(x; \zeta)]^\gamma]^{a(2i+2)+1}}. \end{aligned}$$

Considering the generalized binomial expansions

$$[1 - [G(x; \zeta)]^\gamma]^{a(2i+2)-1} = \sum_{j=0}^{\infty} \binom{a(2i+2)-1}{j} (-1)^j [G(x; \zeta)]^{\gamma j},$$

and

$$[1 + [G(x; \zeta)]^\gamma]^{-(a(2i+2)+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(a(2i+2) + 1 + k)}{\Gamma(a(2i+2) + 1) k!} [G(x; \zeta)]^{\gamma k},$$

we get

$$\begin{aligned} f(x; \gamma, a, b, \zeta) &= 4ab\gamma \sum_{i,j,k=0}^{\infty} \binom{b-1}{i} \binom{a(2i+2)-1}{j} (-1)^{i+j+k} \frac{\Gamma(a(2i+2) + 1 + k)}{\Gamma(a(2i+2) + 1) k!} \\ &\times [G(x; \zeta)]^{\gamma(j+k+1)-1} g(x; \zeta). \end{aligned}$$

Furthermore, noting that

$$[G(x; \zeta)]^{\gamma(j+k+1)-1} = [1 - \overline{G}(x; \zeta)]^{\gamma(j+k+1)-1} = \sum_{l=0}^{\infty} (-1)^l \binom{\gamma(j+k+1)-1}{l} [\overline{G}(x; \zeta)]^l$$

and

$$[1 - G(x; \zeta)]^l = \sum_{p=0}^{\infty} (-1)^p \binom{l}{p} [G(x; \zeta)]^p,$$

the pdf of the TL-TIIEHL-G family of distributions can be written as

$$\begin{aligned} f(x; \gamma, a, b, \zeta) &= 4ab\gamma \sum_{i,j,k,l,p=0}^{\infty} \binom{b-1}{i} \binom{a(2i+2)-1}{j} (-1)^{i+j+k+l+p} \frac{\Gamma(a(2i+2) + 1 + k)}{\Gamma(a(2i+2) + 1) k!} \\ &\times \binom{\gamma(j+k+1)-1}{l} \binom{l}{p} \left(\frac{p+1}{p+1} \right) [G(x; \zeta)]^p g(x; \zeta) \\ &= \sum_{p=0}^{\infty} t_{p+1} h_{p+1}(x; \zeta), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} t_{p+1} &= 4ab\gamma \sum_{i,j,k,l=0}^{\infty} \binom{b-1}{i} \binom{a(2i+2)-1}{j} (-1)^{i+j+k+l+p} \frac{\Gamma(a(2i+2) + 1 + k)}{\Gamma(a(2i+2) + 1) k!} \\ &\times \binom{\gamma(j+k+1)-1}{l} \binom{l}{p} \frac{1}{p+1}, \end{aligned} \tag{2.5}$$

and $h_p(x; \zeta) = (p+1)[G(x; \zeta)]^p g(x; \zeta)$ is the Exp-G density (Gupta et al. (1998)) with power parameter $(p+1)$ and parameter vector ζ . It follows from the results that statistical properties of the TL-TIIEHL-G family of distributions can be directly obtained from those of the Exp-G distribution. The exponentiated-G distribution in this paper refers to the exponentiated type distribution first defined by Gupta et al. (1998), which was further generalized by Cordeiro et al. (2013), as opposed to the distribution defined by Barreto-Souza and Simas (2013). It is worth

noting that a slightly different generalized family of distributions in the literature is also called Exp-G distribution (see Barreto-Souza and Simas (2013) for additional details).

2.3.2 Quantile Function

The quantile function of the TL-TIIEHL-G family of distributions can be obtained by inverting the non-linear equation

$$F(x; \gamma, a, b, \zeta) = \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2 \right]^b = u, \quad \text{for } 0 \leq u \leq 1.$$

This can be written as

$$(1 - u^{\frac{1}{b}})^{\frac{1}{2}} = [K_G(x; \gamma, \zeta)]^a,$$

which simplifies to

$$K_G(x; \gamma, \zeta) = [(1 - u^{\frac{1}{b}})^{\frac{1}{2}}]^{\frac{1}{a}}.$$

Therefore, the quantile function of the TL-TIIEHL-G family of distributions is

$$Q_G(u; \gamma, a, b, \zeta) = G^{-1} \left(\frac{1 - [(1 - u^{\frac{1}{b}})^{\frac{1}{2}}]^{\frac{1}{a}}}{1 + [(1 - u^{\frac{1}{b}})^{\frac{1}{2}}]^{\frac{1}{a}}} \right)^{\frac{1}{\gamma}}. \quad (2.6)$$

2.3.3 Moments and Generating Function

In this subsection, we present the moments and generating function of the TL-TIIEHL-G distribution. The s^{th} moment is obtained as

$$E(X^s) = \int_{-\infty}^{\infty} x^s f_{TL-TIIEHL-G}(x; \gamma, a, b, \zeta) dx = \sum_{p=0}^{\infty} t_{p+1} E(Y_{p+1}^s), \quad (2.7)$$

where $E(Y_{p+1}^s)$ is the s^{th} moment of Y_{p+1} which follows the Exp-G distribution with power parameter $(p+1)$ and t_{p+1} is defined in equation (2.5). The moment generating function (mgf) is given by

$$M_X(h) = E(e^{hX}) = \sum_{p=0}^{\infty} t_{p+1} E(e^{hY_{p+1}}),$$

where $E(e^{hY_{p+1}})$ is the mgf of the Exp-G density with power parameter $(p+1)$ and t_{p+1} is defined in equation (2.5).

2.3.4 Probability Weighted Moments

The PWMs of the TL-TIIEHL-G family of distributions are derived as follows. Note that,

$$\eta_{a,r} = E(X^a [F(X)]^r) = \int_{-\infty}^{\infty} x^a [F(x)]^r f(x) dx.$$

From equation (2.1) and equation (2.2), we can write

$$\begin{aligned} f(x)[F(x)]^r &= 4ab\gamma \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2\right]^{b-1} \\ &\times ([K_G(x; \gamma, \zeta)]^a) [K_G(x; \gamma, \zeta)]^{a-1} \\ &\times [G(x; \zeta)]^{\gamma-1} g(x; \zeta) \left(\left[1 - ([K_G(x; \gamma, \zeta)]^a)^2\right]^b\right)^r \\ &= 4ab\gamma \sum_{i=0}^{\infty} \binom{b(r+1)-1}{i} (-1)^i [K_G(x; \gamma, \zeta)]^{a(2i+2)-1} \\ &\times [G(x; \zeta)]^{\gamma-1} g(x; \zeta). \end{aligned}$$

Considering the generalized binomial expansions

$$[1 - [G(x; \zeta)]^\gamma]^{a(2i+2)-1} = \sum_{j=0}^{\infty} \binom{a(2i+2)-1}{j} (-1)^j [G(x; \zeta)]^{\gamma j},$$

and

$$[1 + (G(x; \zeta))^\gamma]^{-(a(2i+2)+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(a(2i+2)+1+k)}{\Gamma(a(2i+2)+1)k!} [G(x; \zeta)]^{\gamma k},$$

we get

$$\begin{aligned} f(x)[F(x)]^r &= 4ab\gamma \sum_{i,j,k=0}^{\infty} \binom{b(r+1)-1}{i} \binom{a(2i+2)-1}{j} (-1)^{i+j+k} \\ &\times \frac{\Gamma(a(2i+2)+1+k)}{\Gamma(a(2i+2)+1)k!} [G(x; \zeta)]^{\gamma(j+k+1)-1} g(x; \zeta). \end{aligned}$$

Furthermore, with the generalized binomial expressions

$$[G(x; \zeta)]^{\gamma(j+k+1)-1} = [1 - \overline{G}(x; \zeta)]^{\gamma(j+k+1)-1} = \sum_{l=0}^{\infty} (-1)^l \binom{\gamma(j+k+1)-1}{l} [\overline{G}(x; \zeta)]^l$$

and

$$[1 - G(x; \zeta)]^l = \sum_{p=0}^{\infty} (-1)^p \binom{l}{p} [G(x; \zeta)]^p,$$

we obtain

$$\begin{aligned}
 f(x)[F(x)]^r &= 4ab\gamma \sum_{i,j,k,l,p=0}^{\infty} \binom{b(r+1)-1}{i} \binom{a(2i+2)-1}{j} (-1)^{i+j+k+l+p} \\
 &\times \frac{\Gamma(a(2i+2)+1+k)}{\Gamma(a(2i+2)+1)k!} \binom{\gamma(j+k+1)-1}{l} \binom{l}{p} \frac{(p+1)}{(p+1)} \\
 &\times [G(x;\zeta)]^p g(x;\zeta) \\
 &= \sum_{p=0}^{\infty} V_{p+1} h_{p+1}(x;\zeta).
 \end{aligned}$$

Consequently, the PWMs of the TL-TIIEHL-G family of distributions is given by

$$\eta_{a,r} = \sum_{p=0}^{\infty} V_{p+1} \int_{-\infty}^{\infty} x^a h_{p+1}(x;\zeta) dx, \quad (2.8)$$

where $h_{p+1}(x;\zeta) = (p+1)[G(x;\zeta)]^p g(x;\zeta)$ is the Exp-G density with the power parameter $(p+1)$ and parameter vector ζ , and

$$\begin{aligned}
 V_{p+1} &= 4ab\gamma \sum_{i,j,k,l=0}^{\infty} \binom{b(r+1)-1}{i} \binom{a(2i+2)-1}{j} (-1)^{i+j+k+l+p} \\
 &\times \frac{\Gamma(a(2i+2)+1+k)}{\Gamma(a(2i+2)+1)k!(p+1)} \binom{\gamma(j+k+1)-1}{l} \binom{l}{p}.
 \end{aligned}$$

2.3.5 Stochastic Ordering

In this subsection, we present stochastic orders for the TL-TIIEHL-G family of distributions. Let T and Q be random variables with the cumulative distribution functions $F_T(r)$ and $F_Q(r)$, respectively, and $\bar{F}_T(r) = 1 - F_T(r)$ the survival function. Then T is stochastically smaller than Q if $\bar{F}_T(r) \leq \bar{F}_Q(r)$ for all r or $F_T(r) \geq F_Q(r)$ for all r , denoted by $T <_s Q$. Hazard rate order and likelihood ratio order are given by $T <_{hr} Q$ if $h_T(r) \geq h_Q(r)$ for all r and $T <_{lr} Q$ if $\frac{f_T(r)}{f_Q(r)}$ is decreasing in r , (Shaked and Shanthikumar (2007)). It is well known that $T <_{lr} Q \Rightarrow T <_{hr} Q \Rightarrow T <_s Q$. In probability theory, a sequence of random variables or probability distributions is said to be stochastically ordered if larger values in one sequence are more likely to occur than in another sequence.

Theorem 1. Let $X_1 \sim TL-TIIEHL-G(\gamma, a, b_1, \zeta)$ and $X_2 \sim TL-TIIEHL-G(\gamma, a, b_2, \zeta)$, respectively. If $b_1 \leq b_2$, then X_1 and X_2 are stochastically ordered.

Proof: The pdfs of X_1 and X_2 are:

$$\begin{aligned}
 f_1(x; \gamma, a, b_1, \zeta) &= 4ab_1\gamma \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2 \right]^{b_1-1} \\
 &\times ([K_G(x; \gamma, \zeta)]^a) \frac{[1 - [G(x; \zeta)]^\gamma]^{a-1}}{[1 + [G(x; \zeta)]^\gamma]^{a+1}} [G(x; \zeta)]^{\gamma-1} g(x; \zeta)
 \end{aligned}$$

and

$$f_2(x; \gamma, a, b_2, \zeta) = 4ab_2\gamma \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2\right]^{b_2-1} \\ \times ([K_G(x; \gamma, \zeta)]^a) \frac{[1 - [G(x; \zeta)]^\gamma]^{a-1}}{[1 + [G(x; \zeta)]^\gamma]^{a+1}} [G(x; \zeta)]^{\gamma-1} g(x; \zeta),$$

respectively. Note that the ratio of the pdfs is

$$\frac{f_1(x; \gamma, a, b_1, \zeta)}{f_2(x; \gamma, a, b_2, \zeta)} = \frac{b_1}{b_2} \left[1 - \left(\frac{[1 - [G(x; \zeta)]^\gamma]^a}{[1 + [G(x; \zeta)]^\gamma]}\right)^2\right]^{b_1-b_2}, \quad (2.9)$$

and upon differentiating equation (2.9), we obtain

$$\frac{d}{dx} \left(\frac{f_1(x; \gamma, a, b_1, \zeta)}{f_2(x; \gamma, a, b_2, \zeta)} \right) = 4a\gamma \frac{b_1}{b_2} (b_1 - b_2) \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2\right]^{b_1-b_2-1} \\ \times ([K_G(x; \gamma, \zeta)]^a) \frac{[1 - [G(x; \zeta)]^\gamma]^{a-1}}{[1 + [G(x; \zeta)]^\gamma]^{a+1}} \\ \times [G(x; \zeta)]^{\gamma-1} g(x; \zeta). \quad (2.10)$$

Clearly, equation (2.10) is ≤ 0 if $b_1 \leq b_2$. Therefore, $X_1 <_{lr} X_2$, $X_1 <_{hr} X_2$ and $X_1 <_s X_2$, and the random variables X_1 and X_2 are stochastically ordered.

2.3.6 Distribution of Order Statistics

Order Statistics play an important role in several areas of probability and statistics, including reliability, survival analysis, and selection of experiments. Order statistics provide valuable insights into extreme events, helping professionals across various fields to make informed decisions and design systems that can withstand rare but impactful occurrences. In today's world, where complex systems and uncertainties are inherent, the application of order statistics contributes to strong decision making and risk management.

Theorem 2. Let $X_1, X_2, \dots, X_n$ be iid random variables from the TL-TIIEHL-G family of distributions and $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ the corresponding order statistics.

The pdf of the i^{th} order statistic can be written as

$$f_{i:n}(x) = \frac{n!f(x)}{(i-1)!(n-i)!} \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} [F(x)]^{m+i-1}.$$

Proof: Now, from equations (2.1) and (2.2), we have

$$\begin{aligned}
 f(x)[F(x)]^{m+i-1} &= 4ab\gamma \left[1 - ([K_G(x; \gamma, \zeta)]^a)^2\right]^{b(m+i)-1} \\
 &\times ([K_G(x; \gamma, \zeta)]^a) \frac{[1 - [G(x; \zeta)]^\gamma]^{a-1}}{[1 + [G(x; \zeta)]^\gamma]^{a+1}} \\
 &\times [G(x; \zeta)]^{\gamma-1} g(x; \zeta).
 \end{aligned}$$

Applying the generalized binomial expansions and following the steps used in the expansion of the density, we obtain

$$\begin{aligned}
 f(x)[F(x)]^{m+i-1} &= 4ab\gamma \sum_{i,j,k,l,p=0}^{\infty} \binom{b(m+i)-1}{i} \binom{a(2i+2)-1}{j} (-1)^{i+j+k+l+p} \\
 &\times \frac{\Gamma(a(2i+2)+1+k)}{\Gamma(a(2i+2)+1)k!} \binom{\gamma(j+k+1)-1}{l} \binom{l}{p} \frac{(p+1)}{(p+1)} [G(x; \zeta)]^p g(x; \zeta) \\
 &= \sum_{p=0}^{\infty} w_{p+1} h_{p+1}(x; \zeta),
 \end{aligned}$$

where $h_{p+1}(x; \zeta) = (p+1)[G(x; \zeta)]^p g(x; \zeta)$ is the Exp-G density with power parameter $(p+1)$ and parameter vector ζ , and

$$\begin{aligned}
 w_{p+1} &= 4ab\gamma \sum_{i,j,k,l=0}^{\infty} \binom{b(m+i)-1}{i} \binom{a(2i+2)-1}{j} (-1)^{i+j+k+l+p} \\
 &\times \frac{\Gamma(a(2i+2)+1+k)}{\Gamma(a(2i+2)+1)k!(p+1)} \binom{\gamma(j+k+1)-1}{l} \binom{l}{p}.
 \end{aligned}$$

Therefore, the pdf of the i^{th} order statistic for the TL-TIIEHL-G family of distributions is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{m=0}^{n-i} \sum_{p=0}^{\infty} (-1)^m \binom{n-i}{m} w_{p+1} h_{p+1}(x; \zeta).$$

2.3.7 Rényi Entropy

In this subsection, we present Rényi entropy of the new family of distributions. Rényi entropy (Rényi (1961)) is a measure of variation of uncertainty, a mathematical concept that quantifies uncertainty and disorder within a distribution. This is important in today's world as it provides a quantitative measure of uncertainty, enabling more precise modeling and analysis in fields ranging from finance to environmental science. Understanding and quantifying uncertainty is key for making informed decisions in complex systems. The developed families of distributions, including their entropy measures, contribute to advancing statistical methodologies and enhancing our ability to model and interpret diverse real-world phenomena. Rényi entropy is defined

by

$$I_R(\delta) = (1 - \delta)^{-1} \log \left[\int_{-\infty}^{\infty} f^\delta(x) dx \right] \quad \text{for } \delta > 0 \quad \text{and} \quad \delta \neq 1.$$

From equation (2.2), we have

$$\begin{aligned} f^\delta(x) &= (4ab\gamma)^\delta \left[1 - \left(\left[\frac{1 - [G(x; \zeta)]^\gamma}{1 + [G(x; \zeta)]^\gamma} \right]^{a-1} \right)^a \right]^{2\delta(b-1)} \\ &\times \left(\left[\frac{1 - [G(x; \zeta)]^\gamma}{1 + [G(x; \zeta)]^\gamma} \right]^{a-1} \right)^\delta \left[\frac{1 - [G(x; \zeta)]^\gamma}{1 + [G(x; \zeta)]^\gamma} \right]^{\delta(a-1)} [G(x; \zeta)]^{\delta(\gamma-1)} g^\delta(x; \zeta) \\ &= (4ab\gamma)^\delta \sum_{i=0}^{\infty} \binom{\delta(b-1)}{i} (-1)^i \left(\left[\frac{1 - [G(x; \zeta)]^\gamma}{1 + [G(x; \zeta)]^\gamma} \right]^{a-1} \right)^{2i+\delta} \\ &\times \left[\frac{1 - [G(x; \zeta)]^\gamma}{1 + [G(x; \zeta)]^\gamma} \right]^{\delta(a-1)} [G(x; \zeta)]^{\delta(\gamma-1)} g^\delta(x; \zeta). \end{aligned}$$

Considering the generalized binomial expansions

$$[1 - [G(x; \zeta)]^\gamma]^{a(2i+2\delta)-\delta} = \sum_{j=0}^{\infty} \binom{a(2i+2\delta)-\delta}{j} (-1)^j [G(x; \zeta)]^{\gamma j},$$

and

$$[1 + [G(x; \zeta)]^\gamma]^{-(a(2i+2\delta)+\delta)} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(a(2i+2\delta) + \delta + k)}{\Gamma(a(2i+2\delta) + \delta) k!} [G(x; \zeta)]^{\gamma k},$$

we can write

$$\begin{aligned} f^\delta(x) &= (4ab\gamma)^\delta \sum_{i,j,k=0}^{\infty} \binom{\delta(b-1)}{i} \binom{a(2i+2\delta)-\delta}{j} \frac{\Gamma(a(2i+2\delta) + \delta + k)}{\Gamma(a(2i+2\delta) + \delta) k!} \\ &\times (-1)^{i+j+k} [G(x; \zeta)]^{\gamma(j+k+\delta)-\delta} g^\delta(x; \zeta) \\ &= (4ab\gamma)^\delta \sum_{i,j,k,l,p=0}^{\infty} \binom{\delta(b-1)}{i} \binom{a(2i+2\delta)-\delta}{j} \frac{\Gamma(a(2i+2\delta) + \delta + k)}{\Gamma(a(2i+2\delta) + \delta) k!} \\ &\times (-1)^{i+j+k+l+p} \binom{\gamma(j+k+\delta)-\delta}{l} \binom{l}{p} [G(x; \zeta)]^p g^\delta(x; \zeta). \end{aligned}$$

Consequently, Rényi entropy of the TL-TIIEHL-G distribution is expressed as

$$\begin{aligned}
 I_R(\delta) &= \frac{1}{1-\delta} \log \left[(4ab\gamma)^\delta \sum_{i,j,k,l,p=0}^{\infty} \binom{\delta(b-1)}{i} \binom{a(2i+2\delta)-\delta}{j} \right. \\
 &\times \frac{\Gamma(a(2i+2\delta)+\delta+k)}{\Gamma(a(2i+2\delta)+\delta)k!} (-1)^{i+j+k+l+p} \binom{\gamma(j+k+\delta)-\delta}{l} \binom{l}{p} \\
 &\times \left. \frac{1}{(\frac{p}{\delta}+1)^\delta} \int_0^\infty \left(\left[\left(\frac{p}{\delta} + 1 \right) [G(x;\zeta)]^{\frac{p}{\delta}} g(x;\zeta) \right]^\delta dx \right) \right] \\
 &= \frac{1}{1-\delta} \log \left[\sum_{p=0}^{\infty} d_p^* \exp((1-\delta)I_{REG}) \right], \quad \delta > 0 \quad \text{and} \quad \delta \neq 1,
 \end{aligned}$$

where

$$I_{REG} = \frac{1}{1-\delta} \log \left[\int_0^\infty \left(\left[\left(\frac{p}{\delta} + 1 \right) [G(x;\zeta)]^{\frac{p}{\delta}} g(x;\zeta) \right]^\delta dx \right) \right]$$

is the Rényi entropy of the Exp-G density with power parameter $(\frac{p}{\delta} + 1)$ and

$$\begin{aligned}
 d_p^* &= (4ab\gamma)^\delta \sum_{i,j,k,l=0}^{\infty} \binom{\delta(b-1)}{i} \binom{a(2i+2\delta)-\delta}{j} (-1)^{i+j+k+l+p} \\
 &\times \frac{\Gamma(a(2i+2\delta)+\delta+k)}{\Gamma(a(2i+2\delta)+\delta)k!} \binom{\gamma(j+k+\delta)-\delta}{l} \binom{l}{p} \frac{1}{(\frac{p}{\delta}+1)^\delta}.
 \end{aligned}$$

2.4 Maximum Likelihood Estimation

Let $X \sim TL - TIIHL - G(\gamma, a, b, \zeta)$ and $\Delta = (\gamma, a, b, \zeta)^T$ be the parameter vector, the log-likelihood function (ℓ_n) of a random sample of size n from the TL-TIIHL-G distribution is

$$\begin{aligned}
 \ell_n(\Delta) &= n \log(4\gamma ab) + (b-1) \sum_{i=1}^n \log \left[1 - \left(\left[\frac{1 - (G(x_i; \zeta))^\gamma}{1 + (G(x_i; \zeta))^\gamma} \right]^a \right)^2 \right] \\
 &+ \sum_{i=1}^n \log \left(\left[\frac{[G(x_i; \zeta)]^\gamma}{1 + [G(x_i; \zeta)]^\gamma} \right]^a \right) + (a-1) \sum_{i=1}^n \log[1 - (G(x_i; \zeta))^\gamma] \\
 &- (a+1) \sum_{i=1}^n \log[1 + (G(x_i; \zeta))^\gamma] + (\gamma-1) \sum_{i=1}^n \log[G(x_i; \zeta)] \\
 &+ \sum_{i=1}^n \log[g(x_i; \zeta)].
 \end{aligned} \tag{2.11}$$

The elements of the score vector $\Delta = (\gamma, a, b, \zeta)$ are given in the appendix.

The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial \gamma}, \frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \zeta})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The FIM given by $\mathbf{I}(\Delta) = [\mathbf{I}_{\theta_i, \theta_j}]_{(p+3) \times (p+3)} = E(-\frac{\partial^2 \ell_n}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, \dots, (p+3)$, can be numerically obtained by MATLAB or NLMIXED in SAS or R (Team (2010)) software. The

total Fisher information matrix $n\mathbf{I}(\Delta)$ can be approximated by

$$\mathbf{J}_n(\hat{\Delta}) \approx \left[-\frac{\partial^2 \ell_n}{\partial \theta_i \partial \theta_j} \Big|_{\Delta=\hat{\Delta}} \right]_{(p+3) \times (p+3)}, \quad i, j = 1, 2, \dots, (p+3). \quad (2.12)$$

2.5 Special Models

In this section, we present some special models of the TL-TIIEHL-G family of distributions. The baseline distributions considered are Weibull, Uniform and Lindley distributions, respectively.

2.5.1 Topp-Leone Type II Exponentiated Half Logistic-Weibull Distribution

Let the baseline distribution be the Weibull distribution with cdf and pdf given by $G(x; \lambda) = 1 - e^{-x^\lambda}$, and $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$, respectively for $x > 0$ and $\lambda > 0$. The cdf, pdf and hrf of the TL-TIIEHL-W distribution are given by

$$F_{TL-TIIEHL-W}(x; \gamma, a, b, \lambda) = \left[1 - ([K_1(x; \lambda, \gamma)]^a)^2 \right]^b,$$

$$\begin{aligned} f_{TL-TIIEHL-W}(x; \gamma, a, b, \lambda) &= 4ab\gamma \left[1 - ([K_1(x; \lambda, \gamma)]^a)^2 \right]^{b-1} \\ &\times \frac{(1 - [1 - e^{-x^\lambda}]^\gamma)^{a-1} [1 - e^{-x^\lambda}]^{\gamma-1} \lambda x^{\lambda-1} e^{-x^\lambda}}{(1 + [1 - e^{-x^\lambda}]^\gamma)^{a+1}} \\ &\times ([K_1(x; \lambda, \gamma)]^a), \end{aligned}$$

and

$$\begin{aligned} h_{TL-TIIEHL-W}(x; \gamma, a, b, \lambda) &= 4ab\gamma \left[1 - ([K_1(x; \lambda, \gamma)]^a)^2 \right]^{b-1} \\ &\times \frac{(1 - [1 - e^{-x^\lambda}]^\gamma)^{a-1} [1 - e^{-x^\lambda}]^{\gamma-1} \lambda x^{\lambda-1} e^{-x^\lambda}}{(1 + [1 - e^{-x^\lambda}]^\gamma)^{a+1}} \\ &\times ([K_1(x; \lambda, \gamma)]^a) \\ &\times \left(1 - \left[1 - ([K_1(x; \lambda, \gamma)]^a)^2 \right]^b \right)^{-1}, \end{aligned}$$

respectively, for $x, \gamma, a, b, \lambda > 0$, where $K_1(x; \lambda, \gamma) = \frac{1 - [1 - e^{-x^\lambda}]^\gamma}{1 + [1 - e^{-x^\lambda}]^\gamma}$.

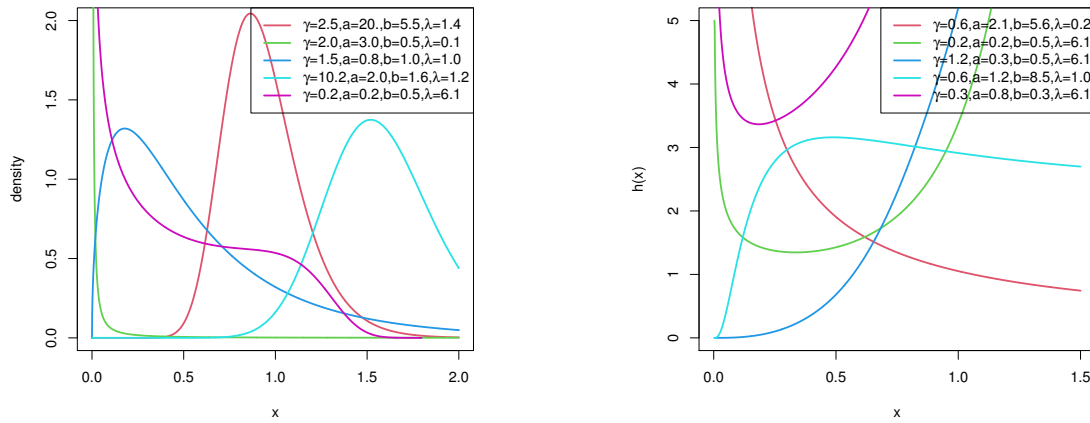


Figure 2.1: Plots of the pdf and hrf for the TL-TIIEHL-W distribution

The pdf of the TL-TIIEHL-W distribution can take several shapes including left-skewed, right-skewed, almost symmetric and reverse-J shapes, whereas the hrf displays bathtub, upside-down bathtub, increasing and decreasing shapes as indicated in Figure 2.1. Table 2.1 gives the table of quantiles for TL-TIIEHL-W distribution for.

Table 2.1: Some quantiles for TL-TIIEHL-W distribution

	(γ, a, b, λ)				
u	(1, 1.5, 1.3, 3.5)	(0.7, 1, 1.5, 1.5)	(2.4, 1, 0.6, 1)	(1.1, 0.5, 1.9, 1.4)	(1.1, 0.5, 1.9, 1.4)
0.1	0.3726	0.0697	0.1209	0.3492	0.3516
0.2	0.4448	0.1177	0.2061	0.4884	0.4439
0.3	0.4987	0.1652	0.2884	0.6134	0.5172
0.4	0.5457	0.2160	0.3741	0.7391	0.5843
0.5	0.5903	0.2730	0.4682	0.8741	0.6512
0.6	0.6357	0.3402	0.5771	1.0275	0.7219
0.7	0.6849	0.4243	0.7119	1.2131	0.8024
0.8	0.7435	0.5402	0.8967	1.4593	0.9032
0.9	0.8266	0.7341	1.2094	1.8519	1.0553

Figures 2.2 and 2.3 shows the plots of skewness and kurtosis for TL-TIIEHL-W distribution. From the plots, we see that:

- when the parameters b and λ are fixed, skewness and kurtosis of the TL-TIIEHL-W distribution increases as γ and a increase.

- when the parameters γ and a are fixed, skewness and kurtosis of the TL-TIIEHL-W distribution increases as b and λ increase.

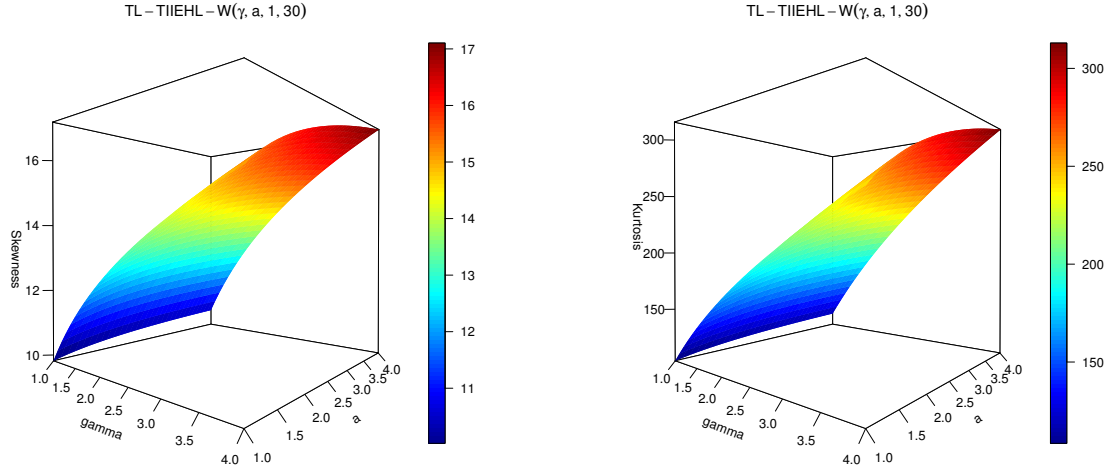


Figure 2.2: Skewness and kurtosis plots for TL-TIIEHL-W distribution

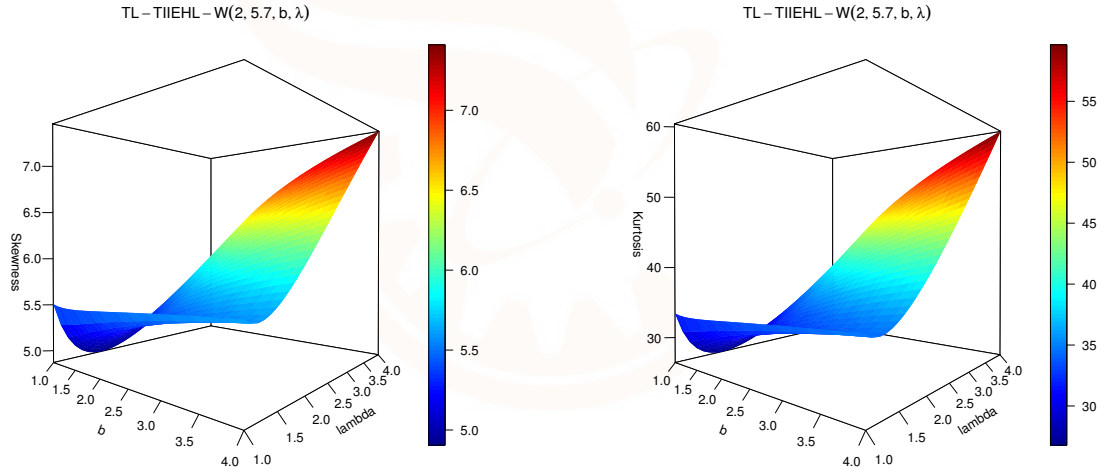


Figure 2.3: Skewness and kurtosis plots for TL-TIIEHL-W distribution

2.5.2 Topp-Leone Type II Exponentiated Half Logistic-Uniform Distribution

Suppose the baseline distribution is the uniform distribution with the cdf and pdf $G(x; \theta) = \frac{x}{\theta}$, and $g(x; \theta) = \frac{1}{\theta}$, respectively for $0 < x < \theta$. Therefore, we obtain the cdf, pdf and hrf of the TL-TIIEHL-U distribution as

$$F_{TL-TIIEHL-U}(x; \gamma, a, b, \theta) = \left[1 - \left(\left[\frac{1 - \left[\frac{x}{\theta} \right]^\gamma}{1 + \left[\frac{x}{\theta} \right]^\gamma} \right]^a \right)^2 \right]^b,$$

$$\begin{aligned}
 f_{TL-TIIEHL-U}(x; \gamma, a, b, \theta) &= 4ab\gamma \left[1 - \left(\left[\frac{1 - \left[\frac{x}{\theta} \right]^\gamma}{1 + \left[\frac{x}{\theta} \right]^\gamma} \right]^a \right)^2 \right]^{b-1} \\
 &\times \frac{(1 - \left[\frac{x}{\theta} \right]^\gamma)^{a-1} \left[\frac{x}{\theta} \right]^{\gamma-1} \frac{1}{\theta}}{(1 + \left[\frac{x}{\theta} \right]^\gamma)^{a+1}} \left(\left[\frac{1 - \left[\frac{x}{\theta} \right]^\gamma}{1 + \left[\frac{x}{\theta} \right]^\gamma} \right)^a \right)
 \end{aligned}$$

and

$$\begin{aligned}
 h_{TL-TIIEHL-U}(x; \gamma, a, b, \theta) &= 4ab\gamma \left[1 - \left(\left[\frac{1 - \left[\frac{x}{\theta} \right]^\gamma}{1 + \left[\frac{x}{\theta} \right]^\gamma} \right]^a \right)^2 \right]^{b-1} \\
 &\times \frac{(1 - \left[\frac{x}{\theta} \right]^\gamma)^{a-1} \left[\frac{x}{\theta} \right]^{\gamma-1} \frac{1}{\theta}}{(1 + \left[\frac{x}{\theta} \right]^\gamma)^{a+1}} \left(\left[\frac{1 - \left[\frac{x}{\theta} \right]^\gamma}{1 + \left[\frac{x}{\theta} \right]^\gamma} \right)^a \right) \\
 &\times \left(1 - \left[1 - \left(\left[\frac{1 - \left[\frac{x}{\theta} \right]^\gamma}{1 + \left[\frac{x}{\theta} \right]^\gamma} \right)^a \right]^2 \right)^b \right)^{-1},
 \end{aligned}$$

respectively, for $x, \gamma, a, b, \theta > 0$.

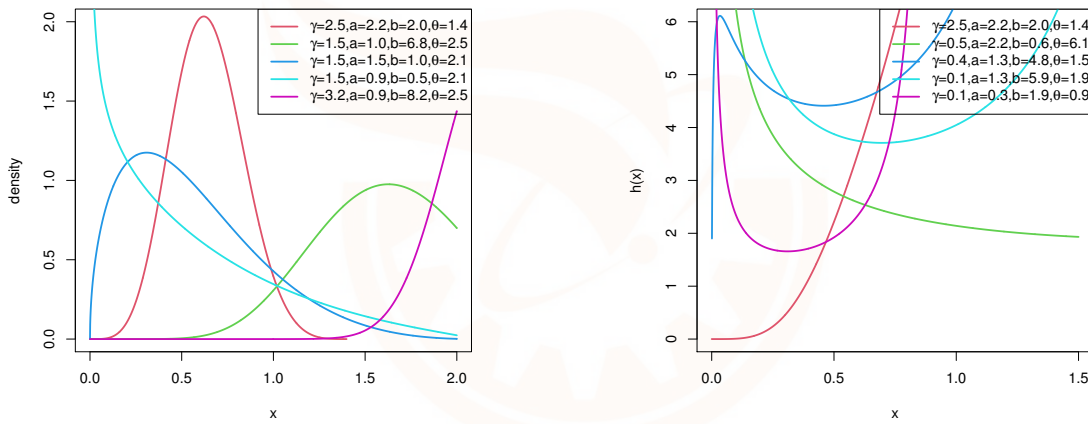


Figure 2.4: Plots of the pdf and hrf for the TL-TIIEHL-U distribution

Figure 2.4 gives the pdf and hrf of the TL-TIIEHL-U distribution. The pdf exhibit right-skewed, almost symmetric, J-shape, and left-skewed shapes. The hrf of the TL-TIIEHL-U distribution on the other hand displays upside-down bathtub followed by bathtub, bathtub, increasing and decreasing shapes. Table of quantiles for TL-TIIEHL-U distribution for selected parameter values is given in Table 2.2.

Table 2.2: Some quantiles for TL-TIIEHL-U distribution

(γ, a, b, θ)					
u	(1, 1.5, 1.3, 3.5)	(0.7, 1, 1.5, 1.5)	(2.4, 1, 0.6, 1)	(0.9, 1, 0.5, 2.5)	(1.5, 1, 1.2, 2)
0.1	0.1107	0.0277	0.1138	0.0098	0.2349
0.2	0.2013	0.0575	0.1862	0.0136	0.3558
0.3	0.2911	0.0972	0.2504	0.0366	0.4692
0.4	0.3974	0.1454	0.3123	0.0778	0.5782
0.5	0.5138	0.2009	0.3735	0.1358	0.6923
0.6	0.6477	0.2707	0.4377	0.2183	0.8116
0.7	0.8169	0.3607	0.5111	0.3424	0.9493
0.8	1.0461	0.4912	0.5931	0.5352	1.1152
0.9	1.4069	0.7002	0.7007	0.8853	1.3437

The 3D plots of skewness and kurtosis for TL-TIIEHL-U distribution are given in Figures 2.5 and 2.6. From the plots, we see that:

- when parameters a and θ are fixed, skewness and kurtosis of the TL-TIIEHL-U distribution decreases as γ and b increase.
- when parameters γ and a are fixed, skewness and kurtosis of the TL-TIIEHL-U distribution decreases as b and θ increase.

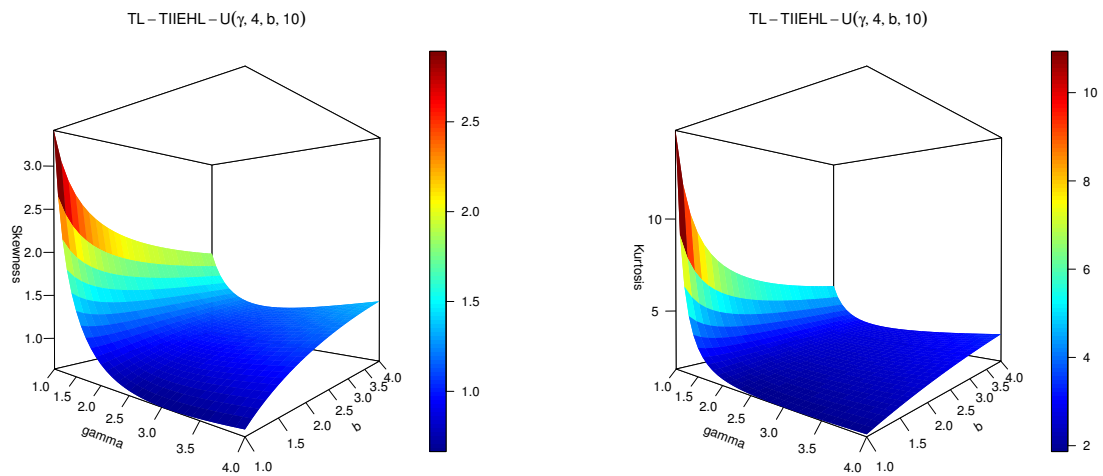


Figure 2.5: Skewness and kurtosis plots for TL-TIIEHL-U distribution

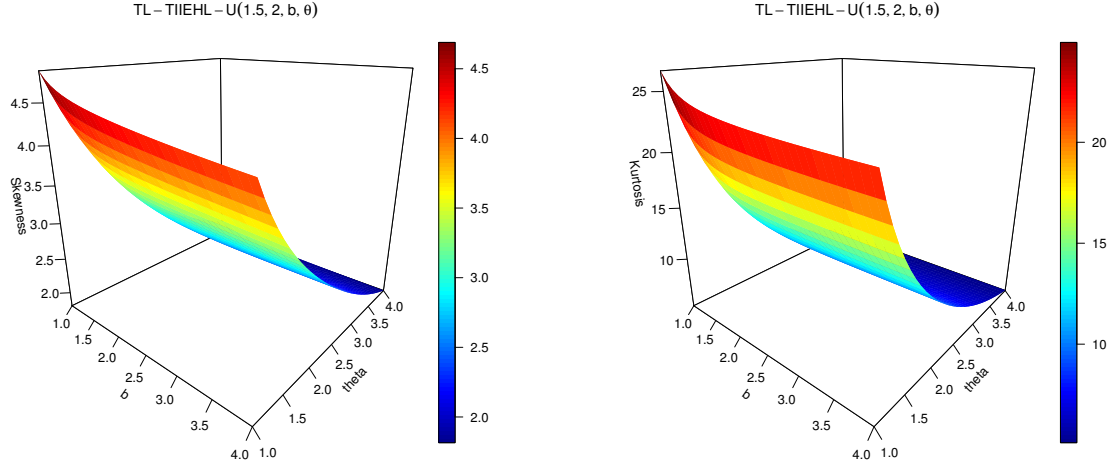


Figure 2.6: Skewness and kurtosis plots for TL-TIIEHL-U distribution

2.5.3 Topp-Leone Type II Exponentiated Half Logistic-Lindley Distribution

If we take the baseline distribution to be the Lindley distribution with the cdf and pdf given by $G(x; \theta) = 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}$, and $g(x; \theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$, respectively for $x > 0$ and $\theta > 0$, then the cdf, pdf and hrf of the TL-TIIEHL-L distribution are given by

$$F_{TL-TIIEHL-L}(x; \gamma, a, b, \theta) = \left[1 - \left(\left[\frac{1 - [1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}]^\gamma}{1 + [1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}]^\gamma} \right]^a \right)^2 \right]^b,$$

$$\begin{aligned} f_{TL-TIIEHL-L}(x; \gamma, a, b, \theta) &= 4ab\gamma \left[1 - \left(\left[\frac{1 - [1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}]^\gamma}{1 + [1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}]^\gamma} \right]^a \right)^2 \right]^{b-1} \\ &\times \frac{(1 - [1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}]^\gamma)^{a-1} [1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}]^{\gamma-1}}{(1 + [1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}]^\gamma)^{a+1}} \\ &\times \frac{\theta^2}{\theta+1} (1+x)e^{-\theta x} \left(\left[\frac{1 - [1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}]^\gamma}{1 + [1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}]^\gamma} \right]^a \right) \end{aligned}$$

and

$$\begin{aligned}
 h_{TL-TIIEHL-L}(x; \gamma, a, b, \theta) &= 4ab\gamma \left[1 - \left(\frac{1 - [1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}]^\gamma}{1 + [1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}]^\gamma} \right)^a \right]^{2b-1} \\
 &\times \frac{(1 - [1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}]^\gamma)^{a-1} [1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}]^{\gamma-1}}{(1 + [1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}]^\gamma)^{a+1}} \\
 &\times \frac{\theta^2}{\theta+1} (1+x) e^{-\theta x} \left(\frac{1 - [1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}]^\gamma}{1 + [1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}]^\gamma} \right)^a \\
 &\times \left(1 - \left[1 - \left(\frac{1 - [1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}]^\gamma}{1 + [1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}]^\gamma} \right)^a \right]^b \right)^{-1},
 \end{aligned}$$

respectively, for $x, \gamma, a, b, \theta > 0$.

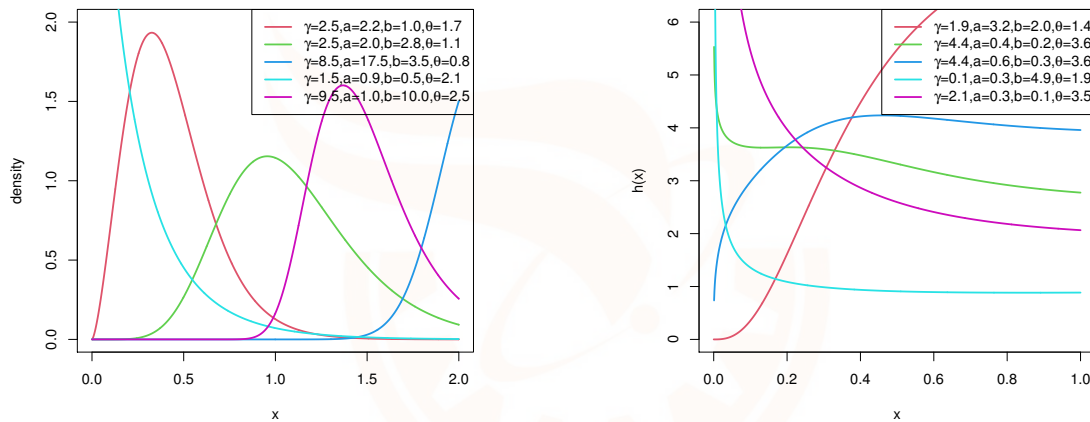


Figure 2.7: Plots of the pdf and hrf for the TL-TIIEHL-L distribution

The pdf of the TL-TIIEHL-L distribution can take several shapes including right-skewed, almost symmetric, J-shape and reverse-J shape, whereas the hrf displays upside-down bathtub, increasing and decreasing shapes as indicated in Figure 2.7. Quantiles for the TL-TIIEHL-L distribution for selected parameter values is given in Table 2.3.

Table 2.3: Some quantiles for TL-TIIEHL-L distribution

	(γ, a, b, θ)				
u	(1, 1.5, 1.3, 3.5)	(0.7, 1, 1.5, 1.5)	(2.4, 1, 0.6, 1)	(2.1, 0.5, 1.9, 2.4)	(1.5, 1, 1.2, 2)
0.1	0.0140	0.0208	0.2300	0.3249	0.0912
0.2	0.0219	0.0435	0.3799	0.4347	0.1443
0.3	0.0326	0.0731	0.5192	0.5313	0.1958
0.4	0.0433	0.1116	0.6597	0.6297	0.2457
0.5	0.0565	0.1545	0.8067	0.7342	0.3049
0.6	0.0750	0.2138	0.9716	0.8581	0.3731
0.7	0.0970	0.2925	1.1735	1.0098	0.4541
0.8	0.1262	0.4095	1.4401	1.2135	0.5704
0.9	0.1831	0.6366	1.8690	1.5541	0.7618

Figures 2.8 and 2.9 shows the 3D plots of skewness and kurtosis for TL-TIIEHL-L distribution. From the plots, we see that:

- by fixing the parameters b and θ , skewness and kurtosis of the TL-TIIEHL-L distribution decreases as γ and a increase.
- by fixing the parameters γ and a , skewness and kurtosis of the TL-TIIEHL-L distribution increases as b and θ increase.

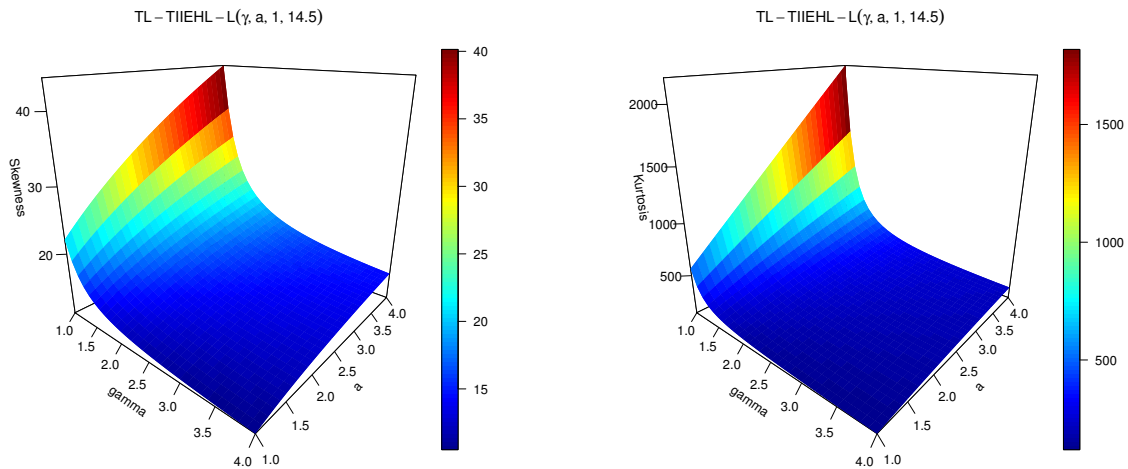


Figure 2.8: Skewness and kurtosis plots for TL-TIIEHL-L distribution

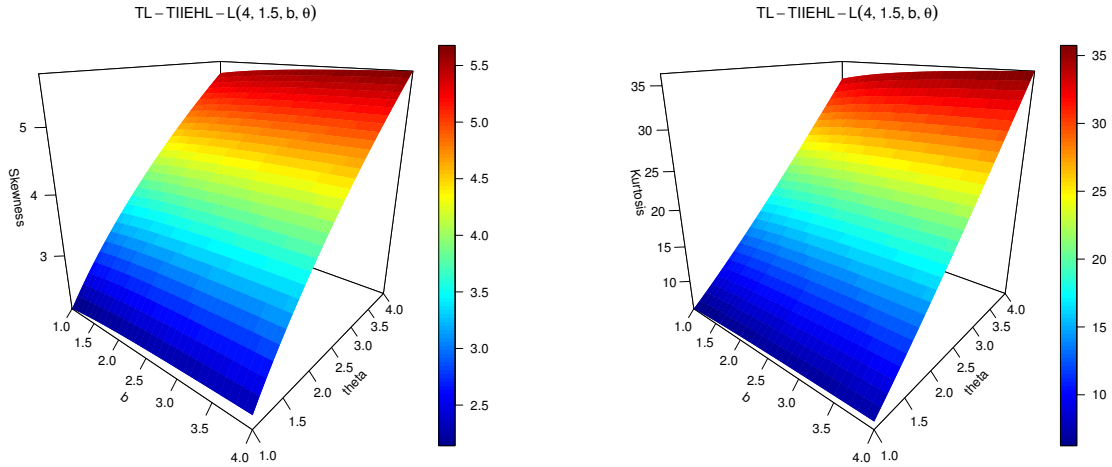


Figure 2.9: Skewness and kurtosis plots for TL-TIIEHL-L distribution

2.6 Simulation Results

In this section, we present the results of a simulation study that was done in order to examine the performance of maximum likelihood estimators of the parameters of the TL-TIIEHL-G family of distributions. We performed various simulations for different sample sizes and for different parameter values. Equation (2.6) was used to generate random samples from TL-TIIEHL-G family of distributions via the bbmle (Bolker and Bolker (2017)) package in R when the baseline cdfs are Weibull and Lindley distributions, respectively. The simulation were repeated for $N = 1000$ times each with sample sizes $n = 50, 100, 200, 400, 800, 1200$ and parameter values as given in Tables 2.4 and 2.5, respectively.

Specifically, the estimated mean, ABIAS and RMSEs of the parameter say, $\hat{\zeta}$, are computed as $\text{Mean} = \frac{\sum_{i=1}^N \hat{\zeta}_i}{N}$, $\text{RMSE} = \sqrt{\frac{\sum_{i=1}^N (\hat{\zeta}_i - \zeta)^2}{N}}$ and $\text{ABIAS}(\hat{\zeta}) = \frac{\sum_{i=1}^N (\hat{\zeta}_i - \zeta)}{N}$, respectively.

Tables 2.4 and 2.5 list the mean MLEs of the parameters along with the respective RMSEs and ABIAS. From the tables, we see that as n increases, the mean estimates of the parameters approximate the true parameter values while the RMSEs decrease and ABIAS decay towards zero. Hence, we can conclude that this model gives consistent estimates.

