

Faculty of Sciences
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**ON THE GENERATION OF NEW GENERALIZED FAMILIES OF
DISTRIBUTIONS WITH APPLICATIONS**

by

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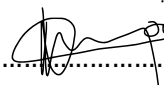
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Dedication

This work is dedicated to the memory of my late father Mr. D. Peter. There is no doubt in my mind that his mentorship, great parental support and guidance have sustained me throughout the hardships of life. You will forever remain in my heart ubaba.

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Abstract

In this thesis, new generalized families of distributions namely, Weibull Odd Burr III-G (WOBIII-G), Ristić-Balakrishnan Odd Burr III-G (RBOBIII-G), Ristić-Balakrishnan Marshall-Olkin-G (RBMO-G), Exponentiated Half-Logistic Odd Burr III-G (EHL-OBIII-G) and Half-Logistic Odd Power Generalized Weibull-G (HLOPGW-G) families of distributions are developed and studied in detail. Furthermore, the mathematical and statistical properties of these new families of distributions such as moments and moment generating functions, order statistics, probability weighted moments, stochastic ordering and Rényi entropy are derived and studied in detail. We use some of the existing known baseline models to generate some special cases for these newly generated families of distributions. The maximum likelihood estimation technique is used to estimate model parameters and simulation experiments are conducted to illustrate the consistency of maximum likelihood estimates. Finally we demonstrate the efficacy and flexibility for some of the special cases under these distributions by fitting them to some real data sets. The proposed models presents better fits than other existing equal parameter models.

LIST OF PUBLICATIONS & SUBMITTED PAPERS

1. Peer Reviewed & Published Articles

- (i) P. O. Peter, B. Oluyede, N. Ndwapi & H. Bindele. **The Weibull Odd Burr III-G Family of Distributions with Properties and Applications.** *Journal of Statistical Modelling: Theory & Applications.* Vol 2(1), 2021, doi:jsmta/jsmta.2021.2021, http://jsm.yazd.ac.ir/article_2021.html
- (ii) P. O. Peter, B. Oluyede, H. Bindele, N. Ndwapi & V. Mabikwa, **The Gamma Odd Burr III-G Family of Distributions: Model, Properties and Applications.** *Revista Columbiana de Estadística.* Vol 44 (2), 2021, 331-368, <https://revistas.unal.edu.co/index.php/estad/article/view/89320/80264>
- (iii) P. O. Peter, B. Oluyede, N. Ndwapi, O. Mabikwa & H. Bindele. **The New Family of Ristić Balakrishnan Marshall Olkin-G Distributions with Properties and Applications.** *International Journal of Mathematics and Operations Research Research (Forthcoming)*, doi:10.1504/IJMOR.2021.10040442, <https://inderscience.com/info/ingeneral/forthcoming.php?jcode=ijmor#101370>
- (iv) P. O. Peter, F. Chipepa, B. Oluyede & B. Makubate. **The Half-Logistic Power Generalized Weibull-G Family of Distributions: Model, Properties and Applications.** *Central European Journal of Economic Modelling and Econometrics* Vol 4 (1), 2022, 1-35, doi: 10.24425/cejeme.2022.140510, <http://cejeme.org/publishedarticles/2022-35-07-637798665004778595-4700.pdf>
- (v) B. Oluyede, P. O. Peter, N. Ndwapi & H. Bindele. **The Exponentiated Half-Logistic Odd Burr III-G: Model, Properties and Applications.** *Pakistan Journal of Statistics & Operations Research.* Vol 18 (1), 2022, 33-57, <https://doi.org/10.18187/pjsor.v18i1.3668>

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- (ii) B. Oluyede, P. O. Peter & N. Ndwapi. **The Exponentiated Half-Logistic-Generalized-Marshall-Olkin-G Family of Distributions with Applications.** (*Thailand Statistician*)
- (iii) P. O. Peter, N. Ndwapi, B. Oluyede & H. Bindele. **The Weibull Exponentiated Half-Logistic-G Family of Distributions with Properties and Applications.** (*South African Statistical Journal*)
- (iv) P. O. Peter, F. Chipepa & B. Oluyede. **The Half-Logistic Odd Power Generalized Weibull-G Power Series Family of Distributions: Properties and Applications.** (*Journal of Statistics and Management Systems*)
- (v) P. O. Peter, B. Oluyede, N. Ndwapi, H. Bindele, & O. Mabikwa. **The New Family of Zografos Balakrishnan Odd Burr III-G Distributions with Properties and Applications.** (*The Philippine Statistician*)

3. Other collaborative work Published during the Course of Study

- (i) Boikanyo Makubate, Fastel Chipepa, Broderick Oluyede & Peter O. Peter. **The Marshall-Olkin Half-Logistic-G Family of Distributions with Applications.** *International Journal of Statistics & Probability*. Vol 10 (2), 2021, 120, <https://doi.org/10.5539/ijsp.v10n2p120>
- (ii) Fastel Chipepa, Broderick Oluyede & Peter O. Peter. **The Burr III-Topp-Leone-G Family of Distributions with Applications.** *The Heliyon*. Vol 7 (4), 2021, <https://doi.org/10.1016/j.heliyon.2021.e06534>

Chapter 1

Introduction

1.1 Introduction

Statistical distributions play an important role when it comes to description and prediction of real data in various areas of research such as engineering, financial modelling, economics, biological and reliability studies. However, there are instances where classical distributions do not provide adequate fits to real data. It is for this reason that several probability distributions have been developed by introducing one or more parameters to generate new distributions. In literature, new families of distributions have been developed by introducing shape parameters to control skewness, kurtosis and tail weights thus providing great flexibility in modelling skewed data. Some of the similar work include the two-piece approach introduced by [61] and also the generators by [56], [44], [10]. Many subsequent articles apply these techniques to induce skewness into well-known symmetric distributions such as the symmetric Student t, see [1] for a detailed review. The recent perculiar works include Odd Burr-G (OB-G) family [12], Gamma-Weibull-G family [81], Generalized odd-log-logistic family [39], New Generalized odd log-logistic family [60], and Odd log-logistic Lindley Poisson family [85]. Following the system of twelve cumulative distribution functions proposed by [28], more work on Burr-XII models and other related distributions were widely studied by various authors, (see [102], [95], [112], [100], [107], [97]). Afify et al. [4] introduced a new four-parameter lifetime model namely the Weibull Burr-XII distribution capable of modelling various shapes of

aging and failure rate. Recent useful extensions of the Burr-XII models were also studied in detail by [15], [49], [109] and [111].

Some attempts have been made to define new families of probability distributions that extend well-known families of distributions providing great flexibility in modelling lifetime data. Some of the well-established generators and other recently developed families of distributions include; the Kumaraswamy generalized [40], gamma-G [91], exponentiated generalized [45], Weibull-G [27], Kumaraswamy transmuted-G [5], Type I half-logistic-G [38], Garhy-G [55], Kumaraswamy Weibull-G [63], exponentiated Weibull-G [64], additive Weibull-G [62], Topp–Leone-G [8], exponentiated extended-G [54], Type II half logistic-G [65], alpha power transformed-G [78] and Kumaraswamy generalized Marshall-Olkin-G [30] among others. Subsequently, some of the new families of distributions were recently developed and the famous ones being the Odd Burr-G family [12], Gamma-Weibull-G family [81], Generalized odd-log-logistic family [39], New Generalized odd log-logistic family [60], and Odd log-logistic Lindley Poisson family [85].

Despite many lifetime modelling distributions studied by different authors in the literature, there is always an opportunity for the discovery of new distributions which provides more flexibility in fitting various real-world problems. Other well known generated families studied by various authors include the Weibull-G family of distributions by [27], exponentiated-generalized-G family of distributions by [45] and two log-gamma-G families by [18]. Another useful generators called gamma-G were introduced by [113] and [91]. Alzaatreh et al. [16] developed a general approach, the transformed-transformer (T-X) family and Alzaghal et al. [17] further extended T-X family and proposed exponentiated-T-X family of distributions. Similar work was done by Aljarrah et al. [14] where the T-X family was introduced using the quantile function approach.

Some of the other well-known generators which played a major role in the extension of generalized families of distributions include the Marshall-Olkin-G [74], the beta-G [56], odd log-logistic-G [13], the transmuted-G [96], the gamma-G [113], the

Kumaraswamy-G [44], the logistic-G [105], exponentiated generalized-G [45], T-X family [16], the Weibull-G [27], the exponentiated half-logistic generated family [42] and the beta odd log-logistic generalized [43] to mention just a few. Burr [28] introduced a system of cumulative distribution functions which were widely used by many researchers, (see [102], [95] and [112] among others) to develop more distributions with great flexibility. The Burr-XII models have wider applicability in other various areas of research such as reliability and engineering, acceptance sampling plans and failure time modelling. Afify et al. [6] developed a new flexible family of distributions called the Odd exponentiated half logistic-G (OEHL-G) family of distributions using the Half-Logistic (HL) distribution as the generator. Other related works include the Weibull-G family [27], the Weibull-X family [16], the exponentiated Weibull-X family [17], the new Weibull-X family [7], the new Weibull-G family [103], the modified odd Weibull-G family [34] and the Weibull Generalized-G [110]. Similarly, [6] developed a new flexible family of distributions called the odd exponentiated half logistic-G (OEHL-G) family of distributions using the Half-Logistic (HL) distribution as the generator. Cordeiro et al. [42] also used the same generator to develop the Exponentiated Half-logistic (EHL) family of distributions. These new generators have been widely applied recently by various researchers to generate new flexible models useful in life time data analysis.

1.2 Some Important Generators

In this section, some of the important generators used in this study to extend some lifetime distributions are briefly discussed.

1.2.1 The Transformed-Transformer (T-X) Generator

Consider the T-X transformation technique proposed by [16] having the cumulative distribution function (cdf) given by

$$F(x) = \int_0^{W(G(x;\varphi))} r(t)dt, \quad x \in \mathbb{R},$$

where $r(t)$ is the probability density function (pdf) of a random variable T and $W(G(x; \underline{\varphi}))$ represent the function of the cdf for the baseline function of a random variable X .

1.2.2 Beta-G Generator

Let $G(x; \underline{\varphi})$ and $g(x; \underline{\varphi})$ denote the cumulative distribution function (cdf) and probability density function (pdf) of a continuous random variable X , then the cdf and pdf of the beta-G distribution are given by

$$F(x) = I_{G(x; \underline{\varphi})}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x; \underline{\varphi})} t^{a-1} (1-t)^{b-1} dt \quad (1.1)$$

and

$$f(x) = \frac{1}{B(a, b)} [G(x; \underline{\varphi})]^{a-1} [1 - G(x; \underline{\varphi})]^{b-1} g(x; \underline{\varphi}), \quad (1.2)$$

respectively, for $a, b > 0$, and $\underline{\varphi}$ denotes the baseline parameter vector. Note that $I_y = B_y(a, b)/B(a, b)$ is the incomplete beta ratio, $B_y(a, b) = \int_0^y t^{a-1} (1-t)^{b-1} dt$ is the incomplete beta function and $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(\cdot)$ is the gamma function.

1.2.3 Gamma-G Generator

Let $G(x; \underline{\varphi})$ and $g(x; \underline{\varphi})$ denote the cumulative distribution function (cdf) and probability density function (pdf) of a continuous random variable X , then the cdf and pdf of the Gamma-G distribution by [113] are given by

$$F(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log(1-G(x; \underline{\varphi}))} t^{\delta-1} e^{-t} dt = \frac{\gamma(\delta, G(x; \underline{\varphi}))}{\Gamma(\delta)}, \quad (1.3)$$

and

$$f(x) = \frac{1}{\Gamma(\delta)} [G(x; \underline{\varphi})]^{\delta-1} g(x; \underline{\varphi}), x \in R, \delta > 0, \quad (1.4)$$

respectively, for $\delta > 0$ and baseline parameter vector $\underline{\varphi}$, where $\gamma(x; \delta) = \int_0^x t^{\delta-1} e^{-t} dt$ is the incomplete gamma function.

1.2.4 Kumaraswamy-G Generator

Cordeiro and de Castro (see [40]) introduced the Kw-G distribution defined from an arbitrary parent cdf $G(x; \underline{\varphi})$, having the cdf $F(x)$ given by

$$F(x) = 1 - (1 - G(x; \underline{\varphi})^a)^b, \quad (1.5)$$

and its pdf is given by

$$f(x) = abg(x)(G(x; \underline{\varphi}))^{a-1}(1 - G(x; \underline{\varphi})^a)^{b-1}, \quad (1.6)$$

for $a, b > 0$, $x > 0$ and baseline parameter vector $\underline{\varphi}$.

1.2.5 Marshall-Olkin-G Generator

Marshall and Olkin (see [74]) developed the Marshall-Olkin (MO) extended family of distributions. For any baseline distribution with pdf $f(x)$ and cdf $F(x)$, Marshall and Olkin [74] defined a new survival function $\bar{F}(x)$ by adding an additional parameter $\alpha > 0$. The survival function is defined by

$$\bar{F}_{MO}(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha} \bar{F}(x)}, \quad (1.7)$$

where $-\infty < x < \infty$, $\alpha > 0$, $\bar{\alpha} = 1 - \alpha$, α is the tilt parameter since the hazard rate function of the new family is shifted either above or below the hazard rate function of the baseline distribution for $0 < \alpha < 1$ and $\alpha > 1$, respectively. The cdf and pdf of the MO-extended distribution are given by

$$F_{MO}(x) = \frac{G(x; \underline{\varphi})}{1 - \bar{\alpha} \bar{G}(x; \underline{\varphi})}, \quad (1.8)$$

and

$$f_{MO}(x) = \frac{\alpha g(x; \underline{\varphi})}{[1 - \bar{\alpha} \bar{G}(x; \underline{\varphi})]^2}, \quad (1.9)$$

respectively, where $g(x; \underline{\varphi})$ and $G(x; \underline{\varphi})$ are the pdf and cdf respectively from any baseline distribution, with $\underline{\varphi}$ as a vector of parameters.

1.2.6 Half-Logistic-G Generator

Cordeiro et al. (see [38]) developed the type I half-logistic family of distributions with the cdf and pdf given by

$$F(x; \lambda, \underline{\varphi}) = \int_0^{-\ln(1-G(x; \underline{\varphi}))} \frac{2\lambda \exp\{-\lambda x\}}{(1 + \exp\{-\lambda x\})} dx = \frac{1 - [1 - G(x; \underline{\varphi})]^\lambda}{1 + [1 - G(x; \underline{\varphi})]^\lambda},$$

where $G(x; \underline{\varphi})$ is the cdf of the baseline distribution, $\lambda > 0$ is the shape parameter and $\underline{\varphi}$ is a vector of parameters. We obtain a special case, namely, half-logistic-G (HL-G) model, with cdf

$$F(x; \underline{\varphi}) = \frac{G(x; \underline{\varphi})}{1 + \overline{G}(x; \underline{\varphi})} \quad (1.10)$$

by setting $\lambda = 1$ in equation (1.10). The corresponding pdf of HL-G model is given by

$$f(x; \underline{\varphi}) = \frac{2g(x; \underline{\varphi})}{(1 + \overline{G}(x; \underline{\varphi}))^2}. \quad (1.11)$$

1.2.7 Competing Risk Approach

Let the lifetime of components in a series system follow $G_1(t)$ and $G_2(t)$ distributions with the corresponding reliability functions given by $R_1(t)$ and $R_2(t)$, respectively, then the reliability $R(t) = P(T > t)$ of the system is given by

$$R(t) = R_1(t)R_2(t).$$

The underlying cdf is given by $G(t) = 1 - R(t)$, and in reliability studies, a series model is also known as the competing risk model and has the pdf given by $g(t) = \frac{d}{dt}G(t)$.

1.2.8 Compounding Approach

Let N be a discrete random variable from a power series distribution assumed to be truncated at zero, whose probability mass function (pmf) is given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \dots, \quad (1.12)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$, and $\{a_n\}_{n \geq 1}$ a sequence of positive real numbers. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions [68]. Table 1.1 presents some useful quantities including a_n , $C(\theta)$, $C^{-1}(\theta)$, $C'(\theta)$ and $C''(\theta)$ for the binomial, Poisson, geometric and logarithmic distributions.

Table 1.1: Useful Quantities for Some Power Series Distributions

Distribution	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	a_n	Parameter Space
Binomial	$(1 + \theta)^m - 1$	$m(1 + \theta)^{m-1}$	$m(m-1)(1 + \theta)^{m-2}$	$(\theta - 1)^{1/m} - 1$	$\binom{m}{n}$	$(0, 1)$
Poisson	$e^\theta - 1$	e^θ	e^θ	$\log(1 + \theta)$	$(n!)^{-1}$	$(0, \infty)$
Geometric	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(1 + \theta)^{-1}$	1	$(0, 1)$
Logarithmic	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	n^{-1}	$(0, 1)$

Some of the power series generalizations available in literature include the Weibull-power series (WPS) distributions by [76], generalized exponential-power series (GEP) distribution by [72], extended Weibull-power series (EWPS) distribution by [98], double bounded Kumaraswamy power series by [24], Burr XII power series by [99], generalized linear failure rate-power series (GLFRP) distribution by [9], Birnbaum-Saunders power series distribution by [26], linear failure rate-power series by [73], complementary extended Weibull-power series by [48] and the new generalized Lindley Weibull power series by [25], just to mention these few.

1.3 Characteristics of Probability Distributions

1.3.1 Survival Function

Survival function ($S(x)$) or reliability function $R(x)$ is important in calculating survival probability or system's reliability. Let X be a continuous random variable having cdf $F(x)$ and pdf $f(x) = \frac{dF(x)}{dx}$, mathematically

$$S(x) = \bar{F}(x) = 1 - F(x).$$

Thus, $S(x)$ is the probability of surviving more than time x and $F(x)$ is the probability of surviving up to time x . $S(x)$ is a monotonic and decreasing function over $[0, \infty)$.

1.3.2 The Hazard Rate, Reverse Hazard Rate and Quantile Functions

The hazard rate function, denoted by $h(x)$, can be defined as the force of mortality or the failure rate. It is derived by dividing the probability density function by the survival function. Mathematically $h(x) = f(x)/\bar{F}(x)$, where $\bar{F}(x) = 1 - F(x)$ and $F(x)$ is the cdf. The hazard rate function can adopt various shapes, that is, it can be monotonic increasing or decreasing, non monotonic (bathtub or upside down bathtub). In case where we have hidden failures, the hazard function may not capture such failures and the reversed hazard rate function becomes a useful tool in such instances. The reversed hazard rate is the conditional probability of failure of a component per unit time interval $(t - \delta t)$, given that the failure occurred in $[0, t]$ Desai et al. [50]. Mathematically it can be expressed as the ratio of the probability density function and the cumulative distribution function, that is, $\tau(x) = \frac{f(x)}{F(x)}$. The reverse hazard rate function has application in maintenance management, especially for condition monitoring [50]. The reversed hazard rate function is always a monotone decreasing function irregardless of the shape of the hazard rate function [50].

The quantile function of a random variable X is the inverse of its distribution function. Quantile function gives the statistical properties of the random variable X , just like the distribution function. The quantile function is the generalized inverse of F , that is, $F^{-1}(u) = \inf \{x : F(x) \geq u\}, 0 < u < 1$.

1.3.3 Ordinary and Incomplete Moments

The mean, variance, skewness and kurtosis are the four commonly studied moments. They are generally used to describe the features of a statistical distribution. It is very useful to derive the moments and incomplete moments of a generalized distribution from a weighted infinite linear combination of established distributions. The ordinary r^{th} moment for the real valued function can be defined as $\mu'_r = E(X^r) = \int x^r f(x)dx$ and the incomplete moment can be obtained as $I_X(t) = \int_0^t x^r f(x)dx$.

The coefficients of variation (CV), skewness (CS) and kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD= σ), coefficient of variation (CV),

coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$

and

$$CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively. Note that the r^{th} cumulant of the random variable X can be readily obtained from the recursive relationship: $\kappa_r = \mu'_r - \sum_{s=1}^{r-1} \binom{r-1}{s-1} \mu'_{r-s} \kappa_s$, where $\mu'_r = E(X - \mu_1')^r$, so that the CS and CK are given by $\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}$ and $\gamma_2 = \frac{\kappa_4}{\kappa_2^2}$.

1.3.4 Order Statistics

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (i.i.d) random variables distributed with pdf $f(x)$ and cdf $F(x)$. The pdf of the i^{th} order statistic $X_{i:n}$, is given by

$$f_{i:n}(x) = \frac{n!f(x)}{(i-1)!(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i}.$$

1.3.5 Entropy

An entropy gives a measure of variation of uncertainty for a random variable X with the pdf given by $f(x)$. The two most popular measures of entropy are Shannon entropy, (see [93]) and Rényi entropy, (see [90]). The Rényi entropy is defined by

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\int_0^\infty f^\nu(x) dx \right], \quad (1.13)$$

where $\nu > 0$ and $\nu \neq 1$. Note that Shannon entropy is a special case of Rényi entropy and is defined as $E \{-\log[f(X)]\}$.

1.3.6 Stochastic Ordering

The three commonly known orders of distribution functions are the stochastic order, the hazard rate order and the likelihood ratio order. These orders are defined as follows;

Let X and Y be the two random variables with the cdfs $F_X(t)$ and $F_Y(t)$, respectively, and $\bar{F}_X(t) = 1 - F_X(t)$ is the reliability or survival function. The random variable X is said to be stochastically smaller than the random variable Y if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all t or $F_X(t) \geq F_Y(t)$ for all t . This is denoted by $X <_{st} Y$. The hazard rate order and likelihood ratio order are stronger and are given by $X <_{hr} Y$ if $h_X(t) \geq h_Y(t)$ for all t , and $X <_{\ell_r} Y$ if $\frac{f_X(t)}{f_Y(t)}$ is decreasing in t . It is well established that $X <_{\ell_r} Y \implies X <_{hr} Y \implies X <_{st} Y$, (see [92] for more details).

1.3.7 Estimation

There are many methods in the literature on parameter estimation. Some of the common methods are the method of moments, maximum likelihood estimation (MLE) and the Bayesian estimation techniques. We used the MLE technique as the method of estimation due to its consistency in yielding unbiased estimators with minimal variance. We computed the total likelihood function $L(\theta; x) = \prod_{i=1}^n f(x_i; \theta)$ for a given pdf and derive the score vector of parameters θ . The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters θ are obtained using iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is computed by solving the nonlinear equation $(\frac{\partial \ell}{\partial \theta})^T = 0$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix given by $\mathbf{I}(\Delta) = [\mathbf{I}_{\theta_i, \theta_j}]_{k \times k} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, \dots$ can be numerically obtained by using either MATLAB, NLMIXED in SAS or R software. The Fisher information matrix $n\mathbf{I}(\Delta)$ can be estimated by

$$\mathbf{J}_n(\hat{\Delta}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \Big|_{\Delta=\hat{\Delta}} \right]_{k \times k}, \quad i, j = 1, 2, \dots \quad (1.14)$$

Note that for a set of observations, equation (1.14) is obtained from Newton-Raphson convergence procedure using R software or NLMIXED in SAS.

1.3.8 Methods for Assessing Goodness-of-Fit

The $-2\log$ -likelihood statistic ($-2\log(L)$), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC) are used for model comparison. Mathematically ($AIC = 2p - 2\log(L)$), ($BIC = p\log(n) - 2\ln(L)$), ($AICC = AIC + 2\frac{p(p+1)}{n-p-1}$) and where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations and p is the number of parameters. The model with smallest values of these statistics is regarded as the best fit model for that particular data set.

The measures of closeness are obtained via the sum of squares

$$SS = \sum_{j=1}^n \left[F(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2,$$

where $j = 1, 2, \dots, n$, and $x_{(j)}$ are the ordered values of the observed data. The goodness-of-fit statistics W^* and A^* , described by Chen and Balakrishnan [33] are also used to assess goodness-of-fit. Note that the smaller the values of W^* , A^* , SS and $K - S$ the better the fit.

1.4 Aim and Objectives

The main aim of this thesis is to use some of the important generators, namely, the Marshall-Olkin-G, Half-Logistic-G, Gamma-G, Exponentiated Half-Logistic-G, Weibull-G to develop new and generalized distributions and specifically

1. To develop the following models:

- (i) Weibull Odd Burr III-G (WOBIII-G) family of distributions.
- (ii) Ristić-Balakhrishnan Odd Burr III-G (RBOBIII-G) family of distributions.
- (iii) Ristić-Balakhrishnan Marshall-Olkin-G (RBMO-G) family of distributions.

- (iv) Exponentiated Half-Logistic Odd Burr III-G (EHL-OBIII-G) family of distributions.
 - (v) Half-Logistic Odd Power Generalized Weibull-G (HLOPGW-G) family of distributions.
2. To derive and extensively study some of the statistical and mathematical properties of these newly developed distributions including their maximum likelihood estimates.
 3. To demonstrate the consistency and accuracy of the maximum likelihood estimates via simulation studies.
 4. To illustrate the usefulness and efficacy of these newly developed models by fitting them to real data sets.

1.5 Motivation

In this thesis, the motivation to develop these new generalized families of distributions arises from their efficiency and flexibility in modeling failure time data with increasing, decreasing, unimodal and bathtub shaped hazard rates. The newly introduced models present a good alternative to most of the failure time distributions that lacks flexibility in modeling various forms of real life data problems. These new families of distributions are more flexible and can adopt various forms of shapes compared to the existing baseline models. Additionally, these models consistently give better fits than some of the available models with the same number of parameters. The generated models yields better fits to various forms of real life data and have interesting tractability properties. These models can also fit data which is heavy-skewed and heavy-tailed with high degree of flexibility. Another important factors is that these generalized families of distributions are quite flexible for fitting data which is evidently shown by the underlying special models that can accommodate left-skewed, right-skewed, symmetrical, and reversed-J shaped densities. The new family of distributions possesses very flexible and desirable properties including hazard rate function that exhibits J, reversed-J, increasing, decreasing, bathtub,

bathtub followed by upside down bathtub, and upside down bathtub shapes.

Furthermore, the development of these new family of distributions is necessitated by the need to model climate and lifetime data with models that takes into account not only shape and scale but also consider skewness, kurtosis and tail variation. We are also motivated by various applications of generalized combination of Weibull and exponentiated half logistic distributions in several areas including reliability, exponential tilting (weighting) in finance and actuarial sciences, as well as economics. We are hopeful that these new families of distributions will find wider applications in various fields of study such as genomic, engineering and survival analysis.

1.6 Outline of Thesis

The structure of this thesis is outlined as follows: Chapter 2 presents development of the Weibull Odd Burr III-G (WOBIII-G) family of distributions, Chapter 3, presents the new Ristić-Balakrishnan Odd Burr III-G (RBOBIII-G) family of distributions, Chapter 4, presents the Ristić-Balakrishnan Marshall-Olkin-G (RBMO-G) family of distributions. In Chapter 5, we present the Exponentiated Half-Logistic Odd Burr III-G (EHL-OBIII-G) family of distributions and Chapter 6, presents Half-Logistic Odd Power Generalized Weibull-G (HLOPGW-G) family of distributions. We finally give concluding remarks and areas of further study under Chapter 7.

1.7 Chapter Summary

This sub section gives a brief summary of Chapter 1 and its contents. In this chapter, a detailed review of literature including the most recent works on the development of families of distributions is given. We discuss different types of generators that are commonly used in the literature including some of the important characteristics of probability distributions as well as different parameter estimation methods. Finally, motivations and justifications for this study including the aim and objectives are clearly stated.

Chapter 2

The Weibull Odd Burr III-G Family of Distributions with Applications

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2.1 Introduction

In this chapter, the new generalized family of distributions namely; Weibull Odd Burr III-G (WOBIII-G) is developed and studied in detail using the T-X transformation technique. Some of the useful mathematical and statistical properties such as the hazard function, quantile function, moments and incomplete moments, moment generating functions, Rényi entropy, order statistics and stochastic orders are derived. The method of maximum likelihood estimation is used to estimate the model parameters. The usefulness of this new family of distributions is demonstrated via simulated experiments and its special case is applied to real life data sets to illustrate flexibility.

2.2 Generating the Weibull Odd Burr III-G Model

We formulate and develop the new model using the T-X generator technique proposed by [16] having the cumulative distribution function (cdf) given by

$$F(x) = \int_0^{W(G(x; \underline{\varphi}))} r(t) dt, \quad x \in \mathbb{R},$$

where $r(t)$ is the probability density function (pdf) of a random variable T and $W(G(x; \underline{\varphi}))$ represent the function of the cdf for the baseline function of a random variable X . To develop the WOBIII-G family using the T-X family technique, we let $r(t) = \beta t^{\beta-1} e^{-t^\beta}$, $t > 0, \beta > 0$ (one parameter Weibull distribution) and $W(G(x; \underline{\varphi})) = -\log(1 - (1 + (\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})})^{-a})^{-b})$, where $G(x; \underline{\varphi})$ is the baseline cdf. From these results, we can write the cdf of the WOBIII-G family of distributions as

$$F(x; a, b, \beta, \underline{\varphi}) = 1 - \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^\beta \right), \quad (2.1)$$

for $a, b, \beta, x > 0$ and parameter vector $\underline{\varphi}$, with the corresponding pdf given by

$$\begin{aligned} f(x; a, b, \beta, \underline{\varphi}) &= ab\beta \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^\beta \right) \\ &\times \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\ &\times \frac{\left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right)} \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a-1} \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}. \end{aligned} \quad (2.2)$$

The hazard rate function (hrf) is given by

$$\begin{aligned} h(x; a, b, \beta, \underline{\varphi}) &= ab\beta \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\ &\times \frac{\left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right)} \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a-1} \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}, \end{aligned} \quad (2.3)$$

for $a, b > 0$, and parameter vector $\underline{\varphi}$.

2.2.1 Quantile Function

The quantile function for the WOBIII-G family of distributions is derived by inverting the following function

$$1 - \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta} \right) = u,$$

for $0 < u < 1$, so that

$$\left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right) = \left(1 - \exp \left(- (-\log(1-u))^{\frac{1}{\beta}} \right) \right)^{\frac{-1}{b}},$$

and

$$\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} = \left(\left(1 - \exp \left(- (-\log(1-u))^{\frac{1}{\beta}} \right) \right)^{\frac{-1}{b}} - 1 \right)^{\frac{-1}{a}}. \quad (2.4)$$

The last expression further simplifies to

$$G(x; \underline{\varphi}) = \left(\left(\left(1 - \exp \left(- (-\log(1-u))^{\frac{1}{\beta}} \right) \right)^{\frac{-1}{b}} - 1 \right)^{\frac{1}{a}} + 1 \right)^{-1},$$

and therefore, we obtain the quantile function as

$$Q_{X_{(n)}}(u) = G^{-1} \left[\left(\left(\left(1 - \exp \left(- (-\log(1-u))^{\frac{1}{\beta}} \right) \right)^{\frac{-1}{b}} - 1 \right)^{\frac{1}{a}} + 1 \right)^{-1} \right], \quad (2.5)$$

which can be solved using iterative methods from any applicable software such as R.

2.2.2 Linear Representation of the Density Function

This section presents results on linear representation of the pdf for the WOBIII-G family of distributions. The expansion of the pdf will further allow us to derive impor-

tant mathematical and statistical properties. By applying several series expansions, (see **Appendix (A.1)** for details), we can write the pdf of W-OBIII-G family of distributions given by equation (2.2) as

$$\begin{aligned}
f(x; a, \beta, b, \underline{\varphi}) &= ab\beta \sum_{k,z,i,p,m,s=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m}(-1)^{k+p} \binom{\beta(k+1)-1}{m} \\
&\times \binom{-a(i+1)-1}{p} \binom{-b(\beta(k+1)+m+s+z)-1}{i} (\overline{G}(x; \underline{\varphi}))^{a(i+1)+p-1} g(x; \underline{\varphi}) \\
&= ab\beta \sum_{k,z,i,p,w,m,s=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m}(-1)^{k+p+w} \binom{\beta(k+1)-1}{m} \\
&\times \binom{a(i+1)+p-1}{w} \binom{-b(\beta(k+1)+m+s+z)-1}{i} \\
&\times \binom{-a(i+1)-1}{p} \left(\frac{w+1}{w+1}\right) (G(x; \underline{\varphi}))^w g(x; \underline{\varphi}) \\
&= \sum_{w=0}^{\infty} c_{w+1} g_{w+1}(x; \underline{\varphi}), \tag{2.6}
\end{aligned}$$

where $g_{w+1}(x; \underline{\varphi}) = (w+1) (G(x; \underline{\varphi}))^w g(x; \underline{\varphi})$ is the exponentiated-G (Exp-G) distribution with power parameter $w+1$ and

$$\begin{aligned}
c_{w+1} &= ab\beta \sum_{k=0}^{\infty} \sum_{z=0}^{\infty} \sum_{i,p=0}^{\infty} \frac{\Gamma(z+1)}{(w+1)\Gamma(1)z!} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} b_{s,m}(-1)^{k+p+w} \binom{\beta(k+1)-1}{m} \\
&\times \binom{-a(i+1)-1}{p} \binom{a(i+1)+p-1}{w} \binom{-b(\beta(k+1)+m+s+z)-1}{i}. \tag{2.7}
\end{aligned}$$

The details of these results are presented under the **Appendix (A.1)**.

2.3 Some Special Cases

This subsection consider some of the special cases from various sub-models obtained by changing the baseline distribution function $G(x; \underline{\varphi})$ to other flexible distributions. The parameter vector space is limited to at most 2 component vectors to avoid redundancy and over parametrization.

2.3.1 WOBIII-Log-Logistic (WOBIII-LLoG) Distribution

Consider the log-logistic distribution as the baseline distribution with parameter $\lambda > 0$ having cdf and pdf $G(x; \lambda) = 1 - (1 + x^\lambda)^{-1}$ and $g(x; \lambda) = \lambda x^{\lambda-1} (1 + x^\lambda)^{-2}$, respectively. The cdf, pdf and hrf of the WOBIII-LLoG distribution are given by

$$F(x; a, b, \beta, \lambda) = 1 - \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a} \right)^{-b} \right) \right]^\beta \right),$$

$$\begin{aligned} f(x; a, b, \beta, \lambda) &= ab\beta \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a} \right)^{-b} \right) \right]^\beta \right) \\ &\times \left[-\log \left(1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\ &\times \frac{\left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a} \right)^{-b} \right)} \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a-1} \frac{\lambda x^{\lambda-1} (1 + x^\lambda)^{-2}}{((1 + x^\lambda)^{-1})^2} \end{aligned}$$

and

$$\begin{aligned} h(x; a, b, \beta, \lambda) &= ab\beta \left[-\log \left(1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\ &\times \frac{\left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a} \right)^{-b} \right)} \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^\lambda} \right)^{-a-1} \frac{\lambda x^{\lambda-1} (1 + x^\lambda)^{-2}}{((1 + x^\lambda)^{-1})^2}, \end{aligned}$$

respectively, for $a, b, \beta, \lambda > 0$.

Plots of the pdf and hrf for the WOBIII-LLoG distribution are given in figure 2.1

below.