Table 2.4: Simulation Results for TL-TIIEHL-W Distribution: Mean, ABIAS and RMSE

Parameter	n	(1.0, 0.1, 1.0, 2.0)				(0.5, 0.2, 1.0, 2.0)				(1.0, 0.1, 0.5, 2.0)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
γ	50	0.4389	0.3297	-0.0610	0.0610	0.4669	0.3914	-0.0330	0.0330	0.7662	0.6627	-0.2338	0.2338
	100	0.4302	0.2579	-0.0698	0.0698	0.5072	0.3822	0.0072	0.0072	0.8272	0.5272	-0.1728	0.1728
	200	0.4671	0.1759	-0.0329	0.0329	0.5233	0.3196	0.0233	0.0233	0.1908	0.4446	-0.1784	0.1784
	400	0.4825	0.1292	-0.0175	0.0175	0.4928	0.2278	-0.0072	0.0072	0.8847	0.2942	-0.1153	0.1153
	800	0.5129	0.0574	0.0129	0.0129	0.5067	0.2017	0.0067	0.0067	0.9061	0.2147	-0.0939	0.0939
	1200	0.5096	0.0556	0.0096	0.0096	0.5497	0.1794	0.0047	0.0047	1.0161	0.1674	0.0161	0.0161
a	50	0.1613	0.1017	0.0613	0.0613	0.2733	0.2525	0.0733	0.0733	0.2158	0.1832	0.1158	0.1158
	100	0.1491	0.0837	0.0491	0.0491	0.2331	0.1055	0.0331	0.0331	0.1804	0.1312	0.0804	0.0804
	200	0.1287	0.0569	0.0287	0.0287	0.2229	0.0886	0.0228	0.0228	0.1678	0.1063	0.0678	0.0678
	400	0.1155	0.0356	0.0155	0.0155	0.2237	0.0716	0.0237	0.0237	0.1408	0.0737	0.0408	0.0408
	800	0.1028	0.0119	0.0028	0.0028	0.2155	0.0608	0.0155	0.0155	0.1243	0.0510	0.0243	0.0243
	1200	0.1001	0.0098	0.0001	0.0001	0.1990	0.0462	-0.0009	0.0009	0.1076	0.0259	0.0076	0.0076
b	50	3.8057	8.8483	2.8057	2.8057	3.8768	7.4948	2.8768	2.8768	2.6115	5.5771	2.1115	2.1115
	100	2.4729	3.9925	1.4729	1.4729	2.5389	4.1212	1.5389	1.5389	1.5671	2.5695	1.0671	1.0671
	200	1.5228	1.4851	0.5228	0.5228	1.9107	3.1424	0.9107	0.9107	1.2106	1.4773	0.7106	0.7106
	400	1.2152	0.5665	0.2152	0.2152	1.5951	1.5459	0.5951	0.5951	0.8145	0.7157	0.3145	0.3145
	800	1.0175	0.1208	0.0175	0.0175	1.3741	1.0420	0.3741	0.3741	0.6692	0.4176	0.1692	0.1692
	1200	1.0049	0.1293	0.0049	0.0049	1.0887	0.5681	0.0887	0.0887	0.5339	0.1519	0.0339	0.0339
λ	50	1.8085	0.3838	-0.1915	0.1915	1.9533	0.5547	-0.04674	0.04674	1.7212	0.5448	-0.2788	0.2788
	100	1.8377	0.3333	-0.1624	0.1624	2.0037	0.4591	0.0037	0.0037	1.7867	0.4584	-0.2133	0.2133
	200	1.8892	0.2465	-0.1108	0.1108	2.0022	0.3891	0.0022	0.0022	1.7702	0.3819	-0.2298	0.2298
	400	1.9355	0.1687	-0.0645	0.0645	1.9503	0.2664	-0.0496	0.0496	1.8458	0.2884	-0.1542	0.1542
	800	1.9809	0.0673	-0.0191	0.0191	1.9766	0.2456	-0.0234	0.0234	1.9074	0.2158	-0.0926	0.0926
	1200	1.9981	0.0553	-0.0019	0.0019	2.0312	0.2049	0.0312	0.0312	1.9774	0.1367	-0.0226	0.0226

Table 2.5: Simulation Results for TL-TIIEHL-L Distribution: Mean, ABIAS and RMSE

Parameter	n	(1.2, 1.2, 1.2, 0.1)				(1.0, 0.1, 1.0, 0.5)				(1.0, 0.3, 0.6, 0.2)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
γ	50	2.3405	2.5896	1.1405	1.1405	1.1243	1.0052	0.1243	0.1243	1.7828	1.7529	0.7828	0.7828
	100	1.9171	1.6579	0.7171	0.7171	1.0819	0.6787	0.0819	0.0819	1.6238	1.3319	0.6238	0.6238
	200	1.489	0.9416	0.2886	0.2886	0.9817	0.4297	-0.0183	0.0183	1.4916	1.0681	0.4916	0.4916
	400	1.3552	0.5953	0.1552	0.1552	0.9432	0.2797	-0.0568	0.0568	1.3316	0.7743	0.3316	0.3316
	800	1.3123	0.5615	0.1123	0.1123	0.9822	0.1548	-0.0178	0.0178	1.1311	0.4984	0.1311	0.1311
	1200	1.3189	0.5373	0.1189	0.1189	0.9931	0.1366	-0.0069	0.0069	1.0915	0.4262	0.0915	0.0915
a	50	1.7678	1.8606	0.5678	0.5678	0.1632	0.1206	0.0632	0.0632	0.5641	1.6113	0.2641	0.2641
	100	1.496	1.3286	0.2961	0.2961	0.1687	0.1224	0.0687	0.0687	0.4042	0.9333	0.1042	0.1042
	200	1.3426	0.7946	0.1426	0.1426	0.1613	0.1135	0.0613	0.0613	0.3434	0.4696	0.0434	0.0434
	400	1.2047	0.4706	0.0047	0.0047	0.1501	0.1012	0.0501	0.0501	0.33461	0.3544	0.0346	0.0346
	800	1.2315	0.3429	0.0315	0.0315	0.1188	0.0537	0.0188	0.0188	0.3359	0.2899	0.0359	0.0359
	1200	1.2452	0.3228	0.0452	0.0452	0.1129	0.0416	0.0129	0.0129	0.3169	0.2073	0.0169	0.0169
b	50	1.1208	0.8383	-0.0792	0.0792	1.9527	3.9995	0.9527	0.9527	0.7386	0.7183	0.1386	0.1386
	100	1.1818	0.8002	-0.0182	0.0182	1.5026	2.1691	0.5026	0.5026	0.7241	0.6348	0.1241	0.1241
	200	1.2507	0.7339	0.0507	0.0507	1.4793	2.3171	0.4793	0.4793	0.6514	0.4043	0.0514	0.0514
	400	1.2499	0.5955	0.0499	0.0499	1.2701	1.2450	0.2701	0.2701	0.6361	0.3146	0.0361	0.0361
	800	1.2019	0.3670	0.0019	0.0019	1.0498	0.1973	0.0498	0.0498	0.6132	0.2209	0.0132	0.0132
	1200	1.1597	0.2333	-0.0403	0.0403	1.0267	0.1440	0.0267	0.0267	0.6007	0.1924	0.0007	0.0007
θ	50	0.1848	0.2440	0.0848	0.0848	0.4427	0.2379	-0.0573	0.0573	0.2672	0.3165	0.0672	0.0672
	100	0.1664	0.1849	0.0664	0.0664	0.3972	0.2029	-0.1028	0.1028	0.2411	0.2175	0.0411	0.0411
	200	0.1298	0.1141	0.0298	0.0298	0.4043	0.1836	-0.0957	0.0957	0.2209	0.1317	0.0209	0.0209
	400	0.1218	0.0774	0.0218	0.0218	0.4095	0.1642	-0.0905	0.0905	0.2074	0.0805	0.0074	0.0074
	800	0.1054	0.0346	0.0054	0.0054	0.4593	0.1041	-0.0407	0.0407	0.2060	0.0602	0.0060	0.0060
	1200	0.1021	0.0223	0.0021	0.0021	0.4674	0.0885	-0.0326	0.0326	0.2052	0.0523	0.0052	0.0052

2.7 Applications

We present two examples to show the flexibility and applicability of the TL-TIIEHL-G family of distributions in this subsection. The TL-TIIEHL-W distribution is compared with other existing models including the odd log-logistic exponentiated Weibull (OLLEW) distribution by Afify et al. (2018), the Beta odd Lindley-exponential (BOL-E) distribution by Chipepa et al. (2019), Weibull-Lomax (WL) distribution by Tahir et al. (2014), Kumaraswamy Weibull (KW) distribution by Cordeiro et al. (2010) and Generalized Weibull-Log-logistic (GW-LLo) distribution by Cordeiro et al. (2015). The pdf's of the four parameter distributions are

$$f_{OLLEW}(x; \alpha, \beta, \gamma, \delta) = \frac{\theta \beta \gamma x^{\beta-1} e^{-(x/\alpha)^\beta} [1 - e^{-(x/\alpha)^\beta}]^{\gamma\theta-1} \{1 - [1 - e^{-(x/\alpha)^\beta}]^\gamma\}^{\theta-1}}{\alpha^\beta \{ [1 - e^{-(x/\alpha)^\beta}]^{\gamma\theta} + \{1 - [1 - e^{-(x/\alpha)^\beta}]^\gamma\}^\theta \}^2},$$

for $x > 0, \alpha, \beta, \gamma, \delta > 0$,

$$\begin{aligned} f_{BOL-E}(x; a, b, \lambda, \theta) &= \frac{1}{B(a, b)} \left[1 - \frac{\lambda + e^{-\theta x}}{(1 + \lambda)e^{-\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\} \right]^{a-1} \\ &\times \left[\frac{\lambda + e^{-\theta x}}{(1 + \lambda)e^{-\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\} \right]^{b-1} \\ &\times \frac{\lambda^2}{(1 + \lambda)} \frac{(\theta e^{-\theta x})}{e^{-3\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\}, \end{aligned}$$

for $x > 0, a, b, \lambda, \theta > 0$,

$$f_{WL}(x; a, b, \alpha, \beta) = \frac{ab\alpha}{\beta} (1 + (x/\beta))^{b\alpha-1} (1 - (1 + (x/\beta)))^{b-1} \exp\{-a[(1 + (x/\beta))^\alpha - 1]^b\},$$

for $x > 0, a, b, \alpha, \beta > 0$,

$$f_{KW}(x; a, b, c, \lambda) = abc\lambda^c x^{c-1} \exp -(\lambda x)^c [1 - \exp(\lambda x)^c]^{a-1} \{1 - [1 - \exp -(\lambda x)^c]^a\}^{b-1},$$

for $x > 0, a, b, c, \lambda > 0$ and

$$\begin{aligned} f_{GW-LLo}(x; \alpha, \beta, a, \theta) &= \frac{\alpha \beta \theta x^{\theta-1}}{a^\theta} \exp \left\{ -\alpha \left[\log \left(1 + \left(\frac{x}{a} \right)^\theta \right) \right]^\beta \right\} \left(1 + \left(\frac{x}{a} \right)^\theta \right)^{-1} \\ &\times \left[\log \left(1 + \left(\frac{x}{a} \right)^\theta \right) \right]^{\beta-1}, \end{aligned}$$

for $x > 0, \alpha, \beta, a, \theta > 0$. Maximum likelihood estimates of the parameters of the TL-TIIEHL-W distribution with (standard errors in parentheses) are presented in Tables 2.6 and 2.7, respectively. The goodness-of-fit statistics including -2logL statistic, AIC, AICC, BIC, (W^*), (A^*) statistic, K-S statistic and its p-value are also presented in the tables.

Plots of the fitted pdfs, the histogram of the data and probability plots (Chambers et al. (1983)) are presented in Figures 2.11 and 2.14. For the probability plots, we plotted $F(x_{(j)}; \hat{\gamma}, \hat{a}, \hat{b}, \hat{\lambda})$

against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness to the diagonal line is given by the sum of squares:

$$SS = \sum_{j=1}^n \left[F(x_{(j)}; \hat{\gamma}, \hat{a}, \hat{b}, \hat{\lambda}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

2.7.1 Prices (in dollars) of 428 New Vehicles Data

The first 428 data points correspond to prices (in dollars) of 428 new vehicles for the 2004 year (Kiplinger's Personal Finance, Dec 2003). The data are listed in the appendix.

The estimated variance-covariance matrix of TL-TIIEHL-W model for the new vehicles data is given by

$$\begin{bmatrix} 1.6927 \times 10^3 & -2.3942 \times 10^{-04} & 1.0318 \times 10^{-05} & 1.3966 \times 10^{-04} \\ -2.3942 \times 10^{-04} & 3.3866 \times 10^{-05} & -1.4594 \times 10^{-06} & -1.9728 \times 10^{-05} \\ 1.0318 \times 10^{-05} & -1.4594 \times 10^{-06} & 6.2893 \times 10^{-08} & 8.5244 \times 10^{-07} \\ 1.3966 \times 10^{-04} & -1.9728 \times 10^{-05} & 8.5244 \times 10^{-07} & 1.9822 \times 10^{-05} \end{bmatrix}$$

and the approximate 95% confidence interval are given by $\gamma \in (6.4603, 6.6215)$, $a \in (15.2576, 15.2804)$, $b \in (108.9095, 108.9105)$ and $\lambda \in (0.1276, 0.1451)$.

The estimated variance measures the extent to which each estimated parameter deviates from its mean, indicating the level of uncertainty in the parameter estimates. The estimated covariance matrix shows the relationship between pairs of estimated parameters, revealing how changes in one correspond to changes in another. Variance-covariance structure helps in assessing model precision, identifiability, and consistency across datasets. This analysis adds depth to understanding the behavior of the distribution and evaluating the reliability of the model.

Table 2.6: Estimates of models for prices of 428 new vehicles data

Model	Estimates				Statistics								
	γ	a	b	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value	SS
TL-TIIEHL-W	$6.5409 \times 10^{+00}$ (4.1143×10^{-02})	$1.5269 \times 10^{+01}$ (5.8195×10^{-03})	$1.0891 \times 10^{+02}$ (2.5079×10^{-04})	1.3636×10^{-01} (4.4523×10^{-03})	1488.9	1496.9	1496.9	1513.1	0.0335	0.2627	0.0212	0.9907	0.0334
OLLEW	α 0.0263 (0.0103)	β 0.2503 (0.0291)	γ 17.0509 (3.9205)	θ 3.1452 (0.5549)	1503.4	1511.4	1511.5	1527.6	0.0945	0.7467	0.0256	0.9415	0.0448
BOL-E	a $4.0847 \times 10^{+00}$ (3.4617×10^{-09})	b $1.5045 \times 10^{+01}$ (9.7052×10^{-10})	λ $9.3264 \times 10^{+01}$ (1.7265×10^{-10})	θ 8.1676×10^{-04} (1.9549×10^{-05})	1555.2	1563.2	1563.3	1579.3	0.6444	4.2958	0.0688	0.0348	0.6975
WL	a $2.1787 \times 10^{+03}$ (8.6543×10^{-04})	b $4.0853 \times 10^{+00}$ ($1.6851 \times 10^{+00}$)	α 7.4109×10^{-02} (4.3511×10^{-03})	β 6.2112×10^{-01} (9.6157×10^{-01})	1566.4	1574.4	1574.5	1590.7	0.7169	4.7396	0.0715	0.0251	0.8312
KW	a 48.8501 (26.1532)	b 0.4339 (0.1867)	c 0.7762 (0.1694)	λ 3.1342 (1.3689)	1490.4	1498.4	1498.5	1514.6	0.0432	0.3324	0.0232	0.9750	0.0425
GW-LLo	α 0.0014 (0.0005)	β 2.6248 (0.1173)	a 0.8952 (0.0463)	θ 9.2255 (0.0075)	1489.7	1497.7	1497.8	1513.9	0.0595	0.4339	0.0302	0.8308	0.0871

From Table 2.6, we can see that the values of the goodness-of-fit statistics: AIC, AICC and BIC are smallest for the TL-TIIEHL-W distribution as compared to the non-nested OLLEW, BOL-E, WL, KW and GW-LLo distributions. The value of SS from the probability plots and the values of the goodness-of-fit statistics A^* , W^* and K-S are smallest for the TL-TIIEHL-W distribution, which indeed showed that this model is a better fit for the new vehicles data. The value of K-S p-value is closer to 1. Profile log-likelihood plots for new vehicles data are given in Figure 2.10. The plots confirms that maximum likelihood estimates of the parameters γ , a , b and λ exist and are unique.

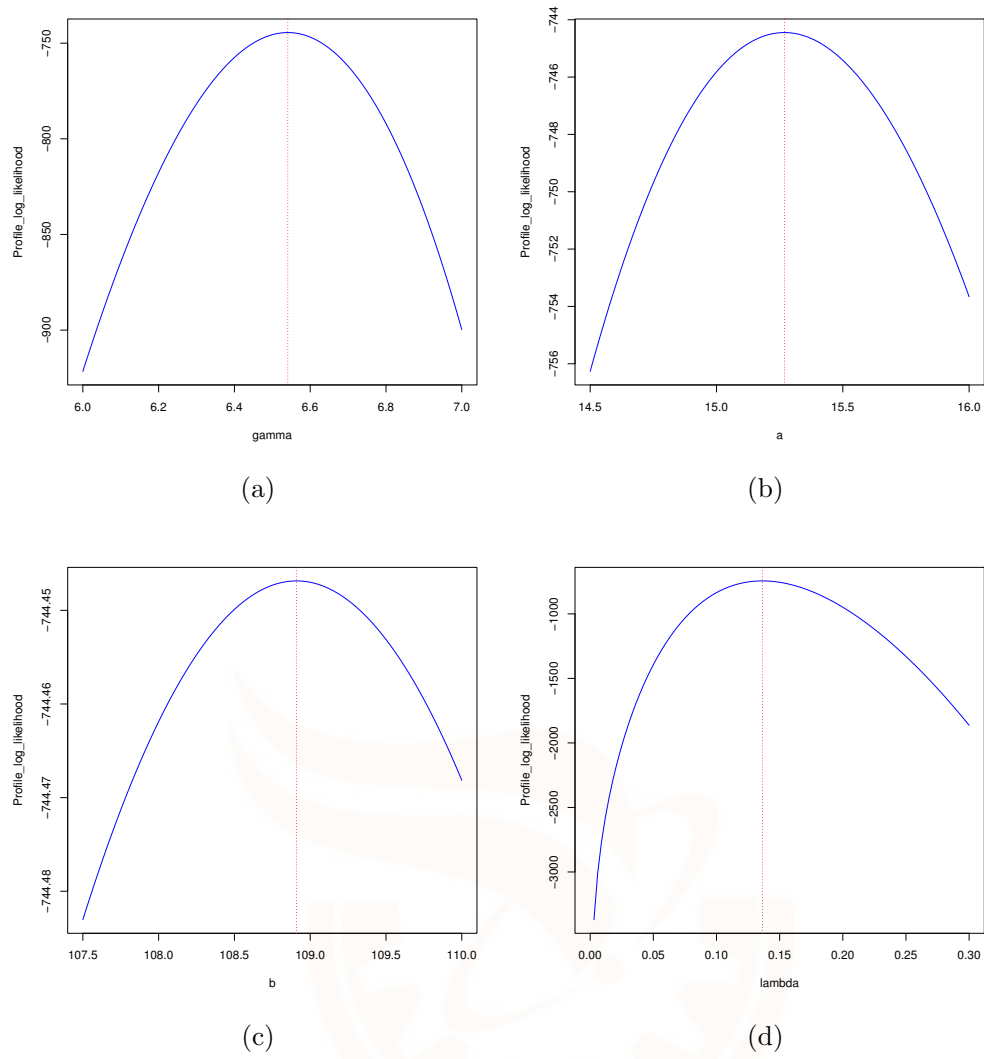


Figure 2.10: Profile log-likelihood plots for new vehicles data

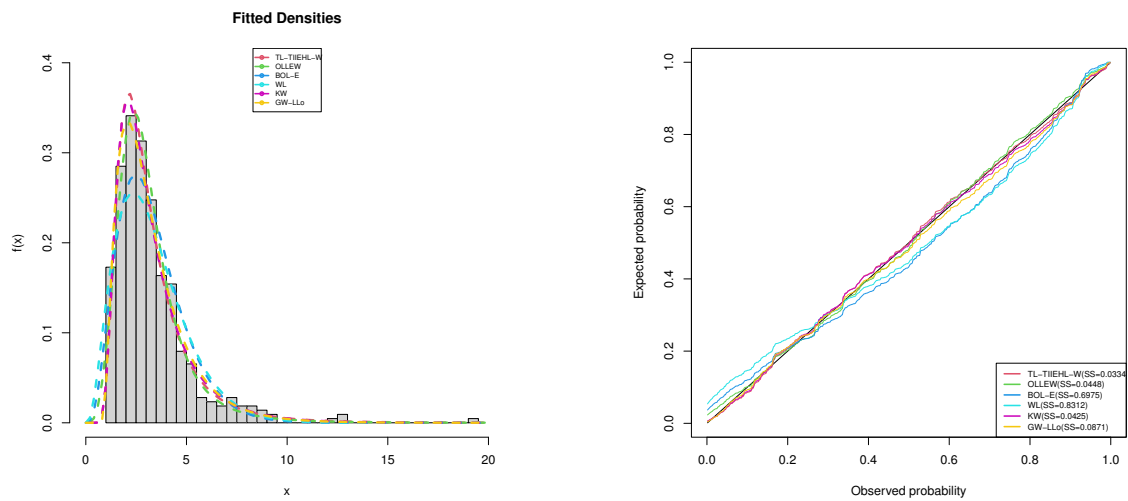


Figure 2.11: Fitted densities and probability plots for prices of 428 new vehicles data

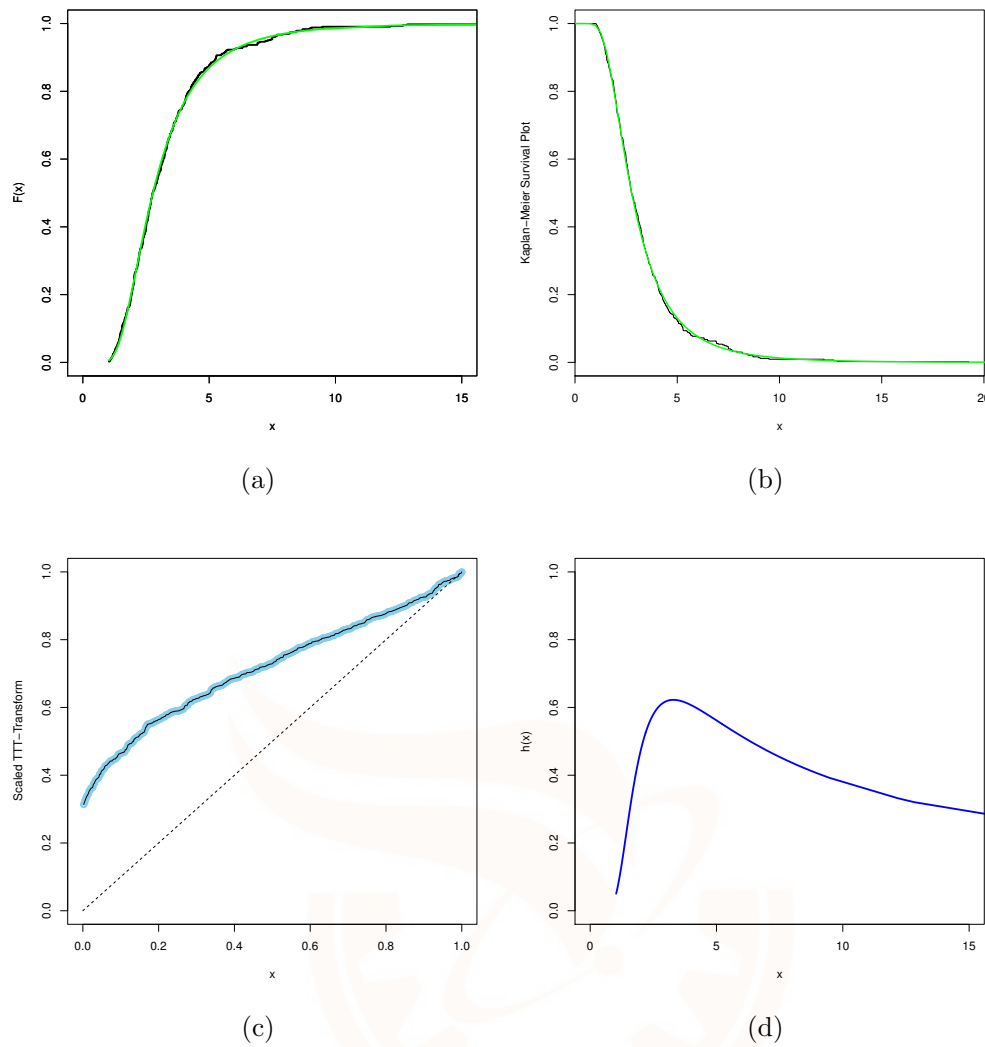


Figure 2.12: Estimated cdf, K-M survival, TTT-Transform and estimated hrt plots of the TL-TIIEHL-W distribution for prices of 428 new vehicles data

The estimated cdf plot, K-M survival plot, TTT-Transform and the estimated hazard rate plots for prices of 428 new vehicles data are given in Figure 2.12. The fitted cdf indicated by the green line is close to the empirical cdf while the survival function in green is closer to the K-M curve which shows that indeed the new proposed model is better in explaining the new vehicles data. The estimated hrf for the new vehicles data has the upside-down bathtub shape.

2.7.2 Growth Hormone Data

The second data points represents the time since the growth hormone medication was given to children until they attained the target height. This data was also analysed by Alizadeh et al. (2018). The data are: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

The estimated variance-covariance matrix of TL-TIIEHL-W model for growth hormone data is

given by

$$\begin{bmatrix} 2.4194 \times 10^{-02} & -1.6952 \times 10^{-03} & 2.4992 \times 10^{-05} & 1.4519 \times 10^{-03} \\ -1.6952 \times 10^{-03} & 1.1878 \times 10^{-04} & -1.7511 \times 10^{-06} & -1.0171 \times 10^{-04} \\ 2.4992 \times 10^{-05} & -1.7511 \times 10^{-06} & 2.5816 \times 10^{-08} & 1.5002 \times 10^{-06} \\ 1.4519 \times 10^{-03} & -1.0171 \times 10^{-04} & 1.5002 \times 10^{-06} & 1.1047 \times 10^{-04} \end{bmatrix}$$

and the approximate 95% confidence interval are given by $\gamma \in (7.3393, 7.9491)$, $a \in (30.8216, 30.8643)$, $b \in (512.8797, 512.8803)$ and $\lambda \in (0.0701, 0.1113)$.

Table 2.7: Model estimates for growth hormone data

Model	Estimates				Statistics								
	γ	a	b	λ	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value	SS
TL-TIIEHL-W	$7.6442 \times 10^{+00}$ (1.5554×10^{-01})	$3.0843 \times 10^{+01}$ (1.0899×10^{-02})	$5.1288 \times 10^{+02}$ (1.6069×10^{-04})	9.0703×10^{-02} (1.0510×10^{-02})	155.2	163.2	164.5	169.4	0.0345	0.2483	0.0864	0.9562	0.0364
OLLEW	α 0.0044 (0.0135)	β 0.0972 (0.0199)	γ 4.5893 (1.7635)	θ 12.3044 (0.4784)	158.7	166.7	167.9	172.9	0.0573	0.4130	0.0984	0.8873	0.0443
BOL-E	a $4.1596 \times 10^{+00}$ (4.3531×10^{-09})	b $3.2395 \times 10^{+01}$ (5.6716×10^{-10})	λ $4.8742 \times 10^{+01}$ (4.0227×10^{-10})	θ 4.8247×10^{-04} (3.9944×10^{-05})	160.2	168.2	169.6	174.5	0.1012	0.6568	0.1243	0.6515	0.1032
WL	a 183.5850 (0.0472)	b 3.8577 (4.2627)	α 0.1287 (0.0796)	β 1.1717 (4.8634)	161.3	169.3	170.7	175.6	0.1135	0.7289	0.12623	0.6325	0.1006
KW	a 18.4540 (9.1082)	b 9261.4000 (0.0020)	c 0.1829 (0.0533)	λ 0.1199 (0.2551)	163.5	171.5	172.8	177.7	0.1445	0.9116	0.1415	0.4851	0.1331
GW-LLo	α 0.0025 (0.0151)	β 9.0895 (0.0082)	a 0.1688 (0.2536)	θ 0.4998 (0.2317)	162.6	170.6	171.9	176.8	0.1322	0.8390	0.1339	0.5563	0.1170

Table 2.7, shows that the statistics: AIC, AICC and BIC are the smallest for the TL-TIIEHL-W distribution as compared to the listed in the table. The value of SS from the probability plots and the values of the goodness-of-fit statistics A^* , W^* and K-S are the smallest for the TL-TIIEHL-W distribution, the K-S p-value is the largest and closer to 1 which indeed showed that the distribution is a better fit for growth hormone data. Figure 2.13 gives profile log-likelihood plots for growth hormone data. The plots indicate that maximum likelihood estimates of the parameters γ, a, b and λ exist and are unique.

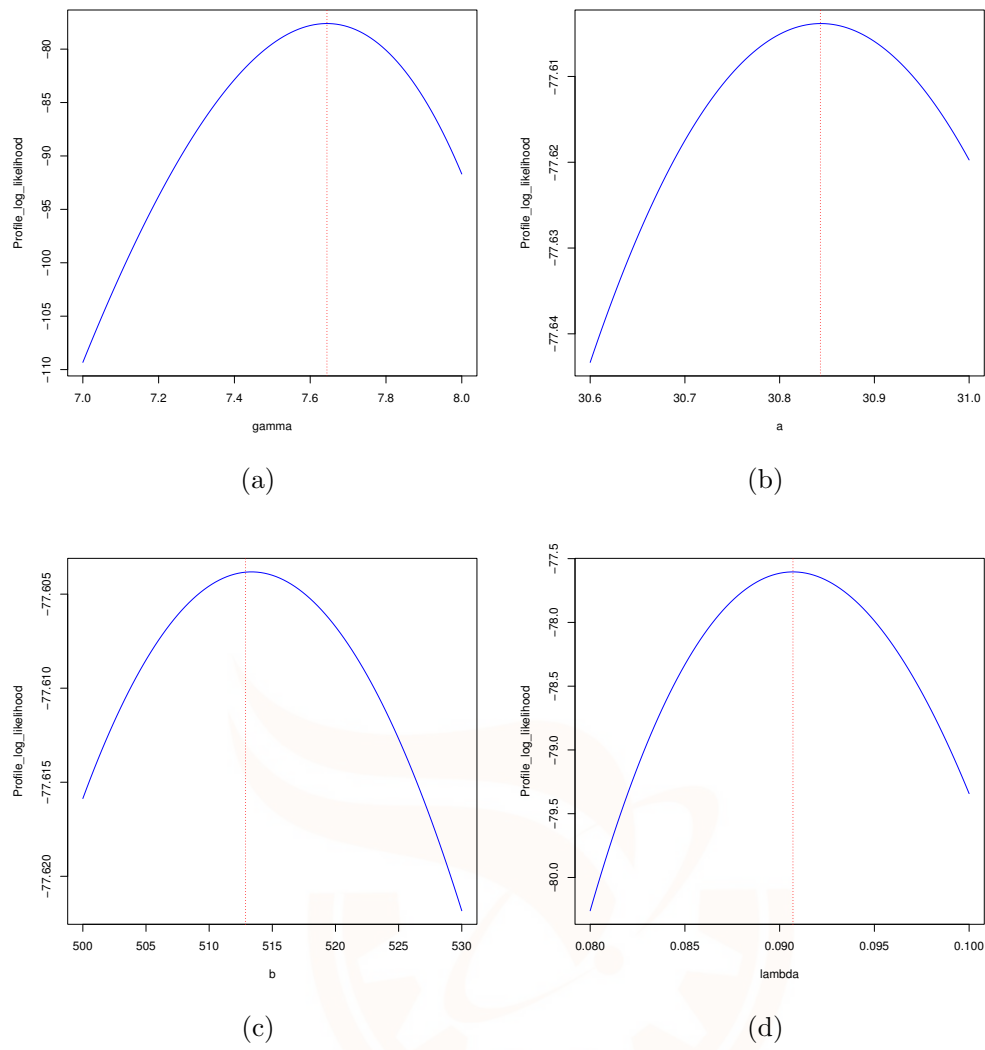


Figure 2.13: Profile log-likelihood plots for growth hormone data

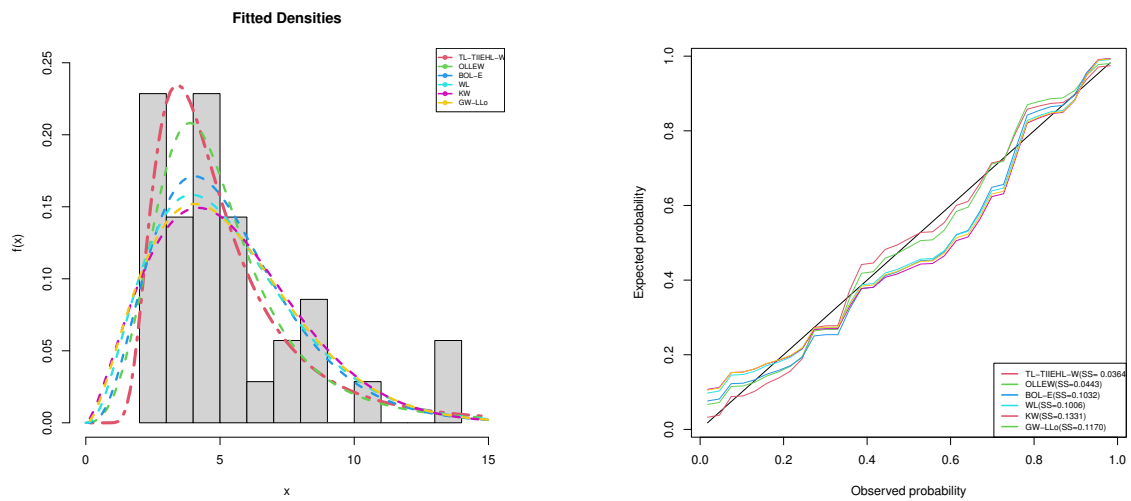


Figure 2.14: Fitted densities and probability plots for growth hormone data

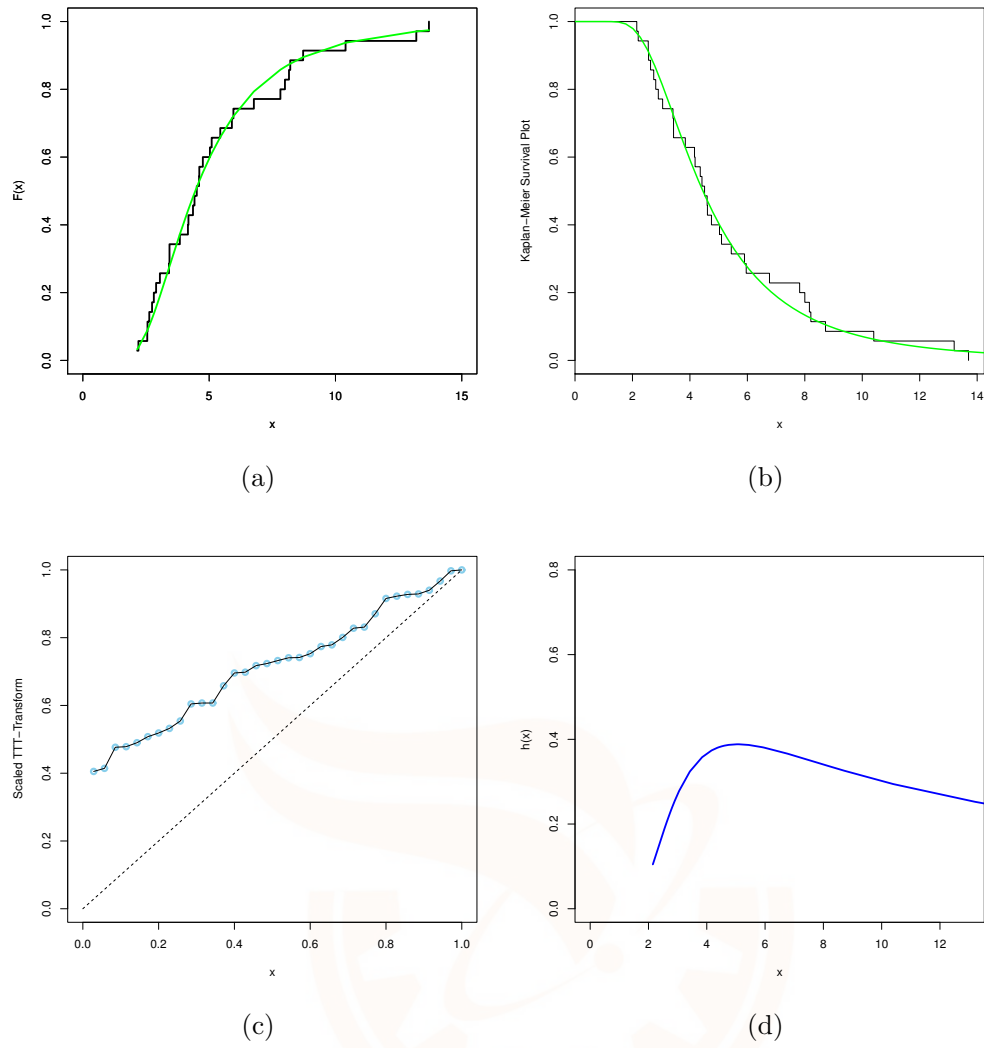


Figure 2.15: Estimated cdf, K-M survival, TTT-Transform and estimated hrt plots of the TL-TIIEHL-W distribution growth hormone data

Figure 2.15 gives the estimated cdf plot, K-M survival plot, TTT-Transform and the estimated hazard rate plots for growth hormone data set. Again, The fitted cdf for the TL-TIIEHL-W distribution indicated by the green line is close to the empirical cdf while the survival function in green is closer to the K-M curve which shows that indeed the proposed model is better in explaining the growth hormone data. The estimated hrf for the growth hormone data gives the upside-down bathtub shape.

2.8 Concluding Remarks

In this chapter, a new family of generalized distributions namely, the Topp-Leone type II exponentiated half logistic-G distribution was introduced. Some statistical properties of the new family of distributions were derived. Maximum likelihood estimation technique was used to estimate the parameters. We also studied three special cases of the new family of distributions namely, the Topp-Leone type II exponentiated half logistic-Weibull, the Topp-Leone type II ex-

ponentiated half logistic-uniform and the Topp-Leone type II exponentiated half logistic-Lindley distributions. The importance and applicability of the new family of distributions for the special case of the Topp-Leone type II exponentiated half logistic-Weibull distribution was examined by means of applications to prices of 428 new vehicles and growth hormone data sets. From these examples, we see that the proposed model can be very useful in the areas of science and engineering among others. Furthermore, more applications can be done using other special cases such as TL-TIIEHL-L and TL-TIIEHL-U distributions. The new family of distributions contain several new sub-models as indicated in subsection 2.2.1.



3. The Topp-Leone Exponentiated Half Logistic Generalized-G Family of Distributions

3.1 Introduction

The Topp-Leone generator is important in generating distributions that are useful in reliability analysis as it produces bathtub as well as other desirable shapes for the hazard rate functions, and the EHL-G can be used to produce flexible models in lifetime data analysis. Hence combining these two generators with the Generalized-G generator which gives models with better fits, we obtain a more flexible family of distributions which is applicable in the areas of economics, reliability and biology, just to name a few.

The Topp-Leone exponentiated half logistic-generalized-G family of distributions is proposed and studied in this chapter. Some mathematical and statistical properties of the new family of distributions presented are, the quantile function, linear representation of the density function, moments, moment generating function, probability weighted moments, distribution of the i^{th} order statistics and Rényi entropy. Some special cases of the proposed distribution are also discussed in details. Maximum likelihood method is used for estimating the parameters and Monte Carlo simulation is conducted to examine the consistency of the parameter estimates. Finally, the usefulness and flexibility of a member of the new family of distributions is determined by means of applications to three real data sets.

3.2 The New and Generalized Family

The TL-EHL-GG distribution and its sub-families are presented in this section. The Topp-Leone generated family of distributions was introduced by Al-Shomrani et al. (2016) with the cdf and pdf given by equation (1.5) and (1.6) respectively, for $b > 0$, $\overline{G}(x; \Psi) = 1 - G(x; \Psi)$ and parameter vector Ψ . Cordeiro et al. (2014) developed the EHL-G family of distributions with cdf

$$F_{EHL-G}(x; \alpha, \beta, \Psi) = [B_G(x; \alpha, \Psi)]^\beta, \quad (3.1)$$

for $\alpha, \beta > 0$ and baseline vector Ψ , where $B_G(x; \alpha, \Psi) = \frac{1 - \overline{G}^\alpha(x; \Psi)}{1 + \overline{G}^\alpha(x; \Psi)}$. Note that the cdf

$$F_{G-G}(x; \alpha, \Psi) = 1 - \overline{G}^\alpha(x; \Psi),$$

for $\alpha > 0$ and parameter vector Ψ , is the generalized-G family of distributions. Now, by combining equation (1.5) and equation (3.1), we obtain the cdf of the TL-EHL-GG distribution as

$$F_{TL-EHL-GG}(x; \alpha, \beta, b, \Psi) = \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^b, \quad (3.2)$$

for $\alpha, \beta, b > 0$ and baseline vector Ψ . The corresponding pdf is given by

$$\begin{aligned} f_{TL-EHL-GG}(x; \alpha, \beta, b, \Psi) &= 4\alpha\beta b \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^{b-1} \\ &\times \left(1 - (B_G(x; \alpha, \Psi))^\beta \right) \frac{(1 - \overline{G}^\alpha(x; \Psi))^{\beta-1}}{(1 + \overline{G}^\alpha(x; \Psi))^{\beta+1}} \\ &\times [\overline{G}(x; \Psi)]^{\alpha-1} g(x; \Psi), \end{aligned} \quad (3.3)$$

for $\alpha, \beta, b > 0$ and parameter vector Ψ .

The hrf for the new family of distribution is

$$\begin{aligned} h_{TL-EHL-GG}(x; \alpha, \beta, b, \Psi) &= 4\alpha\beta b \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^{b-1} g(x; \Psi) \\ &\times \left(1 - (B_G(x; \alpha, \Psi))^\beta \right) \frac{(1 - \overline{G}^\alpha(x; \Psi))^{\beta-1}}{(1 + \overline{G}^\alpha(x; \Psi))^{\beta+1}} \\ &\times [\overline{G}(x; \Psi)]^{\alpha-1} \left(1 - \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^b \right)^{-1}, \end{aligned} \quad (3.4)$$

for $\alpha, \beta, b > 0$ and baseline vector Ψ .

3.2.1 Sub-Families

In this subsection, we present some of the sub-families of the TL-EHL-GG distribution.

- If we let $b = 1$, we get a new distribution with the cdf

$$F(x; \alpha, \beta, \Psi) = 1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2,$$

for $\alpha, \beta > 0$ and baseline vector Ψ .

- When $\beta = 1$, we have a family of distributions with the cdf

$$F(x; \alpha, b, \Psi) = \left[1 - (1 - (B_G(x; \alpha, \Psi)))^2 \right]^b,$$

for $\alpha, b > 0$ and baseline vector Ψ .

- By letting $\alpha = 1$, we have the TL-EHL-G distribution with the cdf

$$F(x; \beta, b, \Psi) = \left[1 - \left(1 - \left(\frac{G(x; \Psi)}{1 + \overline{G}(x; \Psi)} \right)^\beta \right)^2 \right]^b,$$

for $\beta, b > 0$ and baseline vector Ψ .

- When $b = \beta = 1$, we get a new family of distribution with the cdf

$$F(x; \alpha, \Psi) = 1 - (1 - (B_G(x; \alpha, \Psi)))^2,$$

for $\alpha > 0$ and baseline vector Ψ .

- If we let $b = \beta = \alpha = 1$ we have a new family of distributions with the cdf

$$F(x; \Psi) = 1 - \left(1 - \left(\frac{G(x; \Psi)}{1 + \overline{G}(x; \Psi)} \right) \right)^2,$$

with parameter vector Ψ .

3.3 Some Statistical Properties

In this section, we present some statistical properties of the TL-EHL-GG family of distributions.

3.3.1 Quantile Function

We compute the quantile function of the proposed family of distributions by inverting the non-linear equation $F_{TL-EHL-GG}(x; \alpha, \beta, b, \Psi) = u$, for $0 \leq u \leq 1$. That is,

$$\left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^b = u.$$

The above equation can be written as

$$(1 - u^{\frac{1}{b}})^{\frac{1}{2}} = 1 - (B_G(x; \alpha, \Psi))^\beta.$$

which is equivalent to

$$B_G(x; \alpha, \Psi) = \left(1 - (1 - u^{\frac{1}{b}})^{\frac{1}{2}} \right)^{\frac{1}{\beta}},$$

and simplifies to

$$\overline{G}(x; \Psi) = \left(\frac{1 - \left(1 - (1 - u^{\frac{1}{b}})^{\frac{1}{2}}\right)^{\frac{1}{\beta}}}{1 + \left(1 - (1 - u^{\frac{1}{b}})^{\frac{1}{2}}\right)^{\frac{1}{\beta}}} \right)^{\frac{1}{\alpha}}.$$

Hence, the quantile function of the TL-EHL-GG distribution is

$$Q_G(u; \alpha, \beta, b, \Psi) = G^{-1} \left[1 - \left(\frac{1 - \left(1 - (1 - u^{\frac{1}{b}})^{\frac{1}{2}}\right)^{\frac{1}{\beta}}}{1 + \left(1 - (1 - u^{\frac{1}{b}})^{\frac{1}{2}}\right)^{\frac{1}{\beta}}} \right)^{\frac{1}{\alpha}} \right]. \quad (3.5)$$

Utilizing the `bbmle` package (Bolker and Bolker (2017)) in R software, we can obtain random numbers from equation (3.5) for a specified baseline cdf G .

3.3.2 Linear Representation of the Density Function

The series expansion of the density function of the TL-EHL-GG family of distributions is presented in this subsection. We derive the linear representation of the density function by considering the following generalized binomial expansion,

$$(1 - z)^b = \sum_{s=0}^{\infty} (-1)^s \binom{b}{s} z^s, \quad \text{for } |z| < 1.$$

Taking into consideration the following generalized binomial expansions:

$$\left[1 - \left(1 - (B_G(x; \alpha, \Psi))^{\beta}\right)^2 \right]^{b-1} = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \left(1 - (B_G(x; \alpha, \Psi))^{\beta}\right)^{2i},$$

and

$$\left(1 - (B_G(x; \alpha, \Psi))^{\beta}\right)^{2i+1} = \sum_{j=0}^{\infty} (-1)^j \binom{2i+1}{j} (B_G(x; \alpha, \Psi))^{\beta j},$$

we can write the pdf of the TL-EHL-GG family of distributions as

$$\begin{aligned} f_{TL-EHL-GG}(x; \alpha, \beta, b, \Psi) &= 4\alpha\beta b \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{b-1}{i} \binom{2i+1}{j} \\ &\times \frac{(1 - \overline{G}^{\alpha}(x; \Psi))^{\beta(j+1)-1}}{(1 + \overline{G}^{\alpha}(x; \Psi))^{\beta(j+1)+1}} [\overline{G}(x; \Psi)]^{\alpha-1} g(x; \Psi). \end{aligned}$$

Applying the expansions,

$$(1 - \overline{G}^\alpha(x; \Psi))^{\beta(j+1)-1} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta(j+1)-1}{k} \overline{G}^{\alpha k}(x; \Psi),$$

$$(1 + \overline{G}^\alpha(x; \Psi))^{-(\beta(j+1)+1)} = \sum_{l=0}^{\infty} (-1)^l \binom{-(\beta(j+1)+1)}{l} \overline{G}^{\alpha l}(x; \Psi),$$

and

$$(1 - G(x; \Psi))^{\alpha(k+l+1)-1} = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha(k+l+1)-1}{m} G^m(x; \Psi),$$

finally, the pdf of TL-EHL-GG family of distributions is

$$\begin{aligned} f_{TL-EHL-GG}(x; \alpha, \beta, b, \Psi) &= 4\alpha\beta b \sum_{i,j,k,l,m=0}^{\infty} (-1)^{i+j+k+l+m} \binom{b-1}{i} \binom{2i+1}{j} \\ &\times \binom{\beta(j+1)-1}{k} \binom{-(\beta(j+1)+1)}{l} \binom{\alpha(k+l+1)-1}{m} \\ &\times \frac{(m+1)}{(m+1)} G^m(x; \Psi) g(x; \Psi) \\ &= \sum_{m=0}^{\infty} W_{m+1} g_{m+1}^*(x; \Psi), \end{aligned}$$

where

$$\begin{aligned} W_{m+1} &= 4\alpha\beta b \sum_{i,j,k,l=0}^{\infty} (-1)^{i+j+k+l+m} \binom{b-1}{i} \binom{2i+1}{j} (m+1) \\ &\times \binom{\beta(j+1)-1}{k} \binom{-(\beta(j+1)+1)}{l} \binom{\alpha(k+l+1)-1}{m} \end{aligned} \quad (3.6)$$

and $g_{m+1}^*(x; \Psi) = (m+1)g(x; \Psi)G^m(x; \Psi)$ is the exponentiated-G (Exp-G) density having power parameter $(m+1)$. Thus, the pdf of the TL-EHL-GG distribution can be expressed as an infinite linear combination of Exp-G densities. Consequently the mathematical and statistical properties of the TL-EHL-GG family of distributions can be readily obtained from those of the Exp-G distribution.

3.3.3 Moments and Generating Function

We get the n^{th} moment for the TL-EHL-GG family of distributions as follows

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_{TL-EHL-GG}(x; \alpha, \beta, b, \Psi) dx = \sum_{m=0}^{\infty} W_{m+1} E(Y_{m+1}^n),$$

where $E(Y_{m+1}^n)$ is the n^{th} moment of Y_{m+1} which follows an Exp-G distribution with power parameter $(m + 1)$ and W_{m+1} is given in equation (3.6). The moment generating function (MGF) for the TL-EHL-GG family of distributions is given by

$$M_X(t) = E(e^{tX}) = \sum_{m=0}^{\infty} W_{m+1} E(e^{tY_{m+1}}),$$

where $E(e^{tY_{m+1}})$ is the MGF of the Exp-G distribution with power parameter $(m + 1)$ and W_{m+1} is given in equation (3.6).

3.3.4 Probability Weighted Moments

In this subsection, we present the PWMs of the TL-EHL-GG distribution. The PWMs are used to estimate distribution parameters when the inverse form cannot be given explicitly (Greenwood et al. (1979)). The PWMs for a random variable X is given by

$$\eta_{a,r} = E(X^a [F(X)]^r) = \int_{-\infty}^{\infty} x^a f(x) [F(x)]^r dx.$$

Note that, for the TL-EHL-GG family of distributions, we can write

$$\begin{aligned} f_{TL-EHL-GG}(x) [F_{TL-EHL-GG}(x)]^r &= 4\alpha\beta b \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^{b(r+1)-1} \\ &\times \left(1 - (B_G(x; \alpha, \Psi))^\beta \right) \frac{(1 - \overline{G}^\alpha(x; \Psi))^{\beta-1}}{(1 + \overline{G}^\alpha(x; \Psi))^{\beta+1}} \\ &\times [\overline{G}(x; \Psi)]^{\alpha-1} g(x; \Psi). \end{aligned}$$

By applying the generalized binomial expansion used in the expansion of the density function, we obtain

$$\begin{aligned} f(x) [F(x)]^r &= 4\alpha\beta b \sum_{i,j,k,l,m=0}^{\infty} (-1)^{i+j+k+l+m} \binom{b(r+1)-1}{i} \binom{2i+1}{j} \\ &\times \binom{\beta(j+1)-1}{k} \binom{-(\beta(j+1)+1)}{l} \binom{\alpha(k+l+1)-1}{m} \\ &\times \frac{(m+1)}{(m+1)} G^m(x; \Psi) g(x; \Psi) \\ &= \sum_{m=0}^{\infty} V_{m+1} g_{m+1}^*(x; \Psi), \end{aligned}$$

where

$$V_{m+1} = 4\alpha\beta b \sum_{i,j,k,l=0}^{\infty} (-1)^{i+j+k+l+m} \binom{b(r+1)-1}{i} \binom{2i+1}{j} (m+1) \\ \times \binom{\beta(j+1)-1}{k} \binom{-(\beta(j+1)+1)}{l} \binom{\alpha(k+l+1)-1}{m}$$

and $g_{m+1}^*(x; \Psi) = (m+1)g(x; \Psi)G^m(x; \Psi)$ is the Exp-G pdf with power parameter $(m+1)$. Therefore, the PWMs of the TL-EHL-GG family of distributions can be derived from the moments of the Exp-G distribution and is given by

$$\eta_{a,r} = \sum_{m=0}^{\infty} V_{m+1} \int_{-\infty}^{\infty} x^a g_{m+1}^*(x; \Psi) dx. \quad (3.7)$$

3.3.5 Stochastic Ordering

Stochastic orders for the TL-EHL-GG family of distributions are presented in this subsection. Let Y and Q be random variables with the distribution functions $F_Y(r)$ and $F_Q(r)$, respectively, and $\bar{F}_Y(r) = 1 - F_Y(r)$ the survival function. Then Y is stochastically smaller than Q if $\bar{F}_Y(r) \leq \bar{F}_Q(r)$ for all r or $F_Y(r) \geq F_Q(r)$ for all r , denoted by $Y <_s Q$. Hazard rate order and likelihood ratio order are given by $Y <_{hr} Q$ if $h_Y(r) \geq h_Q(r)$ for all r and $Y <_{lr} Q$ if $\frac{f_Y(r)}{f_Q(r)}$ is decreasing in r , (Shaked and Shanthikumar (2007)), respectively. It is well known that $Y <_{lr} Q \Rightarrow Y <_{hr} Q \Rightarrow Y <_s Q$.

Theorem 3. Suppose $X_1 \sim TL-EHL-GG(\alpha, \beta, b_1, \Psi)$ and $X_2 \sim TL-EHL-GG(\alpha, \beta, b_2, \Psi)$, respectively. If $b_1 \leq b_2$, then X_1 and X_2 are stochastically ordered.

proof: The pdfs of X_1 and X_2 are given by

$$f_1(x; \alpha, \beta, b_1, \Psi) = 4\alpha\beta b_1 \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^{b_1-1} \\ \times \left(1 - (B_G(x; \alpha, \Psi))^\beta \right) \frac{(1 - \bar{G}^\alpha(x; \Psi))^{\beta-1}}{(1 + \bar{G}^\alpha(x; \Psi))^{\beta+1}} \\ \times [\bar{G}(x; \Psi)]^{\alpha-1} g(x; \Psi)$$

and

$$f_2(x; \alpha, \beta, b_2, \Psi) = 4\alpha\beta b_2 \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^{b_2-1} \\ \times \left(1 - (B_G(x; \alpha, \Psi))^\beta \right) \frac{(1 - \bar{G}^\alpha(x; \Psi))^{\beta-1}}{(1 + \bar{G}^\alpha(x; \Psi))^{\beta+1}} \\ \times [\bar{G}(x; \Psi)]^{\alpha-1} g(x; \Psi),$$

respectively. Now,

$$\frac{f_1(x; \alpha, \beta, b_1, \Psi)}{f_2(x; \alpha, \beta, b_2, \Psi)} = \frac{b_1}{b_2} \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^{b_1 - b_2}, \quad (3.8)$$

and upon differentiating equation (3.8) with respect to x , we obtain

$$\begin{aligned} \frac{d}{dx} \left(\frac{f_1(x; \alpha, \beta, b_1, \Psi)}{f_2(x; \alpha, \beta, b_2, \Psi)} \right) &= 4\alpha\beta \frac{b_1}{b_2} (b_1 - b_2) \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^{b_1 - b_2 - 1} \\ &\times \left(1 - (B_G(x; \alpha, \Psi))^\beta \right) \left(\frac{1 - \overline{G}^\alpha(x; \Psi)}{1 + \overline{G}^\alpha(x; \Psi)} \right)^{\beta - 1} \\ &\times [\overline{G}(x; \Psi)]^{\alpha - 1} g(x; \Psi), \end{aligned} \quad (3.9)$$

which is ≤ 0 if $b_1 \leq b_2$. Therefore, $X_1 <_{lr} X_2$, $X_1 <_{hr} X_2$ and $X_1 <_s X_2$. Consequently, the random variables X_1 and X_2 are stochastically ordered.

3.3.6 Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from the TL-EHL-GG family of distributions, and $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ the corresponding order statistics. Let $f(x; \alpha, \beta, b, \Psi) = f(x)$ and $F(x; \alpha, \beta, b, \Psi) = F(x)$. The pdf of the i^{th} order statistic, $X_{i:n}$ from the TL-EHL-GG family of distributions is

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} \sum_{s=0}^{n-i} (-1)^s \binom{n-i}{s} f(x; \alpha, \beta, b, \Psi) [F(x; \alpha, \beta, b, \Psi)]^{s+i-1}.$$

Note that

$$\begin{aligned} f(x)[F(x)]^{s+i-1} &= 4\alpha\beta b \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^{b(s+i)-1} \\ &\times \left(1 - (B_G(x; \alpha, \Psi))^\beta \right) \frac{(1 - \overline{G}^\alpha(x; \Psi))^{\beta-1}}{(1 + \overline{G}^\alpha(x; \Psi))^{\beta+1}} \\ &\times [\overline{G}(x; \Psi)]^{\alpha-1} g(x; \Psi). \end{aligned}$$

Applying the following generalized binomial expansions:

$$\begin{aligned} \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^2 \right]^{b(s+i)-1} &= \sum_{r=0}^{\infty} (-1)^r \binom{b(s+i)-1}{r} \\ &\times \left(1 - (B_G(x; \alpha, \Psi))^\beta \right)^{2r}, \end{aligned}$$

$$\left(1 - (B_G(x; \alpha, \Psi))^\beta\right)^{2r+1} = \sum_{j=0}^{\infty} (-1)^j \binom{2r+1}{j} (B_G(x; \alpha, \Psi))^{\beta j},$$

$$(1 - \overline{G}^\alpha(x; \Psi))^{\beta(j+1)-1} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta(j+1)-1}{k} \overline{G}^{\alpha k}(x; \Psi),$$

$$(1 + \overline{G}^\alpha(x; \Psi))^{-(\beta(j+1)+1)} = \sum_{l=0}^{\infty} (-1)^l \binom{-(\beta(j+1)+1)}{l} \overline{G}^{\alpha l}(x; \Psi),$$

and

$$(1 - G(x; \Psi))^{\alpha(k+l+1)-1} = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha(k+l+1)-1}{m} G^m(x; \Psi),$$

we can write

$$\begin{aligned} f(x)[F(x)]^{s+i-1} &= 4\alpha\beta b \sum_{r,j,k,l,m=0}^{\infty} (-1)^{r+j+k+l+m} \binom{b(s+i)-1}{r} \binom{2r+1}{j} \\ &\times \binom{\beta(j+1)-1}{k} \binom{-(\beta(j+1)+1)}{l} \binom{\alpha(k+l+1)-1}{m} \\ &\times \frac{(m+1)}{(m+1)} G^m(x; \Psi) g(x; \Psi) \\ &= \sum_{m=0}^{\infty} b_{m+1} g_{m+1}^*(x; \Psi), \end{aligned}$$

where

$$\begin{aligned} b_{m+1} &= 4\alpha\beta b \sum_{r,j,k,l=0}^{\infty} (-1)^{r+j+k+l+m} \binom{b(s+i)-1}{r} \binom{2r+1}{j} (m+1) \\ &\times \binom{\beta(j+1)-1}{k} \binom{-(\beta(j+1)+1)}{l} \binom{\alpha(k+l+1)-1}{m} \end{aligned}$$

and $g_{m+1}^*(x; \Psi) = (m+1)g(x; \Psi)G^m(x; \Psi)$ is the Exp-G density with power parameter $(m+1)$. Consequently, the pdf of the i^{th} order statistics from TL-EHL-GG family of distributions is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} \sum_{s=0}^{n-i} \sum_{m=0}^{\infty} (-1)^s \binom{n-i}{s} b_{m+1} g_{m+1}^*(x; \Psi). \quad (3.10)$$

3.3.7 Rényi Entropy

In this subsection, we present Rényi entropy of the TL-EHL-GG family of distributions. Rényi entropy (Rényi (1961)) of a random variable X following TL-EHL-GG family of distributions is obtained as:

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\int_0^{\infty} f^{\nu}(x; \alpha, \beta, b, \Psi) dx \right] \text{ for } \nu > 0 \text{ and } \nu \neq 1.$$

Note that

$$\begin{aligned} f^{\nu}(x; \alpha, \beta, b, \Psi) &= (4\alpha\beta b)^{\nu} \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^{\beta} \right)^2 \right]^{\nu(b-1)} \\ &\times \left(1 - (B_G(x; \alpha, \Psi))^{\beta} \right)^{\nu} \frac{(1 - \overline{G}^{\alpha}(x; \Psi))^{\nu(\beta-1)}}{(1 + \overline{G}^{\alpha}(x; \Psi))^{\nu(\beta+1)}} \\ &\times [\overline{G}(x; \Psi)]^{\nu(\alpha-1)} g^{\nu}(x; \Psi). \end{aligned}$$

Applying the generalized binomial expansions:

$$\begin{aligned} \left[1 - \left(1 - (B_G(x; \alpha, \Psi))^{\beta} \right)^2 \right]^{\nu(b-1)} &= \sum_{i=0}^{\infty} (-1)^i \binom{\nu(b-1)}{i} \\ &\times \left(1 - (B_G(x; \alpha, \Psi))^{\beta} \right)^{2i}, \end{aligned}$$

$$\left(1 - (B_G(x; \alpha, \Psi))^{\beta} \right)^{2i+1} = \sum_{j=0}^{\infty} (-1)^j \binom{2i+1}{j} (B_G(x; \alpha, \Psi))^{\beta j},$$

$$(1 - \overline{G}^{\alpha}(x; \Psi))^{\beta(j+\nu)-\nu} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta(j+\nu)-\nu}{k} \overline{G}^{\alpha k}(x; \Psi),$$

$$(1 + \overline{G}^{\alpha}(x; \Psi))^{-(\beta(j+\nu)+\nu)} = \sum_{l=0}^{\infty} (-1)^l \binom{-(\beta(j+\nu)+\nu)}{l} \overline{G}^{\alpha l}(x; \Psi),$$

and

$$(1 - G(x; \Psi))^{\alpha(k+l+\nu)-\nu} = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha(k+l+\nu)-\nu}{m} G^m(x; \Psi),$$

we obtain

$$\begin{aligned} f^\nu(x; \alpha, \beta, b, \Psi) &= (4\alpha\beta b)^\nu \sum_{i,j,k,l,m=0}^{\infty} (-1)^{i+j+k+l+m} \binom{\nu(b-1)}{i} \binom{2i+1}{j} \\ &\times \binom{\beta(j+\nu)-\nu}{k} \binom{-(\beta(j+\nu)+\nu)}{l} \binom{\alpha(k+l+\nu)-\nu}{m} \\ &\times G^m(x; \Psi) g^\nu(x; \Psi). \end{aligned}$$

Finally, Rényi entropy of the TL-EHL-GG family of distributions is

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1} \log \left[(4\alpha\beta b)^\nu \sum_{i,j,k,l,m=0}^{\infty} (-1)^{i+j+k+l+m} \binom{\nu(b-1)}{i} \binom{2i+1}{j} \right. \\ &\times \binom{\beta(j+\nu)-\nu}{k} \binom{-(\beta(j+\nu)+\nu)}{l} \binom{\alpha(k+l+\nu)-\nu}{m} \\ &\times \left. \frac{1}{(\frac{m}{\nu}+1)^\nu} \int_0^\infty \left(\left(\frac{m}{\nu}+1 \right) g(x; \Psi) G^{\frac{m}{\nu}}(x; \Psi) \right)^\nu dx \right] \\ &= (1-\nu)^{-1} \log \left[\sum_{m=0}^{\infty} R_m^* \exp((1-\nu)I_{REG}) \right], \end{aligned}$$

where $I_{REG} = (1-\nu)^{-1} \log \left[\int_0^\infty \left(\left(\frac{m}{\nu}+1 \right) g(x; \Psi) G^{\frac{m}{\nu}}(x; \Psi) \right)^\nu dx \right]$ is Rényi entropy of Exp-G densities with power parameter $(\frac{m}{\nu}+1)$ and

$$\begin{aligned} R_m^* &= (4\alpha\beta b)^\nu \sum_{i,j,k,l=0}^{\infty} (-1)^{i+j+k+l} \binom{\nu(b-1)}{i} \binom{2i+1}{j} \\ &\times \binom{\beta(j+\nu)-\nu}{k} \binom{-(\beta(j+\nu)+\nu)}{l} \binom{\alpha(k+l+\nu)-\nu}{m} \frac{1}{(\frac{m}{\nu}+1)^\nu}. \end{aligned}$$

3.4 Maximum Likelihood Estimation

Suppose $X \sim TL-EHL-GG(\alpha, \beta, b, \Psi)$ and $\Delta = (\alpha, \beta, b, \Psi)^T$ the parameter vector, then the log-likelihood function (ℓ_n) of a random sample of size n from the $TL-EHL-GG(\alpha, \beta, b, \Psi)$ family of distributions is given by

$$\begin{aligned}
\ell_n(\Delta) &= n \log(4\alpha\beta b) + (b-1) \sum_{i=1}^n \log \left[1 - \left(1 - \left(\frac{1 - \overline{G}^\alpha(x_i; \Psi)}{1 + \overline{G}^\alpha(x_i; \Psi)} \right)^\beta \right)^2 \right] \\
&+ \sum_{i=1}^n \log \left[1 - \left(\frac{1 - \overline{G}^\alpha(x_i; \Psi)}{1 + \overline{G}^\alpha(x_i; \Psi)} \right)^\beta \right] + (\beta-1) \sum_{i=1}^n \log[1 - \overline{G}^\alpha(x_i; \Psi)] \\
&- (\beta+1) \sum_{i=1}^n \log[1 + \overline{G}^\alpha(x_i; \Psi)] + (\alpha-1) \sum_{i=1}^n \log[\overline{G}(x_i; \Psi)] + \sum_{i=1}^n \log[g(x_i; \Psi)].
\end{aligned}$$

The MLE's are obtained by solving a system of non-linear equations $\left(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \Psi_k} \right)^T = \mathbf{0}$ by numerical methods. Elements of the score vector $\Delta = (\alpha, \beta, b, \Psi)$ are given in the appendix.

The Fisher Information Matrix of the TL-EHL-GG family of distributions is given by $\mathbf{I}(\Delta) = [\mathbf{I}_{\Delta_i, \Delta_j}]_{(3+q) \times (3+q)} = E \left(-\frac{\partial^2 \ell_n}{\partial \Delta_i \partial \Delta_j} \right)$, $i, j = 1, 2, \dots, (3+q)$, and can be numerically obtained by mle2 package (Bolker and Bolker (2017)) in R software. The total Fisher information matrix $n\mathbf{I}(\Theta)$ can be approximated by

$\mathbf{J}_n(\hat{\Delta}) \approx \left[-\frac{\partial^2 \ell_n}{\partial \Delta_i \partial \Delta_j} \Big|_{\Delta=\hat{\Delta}} \right]_{(3+q) \times (3+q)}$, $i, j = 1, 2, \dots, (3+q)$. The multivariate normal distribution $N_{(3+q)}(\mathbf{0}, \mathbf{J}(\hat{\Delta})^{-1})$ can be used to construct confidence intervals and confidence regions for the individual model parameters. The approximate $100(1-\eta)\%$ two sided confidence intervals for α, β, b and Ψ_k are given by:

$$\hat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\Delta})}, \quad \hat{\beta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\beta\beta}^{-1}(\hat{\Delta})}, \quad \hat{b} \pm Z_{\frac{\eta}{2}} \sqrt{I_{bb}^{-1}(\hat{\Delta})} \quad \text{and} \quad \hat{\Psi}_k \pm Z_{\frac{\eta}{2}} \sqrt{I_{\Psi_k \Psi_k}^{-1}(\hat{\Delta})},$$

respectively, where $I_{\alpha\alpha}^{-1}(\hat{\Delta})$, $I_{\beta\beta}^{-1}(\hat{\Delta})$, $I_{bb}^{-1}(\hat{\Delta})$ and $I_{\Psi_k \Psi_k}^{-1}(\hat{\Delta})$ are the diagonal elements of $I_n^{-1}(\hat{\Delta}) = (n\mathbf{I}(\hat{\Delta}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $(\frac{\eta}{2})^{th}$ percentile of the standard normal distribution.

3.5 Some Special Cases

In this subsection, we specify the baseline cdf $G(x; \Psi)$ and pdf $g(x; \Psi)$ in equations (3.2) and (3.3) to come up with some special cases of TL-EHL-GG family of distributions.

3.5.1 Topp-Leone Exponentiated Half Logistic Generalized-Log-logistic Distribution

Suppose the baseline distribution is the log-logistic distribution with cdf and pdf given by $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$, respectively, for $c, x > 0$, then the cdf and pdf of the TL-EHL-GLLo distribution are given by

$$F_{TL-EHL-GLLo}(x; \alpha, \beta, b, c) = \left[1 - \left(1 - (B_1(x; \alpha, c))^\beta \right)^2 \right]^b$$

and

$$\begin{aligned} f_{TL-EHL-GLLo}(x; \alpha, \beta, b, c) &= 4\alpha\beta b \left[1 - \left(1 - (B_1(x; \alpha, c))^\beta \right)^2 \right]^{b-1} \\ &\times \left(1 - (B_1(x; \alpha, c))^\beta \right) \frac{(1 - (1 + x^c)^{-\alpha})^{\beta-1}}{(1 + (1 + x^c)^{-\alpha})^{\beta+1}} \\ &\times cx^{c-1}(1 + x^c)^{-\alpha-1}, \end{aligned}$$

respectively. The corresponding hrf is given by

$$\begin{aligned} h_{TL-EHL-GLLo}(x; \alpha, \beta, b, c) &= \alpha\beta b \left[1 - \left(1 - (B_1(x; \alpha, c))^\beta \right)^2 \right]^{b-1} (1 + x^c)^{-\alpha-1} \\ &\times \left(1 - (B_1(x; \alpha, c))^\beta \right) \frac{(1 - (1 + x^c)^{-\alpha})^{\beta-1}}{(1 + (1 + x^c)^{-\alpha})^{\beta+1}} \\ &\times cx^{c-1} \left(1 - \left[1 - \left(1 - (B_1(x; \alpha, c))^\beta \right)^2 \right]^b \right)^{-1}, \end{aligned}$$

for $\alpha, \beta, b, c > 0$, and $x > 0$, where $B_1(x; \alpha, c) = \frac{1-(1+x^c)^{-\alpha}}{1+(1+x^c)^{-\alpha}}$.

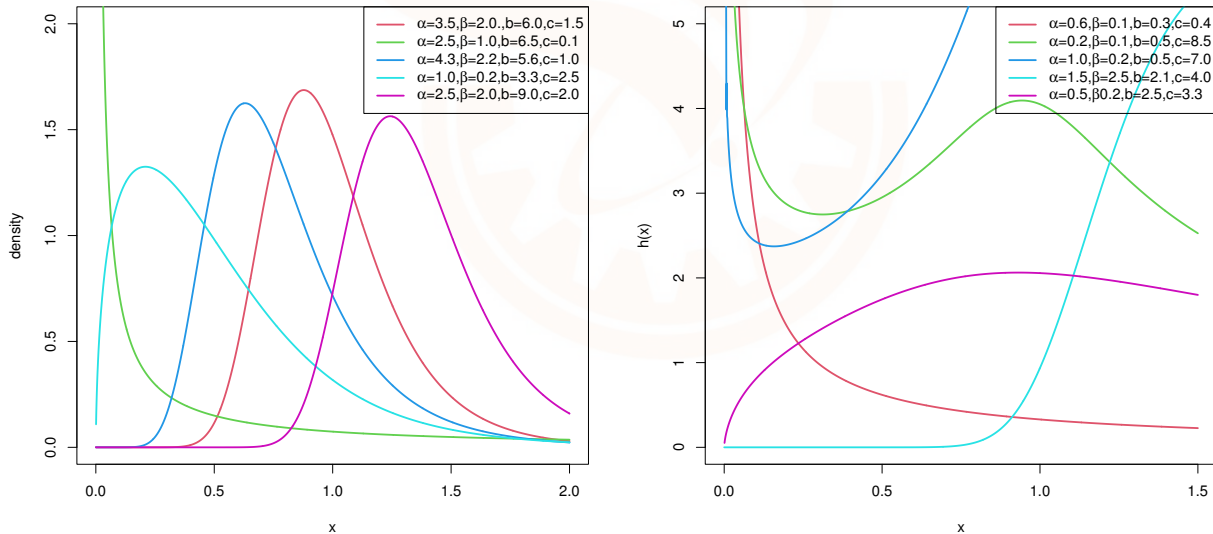


Figure 3.1: Plots of the pdf and hrf for the TL-EHL-GLLo distribution

Figure 3.1 gives the plots of pdf and hrf of the TL-EHL-GLLo distribution. The pdf of the TL-EHL-GLLo distribution can be right-skewed, left-skewed, reverse-J and almost symmetric shapes. The hrf on the other hand displays upside-down bathtub, decreasing, increasing, bathtub, and bathtub followed by upside-down bathtub shapes. Figure 3.2 shows that TL-EHL-GLLo distribution can model data sets with different levels of skewness and kurtosis.

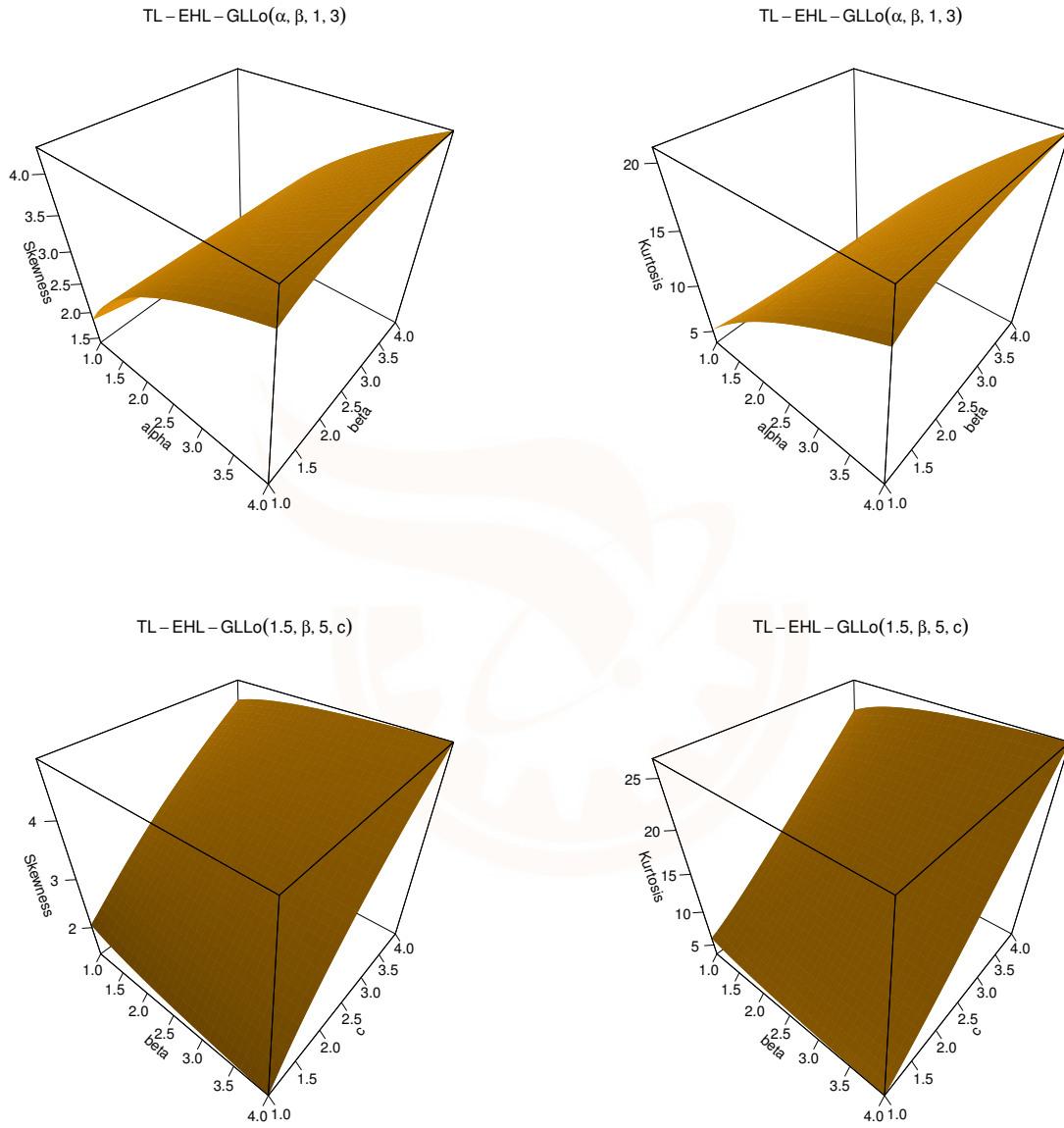


Figure 3.2: Skewness and kurtosis plots for TL-EHL-GLLo distribution

3.5.2 Topp-Leone Exponentiated Half Logistic Generalized-Weibull Distribution

If we let the baseline distribution to be Weibull distribution with cdf and pdf given as $G(x; \lambda) = 1 - e^{-x^\lambda}$ and $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$, respectively, for $\lambda, x > 0$, we obtain the cdf, pdf and hrf of the TL-EHL-GW distribution as

$$F_{TL-EHL-GW}(x; \alpha, \beta, b, \lambda) = \left[1 - \left(1 - (B_2(x; \alpha, \lambda))^\beta \right)^2 \right]^b,$$

$$\begin{aligned} f_{TL-EHL-GW}(x; \alpha, \beta, b, \lambda) &= 4\alpha\beta b \left[1 - \left(1 - (B_2(x; \alpha, \lambda))^\beta \right)^2 \right]^{b-1} \\ &\times \left(1 - (B_2(x; \alpha, \lambda))^\beta \right) \frac{(1 - e^{-\alpha x^\lambda})^{\beta-1}}{(1 + e^{-\alpha x^\lambda})^{\beta+1}} \\ &\times e^{-\alpha x^\lambda} \lambda x^{\lambda-1}, \end{aligned}$$

and

$$\begin{aligned} h_{TL-EHL-GW}(x; \alpha, \beta, b, \lambda) &= 4\alpha\beta b \left[1 - \left(1 - (B_2(x; \alpha, \lambda))^\beta \right)^2 \right]^{b-1} \\ &\times \left(1 - (B_2(x; \alpha, \lambda))^\beta \right) \frac{(1 - e^{-\alpha x^\lambda})^{\beta-1}}{(1 + e^{-\alpha x^\lambda})^{\beta+1}} \\ &\times \left(1 - \left[1 - \left(1 - (B_2(x; \alpha, \lambda))^\beta \right)^2 \right]^b \right)^{-1} \\ &\times e^{-\alpha x^\lambda} \lambda x^{\lambda-1}, \end{aligned}$$

for $\alpha, \beta, b, \lambda > 0$, and $x > 0$, where $B_2(x; \alpha, \lambda) = \frac{1 - e^{-\alpha x^\lambda}}{1 + e^{-\alpha x^\lambda}}$.

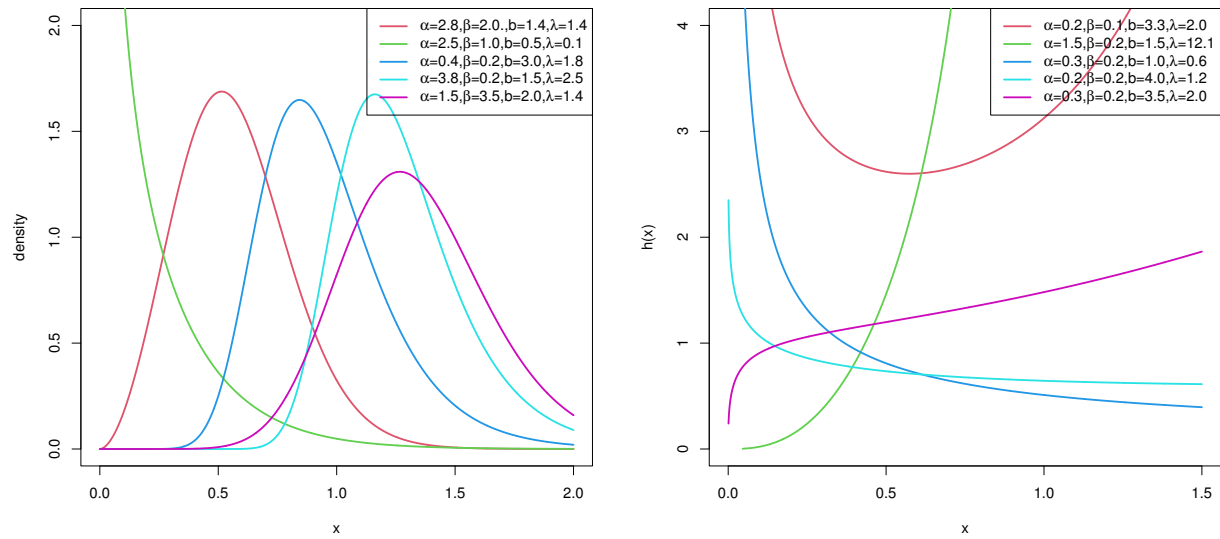


Figure 3.3: Plots of the pdf and hrf for the TL-EHL-GW distribution

Figure 3.3 represents the plots of pdf and hrf of the TL-EHL-GW distribution. The pdf of the TL-EHL-GW distribution can be right-skewed, left-skewed, reverse-J and almost symmetric shapes, whereas the hrf displays bathtub, decreasing and increasing shapes. Figure 3.4 indicate that TL-EHL-GW distribution can model data sets with different levels of skewness and kurtosis.

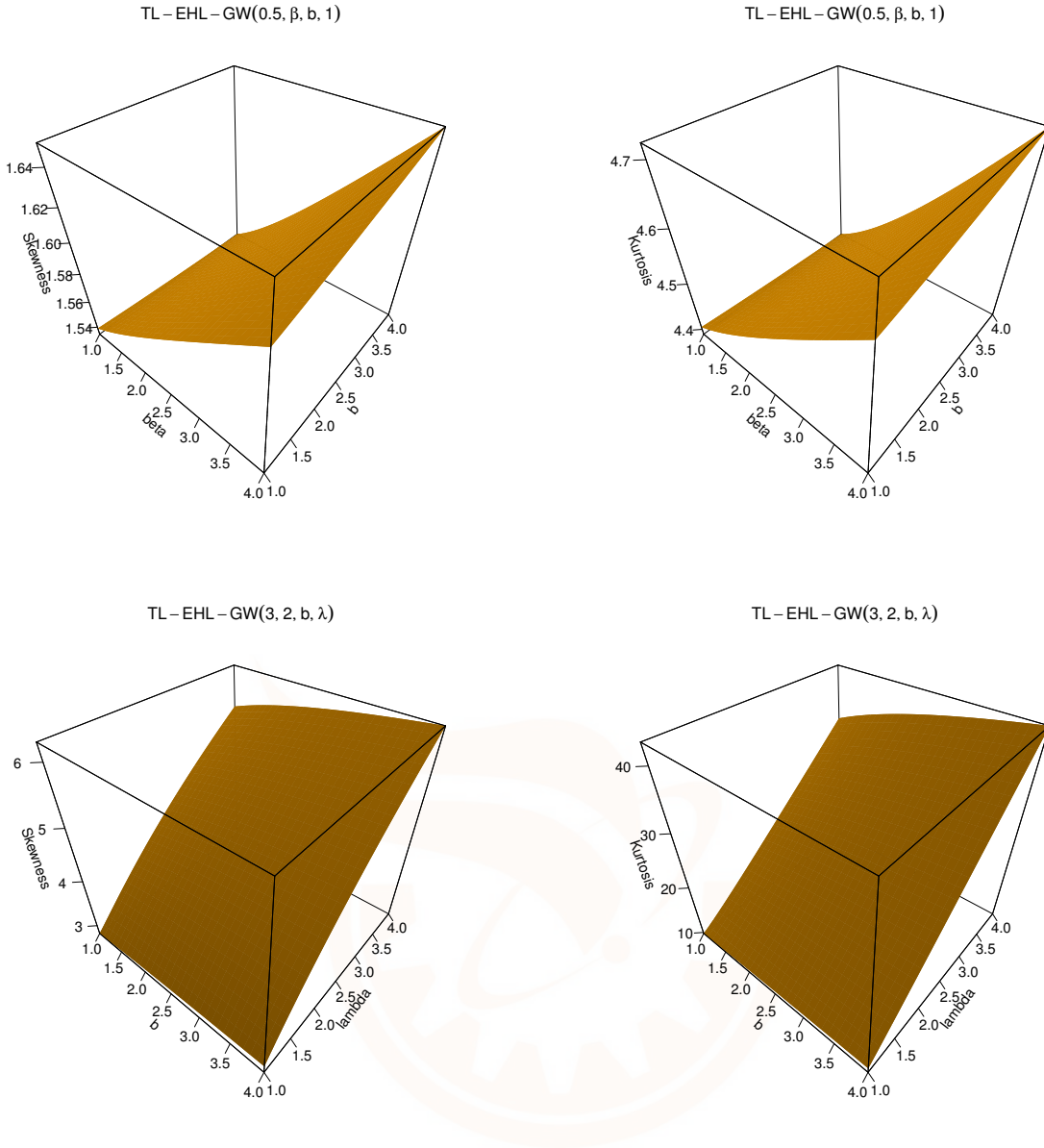


Figure 3.4: Skewness and kurtosis plots for TL-EHL-GW distribution

3.5.3 Topp-Leone Exponentiated Half Logistic Generalized-Power Distribution

If we take the baseline distribution to be power distribution with cdf and pdf given as $G(x; \lambda, \delta) = (\delta x)^\lambda$ and $g(x; \lambda, \delta) = \lambda \delta^\lambda x^{\lambda-1}$, respectively, for $\lambda, \delta > 0$ and $0 < x < \frac{1}{\delta}$. Then the cdf, pdf and hrf of the TL-EHL-GP distribution are given by

$$F_{TL-EHL-GP}(x; \alpha, \beta, b, \lambda, \delta) = \left[1 - \left(1 - (B_3(x; \alpha, \lambda, \delta))^\beta \right)^2 \right]^b,$$

$$\begin{aligned}
 f_{TL-EHL-GP}(x; \alpha, \beta, b, \lambda, \delta) &= 4\alpha\beta b \left[1 - \left(1 - (B_3(x; \alpha, \lambda, \delta))^\beta \right)^2 \right]^{b-1} \\
 &\times \left(1 - (B_3(x; \alpha, \lambda, \delta))^\beta \right) \frac{(1 - (1 - (\delta x)^\lambda)^\alpha)^{\beta-1}}{(1 + (1 - (\delta x)^\lambda)^\alpha)^{\beta+1}} \\
 &\times (\delta x)^{\lambda(\alpha-1)} \lambda \delta^\lambda x^{\lambda-1},
 \end{aligned}$$

and

$$\begin{aligned}
 h_{TL-EHL-GP}(x; \alpha, \beta, b, \lambda, \delta) &= 4\alpha\beta b \left[1 - \left(1 - (B_3(x; \alpha, \lambda, \delta))^\beta \right)^2 \right]^{b-1} \\
 &\times \left(1 - (B_3(x; \alpha, \lambda, \delta))^\beta \right) \frac{(1 - (1 - (\delta x)^\lambda)^\alpha)^{\beta-1}}{(1 + (1 - (\delta x)^\lambda)^\alpha)^{\beta+1}} \\
 &\times \left(1 - \left[1 - \left(1 - (B_3(x; \alpha, \lambda, \delta))^\beta \right)^2 \right]^b \right)^{-1} \\
 &\times (\delta x)^{\lambda(\alpha-1)} \lambda \delta^\lambda x^{\lambda-1}
 \end{aligned}$$

for $\alpha, \beta, b, \lambda, \delta > 0$, where $B_3(x; \alpha, \lambda, \delta) = \frac{1 - (1 - (\delta x)^\lambda)^\alpha}{1 + (1 - (\delta x)^\lambda)^\alpha}$.

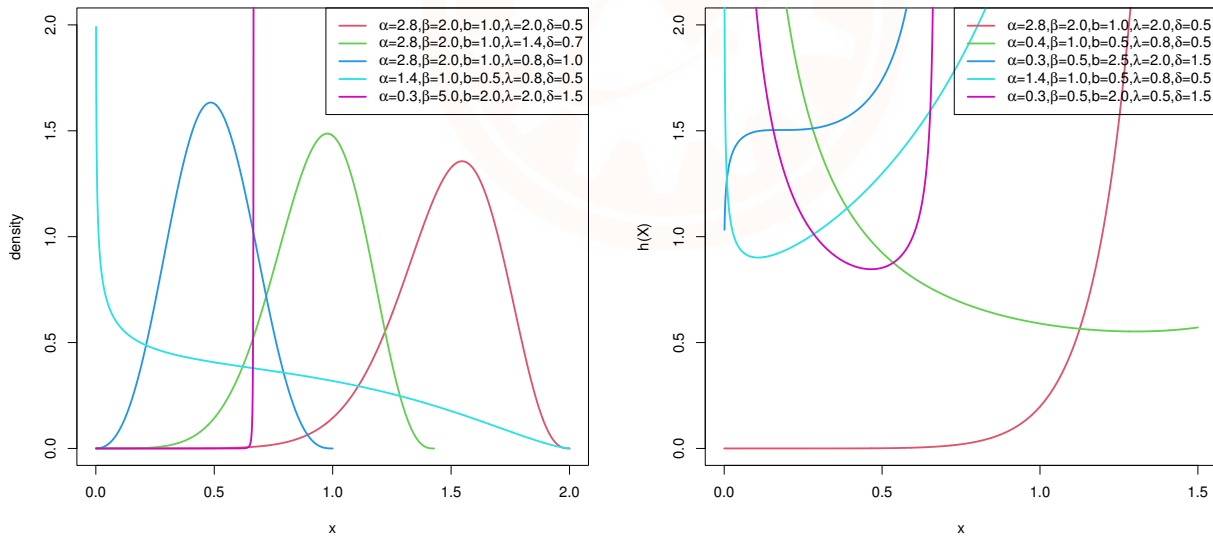


Figure 3.5: Plots of the pdf and hrf for the TL-EHL-GP distribution

Figure 3.5 shows the plots of the pdf and hrf for the TL-EHL-GP distribution. The pdf displays shapes that are left-skewed, almost symmetric, reverse-J and J-shapes, whereas the hrf shows bathtub, upside-down bathtub followed by bathtub, decreasing and increasing shapes. Figure 3.6 gives plots of skewness and kurtosis for the TL-EHL-GP distribution. The plots clearly show the dependence of skewness and kurtosis on the shape parameters.

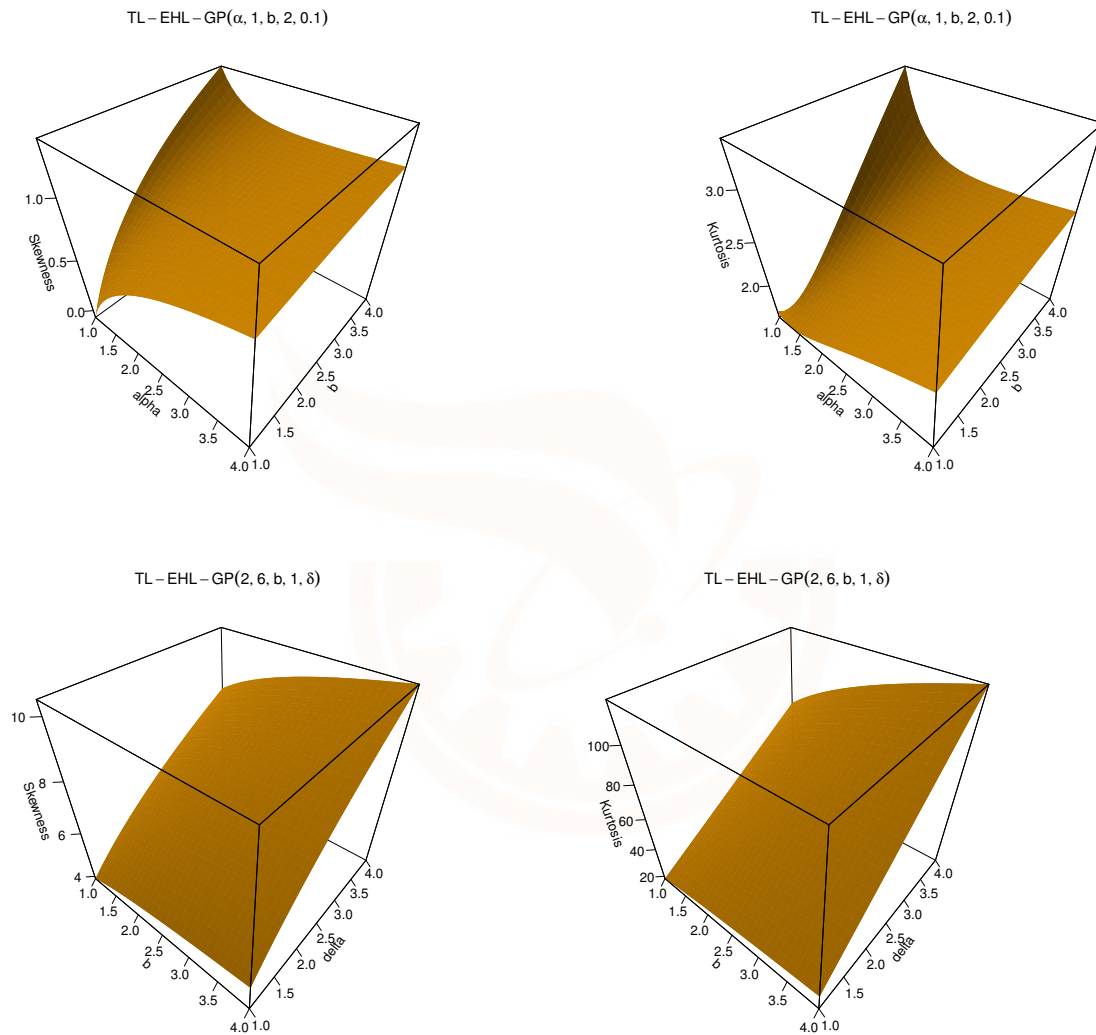


Figure 3.6: Skewness and kurtosis plots for TL-EHL-GP distribution

3.6 Simulation Results

The effectiveness of maximum likelihood estimators of the parameters of the TL-EHL-GW distribution is examined by conducting various simulations for different sample sizes ($n = 50, 100, 200, 400, 800, 1600$) via the mle2 (Bolker and Bolker (2017)) package in R. We simulate $N = 3000$ samples for the selected parameters values of $(\alpha, \beta, b, \lambda)$ as given in Tables 3.1 and 3.2, respectively. The BIAS and RMSEs for the estimated parameter, say, $\hat{\theta}$, are given by:

$$ABIAS(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. From the results, we see that RMSEs decrease and ABIAS decreases towards zero as n increases, and the mean estimates of parameter values are getting closer to the true parameter values as the sample size increases.

Table 3.1: Simulation Results for TL-EHL-GW Distribution: Mean, ABIAS and RMSE

Parameter	n	(0.5, 1.0, 0.5, 2.5)				(0.1, 2.0, 1.0, 4.0)				(0.1, 0.7, 0.7, 3.0)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
α	50	0.5850	0.4902	0.0850	0.0850	0.1598	0.3240	0.0598	0.0598	0.2089	0.2777	0.1089	0.1089
	100	0.5342	0.4001	0.0342	0.0342	0.1267	0.1259	0.0267	0.0267	0.1674	0.2006	0.0674	0.0674
	200	0.4987	0.3385	-0.0013	0.0013	0.1139	0.0685	0.0139	0.0139	0.1378	0.1434	0.0379	0.0379
	400	0.4771	0.2936	-0.0229	0.0229	0.1080	0.0474	0.0080	0.0080	0.1228	0.1172	0.0228	0.0228
	800	0.4771	0.2457	-0.0229	0.0229	0.1033	0.0328	0.0033	0.0033	0.1154	0.0967	0.0154	0.0154
	1600	0.4807	0.2018	-0.0193	0.0193	0.1020	0.0233	0.0020	0.0020	0.1115	0.0753	0.0115	0.0115
β	50	1.5869	1.2846	0.5869	0.5869	2.2705	2.7793	0.2705	0.2705	1.3784	1.3184	0.6784	0.6784
	100	1.4333	1.1795	0.4333	0.4333	2.1514	1.0000	0.1514	0.1514	1.2325	1.1972	0.5325	0.5325
	200	1.3381	1.0934	0.3381	0.3381	2.0938	0.5582	0.0938	0.0938	1.1115	1.0382	0.4115	0.4115
	400	1.2199	0.9503	0.2199	0.2199	2.0709	0.5089	0.0709	0.0709	0.9850	0.8730	0.2850	0.2850
	800	1.1572	0.8274	0.1572	0.157	2.0481	0.3976	0.0481	0.0481	0.9131	0.7308	0.2131	0.2131
	1600	1.1059	0.6832	0.1059	0.1059	2.0266	0.3057	0.0266	0.0266	0.8294	0.5376	0.1294	0.1294
b	50	2.7004	8.9821	2.2004	2.2004	2.9124	9.0313	1.9124	1.9124	3.9604	3.4871	3.2604	3.2604
	100	1.4469	3.8795	0.9469	0.9469	1.6445	3.5929	0.6445	0.6445	1.6764	3.4703	0.9764	0.9764
	200	0.8400	1.1394	0.3400	0.3400	1.2774	2.0256	0.2774	0.2774	1.0252	1.2787	0.3252	0.3252
	400	0.6889	0.5624	0.1889	0.1889	1.0895	0.6098	0.0895	0.0895	0.8672	0.6843	0.1671	0.1671
	800	0.6345	0.4015	0.1345	0.1345	1.0471	0.4086	0.0471	0.0471	0.7897	0.4470	0.0897	0.0897
	1600	0.5815	0.2768	0.0815	0.0815	1.0245	0.2086	0.0245	0.0245	0.7508	0.3087	0.0508	0.0508
λ	50	3.3701	2.2041	0.8701	0.8701	4.3493	1.3711	0.3493	0.3493	3.3312	1.4174	0.3312	0.3312
	100	3.1792	1.7922	0.6792	0.6792	4.1642	1.0253	0.1642	0.1642	3.3279	1.2457	0.3279	0.3279
	200	3.0361	1.4232	0.5361	0.5361	4.0200	0.6594	0.0200	0.0200	3.2881	1.0466	0.2881	0.2881
	400	2.9237	1.1094	0.4237	0.4237	4.0045	0.4968	0.0045	0.0045	3.2571	0.9051	0.2571	0.2571
	800	2.7828	0.8051	0.2828	0.2828	4.0087	0.3631	0.0087	0.0087	3.1839	0.7461	0.1839	0.1839
	1600	2.6845	0.5833	0.1845	0.1845	3.9977	0.2519	-0.0023	0.0023	3.0918	0.5705	0.0918	0.0918

Table 3.2: Simulation Results for TL-EHL-GW Distribution: Mean, ABIAS and RMSE

Parameter	n	(2.0, 2.0, 0.5, 1.5)				(0.5, 1.0, 0.3, 3.5)				(0.1, 1.0, 2.0, 4.0)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
α	50	1.6522	0.9609	-0.3478	0.3478	0.5230	0.4537	0.0230	0.0230	0.2072	0.4051	0.1072	0.1072
	100	1.6256	0.8252	-0.3744	0.3744	0.4855	0.3797	-0.0145	0.0145	0.1397	0.1594	0.0397	0.0397
	200	1.6340	0.7388	-0.3659	0.3659	0.4719	0.3316	-0.0281	0.0281	0.1189	0.0924	0.0189	0.0189
	400	1.7082	0.6256	-0.2918	0.2918	0.4839	0.2930	-0.0161	0.0161	0.1199	0.0716	0.0199	0.0199
	800	1.7879	0.5069	-0.2121	0.2121	0.4946	0.2528	-0.0054	0.0054	0.1108	0.0503	0.0108	0.0108
	1600	1.8445	0.4243	-0.1555	0.1555	0.5067	0.2111	0.0047	0.0047	0.1071	0.0376	0.0071	0.0071
β	50	1.7451	2.8126	-0.2549	0.2549	1.5562	1.4652	0.5562	0.5562	1.7435	4.2186	0.7435	0.7435
	100	1.7161	1.8882	-0.2839	0.2839	1.4893	1.4088	0.4893	0.4893	1.2514	1.1209	0.2514	0.2514
	200	1.7125	1.6476	-0.2875	0.2875	1.4346	1.3457	0.4346	0.4346	1.1925	0.8149	0.1925	0.1925
	400	1.8049	1.5528	-0.1951	0.1951	1.4307	1.2971	0.4307	0.4307	1.2279	0.7963	0.2279	0.2279
	800	1.8759	1.4283	-0.1241	0.1241	1.3875	1.1913	0.3875	0.3875	1.1553	0.6491	0.1553	0.1553
	1600	1.9319	1.3308	-0.0681	0.0681	1.3228	1.0474	0.3228	0.3228	1.1010	0.5256	0.1010	0.1010
b	50	1.7699	5.2073	1.4699	1.4699	1.5273	3.9444	1.2273	1.2273	6.4417	9.2306	4.4417	4.4417
	100	1.0355	2.2889	0.7355	0.7355	0.9374	2.3473	0.6374	0.6374	3.8541	6.9104	1.8541	1.8541
	200	0.7475	1.0467	0.4475	0.4475	0.6036	1.1441	0.3036	0.3036	3.4697	6.1498	1.4697	1.4697
	400	0.6298	0.6894	0.3298	0.3298	0.4536	0.6455	0.1536	0.1536	2.4469	2.8508	0.4469	0.4469
	800	0.5767	0.4976	0.2767	0.2767	0.3654	0.2677	0.0654	0.0654	2.1883	1.5549	0.1883	0.1883
	1600	0.5244	0.3985	0.2244	0.2244	0.3451	0.1763	0.0451	0.0451	2.0934	0.9089	0.0934	0.0934
λ	50	1.8914	0.8928	0.3914	0.3914	4.9874	3.3345	1.4874	1.4874	4.2569	1.5032	0.2569	0.2569
	100	1.8861	0.8176	0.3861	0.3861	4.7101	2.7727	1.2101	1.2101	4.2426	1.2029	0.2426	0.2426
	200	1.8638	0.7341	0.3638	0.3638	4.4446	2.2434	0.9446	0.9446	4.1193	0.8713	0.1193	0.1193
	400	1.8036	0.6249	0.3036	0.3036	4.0902	1.5939	0.5902	0.5902	3.9822	0.6428	-0.0178	0.0178
	800	1.7494	0.4981	0.2494	0.2494	3.8435	1.0876	0.3435	0.3435	3.9883	0.4804	-0.0117	0.0117
	1600	1.6662	0.3621	0.1662	0.1662	3.6637	0.7082	0.1637	0.1637	3.9829	0.3584	-0.0170	0.0170

3.7 Applications

In this subsection, we give examples of applications to real data sets in order to assess the usefulness of the TL-EHL-GW distribution. The TL-EHL-GW distribution is compared to several competing models with the same number of parameters including Topp-Leone Weibull-Lomax distribution by Jamal et al. (2019), Topp-Leone generalized inverted Kumaraswamy distribution by Reyad et al. (2019), Weibull-Lomax distribution by Tahir et al. (2014), the Beta odd Lindley-exponential distribution by Chipepa et al. (2019) and odd log-logistic exponentiated Weibull distribution by Afify et al. (2018). The pdf's of the models listed above are given by

$$\begin{aligned}
 f_{TLWLx}(x; a, b, \alpha, \theta) &= 2\theta\alpha ab(1 + bx)^{a\alpha-1} \{1 - (1 + bx)^{-a}\}^{a-1} \\
 &\times \exp \left\{ -2 \left(\frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right)^\alpha \right\} \\
 &\times \left[1 - \exp \left\{ -2 \left(\frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right)^\alpha \right\} \right]^{\theta-1},
 \end{aligned}$$

for $x > 0, a, b, \alpha, \theta > 0$,

$$f_{TLGIKw}(x; \alpha, \lambda, \beta, \theta) = 2\alpha\beta\lambda\theta x^{\lambda-1} \left[1 - \{1 - [1 - (1 + x^\lambda)^{-\alpha}]^\beta\}^2 \right]^{\theta-1} \\ \times (1 + x^\lambda)^{-\alpha-1} [1 - (1 + x^\lambda)^{-\alpha}]^{\beta-1} \{1 - [1 - (1 + x^\lambda)^{-\alpha}]^\beta\}$$

for $x > 0, \alpha, \lambda, \beta, \theta > 0$,

$$f_{wL}(x; a, b, \alpha, \beta) = \frac{ab\alpha}{\beta} (1 + (x/\beta))^{b\alpha-1} (1 - (1 + (x/\beta)))^{b-1} \exp\{-a[(1 + (x/\beta))^\alpha - 1]^b\},$$

for $x > 0, a, b, \alpha, \beta > 0$,

$$f_{BOL-E}(x; a, b, \lambda, \theta) = \frac{1}{B(a, b)} \left[1 - \frac{\lambda + e^{-\theta x}}{(1 + \lambda)e^{-\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\} \right]^{a-1} \\ \times \left[\frac{\lambda + e^{-\theta x}}{(1 + \lambda)e^{-\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\} \right]^{b-1} \\ \times \frac{\lambda^2}{(1 + \lambda)} \frac{(\theta e^{-\theta x})}{e^{-3\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\},$$

for $x > 0, a, b, \lambda, \theta > 0$, and

$$f_{OLLEW}(x; \alpha, \beta, \gamma, \delta) = \frac{\theta\beta\gamma x^{\beta-1} e^{-(x/\alpha)^\beta} [1 - e^{-(x/\alpha)^\beta}]^{\gamma\theta-1} \{1 - [1 - e^{-(x/\alpha)^\beta}]^\gamma\}^{\theta-1}}{\alpha^\beta \{[1 - e^{-(x/\alpha)^\beta}]^{\gamma\theta} + \{1 - [1 - e^{-(x/\alpha)^\beta}]^\gamma\}^\theta\}^2},$$

for $x > 0, \alpha, \beta, \gamma, \delta > 0$.

The estimated values of parameters of the distributions with standard errors (in parentheses) are given in Tables 3.3 and 3.4, respectively. The goodness-of-fit statistics including -2log-likelihood statistic (-2log(L)), AIC, AICC, BIC, (W^*), (A^*) statistic, K-S statistic and its p-value are also presented in the tables.

Plots of the fitted pdfs, the histogram of the data, probability plots, estimated cdf, K-M survival plots, and TTT plots are given in Figures 3.8, 3.9, 3.11, 3.12, 3.14 and 3.15. For the probability plots, we plotted $F(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{b}, \hat{\lambda})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness to the diagonal line is given by the sum of squares:

$$SS = \sum_{j=1}^n \left[F(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{b}, \hat{\lambda}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2,$$

(see Chambers et al. (1983) for additional details). For the scaled TTT-transform plot, we plotted $\Phi_F(u)$ vs u , where

$$\Phi_F(u) = \frac{H_F^{-1}(u)}{H_F^{-1}(1)} \quad \text{and} \quad H^{-1}(u) = \int_0^{F^{-1}(u)} (1 - F(x))dx, \quad \text{for } 0 \leq u \leq 1.$$

3.7.1 Growth Hormone Data

The first data corresponds to growth hormone data, also cited by Alizadeh et al. (2018). The observations are: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

The estimated variance-covariance matrix for TL-EHL-GW distribution on growth hormone data set is given by

$$\begin{pmatrix} 2.8531 \times 10^{-02} & -6.3355 \times 10^{-04} & -4.2978 \times 10^{-05} & -3.2370 \times 10^{-03} \\ -6.3355 \times 10^{-04} & 1.4068 \times 10^{-05} & 9.5436 \times 10^{-07} & 7.1921 \times 10^{-05} \\ -4.2978 \times 10^{-05} & 9.5436 \times 10^{-07} & 6.4741 \times 10^{-08} & 4.8793 \times 10^{-06} \\ -3.2370 \times 10^{-03} & 7.1921 \times 10^{-05} & 4.8793 \times 10^{-06} & 4.4838 \times 10^{-04} \end{pmatrix}.$$

Table 3.3: Parameter estimates and goodness-of-fit statistics of models for growth hormone data

Model	Estimates				Statistics							
	α	β	b	λ	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
TL-EHL-GW	4.7814 (0.1617)	13.6540 (0.0115)	349.6700 (0.0002)	0.1962 (0.0249)	155.1	163.1	164.5	169.4	0.0352	0.2487	0.0851	0.9619
TLWLx	a (8.0040×10^{-07})	b (2.6926×10^{-04})	α (2.8529×10^{-05})	θ (8.6509×10^{-02})	188.6	196.6	197.9	202.8	0.3762	2.3515	0.2399	0.0355
TLGIKw	α (0.0165)	λ (0.0002)	β (0.6183)	θ (0.1475)	155.3	163.3	164.6	169.5	0.0417	0.2777	0.0949	0.9108
WL	a (0.0071)	b (2.6003)	α (0.0537)	β (2.8892)	161.3	169.3	170.7	175.6	0.1137	0.7300	0.12578	0.6371
BOL-E	a (4.8194×10^{-07})	b (5.1120×10^{-07})	λ (5.6004×10^{-07})	θ (4.1789×10^{-07})	160.2	168.2	169.7	174.5	0.1013	0.6572	0.1229	0.6663
OLLEW	α (0.0197)	β (0.0211)	γ (1.7941)	θ (0.4294)	158.7	166.7	167.9	172.9	0.0574	0.4135	0.0983	0.8880

From Table 3.3, the TL-EHL-GW distribution has the smallest goodness-of-fit statistics compared to other non-nested distributions. Therefore, we can conclude that the TL-EHL-GW model performs better than the non-nested distributions listed in the table for the growth hormone data. Moreover, the TL-EHL-GW model has the highest K-S p-value and smallest SS value from the probability plots in Figure 3.8. Plots of the profile log-likelihood in Figure 3.7

shows that maximum likelihood estimates of α, β, b and λ can be uniquely obtained.