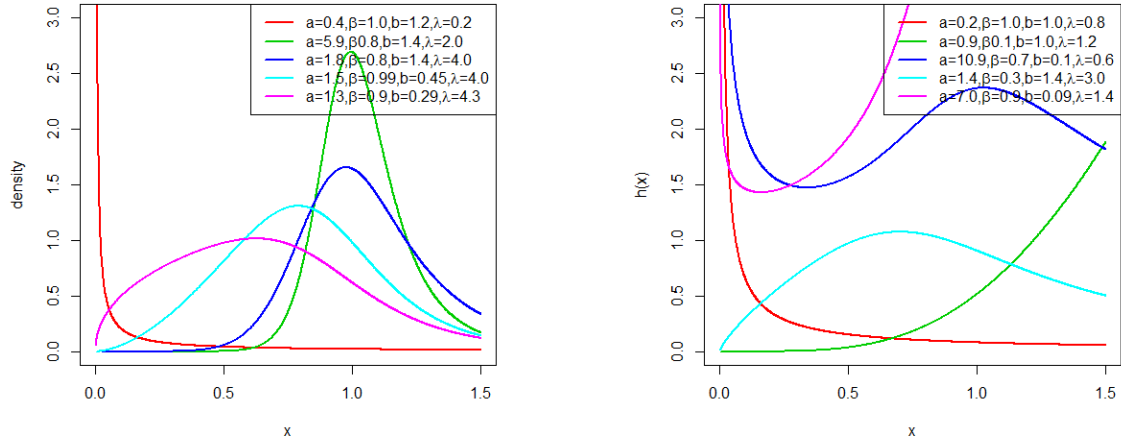


Figure 2.1: Plots of the pdf and hrf for the WOBIII-LLoG Distribution

Figure 2.1 shows the flexibility of the WOBIII-LLoG distribution for selected parameter values. The pdfs of the WOBIII-LLoG distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Additionally, the WOBIII-LLoG distribution hrf plots reveal decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

2.3.2 WOBIII-Logistic (WOBIII-LoG) Distribution

Consider the logistic distribution as the baseline distribution with parameter $\lambda > 0$ having cdf and pdf $G(x; \lambda) = (1 + e^{-\lambda x})^{-1}$ and $g(x; \lambda) = \lambda e^{-\lambda x} (1 + e^{-\lambda x})^{-2}$, respectively. The cdf, pdf and hrf of the WOBIII-LoG distribution are given by

$$F(x; a, b, \beta, \lambda) = 1 - \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a} \right)^{-b} \right) \right]^{\beta} \right),$$

$$\begin{aligned}
f(x; a, b, \beta, \lambda) &= ab\beta \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a} \right)^{-b} \right) \right]^{\beta} \right) \\
&\times \left[-\log \left(1 - \left(1 + \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\
&\times \frac{\left(1 + \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a} \right)^{-b} \right)} \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a-1} \\
&\times \frac{\lambda e^{-\lambda x} (1 + e^{-\lambda x})^{-2}}{((1 + e^{-\lambda x})^{-1})^2}
\end{aligned}$$

and

$$\begin{aligned}
h(x; a, b, \beta, \lambda) &= ab\beta \left[-\log \left(1 - \left(1 + \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\
&\times \frac{\left(1 + \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a} \right)^{-b} \right)} \left(\frac{(1 + e^{-\lambda x})^{-1}}{1 - (1 + e^{-\lambda x})^{-1}} \right)^{-a-1} \frac{\lambda e^{-\lambda x} (1 + e^{-\lambda x})^{-2}}{((1 + e^{-\lambda x})^{-1})^2},
\end{aligned}$$

respectively, for $a, b, \beta, \lambda > 0$.

Plots of the pdf and hrf for the WOBIII-LoG distribution are given in figure 2.2 below.

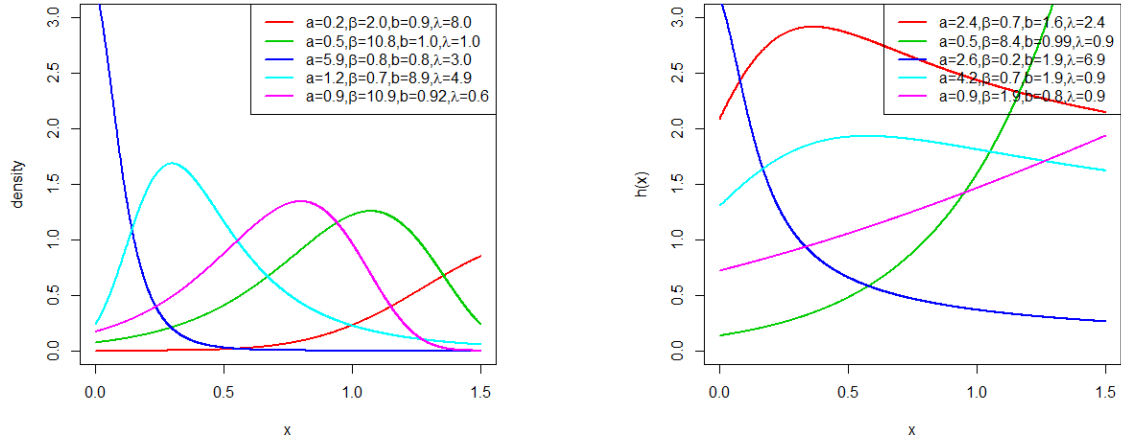


Figure 2.2: Plots of the pdf and hrf for the WOBIII-LoG Distribution

Figure 2.2 demonstrates different shapes adopted by the WOBIII-LoG distribution for selected parameter values. The pdfs of the WOBIII-LoG distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Also, the WOBIII-LoG distribution hrf plots give decreasing, increasing and upside down bathtub shapes.

2.3.3 WOBIII-Lindley (WOBIII-L) Distribution

If we let the Lindley distribution as the baseline distribution with pdf and cdf given by $g(x; \lambda) = \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}$ and $G(x; \lambda) = 1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}$, respectively, for $\lambda > 0$, then the cdf, pdf and hrf of the WOBIII-L distribution are given by

$$F(x; a, b, \beta, \lambda) = 1 - \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a} \right)^{-b} \right) \right]^\beta \right),$$

$$\begin{aligned}
f(x; a, b, \beta, \lambda) &= ab\beta \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a} \right)^{-b} \right) \right]^\beta \right) \\
&\times \left[-\log \left(1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\
&\times \frac{\left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a} \right)^{-b} \right)} \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a-1} \frac{\frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}}{\left((1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x} \right)^2}
\end{aligned}$$

and

$$\begin{aligned}
h(x; a, b, \beta, \lambda) &= ab\beta \left[-\log \left(1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\
&\times \frac{\left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a} \right)^{-b} \right)} \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})}{(1 + \frac{\lambda x}{1+\lambda})} \right)^{-a-1} \frac{\frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}}{\left((1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x} \right)^2},
\end{aligned}$$

respectively, for $a, b, \beta, \lambda > 0$.

Plots of the pdf and hrf for the WOBIII-L distribution are given in figure 2.3 below.

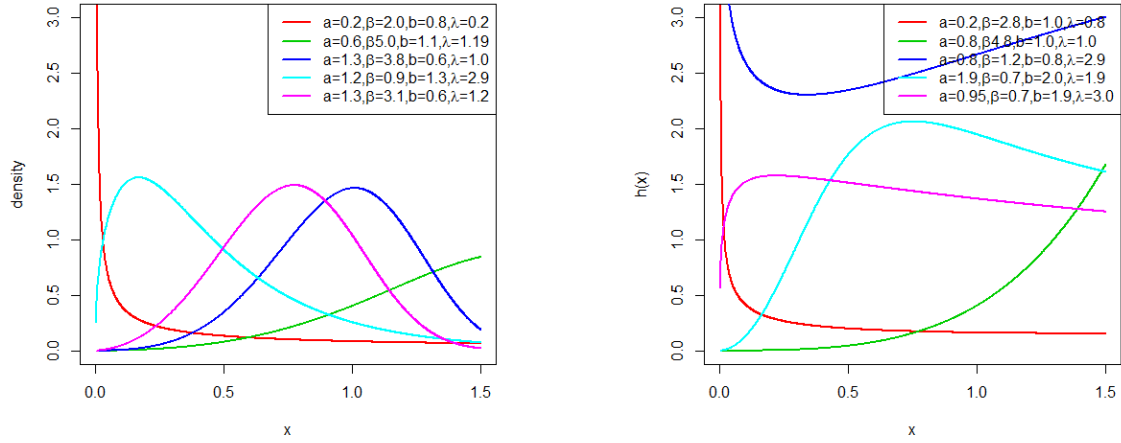


Figure 2.3: Plots of the pdf and hrf for the WOBIII-L Distribution

Figure 2.3 illustrates the flexible nature of the WOBIII-L distribution for selected parameter values. The pdfs of the WOBIII-L distribution can adopt various shapes that include reverse-J, uni-modal, left or right skewed shapes. The WOBIII-L distribution hrf plots also exhibit decreasing, increasing, bathtub and upside down bathtub shapes.

2.3.4 WOBIII-Uniform (WOBIII-U) Distribution

Suppose that we take the baseline distribution to be the uniform distribution with pdf and cdf given by $g(x) = 1/\lambda$ and $G(x; \lambda) = x/\lambda$, respectively, for $0 < x < \lambda$, we obtain the WOBIII-U distribution with cdf, pdf and hrf given by

$$F(x; a, b, \beta, \lambda) = 1 - \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{x/\lambda}{1 - x/\lambda} \right)^{-a} \right)^{-b} \right) \right]^\beta \right),$$

$$\begin{aligned}
f((x; a, b, \beta, \lambda) &= ab\beta \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{x/\lambda}{1-x/\lambda} \right)^{-a} \right)^{-b} \right) \right]^\beta \right) \\
&\times \left[-\log \left(1 - \left(1 + \left(\frac{x/\lambda}{1-x/\lambda} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\
&\times \frac{\left(1 + \left(\frac{x/\lambda}{1-x/\lambda} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{x/\lambda}{1-x/\lambda} \right)^{-a} \right)^{-b} \right)} \left(\frac{x/\lambda}{1-x/\lambda} \right)^{-a-1} \frac{1/\lambda}{(1-x/\lambda)^2}
\end{aligned}$$

and

$$\begin{aligned}
h((x; a, b, \beta, \lambda) &= ab\beta \left[-\log \left(1 - \left(1 + \left(\frac{x/\lambda}{1-x/\lambda} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\
&\times \frac{\left(1 + \left(\frac{x/\lambda}{1-x/\lambda} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{x/\lambda}{1-x/\lambda} \right)^{-a} \right)^{-b} \right)} \left(\frac{x/\lambda}{1-x/\lambda} \right)^{-a-1} \frac{1/\lambda}{(1-x/\lambda)^2},
\end{aligned}$$

respectively, for $a, b, \beta, \lambda > 0$.

Plots of the pdf and hrf for the WOBIII-U distribution are given in figure 2.4 below.

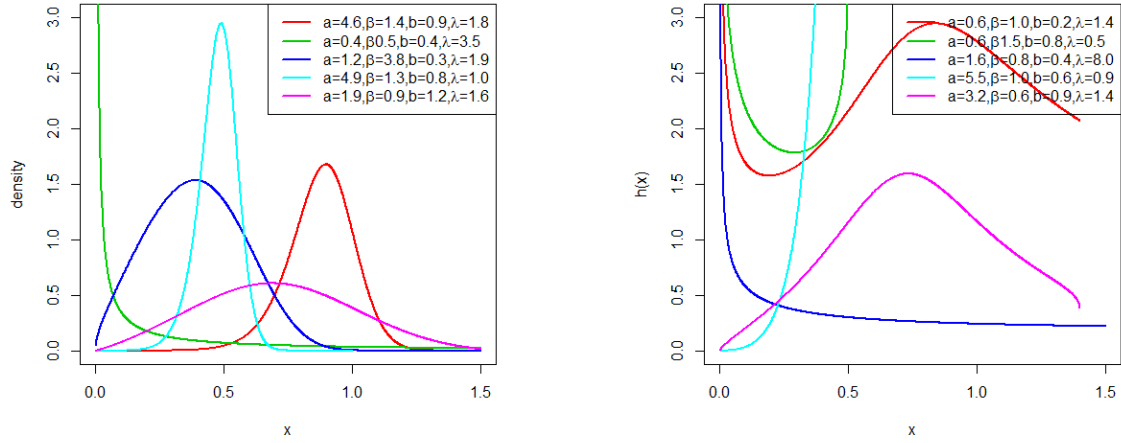


Figure 2.4: Plots of the pdf and hrf for the WOBIII-U Distribution

Figure 2.4 gives different shapes from the WOBIII-U distribution for selected parameter values. The pdfs of the WOBIII-U distribution follow various shapes that include reverse-J, uni-modal, left or right skewed shapes. In addition, plots of the hrf for the WOBIII-U distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

2.4 Some Mathematical Properties

In this section, some useful mathematical and statistical properties for the WOBIII-G family of distributions such as moments, distribution of order statistics and Rényi entropy are derived.

2.4.1 Moments

Let $Y_{w+1} \sim \text{Exp} - G(w+1)$. Then using equation (2.6), the n^{th} raw moment, μ'_n of the WOBIII-G family of distributions is obtained as

$$\mu'_n = E(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx = \sum_{w=0}^{\infty} c_{w+1} E(Y_{w+1}^n).$$

The first five moments for the WOBIII-LoG distribution together with the standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coeffi-

cient of kurtosis (CK) for selected values are presented under table 2.2. The quantiles for the selected parameter values are also given under table 2.1.

Table 2.1: WOBIII-LoG Quantiles for selected parameter values

(a,b, β , λ)					
u	(0.9,3.5,5.5,4.5)	(0.2,1.3,5.2,3.3)	(1.3,1.2,4.5,2.5)	(3.1,4.5,3.5,2.5)	(1.2,2.5,5.5,0.5)
0.1	0.5386	1.6841	0.3263	0.3324	3.0381
0.2	0.5151	1.5168	0.2862	0.3125	2.8764
0.3	0.4968	1.3855	0.2548	0.2973	2.7498
0.4	0.4801	1.2651	0.2262	0.2837	2.6343
0.5	0.4636	1.1451	0.1977	0.2705	2.6343
0.6	0.4460	1.0172	0.1674	0.2568	2.3978
0.7	0.4261	0.8708	0.1328	0.2415	2.2590
0.8	0.4011	0.6859	0.0893	0.2228	2.0848
0.9	0.3635	0.4031	0.0229	0.1954	1.8211

Table 2.2: WOBIII-LoG Moments for selected parameter values

(a,b, β , λ)					
	(0.9,1.5,0.5,0.5)	(0.5,1.9,1.0,0.5)	(0.7,2.0,0.5,0.3)	(0.5,2.0,0.9,0.8)	(0.7,1.6,0.6,0.3)
$E(X)$	0.0343	0.0339	0.0189	0.0528	0.0199
$E(X^2)$	0.0227	0.0228	0.0125	0.0355	0.0132
$E(X^3)$	0.0170	0.0171	0.0094	0.0267	0.0099
$E(X^4)$	0.0135	0.0138	0.0075	0.0214	0.0079
$E(X^5)$	0.0113	0.0115	0.0063	0.0179	0.0066
SD	0.1468	0.1470	0.1104	0.1808	0.1134
CV	1.2742	2.3352	1.8557	3.4217	2.6971
CS	3.6492	4.6897	2.4648	3.6189	3.2820
CK	4.4561	5.7754	3.9514	5.3030	4.4636

2.4.2 Probability Weighted Moments (PWMs)

The $(s, m)^{th}$ Probability Weighted Moment (PWM) of X denoted by $\eta_{s,m}$ is

$$\eta_{s,m} = E(X^s(F(X))^m) = \int_{-\infty}^{\infty} x^s(F(x))^m f(x) dx.$$

By using equations (2.1) and (2.2), we can write

$$\begin{aligned}
f(x)(F(x))^m &= ab\beta \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^\beta \right) \\
&\times \left[-\log \left(1 - \left(1 + \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\
&\times \frac{\left(1 + \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})} \right)^{-a} \right)^{-b} \right)} \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})} \right)^{-a-1} \frac{g(x;\underline{\varphi})}{(\overline{G}(x;\underline{\varphi}))^2} \\
&\times \left(1 - \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^\beta \right) \right)^m.
\end{aligned}$$

Using the series expansion results (see **Appendix (B.2)** for details), we have

$$f(x)(F(x))^m = \sum_{w=0}^{\infty} \phi_{w+1} g_{w+1}(x; \underline{\varphi}), \quad (2.8)$$

where $g_w(x; \underline{\varphi}) = (w+1) \left(G(x; \underline{\varphi}) \right)^w g(x; \underline{\varphi})$ is the exponentiated-G (Exp-G) distribution with power parameter w and

$$\begin{aligned}
\phi_{w+1} &= ab\beta \sum_{q,k,z,i,p,m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k+p+w}}{k!} (q+1)^k \frac{\Gamma(z+1)}{(w+1)\Gamma(1)z!} b_{s,m} \\
&\times \binom{-a(i+1)-1}{p} \binom{\beta(k+1)-1}{m} \binom{a(i+1)+p-1}{w} \\
&\times \binom{-b(\beta(k+1)+m+s+z)-1}{i}.
\end{aligned} \quad (2.9)$$

Finally, the PWMs of the WOBIII-G family of distributions can be written as

$$\eta_{s,m} = \int_{-\infty}^{\infty} x^s \sum_{w=0}^{\infty} \phi_{w+1} g_{w+1}(x; \underline{\varphi}) dx = \sum_{w=0}^{\infty} \phi_{w+1} \int_{-\infty}^{\infty} x^s g_{w+1}(x; \underline{\varphi}) dx.$$

This shows that the $(s, m)^{th}$ PWMs of WOBIII-G family of distributions can be ob-

tained from the moments of the E-G distribution. The details of these results are presented under **Appendix (B.2)**

2.4.3 Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from the WOBIII-G family of distribution and suppose $X_{1:n} < X_{2:n}, \dots < X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l f(x) [F(x)]^{k+l-1}. \quad (2.10)$$

From the results derived under section 2.4.2, (also see **Appendix C.3** for details), we have

$$f(x)(F(x))^{k+l-1} = \sum_{w=0}^{\infty} b_{w+1} g_{w+1}(x; \underline{\varphi}), \quad (2.11)$$

where $g_{w+1}(x; \underline{\varphi}) = (w+1) (G(x; \underline{\varphi}))^w g(x; \underline{\varphi})$ is the exponentiated-G (Exp-G) distribution with power parameter $w+1$ and

$$\begin{aligned} b_{w+1} &= ab\beta \sum_{q,k=0}^{\infty} \sum_{z,i=0}^{\infty} \sum_{p,m,s=0}^{\infty} \binom{k+l-1}{q} (-1)^{q+k+p+w} (q+1)^k \frac{\Gamma(z+1)}{(w+1)\Gamma(1)z!} b_{s,m} \\ &\times \binom{-a(i+1)-1}{p} \binom{\beta(k+1)-1}{m} \binom{a(i+1)+p-1}{w} \\ &\times \binom{-b(\beta(k+1)+m+s+z)-1}{i}. \end{aligned}$$

If we substitute equation (2.11) into (2.10), we obtain

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \sum_{w=0}^{\infty} \binom{n-k}{l} (-1)^l b_{w+1} g_{w+1}(x; \underline{\varphi}), \quad (2.12)$$

where $g_{w+1}(x; \underline{\varphi}) = (w+1) (G(x; \underline{\varphi}))^w g(x; \underline{\varphi})$ is the exponentiated-G (Exp-G) distribution with power parameter $w+1 > 0$ and parameter vector $\underline{\varphi}$.

2.4.4 Rényi Entropy

An entropy gives a measure of variation for the random variable X and the commonly known entropies are Shannon entropy ([93]) and Rényi entropy ([90]). Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f(x; a, b, \beta, \underline{\varphi})]^v dx \right), v \neq 1, v > 0. \quad (2.13)$$

Note that $I_R(v)$ can be written as (see **Appendix D.4** for details):

$$I_R(v) = \frac{1}{1-v} \log \left[\sum_{w=0}^{\infty} \tau_w \exp((1-v)I_{REG}) \right],$$

for $v > 0, v \neq 1$, where $I_{REG} = \frac{1}{1-v} \log \left[\int_0^\infty \left(\left[\frac{w}{v} + 1 \right] (G(x; \underline{\varphi}))^{\frac{w}{v}} g(x; \underline{\varphi}) \right)^v dx \right]$ is the Rényi entropy of Exp-G distribution with power parameter $\frac{w}{v} + 1$, and

$$\begin{aligned} \tau_w &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s,z,i,p=0}^{\infty} (ab\beta)^v b_{s,m} \binom{\beta(k+v)-v}{m} (-1)^{k+p+w} v^k \\ &\times \binom{-a(v+i)-v}{p} \binom{-b(\beta(k+v)+m+s+z)-v}{i} \binom{a(v+i)-v+p}{w} \frac{\Gamma(z+v)}{\Gamma(v)z!} \\ &\times \frac{1}{\left[1 + \frac{w}{v}\right]^v}. \end{aligned}$$

2.4.5 Stochastic Ordering

In this section, the three most commonly applied orders for the WOBIII-G family of distributions namely; the usual stochastic order, the hazard rate order and the likelihood ratio order, (see Section 1.3.6 for definitions and [92] for more details) are derived.

Consider X_1 and X_2 as two independent random variables following $WOBIII - G(a, b, \beta_1, \underline{\varphi})$ and $WOBIII - G(a, b, \beta_2, \underline{\varphi})$ distributions, then the pdfs of X_1 and X_2 are

$$\begin{aligned}
f_1(x) &= ab\beta_1 \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta_1} \right) \\
&\times \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta_1-1} \\
&\times \frac{\left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right)} \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a-1} \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}.
\end{aligned}$$

and

$$\begin{aligned}
f_2(x) &= ab\beta_2 \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta_2} \right) \\
&\times \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta_2-1} \\
&\times \frac{\left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right)} \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a-1} \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}.
\end{aligned}$$

The ratio, $\frac{f_1(x)}{f_2(x)}$ takes the form

$$\begin{aligned}
\frac{f_1(x)}{f_2(x)} &= \frac{\beta_1 \exp(-z^{\beta_1}) z^{\beta_1-1}}{\beta_2 \exp(-z^{\beta_2}) z^{\beta_2-1}} \\
&= \frac{\beta_1}{\beta_2} \exp(-z^{\beta_1} + z^{\beta_2}) z^{\beta_1-\beta_2},
\end{aligned} \tag{2.14}$$

where

$$z = -\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{1 - G(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right).$$

If we differentiate equation (2.14) with respect to x , we get

$$\begin{aligned} \frac{d}{dx} \left(\frac{f_1(x)}{f_2(x)} \right) &= \frac{\beta_1}{\beta_2} [\exp(z^{\beta_2} - z^{\beta_1}) z' (\beta_2 z^{\beta_2-1} - \beta_1 z^{\beta_1-1}) z^{\beta_1-\beta_2}] \\ &\times (\beta_1 - \beta_2) z^{(\beta_1-\beta_2)-1} z' \exp(z^{\beta_2} - z^{\beta_1}), \end{aligned}$$

where $z' = \frac{\left(1 + \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})}\right)^{-a}\right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})}\right)^{-a}\right)^{-b}\right)} \left(\frac{G(x;\underline{\varphi})}{\overline{G}(x;\underline{\varphi})}\right)^{-a-1} \frac{g(x;\underline{\varphi})}{(\overline{G}(x;\underline{\varphi}))^2}$, and finally if $\beta_2 < \beta_1$, then

$\frac{d}{dx} \left(\frac{f_1(x)}{f_2(x)} \right) < 0$, and therefore, likelihood ratio order $X_1 <_{\ell_r} X_2$ exists. As a result, the random variables X_1 and X_2 are stochastically ordered.

2.4.6 Maximum Likelihood Estimation

Let $X \sim \text{WOBIII-G}(a, b, \beta, \underline{\varphi})$ and $\Delta = (a, b, \beta, \underline{\varphi})^T$ be the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the WOBIII-G family of distributions is given by

$$\begin{aligned} \ell_n(\Delta) &= n \ln(a\beta b) - \sum_{i=1}^n \left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^\beta \\ &+ (\beta - 1) \sum_{i=1}^n \ln \left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right] \\ &+ (-b - 1) \sum_{i=1}^n \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right) \\ &+ (-a - 1) \sum_{i=1}^n \ln \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right) - \sum_{i=1}^n \ln \left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \\ &+ \sum_{i=1}^n \ln(g(x_i; \underline{\varphi})) - 2 \sum_{i=1}^n \ln(\overline{G}(x_i; \underline{\varphi})). \end{aligned}$$

The elements of the score vector are given in **appendix E.5**. Through the use of numerical methods such as Newton-Raphson procedure, we can obtain the maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \underline{\varphi}})^T = \mathbf{0}$. The multivariate normal distribution $N_{q+3}(\mathbf{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\mathbf{0} = (0, 0, 0, \mathbf{0})^T$ and $J(\hat{\Delta})^{-1}$ is the observed Fisher information

matrix evaluated at $\hat{\Delta}$, which is critical in the construction of confidence intervals and regions for the model parameters.

2.5 Monte Carlo Simulations

In this section, we present the simulation results for the WOBIII-LoG distribution. Monte Carlo simulations were performed to evaluate the accuracy of the MLEs. We expect the estimates to converge towards the true values consistently as the sample size increases. We generate $N=1000$ random samples of size n and determine the MLEs for each sample. The bias and Root Mean Square Error (RMSE) are obtained from each sample size. The numerical results for the simulation runs are collected and presented under tables 2.3 and 2.4. From these results, for all the considered sets of parameters, we see that the RMSE and bias decreases as the sample size increases as expected which is consistent with the theoretical properties of convergence for the MLEs.

2.6 Applications

We present two data sets to demonstrate the flexibility of the WOBIII-LoG distribution compared to other models with the same number of parameters. In order to show the flexibility of WOBIII-LoG distribution, we compared its performance to other existing models namely, the Kumaraswamy Odd Lindley-Log logistic (KOLL-LoG) by [35], the New Modified Weibull (NMW) distribution introduced by [52], the Beta Generalized Lindley (BGL) distribution by [82], the Marshall Olkin-Kappa (MO-K) distribution by [67], the Kumaraswamy Weibull (KW) distribution by [46] and Generalized Weibull Log-Logistic (GWLLoG) distribution by [47]. The goodness-of-fit statistics that were computed and used to compare the model performances include -2 log-likelihood ($-2 \ln(L)$), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion ($AICC$). The pdfs of the non-nested models used for comparison are given under the **appendix I.9**.

Table 2.3: WOBIII-LoG distribution Simulation Results; Mean, Average Bias and RMSE

Parameter	Sample Size	Set I: a=2.3, b=1.4 , $\beta=0.7$, $\lambda=1.2$			Set II: a=1.4, b=1.3, $\beta=0.6$, $\lambda=2.3$		
		Mean	Bias	RMSE	Mean	Bias	RMSE
a	35	2.84774	0.54774	0.76384	1.84885	0.44885	3.46474
	50	2.74884	0.44884	0.48860	1.63889	0.23889	2.48857
	100	2.70940	0.40940	0.44073	1.60490	0.20490	2.11840
	200	2.58959	0.28959	0.36915	1.53886	0.13886	1.83997
	400	2.50949	0.20949	0.33092	1.47039	0.07039	0.67968
	800	2.35070	0.05070	0.21884	1.42100	0.02100	0.48388
	1000	2.30688	0.00688	0.19489	1.40090	0.00090	0.31674
b	35	1.89749	0.49749	0.68499	1.84776	0.54776	0.85500
	50	1.69706	0.29706	0.63749	1.72678	0.42678	0.73885
	100	1.61940	0.21940	0.58578	1.63856	0.33856	0.70477
	200	1.57690	0.17690	0.53749	1.48970	0.18970	0.65883
	400	1.55970	0.15970	0.51994	1.38548	0.08548	0.61784
	800	1.48960	0.08960	0.43775	1.34299	0.04299	0.57489
	1000	1.40887	0.00887	0.40048	1.30284	0.00284	0.50388
β	35	0.99397	0.29397	0.39858	0.93885	0.33885	0.47600
	50	0.87848	0.17848	0.36488	0.78499	0.18499	0.44859
	100	0.85499	0.15499	0.31094	0.75996	0.15996	0.42638
	200	0.78939	0.08939	0.23848	0.63996	0.03996	0.38549
	400	0.75940	0.05940	0.21995	0.61839	0.01839	0.35489
	800	0.72092	0.02092	0.14886	0.61038	0.01038	0.32775
	1000	0.71084	0.01084	0.11939	0.60821	0.00821	0.28177
λ	35	1.79846	0.59846	0.74848	2.88795	0.58795	0.83774
	50	1.69377	0.49377	0.62774	2.48589	0.18589	0.80588
	100	1.48388	0.28388	0.60377	2.38849	0.08849	0.78499
	200	1.42785	0.22785	0.58677	2.35377	0.05377	0.74889
	400	1.29886	0.09886	0.53883	2.31929	0.01929	0.64884
	800	1.23849	0.03849	0.42775	2.30688	0.00688	0.36044
	1000	1.20179	0.00179	0.34266	2.30160	0.00160	0.31088

Table 2.4: WOBIII-LoG distribution Simulation Results; Mean, Average Bias and RMSE

Parameter	Sample Size	Set III: $a=2.1, b=1.3, \beta=0.6, \lambda=1.6$			Set IV: $a=1.3, b=0.9, \beta=2.2, \lambda=1.2$		
		Mean	Bias	RMSE	Mean	Bias	RMSE
a	35	2.53754	0.43754	0.68499	1.84979	0.54979	0.39425
	50	2.43854	0.33854	0.60377	1.79499	0.49499	0.24627
	100	2.35677	0.25677	0.41887	1.63960	0.33960	0.21885
	200	2.31665	0.21665	0.35886	1.58796	0.28796	0.19377
	400	2.30066	0.20066	0.28849	1.39586	0.09586	0.15299
	800	2.17869	0.07869	0.23998	1.35048	0.05048	0.12939
	1000	2.10065	0.00065	0.21774	1.30849	0.00849	0.09948
b	35	1.95857	0.65857	0.73582	1.75494	0.85494	0.64886
	50	1.93759	0.63759	0.70368	1.58384	0.68384	0.52836
	100	1.87485	0.57485	0.65883	1.37589	0.47589	0.50883
	200	1.81460	0.51460	0.52775	1.03738	0.13738	0.48362
	400	1.64860	0.34860	0.49048	0.99937	0.09937	0.42749
	800	1.30576	0.00576	0.45489	0.95388	0.05388	0.39847
	1000	1.30211	0.00211	0.20478	0.91087	0.01087	0.23038
β	35	0.85899	0.25899	0.47848	2.84600	0.64600	0.46639
	50	0.81126	0.21126	0.34665	2.63288	0.43288	0.36875
	100	0.75869	0.15869	0.31877	2.43855	0.23855	0.34388
	200	0.71649	0.11649	0.26488	2.41904	0.21904	0.29467
	400	0.69848	0.09848	0.23849	2.33495	0.13495	0.24804
	800	0.63996	0.03996	0.21065	2.29940	0.09940	0.20840
	1000	0.60595	0.00595	0.10864	2.21044	0.01044	0.18038
λ	35	1.93737	0.33737	0.84995	1.49955	0.29955	0.48359
	50	1.91079	0.31079	0.73939	1.40360	0.20360	0.37468
	100	1.82744	0.22744	0.62649	1.33959	0.13959	0.32039
	200	1.78294	0.18294	0.60348	1.29299	0.09299	0.29076
	400	1.77929	0.17929	0.56994	1.24740	0.04740	0.23050
	800	1.64937	0.04937	0.38920	1.23095	0.03095	0.18072
	1000	1.60894	0.00894	0.24094	1.20995	0.00995	0.12308

2.6.1 Growth Hormone Data

The data consists of the estimated time since growth hormone medication until the children reached the target age. The dataset was analyzed by [11] to demonstrate the applicability of the exponentiated power Lindley power series (EPLPS) class of distributions. The dataset is: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

Estimates of the parameters of WOBIII-LoG distribution and other competing models with (standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W^* , A^* , Kolmogorov-Smirnov (K-S) and p-values are presented under Table 2.5. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 2.5.

Table 2.5: The models estimates for Growth Hormone Data

Model	Estimates				Statistics							
	a	b	β	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
WOBIII-LoG	0.6985 (0.1183)	38.1163 (26.4480)	0.6053 (0.1326)	1.2416 (0.0672)	156.1368	164.1368	165.4701	170.3582	0.0389	0.2821	0.0943	0.9143
KOLLLoG	9.1324×10^3 (5.7564×10^{-6})	8.6785×10^3 (4.1336×10^{-7})	7.1577 (5.2293×10^{-2})	3.1726×10^{-2} (3.8063×10^{-3})	162.0595	170.0596	171.3929	176.2809	0.1255	0.7996	0.1329	0.5666
NMW	0.0006 (0.0014)	2.8548 (0.8086)	4.0927 (0.7964)	0.0018 (0.0021)	168.0796	176.0796	177.413	182.301	0.0551	0.3873	0.1618	0.3187
BGL	1.1079×10^{-1} (8.9210×10^{-3})	1.7823×10^{-8} (2.0558×10^{-6})	3.0140×10^{-1} (8.5445×10^{-4})	1.0100×10^1 (2.2384×10^{-5})	346.975	354.9765	356.3098	361.1979	0.0676	0.4631	0.5083	2.785×10^{-8}
MO-K	5.6571 (3.7007×10^1)	6.2245×10^{-1} (4.0718)	3.7305×10^3 (6.2911×10^{-2})	4.4175×10^{-1} (1.4705×10^{-1})	158.6262	166.6262	167.9595	172.8476	0.0562	0.4076	0.0978	0.8908
KW	9.8285×10^1 (2.2442×10^{-7})	3.3577×10^6 (4.5271×10^{-13})	6.0563×10^3 (4.9683×10^{-10})	6.3797×10^{-2} (5.0671×10^{-4})	163.6781	171.6781	173.0114	177.8995	0.1475	0.9294	0.1416	0.4839
GWLLoG	0.2962 (0.5048)	2.7285 (1.3142)	2.34793 (1.6100)	1.48310 (0.7834)	159.4803	167.4803	168.8137	173.7017	0.0863	0.5686	0.1158	0.7352

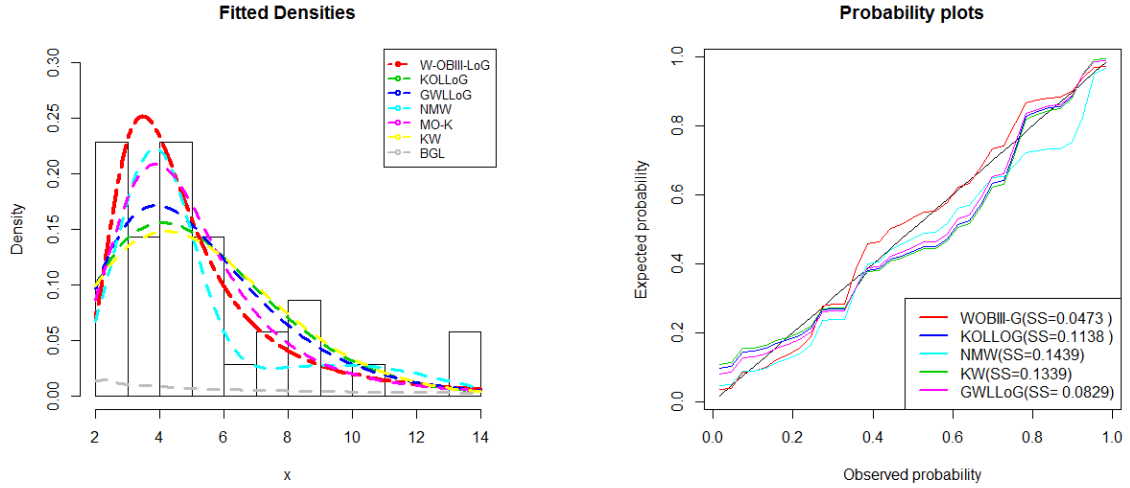


Figure 2.5: Fitted Density and Probability Plots for the Growth Hormone Data

From Figure 2.5 above, it is clear that the proposed model provides better fits compared to other non-nested models used for comparison. We note that the fitted density of the WOBIII-LoG distribution remain closer to the sample histogram and similarly the fitted probability plot is also close to the diagonal line. This shows that the model is consistent in providing a better fit, for both skewed and symmetric data, than the other competing models used for comparison. From the results presented under Table 2.5, it is also shown that the WOBIII-LoG distribution gives a better fit than the competing models due to its smallest values for the goodness-of-fit statistics: W^* , A^* , $K - S$ and a higher p-value.

2.6.2 Tuborcharger Data

The second data set ($n = 40$) below is from [108] and it represents the time to failure (103h) of turbocharger of one type of engine. The data are: 1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0, 8.7, 8.8, 9.0. Similarly, Table 2.6 presents the results on the performance of WOBIII-LoG model in comparison to other existing non-nested models having equal number of parameters. The associated observed probability plots and fitted density curves are given under Figure 2.6.

Table 2.6: The Models estimates for Tuborcharger Data

Model	Estimates				Statistics							
	a	b	β	λ	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
WOBIII-LoG	1.1546 (2.2116×10^{-5})	7.223×10^{-1} (8.7960×10^{-3})	6.735×10^1 (8.4971×10^{-6})	1.4499×10^{-2} (1.7813×10^{-3})	160.7012	168.7012	169.844	175.4567	0.0350	0.2475	0.0877	0.9176
KOLLLoG	7.3457×10^{-1} (9.3756×10^{-2})	1.3032×10^2 (1.0608×10^{-6})	2.8768×10^{-4} (1.0537×10^{-4})	2.6842 (2.6470×10^{-2})	164.983	172.983	174.1258	179.7385	0.0757	0.5644	0.1064	0.7553
NMW	2.4811×10^{-3} (1.0357×10^{-3})	2.7395 (2.0757×10^{-1})	2.4983×10^1 (5.1494×10^{-19})	1.7696×10^1 (7.3026×10^{-19})	285.3333	293.3323	294.4751	300.0878	0.0926	0.5954	0.6354	1.876×10^{-14}
BGL	1.1793×10^{-1} (8.5453×10^{-3})	9.6058×10^{-8} (2.1485×10^{-6})	3.0140×10^{-1} (7.6072×10^{-4})	1.0100×10^{01} (2.4583×10^{-5})	407.2801	415.2771	416.4199	422.0326	0.2630	1.6888	0.4985	4.645×10^{-9}
MO-K	1.3453 (1.9822)	4.8001 (0.6596)	0.0014 (0.0016)	23.2131 (0.0713)	177.3219	185.3219	186.4647	192.0774	0.2138	1.4053	0.1447	0.3718
KW	2.1372×10^2 (4.7822×10^{-8})	5.4862×10^6 (1.2246×10^{-13})	1.3227×10^4 (1.8364×10^{-10})	8.5648×10^{-2} (3.2614×10^{-4})	166.7622	174.7622	175.9051	181.5177	0.0988	0.7164	0.1128	0.6885
GWLLoG	27.73470 (0.5048)	0.4412 (1.3142)	16.3149 (1.6100)	8.7795 (0.7834)	164.9742	172.9742	174.117	179.7297	0.0771	0.5741	0.1077	0.7415

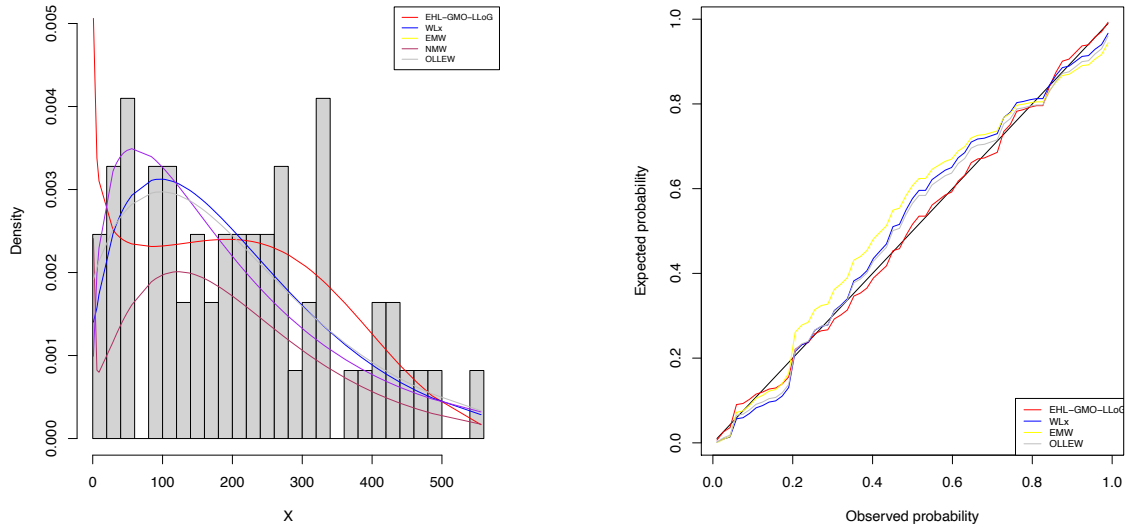


Figure 2.6: Fitted Density and Probability Plots for the Tuborcharger Data

Figure 2.6 shows that the model fits the data well compared to other competing non-nested models. The goodness-of-fit statistics W^* , A^* and K-S presented under Table 2.6 clearly shows the superiority of the WOBIII-LoG distribution over other comparison models. Additionally, the values of AIC and BIC also show that the WOBIII-LoG distribution is better than these competing equal parameter models.

2.7 Conclusions

A new family of distributions called the Weibull Odd Burr III-G (WOBIII-G) has been developed using the T-X transformation technique and studied in details. Its various structural properties have been derived. We applied WOBIII-LoG distribution as a special case to real life dataset to demonstrate its flexibility and goodness-of-fit. The proposed model has been found to provide the best fit in modelling the real life datasets compared to other non-nested models with equal number of parameters.

A similar technique was also used to generate the Weibull Exponentiated Half Logistic-G model, details of which are omitted in this thesis.

Chapter 3

The New Generalized Odd Burr III-G Family of Distributions with Applications

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3.1 Introduction

In this chapter, a new family of distributions namely the Ristić-Balakrishnan Odd Burr III-G (RBOBIII-G) family of distributions is being generated. We obtain some mathematical and statistical properties of this new family of distributions such as hazard and reverse hazard functions, quantile function, moments and incomplete moments, moment generating functions, Rényi entropy, order statistics and stochastic ordering. The model parameters are estimated using maximum likelihood estimation technique. Finally, the efficacy and importance of these newly developed families of distributions is illustrated via simulation experiments. We fit some of the special cases under these models to some real life data sets and compare the results with other existing non-nested models having the same number of parameters. The details on the development of this new family of distributions are given below.

3.2 The Ristić-Balakrishnan Odd Burr III-G Model

Consider the cumulative distribution function (cdf) and probability density function (pdf) of Burr III distribution (see [29]), given by

$$F(x; \alpha, \beta) = (1 + x^{-\alpha})^{-\beta} \quad (3.1)$$

and

$$f(x; \alpha, \beta) = \alpha\beta x^{-\alpha-1}(1 + x^{-\alpha})^{-\beta-1}, \quad x > 0, \quad \alpha, \beta > 0, \quad (3.2)$$

respectively, where α and β are both shape parameters. [12] derived a new family of distributions, namely, Odd Burr III-G by replacing x with the odds $G(x)/[1 - G(x)]$ and defined its cdf by

$$F_{OB-G}(x; \alpha, \beta, \psi) = \int_0^{\frac{G(x)}{1-G(x)}} \alpha\beta t^{-\beta-1}(1 + t^{-\alpha})^{-\beta-1} dt = \left\{ 1 + \left(\frac{1 - G(x; \psi)}{G(x; \psi)} \right)^\alpha \right\}^{-\beta} \quad (3.3)$$

and the corresponding pdf is given by

$$f_{OB-G}(x; \alpha, \beta, \psi) = \alpha\beta \frac{[1 - G(x; \psi)]^{\alpha-1}}{G(x; \psi)^{\alpha+1}} \left[1 + \left(\frac{1 - G(x; \psi)}{G(x; \psi)} \right)^\alpha \right]^{-\beta-1} g(x; \psi), \quad (3.4)$$

where $G(x; \psi)$ and $g(x; \psi)$ is the cdf and pdf respectively of any baseline distribution, and ψ is the vector of parameters. In the remaining part of this section, we present the RBOB III-G model by combining the Odd Burr III-G distribution with the Gamma generator proposed by [91] having the cdf and pdf given by

$$F_{RB}(x; \delta) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log(G(x))} t^{\delta-1} e^{-t} dt, \quad \delta > 0 \quad (3.5)$$

and

$$f_{RB}(x; \delta) = \frac{1}{\Gamma(\delta)} [-\log(G(x))]^{\delta-1} g(x), \quad x \in \mathbb{R}, \quad (3.6)$$

respectively.

If we substitute $G(x) = F_{OB-G}(x; \alpha, \beta, \psi)$ in equations (3.5) and (3.6), then we get a new extended generator called Ristić-Balakhrishnan Odd Burr III-G (RBOB III-G) having the cdf and pdf given

$$\begin{aligned} F_{RBOBIII-G}(x; \alpha, \beta, \delta, \psi) &= 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log \left(\left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right)^{-\beta} \right)} t^{\delta-1} e^{-t} dt \\ &= 1 - \frac{\gamma \left(\beta \log \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right), \delta \right)}{\Gamma(\delta)} \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} f_{RBOBIII-G}(x; \alpha, \beta, \delta, \psi) &= \frac{\alpha \beta^\delta (1 - G(x; \psi))^{\alpha-1}}{\Gamma(\delta) G(x; \psi)^{\alpha+1}} \left[\log \left(1 + \left(\frac{1 - G(x; \psi)}{G(x; \psi)} \right)^\alpha \right) \right]^{\delta-1} \\ &\times \left(1 + \left(\frac{1 - G(x; \psi)}{G(x; \psi)} \right)^\alpha \right)^{-\beta-1} g(x; \psi), \end{aligned} \quad (3.8)$$

respectively, for $\alpha, \beta, \delta > 0$ and parameter vector ψ . The parameters α and β are two positive shape parameters representing different patterns of the RBOB III-G family of distributions. The other additional parameters are from the baseline distribution and their role is to enhance flexibility of these newly family of distributions in modelling any form of lifetime data.

3.2.1 Series Expansion for RBOB III-G Density Function

This section presents series expansion of the RBOB III-G density function.

Let $y = 1 - \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right)^{-\beta}$, then by series representations $-\log(1 - y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, we have

$$\left[-\log(1 - y) \right]^{\delta-1} = y^{\delta-1} \left[\sum_{m=0}^{\infty} \binom{\delta-1}{m} y^m \left(\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^m \right],$$

and using the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s \right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \quad (3.9)$$

where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^s [m(l+1) - s] a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (see [58]), and applying the generalized binomial series representations,

$$(1+z)^{-\beta-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\beta+j+1)}{\Gamma(\beta+1)j!} z^j \quad (3.10)$$

and

$$(1-z)^{k-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(k)}{\Gamma(k-j)\Gamma(j+1)} z^j, \quad (3.11)$$

for $|z| < 1$, $\beta > 0$ and $k > 0$, we can write the RBOB III-G pdf as

$$\begin{aligned} f_{RBOBIII-G}(x; \alpha, \beta, \delta, \psi) &= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} y^{m+s+\delta-1} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \\ &\times \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta-1} g(x;\psi) \\ &= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \\ &\times \left(1 - \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta}\right)^{m+s+\delta-1} \\ &\times \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta-1} g(x;\psi) \\ &= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \frac{(-1)^j \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\ &\times \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta(j+1)-1} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} g(x;\psi) \\ &= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j+i} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\ &\times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} [1 - (1-G(x;\psi))]^{-\alpha(i+1)-1} \\ &\times [1 - G(x;\psi)]^{\alpha(i+1)-1} g(x;\psi) \\ &= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j+i} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\ &\times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} \\ &\times [1 - G(x;\psi)]^{k+\alpha(i+1)-1} g(x;\psi) \end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k,p=0}^{\infty} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} b_{s,m} \\
&\times \frac{(-1)^{j+i+p} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j) \Gamma(j+1)} \frac{\Gamma(\beta(j+1)+i+1) i!}{\Gamma(\beta(i+1)+1)} \\
&\times \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1) k!} [G(x; \psi)]^p g(x; \psi) \\
&= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k,p=0}^{\infty} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} \\
&\times b_{s,m} \frac{(-1)^{j+i+p} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j) \Gamma(j+1)} [G(x; \psi)]^{p+1-1} g(x; \psi) \\
&\times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1) i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1) k!} \frac{p+1}{(p+1)} \\
&= \sum_{p=0}^{\infty} C_{p+1} f_{E-G}(x; \psi, p+1), \tag{3.12}
\end{aligned}$$

where

$$\begin{aligned}
C_{p+1} &= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k=0}^{\infty} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} \frac{(-1)^{j+i+p} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j) \Gamma(j+1)} \\
&\times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1) i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1) k!} \frac{b_{s,m}}{(p+1)}, \tag{3.13}
\end{aligned}$$

and $f_{E-G}(x; \psi, p+1) = (p+1)[G(x; \psi)]^{p+1-1} g(x; \psi)$ is the **Exponentiated-G (E-G)** pdf with power parameter vector $p+1$. The mathematical and statistical properties of the RBOB III-G family of distributions follows directly from those of the E-G distribution.

3.2.2 Sub-families of Distributions

Sub-families of the RBOB III-G family of distributions are presented below. If we fix some of the parameter values then we obtain the reduced form of the RBOB III-G family of distributions as sub-families.

(a) If $\delta = 1$, we obtain the baseline OB-G family of distributions having the pdf

$$f(x; \alpha, \beta, \psi) = \frac{\alpha\beta(1-G(x; \psi))^{\alpha-1}}{G(x; \psi)^{\alpha+1}} \left[1 + \left(\frac{G(x; \psi)}{1-G(x; \psi)} \right)^{-\alpha} \right]^{-\beta-1} g(x; \psi), \tag{3.14}$$

for $\alpha, \beta > 0$ (see [12] for details).

- (b) If $\alpha = \beta = \delta = 1$, we obtain a reduced RBOB III-G family of distributions having the pdf

$$f(x; \psi) = G(x; \psi)^{-2} \left[1 + \left(\frac{G(x; \psi)}{1 - G(x; \psi)} \right)^{-1} \right]^{-2} g(x; \psi). \quad (3.15)$$

- (c) If $\alpha = \beta = 1$, we obtain a reduced RBOB III-G family of distributions having the pdf

$$\begin{aligned} f(x; \delta, \psi) &= \frac{1}{\Gamma(\delta)G(x; \psi)^2} \left[-\log \left(1 + \left(\frac{G(x; \psi)}{1 - G(x; \psi)} \right)^{-1} \right)^{-1} \right]^{\delta-1} \\ &\times \left(1 + \left(\frac{G(x; \psi)}{1 - G(x; \psi)} \right)^{-1} \right)^{-2} g(x; \psi), \end{aligned} \quad (3.16)$$

for $\delta > 0$.

- (d) If $\alpha = \delta = 1$, we obtain a reduced RBOB III-G family of distributions having the pdf

$$f(x; \beta, \psi) = \beta G(x; \psi)^{-2} \left(1 + \left(\frac{G(x; \psi)}{1 - G(x; \psi)} \right)^{-1} \right)^{-\beta-1} g(x; \psi), \quad (3.17)$$

for $\beta > 0$.

- (e) If $\beta = 1$, we obtain a reduced RBOB III-G family of distributions having the pdf

$$\begin{aligned} f(x; \alpha, \delta, \psi) &= \frac{\alpha}{\Gamma(\delta)} \frac{(1 - G(x; \psi))^{\alpha-1}}{G(x; \psi)^{\alpha+1}} \left[-\log \left(1 + \left(\frac{G(x; \psi)}{1 - G(x; \psi)} \right)^{-\alpha} \right)^{-1} \right]^{\delta-1} \\ &\times \left(1 + \left(\frac{G(x; \psi)}{1 - G(x; \psi)} \right)^{-\alpha} \right)^{-2} g(x; \psi), \end{aligned} \quad (3.18)$$

for $\alpha, \delta > 0$.

3.2.3 Hazard, Reverse Hazard and Quantile Functions

This section presents the results on the hazard, reverse hazard and quantile functions for the RBOB III-G distribution. The hazard and reverse hazard functions for the RBOB III-G distribution are given by

$$h_{RBOBIII-G}(x) = \frac{\alpha\beta^\delta g(x; \psi) \frac{(1-G(x; \psi))^{\alpha-1}}{G(x; \psi)^{\alpha+1}} \left[\log \left\{ 1 + \left(\frac{1-G(x; \psi)}{G(x; \psi)} \right)^\alpha \right\} \right]^{\delta-1}}{\gamma \left\{ \beta \log \left(1 + \left(\frac{1-G(x; \psi)}{G(x; \psi)} \right)^\alpha \right), \delta \right\}} \times \left\{ 1 + \left(\frac{1-G(x; \psi)}{G(x; \psi)} \right)^\alpha \right\}^{-\beta-1} \quad (3.19)$$

and

$$\tau_{RBOBIII-G}(x) = \frac{\alpha\beta^\delta g(x; \psi) \frac{(1-G(x; \psi))^{\alpha-1}}{G(x; \psi)^{\alpha+1}} \left[\log \left\{ 1 + \left(\frac{1-G(x; \psi)}{G(x; \psi)} \right)^\alpha \right\} \right]^{\delta-1}}{\Gamma(\delta) - \gamma \left\{ \beta \log \left(1 + \left(\frac{1-G(x; \psi)}{G(x; \psi)} \right)^\alpha \right), \delta \right\}} \times \left\{ 1 + \left(\frac{1-G(x; \psi)}{G(x; \psi)} \right)^\alpha \right\}^{-\beta-1}, \quad (3.20)$$

respectively. Furthermore, the quantile function for the RBOB III-G family of distributions is also derived by solving the non-linear equation

$$F(x; \alpha, \beta, \delta, \psi) = 1 - \frac{\gamma \left\{ \beta \log \left(1 + \left(\frac{1-G(x; \psi)}{G(x; \psi)} \right)^\alpha \right), \delta \right\}}{\Gamma(\delta)} = u, \quad (3.21)$$

for $0 \leq u \leq 1$, that is,

$$\log \left(1 + \left(\frac{G(x; \psi)}{1-G(x; \psi)} \right)^{-\alpha} \right) = \frac{\gamma^{-1} [\Gamma(\delta)(1-u), \delta]}{\beta} \quad (3.22)$$

and

$$\frac{G(x; \psi)}{1-G(x; \psi)} = \left(e^{\frac{\gamma^{-1} [\Gamma(\delta)(1-u), \delta]}{\beta}} - 1 \right)^{-\frac{1}{\alpha}}. \quad (3.23)$$

We can finally write the quantile function for the RBOB III-G family of distributions as

$$Q_{RBOBIII-G}(u; \alpha, \beta, \delta, \psi) = G^{-1} \left[\frac{\left(e^{\frac{\gamma^{-1} [\Gamma(\delta)(1-u), \delta]}{\beta}} - 1 \right)^{\frac{-1}{\alpha}}}{1 + \left(e^{\frac{\gamma^{-1} [\Gamma(\delta)(1-u), \delta]}{\beta}} \right)^{\frac{-1}{\alpha}}} \right]. \quad (3.24)$$

3.3 Some Special Cases

Some special cases of various sub-models are derived in this section. The baseline distribution function $G(x; \psi)$ is changed to other flexible distributions restricting the parameter vector space to at most 2 component vector to avoid over parametrization.

3.3.1 RBOB III - Log Logistic (RBOB III-LLoG) Distribution

Consider the Log Logistic distribution as the baseline distribution with parameter $\lambda > 0$ having cdf and pdf $G(x; \lambda) = 1 - (1 + x^\lambda)^{-1}$ and $g(x; \lambda) = \lambda x^{\lambda-1} (1 + x^\lambda)^{-2}$ respectively. The cdf and pdf of RBOB III-LLoG distribution are given by

$$F_{RBOBIII-LLoG}(x; \alpha, \beta, \delta, \lambda) = 1 - \frac{\gamma \left\{ \beta \log \left(1 + \left(\frac{(1+x^\lambda)^{-1}}{1-(1+x^\lambda)^{-1}} \right)^\alpha \right), \delta \right\}}{\Gamma(\delta)} \quad (3.25)$$

and

$$\begin{aligned} f_{RBOBIII-LLoG}(x; \alpha, \beta, \delta, \lambda) &= \frac{\alpha \beta^\delta}{\Gamma(\delta)} \frac{((1+x^\lambda)^{-1})^{\alpha-1}}{(1 - (1+x^\lambda)^{-1})^{\alpha+1}} \lambda x^{\lambda-1} (1+x^\lambda)^{-2} \\ &\times \left[\log \left(1 + \left(\frac{(1+x^\lambda)^{-1}}{1-(1+x^\lambda)^{-1}} \right)^\alpha \right) \right]^{\delta-1} \\ &\times \left(1 + \left(\frac{(1+x^\lambda)^{-1}}{1-(1+x^\lambda)^{-1}} \right)^\alpha \right)^{-\beta-1}, \end{aligned} \quad (3.26)$$

respectively, for $\alpha, \beta, \delta, \lambda > 0$. The hazard rate function is given by

$$\begin{aligned}
h_{RBOBIII-LLoG}(x) &= \frac{\alpha\beta^\delta}{\Gamma(\delta)} \frac{((1+x^\lambda)^{-1})^{\alpha-1}}{(1-(1+x^\lambda)^{-1})^{\alpha+1}} \lambda x^{\lambda-1} (1+x^\lambda)^{-2} \\
&\times \left[\log \left(1 + \left(\frac{(1+x^\lambda)^{-1}}{1-(1+x^\lambda)^{-1}} \right)^\alpha \right) \right]^{\delta-1} \\
&\times \left(1 + \left(\frac{(1+x^\lambda)^{-1}}{1-(1+x^\lambda)^{-1}} \right)^\alpha \right)^{-\beta-1} \\
&\times \left(\gamma \left\{ \beta \log \left(1 + \left(\frac{(1+x^\lambda)^{-1}}{1-(1+x^\lambda)^{-1}} \right)^\alpha \right), \delta \right\} \right)^{-1}.
\end{aligned} \tag{3.27}$$

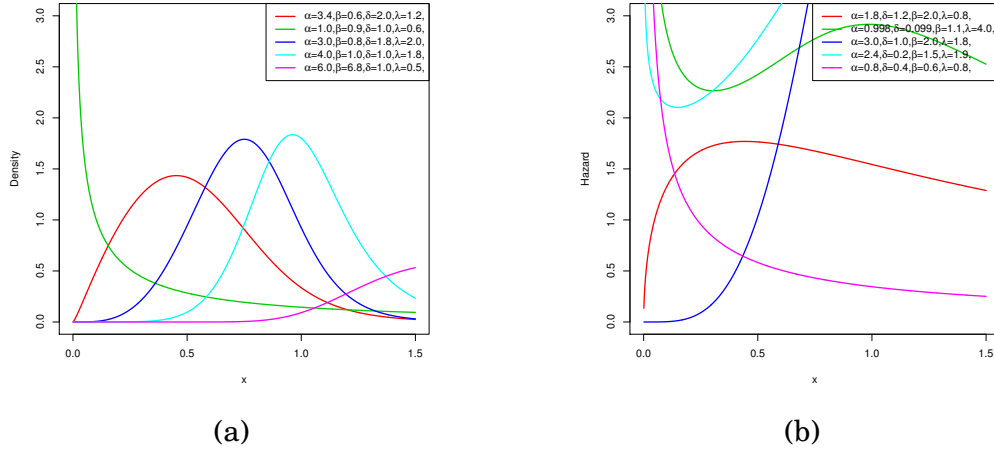


Figure 3.1: Plots of the pdf and hrf for RBOB III-LLoG Distribution

Figure 3.1 describe the flexibility of the RBOB III-LLoG distribution for some of the selected parameter values. The pdfs of the RBOB III-LLoG distribution assume various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the RBOB III-LLoG distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

Tables 3.1 and 3.2, give results on the quantiles and moments, respectively for the RBOB III-LLoG distribution for selected parameter values.

Table 3.1: RBOB III-LLoG quantiles for selected values

u	$(\alpha, \beta, \delta, \lambda)$				
	(0.2,1.2,2.1,1)	(0.1,1.2,2.2,1)	(0.2,1.2,2.3,1)	(0.2,1.2,2.4,1)	(0.1,1.2,2.5,1)
0.1	2.06150	2.18009	2.10302	2.11959	2.19788
0.2	2.24955	2.37087	2.29119	2.30802	2.39000
0.3	2.42861	2.55571	2.47151	2.48902	2.57667
0.4	2.60903	2.74396	2.65386	2.67232	2.76718
0.5	2.79908	2.94383	2.84642	2.86608	2.96978
0.6	3.00849	3.16533	3.05895	3.08010	3.19478
0.7	3.25245	3.42481	3.30692	3.32997	3.45879
0.8	3.56219	3.75578	3.62211	3.64773	3.79618
0.9	4.03111	4.25913	4.09970	4.12949	4.31049

Table 3.2: RBOB III-LLoG Moments for selected parameter values

	(0.1,0.2,0.2,0.5)	(0.7,1.0,1.5,0.5)	(0.2,2.0,2.0,0.2)	(3.4,4.0,0.5,0.9)	(0.2,2.0,1.5,0.5)
E(X)	0.00216	0.04406	0.00434	0.02813	0.01167
$E(X^2)$	0.00118	0.03545	0.00345	0.02451	0.00882
$E(X^3)$	0.00081	0.02963	0.00286	0.02169	0.00708
$E(X^4)$	0.00062	0.02543	0.00245	0.01943	0.00592
$E(X^5)$	0.00050	0.02227	0.00213	0.01759	0.00508
$E(X^6)$	0.00042	0.01981	0.00189	0.01606	0.00445
SD	0.00036	0.01783	0.00170	0.01477	0.00396
CV	0.00032	0.01621	0.00154	0.01366	0.00357
CS	0.00028	0.01486	0.00141	0.01271	0.00324
CK	0.00025	0.01372	0.00130	0.01188	0.00297

3.3.2 RBOB III-Kumaraswamy (RBOB III-K) Distribution

Let the Kumaraswamy distribution be the baseline distribution with paramters $a, b > 0$ having cdf and pdf $G(x) = 1 - (1 - x^a)^b, 0 \leq x \leq 1$, and $g(x) = abx^{a-1}(1 - x^a)^{b-1}$, respectively. The cdf and pdf of RBOB III-K distribution are given by

$$F_{RBOBIII-K}(x; \alpha, \beta, \delta, a, b) = 1 - \frac{\gamma \left\{ \beta \log \left(1 + \left(\frac{(1-x^a)^b}{1-(1-x^a)^b} \right)^\alpha \right), \delta \right\}}{\Gamma(\delta)}$$

and

$$\begin{aligned} f_{RBOBIII-K}(x; \alpha, \beta, \delta, a, b) &= \frac{\alpha \beta^\delta ab}{\Gamma(\delta)} \frac{((1-x^a)^b)^{\alpha-1}}{(1-(1-x^a)^b)^{\alpha+1}} x^{a-1} (1-x^a)^{b-1} \\ &\times \left\{ 1 + \left(\frac{(1-x^a)^b}{1-(1-x^a)^b} \right)^\alpha \right\}^{-\beta-1} \\ &\times \left[\log \left(1 + \left(\frac{(1-x^a)^b}{1-(1-x^a)^b} \right)^\alpha \right) \right]^{\delta-1}, \end{aligned} \quad (3.28)$$

respectively, for $\alpha, \beta, \delta, a, b > 0$. The hazard rate function is given by

$$\begin{aligned} h_{RBOBIII-K}(x) &= \frac{\alpha \beta^\delta ab}{\Gamma(\delta)} \frac{((1-x^a)^b)^{\alpha-1}}{(1-(1-x^a)^b)^{\alpha+1}} \left[\log \left(1 + \left(\frac{(1-x^a)^b}{1-(1-x^a)^b} \right)^\alpha \right) \right]^{\delta-1} \\ &\times x^{a-1} (1-x^a)^{b-1} \left\{ 1 + \left(\frac{(1-x^a)^b}{1-(1-x^a)^b} \right)^\alpha \right\}^{-\beta-1} \\ &\times \left(\gamma \left\{ \beta \log \left(1 + \left(\frac{(1-x^a)^b}{1-(1-x^a)^b} \right)^\alpha \right), \delta \right\} \right)^{-1}. \end{aligned} \quad (3.29)$$

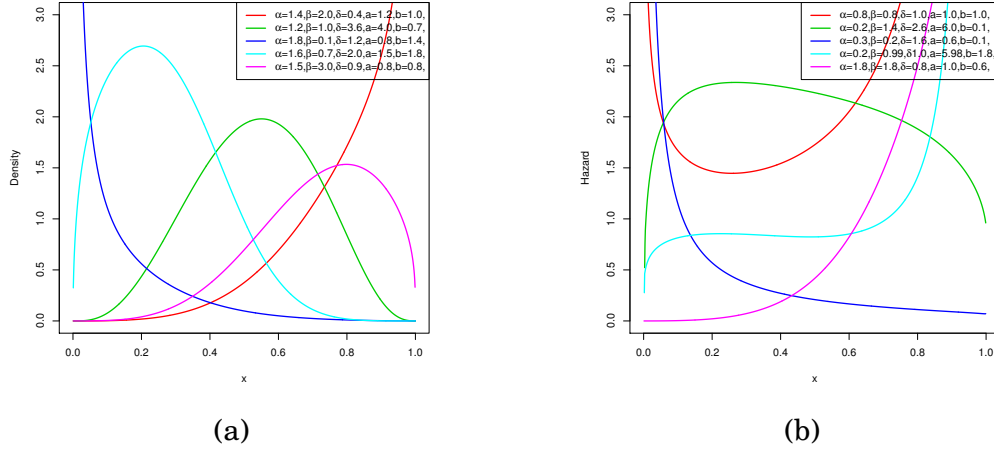


Figure 3.2: Plots of the pdf and hrf for RBOB III-K Distribution

Figure 3.2 shows the behaviour of RBOB III-K distribution plots for selected parameter values. The pdfs of the RBOB III-K distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. In addition, the plots of the hrf for the RBOB III-K distribution exhibit decreasing, increasing, bathtub, upside down bathtub and upside down bathtub followed by bathtub shapes.

Tables 3.3 and 3.4, presents results on the quantiles and moments, respectively for the RBOB III-K distribution on selected parameter values.

Table 3.3: RBOB III-K quantiles for selected values

$(\alpha, \beta, \delta, a, b)$					
u	(1.4,1.3,1.1,1.8,0.5)	(1.5,1.5,1.2,2,0.3)	(1.3,1.2,1.1,0.4)	(1.2,1.1,1.3,1.1,0.5)	(1.8,0.4,1.6,0.6,0.6)
0.1	0.98436	0.99806	0.99301	0.96344	0.50181
0.2	0.96354	0.99371	0.97525	0.91495	0.35243
0.3	0.93952	0.98718	0.94795	0.86016	0.25064
0.4	0.91213	0.97837	0.91067	0.79962	0.17432
0.5	0.88047	0.96673	0.86206	0.73312	0.11598
0.6	0.84323	0.95139	0.79916	0.65990	0.07205
0.7	0.79779	0.93076	0.71699	0.57851	0.04043
0.8	0.73910	0.90170	0.60505	0.48668	0.01931
0.9	0.65516	0.85701	0.43691	0.38025	0.00698

Table 3.4: RBOB III-K Moments for selected parameter values

	(0.1 0.2 0.2 0.5 0.1)	(0.3 0.2 0.2 0.1 0.2)	(0.1 0.2 0.3 0.1 0.5)	(0.9 0.1 0.2 0.1 0.2)	(0.1 0.1 0.2 0.1 0.1)
E(X)	0.03430	0.00957	0.02966	0.01010	0.00726
$E(X^2)$	0.02639	0.00740	0.02089	0.00858	0.00561
$E(X^3)$	0.02159	0.00607	0.01617	0.00749	0.00460
$E(X^4)$	0.01831	0.00516	0.01321	0.00665	0.00391
$E(X^5)$	0.01592	0.00449	0.01118	0.00599	0.00341
$E(X^6)$	0.01410	0.00398	0.00969	0.00545	0.00302
SD	0.01265	0.00358	0.00855	0.00501	0.00271
CV	0.01148	0.00325	0.00765	0.00463	0.00246
CS	0.01051	0.00297	0.00693	0.00431	0.00226
CK	0.00969	0.00274	0.00633	0.00402	0.00208

3.3.3 RBOB III - Beta (RBOB III-B) Distribution

Consider the Beta distribution as the baseline distribution with parameters $a, b > 0$ having the cdf and pdf $G(x) = I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1-t)^{b-1} dt$ and $g(x) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}$, for $0 < x < 1$, respectively. The cdf and pdf of RBOB III-B distribution are given by

$$F_{RBOBIII-B}(x; \alpha, \beta, \delta, a, b) = 1 - \frac{\gamma \left\{ \beta \log \left(1 + \left(\frac{1 - I_x(a, b)}{I_x(a, b)} \right)^\alpha \right), \delta \right\}}{\Gamma(\delta)}$$

and

$$\begin{aligned} f_{RBOBIII-B}(x; \alpha, \beta, \delta, a, b) &= \frac{\alpha \beta^\delta}{\Gamma(\delta) B(a, b)} \frac{(1 - I_x(a, b))^{\alpha-1}}{(I_x(a, b))^{\alpha+1}} x^a (1-x)^{b-1} \\ &\times \left\{ 1 + \left(\frac{1 - I_x(a, b)}{I_x(a, b)} \right)^\alpha \right\}^{-\beta-1} \\ &\times \left[\log \left(1 + \left(\frac{1 - I_x(a, b)}{I_x(a, b)} \right)^\alpha \right) \right]^{\delta-1}, \end{aligned} \quad (3.30)$$

respectively, for $\alpha, \beta, \delta, a, b > 0$. The hrf is given by

$$\begin{aligned} h_{RBOBIII-B}(x) &= \frac{\alpha \beta^\delta}{\Gamma(\delta) B(a, b)} \frac{(1 - I_x(a, b))^{\alpha-1}}{(I_x(a, b))^{\alpha+1}} \left[\log \left(1 + \left(\frac{1 - I_x(a, b)}{I_x(a, b)} \right)^\alpha \right) \right]^{\delta-1} \\ &\times x^a (1-x)^{b-1} \left\{ 1 + \left(\frac{1 - I_x(a, b)}{I_x(a, b)} \right)^\alpha \right\}^{-\beta-1} \\ &\times \left(\gamma \left\{ \beta \log \left(1 + \left(\frac{1 - I_x(a, b)}{I_x(a, b)} \right)^\alpha \right), \delta \right\} \right)^{-1}. \end{aligned} \quad (3.31)$$

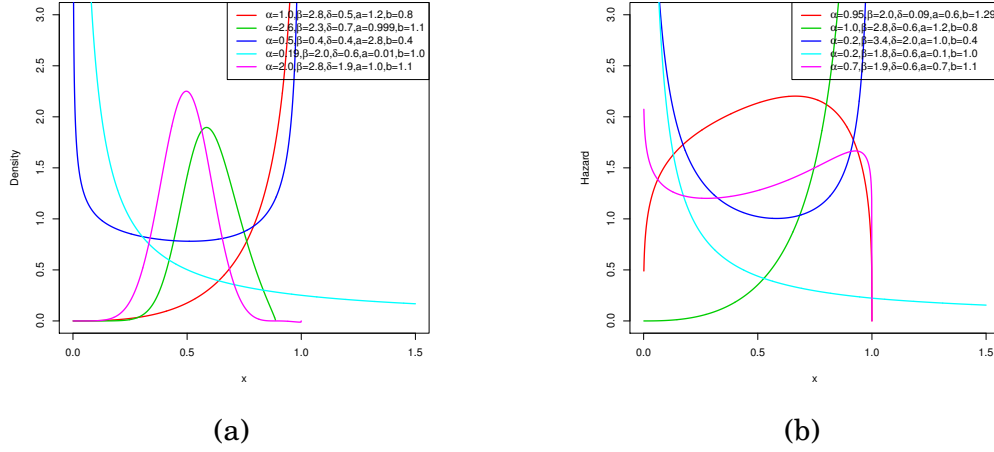


Figure 3.3: Plots of the pdf and hrf for RBOB III-B Distribution

Figure 3.3 gives good illustration on the flexible nature of the RBOB III-B distribution for selected parameter values. The pdfs of the RBOB III-B distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the RBOB III-B distribution exhibit decreasing, increasing, bathtub, upside down bathtub and upside down bathtub followed by bathtub shapes.

Tables 3.5 and 3.6, give results on the quantiles and moments, respectively for the RBOB III-B distribution with selected parameter values.