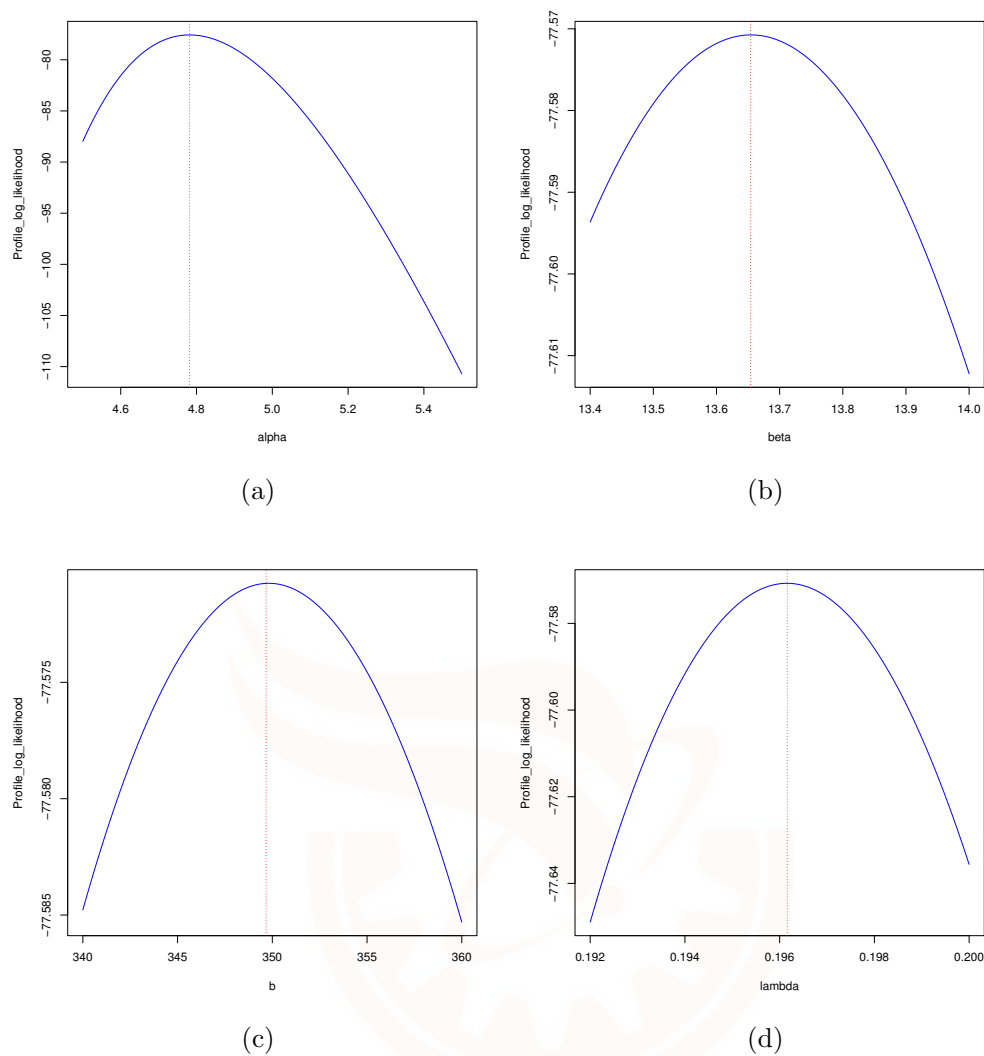


Figure 3.7: Profile log-likelihood plots for growth hormone data

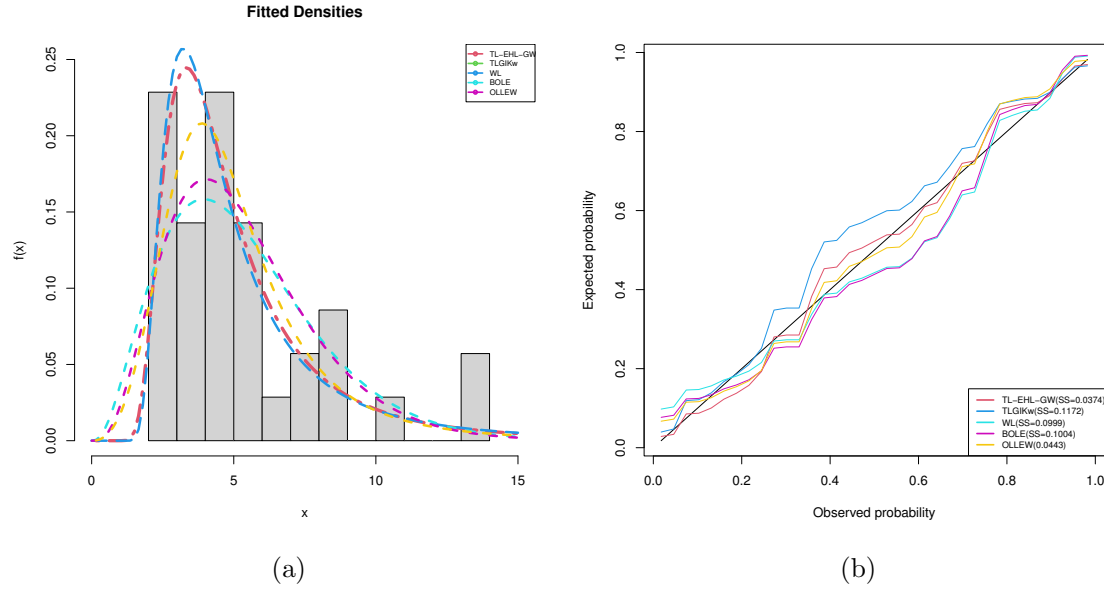


Figure 3.8: Fitted pdfs and probability plots for growth hormone data

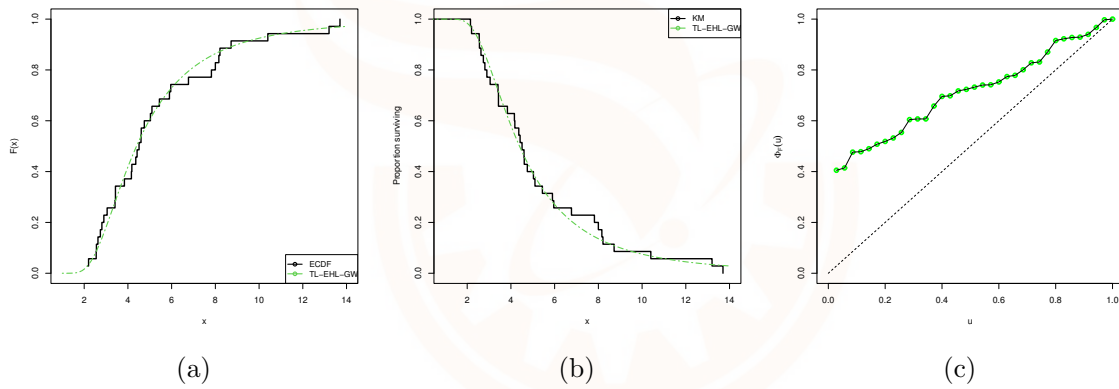


Figure 3.9: Estimated cdf, K-M survival and scaled TTT-Transform plots for growth hormone data

Figure 3.9 shows the estimated cdf, K-M curve and the scaled TTT plots. From the plots we see that the estimated cdf line for the TL-EHL-GW distribution is closer to the empirical cdf while the survival function is also close to the K-M curve, hence our model is best in explaining the growth hormone data. The TTT-transform plot shows that our model can fit data with increasing hrf.

3.7.2 Relief Times of Patients Data

The second data set represent the relief times of 20 patients (Gross and Clark (1975)). The observations are: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

The estimated variance-covariance matrix for TL-EHL-GW distribution on relief times of pa-

tients data set is given by

$$\begin{pmatrix} 0.0252 & -2.1471 \times 10^{-05} & -4.9453 \times 10^{-06} & -0.0034 \\ -2.1471 \times 10^{-05} & 1.8319 \times 10^{-08} & 4.2193 \times 10^{-09} & 2.9325 \times 10^{-06} \\ -4.9453 \times 10^{-06} & 4.2193 \times 10^{-09} & 9.7181 \times 10^{-10} & 6.7553 \times 10^{-07} \\ -0.0034 & 2.9325 \times 10^{-06} & 6.7553 \times 10^{-07} & 8.5287 \times 10^{-04} \end{pmatrix}.$$

Table 3.4: Parameter estimates and goodness-of-fit statistics of models for relief times of patients data

Model	Estimates				Statistics							
	α	β	b	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
TL-EHL-GW	10.8060 (0.1586)	1170.9000 (0.0001)	2589.5000 (3.1174×10^{-05})	0.1714 (0.0292)	30.8013	38.8013	41.4679	42.7843	0.0286	0.1619	0.0927	0.9954
TLWLx	a 3.2376 (1.2582)	b 0.4172 (0.2289)	α 0.4999 (0.2036)	θ 68.0184 (78.9434)	33.4483	41.4483	44.1149	45.4312	0.0693	0.4098	0.1716	0.5979
TLGIKw	α 0.3496 (3.4728)	λ 6.1246 (79.4909)	β 1.6868 (61.4218)	θ 2.8426 (151.8967)	30.8485	38.8842	41.5509	42.8671	0.0332	0.1863	0.1010	0.9868
WL	a 489.9400 (0.0040)	b 5.1301 (4.6914)	α 0.1582 (1.1282)	β 0.4965 (1.6855)	38.9939	46.9939	49.6606	50.9768	0.1519	0.8971	0.1767	0.5605
BOL-E	a 9.6006 (3.1958×10^{-08})	b 110.0100 (2.8056×10^{-08})	λ 21.8890 (1.5272×10^{-08})	θ 0.0021 (1.5198×10^{-04})	35.6567	43.6567	46.3233	47.6396	0.1061	0.6285	0.1721	0.5944
OLLEW	α 0.0041 (4.5122×10^{-03})	β 0.1723 (3.0926×10^{-02})	γ 11.417 (1.1872×10^{-04})	θ 8.4420 (5.8325×10^{-04})	32.9642	40.9642	43.6309	44.9471	0.0523	0.3122	0.1108	0.9669

According to the results in Table 3.4, the values of the goodness-of-fit statistics: AIC, AICC and BIC are smallest for the TL-EHL-GW distribution compared to the other non-nested models. Also, the values of the K-S, W^* and A^* statistics are the least for the proposed model. Thus, we can conclude that the TL-EHL-GW distribution performs better for the relief times of patients data than the non-nested TLWLx, TLGIKw, WL, BOL-E and OLLEW distributions. Again, Figure 3.11 indicates that our model outperforms the competing non-nested models on relief times of patients data. Profile log-likelihood plots in Figure 3.10 shows that maximum likelihood estimates of α , β , b and λ can be uniquely obtained.

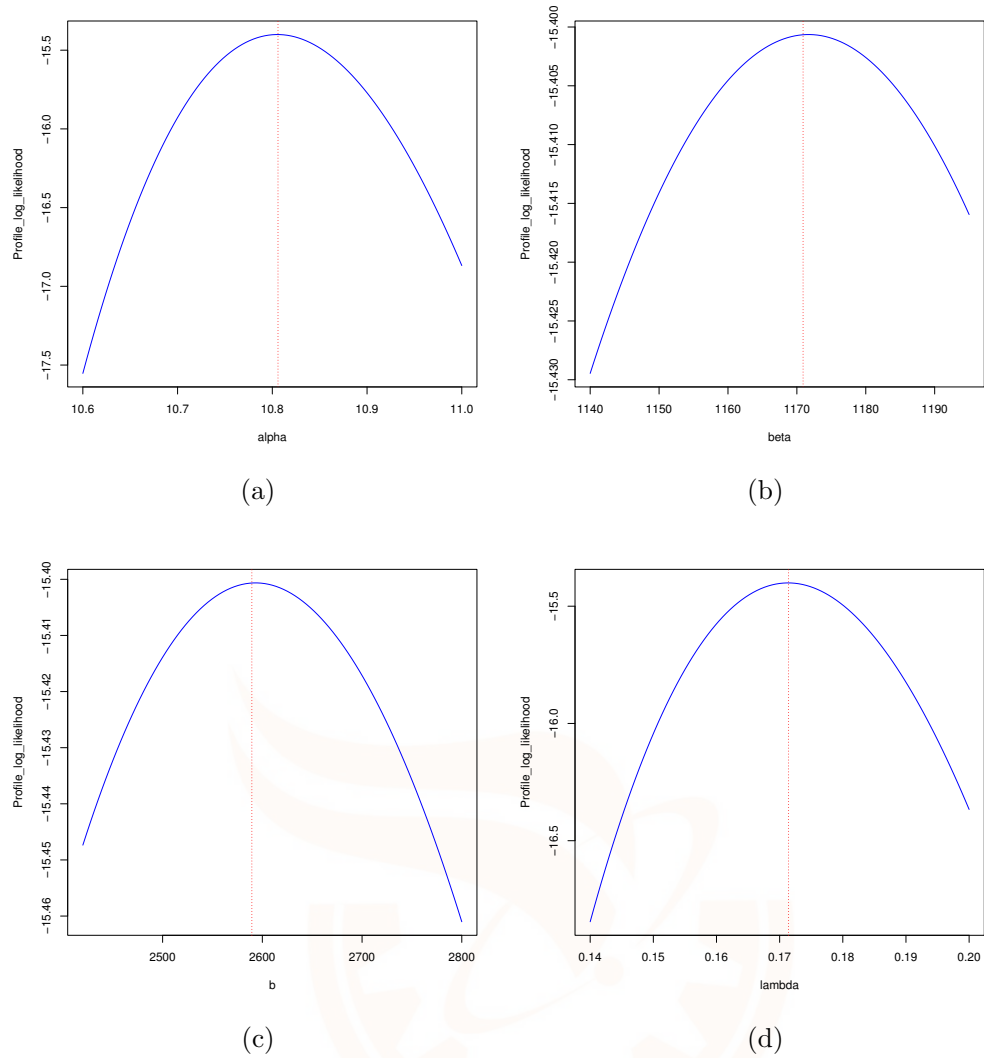


Figure 3.10: Profile log-likelihood plots for patients data

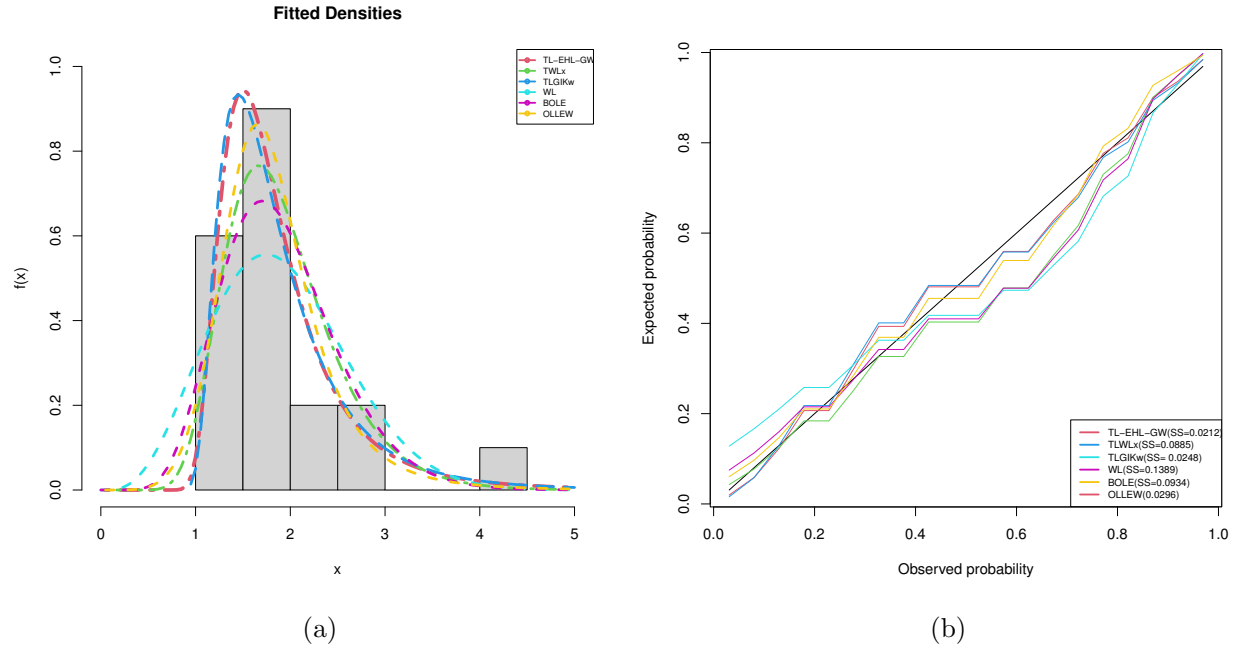


Figure 3.11: Fitted pdfs and probability plots for relief times of patients data

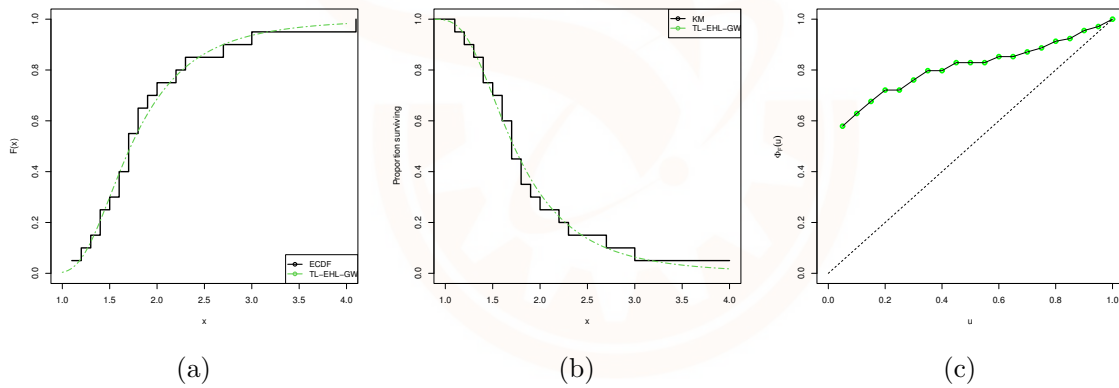


Figure 3.12: Estimated cdf, K-M survival and scaled TTT-Transform plots for patients data

In Figure 3.12, we see that the cdf line for the TL-EHL-GW distribution is superimposed on the empirical cdf while the survival function is also close to the K-M curve indicating that our model is good in explaining the relief times of patients data.

3.7.3 Waiting Times of Bank Customers Data

The following data set corresponds to the waiting times (in minutes) of 100 bank customers. The data was analysed by Ghitany et al. (2008). The observations are as follows: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

The estimated variance-covariance matrix for TL-EHL-GW distribution on bank customers data set is given by

$$\begin{pmatrix} 0.0141 & 7.6168 \times 10^{-03} & 1.5592 \times 10^{-04} & -0.0404 \\ 0.0076 & 4.2077 \times 10^{-03} & 8.4272 \times 10^{-05} & -0.0217 \\ 0.0002 & 8.4272 \times 10^{-05} & 1.7348 \times 10^{-06} & -0.0005 \\ -0.0404 & -2.1749 \times 10^{-05} & -4.5053 \times 10^{-04} & 0.1172 \end{pmatrix}.$$

Table 3.5: Parameter Estimates and Goodness-of-Fit Statistics of Models for Bank Customers Data

Model	Estimates				Statistics							
	α	β	b	λ	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
TL-EHL-GW	0.0921 (0.1186)	0.1412 (0.0649)	32.9208 (0.0013)	0.9614 (0.3423)	633.9	641.9	642.4	652.4	0.0179	0.1322	0.0369	0.9992
TLWLx	a 18.163 (2.8594×10^{-05})	b 0.0041 (2.9104×10^{-04})	α 0.5389 (4.0835×10^{-02})	θ 4.1174 (2.9813×10^{-03})	636.6	644.6	645.1	655.1	0.0543	0.3508	0.0642	0.8050
TLGIKw	α 1.6288 (0.4939)	λ 0.7865 (0.1814)	β 52.8214 (40.0881)	θ 0.2792 (0.1697)	645.8	653.8	654.2	664.2	0.1254	0.8942	0.07389	0.6458
WL	a 2.3751×10^{-05} (2.3750×10^{-06})	b 1.3714 (4.5634×10^{-10})	α 1.0629 (5.8906×10^{-10})	β 7.3688×10^{-03} (1.1148×10^{-08})	637.5	645.5	645.9	655.9	0.0629	0.3961	0.0577	0.8920
BOL-E	a 2.0015 (5.0593×10^{-09})	b 21.3750 (4.8510×10^{-10})	λ 24.7340 (4.4519×10^{-10})	θ 0.0004 (2.7384×10^{-05})	634.6	642.6	643.0	653.04	0.0279	0.1839	0.0434	0.9917
OLLEW	α 2.6990×10^{-05} (2.0210×10^{-05})	β 0.1261 (7.5511×10^{-03})	γ 90.1420 (5.9157×10^{-07})	θ 2.5614 (1.9911×10^{-04})	640.3	648.3	648.7	658.7	0.0693	0.4930	0.0551	0.9214

From Table 3.5, the values of the goodness-of-fit statistics: AIC, AICC and BIC are smallest for the TL-EHL-GW distribution compared to other non-nested models. Again, the values of the goodness-of-fit statistics: K-S, W^* and A^* are smallest for the proposed distribution. Therefore, we conclude that the TL-EHL-GW distribution performs better for the waiting times of customers data than the non-nested distributions listed in the table. Figure 3.14 indicates that our model outperforms the competing non-nested models on waiting times of customers data. Figure 3.13 confirms that maximum likelihood estimates of α, β, b and λ exist and are unique.

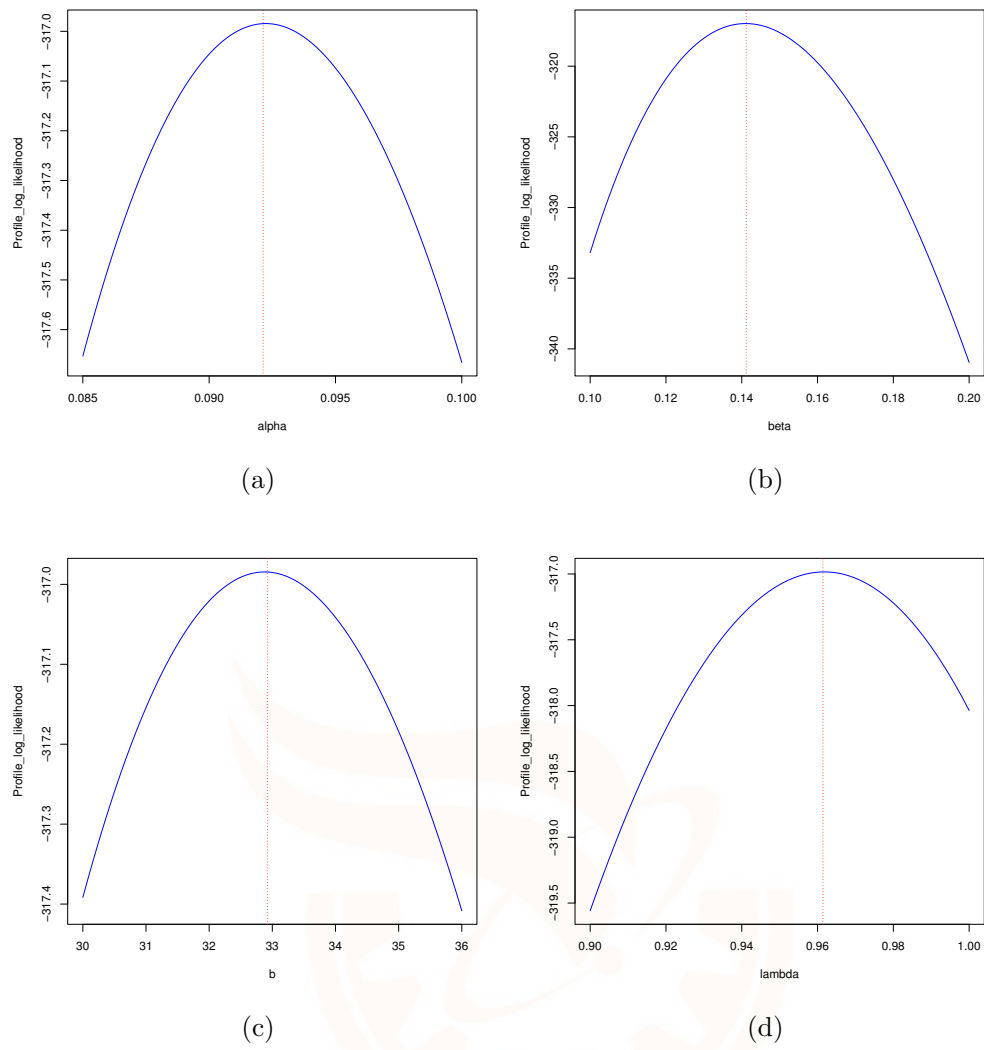


Figure 3.13: Profile log-likelihood plots for Bank Customers Data

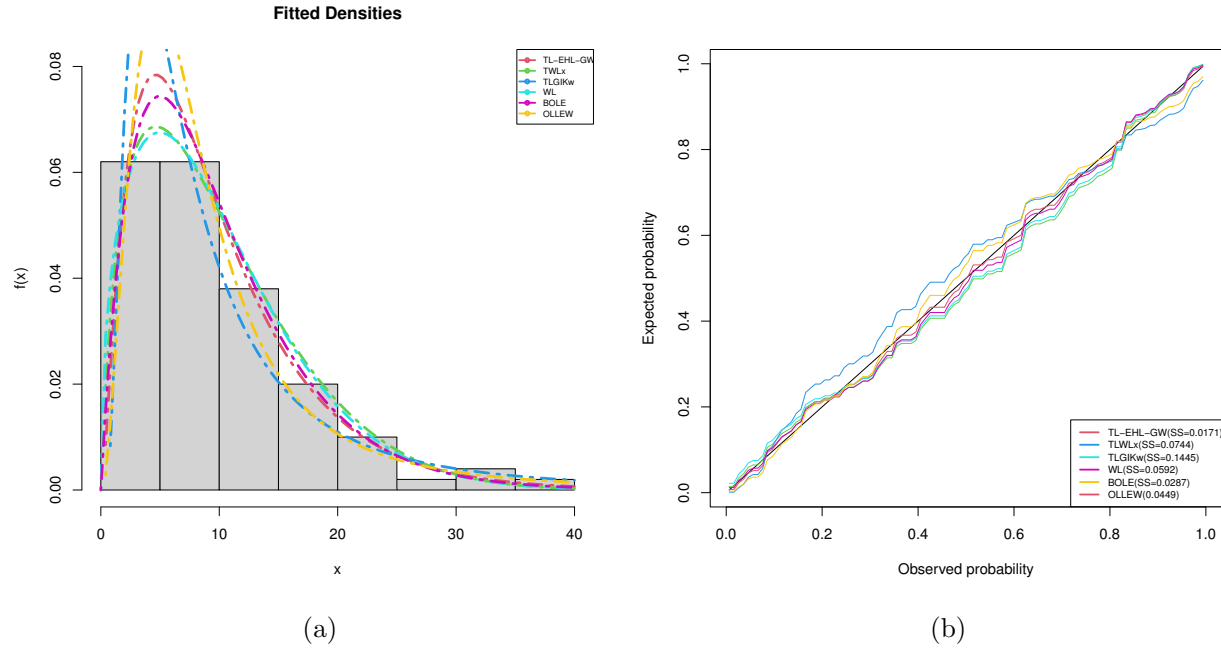


Figure 3.14: Fitted pdfs and probability plots for waiting times of customers data

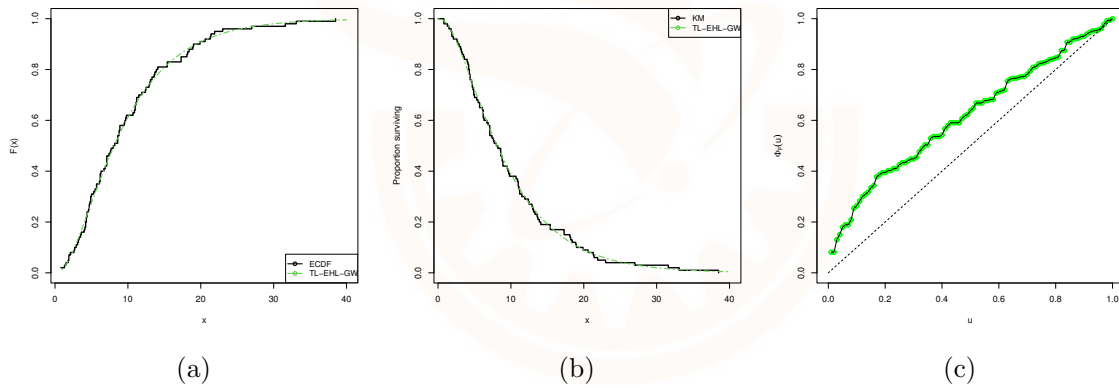


Figure 3.15: Estimated cdf, K-M survival and scaled TTT-Transform plots for customers data

In Figure 3.15, the survival function is close to the K-M curve, demonstrating that our model is effective at describing the waiting times of customers data. The estimated cdf line for the TL-EHL-GW distribution is seen to be on the empirical cdf.

3.8 Concluding Remarks

We proposed a new family of generalized distributions known as the Topp-Leone Exponentiated Half Logistic-Generalized-G distribution. Some important statistical properties of the new family of distributions were derived. Some special cases of the new family of distributions were also studied in details. We conducted a simulation study to examine the performance of the maximum likelihood estimates. Lastly, three real data examples were given to show the importance and flexibility of the proposed family of distributions where we established that indeed the model is useful and performs better than some selected competing models.

4. The Topp-Leone-Gompertz-G Power Series Class of Distributions

4.1 Introduction

The Gompertz generator can be used in the analysis of survival and failure times of computer codes by computer scientists (Eliwa et al. (2020)). It is well known that power series class of distributions are extremely important in the areas of finance, reliability and actuarial sciences. Moreover, the combination of other generators with the Topp-Leone generator can be very useful in reliability analysis and related areas. We combine the Topp-Leone generator with the Gompertz generator, and compound the resulting family of distributions with the power series distribution to obtain new Topp-Leone-Gompertz-G power series class of distributions.

The new generalized class of distributions, the TL-Gom-GPS distribution is presented in this chapter. Some statistical properties of the new class of distributions are explored. For this new class of distributions, we derived the quantile function, moments and generating function, probability weighted moments, distribution of the order statistics and Rényi entropy. The maximum likelihood method is used for estimating the parameters and simulations are conducted to show the performance of the proposed model. Finally, the usefulness and flexibility of the new class of distributions is examined by means of applications to real data sets.

4.2 The New Class of Distributions

In this section, we present the proposed model and its sub-models. Suppose N is a discrete random variable following a power series distribution and assumed to be truncated at zero with the pmf given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \dots, \quad (4.1)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n \geq 1}$ a sequence of positive real numbers. The power series family of distributions includes Poisson, binomial, geometric and logarithmic distributions (Johnson et al. (1994)).

The Topp-Leone Gompertz-G family of distributions was proposed by Oluyede et al. (2022). Let X be a random variable such that X_i , for $i = 1, 2, \dots, N$ denote the time to failure of a device due to the i^{th} defect with the assumption that the X_i 's iid TL-Gom-G random variables and $X_{(1)} = \min(X_1, X_2, \dots, X_N)$, with the distribution of N given by equation (4.1). The TL-Gom-GPS class of distributions is defined by the marginal distribution of $X_{(1)}$, say $F_\theta(x)$, and is given by

$$\begin{aligned}
 F_{\theta}(x) &= 1 - \frac{C(\theta S(x; b, \gamma, \xi))}{C(\theta)} \\
 &= 1 - \frac{C\left(\theta \left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - (1 - G(x; \xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}{C(\theta)}, \quad (4.2)
 \end{aligned}$$

where $S(x; b, \gamma, \xi)$ is the survival function of the TL-Gom-G distribution. The corresponding pdf is given by

$$\begin{aligned}
 f_{\theta}(x) &= 2b\theta g(x; \xi)(1 - G(x; \xi))^{-\gamma-1} \left(\exp\left[\frac{1}{\gamma}(1 - (1 - G(x; \xi))^{-\gamma})\right]\right)^2 \\
 &\times \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - (1 - G(x; \xi))^{-\gamma})\right]\right)^2\right]^{b-1} \\
 &\times \frac{C'\left(\theta \left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - (1 - G(x; \xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}{C(\theta)}, \quad (4.3)
 \end{aligned}$$

for $b, \gamma, \theta > 0$ and parameter vector ξ .

Special classes of the TL-Gom-GPS class of distributions are given in Table 4.1.

Table 4.1: Special classes of the TL-Gom-GPS class of distributions

Distribution	a_n	$C(\theta)$	cdf
TL-Gom-G Poisson	$(n!)^{-1}$	$e^{\theta} - 1$	$1 - \frac{\left(\theta \left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - (1 - G(x; \xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}{e^{\theta} - 1} - 1$
TL-Gom-G Geometric	1	$\theta(1 - \theta)^{-1}$	$1 - \frac{(1 - \theta) \left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - (1 - G(x; \xi))^{-\gamma})\right]\right)^2\right]^b\right)}{\left(1 - \theta \left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - (1 - G(x; \xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}$
TL-Gom-G Logarithmic	n^{-1}	$-\log(1 - \theta)$	$1 - \frac{\log\left(1 - \theta \left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - (1 - G(x; \xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}{\log(1 - \theta)}$
TL-Gom-G Binomial	$\binom{m}{n}$	$(1 + \theta)^m - 1$	$1 - \frac{\left(1 + \theta \left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - (1 - G(x; \xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)^m}{(1 + \theta)^m - 1} - 1$

4.3 Some Statistical Properties

In this section, we present the linear representation of the density function, hazard and quantile functions, moments, generating function, probability weighted moments, distribution of the i^{th}

order statistics and Rényi entropy of the TL-Gom-GPS class of distributions.

4.3.1 Expansion of the Density Function

In this subsection, we obtain the linear representation of the TL-Gom-GPS class of distributions using the following generalized binomial and Taylor series expansions:

$$(1 - z)^s = \sum_{k=0}^{\infty} (-1)^k \binom{s}{k} z^k, \quad \text{for } |z| < 1, \quad \text{and } e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Note that the pdf of the TL-Gom-GPS class of distributions can be written as

$$\begin{aligned} f_{\theta}(x) &= \sum_{n=1}^{\infty} \frac{na_n\theta^n}{C(\theta)} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^b \right)^{n-1} \\ &\times 2b\theta g(x; \xi) (1 - G(x; \xi))^{-\gamma-1} \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^{b-1}. \end{aligned}$$

Considering the following generalized binomial expansions:

$$\begin{aligned} &\left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^b \right)^{n-1} = \sum_{i=0}^{\infty} (-1)^i \binom{n-1}{i} \\ &\times \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^{bi}, \end{aligned}$$

$$\begin{aligned} &\left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^{b(i+1)-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b(i+1)-1}{j} \\ &\times \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^{2j}, \end{aligned}$$

$$\left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^{2j+2} = \sum_{k=0}^{\infty} \frac{(2j+2)^k}{\gamma^k k!} (1 - (1 - G(x; \xi))^{-\gamma})^k,$$

$$(1 - (1 - G(x; \xi))^{-\gamma})^k = \sum_{l=0}^{\infty} (-1)^l \binom{k}{l} (1 - G(x; \xi))^{-\gamma l},$$

and

$$(1 - G(x; \xi))^{-\gamma(l+1)-1} = \sum_{t=0}^{\infty} (-1)^t \binom{-\gamma(l+1)-1}{t} G^t(x; \xi),$$

we can write the pdf of the TL-Gom-GPS class of distributions as

$$\begin{aligned} f_{\theta}(x) &= \sum_{n=1}^{\infty} \sum_{i,j,k,l,t=0}^{\infty} \frac{2b\theta na_n \theta^n}{C(\theta)} \binom{n-1}{i} \binom{b(i+1)-1}{i} \frac{(2j+2)^k (-1)^{i+j+l+t}}{\gamma^k k! (t+1)} \\ &\times \binom{-\gamma(l+1)-1}{t} \binom{k}{l} (t+1) g(x; \xi) G^t(x; \xi) \\ &= \sum_{t=0}^{\infty} W_{t+1} g_{t+1}^*(x; \xi), \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} W_{t+1} &= \sum_{n=1}^{\infty} \sum_{i,j,k,l=0}^{\infty} \frac{2b\theta na_n \theta^n}{C(\theta)} \binom{n-1}{i} \binom{b(i+1)-1}{i} \frac{(2j+2)^k (-1)^{i+j+l+t}}{\gamma^k k! (t+1)} \\ &\times \binom{-\gamma(l+1)-1}{t} \binom{k}{l}, \end{aligned} \quad (4.5)$$

and $g_{t+1}^*(x; \xi) = (t+1)g(x; \xi)G^t(x; \xi)$ is the Exp-G density with power parameter $(t+1)$. Consequently, the pdf of the TL-Gom-GPS class of distributions can be written as a infinite linear combination of Exp-G densities.

4.3.2 Hazard Rate and Quantile Functions

The hrf and quantile function of the TL-Gom-GPS class of distributions are presented in this subsection. The hrf is

$$\begin{aligned} h_{\theta}(x) = \frac{f_{\theta}(x)}{S_{\theta}(x)} &= 2b\theta g(x; \xi) (1 - G(x; \xi))^{-\gamma-1} \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^{b-1} \\ &\times \frac{C' \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^b \right) \right)}{C \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^b \right) \right)}. \end{aligned}$$

The quantile function of the TL-Gom-GPS class of distributions is derived by inverting the non-linear equation $F_{\theta}(x) = u$, $0 \leq u \leq 1$. Note that

$$1 - \frac{C \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^b \right) \right)}{C(\theta)} = u$$

and

$$C \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^b \right) \right) = C(\theta)(1 - u),$$

which is the same as

$$1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x; \xi))^{-\gamma}) \right] \right)^2 \right]^b = \frac{C^{-1}(C(\theta)(1 - u))}{\theta}$$

and

$$1 - (1 - G(x; \xi))^{-\gamma} = \gamma \ln \left(\left[1 - \left[1 - \frac{C^{-1}(C(\theta)(1 - u))}{\theta} \right]^{\frac{1}{b}} \right]^{\frac{1}{2}} \right).$$

Consequently, we obtain the quantile function of the TL-Gom-GPS class of distributions as

$$Q_{\theta}(u) = G^{-1} \left[1 - \left(1 - \gamma \ln \left(\left[1 - \left[1 - \frac{C^{-1}(C(\theta)(1 - u))}{\theta} \right]^{\frac{1}{b}} \right]^{\frac{1}{2}} \right) \right)^{-\frac{1}{\gamma}} \right]. \quad (4.6)$$

By numerical methods with the aid of statistical softwares such as R (utilizing the bbmle (Bolker and Bolker (2017) package), SAS (using NLMIXED procedure in SAS) and MATLAB, random numbers can be generated using equation (4.6) for specified baseline cdf G and $C(\theta)$.

4.3.3 Moments and Generating Function

The moments and generating function for the TL-Gom-GPS class of distributions are presented in this subsection. The n^{th} moment for the TL-Gom-GPS class of distributions is given by

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_{\theta}(x) dx = \sum_{t=0}^{\infty} W_{t+1} E(Y_{t+1}^n),$$

where $E(Y_{t+1}^n)$ is the n^{th} moment of Y_{t+1} which follows an Exp-G distribution with power parameter $(t+1)$ and W_{t+1} is given in equation (4.5). The moment generating function for the

TL-Gom-GPS class of distributions is given by

$$M_X(h) = E\left(e^{hX}\right) = \sum_{t=0}^{\infty} W_{t+1} E\left(e^{hY_{t+1}}\right),$$

where $E\left(e^{hY_{t+1}}\right)$ is the moment generating function of the Exp-G family of distributions with power parameter $(t+1)$ and W_{t+1} is given in equation (4.5).

4.3.4 Probability Weighted Moments

The PWMs for the TL-Gom-GPS class of distributions are presented in this subsection. The PWMs for a random variable X following the TL-Gom-GPS distributions is given by

$$\eta_{a,r} = E(X^a [F(X)]^r) = \int_{-\infty}^{\infty} x^a f_{\theta}(x) [F_{\theta}(x)]^r dx.$$

Note that

$$f_{\theta}(x) [F_{\theta}(x)]^r = \sum_{t=0}^{\infty} A_{t+1} g_{t+1}^*(x; \xi),$$

where

$$\begin{aligned} A_{t+1} &= \sum_{n,z=1}^{\infty} \sum_{q,i,j,k,l=0}^{\infty} \frac{2b\theta n a_n \theta^{n+z} d_{z,q}}{(C(\theta))^{q+1}} (-1)^{q+i+j+l+t} \binom{r}{q} \binom{b(i+1)-1}{j} \\ &\times \frac{(2i+2)^k}{\gamma^k k! (t+1)} \binom{k}{l} \binom{-\gamma(l+1)-1}{t}, \end{aligned}$$

and $g_{t+1}^*(x; \xi) = (t+1)g(x; \xi)G^t(x; \xi)$ is the Exp-G density with the power parameter $(t+1)$ and baseline vector ξ . Therefore, the PWMs of the TL-Gom-GPS distribution are

$$\eta_{a,r} = \sum_{t=0}^{\infty} A_{t+1} \int_{-\infty}^{\infty} x^a g_{t+1}^*(x; \xi) dx.$$

See the appendix for details of the derivation.

4.3.5 Distribution of Order Statistics and Rényi Entropy

We present the pdf of the i^{th} order statistic and Rényi entropy for the TL-Gom-GPS distribution in this subsection.

Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the TL-Gom-GPS class of distributions. The pdf of the i^{th} order statistic is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} \sum_{r=0}^{n-i} \sum_{t=0}^{\infty} (-1)^r \binom{n-i}{r} D_{t+1} g_{t+1}^*(x; \xi), \quad (4.7)$$

where $g_{t+1}^*(x; \xi) = (t+1)g(x; \xi)G^t(x; \xi)$ is the Exp-G density with power parameter $(t+1)$ and

$$\begin{aligned} D_{t+1} &= \sum_{n,z=1}^{\infty} \sum_{q,i,j,k,l=0}^{\infty} \frac{2b\theta n a_n \theta^{n+z} d_{z,q}}{(C(\theta))^{q+1}} (-1)^{q+i+j+l+t} \binom{i+r-1}{q} \binom{b(i+1)-1}{j} \\ &\times \frac{(2i+2)^k}{\gamma^k k! (t+1)} \binom{k}{l} \binom{-\gamma(l+1)-1}{t}. \end{aligned}$$

Details of the derivations can be found in the appendix section.

Rényi Entropy

Rényi entropy (Rényi (1961)) of a random variable X following TL-Gom-GPS class of distributions is defined by

$$I_R(\omega) = (1-\omega)^{-1} \log \left[\int_0^{\infty} f_{\theta}^{\omega}(x) dx \right] \quad \text{for } \omega > 0 \text{ and } \omega \neq 1.$$

Rényi entropy of the TL-Gom-GPS class of distributions is given by

$$I_R(\omega) = (1-\omega)^{-1} \log \left[\sum_{t=0}^{\infty} \Psi_t^* \exp((1-\omega)I_{REG}) \right], \quad (4.8)$$

where $I_{REG} = \frac{1}{1-\omega} \log \left[\int_0^{\infty} \left(\left(\frac{t}{\omega} + 1 \right) g(x; \xi) G^{\frac{t}{\omega}}(x; \xi) \right)^{\omega} dx \right]$ is Rényi entropy of Exp-G density with power parameter $\left(\frac{t}{\omega} + 1 \right)$ and

$$\begin{aligned} \Psi_t^* &= \sum_{n=1}^{\infty} \sum_{i,j,k,l=0}^{\infty} \frac{n d_{\omega,n} (2b)^{\omega} \theta^{\theta+n-1}}{(C(\theta))^{\omega}} \binom{\omega(n-1)}{i} \binom{b(i+\omega)-\omega}{i} \binom{k}{l} \\ &\times \frac{(2i+2\omega)^k (-1)^{i+j+l+t}}{\gamma^k k!} \binom{-\gamma(l+\omega)-\omega}{t} \frac{1}{\left(\frac{t}{\omega} + 1 \right)^{\omega}}. \end{aligned}$$

Details of the derivations are given in the appendix section.

4.4 Maximum Likelihood Estimation

Let $X \sim TL - Gom - GPS(b, \gamma, \theta, \xi)$ and $\Psi = (b, \gamma, \theta, \xi)^T$ be the parameter vector. The log-likelihood function, ℓ_n of a random sample of size n from the $TL - Gom - GPS(b, \gamma, \theta, \xi)$ class of distributions is given by

$$\begin{aligned} \ell_n(\Psi) = & n \log(2b\theta) + \sum_{i=1}^n \log[g(x_i; \xi)] - (\gamma + 1) \sum_{i=1}^n \log[(1 - G(x_i; \xi))] \\ & + (b - 1) \sum_{i=1}^n \log \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x_i; \xi))^{-\gamma}) \right] \right)^2 \right] \\ & + \frac{2}{\gamma} \sum_{i=1}^n [(1 - (1 - G(x_i; \xi))^{-\gamma})] - \sum_{i=1}^n \log[C(\theta)] \\ & + \sum_{i=1}^n \log \left[C' \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 - G(x_i; \xi))^{-\gamma}) \right] \right)^2 \right]^b \right) \right) \right]. \end{aligned}$$

Elements of the score vector $U(\Psi) = \left(\frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \gamma}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \xi_k} \right)$ are presented in the appendix.

The FIM of the TL-Gom-GPS class of distributions is given by $\mathbf{I}(\Psi) = [\mathbf{I}_{\psi_i, \psi_j}]_{(3+q) \times (3+q)} = E \left(-\frac{\partial^2 \ell}{\partial \psi_i \partial \psi_j} \right)$, $i, j = 1, 2, \dots, (3 + q)$, can be numerically obtained by mle2 package (Bolker and Bolker (2017)) in R software or MATLAB. The expectations in the FIM can be numerically obtained and the total Fisher information matrix $n\mathbf{I}(\Delta)$ can be approximated by $\mathbf{J}_n(\hat{\Psi}) \approx \left[-\frac{\partial^2 \ell}{\partial \psi_i \partial \psi_j} \Big|_{\Psi=\hat{\Psi}} \right]_{(3+q) \times (3+q)}$, $i, j = 1, 2, \dots, (3 + q)$. The multivariate normal distribution $N_{(3+q)}(\mathbf{0}, \mathbf{J}(\hat{\Psi})^{-1})$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival hazard rate functions. That is, the approximate $100(1 - \eta)\%$ two sided confidence intervals for b, γ, θ and ξ_k are given by:

$$\hat{b} \pm Z_{\frac{\eta}{2}} \sqrt{I_{bb}^{-1}(\hat{\Psi})}, \quad \hat{\gamma} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\gamma\gamma}^{-1}(\hat{\Psi})}, \quad \hat{\theta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\Psi})} \quad \hat{\xi}_k \pm Z_{\frac{\eta}{2}} \sqrt{I_{\xi_{kk}}^{-1}(\hat{\Psi})},$$

respectively, where $I_{bb}^{-1}(\hat{\Psi})$, $I_{\gamma\gamma}^{-1}(\hat{\Psi})$, $I_{\theta\theta}^{-1}(\hat{\Psi})$ and $I_{\xi_{kk}}^{-1}(\hat{\Psi})$ are the diagonal elements of $I_n^{-1}(\hat{\Psi}) = (n\mathbf{I}(\hat{\Psi}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $(\frac{\eta}{2})^{th}$ percentile of the standard normal distribution.

4.5 Some Special Cases

Some of the special models of the TL-Gom-GPS class of distributions are presented in this section. We consider the cases when the baseline cdf $G(x; \xi)$ is Weibull, log-logistic and exponential distributions.

4.5.1 Topp-Leone-Gompertz-Weibull Power Series Distribution

If we take the baseline distribution to be Weibull distribution with the cdf and pdf given by $G(x; \lambda) = 1 - e^{-x^\lambda}$ and $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$, for $\lambda, x > 0$, we obtain the cdf and pdf of the of the TL-Gom-WPS distribution as

$$F_{\theta}(x) = 1 - \frac{C \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (e^{-x^{\lambda})} - \gamma) \right] \right)^2 \right]^b \right) \right)}{C(\theta)},$$

and

$$\begin{aligned} f_{\theta}(x) &= 2b\theta\lambda x^{\lambda-1}(e^{-x^{\lambda}})^{-\gamma} \left(\exp \left[\frac{1}{\gamma} (1 - (e^{-x^{\lambda}})^{-\gamma}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (e^{-x^{\lambda}})^{-\gamma}) \right] \right)^2 \right]^{b-1} \\ &\times \frac{C' \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (e^{-x^{\lambda}})^{-\gamma}) \right] \right)^2 \right]^b \right) \right)}{C(\theta)}, \end{aligned}$$

respectively, for $b, \gamma, \lambda, \theta > 0$.

4.5.2 Topp-Leone-Gompertz-Weibull Poisson Distribution

We consider when the power series distribution is a Poisson distribution. The cdf and pdf of the TL-Gom-WP distribution are given by

$$F_{TL-Gom-WP}(x; b, \gamma, \lambda, \theta) = 1 - \frac{\exp \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (e^{-x^{\lambda}})^{-\gamma}) \right] \right)^2 \right]^b \right) \right) - 1}{e^{\theta} - 1}$$

and

$$\begin{aligned} f_{TL-Gom-WP}(x; b, \gamma, \lambda, \theta) &= 2b\theta\lambda x^{\lambda-1}(e^{-x^{\lambda}})^{-\gamma} \left(\exp \left[\frac{1}{\gamma} (1 - (e^{-x^{\lambda}})^{-\gamma}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (e^{-x^{\lambda}})^{-\gamma}) \right] \right)^2 \right]^{b-1} \\ &\times \frac{\exp \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (e^{-x^{\lambda}})^{-\gamma}) \right] \right)^2 \right]^b \right) \right)}{e^{\theta} - 1}, \end{aligned}$$

respectively, for $b, \gamma, \lambda, \theta > 0$.

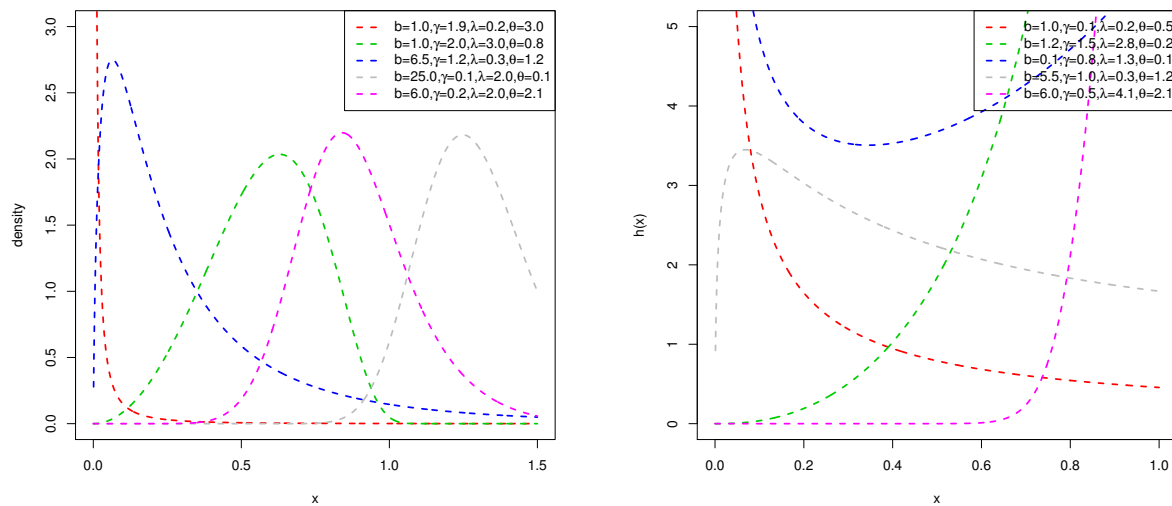


Figure 4.1: Pdf and hrf plots for the TL-Gom-WP distribution

Figure 4.1 gives the pdf and hrf of the TL-Gom-WP distribution. The pdf can be right-skewed, left-skewed, almost symmetric and reverse-J shapes whereas the hrf displays increasing, decreasing, upside-down bathtub and bathtub shapes. Table 4.2 gives some quantile values of the TL-Gom-WP distribution for different parameter values. The 3D plots for TL-Gom-WP distribution are given in Figures 4.2 and 4.3. The plots indicate that TL-Gom-WP distribution can model data with different levels of kurtosis and skewness.