Table 3.5: RBOB III-B Quantiles for selected values

u	$(\alpha, \beta, \delta, a, b)$				
	(1.6, 0.7, 0.9, 0.5, 1.9)	(1.6, 0.7, 0.9, 0.6, 1.9)	(1.5, 0.7, 0.9, 0.5, 1.9)	1.7, 0.7, 0.9, 0.5, 1.9)	(1.6, 0.7, 0.9, 0.5, 1.8)
0.1	0.56175	0.45911	0.62379	0.51395	0.60242
0.2	0.29265	0.31488	0.30989	0.27811	0.30853
0.3	0.18229	0.23067	0.18781	0.17757	0.19047
0.4	0.11812	0.17081	0.11837	0.11789	0.12253
0.5	0.07567	0.12411	0.07355	0.07759	0.07800
0.6	0.04582	0.08571	0.04295	0.04853	0.04695
0.7	0.02442	0.05315	0.02181	0.02694	0.02487
0.8	0.00955	0.02556	0.00794	0.01125	0.00966
0.9	0.00101	0.00419	0.00073	0.00137	0.00102

3.4 Some Mathematical and Statistical Properties

This section presents some mathematical and statistical properties of the RBOB III-G family of distributions, namely; moments and generating functions, conditional

Table 3.6: RBOB III-B Moments for selected parameter values

	(0.1 0.2 0.2 0.5 0.7)	(0.2 1.5 0.5 0.2 2.0)	(0.1 0.3 0.2 0.2 0.5)	(0.3 0.1 0.4 0.2 0.5)	(0.2 0.1 0.3 0.2 0.4)
E(X)	0.02499	0.03390	0.03628	0.23681	0.14712
$E(X^2)$	0.01525	0.01204	0.02387	0.15496	0.10258
$E(X^3)$	0.01145	0.00620	0.01893	0.12265	0.08433
$E(X^4)$	0.00936	0.00379	0.01613	0.10438	0.07374
$E(X^5)$	0.00800	0.00255	0.01427	0.09229	0.06660
$E(X^6)$	0.00704	0.00184	0.01293	0.08356	0.06135
SD	0.00632	0.00139	0.01190	0.07688	0.05728
CV	0.00575	0.00109	0.01108	0.07157	0.05400
CS	0.00529	0.00087	0.01041	0.06720	0.05128
CK	0.00492	0.00072	0.00984	0.06354	0.04898

moments, entropy, distribution of order statistics, stochastic ordering and probability weighted moments.

3.4.1 Moments and Generating Functions

Consider $Y_{p+1} \sim \text{Exponentiated-G}(\psi; p+1)$, then the r^{th} moment can be derived using equation (3.12) as follows

$$\begin{aligned}
 E(X^r) &= \sum_{p=0}^{\infty} C_{p+1} E(Y_{p+1}^r) = \sum_{p=0}^{\infty} C_{p+1} \int_0^{\infty} y^r f_{E-G}(y; \psi, p+1) dy \\
 &= \sum_{p=0}^{\infty} C_{p+1} (p+1) \int_0^1 [Q_F(u; \psi)]^r u^p du.
 \end{aligned} \tag{3.32}$$

The moment generating function (mgf) for $|a| < 1$, is given by

$$M_X(a) = \sum_{p=0}^{\infty} C_{p+1} M_{Y_{p+1}}(a) = \sum_{p=0}^{\infty} \sum_{i=0}^{\infty} C_{p+1} \frac{a^i}{i!} E(Y_{p+1}^i), \tag{3.33}$$

where $M_{Y_{p+1}}(a)$ is the mgf of Y_{p+1} . The coefficient of variation (CV), skewness (CS) and kurtosis (CK) can be readily obtained and the results of these coefficients together with the first six moments for various baseline distributions are presented under section 3.3. Figures 3.4 and 3.5 below presents the 3D plots of skewness and kurtosis for the RBOB III-LLoG distributions for some selected parameter values.

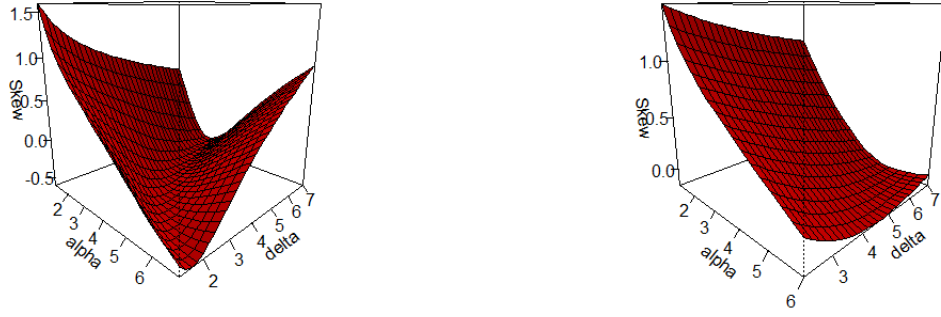


Figure 3.4: 3D Plots of Skewness for RBOB III-LLoG Distribution for some selected Parameter Values



Figure 3.5: 3D Plots of Kurtosis for RBOB III-LLoG Distribution for some selected Parameter Values

3.4.2 Conditional Moments

This subsection present results of the conditional moments for the RBOB III-G distribution. The r^{th} conditional moment of the RBOB III-G distribution is given by

$$\begin{aligned}
 E(X^r | X \geq a) &= \frac{1}{\overline{F}(a; \alpha, \beta, \delta, \psi)} \int_t^\infty x^r f(x; \alpha, \beta, \delta, \psi) dx \\
 &= \frac{1}{\overline{F}(a; \alpha, \beta, \delta, \psi)} \sum_{p=0}^{\infty} C_{p+1} E \left(Y_{p+1}^r I_{\{Y_{p+1}^r \geq t\}} \right), \quad (3.34)
 \end{aligned}$$

where

$$E\left(Y_{p+1}^r I_{\{Y_{p+1}^r \geq t\}}\right) = \int_t^\infty y^r f_{E-G}(y; \psi, p+1) dy = (p+1) \int_{G(u; \xi)}^1 [Q_G(u; \psi)]^r u^q du, \quad (3.35)$$

for $\alpha, \beta, \delta > 0$, and parameter vector ψ . The mean residual life function is given by $E(X-a|X > a) = E(X|X > a) - a = V_F(a) - a$, where $V_F(a)$ is referred to as the vitality function of the distribution function F . Note that the mean deviations, Bonferroni and Lorenz curves can be readily obtained from the conditional and incomplete moments.

3.4.3 Rényi Entropy

Entropy measures variation of uncertainty for a random variable X having the probability distribution $f(x)$. The Rényi Entropy (see [90]) which is an extension of Shannon Entropy, (see [93]) can be defined as

$$I_R(\nu) = \frac{1}{1-\nu} \log \left(\int_0^\infty \left[f(x; \alpha, \beta, \delta, \psi) \right]^\nu dx \right), \quad \nu \neq 1, \nu > 0. \quad (3.36)$$

Rényi Entropy tends to Shannon Entropy as $\nu \rightarrow 1$.

Let $y = 1 - \left\{ 1 + \left(\frac{1-G(x; \psi)}{G(x; \psi)} \right)^\alpha \right\}^{-\beta}$, then using the result on power series raised to a positive integer, with $c_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^\infty c_s y^s \right)^m = \sum_{s=0}^\infty d_{s,m} y^s, \quad (3.37)$$

where $d_{s,m} = (sc_0)^{-1} \sum_{l=1}^s [m(l+1) - s] c_l d_{s-l,m}$, and $d_{0,m} = c_0^m$, (see [58]), we have

$$\begin{aligned} \left[-\log(1-y) \right]^{\nu(\delta-1)} &= y^{\delta-1} \left[\sum_{m=0}^\infty \binom{\nu(\delta-1)}{m} y^m \left(\sum_{s=0}^\infty \frac{y^s}{s+2} \right)^m \right] \\ &= \sum_{m,s=0}^\infty \binom{\nu(\delta-1)}{m} d_{s,m} y^{m+s+\delta-1}, \end{aligned}$$

so that $[f(x; \alpha, \beta, \delta, \psi)]^\nu = f^\nu(x)$ can be written as

$$\begin{aligned}
f^\nu(x) &= \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \frac{(1-G(x;\psi))^{\nu(\alpha-1)}}{G(x;\psi)^{\nu(\alpha+1)}} \sum_{m,s=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \\
&\times \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)} \right)^{-\alpha} \right]^{-\nu(\beta+1)} y^{m+s+\delta-1} (g(x;\psi))^\nu \\
&= \sum_{m,s=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \frac{(1-G(x;\psi))^{\nu(\alpha-1)}}{G(x;\psi)^{\nu(\alpha+1)}} \\
&\times \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)} \right)^{-\alpha} \right]^{-\nu(\beta+1)} \\
&\times \left[1 - \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)} \right)^{-\alpha} \right]^{-\beta} \right]^{(m+s+\delta-1)} g(x;\psi)^\nu \\
&= \sum_{m,s=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \frac{(1-G(x;\psi))^{\nu(\alpha-1)}}{G(x;\psi)^{\nu(\alpha+1)}} \\
&\times \sum_{k=0}^{\infty} \binom{(m+s+\delta-1)}{k} (-1)^k \\
&\times \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)} \right)^{-\alpha} \right]^{-\beta k + \nu(-\beta-1)} g(x;\psi)^\nu \\
&= \sum_{m,s=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \sum_{k=0}^{\infty} \binom{(m+s+\delta-1)}{k} (-1)^k \\
&\times \sum_{i=0}^{\infty} \binom{-\beta k + \nu(-\beta-1)}{i} (1-G(x;\psi))^{\alpha(\nu+i)-\nu} \\
&\times G(x;\psi)^{-\alpha(i+\nu)-\nu} g(x;\psi)^\nu \\
&= \sum_{m,s,i,k=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \sum_{k=0}^{\infty} \binom{(m+s+\delta-1)}{k} (-1)^k \\
&\times \binom{-\beta k + \nu(-\beta-1)}{i} \frac{\Gamma(\alpha(i+\nu)+\nu+k)}{\Gamma(\alpha(i+\nu)+\nu)k!} G(x;\psi)^k g(x;\psi)^\nu.
\end{aligned}$$

(3.38)

Now,

$$\begin{aligned}
\int_0^\infty f^\nu(x; \alpha, \beta, \delta, \psi) dx &= \sum_{m,s,i,k=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \\
&\times \sum_{k=0}^{\infty} \binom{(m+s+\delta-1)}{k} (-1)^k \binom{-\beta k + \nu(-\beta-1)}{i} \\
&\times \frac{\Gamma(\alpha(i+\nu) + \nu + k)}{\Gamma(\alpha(i+\nu) + \nu) k!} \left[\int_0^\infty (G(x; \psi))^k g(x; \psi)^\nu dx \right]. \quad (3.39)
\end{aligned}$$

Finally, the Rényi entropy for the RBOB III-G family of distribution is given by

$$\begin{aligned}
I_R(\nu) &= \frac{1}{1-\nu} \log \left[\sum_{m,s,i,k=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \right. \\
&\times \sum_{k=0}^{\infty} \binom{(m+s+\delta-1)}{k} (-1)^k \binom{-\beta k + \nu(-\beta-1)}{i} \\
&\times \frac{\Gamma(\alpha(i+\nu) + \nu + k)}{\Gamma(\alpha(i+\nu) + \nu) k!} \frac{1}{\left[\frac{k}{\nu} + 1\right]^\nu} \int_0^\infty \left(\left[\frac{k}{\nu} + 1\right] g(x; \psi) G^{\frac{k}{\nu}}(x; \psi) \right)^\nu dx \left. \right] \\
&= (1-\nu)^{-1} \log \left[\sum_{k=0}^{\infty} W_k e^{(1-\nu)I_{REG}} \right], \quad (3.40)
\end{aligned}$$

where $I_{REG} = (1-\nu)^{-1} \log \int_0^\infty \left(\left[\frac{k}{\nu} + 1\right] g(x; \psi) G^{\frac{k}{\nu}}(x; \psi) \right)^\nu dx$ is the Rényi entropy of exponentiated-G distribution with power parameter $\frac{k}{\nu} + 1$ and

$$\begin{aligned}
W_k &= \sum_{m,s,i=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \sum_{k=0}^{\infty} \binom{(m+s+\delta-1)}{k} (-1)^k \binom{-\beta k + \nu(-\beta-1)}{i} \\
&\times \frac{\Gamma(\alpha(i+\nu) + \nu + k)}{\Gamma(\alpha(i+\nu) + \nu) k!} \frac{1}{\left[\frac{k}{\nu} + 1\right]^\nu}. \quad (3.41)
\end{aligned}$$

3.4.4 Distribution of Order Statistics

Consider the binomial expansion

$$(1 - F(x))^{n-i} = \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j [F(x)]^j, \quad (3.42)$$

then we can write the i^{th} order statistic pdf from the RBOB III-G pdf $f(x; \alpha, \beta, \delta, \psi) = f(x)$ as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!f(x)}{(i-1)!(n-i)!} [F(x)]^{i-1} [1-F(x)]^{n-i} \\ &= \frac{n!f(x)}{(i-1)!(n-i)!} \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} [F(x)]^{m+i-1}. \end{aligned} \quad (3.43)$$

From [66], the pdf of the i^{th} order statistic can also be written as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k} g_{r+k+1}^*(x), \quad (3.44)$$

where $g_{r+k+1}^*(x)$ is the exp-G density function with power parameter $r+k+1$,

$$m_{r,k} = \frac{n!(r+1)(i-1)!C_{r+1}}{r+k+1} \sum_{j=0}^{n-i} \frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!j!}, \quad (3.45)$$

and C_{r+1} is defined as C_{p+1} under equation (3.13). The quantities $f_{j+i-1,k}$ are recursively given by

$$f_{j+i-1,k} = (kC_0)^{-1} \sum_{m=1}^k [m(j+i) - k] C_m f_{j+i-1,k-m}. \quad (3.46)$$

3.4.5 Stochastic Ordering

In this section, we present the most commonly applied three orders for the RBOB III-G family of distributions namely; the usual stochastic order, the hazard rate order and the likelihood ratio order, (see [92] for more details).

Let X_1 and X_2 as two independent random variables following $RBOBIII - G(\alpha, \beta, \delta_1, \psi)$ and $RBOBIII - G(\alpha, \beta, \delta_2, \psi)$ distributions, then the pdfs of X_1 and X_2 are

$$\begin{aligned}
f_1(x) &= \frac{\alpha\beta^{\delta_1}}{\Gamma(\delta_1)} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \left[\log \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right) \right]^{\delta_1-1} \\
&\times \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right)^{-\beta-1} g(x;\psi)
\end{aligned}$$

and

$$\begin{aligned}
f_2(x) &= \frac{\alpha\beta^{\delta_2}}{\Gamma(\delta_2)} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \left[\log \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right) \right]^{\delta_2-1} \\
&\times \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right)^{-\beta-1} g(x;\psi).
\end{aligned}$$

The ratio, $\frac{f_1(x)}{f_2(x)}$ takes the form

$$\frac{f_1(x)}{f_2(x)} = \frac{\beta^{\delta_1}\Gamma(\delta_1)}{\beta^{\delta_2}\Gamma(\delta_2)} \left[\log \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right) \right]^{\delta_1-\delta_2}. \quad (3.47)$$

If we differentiate equation (3.47) with respect to x , we get

$$\begin{aligned}
\frac{d}{dx} \left(\frac{f_1(x)}{f_2(x)} \right) &= \frac{\alpha\beta_1^\delta\Gamma(\delta_2)}{\beta_2^\delta\Gamma(\delta_1)} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \left[\log \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right) \right]^{(\delta_1-\delta_2)-1} \\
&\times \frac{(\delta_2 - \delta_1)g(x;\psi)}{\left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)} \right)^\alpha \right)^\alpha},
\end{aligned}$$

and finally $\frac{d}{dx} \left(\frac{f_1(x)}{f_2(x)} \right) < 0$, if $\delta_2 < \delta_1$, and therefore, the likelihood ratio order $X_1 <_{\ell_r} X_2$ exists. As a result, the random variables X_1 and X_2 are stochastically ordered.

3.4.6 Probability Weighted Moments (PWMs)

A distribution function $F \equiv F(x) = P(X \leq x)$ may be characterized by the probability weighted moments defined by

$$M(q, l, w) = E(X^q(F(X))^l(1 - F(X))^w) = \int_{-\infty}^{\infty} x^q(F(x))^l(1 - F(x))^w f(x) dx,$$

where q, l and w are real numbers. If $l = w = 0$ and q is a non-negative integer, then $M_{q,0,0}$ represents the conventional moment about the origin of order q . If $M_{q,0,0}$ exists and X is a continuous function of F , then $M(q, l, w)$ exists for all the non-negative real numbers l and w .

Now, by using equations 3.7 and 3.8, and applying the generalized binomial expansion, we can write

$$\begin{aligned} (F(x))^l(1 - F(x))^w f(x) &= \sum_{z=0}^{\infty} \binom{w}{z} (-1)^z (F(x))^{l+z} f(x) = \sum_{z,r=0}^{\infty} \binom{l+z}{r} (-1)^{z+r} \binom{w}{z} f(x) \\ &\times \left[\frac{\gamma\left(\beta \log\left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^\alpha\right), \delta\right)}{\Gamma(\delta)} \right]^r \\ &= \sum_{z,r,q=0}^{\infty} \binom{w}{z} \binom{l+z}{r} (-1)^{z+r+q} \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \frac{(1 - G(x; \psi))^{\alpha-1}}{(G(x; \psi))^{\alpha+1}} \\ &\times \left[\left(-\log\left(1 + \left(\frac{1 - G(x; \psi)}{G(x; \psi)}\right)^\alpha\right) \right)^{-\beta} \right]^{q+2\delta-1} \\ &\times \left(1 + \left(\frac{1 - G(x; \psi)}{G(x; \psi)}\right)^\alpha \right)^{-\beta-1} g(x; \psi). \end{aligned}$$

Furthermore, by using the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s \right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \quad (3.48)$$

where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^s [m(l+1) - s] a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (see [58]), and applying the generalized binomial series representations,

$$(1+z)^{-\beta-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\beta+j+1)}{\Gamma(\beta+1) j!} z^j \quad (3.49)$$

and

$$(1-z)^{k-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(k)}{\Gamma(k-j) \Gamma(j+1)} z^j, \quad (3.50)$$

for $|z| < 1$, $\beta > 0$ and $k > 0$, we can write $(F(x))^l (1-F(x))^w f(x) = \Omega$, as

$$\begin{aligned} \Omega &= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z} \binom{l+z}{r} (-1)^{z+r+q} \binom{\delta-1}{m} b_{s,m} y^{m+s+\delta-1} \\ &\times \left(1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta-1} g(x;\psi) \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \\ &= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z} \binom{l+z}{r} (-1)^{z+r+q} \binom{\delta-1}{m} b_{s,m} \\ &\times \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \left(1 - \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta}\right)^{m+s+\delta-1} \\ &\times \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta-1} g(x;\psi) \\ &= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z} \binom{l+z}{r} (-1)^{z+r+q} \binom{\delta-1}{m} b_{s,m} \frac{(-1)^j \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j) \Gamma(j+1)} \\ &\times \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta(j+1)-1} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} g(x;\psi) \\ &= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j,i=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z} \binom{l+z}{r} (-1)^{z+r+q} \binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j+i} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j) \Gamma(j+1)} \\ &\times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1) i!} [1 - (1-G(x;\psi))]^{-\alpha(i+1)-1} \\ &\times [1 - G(x;\psi)]^{\alpha(i+1)-1} g(x;\psi) \\ &= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j,i,k=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z} \binom{l+z}{r} (-1)^{z+r+q} \binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j+i} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j) \Gamma(j+1)} \\ &\times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1) i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1) k!} \\ &\times [1 - G(x;\psi)]^{k+\alpha(i+1)-1} g(x;\psi) \\ &= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j,i,k,p=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z} \binom{l+z}{r} (-1)^{z+r+q} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} b_{s,m} \\ &\times \frac{(-1)^{j+i+p} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j) \Gamma(j+1)} \frac{\Gamma(\beta(j+1)+i+1) i!}{\Gamma(\beta(i+1)+1)} \\ &\times \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1) k!} [G(x;\psi)]^p g(x;\psi) \end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j,i,k,p=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z} \binom{l+z}{r} (-1)^{z+r+q} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} \\
&\times b_{s,m} \frac{(-1)^{j+i+p} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j) \Gamma(j+1)} [G(x; \psi)]^{p+1-1} g(x; \psi) \\
&\times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} \frac{p+1}{(p+1)} \\
&= \sum_{p=0}^{\infty} C_{p+1}^{**} f_{E-G}(x; \psi, p+1), \tag{3.51}
\end{aligned}$$

where

$$\begin{aligned}
C_{p+1}^{**} &= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k=0}^{\infty} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} \frac{(-1)^{j+i+p} \Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j) \Gamma(j+1)} \\
&\times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} \frac{b_{s,m}}{(p+1)}, \tag{3.52}
\end{aligned}$$

and $f_{E-G}(x; \psi, p+1) = (p+1)[G(x; \psi)]^{p+1-1} f(x; \psi)$ is the Exponentiated-G (E-G) pdf with power parameter vector $p+1$. Finally, the PWMs of the RBOB III-G family of distributions can be written as

$$M(q, l, w) = \int_{-\infty}^{\infty} x^q \sum_{p=0}^{\infty} C_{p+1}^{**} f_{E-G}(x; \psi, p+1) dx = \sum_{p=0}^{\infty} C_{p+1}^{**} \int_{-\infty}^{\infty} x^q f_{E-G}(x; \psi, p+1) dx,$$

which shows that the $(q, l, w)^{th}$ PWMs of RBOB III-G family of distributions can be obtained from the moments of the Exp-G distribution.

3.5 Maximum Likelihood Estimation

Let $X \sim RBOBIII - G(\alpha, \beta, \delta, \psi)$ and $\Theta = (\alpha, \beta, \delta, \psi)^T$ be the vector of model parameters. The log-likelihood function $\ell_n = \ell_n(\Theta)$ from a random sample of size n is given by

$$\begin{aligned}
\ell_n(\Theta) &= n\ln(\alpha) + n\ln(\beta) - n\ln(\Gamma(\delta)) + (\alpha - 1) \sum_{i=1}^n \ln[1 - G(x; \psi)] \\
&- (\alpha + 1) \sum_{i=1}^n \ln[G(x; \psi)] + \sum_{i=1}^n \ln(g(x_i; \psi)) + (\delta - 1) \\
&\times \sum_{i=1}^n \ln \left[-\log \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{-\beta} \right] \\
&- (\beta + 1) \sum_{i=1}^n \ln \left[1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{\alpha} \right].
\end{aligned} \tag{3.53}$$

The elements of the score vector are given in **appendix F.6**. The maximum likelihood estimates (MLEs) of the parameters, denoted by $\hat{\Theta}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \psi})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. Note that from the multivariate normal distribution $N_{p+3}(\underline{0}, J(\hat{\Theta})^{-1})$, where $\underline{0} = (0, 0, 0, \underline{0})^T$ is the mean vector and $J(\hat{\Theta})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\Theta}$, we can construct confidence intervals and confidence regions for the individual model parameters. The $100(1 - \varphi)\%$ two-sided confidence intervals for α, β, δ and ψ_k are as follows;

$$\hat{\alpha} \pm Z_{\frac{\varphi}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\Theta})}, \quad \hat{\beta} \pm Z_{\frac{\varphi}{2}} \sqrt{I_{\beta\beta}^{-1}(\hat{\Theta})}, \quad \hat{\delta} \pm Z_{\frac{\varphi}{2}} \sqrt{I_{\delta\delta}^{-1}(\hat{\Theta})}, \quad \text{and} \quad \hat{\psi}_k \pm Z_{\frac{\varphi}{2}} \sqrt{I_{\psi_k\psi_k}^{-1}(\hat{\Theta})},$$

respectively, where $I_{\alpha\alpha}^{-1}(\hat{\Theta})$, $I_{\beta\beta}^{-1}(\hat{\Theta})$, $I_{\delta\delta}^{-1}(\hat{\Theta})$ and $I_{\psi_k\psi_k}^{-1}(\hat{\Theta})$ are the diagonal elements of $I_n^{-1}(\hat{\Theta}) = (nI(\hat{\Theta}))^{-1}$, and $Z_{\frac{\varphi}{2}}$ represents the standard normal $(\frac{\varphi}{2})^{th}$ percentile. Note that the asymptotic behaviour still holds when $I(\Theta)$, (which is the expected Fisher information matrix) is replaced by the observed information matrix evaluated at $\hat{\Theta}$.

3.6 Monte Carlo Simulations

In this section, we conduct simulation experiments to evaluate consistency of the maximum likelihood estimators for the RBOB III-LLoG distribution using different parameter values. The simulation study is repeated for $N = 1000$ times with sample size $n = 35, 50, 100, 200, 400, 800$ and 1000 . The simulation results (mean of the MLEs, average bias and root mean square error (RMSE)) are presented under Tables

3.9 and 3.10. The results shows that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, with Average Bias and RMSEs converging towards zero. The Average Bias and RMSEs are given by:

$$\text{Bias}(\hat{\theta}) = \frac{\sum_{i=1}^n \hat{\theta}_i}{n} - \theta \text{ and } \text{RMSE}(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^n (\hat{\theta}_i - \theta)^2}{n}},$$

respectively.

3.7 Applications

In this section, the RBOB III-LLoG distribution was applied to two real data sets to demonstrate its performance compared to other existing models. The first data set represents the lifetimes data relating to relief times (in minutes) for 20 patients receiving an analgesic as reported by [59] and is given by: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0. The second data relates to the effects of variations in airborne exposure on concentration of urinary metabolites studied by [70] and is given by: 1.5, 1.7, 2.1, 2.2, 2.4, 2.5, 2.6, 3.8, 3.8, 4.2, 4.3, 5.6, 6, 7, 7.5, 9.3, 9.9, 10.2, 10.6, 12.3, 12.9, 13.7, 14.1, 17.8, 27.6, 31, 42, 45.6, 51.9, 91.3, 131.8.

The RBOB III-LLoG distribution was compared to other existing models namely, the Burr-Weibull (BW) by [80], Kumaraswamy Weibull (KW) by [46], Weibull-Lomax (WLx) by [104], Beta Generalized Lindley (BGL) by [84] and New Modified Weibull (NMW) by [52] distributions. The goodness-of-fit statistics was used for comparisons, namely, -2 log-likelihood ($-2 \ln(L)$), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion ($AICC$). We obtained results on the Cramer-von Mises (W^*) and Anderson-Darling Statistics (A^*) described by [33], as well as Kolmogorov-Smirnov (KS) statistic and its p-value. Note that for the value of the log-likelihood function at its maximum (ℓ_n), the smaller value is preferred, and for AIC , $AICC$, BIC , and the goodness-of-fit statistics W^* , A^* and $K - S$, smaller values are also preferred. The results from two real lifetime data

sets are presented under Tables 3.7 and 3.8. We used the R software to compute estimates for model parameters and run goodness-of-fit tests. The pdfs of the non-nested models (BW, KW, WLx, BGL and NMW) used for comparisons are

$$f(x; \alpha, \beta, c, k) = e^{-\alpha x^\beta} [1 + x^c]^{-k} \left\{ \alpha x^{\beta-1} \beta + \frac{k c x^{c-1}}{(1 + x^c)} \right\}, \quad (3.54)$$

for $\alpha, \beta, c, k > 0$ and $x > 0$,

$$\begin{aligned} f(x; a, b, c, \lambda) &= abc \lambda^c x^{c-1} \exp\{-(\lambda x)^c\} [1 - \exp\{-(\lambda x)^c\}]^{a-1} \\ &\times \{1 - [1 - \exp\{-(\lambda x)^c\}]^a\}^{b-1}, \end{aligned} \quad (3.55)$$

for $a, b, c, \lambda > 0$ and $x > 0$,

$$\begin{aligned} f(x; a, b, \alpha, \beta) &= \frac{ab\alpha}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{b\alpha-1} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1} \\ &\times \exp\left\{-a \left\{1 + \left(\frac{x}{\beta}\right)^\alpha - 1\right\}^b\right\}, \end{aligned} \quad (3.56)$$

for $a, b, c, \lambda > 0$ and $x > 0$,

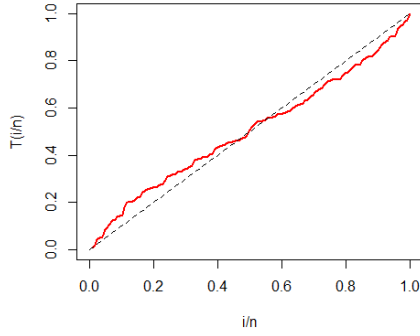
$$\begin{aligned} f(x; \alpha, \theta, a, b) &= \frac{\alpha \theta^2 (1+x) e^{-\theta x}}{B(a, b) (1+\theta)} \left\{1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}\right\}^{a\alpha-1} \\ &\times \left\{1 - \left\{1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}\right\}^\alpha\right\}^{b-1}, \end{aligned} \quad (3.57)$$

for $a, b, \alpha, \theta > 0$ and $x > 0$ and

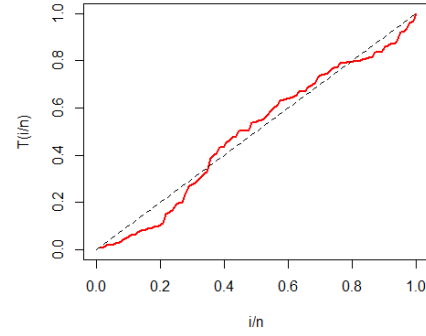
$$f(x; \alpha, \beta, \lambda, \gamma) = (\alpha \gamma x^{\gamma-1} e^{\alpha x^\gamma} + \lambda \beta x^{\lambda-1} e^{-\beta x^\gamma}) e^{-e^{\alpha x^\gamma} + e^{-\beta x^\gamma}}, \quad (3.58)$$

for $\alpha, \beta, \lambda, \gamma > 0$ and $x > 0$, respectively.

The TTT plots for relief times and urinary metabolites data sets are presented under Figures 3.6(a) and 3.6(b), respectively. These plots show that the empirical hazard rate functions for data sets is increasing and therefore the RBOB III-LLoG distribution is appropriate to fit these data. The TTT plots are given by plotting $T(i/n) = [\sum_{k=1}^n y_{k:n} + (n-r)y_{i:n}] / \sum_{k=1}^n y_{k:n}$ against (i/n) , where $i = 1, \dots, n$ and $y_{k:n} (k = 1, \dots, n)$ represent order statistics of the sample.



(a)

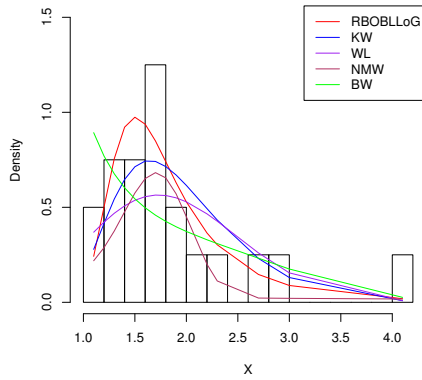


(b)

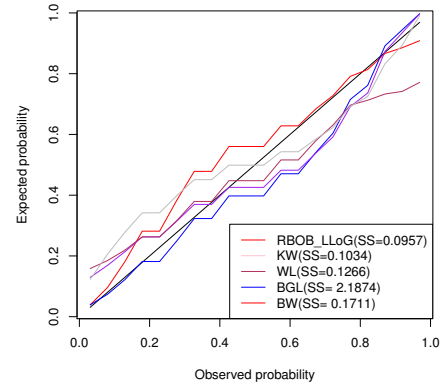
Figure 3.6: TTT Plots for Relief Times and Urinary Metabolites Data

Table 3.7: Parameter estimates and goodness-of-fit statistics for different models fitted in dataset relating to relief times (in minutes) for patients receiving analgesic

Model	Estimates				Statistics							
	α	δ	λ	β	$-2\log L$	AIC	CAIC	BIC	W^*	A^*	K-S	p-value
RBOB III-LLoG	2.2301 (0.6220)	0.8719 (0.8814)	2.0967 (0.6616)	7.4257 (2.8589)	30.8	38.8	41.5	42.8	0.03114	0.1956	0.1469	0.78
KW	a	b	c	λ	33.7	41.7	44.4	45.7	0.0743	0.4419	0.1795	0.54
	9.3780 (11.6562)	0.3073 (0.3462)	1.9780 (0.6181)	1.1543 (0.4181)								
BW	c	k	α	β	37.8	45.8	48.4	49.7	0.1087	0.6373	0.1918	0.45
	165.7300 (2.1×10^{-7})	0.0060 (0.0034)	0.0263 (0.0409)	3.4398 (0.9816)								
NMW	α	γ	λ	β	51.0	59.0	61.7	63.0	0.03811	0.3006	0.2286	0.25
	0.1255 (0.0612)	0.8021 (0.4008)	6.5457 (1.5533)	0.0149 (0.0152)								
WL	a	b	α	β	38.9	46.9	49.5	50.9	0.1506	0.8897	0.1766	0.56
	2806.0 (3.8×10^{-5})	5.8231 (4.0485)	0.1168 (0.0409)	0.3478 (0.9022)								
BGL	α	θ	a	b	162.5	170.5	173.2	174.5	0.0887	0.5230	0.5123	5.5×10^{-5}
	0.1203 (0.0127)	1.5×10^{-7} (4.3×10^{-6})	0.3016 (0.0013)	10.1 (3.5×10^{-5})								



(a)



(b)

Figure 3.7: Fitted Density and Probability Plots of the RBOB III-LLoG Distribution for Relief Times Data

The estimated covariance matrix is given by

$$\begin{bmatrix} 0.38695 & -0.52759 & 0.41157 & -0.57037 \\ -0.52759 & 0.77685 & -0.56115 & 1.26736 \\ 0.41157 & -0.56115 & 0.43775 & -0.60666 \\ -0.57037 & 1.26736 & -0.60666 & 8.17335 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [2.2301 \pm$

1.21923], $\delta \in [0.8719 \pm 1.72752]$, $\lambda \in [2.0967 \pm 1.29679]$ and $\beta \in [7.4257 \pm 5.60346]$.

Table 3.8: Parameter estimates and goodness-of-fit statistics for different models fitted to data relating to effects of variations in airborne exposure on concentration of urinary metabolites

Model	Estimates				Statistics							
	α	δ	λ	β	$-2 \log L$	AIC	CAIC	BIC	W^*	A^*	K-S	p-value
RBOB III-LLoG	1.9844 (0.1205)	0.9653 (1.3223)	0.5658 (0.4225)	6.7645 (3.0050)	233.0	241.0	242.5	246.7	0.0365	0.2754	0.0954	0.94
KW	a	b	c	λ	234.0	242.0	243.6	247.7	0.0488	0.3535	0.1188	0.77
	8.3694 (6.9477)	0.2140 (0.2990)	0.5854 (0.1549)	1.8193 (1.4783)								
BW	c	k	α	β	236.5	244.5	246.0	250.2	0.0824	0.5288	0.1331	0.64
	35.3400 (5.1×10^{-7})	0.0069 (0.0031)	0.0175 (0.0243)	1.0895 (0.2764)								
NMW	α	γ	λ	β	263.6	271.6	273.1	277.3	0.0633	0.5240	0.2488	0.04
	0.0907 (0.0899)	0.2647 (0.6904)	1.1684 (0.7031)	0.0356 (0.0506)								
WL	a	b	α	β	234.8	242.8	244.4	248.6	0.0506	0.3728	0.1038	0.89
	1339.4 (0.0059)	2.3647 (2.9203)	0.0173 (0.0343)	0.0065 (4.8302)								
BGL	α	θ	a	b	347.7	355.7	357.4	361.6	0.0667	0.4774	0.4679	2.54×10^{-6}
	0.1089 (0.0087)	3.4×10^{-8} (1.79×10^{-6})	0.3007 (0.0006)	10.1000 (2.46×10^{-6})								

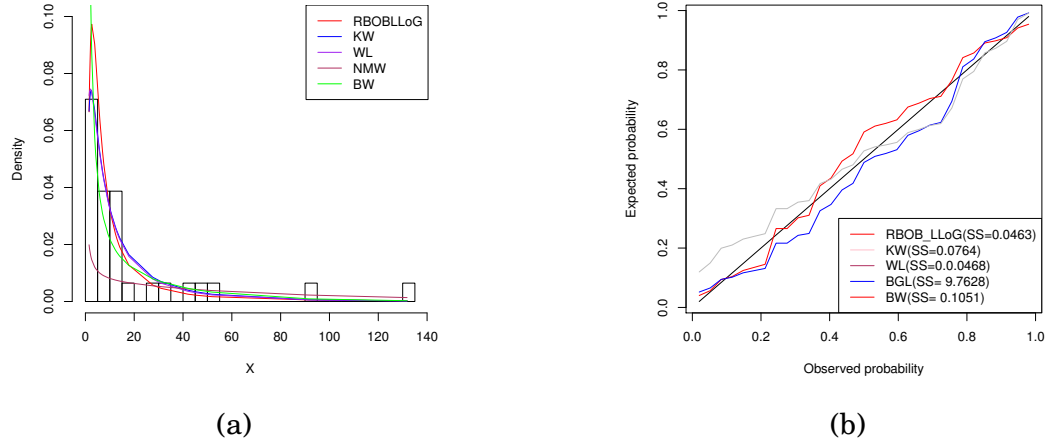


Figure 3.8: Fitted Density and Probability Plots of the RBOB III-LLoG Distribution for Concentration of Urinary Metabolites Data

The estimated covariance matrix is given by

$$\begin{bmatrix} 0.01450 & -0.15722 & 0.05089 & -0.27533 \\ -0.15722 & 1.75030 & -0.55154 & 3.30879 \\ 0.05089 & -0.55154 & 0.17852 & -0.96593 \\ -0.27533 & 3.30879 & -0.96593 & 9.02983 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [1.9844 \pm 0.23607]$, $\delta \in [0.9653 \pm 2.59306]$, $\lambda \in [0.5658 \pm 0.82813]$ and $\beta \in [6.7645 \pm 5.88973]$.

Table 3.9: Monte Carlo Simulation Results for RBOB III-LLoG Distribution; Mean, Average Bias and RMSE

Parameter	Sample Size	Set I: $\alpha=1.0, \beta=0.8, \delta=0.5, \lambda=1.0$			Set II: $\alpha=1.2, \beta=1.0, \delta=0.6, \lambda=1.5$		
		Mean	Bias	RMSE	Mean	Bias	RMSE
α	35	0.95664	-0.04336	0.69133	1.01516	-0.18484	2.35483
	50	0.96133	-0.03867	0.58158	1.03503	-0.16497	0.67562
	100	0.96136	-0.03864	0.34073	1.09029	-0.10971	0.58593
	200	0.96517	-0.03483	0.27915	1.11648	-0.08352	0.51709
	400	0.96678	-0.03322	0.22092	1.11722	-0.08278	0.51518
	800	1.00089	0.00089	0.17599	1.21571	0.01571	0.41114
	1000	1.00012	0.00012	0.15728	1.20051	0.00051	0.28453
β	35	0.51213	-0.28787	0.57845	1.73592	0.73592	0.60866
	50	0.55566	-0.24434	0.54865	1.33972	0.33972	0.60192
	100	0.63118	-0.16882	0.51090	1.23692	0.23692	0.54608
	200	0.67801	-0.12199	0.50636	1.15581	0.15581	0.53129
	400	0.70886	-0.09114	0.50509	1.10924	0.10924	0.52400
	800	0.80053	0.00053	0.37358	1.08134	0.08134	0.50346
	1000	0.80036	0.00036	0.25251	1.00872	0.00872	0.46234
δ	35	0.71279	0.21279	0.43846	0.98798	0.38798	0.57907
	50	0.71109	0.21109	0.42965	0.84478	0.24478	0.56681
	100	0.65505	0.15505	0.33578	0.71697	0.11697	0.53755
	200	0.60439	0.10439	0.26315	0.67810	0.07810	0.49036
	400	0.56590	0.06590	0.20094	0.65597	0.05597	0.41630
	800	0.55032	0.05032	0.15559	0.63246	0.03246	0.34346
	1000	0.50053	0.00053	0.13370	0.60091	0.00091	0.25730
λ	35	0.88513	-0.11487	0.47263	1.75370	0.25370	0.78269
	50	0.88689	-0.11311	0.41593	1.74089	0.24089	0.77227
	100	0.89647	-0.10353	0.39038	1.72322	0.22322	0.75097
	200	0.95618	-0.04382	0.34119	1.63949	0.13949	0.69160
	400	1.00809	0.00809	0.32204	1.61459	0.11459	0.55876
	800	1.00061	0.00061	0.31592	1.51687	0.01687	0.28940
	1000	1.00004	0.00004	0.27577	1.50056	0.00056	0.24761

Table 3.10: Monte Carlo Simulation Results for RBOB III-LLoG Distribution; Mean, Average Bias and RMSE

Parameter	Sample Size	Set III: $\alpha=1.5, \beta=1.2, \delta=0.5, \lambda=2.0$			Set IV: $\alpha=0.6, \beta=1.0, \delta=1.5, \lambda=1.4$		
		Mean	Bias	RMSE	Mean	Bias	RMSE
α	35	1.65846	0.15846	0.72472	0.64134	0.04134	0.24185
	50	1.62834	0.12834	0.62353	0.62120	0.02120	0.20292
	100	1.53509	0.03509	0.46937	0.69012	0.09012	0.13789
	200	1.52670	0.02670	0.39081	0.68045	0.08045	0.10881
	400	1.51122	0.01122	0.30902	0.66008	0.06008	0.08265
	800	1.50808	0.00808	0.26033	0.61033	0.01033	0.06719
	1000	1.50052	0.00052	0.25740	0.60029	0.00029	0.06523
β	35	1.25877	0.05877	0.84088	1.25059	0.25059	0.79295
	50	1.24764	0.04764	0.73485	1.20947	0.20947	0.71019
	100	1.23983	0.03983	0.63747	1.16784	0.16784	0.64150
	200	1.23581	0.03581	0.50057	1.11054	0.11054	0.61578
	400	1.21107	0.01107	0.47207	1.07268	0.07268	0.57400
	800	1.20475	0.00475	0.42575	1.00823	0.00823	0.47615
	1000	1.20014	0.00014	0.14565	1.00106	0.00106	0.46641
δ	35	0.66112	0.16112	0.35865	1.66396	0.16396	0.59323
	50	0.63567	0.13567	0.33348	1.55225	0.05225	0.53846
	100	0.62631	0.12631	0.30234	1.52034	0.02034	0.47229
	200	0.56796	0.06796	0.23245	1.51497	0.01497	0.43555
	400	0.54348	0.04348	0.16918	1.50443	0.00443	0.42873
	800	0.52659	0.02659	0.12928	1.50259	0.00259	0.36730
	1000	0.50072	0.00072	0.11974	1.50028	0.00028	0.31811
λ	35	3.11606	1.11606	0.92648	1.55936	0.15936	0.58519
	50	2.87745	0.87745	0.92164	1.54008	0.14008	0.49292
	100	2.27689	0.27689	0.91941	1.40882	0.00882	0.44626
	200	2.16928	0.16928	0.45793	1.40580	0.00580	0.42997
	400	2.06812	0.06812	0.38234	1.40126	0.00126	0.37838
	800	2.00180	0.00180	0.25083	1.40116	0.00116	0.22376
	1000	2.00058	0.00058	0.20583	1.40065	0.00065	0.21932

3.8 Conclusions

We developed and studied in detail a new family of generalized distributions called the Ristić-Balakrishnan Odd Burr III - G (RBOB III-G) distribution. The new proposed distribution presents a flexible mechanism for fitting a wide spectrum of real world data sets. From the application results and comparison to several known models such as Burr-Weibull (BW) by [80], Kumaraswamy Weibull (KW) by [46], Weibull-Lomax (WLx) by [104], Beta Generalized Lindley (BGL) by [84] and New Modified Weibull (NMW) by [52], we confirm that it provides a better fit, see results under tables 3.7 and 3.8.

Note that in addition to the above model, another model which belongs to the same family called Zografos-Balakrishnan Odd Burr III-G Model was also developed. The model was developed using the Odd Burr III-G generator by [12] which is an extension of the Burr type III distribution first introduced by [29] and the work is omitted in this thesis.

Chapter 4

The New Generalized Marshall-Olkin-G Family of Distributions with Applications

Forthcoming and Available Online at:

<https://inderscience.com/info/ingeneral/forthcoming.php?jcode=ijmor#101370>

4.1 Introduction

We introduce and study in detail a new family of distributions called Ristić-Balakrishnan Marshall Olkin-G (RBMO-G) family of distributions. We obtain some mathematical and statistical properties for this new family of distributions such as hazard and reverse hazard functions, quantile function, moments and incomplete moments, moment generating functions, Rényi entropy and order statistics. The model parameters are obtained through method of maximum likelihood. We finally illustrate the usefulness of these families of distributions via simulation experiments and real life data sets are used for model fittings.

4.2 The Ristić-Balakrishnan Marshall-Olkin-G Model

Consider the Marshall-Olkin-G (MO-G) class of distributions with the cumulative distribution function (cdf) given by

$$F_{MO}(x; \delta, \psi) = 1 - \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)}, \quad (4.1)$$

for $\delta > 0$ (tilt parameter), $\bar{\delta} = 1 - \delta$ and $\bar{G}(x; \psi) = 1 - G(x; \psi)$, where $G(x; \psi)$ is the baseline cdf with parameter vector ψ . Note that when $\delta = 1$, we have the baseline distribution function, $F_{MO}(x; \psi) = G(x; \psi)$. The corresponding probability distribution function (pdf) for MO-G class of distributions is given by

$$f_{MO}(x; \delta, \psi) = \frac{\delta g(x; \psi)}{(1 - \delta \bar{G}(x; \psi))^2}, \quad (4.2)$$

(see [74] for details).

We generate this new family of distributions by combining the Marshall-Olkin-G (MO-G) distribution with the Gamma generator proposed by [91] having the cdf

$$F_{RB}(x; \alpha) = 1 - \frac{1}{\Gamma(\alpha)} \int_0^{-\log(G(x))} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0 \quad (4.3)$$

and the corresponding pdf is given by

$$f_{RB}(x; \alpha) = \frac{1}{\Gamma(\alpha)} (-\log G(x))^{\alpha-1} g(x), \quad x \in \mathbb{R}. \quad (4.4)$$

If we let $G(x) = F_{MO-G}(x; \delta, \psi)$ in equations (4.3) and (4.4) then the cdf and pdf of a new extended generator Ristić-Balakrishnan Marshall-Olkin-G (RBMO-G) are given by

$$\begin{aligned}
F_{RBMO-G}(x; \delta, \alpha, \psi) &= 1 - \frac{1}{\Gamma(\alpha)} \int_0^{-\log \left(1 - \frac{\delta \bar{G}(x, \psi)}{1 - \delta \bar{G}(x, \psi)} \right)} t^{\alpha-1} e^{-t} dt \\
&= 1 - \frac{\gamma \left\{ -\log \left(1 - \frac{\delta \bar{G}(x, \psi)}{1 - \delta \bar{G}(x, \psi)} \right), \alpha \right\}}{\Gamma(\alpha)}
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
f_{RBMO-G}(x; \delta, \alpha, \psi) &= \frac{1}{\Gamma(\alpha)} \left[-\log \left(1 - \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right) \right]^{\alpha-1} \\
&\times \frac{\delta g(x; \psi)}{(1 - \delta \bar{G}(x; \psi))^2},
\end{aligned} \tag{4.6}$$

respectively, for $\alpha, \delta > 0$ and parameter vector ψ .

4.3 Series Expansion of the Density Function

This section presents series expansion of the RBMO-G density function. The series expansion allows for the mathematical and statistical properties of the model to be easily established.

Let $y = \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)}$, then using the series representations $-\log(1 - y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, we have

$$\left[-\log(1 - y) \right]^{\delta-1} = y^{\delta-1} \left[\sum_{m=0}^{\infty} \binom{\delta-1}{m} y^m \left(\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^m \right],$$

and applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s \right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s \tag{4.7}$$

where $b_{s,m} = (s a_0)^{-1} \sum_{l=1}^s [m(l+1) - s] a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, [58],

$$(1 - z)^{-1} = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(a)j!}, \tag{4.8}$$

for $|z| < 1$, $a > 0$ and $j > 0$, then we can write the RBMO-G pdf as

$$\begin{aligned}
f(x; \delta, \alpha, \psi) &= \frac{\delta}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\alpha-1}{m} b_{s,m} y^{m+s+\alpha-1} \frac{g(x; \psi)}{(1 - \delta \bar{G}(x; \psi))^2} \\
&= \frac{\delta}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\alpha-1}{m} b_{s,m} \left(\frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right)^{m+s+\alpha-1} \\
&\quad \times \frac{g(x; \psi)}{(1 - \delta \bar{G}(x; \psi))^2} \\
&= \frac{\delta}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\alpha-1}{m} b_{s,m} (\delta \bar{G}(x; \psi))^{m+s+\alpha-1} \\
&\quad \times \left(1 - \delta \bar{G}(x; \psi) \right)^{-(m+s+\alpha-1)} \frac{g(x; \psi)}{(1 - \delta \bar{G}(x; \psi))^2} \\
&= \frac{\delta^{m+s+\alpha}}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\alpha-1}{m} b_{s,m} (\bar{G}(x; \psi))^{m+s+\alpha-1} \\
&\quad \times \left(1 - \delta \bar{G}(x; \psi) \right)^{-(m+s+\alpha+1)} g(x; \psi) \\
&= \frac{\delta^{m+s+\alpha} \bar{\delta}^k}{\Gamma(\alpha)} \sum_{m,s,k=0}^{\infty} \binom{\alpha-1}{m} b_{s,m} \frac{\Gamma(m+s+\alpha+1+k)}{\Gamma(m+s+\alpha+1)k!} \\
&\quad \times (\bar{G}(x; \psi))^{m+s+\alpha-1} (\bar{G}(x; \psi))^k g(x; \psi) \\
&= \frac{\delta^{m+s+\alpha} \bar{\delta}^k}{\Gamma(\alpha)} \sum_{m,s,k=0}^{\infty} \binom{\alpha-1}{m} b_{s,m} \frac{\Gamma(m+s+\alpha+1+k)}{\Gamma(m+s+\alpha+1)k!} \\
&\quad \times (\bar{G}(x; \psi))^{m+s+\alpha+k-1} g(x; \psi) \\
&= \frac{\delta^{m+s+\alpha} \bar{\delta}^k (-1)^i}{\Gamma(\alpha)} \sum_{m,s,k,i=0}^{\infty} \binom{\alpha-1}{m} b_{s,m} \frac{\Gamma(m+s+\alpha+1+k)}{\Gamma(m+s+\alpha+1)k!} \\
&\quad \times \binom{m+s+\alpha+k-1}{i} (G(x; \psi))^i g(x; \psi) \\
&= \frac{\delta^{m+s+\alpha} \bar{\delta}^k (-1)^i}{\Gamma(\alpha)} \sum_{m,s,k,i=0}^{\infty} \binom{\alpha-1}{m} b_{s,m} \frac{\Gamma(m+s+\alpha+1+k)}{\Gamma(m+s+\alpha+1)k!} \\
&\quad \times \binom{m+s+\alpha+k-1}{i} \frac{i+1}{i+1} (G(x; \psi))^i g(x; \psi) \\
&= \sum_{i=0}^{\infty} C_{i+1} f_{E-G}(x; \psi, i+1), \tag{4.9}
\end{aligned}$$

where $f_{E-G}(x; \psi, i+1) = (i+1) (G(x; \psi))^{i+1-1} g(x; \psi)$ is the exponentiated-G pdf with power parameter $i+1 > 0$ and the parameter vector ψ ,

$$\begin{aligned}
C_{i+1} &= \frac{\delta^{m+s+\alpha} \bar{\delta}^k (-1)^i}{\Gamma(\alpha)} \sum_{m,s,k=0}^{\infty} \binom{\alpha-1}{m} b_{s,m} \frac{\Gamma(m+s+\alpha+1+k)}{\Gamma(m+s+\alpha+1)k!} \\
&\times \binom{m+s+\alpha+k-1}{i} \frac{1}{i+1}.
\end{aligned} \tag{4.10}$$

It follows therefore that the RBMO-G density is an infinite linear combination of the exponentiated-G (E-G) pdf. The statistical and mathematical properties of the RBMO-G distribution can be readily obtained from those of the E-G distribution.

4.4 The Sub-families of Distributions

In this section, sub-families of the proposed family of distributions are given as follows;

- If $\alpha = 1$, then the RBMO-G family of distributions reduces to a distribution with the pdf

$$f(x; \delta, \psi) = \frac{\delta G(x; \psi)}{[1 - \delta \bar{G}(x; \psi)]^2}, \tag{4.11}$$

for $\delta > 0$.

- If $\delta = 1$, the RBMO-G family of distributions reduces to a distribution with the pdf

$$f(x; \alpha, \psi) = \frac{1}{\Gamma(\alpha)} [-\log(1 - G(x; \psi))]^{\alpha-1} g(x; \psi), \tag{4.12}$$

for $\alpha > 0$.

- If $\alpha = \delta = 1$, the RBMO-G family of distributions reduces to the baseline pdf

$$f_{RBMO-G}(x; \psi) = g(x; \psi). \tag{4.13}$$

4.5 The Hazard, Reverse Hazard and Quantile Functions

This section presents results on the hazard rate (hrf), reverse hazard (rhf) and quantile functions for the RBMO-G distribution. The RBMO-G hrf and rhf are given by;

$$h_{RBMO-G}(x) = \frac{\left[-\log \left(1 - \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right) \right]^{\alpha-1} \frac{\delta g(x; \psi)}{(1 - \delta \bar{G}(x; \psi))^2}}{\gamma \left\{ -\log \left(1 - \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right), \alpha \right\}} \quad (4.14)$$

and

$$\tau_{RBMO-G}(x) = \frac{\left[-\log \left(1 - \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right) \right]^{\alpha-1} \frac{\delta g(x; \psi)}{(1 - \delta \bar{G}(x; \psi))^2}}{\Gamma(\delta) - \gamma \left\{ -\log \left(1 - \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right), \alpha \right\}}, \quad (4.15)$$

respectively. Furthermore, the quantile function of RBMO-G family of distributions is obtained by solving the non-linear equation;

$$F(x; \delta, \alpha, \psi) = 1 - \frac{\gamma \left\{ -\log \left(1 - \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right), \alpha \right\}}{\Gamma(\alpha)} = u,$$

for $0 \leq u \leq 1$, that is

$$-\log \left(1 - \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right) = \gamma^{-1} [\Gamma(\alpha)(1-u), \alpha]$$

and

$$\frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} = 1 - \left(e^{-\gamma^{-1} [\Gamma(\alpha)(1-u), \alpha]} \right)$$

such that

$$\frac{1}{\delta \bar{G}(x; \psi)} = \left[1 - \left(e^{-\gamma^{-1} [\Gamma(\alpha)(1-u), \alpha]} \right) \right]^{-1} + \frac{\bar{\delta}}{\delta}.$$

We can therefore write

$$\bar{G}(x; \psi) = \frac{\left(\left[1 - \left(e^{-\gamma^{-1} [\Gamma(\alpha)(1-u), \alpha]} \right) \right]^{-1} + \frac{\bar{\delta}}{\delta} \right)^{-1}}{\delta},$$

and finally the quantile function for the RBMO-G family of distributions is given by

$$Q_{RBMO-G}(u, \delta, \alpha, \psi) = G^{-1} \left[1 - \frac{\left(\left[1 - \left(e^{-\gamma^{-1} [\Gamma(\alpha)(1-u), \alpha]} \right) \right]^{-1} + \frac{\bar{\delta}}{\delta} \right)^{-1}}{\delta} \right]. \quad (4.16)$$

4.6 Some Special Cases

Some of the special cases for the RBMO-G family of distributions are derived and presented in this section.

4.6.1 RBMO-Nadarajah-Haghighi (RBMO-NH) Distribution

Let the Nadarajah-Haghighi distribution be the baseline distribution with parameters $\lambda, \beta > 0$ having the cdf and pdf, $G(x) = 1 - \exp(1 - (1 + \lambda x)^\beta)$ and $g(x) = \lambda\beta(1 + \lambda x)^{\beta-1} \exp(1 - (1 + \lambda x)^\beta)$ respectively. The cdf and pdf of RBMO-NH distribution are given by

$$F_{RBMO-NH}(x; \alpha, \delta, \lambda, \beta) = 1 - \frac{\gamma \left(-\log \left(1 - \frac{\delta(\exp(1 - (1 + \lambda x)^\beta))}{1 - (1 - \delta)(\exp(1 - (1 + \lambda x)^\beta))} \right), \alpha \right)}{\Gamma(\alpha)}$$

and

$$\begin{aligned} f_{RBMO-NH}(x; \alpha, \delta, \lambda, \beta) &= \frac{1}{\Gamma(\alpha)} \left[-\log \left(1 - \frac{\delta(\exp(1 - (1 + \lambda x)^\beta))}{1 - (1 - \delta)(\exp(1 - (1 + \lambda x)^\beta))} \right) \right]^{\alpha-1} \\ &\times \frac{\delta \lambda \beta (1 + \lambda x)^{\beta-1} \exp(1 - (1 + \lambda x)^\beta)}{(1 - (1 - \delta) \exp(1 - (1 + \lambda x)^\beta))^2}, \end{aligned} \quad (4.17)$$

respectively for $\alpha, \delta, \lambda, \beta > 0$. The corresponding hazard rate function is given by

$$\begin{aligned}
 h_{RBMO-NH}(x) &= \left[-\log \left(1 - \frac{\delta(\exp(1 - (1 + \lambda x)^\beta))}{1 - (1 - \delta)(\exp(1 - (1 + \lambda x)^\beta))} \right) \right]^{\alpha-1} \\
 &\times \frac{\delta\lambda\beta(1 + \lambda x)^{\beta-1} \exp(1 - (1 + \lambda x)^\beta)}{(1 - (1 - \delta) \exp(1 - (1 + \lambda x)^\beta))^2} \\
 &\times \left(\gamma \left(-\log \left(1 - \frac{\delta(\exp(1 - (1 + \lambda x)^\beta))}{1 - (1 - \delta)(\exp(1 - (1 + \lambda x)^\beta))} \right), \alpha \right) \right)^{-1}. \quad (4.18)
 \end{aligned}$$

The pdf and hrf plots for the RBMO-NH distribution are given in figures 4.1 (a) and 4.1 (b) below.

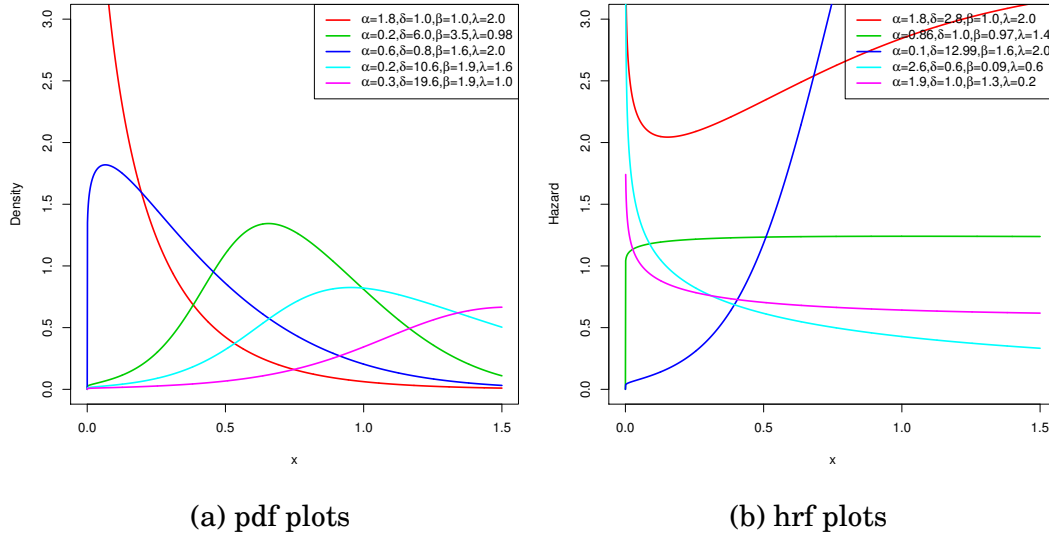


Figure 4.1: Plots of pdf and hrf for the RBMO-NH Distribution

Figures 4.1 (a) and 4.1 (b) illustrates the flexibility of the RBMO-NH distribution for the selected parameter values. From the above pdf plots, it is evident that the RBMO-NH distribution can adopt various shapes that include reverse-J, uni-modal, left or right skewed shapes. In addition, the RBMO-NH hrf plots gives decreasing, increasing, bathtub and upside down bathtub shapes.

4.6.2 RBMO-Burr XII (RBMO-BXII) Distribution

Consider the Burr XII distribution as the baseline distribution with parameters $s, k, \lambda > 0$ having cdf and pdf $G(x; s, k, \lambda) = 1 - (1 + (\frac{x}{s})^\lambda)^{-k}$ and

$g(x; s, k, \lambda) = \lambda k s^{-\lambda} x^{\lambda-1} (1 + (\frac{x}{s})^\lambda)^{-k-1}$, respectively. The cdf and pdf of RBMO-BXII are given by

$$F_{RBMO-BXII}(x; \alpha, \delta, s, k, \lambda) = 1 - \frac{\gamma\left(-\log\left(1 - \frac{\delta((1+(\frac{x}{s})^\lambda)^{-k})}{1 - (1-\delta)((1+(\frac{x}{s})^\lambda)^{-k}}\right), \alpha\right)}{\Gamma(\alpha)}$$

and

$$\begin{aligned} f_{RBMO-BXII}(x; \alpha, \delta, s, k, \lambda) &= \frac{1}{\Gamma(\alpha)} \left[-\log\left(1 - \frac{\delta((1+(\frac{x}{s})^\lambda)^{-k})}{1 - (1-\delta)((1+(\frac{x}{s})^\lambda)^{-k}}\right) \right]^{\alpha-1} \\ &\times \frac{\delta \lambda k s^{-\lambda} x^{\lambda-1} (1 + (\frac{x}{s})^\lambda)^{-k-1}}{(1 - (1-\delta)((1+(\frac{x}{s})^\lambda)^{-k}))^2}, \end{aligned} \quad (4.19)$$

respectively for $\alpha, \delta, s, k, \lambda > 0$. The corresponding hazard rate function is given by

$$\begin{aligned} h_{RBMO-BXII}(x) &= \left[-\log\left(1 - \frac{\delta((1+(\frac{x}{s})^\lambda)^{-k})}{1 - (1-\delta)((1+(\frac{x}{s})^\lambda)^{-k}}\right) \right]^{\alpha-1} \\ &\times \frac{\delta \lambda k s^{-\lambda} x^{\lambda-1} (1 + (\frac{x}{s})^\lambda)^{-k-1}}{(1 - (1-\delta)((1+(\frac{x}{s})^\lambda)^{-k}))^2} \\ &\times \left(\gamma\left(-\log\left(1 - \frac{\delta((1+(\frac{x}{s})^\lambda)^{-k})}{1 - (1-\delta)((1+(\frac{x}{s})^\lambda)^{-k}}\right), \alpha\right) \right)^{-1}. \end{aligned} \quad (4.20)$$

Note that when $s = k = 1$, then the RBMO-BXII distribution reduces to RBMO-log logistic (RBMO-LLoG) distribution having the cdf and pdf given by

$$F_{RBMO-LLoG}(x; \alpha, \delta, \lambda) = 1 - \frac{\gamma\left(-\log\left(1 - \frac{\delta((1+x^\lambda)^{-1})}{1 - (1-\delta)((1+x^\lambda)^{-1}}\right), \alpha\right)}{\Gamma(\alpha)}$$

and

$$\begin{aligned} f_{RBMO-LLoG}(x; \alpha, \delta, \lambda) &= \frac{1}{\Gamma(\alpha)} \left[-\log\left(1 - \frac{\delta((1+x^\lambda)^{-1})}{1 - (1-\delta)((1+x^\lambda)^{-1}}\right) \right]^{\alpha-1} \\ &\times \frac{\delta \lambda x^{\lambda-1} (1 + x^\lambda)^{-2}}{(1 - (1-\delta)((1+x^\lambda)^{-1}))^2}, \end{aligned} \quad (4.21)$$

respectively, for $\alpha, \delta, \lambda > 0$. The corresponding hazard rate function is given by

$$\begin{aligned}
h_{RBMO-LLoG}(x) &= \frac{1}{\Gamma(\alpha)} \left[-\log \left(1 - \frac{\delta((1+x^\lambda)^{-1})}{1 - (1-\delta)((1+x^\lambda)^{-1})} \right) \right]^{\alpha-1} \\
&\times \frac{\delta \lambda x^{\lambda-1} (1+x^\lambda)^{-2}}{(1 - (1-\delta)((1+x^\lambda)^{-1}))^2} \\
&\times \left(\gamma \left(-\log \left(1 - \frac{\delta((1+x^\lambda)^{-1})}{1 - ((1-\delta)((1+x^\lambda)^{-1}))} \right), \alpha \right) \right)^{-1}. \quad (4.22)
\end{aligned}$$

The pdf and hrf plots for the RBMO-BXII distribution are given in figures 4.2 (a) and 4.2 (b) below.

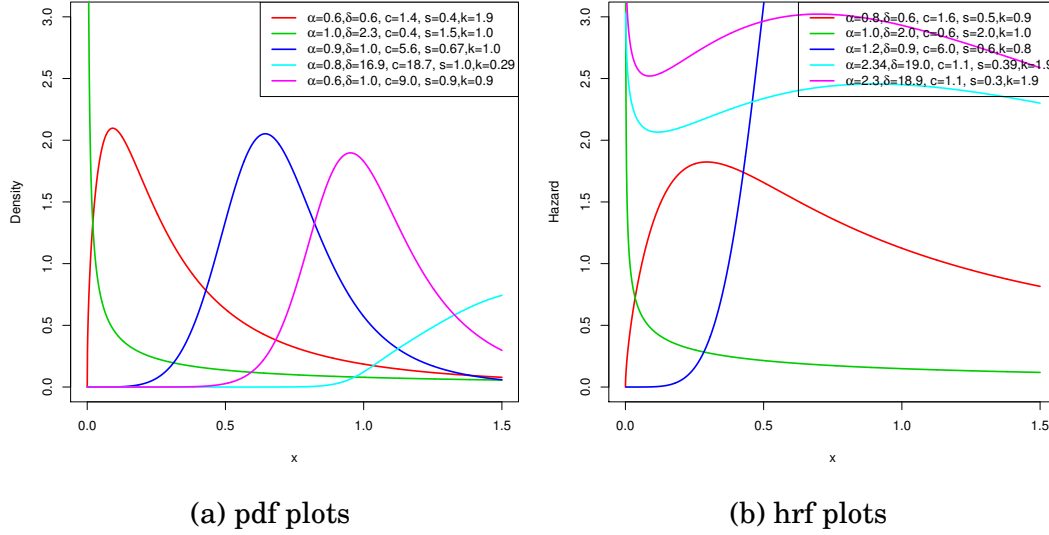


Figure 4.2: Plots of the pdf and hrf for RBMO-BXII Distribution

Figures 4.2 (a) and 4.2 (b) shows that the RBMO-BXII distribution can model different forms of data behaviours for selected parameter values. The pdfs of the RBMO-BXII distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. The hrf plots for RBMO-BXII distribution also gives decreasing, increasing, bathtub, upside down bathtub and upside down bathtub followed by bathtub shapes.

4.6.3 RBMO-Weibull (RBMO-W) Distribution

Let the Weibull distribution be the baseline distribution with shape parameter $\lambda > 0$ and scale parameter $\xi > 0$ having cdf and pdf $G(x; \lambda, \xi) = 1 - e^{-\xi x^\lambda}$ and $g(x; \lambda, \xi) = \xi \lambda x^{\lambda-1} e^{-\xi x^\lambda}$ respectively. The cdf and pdf of RBMO-W distribution are given by

$$F_{RBMO-W}(x; \alpha, \delta, \lambda, \xi) = 1 - \frac{\gamma \left(-\log \left(1 - \frac{\delta e^{-\xi x^\lambda}}{1 - ((1-\delta)e^{-\xi x^\lambda})} \right), \alpha \right)}{\Gamma(\alpha)}$$

and

$$\begin{aligned} f_{RBMO-W}(x; \alpha, \delta, \lambda, \xi) &= \frac{1}{\Gamma(\alpha)} \left[-\log \left(1 - \frac{\delta e^{-\xi x^\lambda}}{1 - ((1-\delta)e^{-\xi x^\lambda})} \right) \right]^{\alpha-1} \\ &\times \frac{\delta \xi \lambda x^{\lambda-1} e^{-\xi x^\lambda}}{(1 - (1-\delta)e^{-\xi x^\lambda})^2}, \end{aligned} \quad (4.23)$$

respectively for $\alpha, \delta, \lambda, \xi > 0$ and $x > 0$. The corresponding hazard rate function is given by

$$\begin{aligned} h_{RBMO-W}(x) &= \left[-\log \left(1 - \frac{\delta e^{-\xi x^\lambda}}{1 - ((1-\delta)e^{-\xi x^\lambda})} \right) \right]^{\alpha-1} \frac{\delta \xi \lambda x^{\lambda-1} e^{-\xi x^\lambda}}{(1 - (1-\delta)e^{-\xi x^\lambda})^2} \\ &\times \left(\gamma \left(-\log \left(1 - \frac{\delta e^{-\xi x^\lambda}}{1 - ((1-\delta)e^{-\xi x^\lambda})} \right), \alpha \right) \right)^{-1}. \end{aligned} \quad (4.24)$$

Note that when $\lambda = 1$, the RBMO-W distribution reduces to RBMO-Exponential distribution with parameters $\alpha, \delta, \xi > 0$ having the cdf and pdf given by

$$F_{RBMO-E}(x; \alpha, \delta, \xi) = 1 - \frac{\gamma \left(-\log \left(1 - \frac{\delta e^{-\xi x}}{1 - ((1-\delta)e^{-\xi x})} \right), \alpha \right)}{\Gamma(\alpha)}$$

and

$$f_{RBMO-E}(x; \alpha, \delta, \xi) = \frac{1}{\Gamma(\alpha)} \left[-\log \left(1 - \frac{\delta e^{-\xi x}}{1 - ((1-\delta)e^{-\xi x})} \right) \right]^{\alpha-1} \times \frac{\delta \xi e^{-\xi x}}{(1 - (1-\delta)e^{-\xi x})^2}, \quad (4.25)$$

respectively. The corresponding hazard rate function is given by

$$h_{RBMO-E}(x) = \frac{\left[-\log \left(1 - \frac{\delta e^{-\xi x}}{1 - ((1-\delta)e^{-\xi x})} \right) \right]^{\alpha-1} \left(\frac{\delta \xi e^{-\xi x}}{(1 - (1-\delta)e^{-\xi x})^2} \right)}{\gamma \left(-\log \left(1 - \frac{\delta e^{-\xi x}}{1 - ((1-\delta)e^{-\xi x})} \right), \alpha \right)}. \quad (4.26)$$

Also, when $\lambda = 2$ we obtain the RBMO-Raleigh (RBMO-R) distribution and for $\alpha = \delta = 1$, we obtain the baseline Weibull distribution.

The pdf and hrf plots for the RBMO-W distribution are given in Figures 4.3 (a) and 4.3 (b) below.

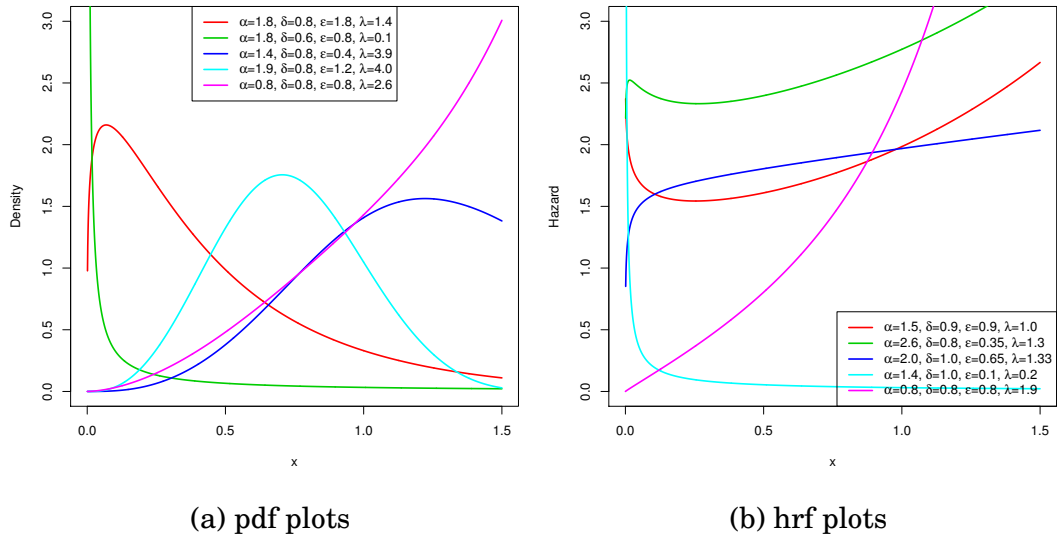


Figure 4.3: Plots of the pdf and hrf for RBMO-W Distribution

As shown by Figures 4.3 (a) and 4.3 (b) above, the shapes of the RBMO-W distribution plots indicate that the distribution can fit various forms of real data for selected

parameter values. The pdfs can flexibly adopt various shapes that include reverse-J, uni-modal, left or right skewed shapes. Moreover, the hrf plots for the RBMO-W distribution exhibit decreasing, increasing, bathtub, upside down bathtub and upside down shapes.