Table 4.2: Quantiles for TL-Gom-WP distribution

$(b, \gamma, \lambda, \theta)$					
u	(1, 2, 1.3, 0.2)	(0.7, 1, 3, 1.5)	(2.4, 1, 1, 2)	(2.1, 1, 1.9, 1)	(1.5, 1, 1.2, 3)
0.1	0.0933	0.1978	0.1492	0.3655	0.0867
0.2	0.1607	0.2816	0.2110	0.4485	0.1335
0.3	0.2226	0.3504	0.2642	0.5117	0.1757
0.4	0.2825	0.4137	0.3158	0.5674	0.2177
0.5	0.3429	0.4756	0.3696	0.6208	0.2622
0.6	0.4058	0.5393	0.4291	0.6752	0.3122
0.7	0.4742	0.6083	0.4999	0.7344	0.3721
0.8	0.5539	0.6884	0.5928	0.8043	0.4518
0.9	0.6607	0.7945	0.7405	0.9006	0.5815

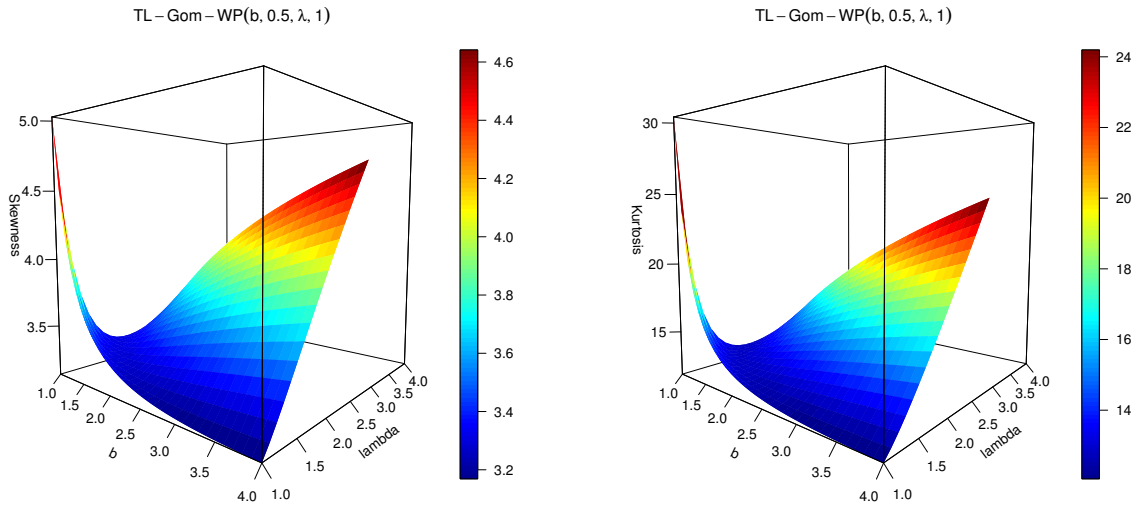


Figure 4.2: Skewness and kurtosis plots for TL-Gom-WP distribution

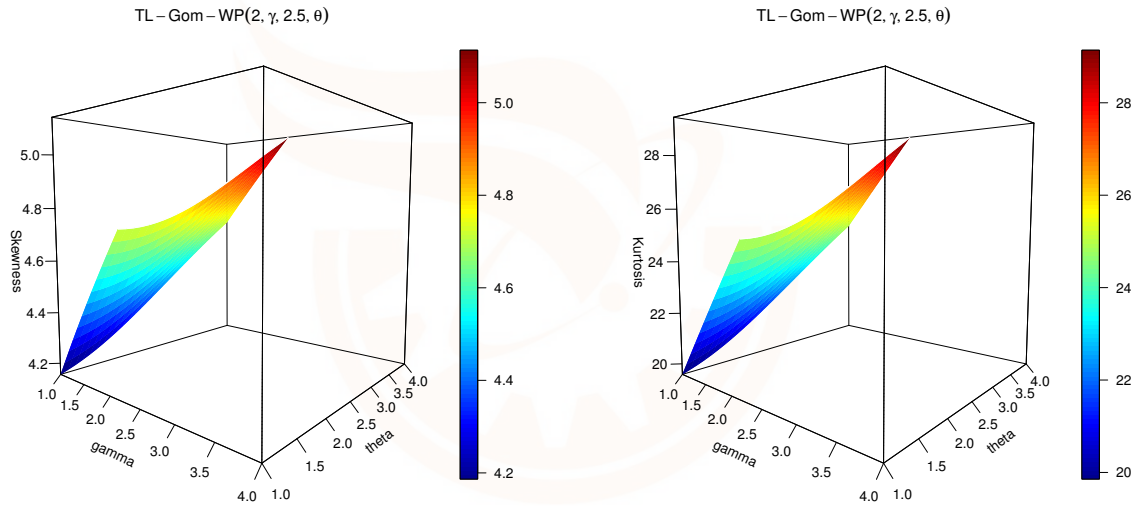


Figure 4.3: Skewness and kurtosis plots for TL-Gom-WP distribution

4.5.3 Topp-Leone-Gompertz-Log-logistic Power Series Distribution

Suppose we take the baseline distribution to be log-logistic distribution with the cdf and pdf $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$, respectively, for $c, x > 0$, we obtain the cdf and pdf of the of the TL-Gom-LLoPS distribution as

$$F_{\theta}(x) = 1 - \frac{C \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 + x^c)^{\gamma}) \right] \right)^2 \right]^b \right) \right)}{C(\theta)},$$

and

$$\begin{aligned} f_{\theta}(x) &= 2b\theta cx^{c-1}(1 + x^c)^{\gamma-1} \left(\exp \left[\frac{1}{\gamma} (1 - (1 + x^c)^{\gamma}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 + x^c)^{\gamma}) \right] \right)^2 \right]^{b-1} \\ &\times \frac{C' \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 + x^c)^{\gamma}) \right] \right)^2 \right]^b \right) \right)}{C(\theta)}, \end{aligned}$$

respectively, for $b, \gamma, c, \theta > 0$.

4.5.4 Topp-Leone-Gompertz-Log-logistic Poisson Distribution

The TL-Gom-LLoP distribution cdf and pdf are

$$F_{TL-Gom-WP}(x; b, \gamma, c, \theta) = 1 - \frac{\exp \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 + x^c)^{\gamma}) \right] \right)^2 \right]^b \right) \right) - 1}{e^{\theta} - 1}$$

and

$$\begin{aligned} f_{TL-Gom-WP}(x; b, \gamma, c, \theta) &= 2b\theta cx^{c-1}(1 + x^c)^{\gamma-1} \left(\exp \left[\frac{1}{\gamma} (1 - (1 + x^c)^{\gamma}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 + x^c)^{\gamma}) \right] \right)^2 \right]^{b-1} \\ &\times \frac{\exp \left(\theta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\gamma} (1 - (1 + x^c)^{\gamma}) \right] \right)^2 \right]^b \right) \right)}{e^{\theta} - 1}, \end{aligned}$$

respectively, for $b, \gamma, c, \theta > 0$.

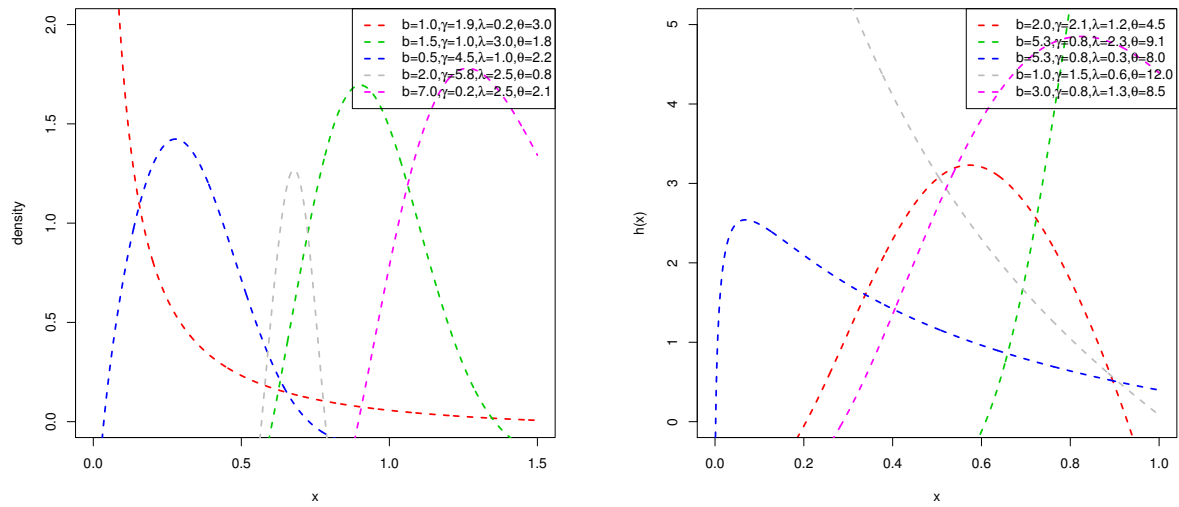


Figure 4.4: Plots of pdf and hrf for the TL-Gom-LLoP distribution

The TL-Gom-LLoP distribution has the pdf that displays right-skewed, left-skewed, almost symmetric and reverse-J shapes, whereas the hrf exhibit upside-down bathtub, increasing and decreasing shapes.

Table 4.3: Some quantiles for TL-Gom-LLoP distribution

(b, γ, c, θ)					
u	(1, 2, 1.3, 0.2)	(0.7, 1, 3, 1.5)	(2.4, 1, 1, 2)	(2.1, 1, 1.9, 1)	(1.5, 1, 1.2, 3)
0.1	0.0949	0.1981	0.1609	0.3802	0.0887
0.2	0.1666	0.2826	0.2349	0.4755	0.1386
0.3	0.2352	0.3529	0.3024	0.5517	0.1851
0.4	0.3047	0.4186	0.3714	0.6222	0.2329
0.5	0.3780	0.4843	0.4471	0.6929	0.2855
0.6	0.4585	0.5538	0.5359	0.7688	0.3468
0.7	0.5512	0.6319	0.6486	0.8559	0.4239
0.8	0.6668	0.7279	0.8091	0.9662	0.5333
0.9	0.8359	0.8668	1.0969	1.1339	0.7295

Quantiles for selected parameters values of the TL-Gom-LLoP distribution are given in Table 4.3. Figures 4.5 and 4.6 shows that TL-Gom-LLoP distribution can model data sets with different

levels of skewness and kurtosis.

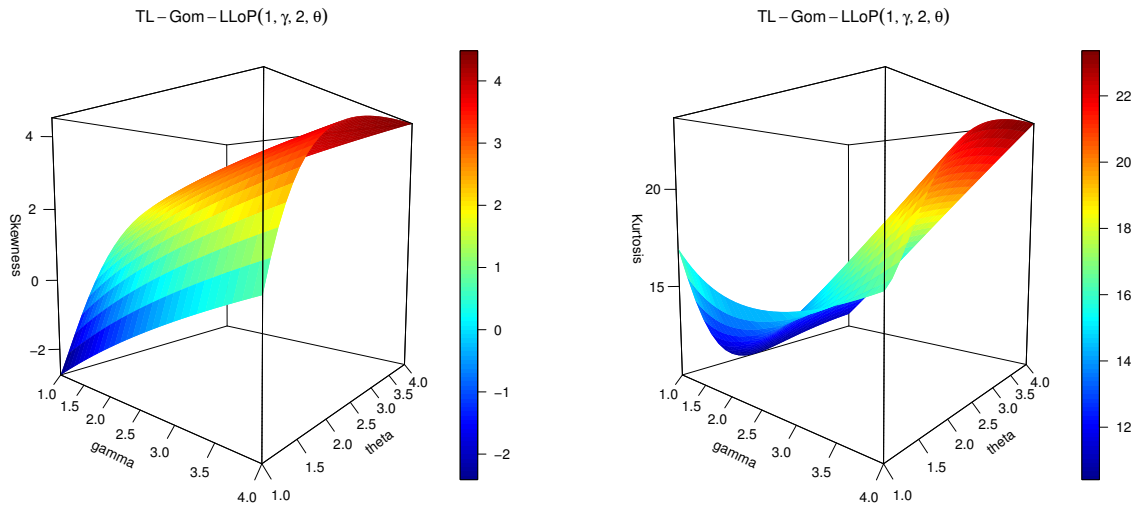


Figure 4.5: Skewness and kurtosis plots for TL-Gom-LLoP distribution

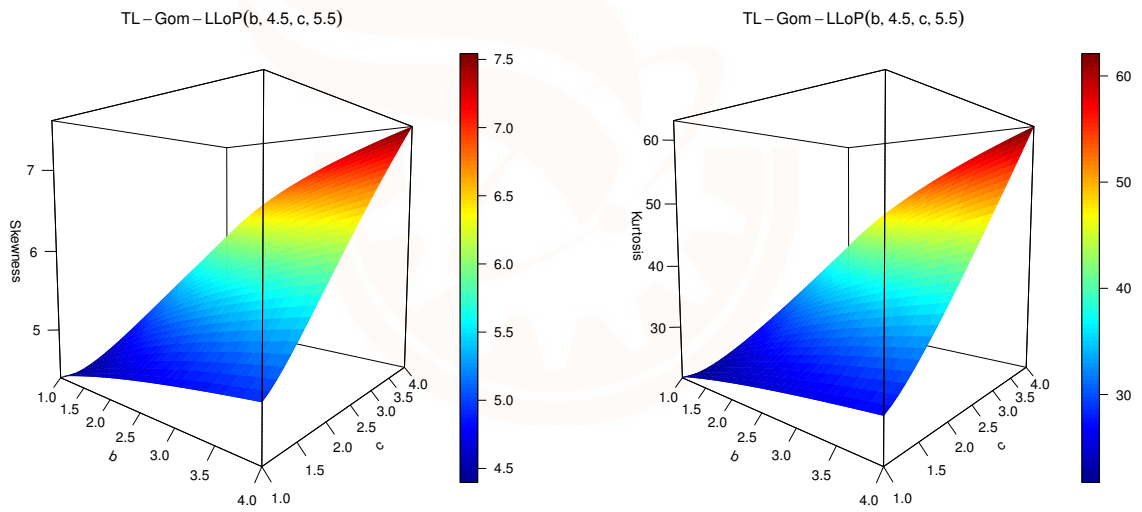


Figure 4.6: Skewness and kurtosis plots for TL-Gom-LLoP distribution

4.5.5 Topp-Leone-Gompertz-Exponential Power Series Distribution

If we consider the baseline distribution to be the exponential distribution with the cdf and pdf given by $G(x; \lambda) = 1 - e^{-\lambda x}$ and $g(x; \lambda) = \lambda e^{-\lambda x}$, for $\lambda, x > 0$, we obtain the cdf and pdf of the TL-Gom-EPS distribution as

$$F_{\theta}(x) = 1 - \frac{C\left(\theta\left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - e^{-\lambda x})^{-\gamma}\right]\right)^2\right]^b\right)\right)}{C(\theta)},$$

and

$$\begin{aligned} f_{\theta}(x) &= 2b\theta\lambda(e^{-\lambda x})^{-\gamma}\left(\exp\left[\frac{1}{\gamma}(1 - e^{-\lambda x})^{-\gamma}\right]\right)^2 \\ &\times \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - e^{-\lambda x})^{-\gamma}\right]\right)^2\right]^{b-1} \\ &\times \frac{C'\left(\theta\left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - e^{-\lambda x})^{-\gamma}\right]\right)^2\right]^b\right)\right)}{C(\theta)}, \end{aligned}$$

respectively, for $b, \gamma, \lambda, \theta > 0$.

4.5.6 Topp-Leone-Gompertz-Exponential Poisson Distribution

The cdf and pdf of the TL-Gom-EP distribution are given by

$$F_{TL-Gom-EP}(x; b, \gamma, \lambda, \theta) = 1 - \frac{\exp\left(\theta\left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - e^{-\lambda x})^{-\gamma}\right]\right)^2\right]^b\right)\right) - 1}{e^{\theta} - 1},$$

and

$$\begin{aligned} f_{TL-Gom-EP}(x; b, \gamma, \lambda, \theta) &= 2b\theta\lambda(e^{-\lambda x})^{-\gamma-1}\left(\exp\left[\frac{1}{\gamma}(1 - e^{-\lambda x})^{-\gamma}\right]\right)^2 \\ &\times \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - e^{-\lambda x})^{-\gamma}\right]\right)^2\right]^{b-1} \\ &\times \frac{\exp\left(\theta\left(1 - \left[1 - \left(\exp\left[\frac{1}{\gamma}(1 - e^{-\lambda x})^{-\gamma}\right]\right)^2\right]^b\right)\right)}{e^{\theta} - 1}, \end{aligned}$$

respectively, for $b, \gamma, \lambda, \theta > 0$.

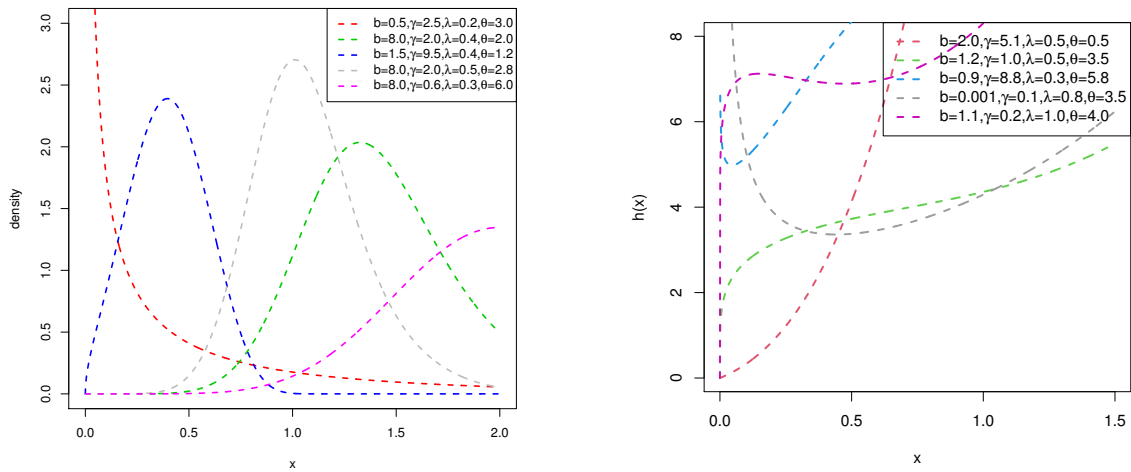


Figure 4.7: Plots of pdf and hrf for the TL-Gom-EP distribution

The TL-Gom-EP distribution has the pdf that displays right-skewed, left-skewed, almost symmetric and reverse-J shapes, whereas the hrf can be upside-down bathtub followed by bathtub, bathtub, decreasing, and increasing shapes. Quantiles for selected parameters values of the TL-Gom-EP distribution are given in Table 4.4. Figures 4.8 and 4.9 shows that TL-Gom-EP distribution can model data sets with different levels of skewness and kurtosis.

Table 4.4: Some quantiles for TL-Gom-EP distribution

$(b, \gamma, \lambda, \theta)$					
u	(1, 2, 1.3, 0.2)	(0.7, 1, 3, 1.5)	(2.4, 1, 1, 2)	(2.1, 1, 1.9, 1)	(1.5, 1, 1.2, 3)
0.1	0.0352	0.0026	0.1492	0.0778	0.0443
0.2	0.0714	0.0075	0.2110	0.1147	0.0743
0.3	0.1091	0.0143	0.2642	0.1473	0.1034
0.4	0.1488	0.0236	0.3158	0.1793	0.1337
0.5	0.1913	0.0359	0.3696	0.2127	0.1672
0.6	0.2381	0.0523	0.4291	0.2496	0.2061
0.7	0.2916	0.0750	0.4999	0.2928	0.2545
0.8	0.3569	0.1087	0.5928	0.3479	0.3212
0.9	0.4488	0.1672	0.7405	0.4314	0.4348

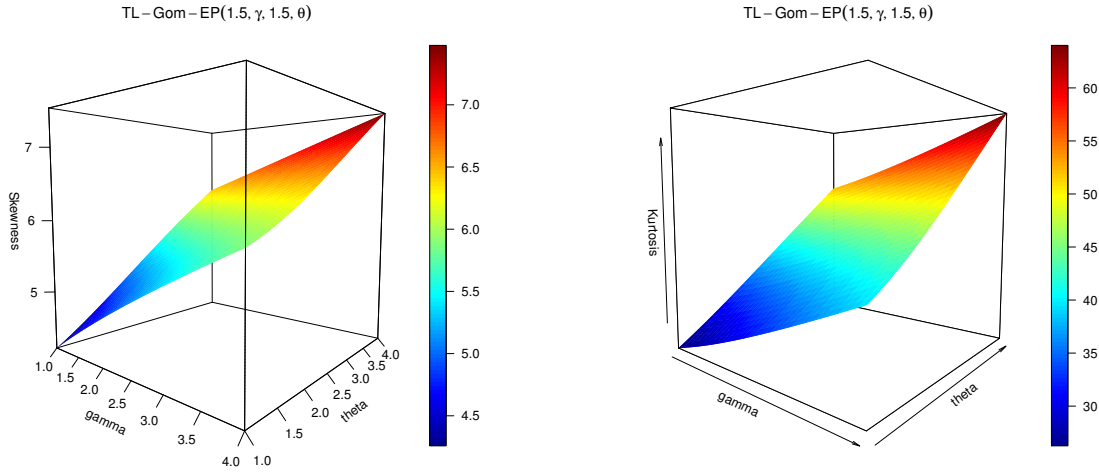


Figure 4.8: Skewness and kurtosis plots for TL-Gom-EP distribution

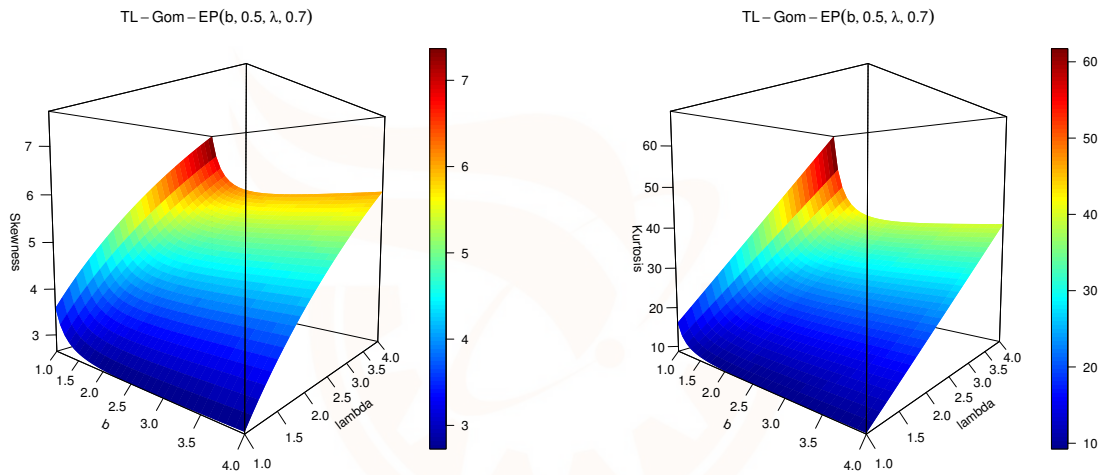


Figure 4.9: Skewness and kurtosis plots for TL-Gom-EP distribution

4.6 Simulation Results

In this subsection, we present some simulation results for the TL-Gom-WP and TL-Gom-EP distributions. The simulation study was performed in order to examine the performance of the mean, RMSEs and ABIAS of maximum likelihood estimators of the parameters of the TL-Gom-WP and TL-Gom-EP distributions. We performed various simulations for different sample sizes and different parameter values. Equation (4.6) was used to generate random samples from the special cases of the TL-Gom-GPS class of distributions via the bbmle package (Bolker and Bolker (2017)) in R. The simulation was repeated for $N = 1000$ times each with sample sizes $n = 50, 100, 200, 400, 800, 1000$. Specifically, the estimated mean, ABIAS and RMSEs of the parameter say, $\hat{\xi}$, given as

$$\text{Mean} = \frac{\sum_{i=1}^N \hat{\xi}_i}{N}, \text{RMSE} = \sqrt{\frac{\sum_{i=1}^N (\hat{\xi}_i - \xi)^2}{N}} \text{ and ABIAS } (\hat{\xi}) = \frac{\sum_{i=1}^N (\hat{\xi}_i - \xi)}{N}, \text{ respectively.}$$

Tables 4.5 to 4.8 list the mean MLEs of the parameters along with the respective RMSEs and

ABIAS. From the results in tables, we can see that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values while the RMSEs and ABIAS decrease with increasing sample size.

Table 4.5: Simulation Results for TL-Gom-WP Distribution: Mean, ABIAS and RMSE

Parameter	n	(0.5, 2.0, 2.0, 1.2)				(0.5, 0.2, 4.0, 0.2)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
b	50	0.4589	0.5826	-0.0410	0.0410	0.4419	0.3597	-0.0581	0.0581
	100	0.4026	0.2969	-0.0974	0.0974	0.4218	0.1649	-0.0782	0.0782
	200	0.4384	0.1909	-0.0616	0.0616	0.4593	0.1236	-0.0407	0.0407
	400	0.4624	0.1522	-0.0376	0.0376	0.4721	0.0846	-0.0279	0.0279
	800	0.4821	0.1237	-0.0179	0.0179	0.4799	0.0594	-0.0201	0.0201
	1000	0.4866	0.1107	-0.0134	0.0134	0.4818	0.0532	-0.0182	0.0182
γ	50	5.2053	9.7615	3.2053	3.2053	0.6034	0.5412	0.4034	0.4034
	100	3.2273	2.7562	1.2273	1.2273	0.5242	0.4476	0.3242	0.3242
	200	2.4837	1.0974	0.4837	0.4837	0.4498	0.3407	0.2498	0.2498
	400	2.1866	0.5214	0.1866	0.1866	0.4242	0.3304	0.2242	0.2242
	800	2.1063	0.2991	0.1063	0.1063	0.4000	0.3036	0.2000	0.2000
	1000	2.0729	0.2440	0.0729	0.0729	0.3636	0.2694	0.1636	0.1636
λ	50	3.6861	2.5691	1.6861	1.6861	7.3821	5.0374	3.3821	3.3821
	100	3.3075	2.0858	1.3075	1.3075	6.3266	3.4492	2.3266	2.3266
	200	2.7399	1.3736	0.7399	0.7399	5.2997	2.0791	1.2997	1.2997
	400	2.4353	0.9887	0.4353	0.4353	4.7499	1.2411	0.7499	0.7499
	800	2.2342	0.6885	0.2342	0.2342	4.4491	0.7429	0.4491	0.4491
	1000	2.1744	0.5995	0.1744	0.1744	4.3752	0.6418	0.3752	0.3752
θ	50	1.9067	1.2679	0.7067	0.7067	1.7286	1.8287	1.5286	1.5286
	100	1.6107	0.9959	0.4107	0.4107	1.4046	1.4814	1.2046	1.2046
	200	1.4600	0.7823	0.2600	0.2600	1.0136	1.0316	0.8136	0.8136
	400	1.3418	0.6229	0.1418	0.1418	0.7275	0.6964	0.5275	0.5275
	800	1.2646	0.4874	0.0646	0.0646	0.5313	0.4591	0.3313	0.3313
	1000	1.2316	0.4583	0.0316	0.0316	0.4743	0.4041	0.2743	0.2743

Table 4.6: Simulation Results for TL-Gom-WP Distribution: Mean, ABIAS and RMSE

Parameter	n	(0.8, 2.0, 1.5, 0.4)				(1.5, 0.3, 1.5, 0.6)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
b	50	0.6406	0.8105	-0.1594	0.1594	3.1672	4.2155	1.6672	1.6672
	100	0.5104	0.4539	-0.2896	0.2896	2.5054	2.8487	1.0054	1.0054
	200	0.5728	0.3373	-0.2272	0.2272	1.9713	1.0082	0.4713	0.4713
	400	0.6421	0.2597	-0.1579	0.1579	1.8534	0.7552	0.3534	0.3534
	800	0.6929	0.1885	-0.1071	0.1071	1.7518	0.5755	0.2518	0.2518
	1000	0.7125	0.1629	-0.0875	0.0875	1.7025	0.5099	0.2518	0.2518
γ	50	5.1672	8.9961	3.1672	3.1672	0.6344	0.4964	0.3344	0.3344
	100	3.1284	2.6054	1.1284	1.1284	0.5402	0.4048	0.2402	0.2402
	200	2.4898	1.1944	0.4898	0.4898	0.4707	0.2861	0.1707	0.1707
	400	2.1225	0.4563	0.1225	0.1225	0.4417	0.2770	0.1417	0.1417
	800	2.0479	0.2221	0.0479	0.0479	0.4148	0.2541	0.1148	0.1148
	1000	2.0269	0.1744	0.0269	0.0269	0.3949	0.2347	0.0949	0.0949
λ	50	3.9503	3.5471	2.4503	2.4503	1.3025	0.6897	-0.1975	0.1975
	100	3.4987	2.7555	1.9987	1.9987	1.3159	0.4667	-0.1840	0.1840
	200	2.7650	1.9073	1.2650	1.2650	1.3411	0.3521	-0.1589	0.1589
	400	2.2214	1.1940	0.7214	0.7214	1.3606	0.3389	-0.1394	0.1394
	800	1.9078	0.7132	0.4078	0.4078	1.3931	0.3153	-0.1069	0.1069
	1000	1.8041	0.5362	0.3041	0.3041	1.4122	0.2942	-0.0878	0.0878
θ	50	1.8438	1.8165	1.4438	1.4438	0.7392	0.6414	0.1392	0.1392
	100	1.6232	1.5812	1.2232	1.2232	0.6473	0.4323	0.0473	0.0473
	200	1.2994	1.2140	0.8994	0.8994	0.6159	0.3924	0.0159	0.0159
	400	1.0109	0.9104	0.6109	0.6109	0.5253	0.3079	-0.0747	0.0747
	800	0.8143	0.6661	0.4143	0.4143	0.5071	0.2663	-0.0929	0.0929
	1000	0.7268	0.5680	0.3268	0.3268	0.5102	0.2529	-0.0898	0.0898

Table 4.7: Simulation Results for TL-Gom-EP Distribution: Mean, ABIAS and RMSE

Parameter	n	(1.0, 1.0, 1.0, 0.6)				(0.8, 0.8, 0.1, 1.6)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
b	50	1.0772	0.2450	0.0772	0.0772	0.8051	0.1341	0.0051	0.0051
	100	1.0337	0.1591	0.0337	0.0337	0.8044	0.1004	0.0044	0.0044
	200	1.0206	0.1145	0.0206	0.0206	0.8110	0.0757	0.0110	0.0110
	400	1.0091	0.0819	0.0091	0.0091	0.8024	0.0530	0.0024	0.0024
	800	1.0074	0.0553	0.0074	0.0074	0.8036	0.0357	0.0036	0.0036
	1000	1.0049	0.0516	0.0049	0.0049	0.8009	0.0347	0.0009	0.0009
γ	50	2.1754	2.0136	1.1754	1.1754	2.0277	1.9549	1.2277	1.2277
	100	2.2388	1.9265	1.2388	1.2388	1.6198	1.4655	0.8198	0.8198
	200	2.1209	1.7406	1.1209	1.1209	1.4371	1.2201	0.6371	0.6371
	400	1.8891	1.3857	0.8891	0.8891	1.3041	0.9626	0.5041	0.5041
	800	1.8484	1.3496	0.8484	0.8484	1.1274	0.6946	0.3274	0.3274
	1000	1.7287	1.1857	0.7287	0.7287	1.0082	0.6616	0.2082	0.2082
λ	50	0.7975	0.3846	-0.2025	0.2025	0.0759	0.0374	-0.0241	0.0241
	100	0.7449	0.3615	-0.2550	0.2550	0.0777	0.0334	-0.0223	0.0223
	200	0.7596	0.3432	-0.2404	0.2404	0.0825	0.0307	-0.0175	0.0175
	400	0.7902	0.3184	-0.2098	0.2098	0.0843	0.0288	-0.0157	0.0157
	800	0.7973	0.3071	-0.2026	0.2026	0.0896	0.0260	-0.0104	0.0104
	1000	0.8190	0.2869	-0.1809	0.1809	0.0962	0.0249	-0.0038	0.0038
θ	50	1.5612	1.1920	0.9612	0.9612	1.9748	0.7886	0.3748	0.3748
	100	1.5522	1.1779	0.9522	0.9522	2.0727	0.9292	0.4727	0.4727
	200	1.4291	1.0663	0.8291	0.8291	2.0181	0.8146	0.4181	0.4181
	400	1.2843	0.9631	0.6843	0.6843	2.0020	0.8459	0.4020	0.4020
	800	1.2544	0.9440	0.6544	0.6544	1.8741	0.6966	0.2741	0.2741
	1000	1.1753	0.8812	0.5753	0.5753	1.6834	0.6255	0.0834	0.0834

Table 4.8: Simulation Results for TL-Gom-EP Distribution: Mean, ABIAS and RMSE

Parameter	n	(1.0, 0.1, 1.5, 2.0)				(0.3, 0.2, 0.7, 0.7)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
b	50	0.9286	0.8763	-0.0714	0.0714	0.4264	0.6133	0.1264	0.1264
	100	0.9002	0.6031	-0.0998	0.0998	0.4199	0.5400	0.1199	0.1199
	200	0.9322	0.3987	-0.0678	0.0678	0.4434	0.5719	0.1434	0.1434
	400	0.9359	0.2756	-0.0641	0.0641	0.4188	0.4585	0.1188	0.1188
	800	0.9822	0.1466	-0.0198	0.0198	0.3931	0.3365	0.0931	0.0931
	1000	0.9819	0.1106	-0.0181	0.0181	0.3643	0.3129	0.0643	0.0643
γ	50	0.1800	0.1327	0.0800	0.0800	0.6142	1.7459	0.4142	0.4142
	100	0.1595	0.1054	0.0595	0.0595	0.3753	0.9249	0.1753	0.1753
	200	0.1415	0.0817	0.0415	0.0415	0.2498	0.1281	0.0498	0.0498
	400	0.1212	0.0529	0.0212	0.0212	0.2491	0.1149	0.0491	0.0491
	800	0.1075	0.0263	0.0075	0.0075	0.2299	0.0859	0.0299	0.0299
	1000	0.1035	0.0191	0.0035	0.0035	0.2150	0.0670	0.0150	0.0150
λ	50	2.0145	1.4624	0.5145	0.5145	0.6039	0.2541	-0.0961	0.0961
	100	1.7925	1.0233	0.2925	0.2925	0.6258	0.1825	-0.0742	0.0742
	200	1.6737	0.7086	0.1738	0.1738	0.6437	0.1389	-0.0563	0.0563
	400	1.5503	0.5323	0.0503	0.0503	0.6590	0.1171	-0.0409	0.0409
	800	1.5162	0.4638	0.0162	0.0162	0.6744	0.0966	-0.0256	0.0256
	1000	1.4856	0.4603	-0.0144	0.0144	0.6816	0.0808	-0.0184	0.0184
θ	50	1.7873	0.4408	-0.2127	0.2127	0.6317	0.2088	-0.0683	0.0683
	100	1.8165	0.3677	-0.1835	0.1835	0.6265	0.1669	-0.0735	0.0735
	200	1.8583	0.2995	-0.1417	0.1417	0.6633	0.1209	-0.0367	0.0367
	400	1.9164	0.2047	-0.0836	0.0836	0.6615	0.1068	-0.0385	0.0385
	800	1.9693	0.1149	-0.0307	0.0307	0.6757	0.0843	-0.0243	0.0243
	1000	1.9803	0.0829	-0.0197	0.0197	0.6884	0.0709	-0.0116	0.0116

4.7 Applications

In this subsection, applications of the TL-Gom-EP distribution to actual data sets are presented to assess its relevance and significance. We compared the new TL-Gom-EP distribution with the non-nested models including the Topp-Leone-Weibull Poisson distribution by Chipepa et al. (2021), Log-logistic-Weibull Poisson distribution by Oluyede et al. (2016), the Marshall-Olkin-Gompertz-Weibull distribution by Chipepa and Oluyede (2021) and the odd log-logistic exponentiated Weibull distribution (Afify et al. (2018)). The pdf's of the above distributions are listed in the appendix.

4.7.1 Breaking Stress of Carbon Fibres Data

The first dataset correspond to breaking stress of carbon fibres data cited by El-Bassiouny et al. (2015), and given as; 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

The estimated variance-covariance matrix of the TL-Gom-EP model for carbon fibres data is given by

$$\begin{bmatrix} 1.2966 \times 10^{-02} & -5.1857 \times 10^{-04} & -9.7543 \times 10^{-06} & 7.0402 \times 10^{-02} \\ -5.1857 \times 10^{-04} & 9.6163 \times 10^{-05} & 6.9438 \times 10^{-06} & -1.2166 \times 10^{-02} \\ -9.7543 \times 10^{-06} & 6.9438 \times 10^{-06} & 7.2676 \times 10^{-07} & -8.6541 \times 10^{-04} \\ 7.0402 \times 10^{-02} & -1.2166 \times 10^{-02} & -8.6541 \times 10^{-07} & 1.5414 \times 10^{+00} \end{bmatrix}$$

and the approximate 95% confidence interval are given by $b \in (1.0116, 1.4579)$, $\gamma \in (93.1229, 93.16122)$, $\lambda \in (0.0083, 0.0117)$ and $\theta \in (2.3361, 7.2029)$.

The maximum likelihood estimates (MLE's) of the TL-Gom-EP distribution with (standard errors in parentheses) are presented in Tables 4.9 and 4.10. The goodness-of-fit statistics are also given in the tables including -2logL, AIC, AICC, BIC, (W^*) , (A^*) statistic, K-S statistic and its P-Value.

Plots of the fitted pdfs, the histogram of the data and probability plots (Chambers et al. (1983)) are given in Figures 4.11 and 4.14. For the probability plots, we plotted $F(x_{(j)}; \hat{b}, \hat{\gamma}, \hat{\lambda}, \hat{\theta})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure

of closeness to the diagonal line is given by the sum of squares

$$SS = \sum_{j=1}^n \left[F(x_{(j)}; \hat{b}, \hat{\gamma}, \hat{\lambda}, \hat{\theta}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

Table 4.9: Estimates of models for carbon fibres data

Model	Estimates				Statistics							
	b	γ	λ	θ	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
TL-Gom-EP	$1.2348 \times 10^{+00}$	$9.3142 \times 10^{+01}$	9.9999×10^{-03}	$4.7695 \times 10^{+00}$	281.63	289.63	290.05	300.05	0.0703	0.6141	0.0558	0.9149
	(1.0473×10^{-01})	(2.5323×10^{-03})	(8.5433×10^{-04})	$(1.5585 \times 10^{+00})$								
TL-WP	α	β	b	θ								
	1.8497×10^{-01}	$1.4227 \times 10^{+00}$	$2.1985 \times 10^{+00}$	1.3279×10^{-08}	294.47	302.47	302.89	312.89	0.1309	0.6679	0.1057	0.2135
	(4.2738×10^{-02})	(1.4246×10^{-01})	(4.0254×10^{-01})	(9.8272×10^{-03})								
LLOGWP	c	α	β	θ								
	$2.6951 \times 10^{+00}$	$0.0000 \times 10^{+00}$	1.0621×10^{-05}	$7.6903 \times 10^{+00}$	308.79	316.79	317.21	327.21	0.4637	2.5406	0.1319	0.0613
	(1.6793×10^{-01})	(7.4506×10^{-09})	$(6.8425 \times 10^{+00})$	(9.4663×10^{-01})								
MO-Gom-W	δ	θ	λ	γ								
	461.0900	0.0230	0.0952	6.8646	284.29	292.29	292.72	302.72	0.0741	0.6366	0.0637	0.812
	(0.0003)	(0.0183)	(0.0143)	(0.9236)								
OLLEW	α	β	γ	θ								
	18.4404	0.5347	0.5614	8.2657	289.19	297.19	297.62	307.62	0.1936	0.9894	0.0899	0.3930
	(9.9159)	(0.2005)	(0.1709)	(2.1152)								

In Table 4.9, we see that the values of the goodness-of-fit statistics: AIC, AICC and BIC are the smallest for the TL-Gom-EP distribution. Also, the values of the goodness-of-fit statistics W^* , A^* and KS indicate that the TL-Gom-EP distribution fits the carbon fibres data better than the selected non-nested models with the same number of parameters. The p-value of the K-S statistic is closer to 1 and the SS value from the probability plots in Figure 4.11 is also the smallest for the TL-Gom-EP distribution. Profile log-likelihood plots are given in Figure 4.10 indicating that maximum likelihood estimates of b, γ, λ and θ are unique and exists.

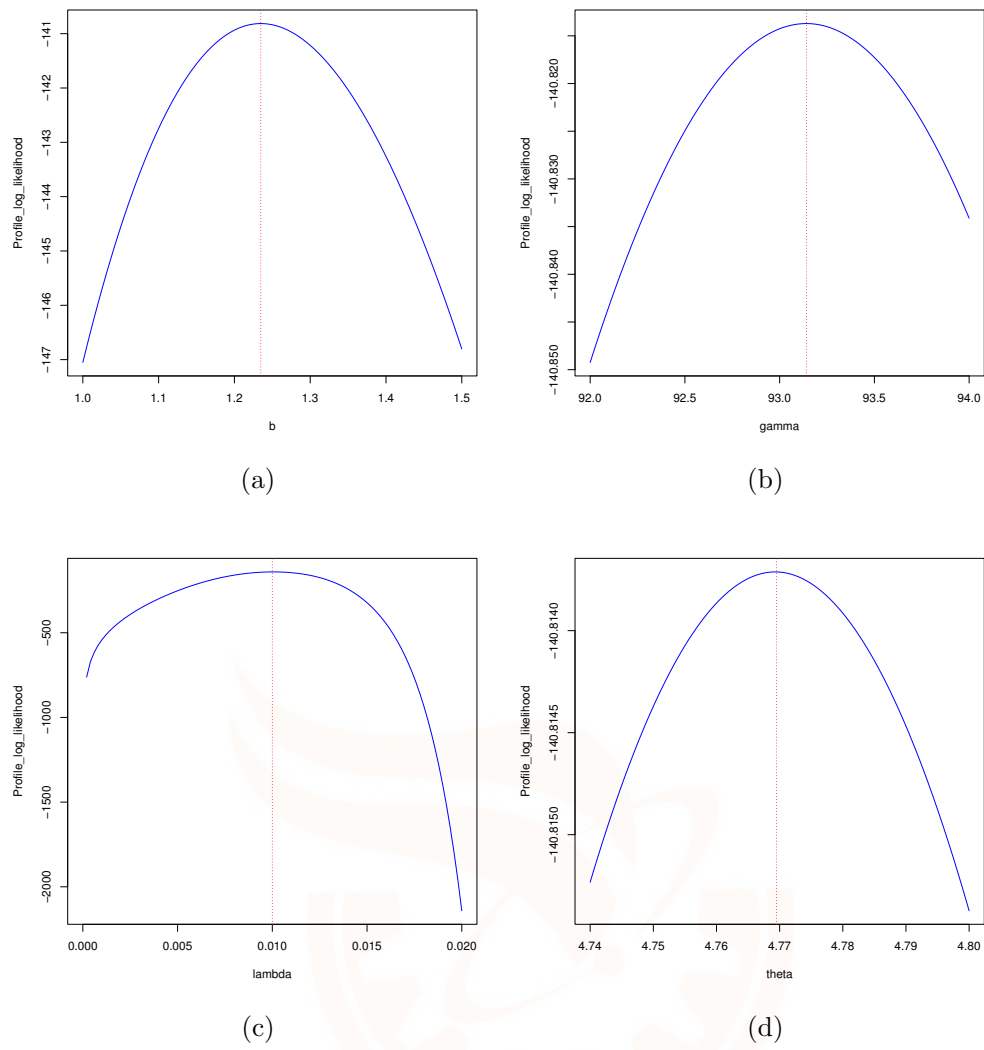


Figure 4.10: Profile log-likelihood plots for carbon fibres data

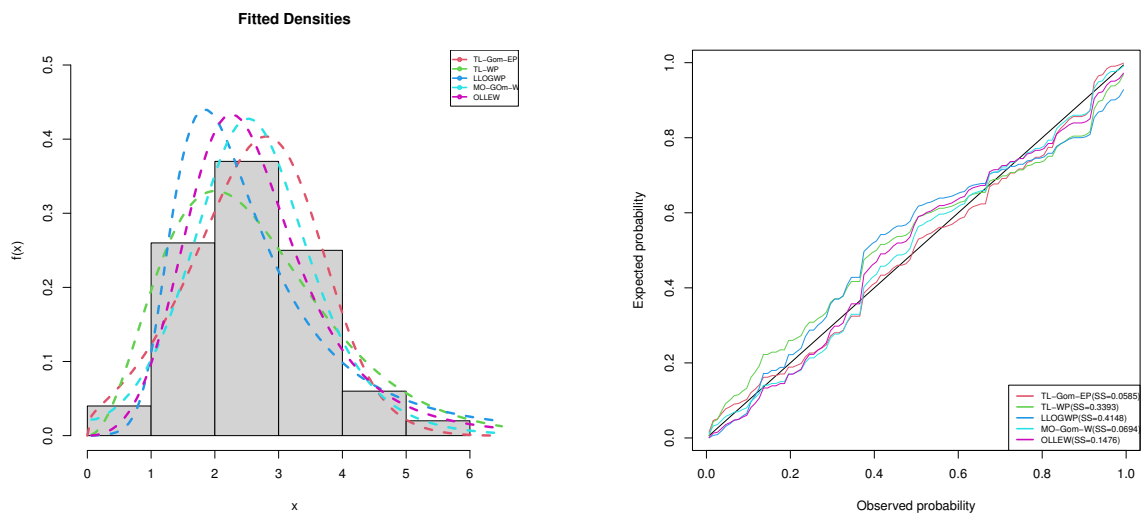


Figure 4.11: Fitted pdf and probability plots for carbon fibres data

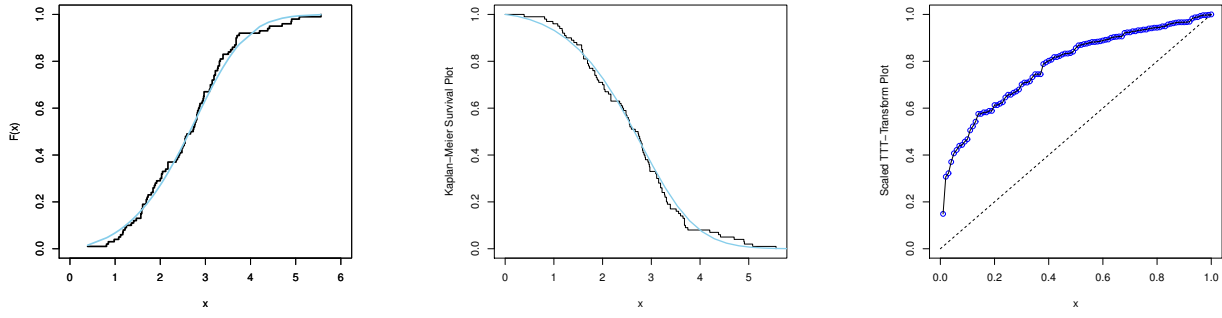


Figure 4.12: Estimated cdf, K-M survival and TTT-Transform plots for the TL-Gom-EP distribution for carbon fibres data

Figure 4.12 gives the fitted cdf, survival function and TTT-Transform plots for the carbon fibres data. The cdf line indicated by the blue line is close to the empirical cdf while the survival function in blue is closer to the K-M curve which shows that indeed our model is better in explaining the carbon fibres data set.

4.7.2 Time-to-Failure (103 h) of Turbocharger Data

The second data is the time-to-failure of turbocharger of one type of engine by Xu et al. (2003). The data points are given by: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

The estimated variance-covariance matrix of the TL-Gom-EP model for turbocharger data is given by

$$\begin{bmatrix} 1.8257 \times 10^{-02} & 1.2041 \times 10^{-05} & 8.6715 \times 10^{-06} & -3.9368 \times 10^{-05} \\ 1.2041 \times 10^{-05} & 7.9419 \times 10^{-09} & 5.7193 \times 10^{-09} & -2.5965 \times 10^{-08} \\ 8.6715 \times 10^{-06} & 5.7193 \times 10^{-09} & 5.2274 \times 10^{-09} & -1.8699 \times 10^{-08} \\ -3.9368 \times 10^{-05} & -2.5965 \times 10^{-09} & -1.8699 \times 10^{-08} & 8.4891 \times 10^{-08} \end{bmatrix}$$

and the approximate 95% confidence interval are given by

$b \in (0.2102, 1.8513)$, $\gamma \in (379.0598, 379.0602)$, $\lambda \in (0.0009, 0.0011)$ and $\theta \in (68.8144, 68.8156)$.

Table 4.10: Estimates of models for turbocharger data

Model	Estimates				Statistics							
	b	γ	λ	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
TL-Gom-EP	$1.5865 \times 10^{+00}$ (1.3509×10^{-01})	$3.7906 \times 10^{+02}$ (8.9162×10^{-04})	1.0004×10^{-03} (7.2287×10^{-04})	$6.8815 \times 10^{+00}$ (2.9133×10^{-05})	159.96	167.96	169.10	174.71	0.0349	0.2669	0.0942	0.8700
TL-WP	2.0318×10^{-02} (5.1960×10^{-02})	$1.9850 \times 10^{+00}$ ($1.0634 \times 10^{+00}$)	$2.6192 \times 10^{+00}$ ($2.7043 \times 10^{+00}$)	8.5500×10^{-09} (1.0776×10^{-02})	171.80	179.80	180.95	186.56	0.1633	1.1144	0.1143	0.6730
LLOGWP	0.6565 (1.9224×10^{-01})	0.0071 (6.7953×10^{-02})	2.7666 (7.9627×10^{-01})	9.2801 ($7.6265 \times 10^{+00}$)	164.88	172.88	174.03	179.64	0.0653	0.4809	0.1043	0.7767
MO-Gom-W	1.3442 (1.3919)	0.0071 (0.0089)	0.8206 (0.2906)	1.0282 (0.8137)	160.18	168.18	169.33	174.94	0.0373	0.2853	0.1347	0.4622
OLLEW	12.5715 (11.2282)	0.3613 (0.3469)	1.1234 (0.5705)	9.0836 (6.7288)	175.78	183.78	2 184.92	190.5305	0.1947	1.2940	0.1369	0.4410

Table 4.10 gives the MLE's and goodness-of-fit statistics of the TL-Gom-EP for turbocharger data. The values of the goodness-of-fit statistics: AIC, AICC and BIC are the smallest for the TL-Gom-EP distribution. The goodness-of-fit statistics W^* , A^* and K-S values are also small for the TL-Gom-EP distribution as compared to the non-nested models listed in the table. This shows that indeed the TL-Gom-EP distribution fits the turbocharger data better. The p-value of the K-S statistic is the largest and the sum of squares value from the probability plots in Figure 4.14 is also the smallest for the TL-Gom-EP distribution. Plots of profile log-likelihood in Figure 4.13 shows that maximum likelihood estimates of b, γ, λ and θ can be uniquely obtained.

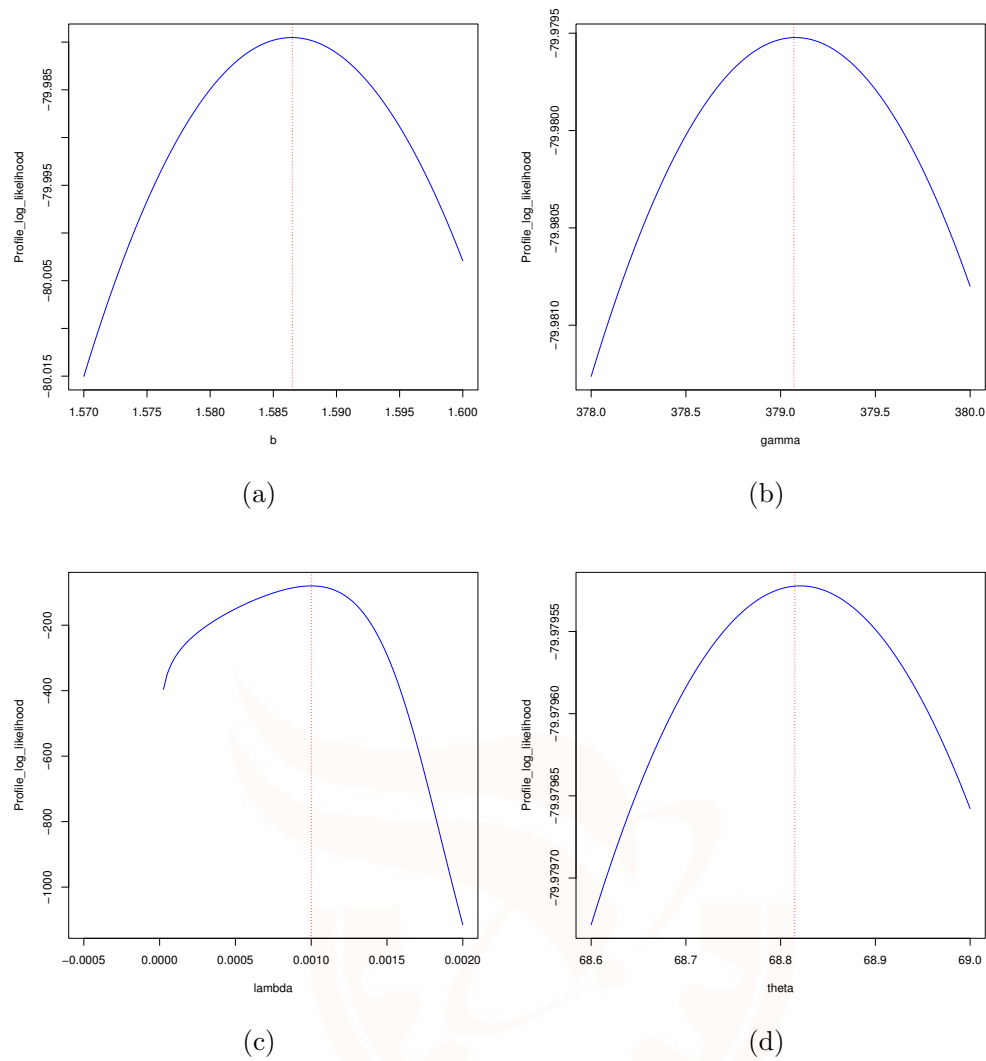


Figure 4.13: Profile log-likelihood plots for turbocharger data

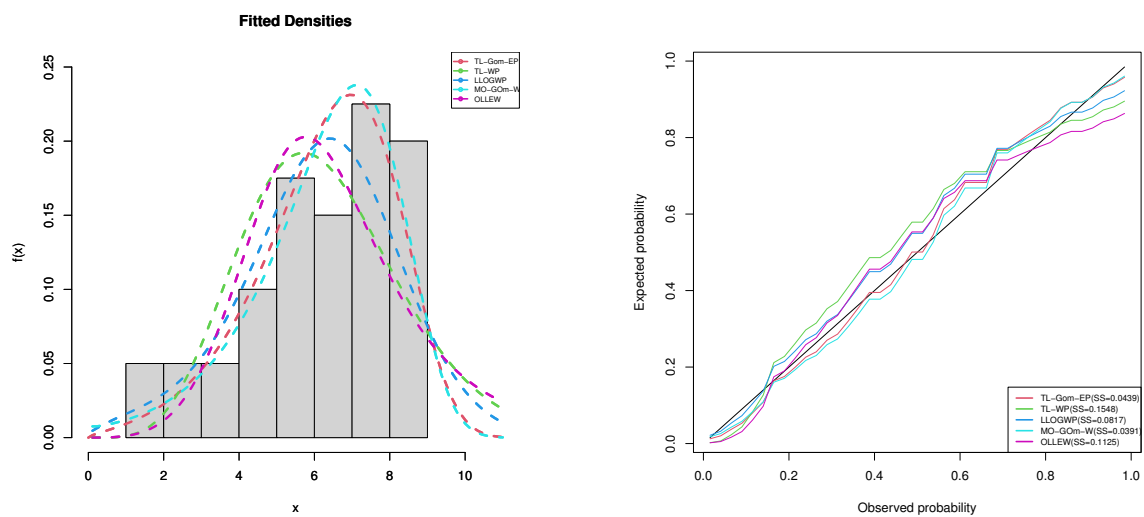


Figure 4.14: Fitted pdf and probability plots for turbocharger data

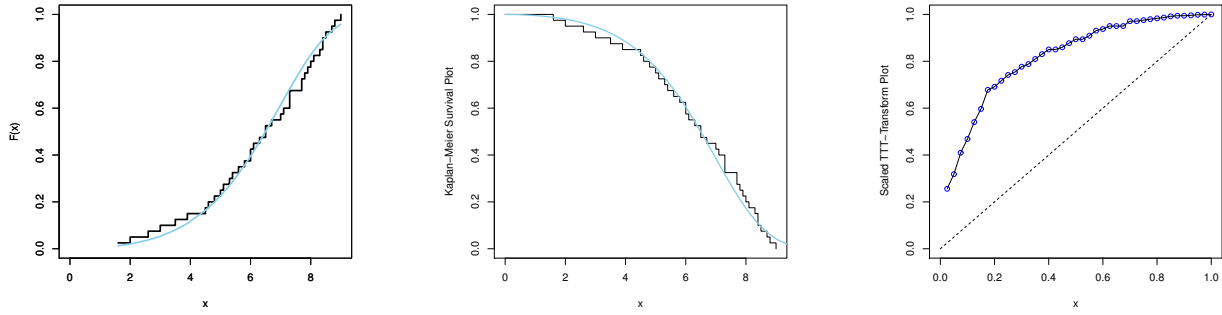


Figure 4.15: Estimated cdf, K-M survival and TTT-Transform plots for the TL-Gom-EP distribution for turbocharger data

Figure 4.15 gives the estimated cdf, K-M survival and TTT-Transform plots for the turbocharger data. The estimated cdf line for the TL-Gom-EP distribution is closer to the empirical cdf while the survival function is also close to the K-M curve, hence our model is better in explaining the turbocharger data. The TTT-transform plot shows that our model can fit data with increasing hrf.

4.8 Concluding Remarks

In this chapter, we presented a new class of generalized distributions called the TL-Gom-GPS distribution and derived some of the statistical properties of the new class of distributions. The method of maximum likelihood estimation was used to estimate the parameters the new class of distributions. We also studied some special cases of the new class of distributions. Applications of the proposed model to real-life data sets has demonstrated its utility and applicability for the special class of the Topp-Leone Gompertz-Exponential Poisson distribution.

5. The Marshall-Olkin-Topp-Leone-Gompertz-G Family of Distributions

5.1 Introduction

Marshall-Olkin generator (Marshall and Olkin (1997)) was introduced as a method of obtaining more flexible distributions by introducing a new parameter into the existing one. The Topp-Leone-Gompertz-G (Oluyede et al. (2022)) family of distributions has been proved to model data which are heavy-tailed and has hazard rate functions that are monotone and non-monotone, hence by combining the two we obtain a more flexible generalized family of distributions that can model data in several areas including economics, relativity and hydrology, just to mention a few.

A new family of distributions called the MO-TL-Gom-G distribution is developed and studied in detail. Some mathematical and statistical properties of the proposed distribution are explored. Statistical properties of the new family of distributions considered are the quantile function, moments and generating function, probability weighted moments, distribution of the order statistics and Rényi entropy. The maximum likelihood technique is used for estimating the model parameters and simulations are conducted in order to observe how the new model performs. Finally, we give examples of real-life data to show the usefulness of the above mentioned Topp-Leone-Gompertz generalization.

5.2 The New Family of Distributions

In this subsection, we present the proposed family of distributions. The Topp-Leone Gompertz-G distribution was introduced by Oluyede et al. (2022) with the cdf and pdf given by

$$F_{TL-Gom-G}(x; b, \theta, \xi) = \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b, \quad (5.1)$$

and

$$\begin{aligned} f_{TL-Gom-G}(x; b, \theta, \xi) &= 2bg(x; \xi)(1 - G(x; \xi))^{-\theta-1} \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^{b-1}, \end{aligned} \quad (5.2)$$

respectively, for $x > 0$, $b, \theta > 0$ and parameter vector ξ . Marshall and Olkin (1997) developed the distribution with the cdf and pdf given by equations (1.15) and (1.16)

respectively. Considering the generalization by Marshall and Oklin, and inserting equations (5.1) and (5.2) into equations (1.15) and (1.16) respectively, we obtain the MO-TL-Gom-G distribution with the cdf and pdf

$$F_{MO-TL-Gom-G}(x; \delta, b, \theta, \xi) = 1 - \frac{\delta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right)}{1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right)}, \quad (5.3)$$

and

$$\begin{aligned} f_{MO-TL-Gom-G}(x; \delta, b, \theta, \xi) &= 2b\delta g(x; \xi) (1 - G(x; \xi))^{-\theta-1} \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^{b-1} \\ &\times \left(1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right) \right)^{-2}, \end{aligned} \quad (5.4)$$

for $\delta, b, \theta > 0$ and parameter vector ξ .

5.2.1 Sub-Families

In this subsection, we present some of the sub-families of the MO-TL-Gom-G family of distributions.

- When $\delta = 1$, we get the TL-Gom-G distribution with the cdf given by

$$F(x; b, \theta, \xi) = \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b,$$

for $b, \theta > 0$, and baseline vector ξ .

- When $b = 1$, we have the distribution with the cdf given by

$$F(x; \delta, \theta, \xi) = 1 - \frac{\delta \left(1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right)}{1 - \bar{\delta} \left(1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right)},$$

for $\delta, \theta > 0$, and parameter vector ξ .

- When $\delta = b = 1$, we get the distribution with the cdf given by

$$F(x; \theta, \xi) = 1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2,$$

for $\theta > 0$ and parameter vector ξ .

In addition, several new distributions can be obtained when the baseline cdf $G(x; \xi)$ is specified.

5.3 Some Statistical Properties

In this section, we present the linear representation of the density function, hazard and quantile functions, moments, generating function, stochastic orders, probability weighted moments, distribution of order statistics and Rényi entropy of the MO-TL-Gom-G distribution.

5.3.1 Expansion of the Density Function

We apply the general results by Barreto-Souza et al. (2013) to obtain the linear representations of the MO-TL-Gom-G family of distributions in this subsection. Equation (5.4) can be written as

$$f_{MO-TL-Gom-G}(x; \delta, b, \theta, \xi) = \frac{f_{TL-Gom-G}(x; b, \theta, \xi)}{\delta \left[1 - \frac{\delta-1}{\delta} F_{TL-Gom-G}(x; b, \theta, \xi)\right]^2}, \quad (5.5)$$

where $f_{TL-Gom-G}(x; b, \theta, \xi)$ and $F_{TL-Gom-G}(x; b, \theta, \xi)$ are given in equations (5.2) and (5.1), respectively.

If $\delta \in (0, 1)$, we obtain the pdf of the MO-TL-Gom-G family of distributions as

$$\begin{aligned} f_{MO-TL-Gom-G}(x; \delta, b, \theta, \xi) &= 2b \sum_{j,i,k,l,m=0}^{\infty} \sum_{q=0}^j \frac{(-1)^{i+l+m} (2i+2)^k}{\theta^k k! (m+1)} \gamma_{j,q} \binom{b(j-q+1)-1}{i} \\ &\times \binom{-\theta(l+1)-1}{m} \binom{k}{l} (m+1) g(x; \xi) G^m(x; \xi) \\ &= \sum_{m=0}^{\infty} V_{m+1} h_{m+1}(x; \xi), \end{aligned} \quad (5.6)$$

where

$$\begin{aligned} V_{m+1} &= 2b \sum_{j,i,k,l=0}^{\infty} \sum_{q=0}^j \frac{(-1)^{i+l+m} (2i+2)^k}{\theta^k k! (m+1)} \gamma_{j,q} \binom{b(j-q+1)-1}{i} \\ &\times \binom{-\theta(l+1)-1}{m} \binom{k}{l}, \end{aligned} \quad (5.7)$$

and $h_{m+1}(x; \xi) = (m+1)g(x; \xi)G^m(x; \xi)$. It follows that for $\delta \in (0, 1)$, the MO-TL-Gom-G family of distributions can be expressed as a linear combination of the Exp-G density with power parameter $(m+1)$.

For $\delta > 1$, the series representation of the pdf of the MO-TL-Gom-G family of distributions is given by

$$\begin{aligned}
 f_{MO-TL-Gom-G}(x; \delta, b, \theta, \xi) &= 2b \sum_{j,i,k,l,q=0}^{\infty} \frac{(-1)^{i+l+q}(2i+2)^k}{\theta^k k! (q+1)} e_j \binom{b(j+1)-1}{i} \\
 &\times \binom{-\theta(l+1)-1}{q} \binom{k}{l} (q+1) g(x; \xi) G^q(x; \xi) \\
 &= \sum_{q=0}^{\infty} W_{q+1} h_{q+1}(x; \xi),
 \end{aligned} \tag{5.8}$$

where

$$\begin{aligned}
 W_{q+1} &= 2b \sum_{j,i,k,l=0}^{\infty} \frac{(-1)^{i+l+q}(2i+2)^k}{\theta^k k! (q+1)} e_j \binom{b(j+1)-1}{i} \\
 &\times \binom{-\theta(l+1)-1}{q} \binom{k}{l},
 \end{aligned} \tag{5.9}$$

and $h_{q+1}(x; \xi) = (q+1)g(x; \xi)G^q(x; \xi)$. Again, for $\delta > 1$, the MO-TL-Gom-G family of distributions can be expressed as an infinite linear combination of the Exp-G densities with power parameter $(q+1)$. Appendix section contains details of the derivations.

5.3.2 Hazard Rate and Quantile Functions

In this subsection, we present the hrf and quantile functions of the MO-TL-Gom-G distribution. The hrf is given by

$$\begin{aligned}
 h_{MO-TL-Gom-G}(x; \delta, b, \theta, \xi) &= 2b\delta g(x; \xi)(1 - G(x; \xi))^{-\theta-1} \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right) \\
 &\times \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^{b-1} \\
 &\times \left(1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right) \right)^{-2} \\
 &\times \left(\frac{\delta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right)}{1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right)} \right)^{-1}.
 \end{aligned}$$

We obtain the quantile function of the MO-TL-Gom-G family of distributions by inverting the non-linear equation

$$1 - \frac{\delta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right)}{1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right)} = u,$$

for $0 \leq u \leq 1$, so that

$$\ln \left[1 - \left(1 - \left(\frac{(1-u)}{\delta + (1-u)\bar{\delta}} \right)^{\frac{1}{b}} \right)^{\frac{1}{2}} \right] = \frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}).$$

Therefore, the quantile function of the MO-TL-Gom-G family of distributions is

$$Q_G(u) = G^{-1} \left[1 - \left(1 - \left(\theta \ln \left[1 - \left(1 - \frac{(1-u)}{\delta + (1-u)\bar{\delta}} \right)^{\frac{1}{b}} \right]^{\frac{1}{2}} \right) \right)^{-\frac{1}{\theta}} \right]. \quad (5.10)$$

5.3.3 Moments and Generating Function

In this subsection, we present the moments and the generating function of the MO-TL-Gom-G family of distributions.

- For $\delta \in (0, 1)$, we obtain the n^{th} moment of the MO-TL-Gom-G family of distributions as

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_{MO-TL-Gom-G}(x; \delta, b, \theta, \xi) dx = \sum_{m=0}^{\infty} V_{m+1} E(Y_{m+1}^n), \quad (5.11)$$

where $E(Y_{m+1}^n)$ is the n^{th} moment of Y_{m+1} which follows an Exp-G distribution with power parameter $(m+1)$ and V_{m+1} is given in equation (5.7).

- For $\delta > 1$, the n^{th} moment of the MO-TL-Gom-G family of distributions is given by

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_{MO-TL-Gom-G}(x; \delta, b, \theta, \xi) dx = \sum_{q=0}^{\infty} W_{q+1} E(Y_{q+1}^n), \quad (5.12)$$

where $E(Y_{q+1}^n)$ is the n^{th} moment of Y_{q+1} which follows an Exp-G distribution with power parameter $(q+1)$ and W_{q+1} is given in equation (5.9). The generating function of the MO-TL-Gom-G family of distributions can be obtained as follows:

- For $\delta \in (0, 1)$, the moment generating function of the MO-TL-Gom-G family of distributions is

$$M_X(h) = E(e^{hX}) = \sum_{m=0}^{\infty} V_{m+1} E(e^{hY_{m+1}}),$$

where $E(e^{hY_{m+1}})$ is the MGF of the Exp-G density having power parameter $(m+1)$.

- For $\delta > 1$, the MGF of the MO-TL-Gom-G family of distributions is

$$M_X(h) = E\left(e^{hX}\right) = \sum_{q=0}^{\infty} W_{q+1} E\left(e^{hY_{q+1}}\right),$$

where $E\left(e^{hY_{q+1}}\right)$ is the mgf of the Exp-G family of distributions with power parameter $(q+1)$.

5.3.4 Stochastic Ordering

Stochastic orders for the MO-TL-Gom-G family of distributions are presented in this subsection. Consider two random variables Z and T having the distribution functions $F_Z(k)$ and $F_T(k)$, respectively, and $\bar{F}_Z(k) = 1 - F_Z(k)$ the survival function. Note that Z is stochastically smaller than T if $\bar{F}_Z(k) \leq \bar{F}_T(k)$ for all k or $F_Z(k) \geq F_T(k)$ for all k . This is denoted by $Z <_s T$. Hazard rate order and likelihood ratio order are stronger and are given by $Z <_{hk} T$ if $h_Z(k) \geq h_T(k)$ for all k , and $Z <_{lk} T$ if $\frac{f_Z(k)}{f_T(k)}$ is decreasing in k , (Shaked and Shanthikumar (2007)). We know that $Z <_{lk} T \Rightarrow Z <_{hk} T \Rightarrow Z <_s T$.

Theorem 4. Suppose X_1 and X_2 are two independent random variables following MO-TL-Gom-G(δ_1, b, θ, ξ) and MO-TL-Gom-G(δ_2, b, θ, ξ) distributions. If $\delta_1 \leq \delta_2$, then X_1 and X_2 are stochastically ordered.

Proof:

The pdfs of X_1 and X_2 are respectively

$$\begin{aligned} f_1(x; \delta_1, b, \theta, \xi) &= 2b\delta_1 g(x; \xi) (1 - G(x; \xi))^{-\theta-1} \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^{b-1} \\ &\times \left(1 - \delta_1 \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right) \right)^{-2}, \end{aligned}$$

and

$$\begin{aligned} f_2(x; \delta_2, b, \theta, \xi) &= 2b\delta_2 g(x; \xi) (1 - G(x; \xi))^{-\theta-1} \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^{b-1} \\ &\times \left(1 - \delta_2 \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right) \right)^{-2}. \end{aligned}$$

Note that the ratio of the pdf $f_1(x; \delta_1, b, \theta, \xi)$ and $f_2(x; \delta_2, b, \theta, \xi)$ is

$$\frac{f_1(x; \delta_1, b, \theta, \xi)}{f_2(x; \delta_2, b, \theta, \xi)} = \frac{\delta_1}{\delta_2} \left(\frac{1 - \bar{\delta}_1 \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right)}{1 - \bar{\delta}_2 \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right)} \right)^{-2}, \quad (5.13)$$

and upon differentiating equation (5.13) with respect to x , we get

$$\begin{aligned} \frac{d}{dx} \left(\frac{f_1(x; \delta_1, b, \theta, \xi)}{f_2(x; \delta_2, b, \theta, \xi)} \right) &= \frac{1 - \bar{\delta}_2 \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right)}{\left[1 - \bar{\delta}_1 \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^b \right) \right]^3} \\ &\times \frac{\delta_1}{\delta_2} (\delta_1 - \delta_2) \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \right)^2 \right]^{b-1} \\ &\times 2bg(x; \xi)(1 - G(x; \xi))^{-\theta-1} \exp \left[\frac{1}{\theta} (1 - (1 - G(x; \xi))^{-\theta}) \right] \end{aligned} \quad (5.14)$$

which is ≤ 0 if $\delta_1 \leq \delta_2$. Therefore, $X_1 <_{lk} X_2$, $X_1 <_{hk} X_2$ and $X_1 <_s X_2$. This proves that the random variables X_1 and X_2 are stochastically ordered.

5.3.5 Probability Weighted Moments

The PWMs of the MO-TL-Gom-G family of distributions are presented in this subsection. The PWMs were introduced by Greenwood et al. (1979). The PWMs for a random variable X following the MO-TL-Gom-G distribution is

$$\eta_{a,r} = E(X^a [F(X)]^r) = \int_{-\infty}^{\infty} x^a f_{MO-TL-Gom-G}(x) [F_{MO-TL-Gom-G}(x)]^r dx. \quad (5.15)$$

If $\delta \in (0, 1)$, we can write

$$f(x) [F(x)]^r = \sum_{m=0}^{\infty} V_{m+1}^* g_{m+1}^*(x; \xi), \quad (5.16)$$

where

$$\begin{aligned}
 V_{m+1}^* &= 2b \sum_{j,r,i,k,l=0}^{\infty} \sum_{q=0}^j \frac{(-1)^{i+l+m}(2i+2)^k}{\theta^k k! (m+1)} B_{j,r,q} \binom{b(j+r-q+1)-1}{i} \\
 &\times \binom{-\theta(l+1)-1}{m} \binom{k}{l},
 \end{aligned}$$

and $g_{m+1}^*(x; \xi) = (m+1)g(x; \xi)G^m(x; \xi)$ is the Exp-G density with power parameter $(m+1)$ and baseline vector ξ . Therefore, the PWMs of the MO-TL-Gom-G family of distributions is given by

$$\eta_{a,r} = \sum_{m=0}^{\infty} V_{m+1}^* \int_{-\infty}^{\infty} x^a g_{m+1}^*(x; \xi) dx. \quad (5.17)$$

For $\delta > 1$, we have

$$f(x)[F(x)]^r = \sum_{m=0}^{\infty} a_{m+1}^* g_{m+1}^*(x; \xi),$$

where

$$\begin{aligned}
 a_{m+1}^* &= 2b \sum_{j,r,i,k,l=0}^{\infty} \frac{(-1)^{i+l+m}(2i+2)^k}{\theta^k k! (m+1)} A_{j,r} \binom{b(j+r+1)-1}{i} \\
 &\times \binom{-\theta(l+1)-1}{m} \binom{k}{l},
 \end{aligned} \quad (5.18)$$

and $g_{m+1}^*(x; \xi) = (m+1)g(x; \xi)G^m(x; \xi)$ is the Exp-G density with the power parameter $(m+1)$ and baseline vector ξ . Therefore, the PWMs of the MO-TL-Gom-G family of distributions is given by

$$\eta_{a,r} = \sum_{m=0}^{\infty} a_{m+1}^* \int_{-\infty}^{\infty} x^a g_{(m+1)}^*(x; \xi) dx. \quad (5.19)$$

Derivations follow in the appendix section.

5.3.6 Distribution of Order Statistics and Rényi Entropy

In this subsection, we present the pdf of the i^{th} order statistic and Rényi entropy for the MO-TL-Gom-G family of distributions.

Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be iid sample from MO-TL-Gom-G family of distributions, then the pdf of the i^{th} order statistic is given by

$$f_{i:n}(x) = \delta n! f_{TL-Gom-G}(x; b, \theta, \xi) \sum_{r=0}^{n-i} \frac{(-1)^r}{(i-1)!(n-i)!} \frac{F_{TL-Gom-G}^{r+i-1}(x; b, \theta, \xi)}{[1 - \delta F_{TL-Gom-G}(x; b, \theta, \xi)]^{r+i-1}}.$$

For $\delta \in (0, 1)$, we obtain the pdf of the i^{th} order statistic as

$$f_{i:n}(x) = \sum_{m=0}^{\infty} \Omega_{m+1} g_{m+1}^*(x; \xi), \quad (5.20)$$

where

$$\begin{aligned} \Omega_{m+1} &= 2b \sum_{j,q,p=0}^{\infty} \sum_{r=0}^{n-i} \sum_{k=0}^j \frac{(-1)^{q+l+m} (2q+2)^p}{\theta^p p! (m+1)} \omega_{j,r,k} \binom{b(j+r-k+i)-1}{q} \\ &\times \binom{-\theta(l+1)-1}{m} \binom{p}{l}, \end{aligned}$$

and $g_{m+1}^*(x; \xi) = (m+1)g(x; \xi) [G(x; \xi)]^m$ is the Exp-G pdf with power parameter (m+1) and parameter vector ξ .

If $\delta > 1$, we have

$$f_{i:n}(x) = \sum_{m=0}^{\infty} D_{m+1} g_{m+1}^*(x; \xi), \quad (5.21)$$

where $g_{m+1}^*(x; \xi) = (m+1)g(x; \xi) G^m(x; \xi)$ and

$$\begin{aligned} D_{m+1} &= 2b \sum_{j,q,p=0}^{\infty} \sum_{r=0}^{n-i} \frac{(-1)^{q+l+m} (2q+2)^p}{\theta^p p! (m+1)} D_{j,r} \binom{b(j+r+i)-1}{q} \\ &\times \binom{-\theta(l+1)-1}{m} \binom{p}{l}. \end{aligned} \quad (5.22)$$

Also, for $\delta > 1$, the pdf of the i^{th} order statistic of the MO-TL-Gom-G family of distributions can also be expressed as an infinite linear combination of the Exp-G density with power parameter (m+1) and parameter vector ξ . See appendix section for derivations.

Rényi Entropy

Rényi entropy (Rényi (1961)) of a random variable X following the MO-TL-Gom-G distribution is

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\int_0^\infty f_{MO-TL-Gom-G}^\nu(x; \delta, b, \theta, \xi) dx \right] \quad \text{for } \nu > 0 \text{ and } \nu \neq 1.$$

For $\delta \in (0, 1)$, we obtain Rényi entropy of the MO-TL-Gom-G family of distributions as

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\sum_{p=0}^{\infty} C_m^* \exp((1 - \nu)I_{REG}) \right], \quad (5.23)$$

where $I_{REG} = \frac{1}{1-\nu} \log \left[\int_0^\infty \left(\left(\frac{m}{\nu} + 1 \right) g(x; \xi) G_{\frac{m}{\nu}}(x; \xi) \right)^\nu dx \right]$ is Rényi entropy of Exp-G density with power parameter $\left(\frac{m}{\nu} + 1 \right)$ and

$$\begin{aligned} C_m^* &= \sum_{j,s,q,p,l=0}^{\infty} C_j \int_0^\infty (2b)^\nu (-1)^{s+q+l+m} \binom{j}{s} \binom{b(s+\nu)-\nu}{q} \frac{(2q+2\nu)^p}{\theta^p p! \left(\frac{m}{\nu} + 1 \right)^\nu} \\ &\times \binom{p}{l} \binom{-\theta(l+\nu)-\nu}{m}. \end{aligned}$$

If $\delta > 1$, Rényi entropy is given by

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\sum_{p=0}^{\infty} e_m^* \exp((1 - \nu)I_{REG}) \right], \quad (5.24)$$

where $I_{REG} = \frac{1}{1-\nu} \log \left[\int_0^\infty \left(\left(\frac{m}{\nu} + 1 \right) g(x; \xi) G_{\frac{m}{\nu}}(x; \xi) \right)^\nu dx \right]$ is Rényi entropy of Exp-G density with power parameter $\left(\frac{m}{\nu} + 1 \right)$ and

$$\begin{aligned} e_m^* &= \sum_{j,q,p,l=0}^{\infty} e_j \int_0^\infty (2b)^\nu (-1)^{q+l+m} \binom{b(j+\nu)-\nu}{q} \frac{(2q+2\nu)^p}{\theta^p p! \left(\frac{m}{\nu} + 1 \right)^\nu} \\ &\times \binom{p}{l} \binom{-\theta(l+\nu)-\nu}{m}. \end{aligned}$$

The full derivations are given in the appendix section.

5.4 Maximum Likelihood Estimation

Let $X \sim MO-TL-Gom-G(\delta, b, \theta, \xi)$ and $\Theta = (\delta, b, \theta, \xi)^T$ be the parameter vector, then the log-likelihood function (ℓ_n) of a random sample of size n from the $MO-TL-Gom-G(\delta, b, \theta, \xi)$

family of distributions is given by

$$\begin{aligned}
 \ell_n(\Theta) &= n \log(2b\delta) + \sum_{i=1}^n \log[g(x_i; \xi)] - (\theta + 1) \sum_{i=1}^n \log[(1 - G(x_i; \xi))] \\
 &+ \frac{2}{\theta} \sum_{i=1}^n \left[(1 - (1 - G(x_i; \xi))^{-\theta}) \right] \\
 &+ (b - 1) \sum_{i=1}^n \log \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x_i; \xi))^{-\theta}) \right] \right)^2 \right] \\
 &- 2 \sum_{i=1}^n \log \left[1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 - G(x_i; \xi))^{-\theta}) \right] \right)^2 \right]^b \right) \right].
 \end{aligned}$$

Elements of the score vector $U(\Theta) = \left(\frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \xi_k} \right)$ can be readily obtained. The MLE's of the parameters can be obtained by solving a system of non-linear equations $\left(\frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \xi_k} \right)^T = 0$ by numerical methods. Elements of the score vector are presented in the appendix.

5.5 Some Special Cases

In this section, we present some of the special models of the MO-TL-Gom-G family of distributions. We consider when the baseline cdf $G(x; \xi)$ are log-logistic, Weibull and Burr III distributions.

5.5.1 Marshall-Olkin-Topp-Leone-Gompertz-Log-Logistic Distribution

Suppose the baseline distribution is the log-logistic distribution with the cdf and pdf given by $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$, respectively, for $c, x > 0$. Then the cdf and pdf of the MO-TL-Gom-LLo distribution are given by

$$F(x; \delta, b, \theta, c) = 1 - \frac{\delta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - ((1 + x^c)^{-1})^{-\theta}) \right] \right)^2 \right]^b \right)}{1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - ((1 + x^c)^{-1})^{-\theta}) \right] \right)^2 \right]^b \right)},$$

and

$$\begin{aligned}
 f(x; \delta, b, \theta, c) &= 2b\delta cx^{c-1}(1 + x^c)^{-2}((1 + x^c)^{-1})^{-\theta-1} \left(\exp \left[\frac{1}{\theta} (1 - ((1 + x^c)^{-1})^{-\theta}) \right] \right)^2 \\
 &\times \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - ((1 + x^c)^{-1})^{-\theta}) \right] \right)^2 \right]^{b-1} \\
 &\times \left(1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - ((1 + x^c)^{-1})^{-\theta}) \right] \right)^2 \right]^b \right) \right)^{-2},
 \end{aligned}$$

respectively, for $\delta, b, \theta, c > 0$.

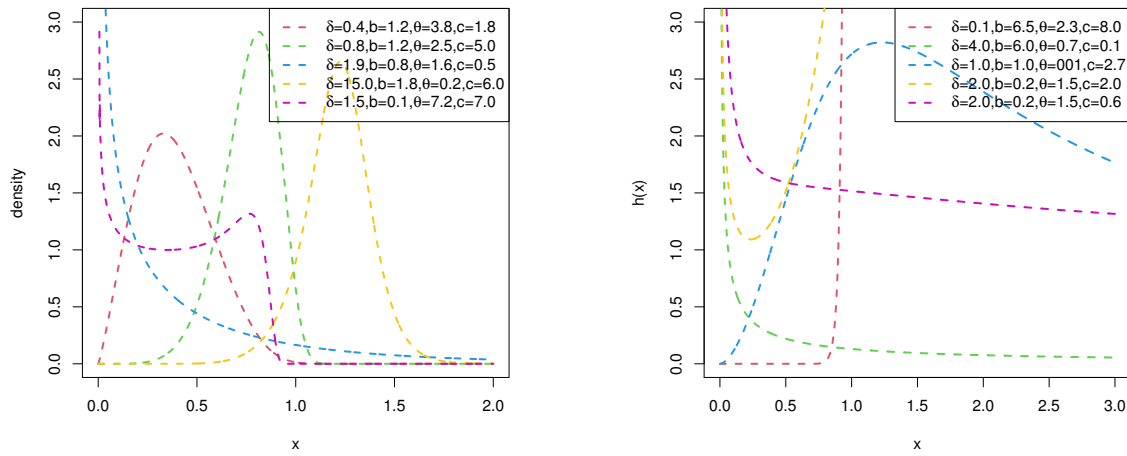


Figure 5.1: Pdf and hrf plots for the MO-TL-Gom-LLo distribution

Figure 5.1 gives the plots of the pdf and hrf of the MO-TL-Gom-LLo distribution. The pdf can take several shapes including right-skewed, left-skewed, bathtub followed by upside-down bathtub and almost symmetric shapes whereas the hrf displays increasing, decreasing, upside-down bathtub and bathtub shapes. In Tables 5.1 and 5.2, some quantile values and moments for MO-TL-Gom-LLo distribution are presented for different parameter values. The 3D plots for MO-TL-Gom-LLo distribution are given in Figures 5.2 and 5.3. The plots indicate that MO-TL-Gom-LLo distribution can model data with different levels of kurtosis and that the skewness can either be positive and negative.

Table 5.1: Quantiles for MO-TL-Gom-LLo distribution

	(δ, b, θ, c)				
u	(1, 2, 1.3, 0.2)	(0.7, 1, 3, 1.5)	(2.4, 1, 1, 2)	(2.1, 1, 1.9, 1)	(1.5, 1, 1.2, 3)
0.1	0.0009	0.1092	0.3439	0.1004	0.4244
0.2	0.0017	0.1778	0.4849	0.1943	0.5394
0.3	0.0078	0.2391	0.5949	0.2848	0.6236
0.4	0.0231	0.2987	0.6912	0.3751	0.6953
0.5	0.0605	0.3597	0.7822	0.4685	0.7609
0.6	0.1491	0.4237	0.8735	0.5685	0.8251
0.7	0.3700	0.4952	0.9714	0.6822	0.8917
0.8	0.9973	0.5809	1.0865	0.8226	0.9672
0.9	3.5238	0.6996	1.2486	1.0304	1.0686

Table 5.2: Moments for MO-TL-Gom-LLo distribution

	(δ, b, θ, c)				
μ'_s	(1, 0.5, 0.3, 1.2)	(0.5, 1.5, 0.5, 1.5)	(1.3, 0.1, 1.8, 0.6)	(1.1, 1, 0.1, 2)	(0.2, 0.8, 0.6, 0.9)
μ'_1	0.4249	0.6168	0.0493	0.7819	0.1679
μ'_2	0.5715	0.5877	0.0360	0.9116	0.1812
μ'_3	1.5646	0.7969	0.0499	1.5415	0.4835
μ'_4	7.6037	1.4462	0.1006	3.9038	2.2387
μ'_5	60.8931	3.3474	0.2652	15.9183	15.5135
μ'_6	760.8010	9.5244	0.8615	114.2684	148.7158
SD	0.6253	0.4553	0.1832	0.5479	0.3912
CV	1.4716	0.7381	3.7164	0.7008	2.3297
CS	4.0479	1.8943	7.2863	2.1836	6.7115
CK	35.7593	9.0185	80.9886	14.4801	82.9597

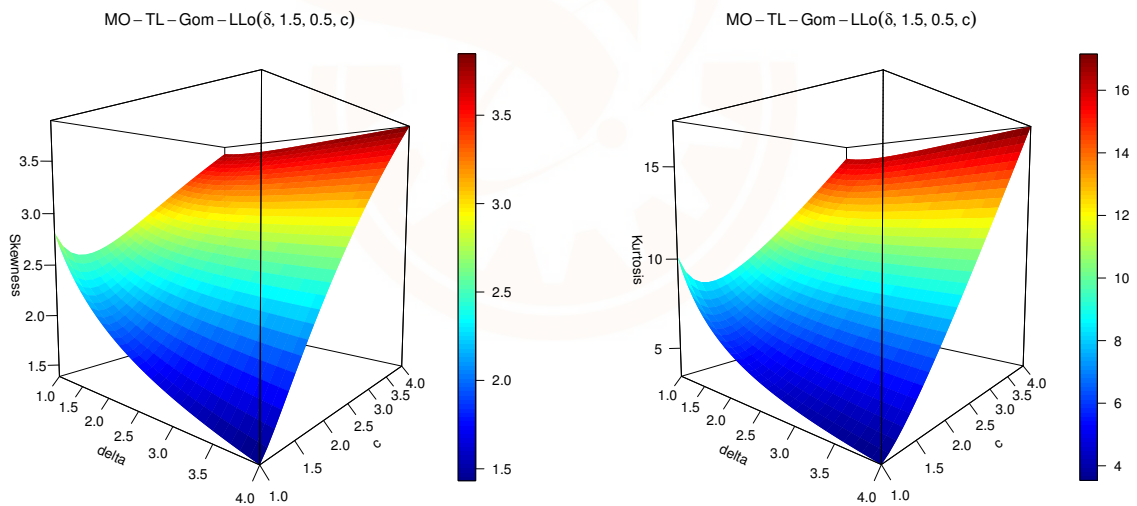


Figure 5.2: Skewness and kurtosis plots for MO-TL-Gom-LLo distribution.

5.5.2 Marshall-Olkin-Topp-Leone-Gompertz-Weibull Distribution

If we take the baseline distribution to be the Weibull distribution with the cdf and pdf given by $G(x; \lambda) = 1 - e^{-x^\lambda}$ and $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$, for $\lambda, x > 0$, then we obtain the cdf and pdf of the

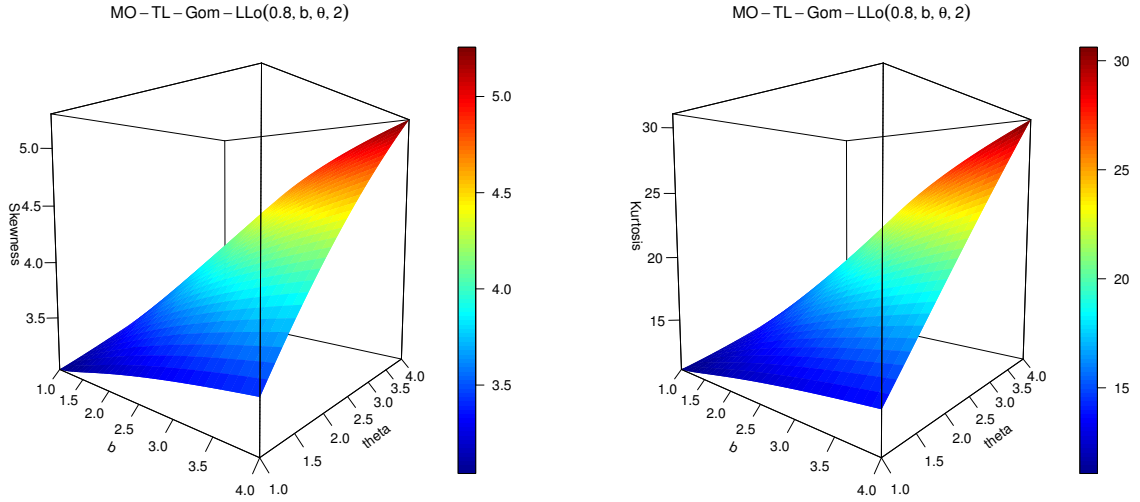


Figure 5.3: Skewness and kurtosis plots for MO-TL-Gom-LLo distribution.

MO-TL-Gom-W distribution as

$$F(x; \delta, b, \theta, \lambda) = 1 - \frac{\delta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - e^{\theta x^\lambda}) \right] \right)^2 \right]^b \right)}{1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - e^{\theta x^\lambda}) \right] \right)^2 \right]^b \right)},$$

and

$$\begin{aligned} f(x; \delta, b, \theta, \lambda) &= 2b\delta\lambda x^{\lambda-1} e^{-x^\lambda} (e^{-x^\lambda})^{-\theta-1} \left(\exp \left[\frac{1}{\theta} (1 - e^{\theta x^\lambda}) \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - e^{\theta x^\lambda}) \right] \right)^2 \right]^{b-1} \\ &\times \left(1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - e^{\theta x^\lambda}) \right] \right)^2 \right]^b \right) \right)^{-2}, \end{aligned}$$

respectively, for $\delta, b, \theta, \lambda > 0$.

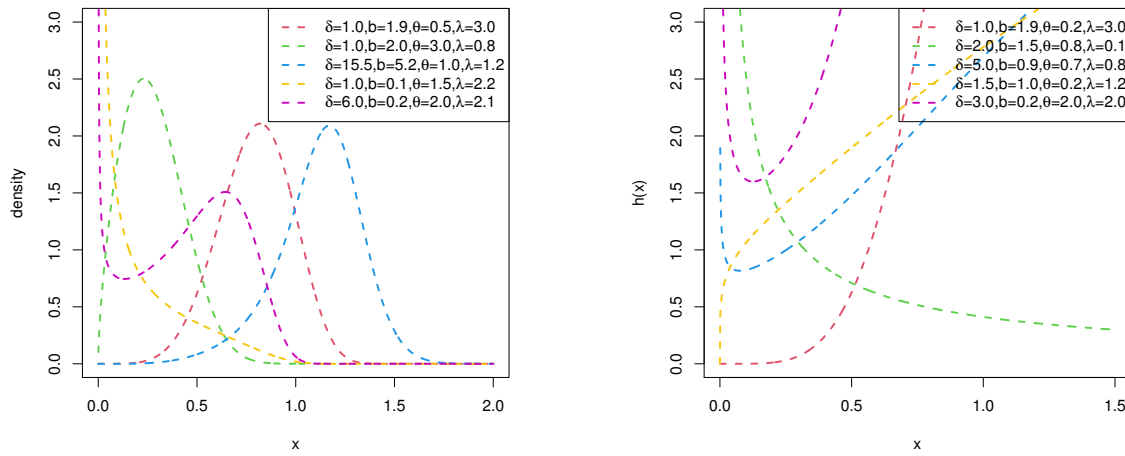


Figure 5.4: Plots of pdf and hrf for the MO-TL-Gom-W distribution

The pdf of the MO-TL-Gom-W distribution can be right-skewed, left-skewed, almost symmetric and bathtub followed by upside down bathtub shapes, whereas the hrf displays bathtub, decreasing, and decreasing shapes. In Tables 5.3 and 5.4, some quantile values and moments for MO-TL-Gom-W distribution are presented for different parameter values. Figures 5.5 and 5.6 shows that MO-TL-Gom-W distribution can model data sets with different levels of skewness and kurtosis.

Table 5.3: Quantiles for MO-TL-Gom-W distribution

$(\delta, b, \theta, \lambda)$					
u	(1, 2, 1.3, 0.2)	(0.7, 1, 3, 1.5)	(2.4, 1, 1, 2)	(2.1, 1, 1.9, 1)	(1.5, 1, 1.2, 3)
0.1	0.0005	0.1079	0.3342	0.0957	0.4193
0.2	0.0009	0.1734	0.4595	0.1774	0.5263
0.3	0.0035	0.2307	0.5503	0.2505	0.6011
0.4	0.0083	0.2839	0.6247	0.3185	0.6618
0.5	0.0187	0.3365	0.6909	0.3841	0.7146
0.6	0.0382	0.3901	0.7530	0.4501	0.7638
0.7	0.0769	0.4473	0.8151	0.5200	0.8123
0.8	0.1597	0.5121	0.8831	0.6003	0.8639
0.9	0.3870	0.5964	0.9692	0.7082	0.9274

Table 5.4: Moments for MO-TL-Gom-W distribution

	$(\delta, b, \theta, \lambda)$				
μ'_s	(0.1, 1.5, 0.5, 1.5)	(1.1, 0.1, 0.2, 1.5)	(3, 1.2, 0.1, 3)	(0.5, 0.2, 0.7, 0.9)	(0.5, 1.5, 3, 0.6)
μ'_1	0.3233	0.1360	1.0740	0.0837	0.1733
μ'_2	0.1929	0.1274	1.3238	0.0906	0.0699
μ'_3	0.2000	0.2490	1.8508	0.2110	0.0439
μ'_4	0.3192	0.8665	2.9370	0.7864	0.0366
μ'_5	0.6986	5.1000	5.3433	4.1553	0.0375
μ'_6	1.9404	49.2592	11.3559	29.0809	0.0449
SD	0.2973	0.3300	0.4127	0.2891	0.1997
CV	0.9195	2.4265	0.3843	3.4543	1.1521
CS	3.0636	5.6221	0.8988	7.8373	2.2573
CK	19.0461	62.7446	5.3868	102.9504	10.1035

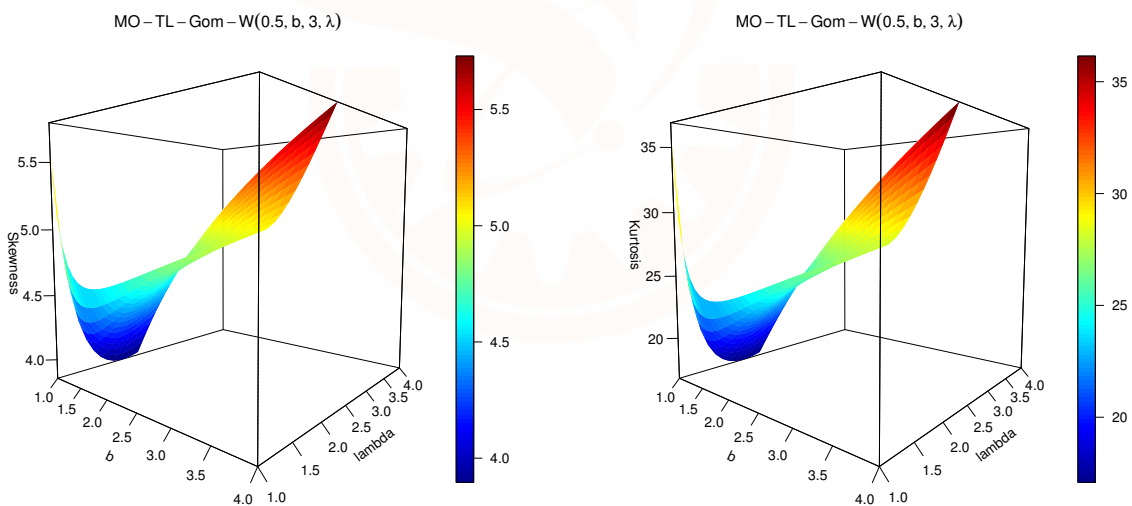


Figure 5.5: Skewness and kurtosis plots for MO-TL-Gom-W distribution

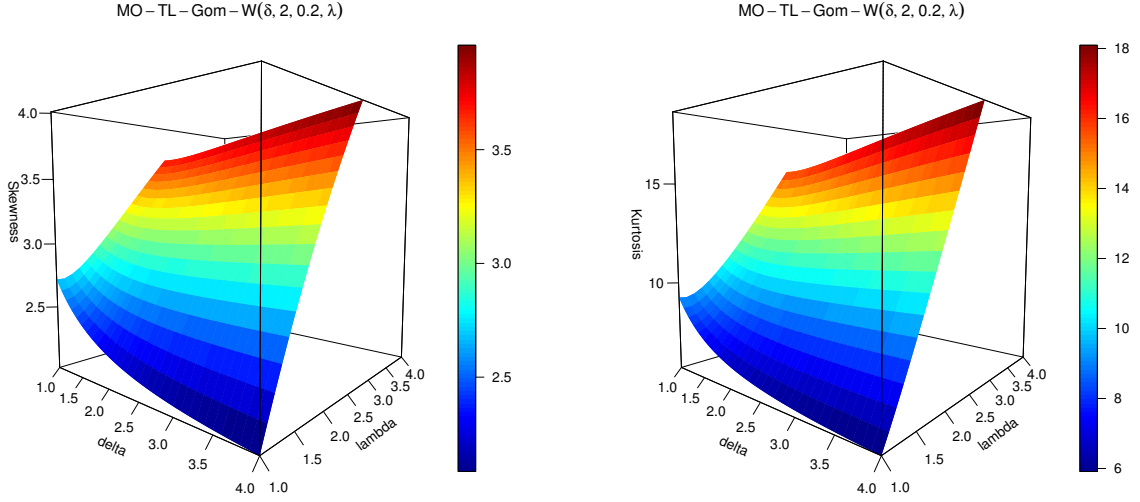


Figure 5.6: Skewness and kurtosis plots for MO-TL-Gom-W distribution

5.5.3 Marshall-Olkin-Topp-Leone-Gompertz-Burr III Distribution

Suppose the baseline distribution is Burr III distribution with the cdf and pdf given by $G(x; \lambda, \beta) = (1 + x^{-\beta})^{-\lambda}$ and $g(x; \lambda, \beta) = \beta x^{-\beta-1} (1 + x^{-\beta})^{-\lambda-1}$, for $\lambda, \beta, x > 0$, then the cdf and pdf of the MO-TL-Gom-BIII distribution are given by

$$F(x; \delta, b, \theta, \lambda, \beta) = 1 - \frac{\delta \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 + x^{-\beta})^{-\lambda})^{-\theta} \right] \right)^2 \right]^b \right)}{1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 + x^{-\beta})^{-\lambda})^{-\theta} \right] \right)^2 \right]^b \right)},$$

and

$$\begin{aligned} f(x; \delta, b, \theta, \lambda, \beta) &= 2b\delta\lambda\beta x^{-\beta-1} (1 + x^{-\beta})^{-\lambda-1} (1 - (1 + x^{-\beta})^{-\lambda})^{-\theta-1} \\ &\times \left(\exp \left[\frac{1}{\theta} (1 - (1 + x^{-\beta})^{-\lambda})^{-\theta} \right] \right)^2 \\ &\times \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 + x^{-\beta})^{-\lambda})^{-\theta} \right] \right)^2 \right]^{b-1} \\ &\times \left(1 - \bar{\delta} \left(1 - \left[1 - \left(\exp \left[\frac{1}{\theta} (1 - (1 + x^{-\beta})^{-\lambda})^{-\theta} \right] \right)^2 \right]^b \right) \right)^{-2}, \end{aligned}$$

respectively, for $\delta, b, \theta, \lambda, \beta > 0$.

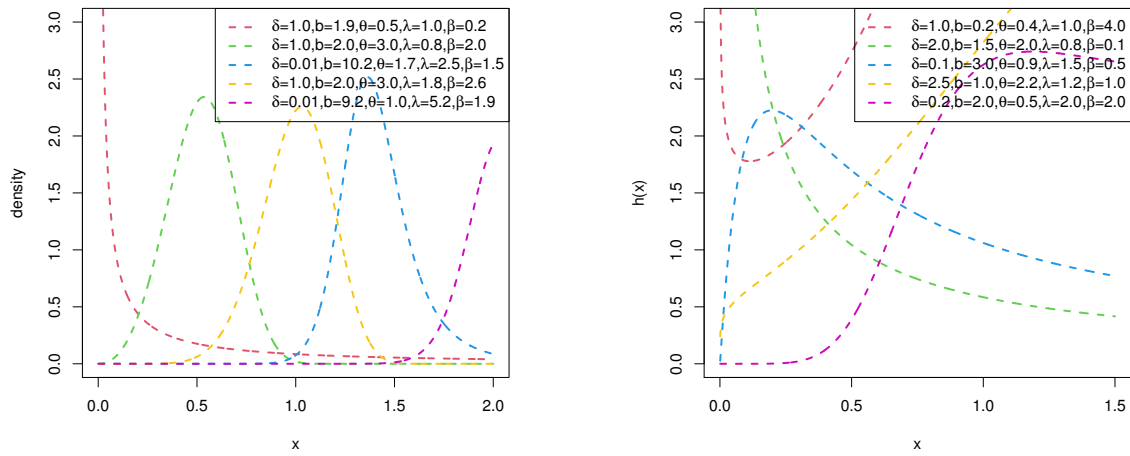


Figure 5.7: Plots of pdf and hrf for the MO-TL-Gom-BIII distribution

The MO-TL-Gom-BIII distribution has the pdf that displays right-skewed, left-skewed, and decreasing shapes, whereas the hrf exhibit bathtub, upside-down bathtub, increasing and decreasing shapes. In Tables 5.5 and 5.6, some quantile values and moments for MO-TL-Gom-BIII distribution are presented for different parameter values.

Table 5.5: Quantiles for MO-TL-Gom-BIII distribution

$(\delta, b, \theta, \lambda, \beta)$					
u	(1, 2, 1.3, 0.2, 2)	(0.7, 1, 3, 1, 1.5)	(2.4, 1, 1, 2.5, 2)	(2.1, 1, 2, 1.9, 1)	(1.5, 1, 2, 1.2, 3)
0.1	0.0097	0.1092	0.8285	0.3945	0.4946
0.2	0.0232	0.1778	1.0303	0.6205	0.6055
0.3	0.0392	0.2391	1.1862	0.8191	0.6839
0.4	0.0579	0.2987	1.3234	1.0073	0.7486
0.5	0.0798	0.3597	1.4537	1.1954	0.8059
0.6	0.1058	0.4237	1.5853	1.3925	0.8607
0.7	0.1375	0.4952	1.7273	1.6118	0.9164
0.8	0.1796	0.5809	1.8959	1.8790	0.9772
0.9	0.2444	0.6996	2.1356	2.2669	1.0561

Table 5.6: Moments for MO-TL-Gom-BIII distribution

	$(\delta, b, \theta, \lambda, \beta)$				
μ'_s	(0.8, 1, 2, 0.5, 2)	(2, 0.5, 1, 1.5, 3, 1)	(1, 2, 0.8, 0.4, 0.8)	(0.5, 0.5, 0.3, 0.2, 0.9)	(1.5, 2, 1.3, 3, 2.5)
μ'_1	0.2391	0.8348	0.1383	0.0100	1.5255
μ'_2	0.0860	0.7966	0.0633	0.0051	2.4252
μ'_3	0.0375	0.8318	0.0553	0.0083	4.0023
μ'_4	0.0188	0.9297	0.0758	0.0313	6.8343
μ'_5	0.0103	1.0977	0.1470	0.2345	12.0417
μ'_6	0.0061	1.3575	0.3784	3.2109	40.7003
SD	0.1698	0.3158	0.2102	0.0707	0.3131
CV	0.7102	0.3783	1.5197	7.0711	0.2053
CS	0.6435	0.0103	3.6975	23.0489	0.1142
CK	3.1631	2.6195	26.3291	1238.8412	2.9343

5.6 Simulation Results

The performance of the MO-TL-Gom-LLo distribution is examined by performing various simulations for different sample sizes ($n=25, 50, 100, 200, 400, 800, 1000$) via the bbmle package (Bolker and Bolker (2017)) in R package. We simulate $N = 1000$ samples for the true parameters values given in Table 5.7. Additional simulation results are available upon request or in the appendix. The tables list the mean MLEs along with the respective ABIAS and root RMSEs. The ABIAS and RMSEs for the estimated parameter, say, $\hat{\theta}$, are given by:

$$ABIAS(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. As the sample size n increases, RMSEs decreases, ABIAS decay towards zero and the mean estimates of parameter values are getting closer to the true parameter values.

Table 5.7: Simulation Results for MO-TL-Gom-LLo Distribution: Mean, ABIAS and RMSE

Parameter	n	$\delta = 1.5, b = 0.2, \theta = 1.0, c = 0.1$				$\delta = 0.9, b = 0.2, \theta = 0.2, c = 2.0$			
		Mean	ABIAS	AABIAS	RMSE	Mean	ABIAS	AABIAS	RMSE
δ	25	1.7891	0.2891	0.2891	2.1656	1.1146	0.2146	0.2146	1.4260
	50	1.7868	0.2868	0.2868	2.0244	0.9730	-0.0730	0.0730	0.8375
	100	1.5696	0.0696	0.0696	0.6909	0.8871	-0.0129	0.0129	0.2656
	200	1.5205	0.0205	0.0205	0.3177	0.8926	-0.0074	0.0074	0.1610
	400	1.4939	-0.0061	0.0061	0.1312	0.8932	-0.0068	0.0068	0.0677
	800	1.4949	-0.0051	0.0051	0.0830	0.8998	-0.0002	0.0002	0.0390
	1000	1.4994	-0.0006	0.0006	0.0698	0.9000	0.0000	0.0000	0.0304
b	25	0.4356	0.2356	0.2356	0.3218	0.7657	0.5657	0.5657	3.1847
	50	0.4068	0.2068	0.2068	0.3041	0.4294	0.2294	0.2294	0.6390
	100	0.3384	0.1384	0.1384	0.2306	0.2965	0.0965	0.0965	0.2003
	200	0.2582	0.0582	0.0582	0.1344	0.2231	0.0231	0.0231	0.0821
	400	0.2282	0.0282	0.0282	0.0844	0.2050	0.0050	0.0050	0.0396
	800	0.2089	0.0089	0.0089	0.0404	0.2011	0.0011	0.0011	0.0217
	1000	0.2055	0.0055	0.0055	0.0346	0.2010	0.0010	0.0010	0.0205
θ	25	1.2173	0.2173	0.2173	0.7363	1.2219	1.0219	1.0219	1.7382
	50	1.1851	0.1851	0.1851	0.6504	0.8872	0.6872	0.6872	1.1212
	100	1.1452	0.1452	0.1452	0.4890	0.5015	0.3015	0.3015	0.5559
	200	1.0595	0.0595	0.0595	0.2990	0.2716	0.0716	0.0716	0.2441
	400	1.0205	0.0205	0.0205	0.1638	0.2158	0.0158	0.0158	0.1035
	800	1.0040	0.0040	0.0040	0.0977	0.2028	0.0028	0.0028	0.0419
	1000	0.9992	-0.0008	0.0008	0.0873	0.2014	0.0014	0.2014	0.0312
c	25	0.5795	0.4795	0.4795	0.6727	1.3333	-0.6667	0.6667	0.9528
	50	0.5393	0.4393	0.4393	0.6431	1.4927	-0.5073	0.5073	0.8137
	100	0.3763	0.2763	0.2763	0.5010	1.7227	-0.2773	0.2773	0.5343
	200	0.1973	0.0973	0.0973	0.2588	1.9327	-0.0673	0.0673	0.2608
	400	0.1337	0.0337	0.0337	0.1150	1.9969	-0.0031	0.0031	0.1157
	800	0.1075	0.0075	0.0075	0.0350	2.0022	0.0022	0.0022	0.0517
	1000	0.1033	0.0033	0.0033	0.0166	1.9999	-0.0001	0.0001	0.0121

5.7 Applications

This section provides illustrations of the flexible nature and usefulness of the MO-TL-Gom-LLo distributions in data modeling. The fits of the MO-TL-Gom-LLo distributions are contrasted with several competing non-nested distributions with the same number of parameters. MO-TL-Gom-LLo distribution is compared with Topp-Leone generated Weibull (Aryal et al. (2017)), Topp-Leone-Marshall-Olkin Weibull (Chipepa et al. (2020)), beta-Gompertz (Jafari et al. (2014)), Kumaraswamy Gompertz (Silva et al. (2022)), Marshall-Olkin extended generalized Gompertz (Benkhelifa (2017)), generalized Weibull Gompertz (El-Damcese et al. (2014)), extended generalized Gompertz (Karamikabir et al. (2021)) and odd generalized exponential Gompertz (El-Damcese et al. (2015)) distributions. The pdfs of the TLGW, TLMOW, BGom, KGom, MOEGGom, GWGom, EGGom and OGGom distributions are given in appendix.

Our model parameters were estimated using NLMixed in SAS and our goodness-of-fit test was conducted using the package AdequacyModel in R software. The parameter estimates (standard error in parenthesis), $(-2 \ln(L))$, Akaike Information Criterion ($AIC = 2p - 2 \ln(L)$), Bayesian Information Criterion ($BIC = p \ln(n) - 2 \ln(L)$) and Consistent Akaike Information Criterion ($AICC = AIC + 2 \frac{p(p+1)}{n-p-1}$), where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated

parameters are presented.