4.7 Mathematical and Statistical Properties

In this section, some of the mathematical and statistical properties of the RBMO-G family of distributions are presented.

4.7.1 Moments and Generating Functions

Let $Y_{i+1} \sim \text{Exponentiated-G}(\psi, i+1)$, then the r^{th} moment of the RBMO-G family of distributions is given by

$$\begin{aligned} E(X^r) &= \sum_{i=0}^{\infty} C_{i+1} E(Y_{i+1}^r) = \sum_{i=0}^{\infty} C_{i+1} \int_0^{\infty} y^r f_{E-G}(y; \psi, i+1) dy \\ &= \sum_{i=0}^{\infty} C_{i+1} (i+1) \int_0^1 [Q_F(u; \psi)]^r u^i du \end{aligned} \quad (4.27)$$

and the moment generating function (mgf) for $|a| < 1$, is given by

$$M_X(a) = \sum_{i=0}^{\infty} C_{i+1} M_{Y_{i+1}}(a) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{i+1} \frac{a^j}{j!} E(Y_{i+1}^j), \quad (4.28)$$

where $M_{Y_{i+1}}(a)$ is the mgf of Y_{i+1} . We present the first five moments for the RBMO-LLoG distribution together with the standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for selected values, (see Table 4.1) below. The quantiles for the selected parameter values are also presented under Table 4.2.

Table 4.1: RBMO-LLoG Moments for selected parameter values

	1.5,1.3,0.6	0.7,1.1,1.0	0.5,1.0,1.1	0.6,1.3,1.0	(1.0,0.5,1.2)
$E(X)$	0.1469	0.1407	0.1136	0.1106	0.2477
$E(X^2)$	0.0771	0.0868	0.0725	0.0698	0.1393
$E(X^3)$	0.0562	0.0621	0.0528	0.0506	0.0944
$E(X^4)$	0.0387	0.0482	0.0414	0.0395	0.0707
$E(X^5)$	0.0309	0.0394	0.334	0.0324	0.0563
SD	0.2356	0.2588	0.2441	0.2398	0.2792
CV	1.6043	1.8394	2.1494	2.1682	1.1272
CS	1.8345	1.7930	2.1319	2.1836	0.975
CK	5.4940	4.9905	6.334	6.6021	2.8232

Table 4.2: RBMO-LLoG Quantiles for selected parameter values

	$(\alpha, \delta, \lambda)$				
u	(1.0,0.8,1.3)	(1.4,1.2,0.7)	(1.0,1.3,1.4)	(1.1,1.2,1.3)	(1.3,1.5,0.8)
1	0.1554	0.0691	0.251	0.2532	0.1444
2	0.2899	0.1403	0.4481	0.4113	0.2735
3	0.4389	0.2494	0.6585	0.5899	0.4568
4	0.6166	0.4158	0.9028	0.8035	0.7199
5	0.8423	0.6761	1.2061	1.0735	1.1098
6	1.1506	1.106	1.6113	1.4384	1.722
7	1.6163	1.8877	2.2092	1.9807	2.7804
8	2.4467	3.5855	3.2466	2.9243	4.959
9	4.5455	9.1348	5.7942	5.2353	11.6166

4.7.2 Conditional Moments

For lifetime models and measures of inequality such as Bonferonni and Lorenz curves, it is of particular interest to find the conditional and incomplete moments. The r^{th} conditional moments of the RBMO-G distribution is given by

$$\begin{aligned} E(X^r|X \geq a) &= \frac{1}{\overline{F}(a; \delta, \alpha, \psi)} \int_a^\infty x^r f(x; \gamma, \delta, \psi) dx \\ &= \frac{1}{\overline{F}(a; \delta, \alpha, \psi)} \sum_{i=0}^\infty C_{i+1} E\left(Y_{i+1}^r I_{\{Y_{i+1}^r \geq t\}}\right), \end{aligned} \quad (4.29)$$

where

$$E\left(Y_{i+1}^r I_{\{Y_{i+1}^r \geq t\}}\right) = \int_t^\infty y^r f_{E-G}(y; \psi, i+1) dy = (i+1) \int_{G(u; \xi)}^1 [Q_G(u; \psi)]^r u^{i+1} du, \quad (4.30)$$

for $\delta, \alpha > 0$, and parameter vector ψ . The mean residual life function is given by $E(X - a|X > a) = E(X|X > a) - a = V_F(a) - a$, where $V_F(a)$ is referred to as the vitality function of the distribution function F . The mean deviations, Bonferonni and Lorenz curves can be readily obtained from the conditional and incomplete moments.

4.7.3 Rényi Entropy

The Rényi entropy, (see [90]), is an extension of Shannon entropy. An entropy measures variation of uncertainty for a random variable X having the probability distribution $f(x)$ and it is derived as follows

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f_{RBMO-G}(x; \delta, \alpha, \psi)]^v dx \right), v \neq 1, v > 0. \quad (4.31)$$

Rényi entropy tends to Shannon entropy as $v \rightarrow 1$. Note that

$$\begin{aligned} \int_0^\infty f_{RBMO-G}^v(x) dx &= \left(\frac{1}{\Gamma(\alpha)} \right)^v \int_0^\infty \left[-\log \left(1 - \frac{\delta \overline{G}(x; \psi)}{1 - \delta \overline{G}(x; \psi)} \right) \right]^{v(\alpha-1)} \\ &\quad \times \frac{\delta^v g(x; \psi)^v}{(1 - \delta \overline{G}(x; \psi))^{2v}} dx. \end{aligned} \quad (4.32)$$

Let $y = \left(\frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right)$. Note that $(1 - \delta \bar{G}(x; \psi))^{-2v} = \sum_{k=0}^{\infty} \frac{\Gamma(2v+k)}{\Gamma(2v)k!} \bar{\delta}^k \bar{G}(x; \psi)^k$ and

$$\left[-\log \left(1 - \frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right) \right]^{v(\alpha-1)} = \sum_{m,s=0}^{\infty} \binom{v\alpha - v}{m} d_{s,m} y^{m+s+\alpha-1},$$

by applying the result on power series raised to a positive integer, with $c_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty} c_s y^s \right)^m = \sum_{s=0}^{\infty} d_{s,m} y^s, \quad (4.33)$$

where $d_{s,m} = (s c_0)^{-1} \sum_{l=1}^s [m(l+1) - s] c_l d_{s-l,m}$, and $d_{0,m} = c_0^m$, [58], so that

$$\begin{aligned} \int_0^{\infty} f_{RBMO-G}^v(x) dx &= \left(\frac{1}{\Gamma(\alpha)} \right)^v \sum_{k=0}^{\infty} \sum_{m,s=0}^{\infty} \binom{v\alpha - v}{m} d_{s,m} \frac{\Gamma(2v+k)}{\Gamma(2v)k!} \bar{\delta}^v \bar{\delta}^k \\ &\times \int_0^{\infty} \left[\frac{\delta \bar{G}(x; \psi)}{1 - \delta \bar{G}(x; \psi)} \right]^{m+s+\alpha-1} \bar{G}(x; \psi)^k g(x; \psi)^v dx \\ &= \left(\frac{1}{\Gamma(\alpha)} \right)^v \sum_{k=0}^{\infty} \sum_{m,s=0}^{\infty} \binom{v\alpha - v}{m} d_{s,m} \frac{\Gamma(2v+k)}{\Gamma(2v)k!} \bar{\delta}^{v+m+s+\alpha-1} \\ &\times \int_0^{\infty} \left(\bar{G}(x; \psi) \right)^{m+s+\alpha-1} \left(1 - \delta \bar{G}(x; \psi) \right)^{-(m+s+\alpha-1)} \\ &\times \bar{\delta}^k \bar{G}(x; \psi)^k g(x; \psi)^v dx \\ &= \left(\frac{1}{\Gamma(\alpha)} \right)^v \sum_{k=0}^{\infty} \sum_{m,s,j=0}^{\infty} \binom{v\alpha - v}{m} d_{s,m} \frac{\Gamma(2v+k)}{\Gamma(2v)k!} \\ &\times \frac{\Gamma(m+s+\alpha-1+j)}{\Gamma(m+s+\alpha-1)j!} \bar{\delta}^{k+j} \bar{\delta}^{v+m+s+\alpha-1} \\ &\times \int_0^{\infty} \left(\bar{G}(x; \psi) \right)^{m+s+\alpha+k+j-1} g(x; \psi)^v dx \\ &= \left(\frac{1}{\Gamma(\alpha)} \right)^v \sum_{k=0}^{\infty} \sum_{m,s,j,p=0}^{\infty} \binom{v\alpha - v}{m} d_{s,m} \frac{\Gamma(2v+k)}{\Gamma(2v)k!} \\ &\times \frac{\Gamma(m+s+\alpha-1+j)}{\Gamma(m+s+\alpha-1)j!} \bar{\delta}^{v+m+s+\alpha-1} \bar{\delta}^{k+j} (-1)^p \\ &\times \binom{m+s+\alpha+k+j-1}{p} \int_0^{\infty} \left(G(x; \psi) \right)^p g(x; \psi)^v dx \quad (4.34) \end{aligned}$$

and finally, the Rényi for the RBMO-G family of distributions is given by

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v} \log \left[\left(\frac{1}{\Gamma(\alpha)} \right)^v \sum_{k=0}^{\infty} \sum_{m,s,j,p=0}^{\infty} \binom{v\alpha - v}{m} d_{s,m} \frac{\Gamma(2v+k)}{\Gamma(2v)k!} \right. \\
&\times \frac{\Gamma(m+s+\alpha-1+j)}{\Gamma(m+s+\alpha-1)j!} \binom{m+s+\alpha+k+j-1}{p} \delta^{v+m+s+\alpha-1} \delta^{k+j} (-1)^p \\
&\times \left. \int_0^{\infty} \left(G(x; \psi) \right)^p g(x; \psi)^v dx \right] \\
&= \frac{\Gamma(m+s+\alpha-1+j)}{\Gamma(m+s+\alpha-1)j!} \frac{1}{\left[\frac{p}{v} + 1 \right]^v} \int_0^{\infty} \left(\left[\frac{p}{v} + 1 \right] g(x; \psi) G^{\frac{p}{v}}(x; \psi) \right)^v dx. \quad (4.35)
\end{aligned}$$

Finally, the Rényi entropy for the RBMO-G family of distribution can be written as

$$I_R(v) = (1-v)^{-1} \log \left[\sum_{p=0}^{\infty} U_p e^{(1-v)I_{REG}} \right],$$

for $v > 0, v \neq 1$, where U_p is a linear component and $I_{REG} = (1-v)^{-1} \log \int_0^{\infty} \left(\left[\frac{p}{v} + 1 \right] g(x; \psi) G^{\frac{p}{v}}(x; \psi) \right)^v dx$ is the Rényi entropy of exponentiated-G distribution with power parameter $\frac{p}{v} + 1$.

4.7.4 Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed RBMO-G random variables. The k^{th} order statistic can be expressed as

$$\begin{aligned}
f_{k:n}(x) &= \frac{n!}{(k-1)!(n-k)!} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} [F(x)]^{j+k-1} f(x) \\
&= \frac{n!}{(k-1)!(n-k)!} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \left[\sum_{p=0}^{\infty} C_{i+1}(i+1) G(x)^{i+1-1} g(x) \right] \\
&\times \left[\sum_{i=0}^{\infty} C_p G(x)^p \right]^{j+k-1}, \quad (4.36)
\end{aligned}$$

where $p = i + 1$. We can write

$$\left[\sum_{p=0}^{\infty} C_p G(x)^p \right]^{j+k-1} = \sum_{p=0}^{\infty} f_{j+k-1,p} G(x)^p$$

where

$$f_{j+k-1,0} = (C_0)^{j+k-1},$$

$f_{j+k-1,p} = (pC_0)^{-1} \sum_{m=1}^p [m(j+k) - p] C_m f_{j+k-1,p-m}$, for $k = 1, 2, \dots$ and C_p can be obtained from equation (4.10). Hence,

$$f_{k:n}(x) = \sum_{j=0}^{n-k} \sum_{i,p=0}^{\infty} m_{j,i,p} f_{E-G}(x; \psi, i + p + 1), \quad (4.37)$$

where

$$m_{j,i,p} = \frac{n! C_{i+1} (i+1) f_{j+k-1,p}}{(k-1)! (n-k)! (i+p+1)} (-1)^j \binom{n-k}{j}. \quad (4.38)$$

4.7.5 Maximum Likelihood Estimation

Let $X \sim RBMO - G(\alpha, \delta, \psi)$ and $\Delta = (\alpha, \delta, \psi)^T$ be the vector of model parameters. The log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the RBMO-G family of distributions is given by

$$\begin{aligned} \ell_n(\Delta) &= -n \ln \Gamma(\alpha) + (\alpha - 1) \sum_{i=1}^n \ln \left[-\log \left(1 - \frac{\delta \bar{G}(x_i; \psi)}{1 - \bar{\delta} \bar{G}(x_i; \psi)} \right) \right] + n \ln(\delta) \\ &+ \sum_{i=1}^n \ln[g(x_i; \psi)] - 2 \sum_{i=1}^n \ln[1 - \bar{\delta} \bar{G}(x_i; \psi)]. \end{aligned} \quad (4.39)$$

The elements of score vector are given in **appendix G.7**. The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \psi})^T = 0$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{p+2}(\underline{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\underline{0} = (0, 0, 0)^T$ and $J(\hat{\Delta})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$, can be used to construct confidence intervals and confidence regions for the individual model parameters.

4.8 Monte Carlo Simulations

This section presents simulation experiments for the RBMO-LLoG model to examine the reliability of maximum likelihood estimators using different parameter values. The summary of these results for simulation studies are presented under Table 4.3 below, repeated for $N = 1000$ times with sample size $n = 35, 50, 100, 200, 400, 800$ and 1000 . The table present results on the mean of the MLEs, average bias and root mean square error (RMSE). From these results, we can confirm that as the sample size n increases, the mean estimates of the parameters reduces towards the true parameter values, with Average Bias and RMSEs converging towards zero. The Average Bias and RMSEs can be computed as follows;

$$\text{Bias}(\hat{\theta}) = \frac{\sum_{i=1}^n \hat{\theta}_i}{n} - \theta \text{ and } \text{RMSE}(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^n (\hat{\theta}_i - \theta)^2}{n}}.$$

Table 4.3: RBMO-LLoG Distribution Simulation results; Mean, RMSE and Average Bias

Parameter	Sample Size	Set I: $\alpha=1.0, \delta=0.5, \lambda=1.0$			Set II: $\alpha=1.0, \delta=1.0, \lambda=0.5$			Set III: $\alpha=1.8, \delta=1.0, \lambda=1.0$		
		Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	2.7079	3.4885	1.7079	2.6548	3.4614	1.6548	2.6398	3.3566	0.8398
	50	1.5561	1.9338	0.5561	1.4746	1.8870	0.4746	2.5505	1.8901	1.7505
	100	1.1635	0.9376	0.1635	1.2886	0.9661	0.2886	1.9747	0.9688	0.1747
	200	1.0765	0.5722	0.0765	1.2457	0.5030	0.2457	1.9245	0.5432	0.1245
	400	1.0130	0.1554	0.0130	1.1176	0.1592	0.1176	1.8863	0.1592	0.0863
	800	1.0103	0.1140	0.0103	1.1023	0.1281	0.1023	1.8550	0.1140	0.0550
	1000	1.0063	0.0967	0.0063	1.0082	0.1012	0.0082	1.8009	0.0972	0.0009
δ	25	9.4509	2.7300	8.9509	8.6860	1.7000	7.6860	6.8401	4.7940	5.8401
	50	8.1605	1.9450	7.6605	4.8260	0.9220	3.8260	2.2803	3.2600	1.2803
	100	1.3705	0.9184	0.8705	2.3369	0.4030	1.3369	1.9122	1.3530	0.9122
	200	1.0482	0.1977	0.5482	2.0406	0.3928	1.0406	1.7954	1.1978	0.7954
	400	0.5331	0.1725	0.0331	1.1080	0.3306	0.1080	1.0662	0.3824	0.0662
	800	0.5199	0.1410	0.0199	1.0151	0.2959	0.0151	1.0385	0.2608	0.0385
	1000	0.5027	0.1171	0.0027	1.0049	0.2107	0.0049	1.0275	0.2192	0.0275
λ	25	1.2372	0.4404	0.2372	0.6349	0.2311	0.1349	1.2365	0.4357	0.2365
	50	1.1032	0.2540	0.1032	0.5660	0.1318	0.0660	1.1019	0.2507	0.1019
	100	1.0466	0.1554	0.0466	0.5346	0.0798	0.0346	1.0463	0.1563	0.0463
	200	1.0214	0.1017	0.0214	0.5219	0.0535	0.0219	1.0199	0.0983	0.0199
	400	1.0102	0.0581	0.0102	0.5201	0.0373	0.0201	1.0105	0.0595	0.0105
	800	1.0044	0.0408	0.0044	0.5096	0.0267	0.0096	1.0040	0.0407	0.0040
	1000	1.0043	0.0365	0.0043	0.5002	0.0211	0.0002	1.0037	0.0365	0.0037

4.9 Applications

The RBMO-LLoG model was fitted to two real data examples to demonstrate usefulness compared to other known non-nested distributions. The goodness-of-fit statistics, namely, -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer-von Mises (W^*) and Andersen-Darling (A^*) was used to evaluate the model fits as described by [33]. The model with lower values of these statistics is deemed to be the best. The results on the estimates of Model parameters (with standard errors in parenthesis) and the goodness-of-fit-statistics for the two data sets are shown in Tables 4.4 and 4.5. The probability plots and fitted densities are also presented under Figures 4.4 and 4.5.

Some of the existing non-nested models that were fitted and their performances compared with the RBMO-LLoG model include; the Weibull Exponential (WE), Marshall-Olkin Extended Fréchet (MOEFr), Generalized Weibull (GW), Half-Logistic Generalized Weibull distribution (HLGW), Topp-Leone-Marshall-Olkin-Log-Logistic (TLMO-LLoG) and Marshall-Olkin Extended Inverse Weibull (IWMO) distributions proposed by [79], [69], [51], [19], [36] and [86] respectively. The pdfs of these models are as follows;

$$f_{WE}(x; \alpha, \beta, \lambda) = \alpha\beta(\lambda e^{-\lambda x}) \left[\frac{(1 - e^{-\lambda x})^{\beta-1}}{(e^{-\lambda x})^{\beta+1}} \right] \exp \left[\left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{\beta} \right], \quad (4.40)$$

for $\alpha, \beta, \lambda > 0$ and $x > 0$,

$$f_{MOEFr}(x; \delta, \beta, \alpha) = \frac{\alpha\beta\delta^{\beta}e^{-(\frac{\delta}{x})^{\beta}}}{x^{\beta+1}(\alpha + (1 - \alpha)e^{(\frac{\delta}{x})^{\beta}})^2}, \quad (4.41)$$

for $\delta, \beta, \alpha > 0$ and $x > 0$,

$$f_{GW}(x; \omega, \lambda, \gamma) = \omega \lambda \gamma (x^{\lambda-1}) (1 + \gamma x^\lambda)^{\omega-1} \exp(1 - (1 + \gamma x^\lambda)^\omega), \quad (4.42)$$

for $\omega, \lambda, \gamma > 0$ and $x > 0$,

$$f_{HLGW}(x; \omega, \lambda, \gamma) = \frac{2\omega \lambda \gamma (x^{\lambda-1}) (1 + \gamma x^\lambda)^{\omega-1} \exp(1 - (1 + \gamma x^\lambda)^\omega)}{(1 + \exp(1 - (1 + \gamma x^\lambda)^\omega))^2}, \quad (4.43)$$

for $\omega, \lambda, \gamma > 0$ and $x > 0$,

$$f_{TLMO-LLoG}(x; b, \delta, c) = \frac{2b\delta^2 c x^{c-1} (1 + x^c)^{-3}}{(1 - \delta(1 + x^c)^{-1})^3} \left[\frac{\delta^2 (1 + x^c)^{-2}}{(1 - \delta(1 + x^c)^{-1})^2} \right]^{b-1}, \quad (4.44)$$

for $b, \delta, c > 0$ and $x > 0$

and

$$f_{IWMO}(x; \alpha, \theta, \lambda) = \frac{\alpha \lambda \theta^{-\lambda} x^{-\lambda-1} e^{-(\theta x)^{-\lambda}}}{(\alpha - (\alpha - 1)e^{-(\theta x)^{-\lambda}})^2}, \quad (4.45)$$

for $\alpha, \theta, \lambda > 0$ and $x > 0$.

4.9.1 Carbon-Fibre Strength Data

The first real data set was originally reported by [20] which represents the strength measured in GPa for single carbon fibres at gauge lengths of 1, 10, 20 and 50 mm. The dataset is given by: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 0.27493 & 0.00004 & -0.54367 \\ 0.00004 & 6.05 \times 10^{-9} & -0.00008 \\ -0.54367 & -0.00008 & 1.11428 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$\alpha \in [1.0012 \pm 1.02771]$, $\delta \in [1.31 \times 10^4 \pm 0.00015]$ and $\lambda \in [8.6433 \pm 2.06896]$.

Table 4.4: Goodness-of-fit statistics and estimates of parameters for different models relating to carbon fibre strength (GPa) data

Model	Estimates			Statistics							
	α	δ	λ	$-2\log L$	AIC	CAIC	BIC	W^*	A^*	K-S	p-value
RBMO-LLoG	1.0012 (0.5243)	1.31×10^4 (0.0001)	8.6433 (1.0556)	115.7	121.7	122.1	128.1	0.0884	0.4986	0.0912	0.67
WE	α	β	λ	125.1	131.1	131.5	137.5	0.1374	0.9514	0.0923	0.65
	7.36×10^2 (4.21×10^{-5})	4.5168 (0.3631)	0.0627 (0.0062)								
IWMO	α	λ	θ	116.5	122.5	122.9	128.9	0.1116	0.5925	0.0940	0.63
	5.10×10^{-5} (5.28×10^{-5})	0.8774 (3.06×10^{-4})	0.0248 (0.0029)								
MOEFr	α	δ	λ	115.8	121.8	122.2	128.2	22.0947	126.4542	0.9871	2.20×10^{-16}
	4.82×10^6 (1.68×10^{-9})	0.4750 (0.0913)	8.3642 (0.8664)								
GW	ω	λ	γ	137.1	143.1	143.5	149.5	0.2228	1.4917	0.1421	0.16
	13.1 (0.0031)	3.249 (0.2953)	0.0010 (0.0004)								
HLGW	ω	λ	γ	139.7	145.7	146.1	152.1	0.2614	1.7069	0.1599	0.08
	1.43×10^3 (4.69×10^{-14})	2.4245 (9.47×10^{-11})	3.38×10^{-5} (1.98×10^{-6})								
TLMO-LLoG	b	δ	c	117.3	123.3	123.7	129.7	0.1114	0.6166	0.0985	0.57
	1.16×10^3 (3.86×10^{-5})	0.5 (0.1249)	2.7980 (0.2611)								

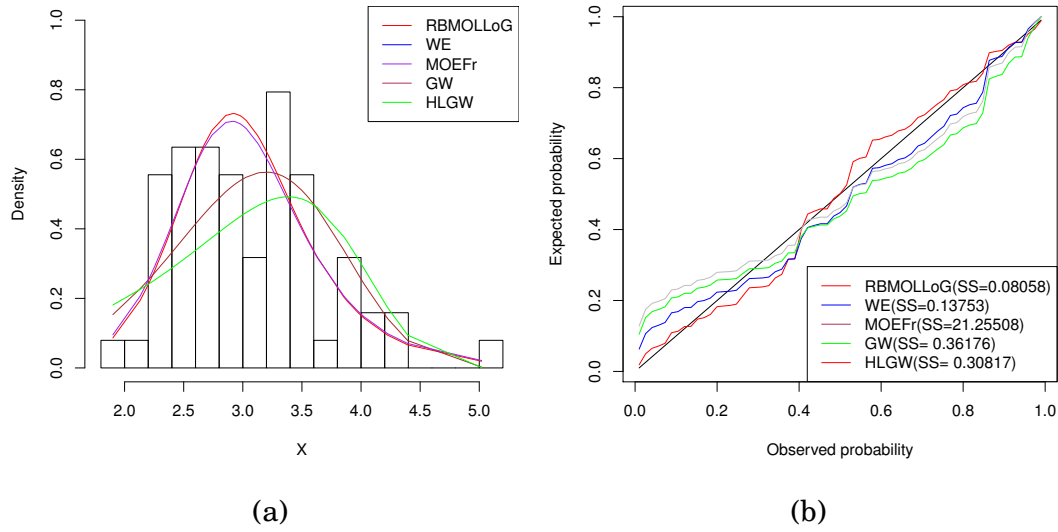


Figure 4.4: Fitted Density and Probability Plots of RBMO-LLoG Distribution for Carbon Fibre Strength (GPa) Data

From Table 4.4 and the fitted density plots (see Figure 4.4 above), we can confidently conclude that the RBMO-LLoG distribution provides a better fit to the carbon fibre data set better than the non-nested models used for comparison in this paper.

4.9.2 Toluene Exposure Concentrations Data

The second dataset represent time series data for toluene exposure concentrations (8 hr TWAs) for a worker doing stain removing as reported by [71]. The data is presented as follows: 0.9,1.1,1.9,2.1,2.6,2.9,3.1,3.2,4.9,4.9,5.2,5.8, 6.2,6.9,7.8,8.3,8.7,10.5,11.1,13.6,16.6,17.4,20.4,21.9,22.4,50.9,57.4,58.3,5 8.6,66.9.

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 0.61077 & 18.46410 & -0.27388 \\ 18.46410 & 758.64392 & -5.52854 \\ -0.27388 & -5.52854 & 0.17725 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$$\alpha \in [1.00150 \pm 1.53178], \delta \in [22.14470 \pm 53.98524] \text{ and } \lambda \in [1.44592 \pm 0.82518].$$

Table 4.5: Goodness-of-fit statistics and estimates of parameters for different models relating to time series data for toluene exposure concentrations

Model	Estimates			Statistics							
	α	δ	λ	$-2\log L$	AIC	CAIC	BIC	W^*	A^*	K-S	p-value
RBMO-LLoG	1.0055 (0.9009)	22.1447 (28.8452)	1.4459 (0.4204)	226.4	232.4	233.3	236.6	0.0315	0.2955	0.0968	0.94
WE	α	β	λ	228.4	234.4	235.3	238.6	0.1079	0.7919	0.1227	0.76
	280.2500 (3.23×10^{-6})	0.8921 (0.1201)	0.0001 (0.0001)								
IWMO	α	λ	θ	227.6	233.6	234.5	237.8	0.0461	0.3770	0.1042	0.90
	0.6481 (1.0675)	0.8202 (0.3589)	0.1502 (0.1907)								
MOEFr	α	δ	λ	231.6	237.6	238.5	241.8	10.4368	60.4128	0.9057	2.20×10^{-16}
	988.1 (8.54×10^{-8})	0.0087 (0.0097)	1.0001 (0.1561)								
GW	ω	λ	γ	231.7	237.7	238.7	241.9	0.1656	1.1362	0.1245	0.74
	20.9930 (4.56×10^{-5})	0.5915 (0.0895)	0.0063 (0.0022)								
HLGW	ω	λ	γ	318.4	324.4	325.3	328.6	0.0409	0.3598	0.6071	4.98×10^{-10}
	635.8300 (4.42×10^{-10})	0.0624 (0.0113)	0.0014 (0.0001)								
TLMO-LLoG	b	δ	c	226.6	232.6	233.5	236.8	0.0360	0.3154	0.0982	0.92
	3.39×10^2 (5.54×10^{-6})	0.1250 (0.0157)	0.0488 (0.0669)								

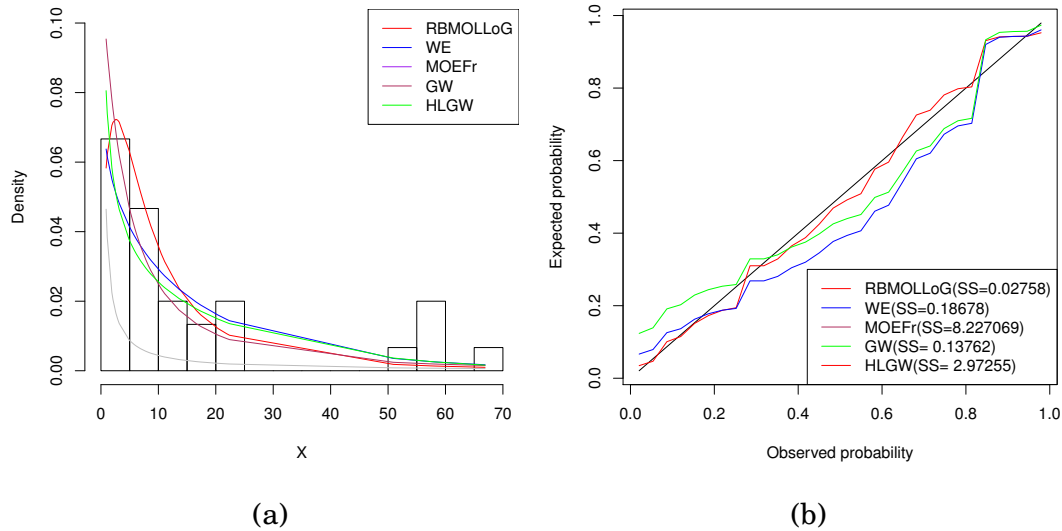


Figure 4.5: Fitted Density and Probability Plots of RBMO-LLoG Distribution for Toluene Exposure Concentrations Data

Table 4.5 also presents model estimates and from these results it is evident that the RBMO-LLoG distribution fits the time series dataset on toluene exposure concentrations better than other non-nested models considered. Furthermore, the fitted

plots (see Figure 4.5) indicate that the RBMO-LLoG model outperforms the competing models with a better fit.

4.10 Conclusions

In this chapter, a new family of distributions (RBMO-G) is introduced and some of its mathematical and statistical properties have been examined. Some of the special cases for this family of distributions are also discussed including the RBMO-LLoG distribution which has been applied to lifetime data examples and has been found to yield better fits compared to other existing equal parameter models.

Similarly, another new model namely, the Zografos-Balakrishnan Marshall Olkin-G family of distributions was also developed. The details on the development of this model are omitted in this thesis.

Chapter 5

The Exponentiated Half-logistic Odd Burr III-G Family of Distributions with Applications

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5.1 Introduction

This chapter presents the newly developed family of distributions namely the exponentiated half-logistic Odd Burr III-G (EHL-OBIII-G) family of distributions. Mathematical and statistical properties such as the hazard function, quantile function, moments, probability weighted moments, Rényi entropy and stochastic orders are derived. The model parameters for this new family of distributions are estimated based on the maximum likelihood estimation method. The usefulness of the newly proposed distributions is demonstrated via extensive simulation studies. Finally the proposed models and their special case is applied to real data sets to illustrate flexibility in fitting the data.

5.2 The Exponentiated Half-logistic Odd Burr III-G Model

Consider a family of distributions called the exponentiated half-logistic (EHL) family by [42] derived from the gamma-generator by [113]. The cumulative distribution function (cdf) of the EHL-G family of distributions is given by

$$F(x; \alpha, \lambda, \underline{\xi}) = \left\{ \frac{1 - [1 - G(x; \underline{\xi})]^\lambda}{1 + [1 - G(x; \underline{\xi})]^\lambda} \right\}^\alpha \quad (5.1)$$

and the corresponding probability distribution function (pdf) is given by

$$f(x; \alpha, \lambda, \underline{\xi}) = 2\alpha\lambda g(x; \underline{\xi}) [1 - G(x; \underline{\xi})]^{\lambda-1} \frac{(1 - [1 - G(x; \underline{\xi})]^\lambda)^{\alpha-1}}{(1 + [1 - G(x; \underline{\xi})]^\lambda)^{\alpha+1}}, \quad (5.2)$$

where $G(x; \underline{\xi})$ and $g(x; \underline{\xi})$ is the cdf and pdf respectively for any baseline distribution, and $\alpha, \lambda > 0$ are additional shape parameters with $\underline{\xi}$ as the vector of parameters. [12] proposed a new family of distributions called the Odd Burr III-G (OBIII-G) family having the cdf and pdf given by

$$F(x; a, b, \underline{\psi}) = \int_0^{\frac{G(x; \underline{\psi})}{1 - G(x; \underline{\psi})}} abt^{-b-1} (1 + t^{-a})^{-b-1} dt = \left\{ 1 + \left(\frac{1 - G(x; \underline{\psi})}{G(x; \underline{\psi})} \right)^a \right\}^{-b} \quad (5.3)$$

and

$$f(x; a, b, \underline{\psi}) = ab \frac{[1 - G(x; \underline{\psi})]^{a-1}}{G(x; \underline{\psi})^{a+1}} \left[1 + \left(\frac{1 - G(x; \underline{\psi})}{G(x; \underline{\psi})} \right)^a \right]^{-b-1} g(x; \underline{\psi}), \quad (5.4)$$

respectively, for $a, b > 0$ and the parameter vector $\underline{\psi}$.

If we let equation (5.3) to be the baseline cdf in equation (5.1), then the cdf of the EHL-OBIII-G family of distributions can be written as

$$F(x; a, b, \alpha, \underline{\psi}) = \left(\frac{\left(1 + \left(\frac{G(x; \underline{\psi})}{1 - G(x; \underline{\psi})} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{1 - G(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right]} \right)^\alpha, \quad (5.5)$$

and the corresponding pdf is given by

$$\begin{aligned}
f(x; a, b, \alpha, \underline{\psi}) &= 2\alpha ab \left(\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b-1} \\
&\times \left(1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \\
&\times \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a-1} \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^2},
\end{aligned} \tag{5.6}$$

for $a, b, \alpha, x > 0$ and parameter vector $\underline{\psi}$. The hazard rate function (hrf) is given by

$$\begin{aligned}
h(x; a, b, \alpha, \underline{\psi}) &= 2\alpha ab \left(\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b-1} \\
&\times \left(1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a-1} \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^2} \\
&\times \left(1 - \left(\frac{\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right]} \right)^{\alpha} \right)^{-1}.
\end{aligned} \tag{5.7}$$

5.2.1 Quantile Function

The quantile function for the EHL-OBIII-G family of distributions is derived by inverting the following function

$$\left(\frac{\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right]} \right)^{\alpha} = u,$$

for $0 < u < 1$. Note that

$$\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-a}\right)^{-b} = \frac{2u^{\frac{1}{\alpha}}}{1 + u^{\frac{1}{\alpha}}},$$

that is,

$$\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} = \left(\left(\frac{2u^{\frac{1}{\alpha}}}{1 + u^{\frac{1}{\alpha}}}\right)^{-\frac{1}{b}} - 1\right)^{-\frac{1}{a}}. \quad (5.8)$$

The expression further simplifies to

$$G(x; \underline{\psi}) = \left(\left(\left(\frac{2u^{\frac{1}{\alpha}}}{1 + u^{\frac{1}{\alpha}}}\right)^{-\frac{1}{b}} - 1\right)^{-\frac{1}{a}} + 1\right)^{-1},$$

and therefore, the quantile function is given by

$$Q_{X_{(n)}}(u) = G^{-1} \left[\left(\left(\left(\frac{2u^{\frac{1}{\alpha}}}{1 + u^{\frac{1}{\alpha}}} \right)^{-\frac{1}{b}} - 1 \right)^{-\frac{1}{a}} + 1 \right)^{-1} \right], \quad (5.9)$$

which can be solved using iterative methods.