The following goodness-of-fit statistics: W^* and A^* described by Chen and Balakrishnan (1995), as well as K-S statistic and its p-value were obtained. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred, and for AIC, AICC, BIC, and the goodness-of-fit statistics W^* , A^* and K-S, smaller values are preferred. The results are shown in Tables 5.8 and 5.9.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. (1983)) are given for each example to show how well the new model fits the observed data sets. For the probability plot, we plotted $F_{MO-TL-Gom-G}(x_{(j)}; \hat{\delta}, \hat{b}, \hat{\theta}, \hat{\xi})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares:

$SS = \sum_{j=1}^n \left[F_{MO-TL-Gom-G}(x_{(j)}; \hat{\delta}, \hat{b}, \hat{\theta}, \hat{\xi}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2$. These plots are shown in Figures 5.8 and 5.12.

5.7.1 Fatigue Fracture Data

The first data set represents the life of fatigue fracture of Kevlar 373/epoxy. The data set has previously been used by Barlow et al. (1984). The observations are as follows:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

The estimated variance-covariance matrix for MO-TL-Gom-LLo model on fatigue fracture data set is given by

$$\begin{pmatrix} 0.0726 & 6.3647 & 0.2039 & -0.0192 \\ 6.3647 & 558.1588 & 17.8829 & -1.6819 \\ 0.2039 & 17.8829 & 0.5781 & -0.0547 \\ -0.0192 & -1.6819 & -0.0547 & 0.0053 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for δ, b, θ and c are given by (119.39 ± 0.5280) , (10.5298 ± 46.3058) , (3.6039 ± 1.4902) and (0.1531 ± 0.1431) , respectively.

Table 5.8: Estimates of models for fatigue fracture data

Model	Estimates				Statistics							
	δ	b	θ	c	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
MO-TL-Gom-LLo	119.3900 (0.2694)	10.5298 (23.6254)	3.6039 (0.7603)	0.1531 (0.0730)	240.2	248.2	248.8	257.7	0.0575	0.3359	0.0754	0.7513
TLGW	α 0.8382 (1.1722)	θ 1.9633 (3.3561)	β 0.9636 (0.3835)	η 0.4227 (0.3173)	244.2	252.2	252.7	261.5	0.1144	0.6778	0.0976	0.4358
TLMOW	b 0.6326 (7.46×10^{-10})	δ 0.0006 (7.05×10^{-7})	λ 4.48×10^{-5} (8.68×10^{-6})	γ 2.2203 (3.09×10^{-10})	242.1	250.1	250.7	259.4	0.0785	0.4713	0.0813	0.6661
BGom	α 1.6797 (0.3153)	β 1.5086 (0.0020)	γ 7.06×10^{-8} (0.0409)	θ 0.4849 (1.2715)	244.5	252.5	253.0	261.8	0.1169	0.6938	0.0957	0.4608
KGom	a 1.5869 (0.3708)	b 2.0031 (3.9509)	γ 1.48×10^{-8} (0.0217)	θ 0.3896 (0.6509)	244.2	252.2	252.8	261.5	0.1146	0.6791	0.0975	0.4374
MOEGGom	α 1.4534 (0.2919)	β 0.0084 (0.0242)	λ 0.4669 (0.2195)	θ 0.0033 (0.0114)	242.3	250.3	250.8	259.6	0.0900	0.5318	0.0875	0.5756
GWGom	a 0.4482 (0.0380)	b 1.3256 (0.7974)	c 0.5974 (0.0379)	d 1.25×10^{-7} (0.5942)	245.0	253.0	253.6	262.4	0.1306	0.7672	0.1099	0.2953
EGGom	α 1.2974 (0.2314)	β 0.0173 (0.0706)	λ 0.3058 (0.2185)	σ 0.0441 (0.1192)	242.2	250.2	250.8	259.5	0.0891	0.5251	0.0870	0.5819
OEGGom	α 418.97 (2.26×10^{-6})	β 1.7006 (0.3043)	λ 0.0017 (0.0003)	c 1.55×10^{-7} (0.0427)	244.5	252.5	253.1	261.8	0.1169	0.6943	0.0946	0.4754

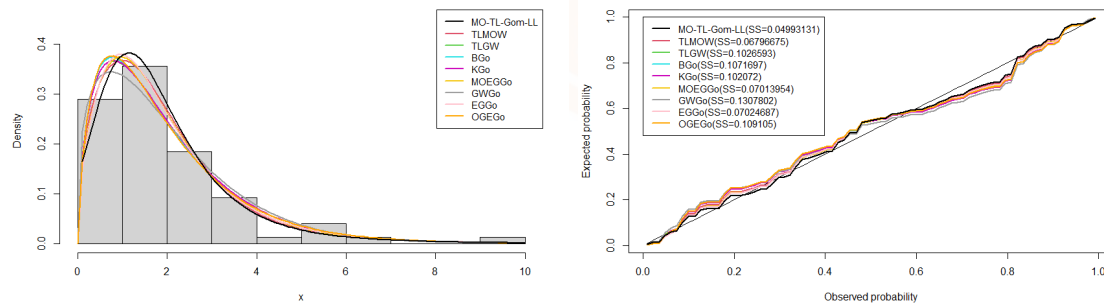


Figure 5.8: Histogram, fitted density and probability plots for fatigue fracture data

Table 5.8 indicates that MO-TL-Gom-LLo distribution has the highest p-value for the K-S statistic and the least values for all goodness-of-fit statistics. Thus, we conclude that the MO-TL-Gom-LLo model performs better on fatigue fracture data than non-nested TLGW, TLMOW, BGom, KGom, MOEGGom, GWGom, EGGom and OEGGom models. Further, Figure 5.8 shows that our model outperforms the competing non-nested models listed in the table. Additionally, the findings of Khan et al. (2020), who analyzed fatigue fracture data using the Transmuted Burr type X (TBX) distribution, were examined. Upon comparing the goodness-of-fit statistics, including AIC, BIC, and K-S, presented in their study with our results derived

from the MO-TL-Gom-LLo distribution, our model demonstrated superior performance, indicated by consistently smaller values in the goodness-of-fit statistics. Profile log-likelihood plots are given in Figure 5.9. The plots shows that maximum likelihood estimates of δ , b , θ and c can be uniquely obtained.

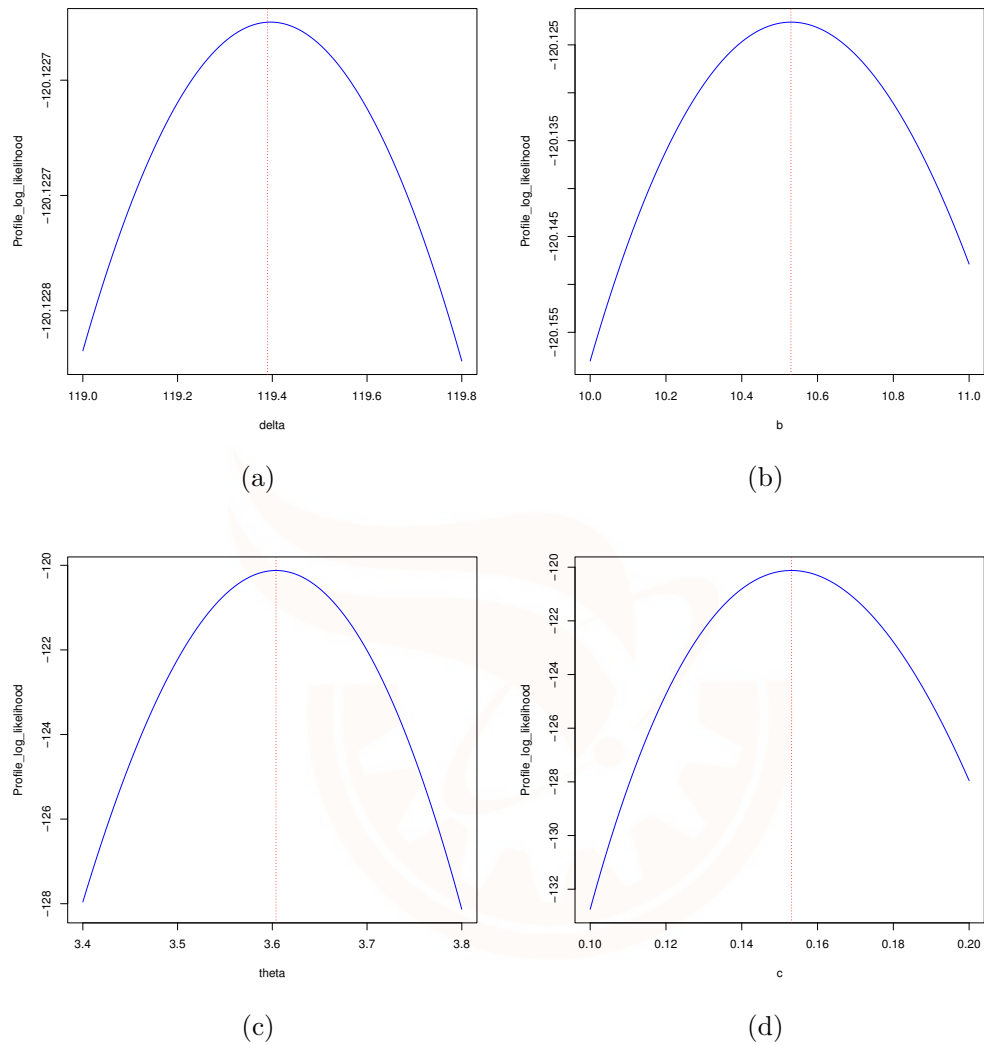


Figure 5.9: Profile log-likelihood plots for fatigue fracture data

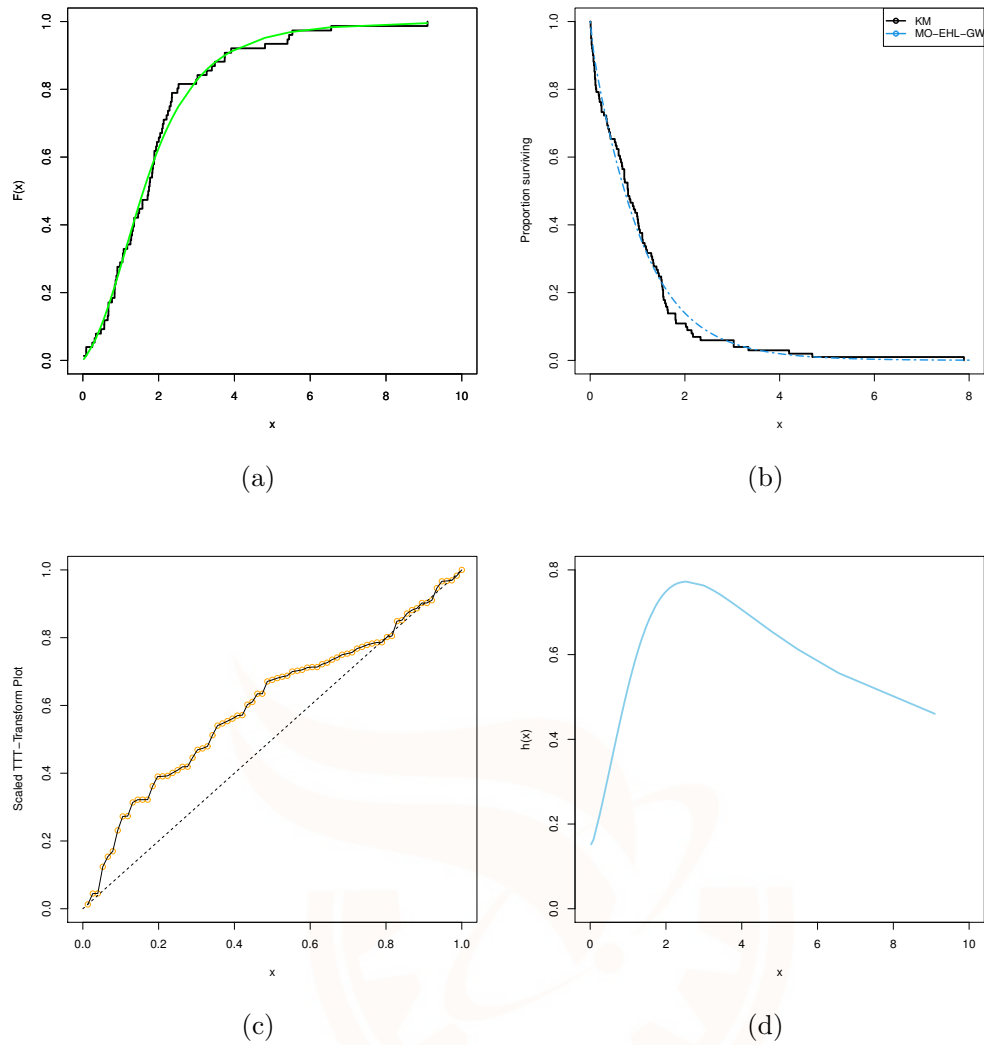


Figure 5.10: Estimated cdf, K-M survival, TTT-Transform and estimated hrf plots for fatigue fracture data

Figure 5.10 gives the estimated cdf, K-M survival, the scaled TTT-Transform plots and the estimated hrf plot for fatigue fracture data. We see that the cdf line for the MO-TL-Gom-LLo distribution indicated by the green line in (a) is close to the empirical cdf while the survival function (b) in green is also close to the K-M curve which shows that indeed our model is better in explaining the fatigue fracture data. The scaled TTT-Transform plot for fatigue fracture with the upside-down bathtub shaped hazard rate function are also given in (c) and (d), respectively.

5.7.2 Bladder Cancer Data

As the second example, we consider remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003). The observations are as follows:

0.08, 4.98, 25.74, 3.70, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.50, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.40, 11.98, 6.54, 6.93, 6.94, 0.40, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26, 26.31, 5.32, 15.96,

2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.50, 2.02, 0.20, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26, 4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.90, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07.

The estimated variance-covariance matrix for MO-TL-Gom-LLo model on remission times data set is given by

$$\begin{pmatrix} 6.52 \times 10^{-10} & 1.17 \times 10^{-11} & -9.54 \times 10^{-7} & 7.47 \times 10^{-8} \\ 1.17 \times 10^{-11} & 2.08 \times 10^{-13} & -1.71 \times 10^{-8} & 1.34 \times 10^{-9} \\ -9.54 \times 10^{-7} & -1.71 \times 10^{-8} & 1.40 \times 10^{-3} & -1.09 \times 10^{-4} \\ 7.47 \times 10^{-8} & 1.34 \times 10^{-9} & -1.09 \times 10^{-4} & 1.12 \times 10^{-5} \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for δ, b, θ and c are given by $(219.84 \pm 5.00 \times 10^{-5})$, $(13074 \pm 8.94 \times 10^{-7})$, (4.9427 ± 0.0733) and (0.0444 ± 0.0067) , respectively.

Table 5.9: Estimates of models for bladder cancer data

Model	Estimates				Statistics							
	δ	b	θ	c	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
MO-TL-Gom-LLo	219.84 (2.55×10^{-5})	13074.0 (4.56×10^{-7})	4.9427 (0.0374)	0.0444 (0.0034)	818.9	826.9	827.2	838.3	0.0140	0.0991	0.0293	0.9999
TLGW	α	θ	β	η	821.2	829.2	829.5	840.6	0.0414	0.2735	0.0444	0.9627
	1.1652 (1.9707)	2.9000 (6.2302)	0.5427 (0.2222)	0.2039 (0.3177)								
TLMOW	b	δ	λ	γ	819.3	827.3	827.6	838.7	0.0190	0.1221	0.0338	0.9986
	1.7215 (0.5055)	6516.74 (5.8×10^{-5})	6.5826 (0.4824)	0.1371 (0.0206)								
BGom	α	β	γ	θ	824.7	832.7	833.0	844.1	0.5276	3.3945	0.0674	0.6053
	1.4485 (0.3102)	0.1788 (0.1412)	1.27×10^{-9} (0.0062)	0.6454 (0.4809)								
KGom	a	b	γ	θ	825.1	833.1	833.4	844.5	0.0925	0.5632	0.0705	0.5487
	1.4165 (0.3490)	0.3708 (0.3813)	6.58×10^{-7} (0.0064)	0.3190 (0.3124)								
MOEGGom	α	β	λ	θ	819.3	827.3	827.7	838.8	0.0218	0.1534	0.0322	0.9994
	1.4920 (0.1772)	0.0024 (0.0067)	0.0387 (0.0215)	0.0022 (0.0084)								
GWGom	a	b	c	d	828.2	836.2	836.5	847.6	0.1314	0.7865	0.0701	0.5552
	0.6474 (0.0049)	1.0478 (0.4798)	0.1354 (0.0248)	5.57×10^{-8} (0.4443)								
EGGom	α	β	λ	σ	819.2	827.2	827.5	838.6	0.0189	0.1335	0.0313	0.9996
	1.2844 (0.2531)	0.2173 (1.0982)	2.97×10^{-8} (0.0502)	0.0600 (0.1756)								
OEGGom	α	β	λ	c	827.7	835.7	836.1	847.1	0.1283	0.7696	0.0829	0.3420
	45.7510 (0.0001)	1.2054 (0.1702)	0.0025 (0.0004)	3.00×10^{-10} (0.0062)								

Based on Table 5.9, MO-TL-Gom-LLo distribution has the highest p-value for the K-S statistic and the lowest goodness-of-fit statistics compared to other non-nested models. Thus, we conclude that the MO-TL-Gom-LLo model performs better with bladder cancer data than non-nested TLGW, TLMOW, BGom, KGom, MOEGGom, GWGom, EGGom and OGGom models. Moreover, Figure 5.12 shows that our model outperforms the competing non-nested models on bladder cancer data. Profile log-likelihood plots are given in Figure 5.11. The plots show that maximum likelihood estimates of δ , b , θ and c exits are distinct.

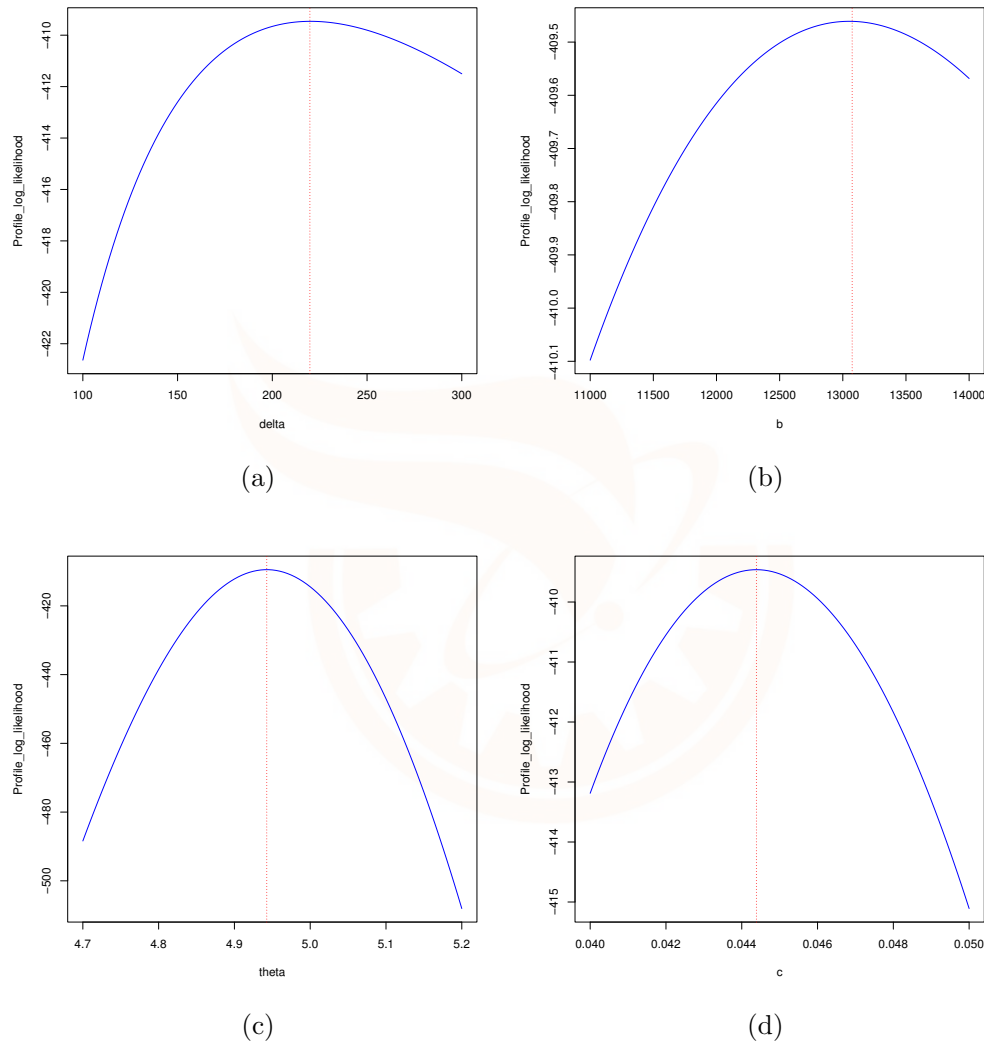


Figure 5.11: Profile log-likelihood plots for bladder cancer data

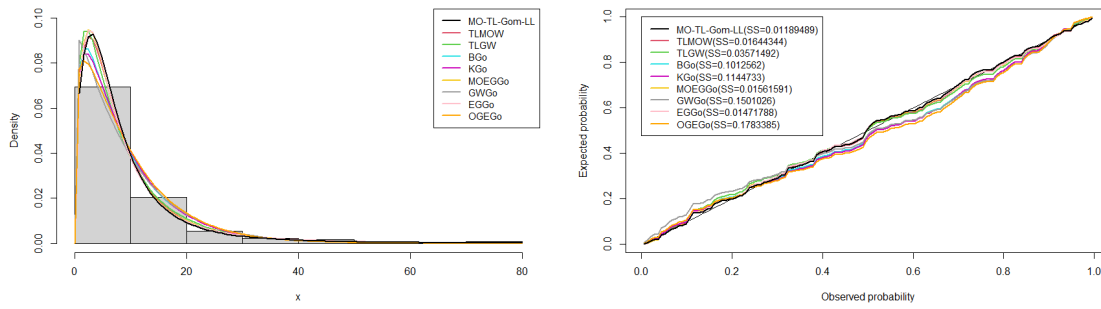


Figure 5.12: Histogram, fitted density and probability plots for bladder cancer data

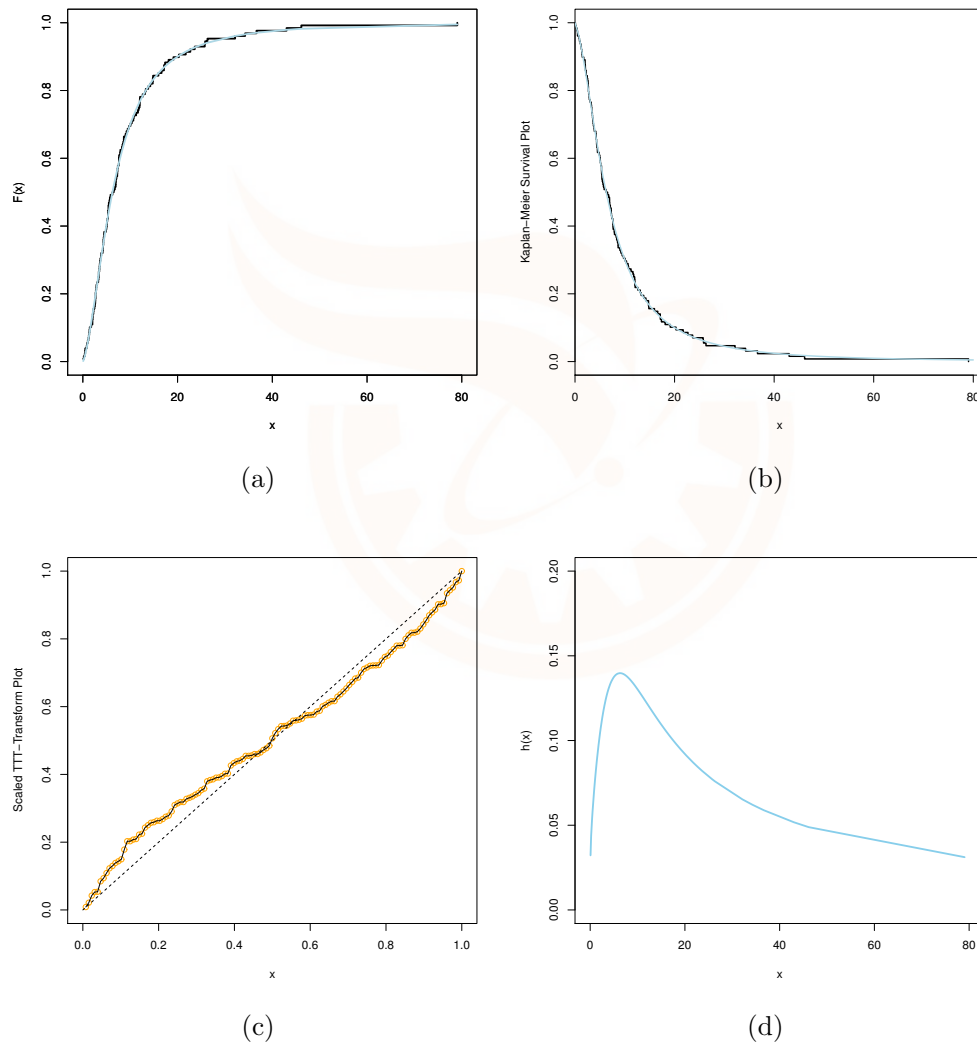


Figure 5.13: Estimated cdf, K-M survival, TTT-Transform and estimated hrf plots for bladder cancer data

In Figure 5.13 again, we see that the cdf line for the MO-TL-Gom-LLo distribution indicated by the blue line in (a) is closer to the empirical cdf while the survival function in blue (b) is also close to the K-M curve which indicate that our model is the best in explaining the bladder cancer data. The scaled TTT-Transform plot for bladder cancer is also presented in (c) and a

uni-modal hazard rate function in (d).

5.8 Concluding Remarks

A new generalized distribution called the Marshall-Olkin-Topp-Leone-Gompertz-G distribution is developed. The MO-TL-Gom-G distribution has several new distributions as special cases or sub-models. The behaviour of the hazard rate function of the MO-TL-Gom-G family of distributions is flexible. We also obtained closed form expressions for the moments and generating function, distribution of order statistics and entropy. Maximum likelihood estimation technique was used to estimate the model parameters. Performance of the MO-TL-Gom-G family of distributions was assessed by conducting various simulations for different sample sizes. Finally, we illustrated that the MO-TL-Gom-G distribution is useful for by fitting its special case, MO-TL-Gom-LLo distribution to two real data sets.



6. The Gamma Odd Power Generalized Weibull-G Family of Distributions

6.1 Introduction

The odd power generalized Weibull generator (Oluyede et al. (2021b)) was developed based on its flexibility when fitted to data in practice. The gamma-G (Zografos and Balakrishnan (2009)) generator helps the researchers to come up with heavy-tailed distributions and to get skewness for symmetrical distributions, therefore combining the two generators we are confident to obtain more flexible distributions to model real life data.

A new generalized distribution called the Gamma odd power generalized Weibull-G family of distributions is developed and studied. Some special models of the new family of distribution are explored. Statistical properties of the new family of distributions including the quantile function, ordinary and incomplete moments, probability weighted moments, stochastic ordering, distribution of the order statistics and Rényi entropy are presented. The maximum likelihood method is used for estimating model parameters and Monte Carlo simulation is conducted to examine the performance of the model. The flexibility of the new family of distributions is demonstrated by means of two applications to real data sets.

6.2 The New Family of Distributions

In a recent note, Oluyede et al. (2021b) developed the odd power generalized Weibull-G family of distributions with the cdf and pdf given by

$$F_{OPGW-G}(x; \alpha, \beta, \xi) = 1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^\beta \right\},$$

and

$$\begin{aligned} f_{OPGW-G}(x; \alpha, \beta, \xi) &= \alpha \beta \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{1 - G(x; \xi)} \right)^\alpha \right]^\beta \right\} \left(\frac{G(x; \xi)}{1 - G(x; \xi)} \right)^\alpha \\ &\times \left[1 + \left(\frac{G(x; \xi)}{1 - G(x; \xi)} \right)^\alpha \right]^{\beta-1} \frac{g(x; \xi)}{(1 - G(x; \xi))^2}, \end{aligned}$$

respectively, for $\alpha, \beta > 0$, where $\overline{G}(x; \xi) = 1 - G(x; \xi)$ is the survival function with parameter vector ξ , and $g(x; \xi) = \frac{dG(x; \xi)}{dx}$. The cdf and pdf of the proposed GOPGW-G family of distributions are given by

$$\begin{aligned}
 F_{GOPGW-G}(x; \delta, \alpha, \beta, \xi) &= 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log(F_{OPGW-G}(x; \alpha, \beta, \xi))} t^{\delta-1} e^{-t} dt \\
 &= 1 - \frac{\gamma(-\log(F_{OPGW-G}(x; \alpha, \beta, \xi)), \delta)}{\Gamma(\delta)},
 \end{aligned} \tag{6.1}$$

and

$$\begin{aligned}
 f_{GOPGW-G}(x; \delta, \alpha, \beta, \xi) &= \frac{1}{\Gamma(\delta)} \beta \alpha \left\{ -\log \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \right) \right\}^{\delta-1} \\
 &\times \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{\alpha-1} \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^{\beta-1} \\
 &\times \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \frac{g(x; \xi)}{(\overline{G}(x; \xi))^2},
 \end{aligned} \tag{6.2}$$

respectively. The hrf is given by

$$\begin{aligned}
 h_{GOPGW-G}(x; \delta, \alpha, \beta, \xi) &= \beta \alpha \left\{ -\log \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \right) \right\}^{\delta-1} \\
 &\times \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{\alpha-1} \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^{\beta-1} \\
 &\times \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \frac{g(x; \xi)}{(\overline{G}(x; \xi))^2} \\
 &\times \left(\gamma \left(-\log \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \right), \delta \right) \right)^{-1},
 \end{aligned}$$

for $x, \delta, \alpha, \beta > 0$, and parameter vector ξ , where $\gamma(x, \delta)$ is the lower incomplete gamma function and $\Gamma(\cdot)$ is the gamma function. If a random variable X has the GOPGW-G distribution, we write $X \sim GOPGW - G(\delta, \alpha, \beta, \xi)$.

6.2.1 Sub-Families of the GOPGW-G Distribution

Some of the sub-families of the GOPGW-G family of distributions are given in this subsection.

- When $\delta = 1$, we obtain the OPGW-G family of distributions with the cdf given in equation (6.1). (See Oluyede et al. (2021b) for additional details).
- When $\delta = \beta = 1$, we obtain the Weibull-G (W-G) family of distributions (Bourguignon et al. (2014)) with the cdf

$$F(x; \alpha, \xi) = 1 - \exp \left\{ - \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right\},$$

for $\alpha > 0$, and parameter vector ξ .

- If $\alpha = \delta = 1$, the GOPGW-G reduces to the Odd Nadarajah-Haghighi family of distributions, with the cdf

$$F(x; \beta, \xi) = 1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right\},$$

for $\beta > 0$, and parameter vector ξ .

- When $\beta = 1$, the GOPGW-G reduces to the gamma odd Exponential-G family of distributions with the cdf

$$F(x; \delta, \alpha, \xi) = 1 - \frac{\gamma \left(-\log \left(1 - \exp \left\{ - \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right\} \right), \delta \right)}{\Gamma(\delta)},$$

for $\delta, \alpha > 0$, and parameter vector ξ .

- When $\alpha = 1$, we obtain gamma odd Nadarajah Haghighi-G distributions with cdf given as

$$F(x; \delta, \beta, \xi) = 1 - \frac{\gamma \left(-\log \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right\} \right), \delta \right)}{\Gamma(\delta)}.$$

- When $\beta = 1, \alpha = 2$, we have the gamma odd Rayleigh-G distributions with the cdf given by

$$F(x; \delta, \xi) = 1 - \frac{\gamma \left(-\log \left(1 - \exp \left\{ - \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right\} \right), \delta \right)}{\Gamma(\delta)}.$$

- If $\delta = \beta = 1, \alpha = 2$, the GOPGW-G reduces to odd Rayleigh-G distributions with the cdf

$$F(x) = 1 - \exp \left\{ - \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right\}.$$

6.3 Statistical Properties

In this section, expansion of the density function, quantile function, moments, incomplete moments, generating function, mean deviations, probability weighted moments, stochastic ordering, order statistics and Rényi entropy are presented.

6.3.1 Expansion of the GOPGW-G Density Function

We obtain the series expansion of the GOPGW-G family of distributions in this section. The GOPGW-G family of distributions pdf is given by

$$\begin{aligned}
 f(x) &= \frac{1}{\Gamma(\delta)} \alpha \beta \sum_{i,j,k,l,p,r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \binom{\beta-1}{j} \binom{\alpha(i+j+1)}{r} \\
 &\times \frac{(m+s+\delta)^k (-1)^{l+p+r}}{k!} \binom{k}{l} \binom{\beta l}{i} \binom{p-\alpha(i+j+1)+1}{r} [G(x;\xi)]^r g(x;\xi) \\
 &= \sum_{r=0}^{\infty} w_{r+1} h_{r+1}(x;\xi),
 \end{aligned} \tag{6.3}$$

where

$$\begin{aligned}
 w_{r+1} &= \frac{1}{\Gamma(\delta)} \alpha \beta \sum_{i,j,k,l,p=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \binom{\beta-1}{j} \binom{\alpha(i+j+1)}{r} \\
 &\times \frac{(m+s+\delta)^k (-1)^{l+p+r}}{k!(r+1)} \binom{k}{l} \binom{\beta l}{i} \binom{p-\alpha(i+j+1)+1}{r}
 \end{aligned} \tag{6.4}$$

and

$$h_{r+1}(x;\xi) = (r+1)[G(x;\xi)]^r g(x;\xi).$$

Equation (6.3) express the GOPGW-G density function as an infinite linear mixture of Exp-G (Cordeiro et al. (2013)) densities, therefore, mathematical properties of the GOPGW-G family of distributions can be directly obtained from those of the Exp-G distribution. See appendix for more details of the derivations.

6.3.2 Quantile Function

The quantile function can be obtained by converting $F_{GOPGW-G}(x;\delta,\alpha,\beta,\xi) = u$, where $F_{GOPGW-G}(x;\delta,\alpha,\beta,\xi)$ is given in equation (6.1), and $0 \leq u \leq 1$. Note that

$$F_{GOPGW-G}(x;\delta,\alpha,\beta,\xi) = 1 - \frac{\gamma(-\log(F_{OPGW-G}(x;\alpha,\beta,\xi)),\delta)}{\Gamma(\delta)} = u,$$

so that

$$\gamma(-\log(F_{OPGW-G}(x;\alpha,\beta,\xi)),\delta) = (1-u)\Gamma(\delta)$$

and

$$1 - \exp\left\{1 - \left[1 + \left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)^\alpha\right]^\beta\right\} = \exp\{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)\}.$$

That is,

$$\left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)^\alpha\right]^\beta = 1 - \log(1 - \exp\{-\gamma^{-1}((1-u)\Gamma(\delta), \delta)\}).$$

Furthermore,

$$\frac{G(x; \xi)}{\overline{G}(x; \xi)} = \left([1 - \log(1 - \exp\{-\gamma^{-1}((1-u)\Gamma(\delta), \delta)\})]^\frac{1}{\beta} - 1\right)^\frac{1}{\alpha},$$

thus the quantile function of the GOPGW-G family of distributions reduces to

$$Q_{G(u; \delta, \alpha, \beta, \xi)} = G^{-1} \left(\frac{\left([1 - \log(1 - \exp\{-\gamma^{-1}((1-u)\Gamma(\delta), \delta)\})]^\frac{1}{\beta} - 1\right)^\frac{1}{\alpha}}{1 + \left([1 - \log(1 - \exp\{-\gamma^{-1}((1-u)\Gamma(\delta), \delta)\})]^\frac{1}{\beta} - 1\right)^\frac{1}{\alpha}} \right). \quad (6.5)$$

6.3.3 Moments, Incomplete Moments and Generating Function

The q^{th} moment of the GOPGW-G family of distributions is obtained from equation (6.3) as

$$E(X^q) = \int_{-\infty}^{\infty} x^q f_{GOPGW-G}(x; \alpha, \beta, \delta, \xi) dx = \sum_{r=0}^{\infty} w_{r+1} E(Y_{r+1}^q), \quad (6.6)$$

where $E(Y_{r+1}^q)$ is the q^{th} moment of Y_{r+1} which follows the Exp-G distribution with power parameter $(r+1)$ and w_{r+1} is defined in equation (6.4). The incomplete moment and the MGF are given by

$$I_X(t) = \int_{-\infty}^t x^q f_{GOPGW-G}(x; \alpha, \beta, \delta, \xi) dx = \sum_{r=0}^{\infty} w_{r+1} I_{r+1}(t),$$

and

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} w_{r+1} E(e^{tY_{r+1}}),$$

where $I_{r+1}(t) = \int_{-\infty}^t x^q (r+1)[G(x; \xi)]^r g(x; \xi) dx$ and $E(e^{tY_{r+1}})$ is the MGF of Exp-G family of distributions with power parameter $(r+1)$ and w_{r+1} is defined in equation (6.4).

6.3.4 Probability Weighted Moments

The PWMs of the GOPGW-G family of distributions are presented in this subsection. The PWMs were introduced by Greenwood et al. (1979) which can be an alternative method for

parameter estimation of probability distributions. From equations (6.1) and (6.2), we have

$$\begin{aligned}
 f(x)(F(x))^n &= \frac{1}{\Gamma(\delta)} \beta \alpha \left\{ -\log \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \right) \right\}^{\delta-1} \\
 &\times \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^{\alpha-1} \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^{\beta-1} \\
 &\times \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \frac{g(x; \xi)}{(\overline{G}(x; \xi))^2} \\
 &\times \left[\frac{\gamma \left(-\log \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \right), \delta \right)}{\Gamma(\delta)} \right]^n,
 \end{aligned}$$

which can be rewritten as

$$\begin{aligned}
 f(x)(F(x))^n &= \beta \alpha \sum_{z,s,m,p=0}^{\infty} \binom{\delta+m-1}{z} b_{s,z} d_{m,p} \frac{(-1)^p}{(\Gamma(\delta))^{p+1}} \binom{n}{p} \\
 &\times \sum_{k,l,i,j,r,q=0}^{\infty} \frac{(z+s+\delta+m)^k (-1)^{l+j+q+r}}{k!} \binom{k}{l} \binom{\beta l}{i} \binom{\beta-1}{j} \\
 &\times \binom{\alpha(j+i+1)-1}{q} \binom{q-\alpha(j+i+1)-1}{r} [G(x; \xi)]^r g(x; \xi) \\
 &= \sum_{r=0}^{\infty} v_{r+1} h_{r+1}(x; \xi),
 \end{aligned}$$

where

$$\begin{aligned}
 v_{r+1} &= \beta \alpha \sum_{z,s,m,p=0}^{\infty} \binom{\delta+m-1}{z} b_{s,z} d_{m,p} \frac{(-1)^p}{(\Gamma(\delta))^{p+1}} \binom{n}{p} \\
 &\times \sum_{k,l,i,j,q=0}^{\infty} \frac{(z+s+\delta+m)^k (-1)^{l+j+q+r}}{k!} \binom{k}{l} \binom{\beta l}{i} \binom{\beta-1}{j} \\
 &\times \binom{\alpha(j+i+1)-1}{q} \binom{q-\alpha(j+i+1)-1}{r}
 \end{aligned}$$

and $h_{r+1}(x; \xi) = (r+1) (G(x; \xi))^r g(x; \xi)$ is the Exp-G density with the power parameter (r+1) and parameter vector ξ . Consequently the PWMs of the GOPGW-G family of distributions are given by

$$\eta_{a,n} = \int_0^\infty x^a \sum_{r=0}^{\infty} v_{r+1} h_{r+1}(x; \xi) dx = \sum_{r=0}^{\infty} v_{r+1} \int_0^\infty x^a h_{r+1}(x; \xi) dx.$$

See Appendix for more details of the derivations.

6.3.5 Stochastic Ordering

In this subsection, the stochastic orders for the GOPGW-G family of distribution are presented. Suppose we have two random variables X and Y with distribution functions $F_X(r)$ and $F_Y(r)$, respectively, and $\bar{F}_X(r) = 1 - F_X(r)$ the survival function, X is stochastically smaller than Y if $\bar{F}_X(r) \leq \bar{F}_Y(r)$ for all r or $F_X(r) \geq F_Y(r)$ for all r . This is denoted by $X <_s Y$. Hazard rate order and likelihood ratio order are stronger and are given by $X <_{hr} Y$ if $h_X(r) \geq h_Y(r)$ for all r , and $X <_{lr} Y$ if $\frac{f_X(r)}{f_Y(r)}$ is decreasing in r , (Shaked and Shanthikumar (2007)). We know that $X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_s Y$.

Theorem 5. *Let X_1 and X_2 be two independent random variables following $GOPGW-G(\alpha, \beta, \delta_1, \xi)$ and $GOPGW-G(\alpha, \beta, \delta_2, \xi)$ distributions. If $\delta_1 \leq \delta_2$, then the random variables X_1 and X_2 are stochastically ordered.*

Proof: Note that

$$\begin{aligned} f_i(x) &= \frac{1}{\Gamma(\delta_i)} \beta \alpha \left\{ -\log \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \right) \right\}^{\delta_i - 1} \\ &\times \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^{\alpha - 1} \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^\alpha \right]^{\beta - 1} \\ &\times \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \frac{g(x; \xi)}{(\bar{G}(x; \xi))^2}, \quad \text{for } i = 1, 2. \end{aligned}$$

The ratio, $\frac{f_1(x; \alpha, \beta, \delta_1, \xi)}{f_2(x; \alpha, \beta, \delta_2, \xi)}$ takes the form

$$\frac{f_1(x; \alpha, \beta, \delta_1, \xi)}{f_2(x; \alpha, \beta, \delta_2, \xi)} = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)} \left\{ -\log \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \right) \right\}^{\delta_1 - \delta_2}. \quad (6.7)$$

If we differentiate equation (6.7) with respect to x , we get

$$\begin{aligned} \frac{d}{dx} \left(\frac{f_1(x; \alpha, \beta, \delta_1, \xi)}{f_2(x; \alpha, \beta, \delta_2, \xi)} \right) &= \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)} (\delta_1 - \delta_2) \alpha \beta \\ &\times \left\{ -\log \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \right) \right\}^{\delta_1 - \delta_2 - 1} \\ &\times \left(1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \right)^{-1} \\ &\times \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^{\alpha - 1} \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^\alpha \right]^{\beta - 1} \\ &\times \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^\alpha \right]^\beta \right\} \frac{g(x; \xi)}{(\bar{G}(x; \xi))^2}, \end{aligned}$$

which is ≤ 0 if $\delta_1 \leq \delta_2$. Therefore, $X_1 <_{lr} X_2$, $X_1 <_{hr} X_2$ and $X_1 <_s X_2$ and the random variables X_1 and X_2 are stochastically ordered.

6.3.6 Order Statistics and Rényi Entropy

In this subsection, we present the pdf of the i^{th} order statistic and the Rényi entropy for the GOPGW-G family of distributions.

6.3.7 Order Statistics

The pdf of the i^{th} order statistic of a distribution with cdf $F(x)$ and pdf $f(x)$ is defined as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!f(x)}{(i-1)!(n-i)!} [F(x)]^{i-1} [1-F(x)]^{n-i} \\ &= \frac{n!f(x)}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j [1-F(x)]^{n-i+j}. \end{aligned}$$

Thus, the pdf of the i^{th} order statistic of the GOPGW-G family of distributions is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j b_{m,n-i+j} \frac{\Gamma(\delta(n-i+j) + \delta + m)}{\Gamma(\delta)^{n-i+j}} \\ &\times \frac{1}{\Gamma(\delta)} \alpha \beta \sum_{i,j,k,l,p,r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} \frac{(-1)^{j+l} \Gamma(1-\beta+j)}{\Gamma(1-\beta) \Gamma(j+1) k!} \binom{\alpha(i+j+1)}{r} \\ &\times (m+s+\delta)^k \binom{k}{l} \binom{\beta l}{i} \sum_{r=0}^{\infty} \frac{\Gamma(p-\alpha(i+j+1)+r-1)}{r! \Gamma(p-\alpha(i+j+1)-1)(r+1)} [G(x;\xi)]^r g(x;\xi) \\ &= \sum_{r=0}^{\infty} d_{r+1} h_{r+1}(x;\xi), \end{aligned}$$

where

$$\begin{aligned} d_{r+1} &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j b_{m,n-i+j} \frac{\Gamma(\delta(n-i+j) + \delta + m)}{\Gamma(\delta)^{n-i+j}} \\ &\times \frac{1}{\Gamma(\delta)} \alpha \beta \sum_{i,j,k,l,p,r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} \frac{(-1)^{j+l} \Gamma(1-\beta+j)}{\Gamma(1-\beta) \Gamma(j+1) k!} \binom{\alpha(i+j+1)}{r} \\ &\times (m+s+\delta)^k \binom{k}{l} \binom{\beta l}{i} \sum_{r=0}^{\infty} \frac{\Gamma(p-\alpha(i+j+1)+r-1)}{r! \Gamma(p-\alpha(i+j+1)-1)(r+1)} \end{aligned}$$

and $h_{r+1}(x;\xi) = (r+1)[G(x;\xi)]^r g(x;\xi)$. The pdf of the i^{th} order statistics can be expressed as a linear combination of the Exp-G densities with a power parameter $(r+1)$ and parameter vector ξ . More derivations are given in the Appendix.

6.3.8 Rényi Entropy

Rényi entropy (Rényi (1961)) for the GOPGW-G family of distributions is given by

$$\begin{aligned}
 I_R(\nu) &= \frac{1}{1-\nu} \log \left[\frac{1}{(\Gamma(\delta))^\nu} \alpha^\nu \beta^\nu \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \binom{\nu(\delta-1)}{m} b_{s,m} \right. \\
 &\quad \times \sum_{i,j,k,l,r,p=0}^{\infty} \frac{(-1)^{j+l} \Gamma((1-\beta)\nu+j)}{\Gamma((1-\beta)\nu) \Gamma(j+1) k!} \binom{\alpha(i+j+\nu)-\nu}{p} \\
 &\quad \times \frac{\Gamma(p-\alpha(i+j+\nu)+r-\nu)}{r! \Gamma(p-\alpha(i+j+\nu)-\nu)} (s+m+\nu\delta)^k \binom{k}{l} \binom{\beta l}{i} \left(\frac{1}{\frac{r}{\nu}+1} \right) \\
 &\quad \times \int_0^\infty \left[\left(\frac{r}{\nu} + 1 \right) [G(x; \xi)]^{\frac{r}{\nu}} g(x) \right]^\nu dx \Big] \\
 &= \frac{1}{1-\nu} \log \left[\sum_{r=0}^{\infty} w_r^* \exp((1-\nu) I_{REG}) \right],
 \end{aligned}$$

for $\nu > 0, \nu \neq 1$, where $I_{REG} = \frac{1}{1-\nu} \log \left[\int_0^\infty \left(\frac{r}{\nu} + 1 \right) [G(x; \xi)]^{\frac{r}{\nu}} g(x) dx \right]^\nu$, is the Rényi entropy of the Exp-G distributions with power parameter $(\frac{r}{\nu} + 1)$, and

$$\begin{aligned}
 w_r^* &= \frac{1}{(\Gamma(\delta))^\nu} \alpha^\nu \beta^\nu \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \binom{\nu(\delta-1)}{m} b_{s,m} \\
 &\quad \times \sum_{i,j,k,l,p=0}^{\infty} \frac{(-1)^{j+l} \Gamma((1-\beta)\nu+j)}{\Gamma((1-\beta)\nu) \Gamma(j+1) k!} \binom{\alpha(i+j+\nu)-\nu}{p} \\
 &\quad \times \frac{\Gamma(p-\alpha(i+j+\nu)+r-\nu)}{r! \Gamma(p-\alpha(i+j+\nu)-\nu)} (s+m+\nu\delta)^k \binom{k}{l} \binom{\beta l}{i} \left(\frac{1}{\frac{r}{\nu}+1} \right).
 \end{aligned}$$

Derivations are given in the Appendix section.

6.4 Maximum Likelihood Estimation

In this subsection, the maximum likelihood technique to estimate the model parameters for the GOPGW-G family of distributions is presented. Let $X \sim GOPGW - G(\alpha, \beta, \delta, \xi)$ and $\Theta = (\alpha, \beta, \delta, \xi)^T$ be the parameter vector, then the log-likelihood function (ℓ_n) of a random sample of size n from the GOPGW-G family of distributions is given by

$$\begin{aligned}
 \ell_n(\Theta) &= n \log \beta + n \log \alpha - n \log(\Gamma(\delta)) \\
 &\quad + (\delta-1) \sum_{i=1}^n \ln \left[-\log \left\{ 1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x_i; \xi)}{\overline{G}(x_i; \xi)} \right)^\alpha \right]^\beta \right\} \right\} \right] \\
 &\quad + \sum_{i=1}^n \ln \left[\exp \left\{ 1 - \left[1 + \left(\frac{G(x_i; \xi)}{\overline{G}(x_i; \xi)} \right)^\alpha \right]^\beta \right\} \right] \\
 &\quad + (\beta-1) \sum_{i=1}^n \ln \left[1 + \left(\frac{G(x_i; \xi)}{\overline{G}(x_i; \xi)} \right)^\alpha \right] + (\alpha-1) \sum_{i=1}^n \ln \left[\left(\frac{G(x_i; \xi)}{\overline{G}(x_i; \xi)} \right)^\alpha \right] \\
 &\quad + \sum_{i=1}^n \ln[g(x_i; \xi)] - 2 \sum_{i=1}^n \ln[1 - G(x_i; \xi)].
 \end{aligned}$$

Elements of the score vector $U(\Theta) = \left(\frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \xi_k} \right)$ can be easily obtained. The mle's can be obtained by equating a system of equations $\frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \delta}$ and $\frac{\partial \ell_n}{\partial \xi_k}$ to zero and solve them simultaneously.

The FIM of the GOPGW-G family of distributions is given by $\mathbf{I}(\Theta) = [\mathbf{I}_{\theta_i, \theta_j}]_{(3+q) \times (3+q)} = E \left(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \right), i, j = 1, 2, \dots, (3+q)$, can be numerically obtained by mle2 package (Bolker and Bolker (2017)) in R software or MATLAB. The expectations in the FIM can be numerically obtained and the total Fisher information matrix $n\mathbf{I}(\Delta)$ can be approximated by $\mathbf{J}_n(\hat{\Theta}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \Big|_{\Theta=\hat{\Theta}} \right]_{(3+q) \times (3+q)}, i, j = 1, 2, \dots, (3+q)$. Elements of the score vector $U(\Theta)$, are given in the appendix section.

6.5 Some Special Models

In this section we present some of the special models of the GOPGW-G family of distributions. We considered the case where the baseline cdf G are the Burr XII, Weibull and Lomax distributions, respectively.

6.5.1 The Gamma Odd Power Generalized Weibull-Burr XII distribution

Suppose the baseline distribution is Burr XII with the cdf and pdf given by $G(x; c, k) = 1 - (1 + x^c)^{-k}$ and $g(x; c, k) = kcx^{c-1}(1 + x^c)^{-k-1}$, for $c, k, x > 0$. When $c = 1$ we obtain the Lomax distribution. The cdf and pdf of the of the GOPGW-BXII distribution are given by

$$F(x; \alpha, \beta, \delta, c, k) = 1 - \frac{\gamma \left(-\log \left(1 - \exp \left\{ 1 - \left[1 + ((1 + x^c)^k - 1)^\alpha \right]^\beta \right\} \right), \delta \right)}{\Gamma(\delta)},$$

and

$$\begin{aligned} f(x; \alpha, \beta, \delta, \lambda) &= \frac{1}{\Gamma(\delta)} \beta \alpha \left\{ -\log \left(1 - \exp \left\{ 1 - \left[1 + ((1 + x^c)^k - 1)^\alpha \right]^\beta \right\} \right) \right\}^{\delta-1} \\ &\times \left((1 + x^c)^k - 1 \right)^{\alpha-1} \left[1 + ((1 + x^c)^k - 1)^\alpha \right]^{\beta-1} \\ &\times \exp \left\{ 1 - \left[1 + ((1 + x^c)^k - 1)^\alpha \right]^\beta \right\} \frac{kcx^{c-1}(1 + x^c)^{-k-1}}{((1 + x^c)^{-k})^2}, \end{aligned}$$

respectively, for $\alpha, \beta, \delta, c, k > 0$.

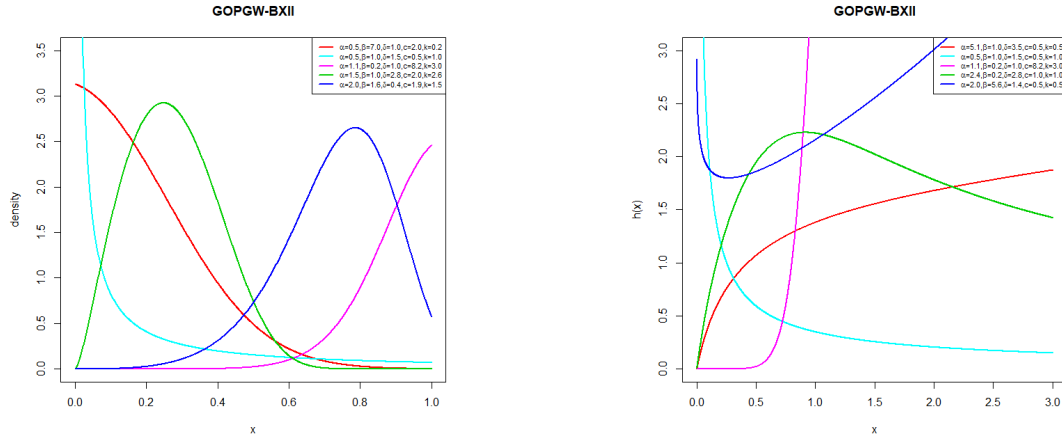


Figure 6.1: Plots of the density and hazard functions for the GOPGW-BXII distribution

The pdf of the GOPGW-BXII distribution can be right skewed, left skewed, reverse-J shape and decreasing shapes whereas the hazard rate function displays increasing, decreasing, bathtub, and upside down bathtub shapes. Quantiles for selected parameters values of the GOPGW-BXII distribution are given in Table 6.1.