5.2.2 Series Expansion of the Density Function

In this section, series expansion of the pdf is presented. The linear representation of the pdf allows for useful mathematical and statistical properties to be derived. Recall that the pdf of EHL-OBIII-G family of distributions is given by

$$\begin{aligned} f(x; a, b, \alpha, \underline{\psi}) &= 2\alpha ab \left(\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b-1} \\ &\times \left(1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a-1} \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^2}, \end{aligned}$$

and using the expansion

$$\begin{aligned}
\left(1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)} &= \sum_{p=0}^{\infty} (-1)^p \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \\
&\times \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-a}\right)^{-b}\right]^p \\
&= \sum_{p,q=0}^{\infty} (-1)^{p+q} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \\
&\times \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-a}\right)^{-bq},
\end{aligned}$$

we can write the EHL-OBIII-G pdf as

$$\begin{aligned}
f(x; a, b, \alpha, \underline{\psi}) &= 2\alpha ab \sum_{p,q=0}^{\infty} (-1)^{p+q} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \\
&\times \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-a}\right)^{-b(q+\alpha)-1} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-a-1} \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^2}.
\end{aligned}$$

Furthermore, by applying the expansion

$$\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-a}\right)^{-b(q+\alpha)-1} = \sum_{l=0}^{\infty} \binom{-b(q+\alpha)-1}{l} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-al},$$

we get

$$\begin{aligned}
f(x; a, b, \alpha, \underline{\psi}) &= 2\alpha ab \sum_{p,q,l=0}^{\infty} (-1)^{p+q} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \binom{-b(q+\alpha)-1}{l} \\
&\times (G(x; \underline{\psi}))^{-a(l+1)-1} \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^{-a(l+1)+1}}.
\end{aligned}$$

Using the expansion

$$(G(x; \underline{\psi}))^{-a(l+1)-1} = (1 - \overline{G}(x; \underline{\psi}))^{-a(l+1)-1} = \sum_{j=0}^{\infty} \binom{-a(l+1)-1}{j} (-1)^j \overline{G}^j(x; \underline{\psi}),$$

we have

$$\begin{aligned}
f(x; a, b, \alpha, \underline{\psi}) &= 2\alpha ab \sum_{p,q,l,j=0}^{\infty} (-1)^{p+q+j} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \binom{-b(q+\alpha)-1}{l} \\
&\times \binom{-a(l+1)-1}{j} g(x; \underline{\psi}) (\overline{G}(x; \underline{\psi}))^{a(l+1)-1+j} \\
&= 2\alpha ab \sum_{p,q,l,j,k=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \binom{-b(q+\alpha)-1}{l} \\
&\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} g(x; \underline{\psi}) (G(x; \underline{\psi}))^k \\
&= 2\alpha ab \sum_{p,q,l,j,k=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \binom{-b(q+\alpha)-1}{l} \\
&\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} \frac{k+1}{k+1} g(x; \underline{\psi}) (G(x; \underline{\psi}))^k \\
&= \sum_{k=0}^{\infty} w_k g_k(x; \underline{\psi}), \tag{5.10}
\end{aligned}$$

where $g_k(x; \underline{\psi}) = (k+1) (G(x; \underline{\psi}))^k g(x; \underline{\psi})$ is the exponentiated-G (Exp-G) distribution with power parameter k and

$$\begin{aligned}
w_k &= \sum_{p,q,l,j=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \binom{-b(q+\alpha)-1}{l} \\
&\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} \frac{2\alpha ab}{k+1}. \tag{5.11}
\end{aligned}$$

5.2.3 Some Special Cases

In this subsection, some of the special cases for the EHL-OBIII-G family of distributions are presented by changing the baseline distribution function $G(x; \underline{\psi})$ to some flexible distributions. The parameter vector space is limited to at most 2 component vector to avoid over parametrization and redundancy.

5.2.3.1 EHL-OBIII-Log-Logistic (EHL-OBIII-LLoG) Distribution

Consider the log-logistic distribution as the baseline distribution with parameter $\lambda > 0$ having cdf and pdf $G(x; \lambda) = 1 - (1 + x^\lambda)^{-1}$ and $g(x; \lambda) = \lambda x^{\lambda-1} (1 + x^\lambda)^{-2}$, respectively. The cdf, pdf and hrf of EHL-OBIII-LLoG distribution are given by

$$F(x; a, b, \alpha, \lambda) = \left(\frac{\left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b} \right]} \right)^\alpha, \quad (5.12)$$

$$\begin{aligned} f(x) &= 2\alpha ab \left(\left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b-1} \\ &\times \left(1 + \left[1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a-1} \\ &\times \frac{\lambda x^{\lambda-1} (1 + x^\lambda)^{-2}}{((1 + x^\lambda)^{-1})^2}, \end{aligned} \quad (5.13)$$

and

$$\begin{aligned} h(x) &= 2\alpha ab \left(\left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b-1} \\ &\times \left(1 + \left[1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a-1} \\ &\times \frac{\lambda x^{\lambda-1} (1 + x^\lambda)^{-2}}{((1 + x^\lambda)^{-1})^2} \left(1 - \left(\frac{\left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-a} \right)^{-b} \right]} \right)^\alpha \right)^{-1}, \end{aligned}$$

respectively, for $a, b, \alpha, \lambda > 0$.

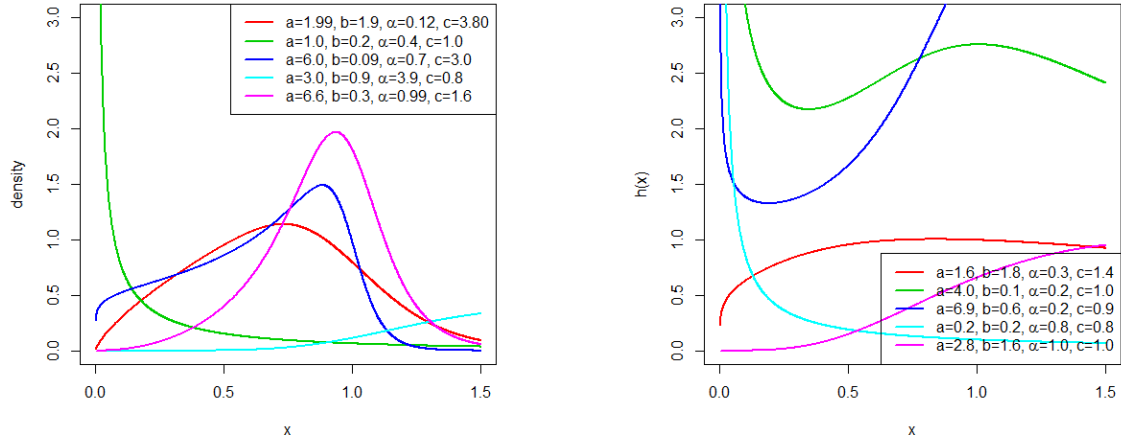


Figure 5.1: Plots of the pdf and hrf for EHL-OBIII-LLoG Distribution

Figure 5.1 illustrates the flexible nature of the EHL-OBIII-LLoG distribution for selected parameter values. The pdfs of the EHL-OBIII-LLoG distribution can adopt various shapes that include reverse-J, uni-modal, left or right skewed shapes. In addition, the EHL-OBIII-LLoG distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

5.2.3.2 EHL-OBIII-Exponential (EHL-OBIII-E) Distribution

Let the exponential distribution be the baseline distribution with pdf and cdf given by $g(x; \lambda) = \lambda e^{-\lambda x}$ and $G(x; \lambda) = 1 - e^{-\lambda x}$, respectively, for $\lambda > 0$, then the cdf, pdf and hrf of the EHL-OBIII-E distribution are given by

$$F(x; a, b, \alpha, \lambda) = \left(\frac{\left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b} \right]} \right)^{\alpha}, \quad (5.14)$$

$$\begin{aligned}
f(x) &= 2\alpha ab \left(\left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b-1} \\
&\times \left(1 + \left[1 - \left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a-1} \\
&\times \frac{\lambda e^{-\lambda x}}{(e^{-\lambda x})^2}
\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
h(x) &= 2\alpha ab \left(\left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b-1} \\
&\times \left(1 + \left[1 - \left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a-1} \frac{\lambda e^{-\lambda x}}{(e^{-\lambda x})^2} \\
&\times \left(1 - \frac{\left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right)^{-a} \right)^{-b} \right]} \right)^{\alpha-1},
\end{aligned}$$

respectively, for $a, b, \alpha, \lambda > 0$.

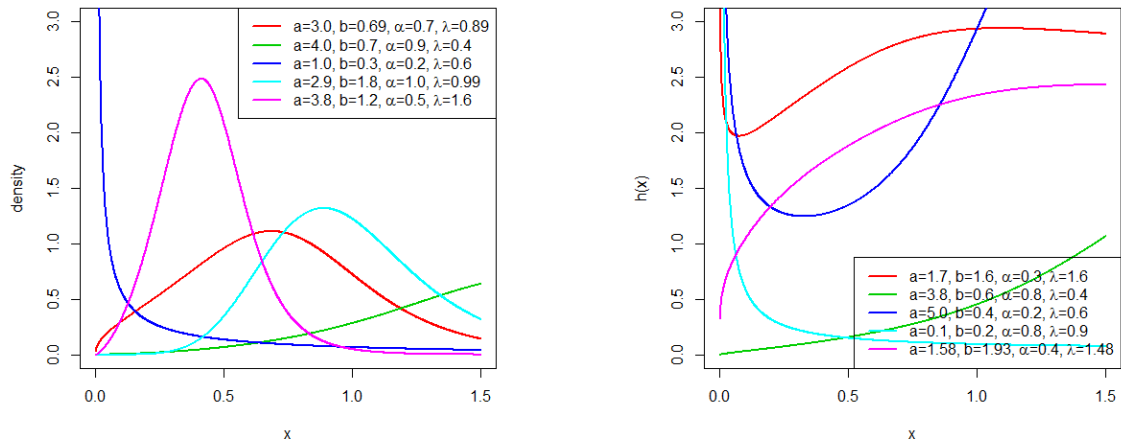


Figure 5.2: Plots of the pdf and hrf for EHL-OBIII-E Distribution

Figure 5.2 demonstrate the efficacy of the EHL-OBIII-E distribution for selected parameter values. The pdfs of the EHL-OBIII-E distribution takes various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the EHL-OBIII-E distribution gives decreasing, increasing and bathtub followed by upside down bathtub shapes.

5.2.3.3 EHL-OBIII-Lomax (EHL-OBIII-Lx) Distribution

Suppose that we take the baseline distribution to be a Lomax distribution with cdf and pdf given by $G(x; \gamma, \lambda) = 1 - (1 + \lambda x)^{-\gamma}$ and $g(x; \gamma, \lambda) = \gamma \lambda (1 + \lambda x)^{-\gamma-1}$, for $\gamma, \lambda > 0$, and $x > 0$, then we obtain the EHL-OBIII-Lx distribution with cdf, pdf and hrf given by

$$F(x; a, b, \alpha, \gamma, \lambda) = \left(\frac{\left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b} \right]} \right)^{\alpha}, \quad (5.16)$$

$$\begin{aligned} f(x) &= 2\alpha ab \left(\left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b-1} \\ &\times \left(1 + \left[1 - \left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a-1} \\ &\times \frac{\gamma \lambda (1 + \lambda x)^{-\gamma-1}}{((1 + \lambda x)^{-\gamma})^2} \end{aligned} \quad (5.17)$$

and

$$\begin{aligned}
h(x) &= 2\alpha ab \left(\left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b-1} \\
&\times \left(1 + \left[1 - \left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a-1} \\
&\times \frac{\gamma \lambda (1 + \lambda x)^{-\gamma-1}}{((1 + \lambda x)^{-\gamma})^2} \left(1 - \left(\frac{\left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{1 - (1 + \lambda x)^{-\gamma}}{(1 + \lambda x)^{-\gamma}} \right)^{-a} \right)^{-b} \right]} \right)^{\alpha} \right)^{-1},
\end{aligned}$$

respectively, for $a, b, \alpha, \gamma, \lambda > 0$.

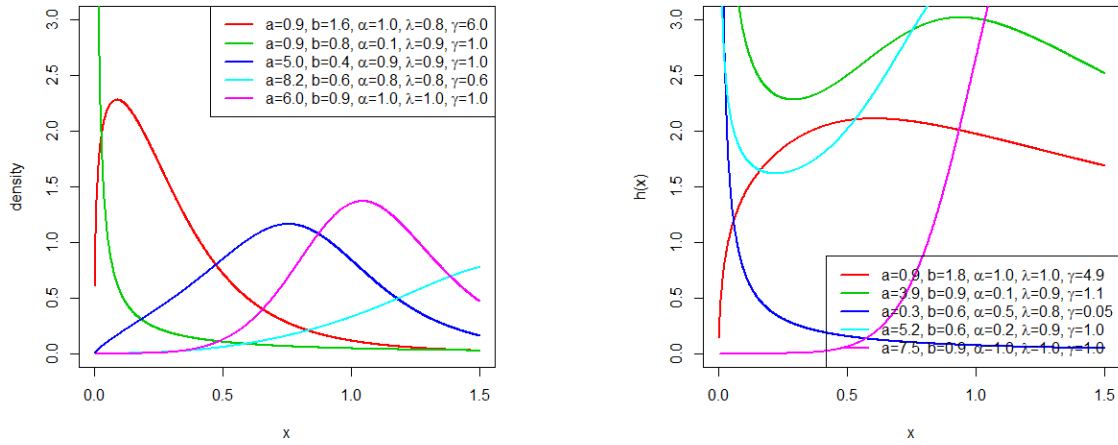


Figure 5.3: Plots of the pdf and hrf for EHL-OBIII-Lx Distribution

From Figure 5.3 above, we note that the EHL-OBIII-Lx distribution can adopt various flexible shapes and these include reverse-J, uni-modal, left or right skewed shapes. It can also be noted that the hrf of the EHL-OBIII-Lx distribution gives decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

5.2.4 EHL-OBIII-Lindley (EHL-OBIII-L) Distribution

If we let Lindley distribution be the baseline distribution with pdf and cdf given by $g(x; \lambda) = \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}$ and $G(x; \lambda) = 1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}$, respectively, for $\lambda > 0$, then

the cdf, pdf and hrf of the EHL-OBIII-L distribution are given by

$$F(x; a, b, \alpha, \lambda) = \left(\frac{\left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a} \right)^{-b} \right]} \right)^{\alpha}, \quad (5.18)$$

$$\begin{aligned} f(x; a, b, \alpha, \lambda) &= 2\alpha ab \left(\left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \\ &\times \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a} \right)^{-b-1} \\ &\times \left(1 + \left[1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \\ &\times \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a-1} \frac{\lambda^2 (1+x)e^{-\lambda x}}{\left((1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x} \right)^2} \end{aligned} \quad (5.19)$$

and

$$\begin{aligned} h(x; a, b, \alpha, \lambda) &= 2\alpha ab \left(\left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a} \right)^{-b} \right)^{\alpha-1} \\ &\times \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a} \right)^{-b-1} \\ &\times \left(1 + \left[1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha+1)} \\ &\times \left(\frac{1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}} \right)^{-a-1} \frac{\lambda^2 (1+x)e^{-\lambda x}}{\left((1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x} \right)^2} \end{aligned}$$

$$\times \left(1 - \frac{\left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1 + \lambda}) e^{-\lambda x}}{(1 + \frac{\lambda x}{1 + \lambda}) e^{-\lambda x}} \right)^{-a} \right)^{-b}}{1 + \left[1 - \left(1 + \left(\frac{1 - (1 + \frac{\lambda x}{1 + \lambda}) e^{-\lambda x}}{(1 + \frac{\lambda x}{1 + \lambda}) e^{-\lambda x}} \right)^{-a} \right)^{-b} \right]} \right)^{\alpha} \right)^{-1},$$

respectively, for $a, b, \beta, \lambda > 0$.

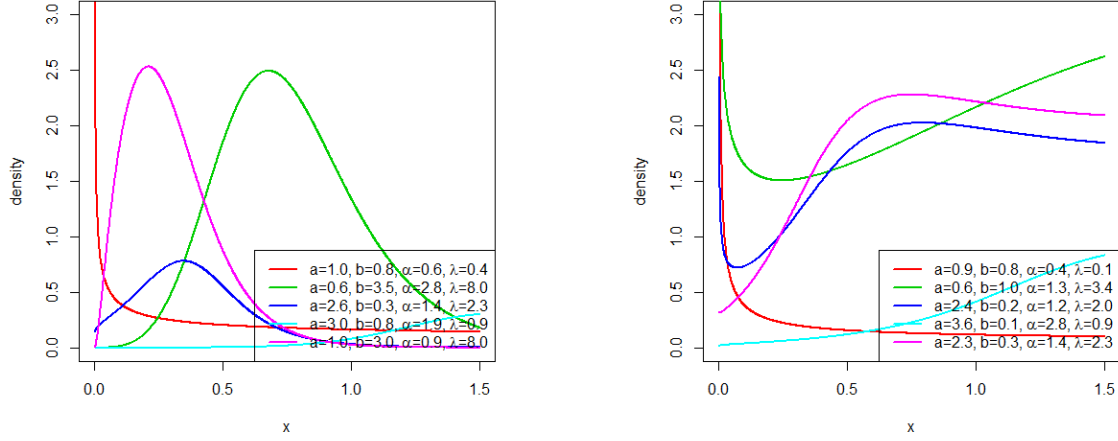


Figure 5.4: Plots of the pdf and hrf for EHL-OBIII-L Distribution

Figure 5.4 reveals the flexibility of the EHL-OBIII-L distribution. The pdfs of the EHL-OBIII-L distribution can flexibly take different shapes that include reverse-J, uni-modal, left or right skewed shapes. Additionally, the EHL-OBIII-L distribution hrf plots also give decreasing, increasing, bathtub and upside down bathtub shapes and decreasing-increasing-decreasing shapes.

5.3 Some Mathematical Properties

In this section, we derive some useful mathematical and statistical properties for the EHL-OBIII-G family of distributions such as moments, conditional moments, Lorenz and Bonferroni curves, probability weighted moments, Rényi entropy and maximum likelihood estimates.

5.3.1 Moments

Let $Y_k \sim \text{Exp-G}(k)$, then using equation (5.10) the s^{th} raw moment, μ'_s of the EHL-OBIII-G family of distributions is obtained as

$$\mu'_s = E(X^s) = \int_{-\infty}^{\infty} x^s f(x) dx = \sum_{k=0}^{\infty} w_k E(Y_k^s),$$

where w_k is given by equation (5.11). We present the first five moments for the EHL-OBIII-E distribution and the corresponding standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) from selected values. These results are given under Table 5.2. The quantile values are also presented under Table 5.1.

Table 5.1: Quantiles for EHL-OBIII-E Distribution for selected parameter values

	(a,b, α , λ)					
u	(1.4,8.5,6.1,0.4)	(1.3,8.3,6.4,0.6)	(1.1,9.9,5.8,0.8)	(1.6,9.1,6,0.3)	(1.2,8.2,6.2,0.6)	(1.1,8.9,5.7,0.2)
0.1	3.0531	2.0120	1.7548	3.7952	2.1975	7.8777
0.2	2.3981	1.6071	1.3186	3.0811	1.6747	5.3712
0.3	2.1813	1.4615	1.1694	2.8329	1.5039	4.7265
0.4	2.0543	1.3753	1.0823	2.6859	1.4046	4.3598
0.5	1.9664	1.3154	1.0224	2.5836	1.3362	4.1103
0.6	1.9001	1.2702	0.9775	2.5062	1.2847	3.9244
0.7	1.8474	1.2341	0.9419	2.4444	1.2440	3.7779
0.8	1.8040	1.2043	0.9128	2.3933	1.2104	3.6580
0.9	1.7672	1.1791	0.8882	2.3499	1.1821	3.5572

Table 5.2: Moments for EHL-OBIII-E Distribution for selected parameter values

	(a,b, α , λ)				
	(0.8,1.4,0.6,1.5)	(0.3,1.7,1.1,1.8)	(0.8,2.2,0.7,2.4)	(0.4,2.3,0.7,1.7)	(0.5,1.5,0.7,0.9)
$E(X)$	0.2298	0.0843	0.3063	0.1210	0.1169
$E(X^2)$	0.1378	0.0525	0.1930	0.0746	0.0692
$E(X^3)$	0.0980	0.0384	0.1395	0.0543	0.0493
$E(X^4)$	0.0759	0.0304	0.1088	0.0427	0.0383
$E(X^5)$	0.0619	0.0252	0.0889	0.0352	0.0313
SD	0.2916	0.2131	0.3149	0.2449	0.2356
CV	1.2691	2.5277	1.0280	2.0239	2.0157
CS	1.1002	2.7252	0.6298	2.0916	2.1548
CK	2.9224	9.4740	2.0402	6.2188	6.6049

5.3.2 Probability Weighted Moments (PWMs)

The $(i, \eta)^{th}$ Probability Weighted Moment (PWM) of X denoted by $\kappa_{i,\eta}$ is

$$\kappa_{i,\eta} = E(X^i (F(X))^\eta) = \int_{-\infty}^{\infty} x^i (F(x))^\eta f(x) dx.$$

Using equations (5.5) and (5.6), we can write

$$\begin{aligned} f(x)(F(x))^\eta &= 2\alpha ab \left(\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right)^{\alpha(\eta+1)-1} \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b-1} \\ &\times \left(1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha(\eta+1)+1)} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a-1} \\ &\times \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^2}. \end{aligned}$$

Using the series expansion

$$\begin{aligned} \left(1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha(\eta+1)+1)} &= \sum_{p=0}^{\infty} (-1)^p \frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!} \\ &\times \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right]^p \\ &= \sum_{p,q=0}^{\infty} (-1)^{p+q} \frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!} \binom{p}{q} \\ &\times \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-bq}, \end{aligned}$$

we can write

$$\begin{aligned} f(x)(F(x))^\eta &= 2\alpha ab \sum_{p,q=0}^{\infty} (-1)^{p+q} \frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!} \binom{p}{q} \\ &\times \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b(q+(\alpha(\eta+1)))-1} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a-1} \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^2}. \end{aligned}$$

Furthermore, applying the series expansion

$$\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-a}\right)^{-b(q + (\alpha(\eta + 1))) - 1} = \sum_{l=0}^{\infty} \binom{-b(q + (\alpha(\eta + 1))) - 1}{l} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})}\right)^{-al},$$

we get

$$\begin{aligned} f(x)(F(x))^\eta &= 2\alpha ab \sum_{p,q,l=0}^{\infty} (-1)^{p+q} \frac{\Gamma(\alpha(\eta + 1) + 1 + p)}{\Gamma(\alpha(\eta + 1) + 1)p!} \binom{p}{q} \binom{-b(q + (\alpha(\eta + 1))) - 1}{l} \\ &\times (G(x; \underline{\psi}))^{-a(l+1)-1} \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^{-a(l+1)+1}}. \end{aligned}$$

Using the expansion

$$(G(x; \underline{\psi}))^{-a(l+1)-1} = (1 - \overline{G}(x; \underline{\psi}))^{-a(l+1)-1} = \sum_{j=0}^{\infty} \binom{-a(l+1)-1}{j} (-1)^j \overline{G}^j(x; \underline{\psi}),$$

we have

$$\begin{aligned} f(x)(F(x))^\eta &= 2\alpha ab \sum_{p,q,l,j=0}^{\infty} (-1)^{p+q+j} \frac{\Gamma(\alpha(\eta + 1) + 1 + p)}{\Gamma(\alpha(\eta + 1) + 1)p!} \binom{p}{q} \binom{-b(q + (\alpha(\eta + 1))) - 1}{l} \\ &\times \binom{-a(l+1)-1}{j} g(x; \underline{\psi}) (\overline{G}(x; \underline{\psi}))^{a(l+1)-1+j} \\ &= 2\alpha ab \sum_{p,q,l,j,k=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha(\eta + 1) + 1 + p)}{\Gamma(\alpha(\eta + 1) + 1)p!} \binom{p}{q} \binom{-b(q + (\alpha(\eta + 1))) - 1}{l} \\ &\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} g(x; \underline{\psi}) (G(x; \underline{\psi}))^k \\ &= \sum_{w=0}^{\infty} w_k^* g_k(x; \underline{\psi}), \end{aligned} \tag{5.20}$$

where $g_k(x; \underline{\psi}) = (k+1) (G(x; \underline{\psi}))^k g(x; \underline{\psi})$ is the exponentiated-G (Exp-G) distribution with power parameter k and

$$\begin{aligned} w_k^* &= \sum_{p,q,l,j=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha(\eta + 1) + 1 + p)}{\Gamma(\alpha(\eta + 1) + 1)p!} \binom{p}{q} \binom{-b(q + (\alpha(\eta + 1))) - 1}{l} \\ &\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} \frac{2\alpha ab}{k+1}. \end{aligned} \tag{5.21}$$

Finally, the PWMs of the EHL-OBIII-G family of distributions can be written as

$$\kappa_{i,\eta} = \int_0^\infty x^i \sum_{k=0}^\infty w_k^* g_k(x; \underline{\psi}) dx = \sum_{k=0}^\infty w_k^* \int_{-\infty}^\infty x^i g_k(x; \underline{\psi}) dx,$$

which shows that the $(i, \eta)^{th}$ PWMs of EHL-OBIII-G family of distributions can be obtained from the moments of the E-G distribution.

5.3.2.1 Rényi Entropy

The Rényi entropy ([90]), is an extension of Shannon Entropy ([93]) and is defined as

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f(x; a, b, \alpha, \underline{\psi})]^v dx \right), v \neq 1, v > 0. \quad (5.22)$$

Note that as $v \rightarrow 1$, the Rényi entropy tends to Shannon entropy and $I_R(v)$ for the EHL-OBIII-G distribution can be written as

$$\begin{aligned} I_R(v) = & \frac{1}{1-v} \log \left(\int_0^\infty \left[(2\alpha ab)^v \left(\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right)^{v\alpha-v} \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-vb-v} \right. \right. \\ & \times \left. \left. \left(1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)^{-v(\alpha+1)} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-va-v} \frac{g^v(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^{2v}} \right] dx \right). \end{aligned}$$

Using the generalized binomial series expansion

$$\begin{aligned} \left(1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)^{-v(\alpha+1)} &= \sum_{p=0}^\infty (-1)^p \frac{\Gamma(v(\alpha+1) + p)}{\Gamma(v(\alpha+1)) p!} \\ &\times \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right]^p \\ &= \sum_{p,q=0}^\infty (-1)^{p+q} \frac{\Gamma(v(\alpha+1) + p)}{\Gamma(v(\alpha+1)) p!} \binom{p}{q} \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-bq}, \end{aligned}$$

we can write

$$\begin{aligned}
f^v(x; a, b, \alpha, \underline{\psi}) &= (2\alpha ab)^v \sum_{p,q=0}^{\infty} (-1)^{p+q} \frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!} \binom{p}{q} \\
&\times \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b(q+v+(v\alpha-v))-1} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-va-v} \frac{g^v(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^{2v}}.
\end{aligned}$$

Furthermore, applying the expansion

$$\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b(q+v+(v\alpha-v))-1} = \sum_{l=0}^{\infty} \binom{-b(q+v+(v\alpha-v))-1}{l} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-al},$$

we get

$$\begin{aligned}
f^v(x; a, b, \alpha, \underline{\psi}) &= (2\alpha ab)^v \sum_{p,q=0}^{\infty} (-1)^{p+q} \frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!} \binom{p}{q} \binom{-b(q+v+(v\alpha-v))-1}{l} \\
&\times (G(x; \underline{\psi}))^{-a(l+v)-v} \frac{g^v(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^{-a(l+v)+v}}.
\end{aligned}$$

Using the generalized binomial series expansion

$$(G(x; \underline{\psi}))^{-a(l+v)-v} = (1 - \overline{G}(x; \underline{\psi}))^{-a(l+v)-v} = \sum_{j=0}^{\infty} \binom{-a(l+v)-v}{j} (-1)^j \overline{G}^j(x; \underline{\psi}),$$

we have

$$\begin{aligned}
f^v(x; a, b, \alpha, \underline{\psi}) &= (2\alpha ab)^v \sum_{p,q,j=0}^{\infty} (-1)^{p+q+j} \frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!} \binom{p}{q} \binom{-b(q+v+(v\alpha-v))-1}{l} \\
&\times \binom{-a(l+v)-v}{j} g^v(x; \underline{\psi}) (\overline{G}(x; \underline{\psi}))^{a(l+v)-v+j} \\
&= (2\alpha ab)^v \sum_{p,q,j,k=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!} \binom{p}{q} \binom{-b(q+v+(v\alpha-v))-1}{l} \\
&\times \binom{a(l+v)-v+j}{k} \binom{-a(l+v)-v}{j} g^v(x; \underline{\psi}) (G(x; \underline{\psi}))^k.
\end{aligned} \tag{5.23}$$

Finally, we can write the Rényi entropy for the EHL-OBIII-G family of distributions as

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v} \log \left((2\alpha ab)^v \sum_{p,q,j,k=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!} \binom{p}{q} \binom{-b(q+v+(v\alpha-v))-1}{l} \right. \\
&\quad \times \left. \binom{a(l+v)-v+j}{k} \binom{-a(l+v)-v}{j} \frac{1}{\left[1+\frac{k}{v}\right]^v} \int_0^{\infty} \left(\left[1+\frac{k}{v}\right] G(x;\underline{\psi})^{\frac{k}{v}} g(x;\underline{\psi}) dx \right)^v \right) \\
&= \frac{1}{1-v} \log \left[\sum_{k=0}^{\infty} w_k^{**} \exp((1-v)I_{REG}) \right],
\end{aligned}$$

for $v > 0$, $v \neq 1$, where $I_{REG} = \frac{1}{1-v} \log \left[\int_0^{\infty} \left(\left[\frac{k}{v} + 1 \right] (G(x;\underline{\psi}))^{\frac{k}{v}} g(x;\underline{\psi}) \right)^v dx \right]$ is the Rényi entropy of Exp-G distribution with power parameter $(\frac{k}{v} + 1)$, and

$$\begin{aligned}
w_k^{**} &= \frac{1}{1-v} \log \left((2\alpha ab)^v \sum_{p,q,j,k=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!} \binom{p}{q} \binom{-b(q+v+(v\alpha-v))-1}{l} \right. \\
&\quad \times \left. \binom{a(l+v)-v+j}{k} \binom{-a(l+v)-v}{j} \frac{1}{\left[1+\frac{k}{v}\right]^v} \right).
\end{aligned}$$

5.3.3 Stochastic Ordering

The stochastic ordering for random variables is a common and widely used concept. The usual stochastic order, the hazard rate order and the likelihood ratio order are the commonly known orders of distribution functions. These three orders are defined under section 1.3.6 in chapter 1 and also see [92] for additional details.

Let X_1 and X_2 be the two independent random variables following $EHL - OBIII - G(a, b, \alpha_1, \underline{\varphi})$ and $EHL - OBIII - G(a, b, \alpha_2, \underline{\varphi})$ distributions, then the pdfs are given by

$$\begin{aligned}
f_1(x) &= 2\alpha_1 ab \left(\left(1 + \left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})} \right)^{-a} \right)^{-b} \right)^{\alpha_1-1} \left(1 + \left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})} \right)^{-a} \right)^{-b-1} \\
&\quad \times \left(1 + \left[1 - \left(1 + \left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha_1+1)} \left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})} \right)^{-a-1} \frac{g(x;\underline{\psi})}{(\overline{G}(x;\underline{\psi}))^2}
\end{aligned}$$

and

$$f_2(x) = 2\alpha_2 ab \left(\left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right)^{\alpha_2 - 1} \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b-1} \\ \times \left(1 + \left[1 - \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)^{-(\alpha_2 + 1)} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a-1} \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^2}.$$

The ratio, $\frac{f_1(x)}{f_2(x)}$ takes the form

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha_1}{\alpha_2} z^{\alpha_1 - \alpha_2} (1 + [1 - z])^{-(\alpha_1 - \alpha_2 + 2)}, \quad (5.24)$$

where

$$z = \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b}.$$

If we differentiate equation (5.24) with respect to x , we get

$$\frac{d}{dx} \left(\frac{f_1(x)}{f_2(x)} \right) = \frac{\alpha_1}{\alpha_2} (\alpha_1 - \alpha_2) z^{\alpha_1 - \alpha_2} (1 + [1 - z])^{-(\alpha_1 - \alpha_2 + 2)} \left[\frac{1}{z} + \frac{(1 + 2(\alpha_1 - \alpha_2))}{(1 + [1 - z])} \right],$$

where $z' = ab \left(1 + \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a} \right)^{-b-1} \left(\frac{G(x; \underline{\psi})}{\overline{G}(x; \underline{\psi})} \right)^{-a-1} \frac{g(x; \underline{\psi})}{(\overline{G}(x; \underline{\psi}))^2}$, and finally if $\alpha_2 < \alpha_1$, then $\frac{d}{dx} \left(\frac{f_1(x)}{f_2(x)} \right) < 0$, and therefore, the likelihood ratio order $X_1 <_{\ell_r} X_2$ exists. Consequently, $X_1 <_{st} X_2$, since $X_1 <_{\ell_r} X_2 \implies X_1 <_{hr} X_2 \implies X_1 <_{st} X_2$, where $X_1 <_{hr} X_2$ and $X_1 <_{st} X_2$ denote hazard rate order and stochastic order, respectively.

5.3.4 Maximum Likelihood Estimation

In this section, the method of maximum likelihood estimation (MLE) is used to estimate the model parameters. Let $X \sim EHL-OBIII-G(a, b, \alpha, \underline{\psi})$ and $\Delta = (a, b, \alpha, \underline{\psi})^T$ be the vector of model parameters. The log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the EHL-OBIII-G family of distributions is given by

$$\begin{aligned}
\ell_n(\Delta) = & n \ln(2\alpha ab) + (\alpha - 1) \sum_{i=1}^n \ln \left(\left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right) \\
& + (-b - 1) \sum_{i=1}^n \ln \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right) + (-a - 1) \sum_{i=1}^n \ln \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right) \\
& - (\alpha + 1) \sum_{i=1}^n \ln \left(1 + \left[1 - \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right) \\
& + \sum_{i=1}^n \ln(g(x_i; \underline{\psi})) - 2 \sum_{i=1}^n \ln(\overline{G}(x_i; \underline{\psi})).
\end{aligned}$$

The elements of the score vector are given in **appendix H.8**. To obtain the estimates of model parameters denoted by $\hat{\Delta}$, the Newton-Raphson procedure is used to solve the system of non-linear equations $(\frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \underline{\psi}_k})^T = \mathbf{0}$. The multivariate normal distribution $N_{q+3}(\underline{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\underline{0} = (0, 0, 0, \underline{0})^T$ and $J(\hat{\Delta})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$, is critical in approximating confidence intervals for model parameters.

5.4 Monte Carlo Simulations

In order to evaluate consistency of the MLEs, simulation experiments were conducted using the EHL-OBIII-E distribution. The MLEs are computed for each sample by generating $N=1000$ random samples of size n obtaining average bias and Root Mean Square Error (RMSE) from each sample. The results are presented under table 5.3. We note that as the sample size n increase the values of average bias and RMSE decreases which satisfies convergence propoerties of consistent and unbiased estimators.