Table 6.1: Some quantiles for GOPGW-BXII distribution

	$(\alpha, \beta, \delta, c, k)$				
u	(1, 1.5, 1.3, 0.5, 1.2)	(0.7, 1, 1.5, 1.5, 2.1)	(0.4, 1, 1, 1.3, 2)	(2.1, 0.5, 1.9, 0.4, 1.9)	(1.5, 1, 1.2, 2, 1.6)
0.1	0.0052	0.0707	0.0078	0.0074	0.3988
0.2	0.0127	0.1097	0.0326	0.0168	0.4678
0.3	0.0253	0.1541	0.0797	0.0296	0.5263
0.4	0.0450	0.2051	0.1559	0.0466	0.5797
0.5	0.0748	0.2643	0.2711	0.0696	0.6313
0.6	0.1199	0.3348	0.4391	0.1015	0.6834
0.7	0.1909	0.4218	0.6833	0.1479	0.7389
0.8	0.3124	0.5371	1.0536	0.2232	0.8033
0.9	0.5676	0.7156	1.7093	0.3759	0.8899

6.5.2 Some-Sub Models of the GOPGW-BXII distribution

- If we let $\delta = 1$, we obtain the odd power generalized Burr XII distribution with the cdf given by

$$F(x; \alpha, \beta, c, k) = 1 - \exp \left\{ 1 - \left[1 + \left((1 + x^c)^k - 1 \right)^\alpha \right]^\beta \right\}.$$

- When $\delta = \beta = 1$, we get the Odd Weibull-Burr XII distribution with the cdf given as

follows

$$F(x; \alpha, c, k) = 1 - \exp \left\{ \left((1 + x^c)^k - 1 \right)^\alpha \right\}.$$

- By letting $\beta = 1$ and $\alpha = 2$ we obtain the gamma odd Rayleigh-Burr XII distribution with the pdf given by

$$F(x; \delta, c, k) = 1 - \frac{\gamma \left(-\log \left(1 - \exp \left\{ - \left((1 + x^c)^k - 1 \right)^2 \right\} \right), \delta \right)}{\Gamma(\delta)}.$$

6.5.3 Moments

We can obtain the q^{th} moment of the GOPGW-BXII from equation (6.6) as

$$E(X^q) = \int_{-\infty}^{\infty} x^q f(x) dx = \sum_{r=0}^{\infty} w_{r+1} E(Y_{r+1}^q),$$

where $E(Y_{r+1}^q)$ is the q^{th} moment of Y_{r+1} which follows the Exp-BXII distribution with power parameter $(r+1)$ and w_{r+1} is defined in equation (6.4). The first six moments, SD, CV, CS and CK for selected parameter values of the GOPGW-BXII distribution are given in Table 6.2. Figure 6.2 displays the plots of skewness and kurtosis for the GOPGW-BXII distribution. The plots show that the skewness can either be positive and negative while kurtosis shows that GOPGW-BXII distribution is highly flexible that it can model data for different levels of skewness and kurtosis.

Table 6.2: Table of moments for selected parameters for GOPGW-BXII distribution

	$(\alpha, \beta, \delta, c, k)$				
	(1,1.5,1.3,0.5,1.2)	(0.7,1,1.5,1.5,3)	(0.7,2,0.5,1.3,0.6)	(0.4,2.5,0.9,0.7,0.4)	(1.2,2.5,0.5,0.4,1.2)
EX	0.1345	0.2271	0.2393	0.1022	0.1856
EX ²	0.0568	0.0896	0.1523	0.0476	0.0794
EX ³	0.0338	0.0458	0.1107	0.0305	0.0461
EX ⁴	0.0235	0.0274	0.0867	0.0223	0.0313
EX ⁵	0.0179	0.0182	0.0711	0.0175	0.0233
EX ⁶	0.0144	0.0129	0.0602	0.0144	0.0184
SD	0.1967	0.1950	0.3082	0.1928	0.2120
CV	1.4633	0.8587	1.2877	1.8874	1.1422
CS	2.0669	1.1062	0.9835	2.5095	1.5389
CK	7.0425	3.7975	2.5665	9.0339	4.9102

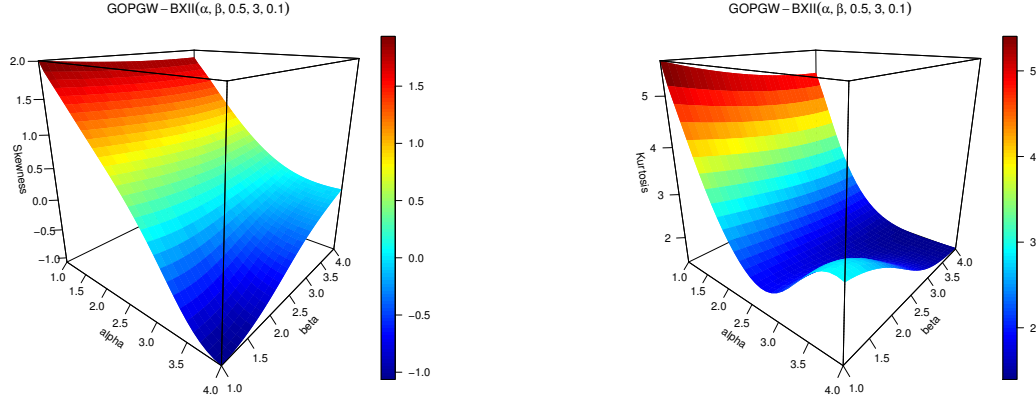


Figure 6.2: Skewness and kurtosis plots for the GOPGW-BXII distribution

6.5.4 Gamma Odd Power Generalized Weibull-Weibull Distribution

Let the baseline distribution be the Weibull distribution with the cdf and pdf given by $G(x; \lambda) = 1 - e^{-x^\lambda}$ and $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$, for $x > 0$ and $\lambda > 0$. The cdf and the pdf of GOPGW-W distribution are given as

$$F(x; \alpha, \beta, \delta, \lambda) = 1 - \frac{\gamma \left(-\log \left(1 - \exp \left\{ 1 - \left[1 + (e^{x^\lambda} - 1)^\alpha \right]^\beta \right\} \right), \delta \right)}{\Gamma(\delta)},$$

and

$$\begin{aligned} f(x; \alpha, \beta, \delta, \lambda) &= \frac{1}{\Gamma(\delta)} \beta \alpha \left\{ -\log \left(1 - \exp \left\{ 1 - \left[1 + (e^{x^\lambda} - 1)^\alpha \right]^\beta \right\} \right) \right\}^{\delta-1} \\ &\times (e^{x^\lambda} - 1)^{\alpha-1} \left[1 + (e^{x^\lambda} - 1)^\alpha \right]^{\beta-1} \\ &\times \exp \left\{ 1 - \left[1 + (e^{x^\lambda} - 1)^\alpha \right]^\beta \right\} \frac{\lambda x^{\lambda-1} e^{-x^\lambda}}{(e^{-x^\lambda})^2}, \end{aligned}$$

respectively, for $\alpha, \beta, \delta, \lambda > 0$.

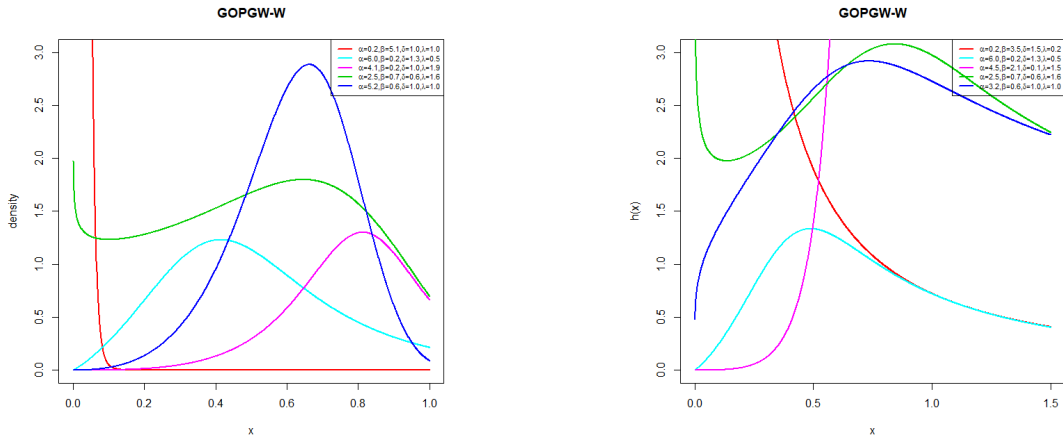


Figure 6.3: Plots of the density and hazard functions for the GOPGW-W distribution

Figure 6.3 shows that the pdf of the GOPGW-W distribution can take several shapes including right skewed, left skewed, uni-modal, decreasing and bathtub shapes whereas the hazard rate function displays increasing, decreasing, bathtub followed by upside down bathtub and upside down bathtub shapes. Table 6.3 gives the quantiles of the GOPGW-W distribution for selected parameter values. The plots of skewness and kurtosis for GOPGW-W distribution are given in Figure 6.4. From the plots, we see that GOPGW-W distribution can model data for different levels of skewness and kurtosis.

Table 6.3: Quantiles for GOPGW-W distribution

$(\alpha, \beta, \delta, c, \lambda)$					
u	(1, 1.5, 1.3, 0.5)	(0.7, 1, 1.5, 1.5)	(0.4, 1, 1, 1.3)	(2.1, 0.5, 1.9, 0.4)	(1.5, 1, 1.2, 2)
0.1	0.0069	0.1153	0.0132	0.0312	0.4859
0.2	0.0165	0.1778	0.0554	0.0666	0.5627
0.3	0.0314	0.2479	0.1339	0.1114	0.6255
0.4	0.0533	0.3264	0.2570	0.1671	0.6809
0.5	0.0841	0.4153	0.4326	0.2372	0.7325
0.6	0.1273	0.5171	0.6663	0.3275	0.7830
0.7	0.1893	0.6371	0.9628	0.4497	0.8350
0.8	0.2839	0.7861	1.3347	0.6301	0.8926
0.9	0.4541	0.9961	1.8354	0.9541	0.9660

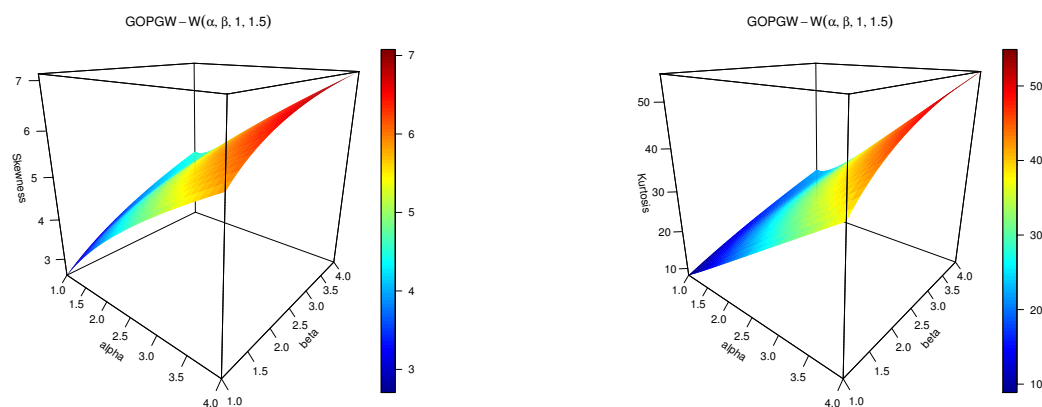


Figure 6.4: Skewness and kurtosis plots for the GOPGW-W distribution

6.6 Simulation Results

In this subsection, a simulation study is carried out to examine the performance of the mean, RMSEs and ABIAS of maximum likelihood estimators of the parameters of the GOPGW-Lx and GOPGW-W distributions. The experiment was conducted through various simulations for different sample sizes and different parameter values. The quantile function given in equation (6.5) was used to generate random samples from GOPGW-Lx and GOPGW-W distributions via the `bbmle` (Bolker and Bolker (2017)) package in R. The simulation was repeated for $N = 3000$ times each with sample sizes $n = 50, 100, 200, 400, 800, 1600$ and parameter values as given in Table 6.4 and 6.5. Specifically, the estimated mean, average bias and root mean square error of the parameter say $\xi = \alpha, \beta, \delta, k$ are computed as:

$$\text{Mean} = \frac{\sum_{i=1}^N \hat{\xi}_i}{N}, \text{RMSE} = \sqrt{\frac{\sum_{i=1}^N (\hat{\xi}_i - \xi)^2}{N}} \text{ and } \text{ABIAS}(\hat{\xi}) = \frac{\sum_{i=1}^N (\hat{\xi}_i - \xi)}{N}, \text{ respectively.}$$

The tables list the mean MLEs of the parameters along with the respective root RMSEs and ABIAS. From the results in Table 6.4 and Table 6.5, we can see that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values while the RMSEs and ABIAS decrease.

Table 6.4: Simulation Results for GOPGW-Lx Distribution: Mean, ABIAS and RMSE

Parameter	n	(1.0, 1.0, 0.4, 1.5)				(2.0, 0.4, 0.8, 0.5)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
α	50	1.5612	1.9211	0.5612	0.5612	2.8382	2.4264	0.8382	0.8382
	100	1.2858	0.7639	0.2858	0.2858	2.5851	1.0930	0.5851	0.5851
	200	1.1421	0.3711	0.1421	0.1421	2.4149	0.8278	0.4149	0.4149
	400	1.0699	0.2348	0.0699	0.0699	2.2408	0.6246	0.2408	0.2408
	800	1.0276	0.1563	0.0276	0.0276	2.1537	0.4647	0.1537	0.1537
	1600	1.0155	0.1080	0.0155	0.0155	2.1029	0.3624	0.1029	0.1029
β	50	1.7004	2.8552	0.7004	0.7004	0.5973	0.6714	0.1973	0.1973
	100	1.3307	1.6846	0.3307	0.3307	0.5541	0.5827	0.1541	0.1541
	200	1.2889	1.3786	0.2889	0.2889	0.4822	0.4582	0.0822	0.0822
	400	1.1360	0.8647	0.1360	0.1360	0.4695	0.3987	0.0695	0.0695
	800	1.0455	0.5027	0.0455	0.0455	0.4276	0.2697	0.0276	0.0276
	1600	1.0347	0.3631	0.0347	0.0347	0.4153	0.1797	0.0153	0.0153
δ	50	0.8407	0.9589	0.4407	0.4407	1.4829	1.3475	0.6829	0.6829
	100	0.7578	0.8329	0.3578	0.3578	1.4235	1.2738	0.6235	0.6235
	200	0.6320	0.5901	0.2320	0.2320	1.2955	1.0176	0.4955	0.4955
	400	0.5172	0.3610	0.1172	0.1172	1.0974	0.7641	0.2974	0.2974
	800	0.4507	0.2259	0.0507	0.0507	0.9954	0.5678	0.1954	0.1954
	1600	0.4320	0.1495	0.0320	0.0320	0.9357	0.4441	0.1357	0.1357
k	50	2.0623	2.1385	0.0320	0.0320	0.4195	0.3648	-0.0805	0.0805
	100	2.0962	2.4326	0.5962	0.5962	0.4332	0.3259	-0.0668	0.0668
	200	1.9169	1.8017	0.4169	0.4169	0.4290	0.2464	-0.0709	0.0709
	400	1.8118	1.3710	0.3118	0.3118	0.4574	0.2145	-0.0426	0.0426
	800	1.7004	0.9556	0.2004	0.2004	0.4721	0.1752	-0.0279	0.0279
	1600	1.5729	0.6500	0.0729	0.0729	0.4801	0.1306	-0.0199	0.0199

Table 6.5: Simulation Results for GOPGW-W Distribution: Mean, ABIAS and RMSE

Parameter	n	(1.5, 1.5, 0.4, 0.7)				(0.4, 0.4, 2.0, 3.0)			
		Mean	RMSE	ABIAS	AABIAS	Mean	RMSE	ABIAS	AABIAS
α	50	1.1831	2.7143	-0.3169	0.3169	0.3627	0.1381	-0.0373	0.0373
	100	1.1551	0.8479	-0.3449	0.3449	0.3664	0.0897	-0.0336	0.0336
	200	1.2221	0.6046	-0.2779	0.2779	0.3788	0.0625	-0.0212	0.0212
	400	1.2991	0.4519	-0.2009	0.2009	0.3850	0.0440	-0.0149	0.0149
	800	1.3644	0.3362	-0.1356	0.1356	0.3915	0.0316	-0.0085	0.0085
	1600	1.4204	0.2551	-0.0796	0.0796	0.3943	0.0219	-0.0057	0.0057
β	50	1.1311	0.8449	-0.3689	0.3689	0.5759	0.5952	0.1759	0.1759
	100	1.1284	0.7907	-0.3716	0.3716	0.5388	0.4843	0.1388	0.1388
	200	1.1415	0.6857	-0.3585	0.3585	0.4623	0.3439	0.0623	0.0623
	400	1.2054	0.5612	-0.2946	0.2946	0.4279	0.2425	0.0278	0.0278
	800	1.2265	0.4827	-0.2735	0.2735	0.4099	0.1836	0.0099	0.0099
	1600	1.2942	0.3799	-0.2057	0.2057	0.3943	0.1222	-0.0057	0.0057
δ	50	1.3804	1.6304	0.9804	0.9804	2.2178	1.3855	0.2178	0.2178
	100	1.2722	1.4396	0.8722	0.8722	2.1277	1.1768	0.1277	0.1277
	200	1.0719	1.0969	0.6719	0.6719	2.1568	1.0014	0.1568	0.1568
	400	0.8606	0.7724	0.4606	0.4606	2.1119	0.7733	0.1119	0.1119
	800	0.7534	0.5741	0.3534	0.3534	2.0743	0.5745	0.0743	0.0743
	1600	0.6289	0.3899	0.2289	0.2289	2.0625	0.3924	0.0625	0.0625
λ	50	2.5779	2.9130	1.8779	1.8779	3.4001	1.1303	0.4001	0.4001
	100	1.9379	1.9606	1.2379	1.2379	3.2186	0.8905	0.2186	0.2186
	200	1.4339	1.1852	0.7339	0.7339	3.1544	0.7369	0.1544	0.1544
	400	1.1325	0.7325	0.4325	0.4325	3.1073	0.5811	0.1073	0.1073
	800	0.9776	0.4649	0.2776	0.2776	3.0641	0.4250	0.0641	0.0641
	1600	0.8633	0.2979	0.1633	0.1633	3.0502	0.2947	0.0502	0.0502

6.7 Applications

In this section, the GOPGW-Lx and GOPGW-W distributions are fitted to data sets and these fits are compared to the fits of other models, including type II general inverse exponential Lomax distribution by Hamedani et al. (2019), exponentiated modified Weibull distribution by Elbatal (2011), and Kumaraswamy Weibull distribution by Cordeiro et al. (2010).

The pdf's of the above distributions are given by

$$f_{TII GIE-Lx}(x; \lambda, \alpha, a, b) = \lambda \alpha \frac{a}{b} \left(1 + \frac{x}{b}\right)^{-(a+1)} \left(1 + \frac{x}{b}\right)^{a(\alpha+1)} \\ \times \exp\left(\lambda \left(1 - \left(1 + \frac{x}{b}\right)^{a\alpha}\right)\right),$$

for $x > 0, \lambda, \alpha, a, b > 0$,

$$f_{EMW}(x; \gamma, \delta, \lambda, \theta) = \gamma [\delta + \lambda \theta^\lambda x^{\lambda-1}] \exp(-\delta x + (\theta x)^\lambda) [1 - \exp(-(\delta x + (\theta x)^\lambda))]^{\gamma-1},$$

for $x > 0, \gamma, \delta, \lambda, \theta > 0$,

and

$$f_{KW}(x; a, b, c, \lambda) = abc\lambda^c x^{c-1} \exp(-(\lambda x)^c) [1 - \exp(-(\lambda x)^c)]^{a-1} \{1 - [1 - \exp(-(\lambda x)^c)]^a\}^{b-1},$$

for $x > 0, a, b, c, \lambda > 0$.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. (1983)) are given in Figures 6.6 and 6.8. For the probability plot, we plotted $F(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\xi})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness is given by the sum of squares:

$$SS = \sum_{j=1}^n \left[F(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\xi}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The goodness-of-fit statistics: Cramér-von Mises and Anderson-Darling described by Chen and Balakrishnan (1995) and K-S statistic are also presented in the tables. In general, the smaller the values of W^* and A^* and K-S, the better the fit.

6.7.1 Kevlar 49/Epoxy Strands Failure Data

The first data set consists of 101 data points of the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90% stress level until all had failed, so that we have complete data with exact times of failure (Andrews and Herzberg (1985); Barlow et al. (1984)). The failure times in hours are:

0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

The estimated variance-covariance matrix for GOPGW-Lx model of Kevlar data is given by

$$\begin{bmatrix} 0.0236 & -0.0041 & 0.0176 & -0.1353 \\ -0.0041 & 0.0368 & -0.0297 & -0.5979 \\ 0.0176 & -0.0297 & 0.0568 & 0.2728 \\ -0.1353 & -0.5979 & 0.2728 & 11.8747 \end{bmatrix}.$$

Table 6.6: Estimates of models for kevlar 49/epoxy strands failure data

Model	Estimates				Statistics								
	α	β	δ	k	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value	SS
GOPGW-Lx	0.7667	0.2579	0.9525	5.0782	204.3	212.3	212.7	222.8	0.1239	0.7661	0.0731	0.6534	0.1629
	(0.1537)	(0.1917)	(0.2383)	(3.4426)									
GOPGW-W	α	β	δ	λ									
	2.11986	0.76310	0.5569	0.2919	206.3	214.3	214.7	224.8	0.1644	0.9637	0.0844	0.4677	0.1639
	(0.9227)	(0.4673)	(0.6285)	(0.1813)									
TIIGIE-Lx	λ	α	a	b									
	69.0667	4.3659	0.0439	12.7699	206.5	214.5	214.9	224.9	0.2069	1.1486	0.0831	0.4887	0.1934
	(0.6552)	(0.0084)	(0.0560)	(17.3383)									
EMW	γ	δ	λ	θ									
	0.9289	1.0000	170.0000	0.0001	206.5	214.5	214.9	224.9	0.1753	1.0032	0.1025	0.2389	0.2509
	(1.1739×10^{-01})	(1.3055×10^{-01})	(9.7521×10^{-20})	(1.6581×10^{-13})									
KW	a	b	c	λ									
	1.0329	0.2886	0.8731	4.2658	206.5	214.5	214.9	224.9	0.2172	1.1998	0.1000	0.2646	0.2215
	(0.4130)	(0.1929)	(0.1411)	(2.7151)									
W	β	λ											
	0.9758	1.0000			206.9	210.9	211.1	216.2	0.1987	1.0283	0.0888	0.4034	0.1645
	(0.1036)	(0.0766)											

From Table 6.6, the values of the goodness-of-fit statistics: AIC, AICC and BIC for the GOPGW-Lx and GOPGW-W distributions are the smallest as compared to TIIGIE-Lx, EMW, KW and Weibull distributions hence the GOPGW-Lx and GOPGW-W distributions are better in fitting for Epoxy Strands Failure data. Also, the value of SS from the probability plots and the values of the statistics A^* , W^* and K-S are the smallest for the GOPGW-Lx and GOPGW-W distributions, which indeed showed that this model can be useful. Profile log-likelihood plots in Figure 6.5 shows that maximum likelihood estimates of α, β, θ and k exits.

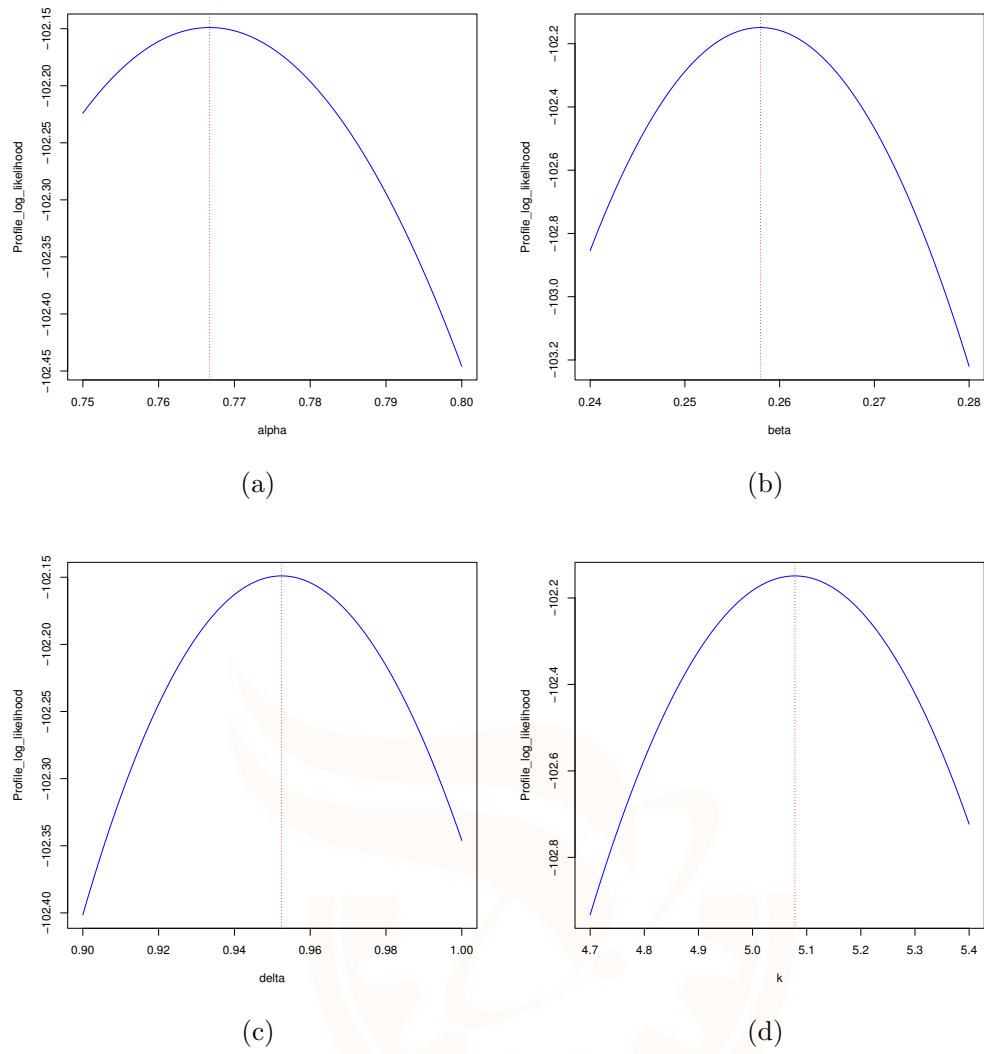


Figure 6.5: Profile log-likelihood plots for epoxy strands failure data

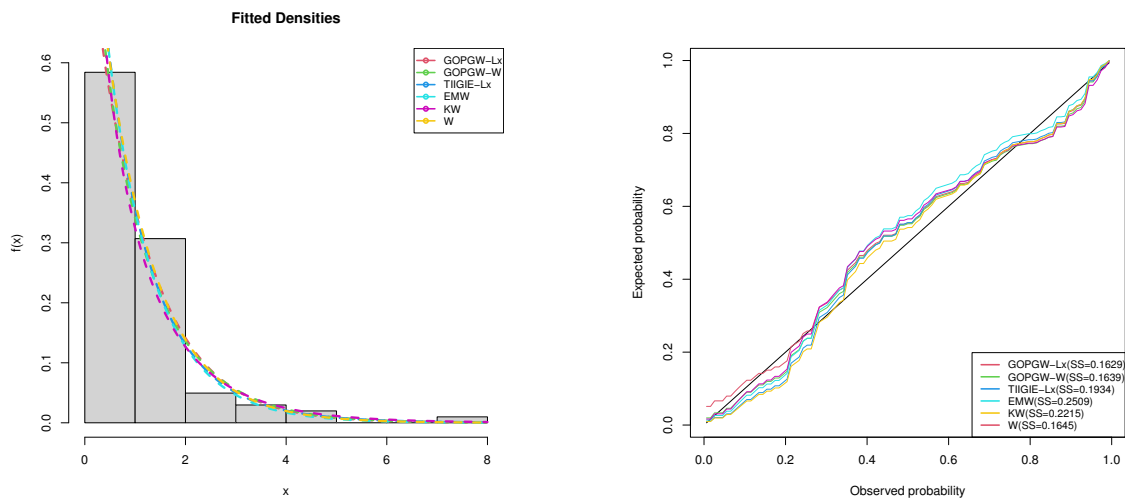


Figure 6.6: Fitted densities and probability plots of epoxy strands failure data

6.7.2 Airborne Communication Transceiver Data

The second data correspond to maintenance on active repair times (in hours) for an airborne communication transceiver with size $n=46$ cited by Balakrishnan et al. (2009) and Chhikara and Folks (1989). The observations are: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

The estimated variance-covariance matrix for GOPGW-Lx distribution of airborne communication transceiver data is given by

$$\begin{bmatrix} 1.7952 \times 10^{+00} & -5.0520 \times 10^{-02} & 1.5764 \times 10^{-14} & 2.1629 \times 10^{-01} \\ -5.0520 \times 10^{-02} & 1.7739 \times 10^{-03} & -4.5536 \times 10^{-16} & -1.4044 \times 10^{-02} \\ 1.6862 \times 10^{-14} & -4.8636 \times 10^{-16} & 1.4852 \times 10^{-28} & 2.3406 \times 10^{-15} \\ 2.1629 \times 10^{-01} & -1.4044 \times 10^{-02} & 2.2053 \times 10^{-15} & 2.3387 \times 10^{-01} \end{bmatrix}.$$

Table 6.7: Estimates of models for airborne communication transceiver data

Model	Estimates				Statistics								
	α	β	δ	k	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value	SS
GOPGW-Lx	3.2776	0.0772	1.0000	2.0930	197.8	205.8	206.8	213.2	0.0248	0.1866	0.0733	0.9658	0.0289
	(1.3399)	(4.2118×10^{-02})	(1.2187×10^{-14})	$(4.83601 \times 10^{-01})$									
GOPGW-W	α	β	δ	λ									
	141.2300	0.0100	0.3946	0.2194	199.2	207.2	208.2	214.5	0.0427	0.3455	0.0984	0.7645	0.0395
	(0.0001)	(0.0022)	(0.1773)	(0.0119)									
TIIGIE-Lx	λ	α	a	b									
	92.3050	3.2100	0.0114	8.7477	205.9	213.9	214.9	221.2	0.0924	0.6092	0.1273	0.4449	0.0668
	(3.4015×10^{-02})	(3.6148×10^{-05})	(7.0061×10^{-03})	$(6.7979 \times 10^{+00})$									
EMW	γ	δ	λ	θ									
	0.9583	0.2694	17.0010	0.0001	209.9	217.9	218.9	225.3	0.1442	1.0004	0.1519	0.2385	0.1889
	(1.8975×10^{-01})	(5.4358×10^{-02})	(2.5315×10^{-18})	(4.3037×10^{-12})									
KW	a	b	c	λ									
	4.3024	0.2470	0.6846	3.8951	201.1	209.1	210.0	216.4	0.0575	0.3988	0.1181	0.5418	0.0874
	(2.6279)	(0.1972)	(0.1451)	(2.7323)									
W	β	λ											
	0.3337	0.8986			208.9	212.9	213.2	216.6	0.1298	0.9009	0.1204	0.5170	0.1157
	(0.0750)	(0.0958)											

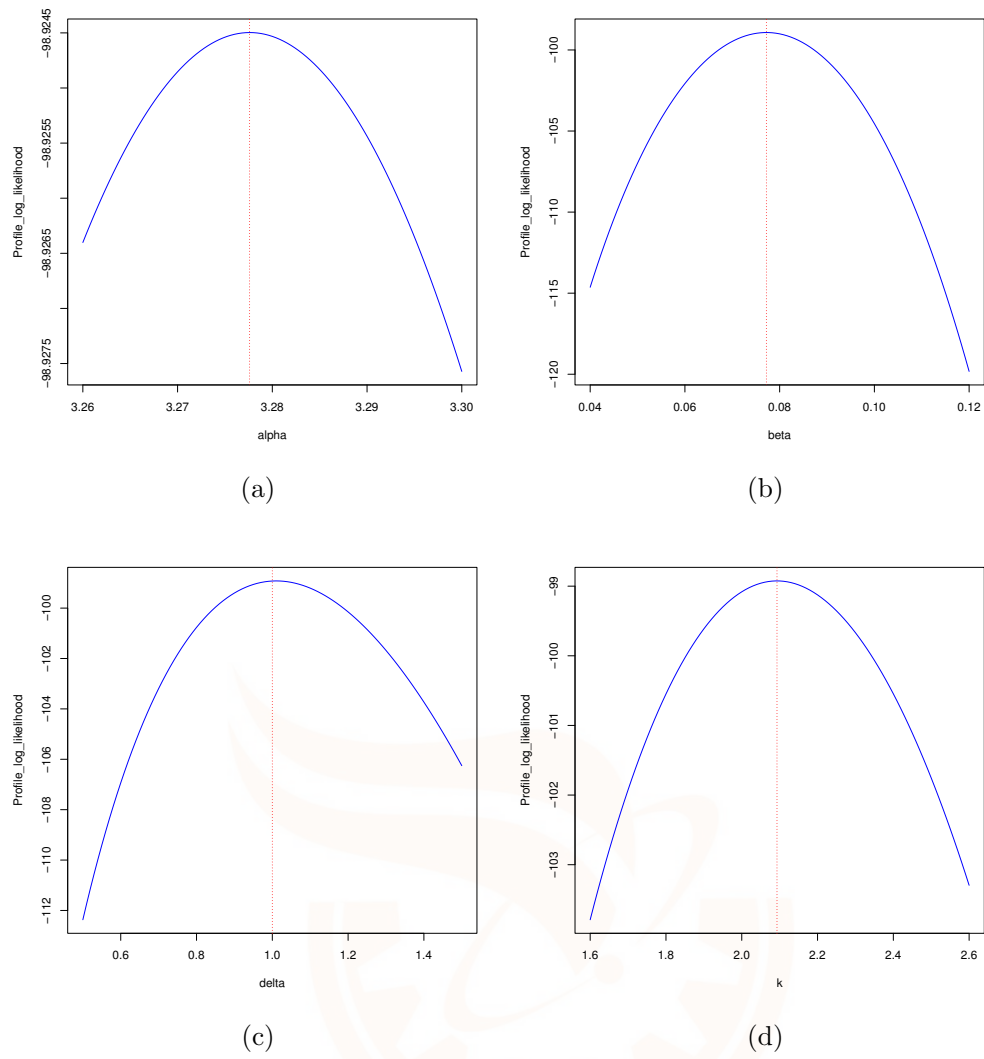


Figure 6.7: Profile log-likelihood plots for epoxy communication transceiver data

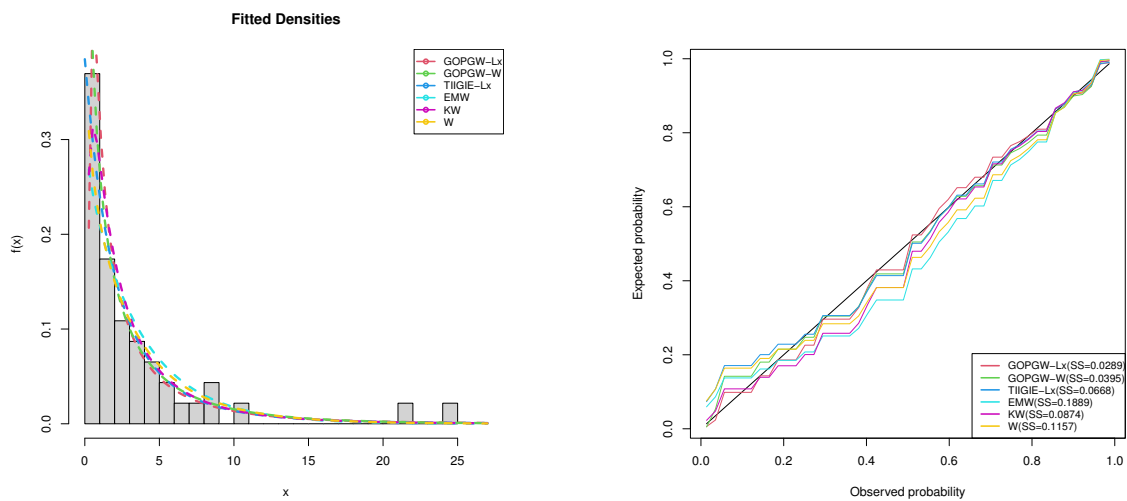


Figure 6.8: Fitted densities and probability plots of airborne communication transceiver data

Table 6.7 gives the MLE's of the model parameters with (standard errors in parentheses), AIC,

AICC, BIC, and the goodness-of-fit statistics W^* , A^* , K-S and its p-value as well as SS. The values of the goodness-of-fit statistics: AIC, AICC and BIC for the GOPGW-Lx and GOPGW-W distributions are the smallest as compared to those of the models listed in the table hence the GOPGW-Lx and GOPGW-W distributions are better in fitting for the airborne communication transceiver data. Also, the value of SS from the probability plots and the values of the statistics A^* , W^* and K-S are the smallest for the GOPGW-Lx and GOPGW-W distributions. Furthermore, the p-value for the K-S statistic is very close to one indicating that this models are the best in fitting Airborne Communication Transceiver data. Profile log-likelihood plots in Figure 6.7 shows that maximum likelihood estimates of α, β, θ and k can be uniquely obtained.

6.8 Concluding Remarks

In this chapter, we presented a new distribution called the Gamma odd power generalized Weibull-G family of distributions. Statistical properties of the new generalized distribution such as quantile function, ordinary and incomplete moments, probability weighted moments, stochastic ordering, distribution of the order statistics and Rényi entropy were derived. The maximum likelihood technique was used to estimate the model parameters. Two special cases of the new family of distribution including Gamma Odd Power Generalized Weibull-Burr XII and Gamma Odd Power Generalized Weibull-Weibull distributions are discussed in details. The usefulness of the new family of distributions was examined by means of applications to two real data.

7. Conclusions and Areas of Further Studies

We studied and proposed several new families of distributions including the Topp-Leone type II exponentiated half logistic-G family of distributions, the Topp-Leone exponentiated half logistic generalized-G family of distributions, the Topp-Leone-Gompertz-G power series class of distributions, the Marshall-Olkin-Topp-Leone-Gompertz-G family of distributions and the Gamma odd power generalized Weibull-G family of distributions. Some special cases of the new families of distributions for the baseline distribution $G(x; \xi)$ were studied in details. We also considered some mathematical and statistical properties of the new families of distributions. The statistical properties studied include the expansion of the density function, hazard rate function and quantile function, moments, moment generating functions, probability weighted moments, stochastic ordering, distribution of order statistics and Rényi entropy. The method of maximum likelihood estimation was used to estimate the model parameters and Monte Carlo simulations were conducted to examine the consistency of the parameter estimates. Applications in several areas are presented to illustrate the importance and the flexibility of these new families of distributions. We discovered that our proposed distributions performed really well compared to some distributions in the literature for different data sets in the areas of finance, survival analysis, science, engineering and economics, to mention a few.

For future research, we plan to explore more of statistical properties that were not considered in this thesis. Also, we intend to consider different methods of estimation such as maximum product spacing, least squares, weighted least squares, Cramér-von Mises, Anderson-Darling and Bayesian techniques, and compare them with the maximum likelihood technique based on simulations. Bayesian techniques are of interest because they are increasingly used in global health modeling, and the maximum product spacing approach relies on maximizing the geometric mean of the probability spacings in the data.

We are also interested in incorporating covariates or concomitant information to these new generalized distributions, as well as the development of bivariate and multivariate extensions of the proposed families of distributions.

A. Chapter 2 Appendix

A.1 The Topp-Leone Type II Exponentiated Half Logistic-G Family of Distributions

The elements of the score vector $U(\Delta)$ are

$$\begin{aligned} \frac{\partial \ell_n}{\partial a} &= \frac{n}{a} - 2(b-1) \sum_{i=1}^n \frac{\left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right]^a \ln \left(\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right)}{\left[1 - \left(\left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right]^a \right)^2 \right]} \\ &+ \sum_{i=1}^n \frac{\left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right]^a \ln \left(\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right)}{\left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right]^a} + \sum_{i=1}^n \ln[1 - [G(x_i; \zeta)]^\gamma] \\ &- \sum_{i=1}^n \ln[1 + [G(x_i; \zeta)]^\gamma], \end{aligned}$$

$$\frac{\partial \ell_n}{\partial b} = \frac{n}{b} + \sum_{i=1}^n \ln \left[1 - \left(\left[\frac{1 - [G(x_i; \zeta)]^\gamma}{1 + [G(x_i; \zeta)]^\gamma} \right]^a \right)^2 \right],$$

$$\begin{aligned} \frac{\partial \ell_n}{\partial \gamma} &= \frac{n}{\gamma} - 2a(b-1) \sum_{i=1}^n \frac{\left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right]^{a-1} \left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right] \ln \left(\frac{\bar{G}(x_i; \zeta)}{1+[G(x_i; \zeta)]^\gamma} \right)}{\left[1 - \left(\left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right]^a \right)^2 \right]} \\ &+ \sum_{i=1}^n \frac{a \left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right]^{a-1} \left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right] \ln \left(\frac{\bar{G}(x_i; \zeta)}{1+[G(x_i; \zeta)]^\gamma} \right)}{\left[\frac{1-[G(x_i; \zeta)]^\gamma}{1+[G(x_i; \zeta)]^\gamma} \right]^a} + \sum_{i=1}^n \ln[G(x_i; \zeta)] \\ &- (a-1) \sum_{i=1}^n \frac{[G(x_i; \zeta)]^\gamma \ln[G(x_i; \zeta)]}{[1 - [G(x_i; \zeta)]^\gamma]} - (a+1) \sum_{i=1}^n \frac{[G(x_i; \zeta)]^\gamma \ln[G(x_i; \zeta)]}{[1 + [G(x_i; \zeta)]^\gamma]}, \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \zeta_k} = & -(b-1) \sum_{i=1}^n \frac{2a\gamma \left[\frac{1-[G(x_i;\zeta)]^\gamma}{1+[G(x_i;\zeta)]^\gamma} \right]^a \left[\frac{1-[G(x_i;\zeta)]^\gamma}{1+[G(x_i;\zeta)]^\gamma} \right]^{a-1} \left(\frac{1-G(x_i;\zeta)}{1+[G(x_i;\zeta)]^\gamma} \right)^{\gamma-1} \frac{\partial G(x_i;\zeta)}{\partial \zeta_k}}{\left[1 - \left(\left[\frac{1-[G(x_i;\zeta)]^\gamma}{1+[G(x_i;\zeta)]^\gamma} \right]^a \right)^2 \right] [1 + [G(x_i;\zeta)]^\gamma]^2} \\
& + \sum_{i=1}^n \frac{a\gamma \left[\frac{1-[G(x_i;\zeta)]^\gamma}{1+[G(x_i;\zeta)]^\gamma} \right]^{a-1} \left(\frac{1-[G(x_i;\zeta)]^\gamma}{1+[G(x_i;\zeta)]^\gamma} \right)^{\gamma-1} \frac{\partial G(x_i;\zeta)}{\partial \zeta_k}}{\left[\frac{1-[G(x_i;\zeta)]^\gamma}{1+[G(x_i;\zeta)]^\gamma} \right]^a [1 + [G(x_i;\zeta)]^\gamma]^2} \\
& - \gamma(a-1) \sum_{i=1}^n \frac{(G(x_i;\zeta))^{\gamma-1} \frac{\partial G(x_i;\zeta)}{\partial \zeta_k}}{[1 - [G(x_i;\zeta)]^\gamma]} - \gamma(a+1) \sum_{i=1}^n \frac{(G(x_i;\zeta))^{\gamma-1} \frac{\partial G(x_i;\zeta)}{\partial \zeta_k}}{[1 + [G(x_i;\zeta)]^\gamma]} \\
& + (\gamma-1) \sum_{i=1}^n \frac{\partial G(x_i;\zeta)}{\partial \zeta_k} \frac{1}{G(x_i;\zeta)} + \frac{\partial g(x_i;\zeta)}{\partial \zeta_k} \frac{1}{g(x_i;\zeta)}.
\end{aligned}$$

This 428 data points correspond to prices (in dollars) of 428 new vehicles for the 2004 year (Kiplinger's Personal Finance, Dec 2003).

Table A.1: Data set for prices (in 10 of thousands of dollars) of 428 new vehicles

1.028	1.0539	1.076	1.0995	1.1155	1.129	1.156	1.169	1.1839	1.1905	1.1939	1.2269	1.236
1.2585	1.274	1.28	1.2884	1.2965	1.327	1.327	1.358	1.367	1.373	1.3839	1.4085	1.4165
1.417	1.43	1.4385	1.45	1.461	1.4622	1.463	1.474	1.481	1.484	1.485	1.503	1.504
1.5295	1.5389	1.5389	1.546	1.5495	1.55	1.5568	1.558	1.5825	1.585	1.604	1.635	1.6385
1.6495	1.6497	1.653	1.6695	1.6722	1.6999	1.7045	1.7163	1.72	1.7232	1.7262	1.7475	1.7495
1.763	1.764	1.7735	1.775	1.7985	1.8345	1.8435	1.869	1.8715	1.8739	1.876	1.882	1.8825
1.8892	1.8995	1.9005	1.909	1.911	1.9135	1.924	1.927	1.9312	1.9339	1.9479	1.949	1.956
1.9635	1.9635	1.9825	1.986	1.986	1.9945	1.9999	2.013	2.014	2.0215	2.022	2.0255	2.029
2.03	2.031	2.032	2.0339	2.037	2.0445	2.0449	2.051	2.0585	2.0615	2.0939	2.1055	2.1055
2.1087	2.141	2.1445	2.1589	2.1595	2.1595	2.1795	2.1825	2.184	2.19	2.1965	2.2	2.201
2.2035	2.218	2.2225	2.226	2.229	2.235	2.2388	2.2395	2.245	2.2515	2.257	2.2595	2.2735
2.2775	2.3215	2.329	2.3495	2.3495	2.356	2.3675	2.3699	2.376	2.3785	2.382	2.3845	2.3895
2.3895	2.3955	2.413	2.4225	2.4295	2.4345	2.452	2.4589	2.4695	2.478	2.4885	2.4895	2.495
2.4955	2.5	2.5045	2.5092	2.513	2.5135	2.5193	2.5215	2.5395	2.552	2.564	2.5645	2.57
2.57	2.5717	2.592	2.5935	2.594	2.5955	2.5995	2.6	2.606	2.6135	2.6189	2.6395	2.647
2.651	2.6545	2.656	2.665	2.686	2.691	2.693	2.696	2.699	2.6992	2.702	2.7145	2.72
2.7339	2.73	2.745	2.749	2.749	2.756	2.771	2.7905	2.793	2.7995	2.8345	2.837	2.8495
2.8495	2.8739	2.875	2.879	2.88	2.9282	2.9322	2.9345	2.938	2.944	2.9562	2.9595	2.967
2.9795	2.9865	2.9995	2.9995	3.0245	3.0295	3.0315	3.0492	3.0795	3.0835	3.086	3.0895	3.092
3.095	3.1045	3.1145	3.123	3.137	3.1545	3.1545	3.1745	3.184	3.1849	3.189	3.2235	3.2245
3.228	3.235	3.2415	3.2445	3.2455	3.2495	3.266	3.278	3.2845	3.3112	3.318	3.3195	3.326
3.3295	3.336	3.343	3.348	3.35	3.354	3.378	3.384	3.3895	3.3995	3.3995	3.439	3.448
3.4495	3.4495	3.456	3.4845	3.4895	3.5105	3.5145	3.5495	3.5515	3.5545	3.5695	3.5725	3.592
3.594	3.594	3.5995	3.61	3.6395	3.664	3.6895	3.6945	3.6995	3.7	3.7245	3.739	3.753
3.756	3.763	3.773	3.7885	3.7895	3.7995	3.838	3.883	3.9195	3.9235	3.925	3.9465	3.964
3.9995	4.0095	4.0235	4.032	4.034	4.0565	4.059	4.067	4.072	4.084	4.0845	4.101	4.1045
4.125	4.1465	4.1475	4.1815	4.1995	4.249	4.2565	4.2735	4.284	4.2845	4.2915	4.3175	4.3365
4.3365	4.3495	4.3755	4.3895	4.424	4.4295	4.4535	4.4925	4.4995	4.521	4.5445	4.57	4.5707
4.61	4.6265	4.647	4.6995	4.7955	4.804	4.817	4.8195	4.845	4.909	4.969	4.9995	4.9995
5.047	5.0595	5.067	5.1535	5.212	5.2195	5.2365	5.2545	5.2775	5.2795	5.28	5.2975	5.4765
5.4995	5.575	5.617	5.6595	5.6665	5.727	5.9995	6.067	6.312	6.32	6.48	6.5	6.8995
6.919	6.9195	6.9995	7.225	7.3195	7.432	7.4995	7.4995	7.5	7.62	7.6765	7.687	7.9165
8.1795	8.1995	8.4165	8.46	8.697	8.6995	8.9765	9.052	9.482	12.177	12.667	12.842	19.2465

B. Chapter 3 Appendix

B.1 The Topp-Leone-Exponentiated Half Logistic-Generalized-G Family of Distributions with Applications

The elements of the score vector $U(\Delta)$ are given by

$$\begin{aligned} \frac{\partial \ell_n(\Delta)}{\partial \alpha} &= \frac{n}{\alpha} - 2\beta(b-1) \sum_{i=1}^n \frac{\left(\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right)^{\beta-1} \left[\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right] \ln \left[\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right]}{\left[1 - \left(1 - \left(\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right)^\beta\right)^2\right]} \\ &\quad - \beta \sum_{i=1}^n \frac{\left(\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right)^{\beta-1} \left[\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right] \ln \left[\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right]}{\left(1 - \left(\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right)^\beta\right)} + \sum_{i=1}^n \ln[\bar{G}^\alpha(x_i; \Psi)] \\ &\quad + (\beta-1) \sum_{i=1}^n \frac{\bar{G}^\alpha(x_i; \Psi) \ln[\bar{G}^\alpha(x_i; \Psi)]}{[1 - \bar{G}^\alpha(x_i; \Psi)]} - (\beta+1) \sum_{i=1}^n \frac{\bar{G}^\alpha(x_i; \Psi) \ln[\bar{G}^\alpha(x_i; \Psi)]}{[1 + \bar{G}^\alpha(x_i; \Psi)]}, \end{aligned}$$

$$\begin{aligned} \frac{\partial \ell_n(\Delta)}{\partial \beta} &= \frac{n}{\beta} - 2(b-1) \sum_{i=1}^n \frac{\left(\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right)^\beta \ln \left[\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right]}{\left[1 - \left(1 - \left(\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right)^\beta\right)^2\right]} \\ &\quad - \sum_{i=1}^n \frac{\left(\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right)^\beta \ln \left[\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right]}{\left[1 - \left(\frac{1-\bar{G}^\alpha(x_i; \Psi)}{1+\bar{G}^\alpha(x_i; \Psi)}\right)^\beta\right]} + \sum_{i=1}^n \ln[1 - \bar{G}^\alpha(x_i; \Psi)] \\ &\quad - \sum_{i=1}^n \ln[1 + \bar{G}^\alpha(x_i; \Psi)], \end{aligned}$$

$$\frac{\partial \ell_n(\Delta)}{\partial b} = \frac{n}{b} + \sum_{i=1}^n \ln \left[1 - \left(1 - \left(\frac{1 - \bar{G}^\alpha(x_i; \Psi)}{1 + \bar{G}^\alpha(x_i; \Psi)} \right)^\beta \right)^2 \right],$$

and

$$\begin{aligned}
 \frac{\partial \ell_n(\Delta)}{\partial \Psi_k} &= 2\beta\alpha \sum_{i=1}^n \frac{\left(\frac{1-\bar{G}^\alpha(x_i;\Psi)}{1+\bar{G}^\alpha(x_i;\Psi)}\right)^\beta \left(\frac{1-\bar{G}^\alpha(x_i;\Psi)}{1+\bar{G}^\alpha(x_i;\Psi)}\right)^{\beta-1} \frac{1-[\bar{G}(x_i;\Psi)]^{\alpha-1}}{1+[\bar{G}(x_i;\Psi)]^{\alpha-1}} \frac{\partial \bar{G}(x_i;\Psi)}{\partial \Psi_k}}{\left[1 - \left(1 - \left(\frac{1-\bar{G}^\alpha(x_i;\Psi)}{1+\bar{G}^\alpha(x_i;\Psi)}\right)^\beta\right)^2\right]} \\
 &+ \beta\alpha \sum_{i=1}^n \frac{\left(\frac{1-\bar{G}^\alpha(x_i;\Psi)}{1+\bar{G}^\alpha(x_i;\Psi)}\right)^{\beta-1} \frac{1-[\bar{G}(x_i;\Psi)]^{\alpha-1}}{1+[\bar{G}(x_i;\Psi)]^{\alpha-1}} \frac{\partial \bar{G}(x_i;\Psi)}{\partial \Psi_k}}{\left[1 - \left(\frac{1-\bar{G}^\alpha(x_i;\Psi)}{1+\bar{G}^\alpha(x_i;\Psi)}\right)^\beta\right]} - \alpha(\beta-1) \sum_{i=1}^n \frac{[\bar{G}(x_i;\Psi)]^{\alpha-1} \frac{\partial \bar{G}(x_i;\Psi)}{\partial \Psi_k}}{[1 - \bar{G}^\alpha(x_i;\Psi)]} \\
 &- \alpha(\beta+1) \sum_{i=1}^n \frac{[\bar{G}(x_i;\Psi)]^{\alpha-1} \frac{\partial \bar{G}(x_i;\Psi)}{\partial \Psi_k}}{[1 + \bar{G}^\alpha(x_i;\Psi)]} + (\alpha-1) \sum_{i=1}^n \frac{\frac{\partial \bar{G}(x_i;\Psi)}{\partial \Psi_k}}{[\bar{G}(x_i;\Psi)]} + \frac{\frac{\partial g(x_i;\Psi)}{\partial \Psi_k}}{g(x_i;\Psi)}.
 \end{aligned}$$

C. Chapter 4 Appendix

C.1 The Topp-Leone-Gompertz-G Power Series Class of Distribution

The following link contains the Appendix for the TL-Gom-GPS class of distributions:

https://drive.google.com/file/d/1MSC2iFm4NNjxvqMbDjjrRDDEF0zZr4R_/view?usp=sharing



D. Chapter 5 Appendix

D.1 The Marshall-Olkin-Topp-Leone-Gompertz-G Family of Distribution

The link below contains the Appendix for the MO-TL-Gom-G family of distributions:

<https://drive.google.com/file/d/1BLQjGat89t5VbGJyaQibDK66WmC8xIol/view?usp=sharing>



E. Chapter 6 Appendix

E.1 The Gamma Odd Power Generalized Weibull-G Family of Distributions

The following link contains the Appendix for the GOPGW-G family of distributions:

<https://drive.google.com/file/d/1i7a0WcTg6Xw-iRzRj3dwv2dssDgSrxFt/view?usp=sharing>



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