5.5 Applications

We present applications based on the two real-life data sets to evaluate the efficacy of the proposed model and measure its performance in comparison to other non-nested models. The first data set is reported by [23] and relates to the survival times (in

Table 5.3: Simulation Results for EHL-OBIII-E Distribution; Mean, Average Bias and RMSE

Parameter	n	Set I: $a=1.4, b=0.6, \alpha=1.8, \lambda=2.1$			Set II: $a=2.6, b=0.9, \alpha=1.4, \lambda=0.7$			Set III: $a=1.8, b=1.3, \alpha=0.8, \lambda=2.3$		
		Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE
a	50	4.3848	2.9848	0.5194	4.1738	1.5738	3.0836	3.7459	1.9459	0.3548
	100	4.2985	2.8985	0.3322	3.4377	0.8377	2.2148	3.3726	1.5726	0.2216
	200	3.6489	2.2489	0.2997	3.1527	0.5527	1.8854	2.7388	0.9388	0.1970
	400	3.4375	2.0375	0.2510	2.9376	0.3376	1.6376	2.4628	0.6628	0.1744
	800	2.3385	0.9385	0.2250	2.8365	0.2365	0.6049	1.9285	0.1285	0.1377
	1200	1.6028	0.2028	0.1488	2.6838	0.0838	0.4307	1.8827	0.0827	0.1164
	1800	1.4204	0.0204	0.1325	2.6184	0.0184	0.2819	1.8327	0.0327	0.0895
b	50	1.7489	1.1489	0.4658	1.9264	1.0264	0.7609	2.6374	1.3374	0.5840
	100	1.3385	0.7385	0.4335	1.7256	0.8256	0.6576	2.3787	1.0787	0.4755
	200	1.2538	0.6538	0.3983	1.2737	0.3737	0.6272	1.8266	0.5266	0.4579
	400	1.1439	0.5439	0.3655	1.1828	0.2828	0.5864	1.5218	0.2218	0.4353
	800	0.8399	0.2399	0.3536	1.1029	0.2029	0.5499	1.4375	0.1375	0.3847
	1200	0.7039	0.1039	0.2977	0.9938	0.0938	0.5116	1.3848	0.0848	0.3586
	1800	0.6298	0.0298	0.2723	0.9204	0.0204	0.4485	1.3265	0.0265	0.2073
α	50	3.1939	1.3939	0.2710	2.1654	0.7654	0.4236	1.6267	0.8267	0.4197
	100	2.8399	1.0399	0.2481	2.0918	0.6918	0.3992	1.4878	0.6878	0.3319
	200	2.8028	1.0028	0.2114	1.8928	0.4928	0.3795	1.2528	0.4528	0.3095
	400	2.2739	0.4739	0.1622	1.7628	0.3628	0.3431	1.0266	0.2266	0.2652
	800	2.0177	0.2177	0.1496	1.5277	0.1277	0.3158	0.9827	0.1827	0.2232
	1200	1.9274	0.1274	0.1012	1.4829	0.0829	0.2917	0.8928	0.0928	0.1876
	1800	1.8199	0.0199	0.0812	1.4593	0.0593	0.2508	0.8478	0.0478	0.1623
λ	50	3.6488	1.5488	0.5090	1.6749	0.9749	0.7456	5.3737	3.0737	0.4352
	100	3.5388	1.4388	0.4269	1.5784	0.8784	0.7172	3.2668	0.9668	0.3372
	200	2.9366	0.8366	0.4106	1.5187	0.8187	0.6986	2.8266	0.5266	0.2883
	400	2.6313	0.5313	0.3990	1.1739	0.4739	0.6665	2.6527	0.3527	0.2617
	800	2.2885	0.1885	0.3664	0.9277	0.2277	0.5775	2.4726	0.1726	0.2074
	1200	2.1938	0.0938	0.2909	0.7828	0.0828	0.3208	2.3744	0.0744	0.1627
	1800	2.1084	0.0084	0.2330	0.7256	0.0256	0.2767	2.3018	0.0018	0.1108

years) of a group of patients given chemotherapy treatment alone. The data consists of survival times (in years) for 46 patients and is as follows; 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033. The second data set is a real life example presented by [37] also reported by [83], and consists of 101 data points representing the stress-rupture life of kevlar 49/epoxy strands that are subjected to constant sustained pressure at the 90% stress level. The data set is as follows; 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20,

4.69, 7.89.

The EHL-OBIII-E distribution is compared to other known non-nested models namely, the Kumaraswamy Odd Lindley-log logistic (KOLLLoG) distribution by [35], Weibull-Lomax (WLx) distribution by [104], the Beta Generalized Exponential (BGE) distribution by [22], the Exponentiated Modified Weibull (EMW) distribution by [53], the Kumaraswamy Weibull (KW) distribution by [46] and Generalized Weibull Log-Logistic (GWLLoG) distribution by [47]. To evaluate and measure the model performances, we used the goodness-of-fit statistics namely, -2 log-likelihood ($-2 \ln(L)$), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion ($AICC$). We computed results on the Cramer-von Mises (W^*) and Anderson-Darling Statistics (A^*) proposed by [33], including the Kolmogorov-Smirnov (K-S) statistic and the associated P-values. Note that smaller values are preferred for the log-likelihood function at its maximum (ℓ_n) and similarly for AIC, AICC, BIC, and the goodness-of-fit statistics W^* , A^* and K-S, smaller values are also preferred. The results from two real lifetime data sets are presented under Tables 5.4 and 5.5. The R software was used to compute estimates for model parameters and run goodness-of-fit tests. The pdfs of the non-nested models used for comparisons are;

Kumaraswamy Odd Lindley-log-logistic (KOLLLoG) distribution

$$\begin{aligned} f(x; a, b, \lambda, c) &= ab \left[\frac{\lambda^2}{(1+\lambda)} \frac{cx^{c-1}}{(1+x^c)^{-1}} \exp \left\{ -\lambda \frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right\} \right] \\ &\times \left[1 - \frac{\lambda + (1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}} \exp \left\{ -\lambda \frac{(1 - (1+x^c)^{-1})}{(1+x^c)^{-1}} \right\} \right]^{a-1} \\ &\times \left(1 - \left[1 - \frac{\lambda + (1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}} \exp \left\{ -\lambda \frac{(1 - (1+x^c)^{-1})}{(1+x^c)^{-1}} \right\} \right]^a \right)^{b-1}, \end{aligned}$$

for $a, b, \lambda, c > 0$ and $x > 0$,

Weibull-Lomax (WLx) distribution

$$f(x; a, b, \alpha, \beta) = \frac{ab\alpha}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{b\alpha-1} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1} \\ \times \exp\left\{-a\left\{1 + \left(\frac{x}{\beta}\right)^\alpha - 1\right\}^b\right\},$$

for $a, b, c, \lambda > 0$ and $x > 0$,

Beta Generalized Exponential (BGE) distribution

$$f(x; a, b, \alpha, \lambda) = \frac{\alpha\lambda}{B(a, b)} e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha a-1} \{1 - (1 - e^{-\lambda x})^\alpha\}^{b-1},$$

for $a, b, \alpha, \lambda > 0$ and $x > 0$,

Exponentiated Modified Weibull (EMW) distribution

$$f(x; \alpha, \beta, \lambda, \theta) = \theta(\alpha + \lambda\beta x^{\lambda-1}) e^{-(\alpha x + \beta x^\lambda)} (1 - e^{\alpha x + \beta x^\lambda})^{\theta-1},$$

for $\alpha, \beta, \lambda, \theta > 0$ and $x > 0$,

Kumaraswamy Weibull (KW) distribution

$$f(x; a, b, c, \lambda) = abc\lambda^c x^{c-1} \exp\{-(\lambda x)^c\} [1 - \exp\{-(\lambda x)^c\}]^{a-1} \\ \times \{1 - [1 - \exp\{-(\lambda x)^c\}]^a\}^{b-1},$$

for $a, b, c, \lambda > 0$ and $x > 0$

and

Generalized Weibull Log-Logistic (GWLLoG) distribution.

$$f(x; \alpha, \beta, \gamma, a) = \frac{\alpha\beta\gamma x^{\gamma-1}}{a^\gamma} \left(1 + \left(\frac{x}{a}\right)^\gamma\right)^{-1} \left[\log\left(1 + \left(\frac{x}{a}\right)^\gamma\right)\right]^{\beta-1} \\ \times \exp\left\{-\alpha \left[\log\left(1 + \left(\frac{x}{a}\right)^\gamma\right)\right]^\beta\right\},$$

for $\alpha, \beta, \gamma, a > 0$ and $x > 0$.

The estimates of model parameters for the EHL-OBIII-E distribution and the other non-nested models (with standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W^* , A^* , Kolmogorov-Smirnov (K-S) and the associated p-values are given in Table 5.4. The plots of the fitted densities and observed probabilities are given in Figure 5.5.

Table 5.4: Model estimates for Survival Times Data

Model	Estimates				Statistics							
	a	b	α	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
EHL-OBIII-E	0.8594 (0.2946)	10.4693 (12.0444)	0.1369 (0.1283)	1.1934 (0.5663)	114.5943	122.5943	123.5943	129.821	0.0694	0.4713	0.1059	0.6541
KOLLLoG	a 1.2521 (0.8038)	b 0.1422 (0.0839)	λ 5.8092 (3.1117)	c 0.9775 (0.1733)	115.8898	123.8898	124.8898	131.1164	0.0735	0.4963	0.1150	0.5521
WLx	a 5.4516 (1.1398)	b 0.9734 (0.1136)	α 1.969×10^3 (3.8605×10^{-3})	β 17.2670 (4.4017×10^{-4})	115.9419	123.9419	124.9419	131.1685	0.0887	0.5897	0.1114	0.5924
BGE	α 13.8679 (14.8578)	λ 0.4085 (0.2297)	a 0.0617 (0.0675)	b 11.0365 (39.0023)	115.273	123.273	124.273	130.4996	0.1080	0.7146	0.12763	0.421
EMW	γ 1.1049 (0.2196)	δ 0.7943 (0.1511)	λ 17.001 (1.8250×10^{-18})	θ 1.0×10^{-4} (3.1034×10^{-13})	116.1897	124.1897	125.1897	131.4163	0.0785	0.5268	0.1099	0.6094
KW	a 0.2060 (1.2986)	b 3.6323 (16.9361)	β 4.4705 (25.1920)	α 0.1451 (0.4282)	115.5601	123.5601	124.5601	130.7867	0.0974	0.6443	0.1157	0.5439
GWLLoG	α 31.7894 (6.6608)	β 0.3602 (1.2399)	a 36.5270 (6.1217)	θ 2.9245 (10.0601)	116.2522	124.2522	125.2522	131.4788	0.0813	0.5437	0.1094	0.6143

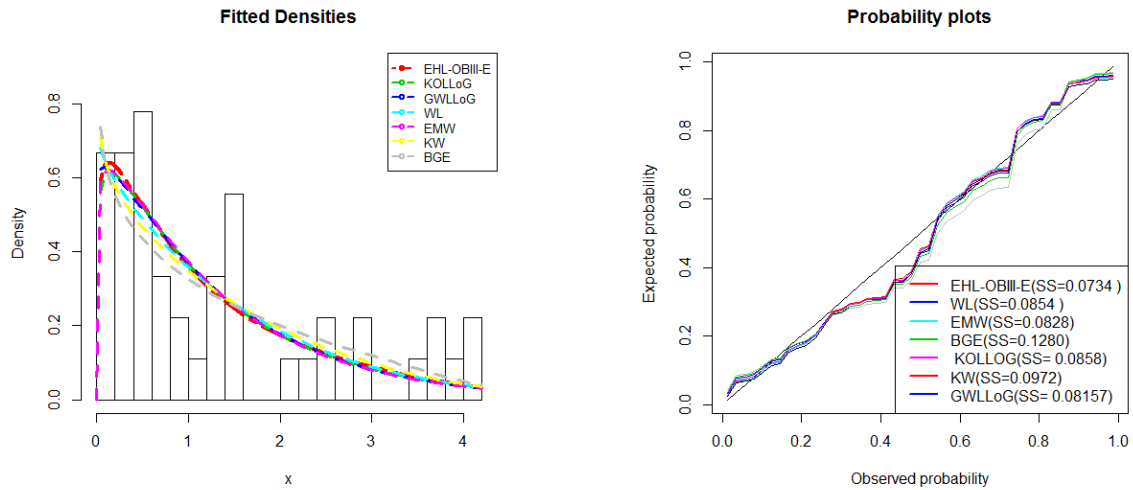


Figure 5.5: Fitted Density and Probability Plots for Survival Times Data

From Table 5.4 and Figure 5.5 above, we note that the EHL-OBIII-E model performs better than other non-nested models looking at its smallest values of W^* , A^* , AIC, AICC, BIC and the higher p-value for the K-S statistic. The fitted density of the EHL-OBIII-E distribution also remain closer to the sample histogram and similarly its fitted probability plot is also closer to the empirical line. Basing on these results, the EHL-OBIII-E provides a better fit compared to these other known non-nested models used for comparison.

The model parameter estimates for the EHL-OBIII-E distribution and other non-nested models (with standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W^* , A^* , Kolmogorov-Smirnov (K-S) with p-values are presented under Table 5.5. Plots of the fitted densities and observed probability are given in Figure 5.6.

Table 5.5: Model Estimates for Kevlar Epoxy Data

Model	Estimates				Statistics							
	a	b	α	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
EHL-OBIII-E	1.7704 (0.5205)	0.7699 (0.6396)	0.4634 (0.3549)	0.5852 (0.1578)	203.2637	211.2637	211.6804	221.7242	0.0650	0.5220	0.0648	0.7886
KOLLLoG	a 0.9437 (0.4861)	b 3.1448 (12.7460)	λ 0.6028 (1.5014)	c 0.8524 (0.4004)	205.0603	213.0603	213.477	223.5208	0.1480	0.8781	0.0786	0.5591
WLx	a 0.2506 (0.4172)	b 0.7860 (0.1803)	α 1.3580 (0.4580)	β 0.3302 (0.6282)	205.1976	213.1976	213.6143	223.6581	0.14402	0.8627	0.0787	0.5587
BGE	α 2.8508 (3.8184)	λ 0.5761 (0.5246)	a 0.2831 (0.3695)	b 0.3695 (1.8191)	205.0185	213.0185	213.4351	223.479	0.1414	0.8493	0.0775	0.5781
EMW	γ 0.8663 (0.1098)	δ 0.8883 (0.1201)	λ 17.8940 (2.0310×10^{-19})	θ 1.0×10^{-4} (3.6356×10^{-14})	205.6399	213.6399	214.0566	224.1004	0.1786	1.0183	0.0887	0.4045
KW	a 1.2800 (0.0245)	b 2.0691×10^3 (7.6348×10^{-8})	α 2.6750×10^{-4} (1.6886×10^{-4})	β 0.7239 (0.0438)	205.9592	213.9592	214.3758	224.4197	0.1991	1.1134	0.0911	0.3714
GWLLoG	α 3.1309 (0.3872)	β 0.1734 (0.0166)	a 3.0819 (0.1552)	θ 5.6875 (0.2913)	207.0122	215.0122	215.4289	225.4727	0.2240	1.2327	0.0900	0.3858

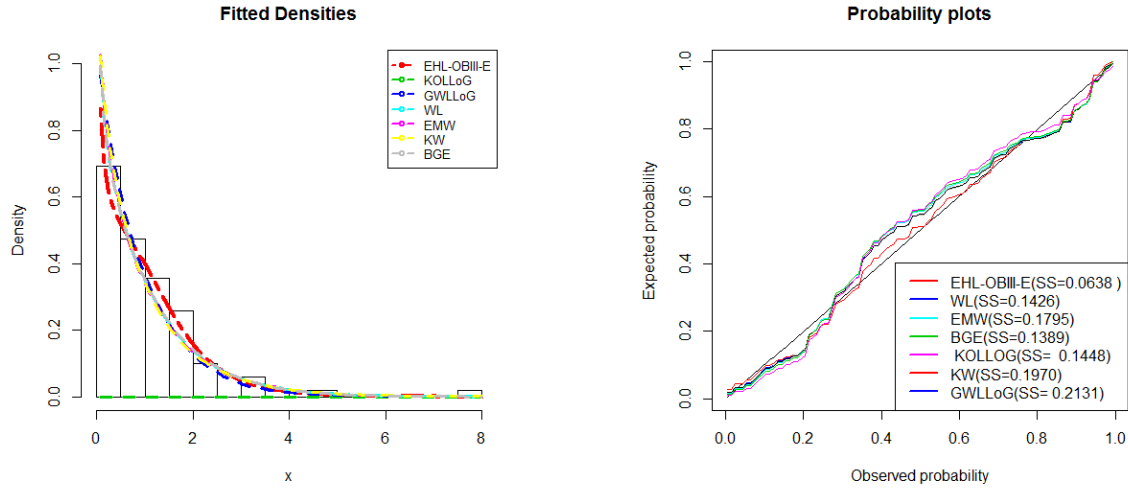


Figure 5.6: Fitted Density and Probability Plots for Kevlar Epoxy Data

From the results presented under Table 5.5, the EHL-OBIII-E distribution gives the smallest values of W^* , A^* , K-S and higher p-value compared to other competing models which shows its superiority over the other models. Figure 5.6, also show that the EHL-OBIII-E distribution provides a better fit to the real life data compared with the other non-nested models.

5.6 Conclusions

In this chapter, a new generalized family of distributions called exponentiated half-logistic Odd Burr III-G (EHL-OBIII-G) was developed. The structural properties

of this new family of distributions have been derived and studied. Some of its special cases have been discussed and the EHL-OBIII-Exponential distribution is applied to two real life data examples together with the other known non-nested models for comparison. Based on the results, the model provides better fits and performs better than the other non-nested models in fitting real life data. From the simulation results, the consistency of the model estimators is indicated by bias and RMSE converging towards zero as the sample size increases. We hope that these new generated family of distributions will find wider applicability in different disciplines such as financial modelling, economics, agriculture, engineering, genomics and operations research, to mention just a few.

Furthermore, another model namely; the Exponentiated Half-Logistic Generalized Marshall-Olkin-G model was also generated using the exponentiated half-logistic generator. The development of this work is omitted in this thesis.

Chapter 6

The Half-Logistic Odd Power Generalized Weibull-G Family of Distributions with Applications

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6.1 Introduction

In this chapter, another new family of distributions namely the Half-logistic odd Power Generalized Weibull-G (HLOPGW-G) family of distributions is developed. Furthermore, some of the structural properties for this newly generated family of distributions such as the quantile function, hazard function, moments and moment generating functions, Rényi entropy and order statistics are derived. In addition, to illustrate its ability to fit heavy tailed data and its non-monotonic hazard rate functions, some special cases under these distributions are studied. Finally, the performance of this family of distributions is assessed via simulation experiments. Real data examples are given to show the applicability of the proposed models compared to several other existing baseline models.

6.2 The Half-Logistic Odd Power Generalized Weibull-G Model

A new family of distributions, namely, the Half Logistic odd Power Generalized Weibull-G (HLOPGW-G) distributions is developed and studied in this section. Consider a random variable X having the cdf given by $G(x; \underline{\xi})$, where $\underline{\xi}$, is a vector of parameters, then the survival function of X is given by $\bar{G}(x; \underline{\xi}) = 1 - G(x; \underline{\xi})$ with the pdf defined by $g(x; \underline{\xi}) = \frac{dG(x; \underline{\xi})}{dx}$. The odd power generalized Weibull-G (OPGW-G) family has its cdf and pdf defined by

$$F(x) = 1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\bar{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\} \quad (6.1)$$

and

$$\begin{aligned} f(x) &= \alpha\beta \left[1 + \left(\frac{G(x; \underline{\xi})}{\bar{G}(x; \underline{\xi})} \right)^\alpha \right]^{\beta-1} \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\bar{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\} \\ &\times (G(x; \underline{\xi}))^{\alpha-1} (\bar{G}(x; \underline{\xi}))^{-(\alpha+1)} g(x; \underline{\xi}), \end{aligned} \quad (6.2)$$

respectively for $\alpha, \beta > 0$ and $\underline{\xi}$ as vector of parameters from the baseline distribution, (see [75]). We combine and extend the results by [41] (see equations (1.10) and (1.11)) and equations (6.1) and (6.2), to derive the HLOPGW-G family of distributions with cdf and pdf given by

$$F(x; \alpha, \beta, \underline{\xi}) = \frac{1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\bar{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\}}{1 + \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\bar{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\}} \quad (6.3)$$

and

$$\begin{aligned}
f(x; \alpha, \beta, \underline{\xi}) &= 2\alpha\beta \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^{\beta-1} \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\} (G(x; \underline{\xi}))^{\alpha-1} \\
&\times (\overline{G}(x; \underline{\xi}))^{-(\alpha+1)} \left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\} \right)^{-2} g(x; \underline{\xi}), \quad (6.4)
\end{aligned}$$

respectively, for $\alpha, \beta > 0$ and parameter vector $\underline{\xi}$. Note that by fixing some of the parameters, we obtain new sub-families of the HLOPGW-G family of distributions given by Table 6.1 below.

Table 6.1: New Sub-families of HLOPGW-G family of Distributions

Distribution	α	β
Half-Logistic Odd Weibull-G (HLOW-G)	-	1
Half-Logistic Odd Nadarajah Haghighi-G (HLONH-G)	1	-
Half-Logistic Odd Exponential-G (HLOE-G)	1	1
Half-Logistic Odd Rayleigh-G (HLOR-G)	2	1

6.2.1 The Hazard Rate and Quantile functions

This subsection presents the hazard rate function (hrf) and quantile function (qf) for the HLOPGW-G family of distributions. The hrf for the HLOPGW-G family of distributions is given by

$$h(x; \alpha, \beta, \underline{\xi}) = \frac{\alpha\beta \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^{\beta-1} (G(x; \underline{\xi}))^{\alpha-1} g(x; \underline{\xi})}{(\overline{G}(x; \underline{\xi}))^{\alpha+1} \left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\} \right)},$$

for $\alpha, \beta > 0$ and parameter vector $\underline{\xi}$. The qf is derived by inverting the cdf given by equation (6.3). We invert the function

$$\frac{1 - \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\}}{1 + \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\}} = u,$$

for $0 \leq u \leq 1$, which simplifies to

$$1 - u = (1 + u) \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\},$$

and can be written as

$$\left([1 + \ln(1 + u) - \ln(1 - u)]^{1/\beta} - 1 \right)^{\frac{1}{\alpha}} = \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right),$$

which simplifies to

$$G(x; \underline{\xi}) = \left([(1 + \ln(1 + u) - \ln(1 - u))]^{1/\beta} - 1 \right]^{\frac{-1}{\alpha}} + 1 \Big)^{-1}.$$

The quantiles of the HLOPGW-G family of distributions may be determined by solving the equation

$$x(u) = G^{-1} \left[\left([(1 + \ln(1 + u) - \ln(1 - u))]^{1/\beta} - 1 \right)^{\frac{-1}{\alpha}} + 1 \right]^{-1}, \quad (6.5)$$

using iterative methods.

6.2.2 Expansion of the Density Function

Series expansion of the HLOPGW-G family of distributions is presented under this subsection. By applying equation (6.4) and using series representation $(1 - x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1}$, for $|x| < 1$, we obtain

$$\left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\} \right)^{-2} = \sum_{n=1}^{\infty} (-1)^{n-1} n \left[\exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\} \right]^{n-1}$$

so that

$$f(x; \alpha, \beta, \underline{\xi}) = 2\alpha\beta \sum_{n=1}^{\infty} (-1)^{n-1} n \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^{\alpha} \right]^{\beta-1} \exp \left\{ n \left(1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^{\alpha} \right]^{\beta} \right) \right\} \\ \times (G(x; \underline{\xi}))^{\alpha-1} (\overline{G}(x; \underline{\xi}))^{-(\alpha+1)} g(x; \underline{\xi}).$$

Applying the exponential series expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the binomial expansion $(x+y)^n = \sum_{m=0}^{\infty} \binom{n}{m} x^m y^{n-m}$, $n \geq 0$, an integer or $|x/y| < 1$, we can write

$$\exp \left\{ n \left(1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^{\alpha} \right]^{\beta} \right) \right\} = \sum_{q=0}^{\infty} \frac{n^q}{q!} \left[1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^{\alpha} \right]^{\beta} \right]^q \\ = \sum_{q,k=0}^{\infty} \frac{n^q}{q!} \binom{q}{k} (-1)^k \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^{\alpha} \right]^{\beta k},$$

so that

$$f(x; \alpha, \beta, \underline{\xi}) = 2\alpha\beta \sum_{n=1}^{\infty} \sum_{q,k=0}^{\infty} \frac{n^{q+1}}{q!} \binom{q}{k} (-1)^{k+n-1} \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^{\alpha} \right]^{\beta(k+1)-1} \\ \times (G(x; \underline{\xi}))^{\alpha-1} (\overline{G}(x; \underline{\xi}))^{-(\alpha+1)} g(x; \underline{\xi}).$$

Considering the generalized binomial expansion, we get

$$\left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^{\alpha} \right]^{\beta(k+1)-1} = \sum_{m=0}^{\infty} \binom{\beta(k+1)-1}{m} \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^{\alpha m},$$

for $\beta(k+1) > 1$, such that

$$\begin{aligned} f(x; \alpha, \beta, \underline{\xi}) &= 2\alpha\beta \sum_{n=1}^{\infty} \sum_{q,k,m=0}^{\infty} \binom{\beta(k+1)-1}{m} \frac{n^{q+1}}{q!} \binom{q}{k} (-1)^{k+n-1} \\ &\times (G(x; \underline{\xi}))^{\alpha(m+1)-1} (\overline{G}(x; \underline{\xi}))^{-(\alpha(m+1)+1)} g(x; \underline{\xi}). \end{aligned}$$

Now applying the series expansion

$$[\overline{G}(x; \underline{\xi})]^{-(\alpha(m+1)+1)} = \sum_{l=0}^{\infty} \binom{-\alpha((m+1)+1)}{l} (-1)^l G^l(x; \underline{\xi}),$$

we have

$$\begin{aligned} f(x; \alpha, \beta, \underline{\xi}) &= 2\alpha\beta \sum_{n=1}^{\infty} \sum_{q,k,m,l=0}^{\infty} \binom{-\alpha((m+1)+1)}{l} \binom{\beta(k+1)-1}{m} \\ &\times \frac{n^{q+1}}{q!} \binom{q}{k} (-1)^{l+k+n-1} (G(x; \underline{\xi}))^{l+\alpha(m+1)-1} g(x; \underline{\xi}). \end{aligned}$$

We can therefore write the series expansion of equation (6.4) as

$$f(x; \alpha, \beta, \underline{\xi}) = \sum_{p=0}^{\infty} v_p g_p(x; \underline{\xi}), \quad (6.6)$$

where

$$\begin{aligned} v_p &= 2\alpha\beta \sum_{n=1}^{\infty} \sum_{q,k,m=0}^{\infty} \binom{-\alpha((m+1)+1)}{l} \binom{\beta(k+1)-1}{m} \\ &\times \frac{n^{q+1}}{q!} \binom{q}{k} \frac{(-1)^{l+k+n-1}}{p+\alpha(m+1)}. \end{aligned} \quad (6.7)$$

and $g_p(x; \underline{\xi}) = pg(x; \underline{\xi})[G(x; \underline{\xi})]^{p-1}$ is an Exp-G with power parameter p . The HLOPGW-G distribution is an infinite linear combination of Exp-G densities and the mathematical properties of the HLOPGW-G family of distributions can be readily obtained directly from the Exp-G family of distributions.

6.3 Some Special Cases

This section presents some special cases of the HLOPGW-G family of distributions. The baseline distributions considered are restricted to at most two parameter models to avoid the problem of over-parametrization.

6.3.1 Half-Logistic Odd Power Generalized Weibull-Uniform (HLOPGW-U) Distribution

Let the uniform distribution be the baseline distribution with pdf and cdf given by $g(x) = 1/\lambda$ and $G(x; \lambda) = x/\lambda$, respectively, for $0 < x < \lambda$. We obtain the cdf, pdf and hrf of the HLOPGW-U distribution which are given by

$$F(x; \alpha, \beta, \lambda) = \frac{1 - \exp \left\{ 1 - \left[1 + \left(\frac{x}{\lambda-x} \right)^\alpha \right]^\beta \right\}}{1 + \exp \left\{ 1 - \left[1 + \left(\frac{x}{\lambda-x} \right)^\alpha \right]^\beta \right\}}, \quad (6.8)$$

$$\begin{aligned} f(x; \alpha, \beta, \lambda) &= \frac{2\alpha\beta}{\lambda} \left[1 + \left(\frac{x}{\lambda-x} \right)^\alpha \right]^{\beta-1} \exp \left\{ 1 - \left[1 + \left(\frac{x}{\lambda-x} \right)^\alpha \right]^\beta \right\} \left(\frac{x}{\lambda} \right)^{\alpha-1} \\ &\times \left(1 - \frac{x}{\lambda} \right)^{-(\alpha+1)} \left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{x}{\lambda-x} \right)^\alpha \right]^\beta \right\} \right)^{-2} \end{aligned} \quad (6.9)$$

and

$$h(x; \alpha, \beta, \lambda) = \frac{\alpha\beta \left[1 + \left(\frac{x}{\lambda-x} \right)^\alpha \right]^{\beta-1} \left(\frac{x}{\lambda} \right)^{\alpha-1}}{\lambda \left(1 - \frac{x}{\lambda} \right)^{\alpha+1} \left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{x}{\lambda-x} \right)^\alpha \right]^\beta \right\} \right)},$$

respectively for $\alpha, \beta, \lambda > 0$. The pdf and hrf plots for the HLOPGW-U distribution are given in figure 6.1 below.

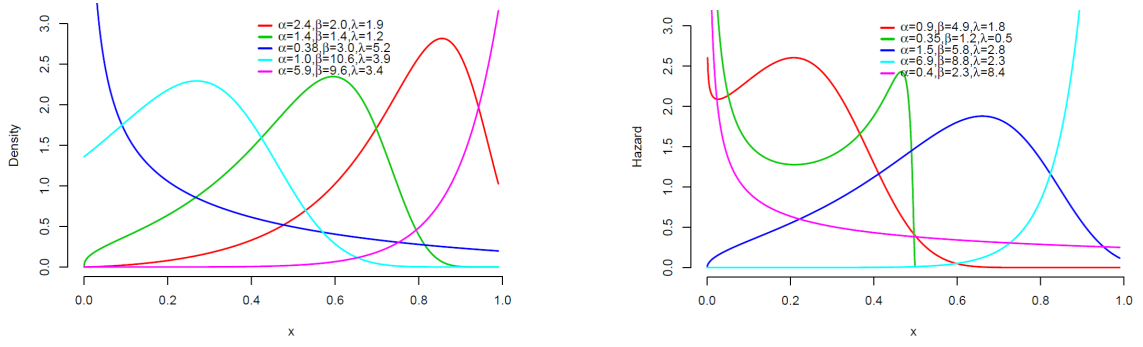


Figure 6.1: Plots of the pdf and hrf for the HLOPGW-U Distribution

Figure 6.1 above demonstrate the flexible nature of the HLOPGW-U distribution for some parameter values. The pdfs of the HLOPGW-U distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the HLOPGW-U distribution exhibit decreasing, increasing, upside down bathtub, bathtub followed by upside down bathtub shapes for the hrf.

The corresponding quantile function (qf) for the half logistic odd power generalized Weibull-Uniform (HLOPGW-U) distribution is derived by inverting the cdf given by equation 6.8 as follows;

$$F(x; \alpha, \beta, \lambda) = \frac{1 - \exp \left\{ 1 - \left[1 + \left(\frac{x}{\lambda - x} \right)^\alpha \right]^\beta \right\}}{1 + \exp \left\{ 1 - \left[1 + \left(\frac{x}{\lambda - x} \right)^\alpha \right]^\beta \right\}} = u,$$

for $0 \leq u \leq 1$, which implies

$$1 - u = (1 + u) \exp \left\{ 1 - \left[1 + \left(\frac{x}{\lambda - x} \right)^\alpha \right]^\beta \right\},$$

and we can write

$$\left(\frac{1 - u}{1 + u} \right) = \exp \left\{ 1 - \left[1 + \left(\frac{x}{\lambda - x} \right)^\alpha \right]^\beta \right\},$$

such that

$$\ln \left(\frac{1-u}{1+u} \right) = \left(1 - \left[1 + \left(\frac{x}{\lambda-x} \right)^\alpha \right]^\beta \right).$$

Consequently, after some few algebra, we can write the quantile function for HLOPGW-U distribution as

$$x(u) = \lambda \left(\left(\left(\left[1 - \ln \left(\frac{1-u}{1+u} \right) \right]^{\frac{1}{\beta}} - 1 \right)^{\frac{1}{\alpha}} \right)^{-1} + 1 \right)^{-1},$$

and may be determined using iterative methods. The quantile values for the HLOPGW-U distribution are given in Table 6.2.

Table 6.2: Table of Quantiles for Selected Parameters of the HLOPGW-U Distribution

u	(0.4,0.6,1.0)	(0.5,0.6,1.0)	(0.2,0.8,1.0)	(0.5,0.5,1.0)	(0.9,0.6,1.0)
0.1	0.0704	0.1126	0.0010	0.1632	0.2411
0.2	0.3374	0.3682	0.0402	0.4875	0.4256
0.3	0.6276	0.6029	0.2780	0.7244	0.5577
0.4	0.8089	0.7603	0.6713	0.8533	0.6550
0.5	0.9029	0.8561	3.8921	0.9205	0.7292
0.6	0.9505	0.9140	0.9670	0.9565	0.7881
0.7	0.9752	0.9497	0.9902	0.9767	0.8365
0.8	0.9884	0.9724	0.9973	0.9883	0.8786
0.9	0.9957	0.9873	0.9994	0.9953	0.9185

6.3.2 Half-Logistic Odd Power Generalized Weibull-Lomax (HLOPGW-Lx) Distribution

Consider the Lomax distribution as the baseline distribution having shape parameters $a > 0, b > 0$ having cdf and pdf $G(x) = [1 + \frac{x}{b}]^{-a}$ and $g(x) = (\frac{a}{b})[1 + (\frac{x}{b})]^{-a-1}$, respectively. The cdf, pdf and hrf of the HLOPGW-Lx distribution are given by

$$F(x; \alpha, \beta, a, b) = \frac{1 - \exp \left\{ 1 - \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^\beta \right\}}{1 + \exp \left\{ 1 - \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^\beta \right\}}, \quad (6.10)$$

$$\begin{aligned} f(x; \alpha, \beta, a, b) &= \frac{2\alpha\beta \left[(1 + (\frac{x}{b}))^{-a} \right]^{\alpha-1} \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^{\beta-1}}{\left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^\beta \right\} \right)^2} \\ &\times \exp \left\{ 1 - \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^\beta \right\} \\ &\times \left[1 - \left(1 + \left(\frac{x}{b} \right) \right)^{-a} \right]^{-(\alpha+1)} \left(\frac{a}{b} \right) \left(1 + \left(\frac{x}{b} \right) \right)^{-a-1} \end{aligned}$$

and

$$\begin{aligned} h(x; \alpha, \beta, a, b) &= \frac{\alpha\beta \left(\frac{a}{b} \right) \left[(1 + (\frac{x}{b}))^{-a} \right]^{\alpha-1} \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^{\beta-1}}{\left[1 - (1 + (\frac{x}{b}))^{-a} \right]^{\alpha+1} \left(1 + \left(\frac{x}{b} \right) \right)^{a+1}} \\ &\times \left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^\beta \right\} \right)^{-1}, \end{aligned}$$

respectively for $\alpha, \beta, a, b > 0$. The pdf and hrf plots for the HLOPGW-Lx distribution are given in Figure 6.2 below.

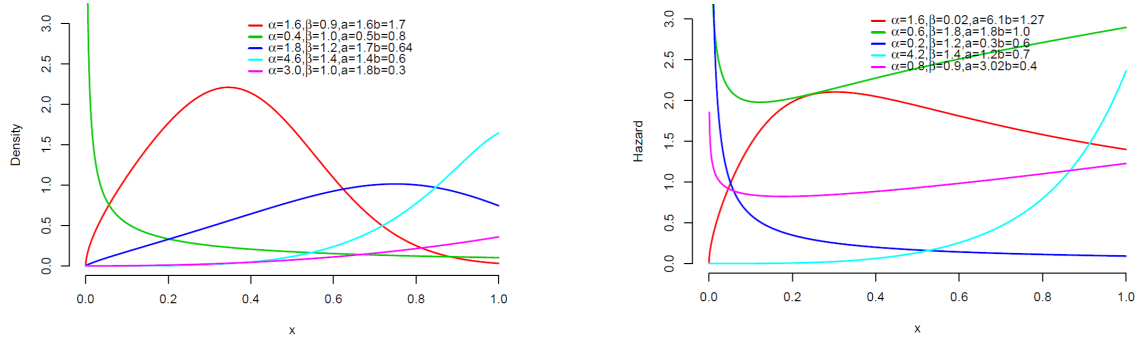


Figure 6.2: Plots of the pdf and hrf for the HLOPGW-Lx Distribution

The above Figure 6.2 illustrates the flexible nature of the HLOPGW-Lx distribution for some parameter values. The pdfs of the HLOPGW-Lx distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the HLOPGW-Lx distribution exhibit reverse-J, decreasing, increasing, bathtub and upside down bathtub shapes for the hazard rate function.

Similarly, we derive the quantile function for the HLOPGW-Lx distribution by inverting equation 6.10 as follows;

$$F(x; \alpha, \beta, a, b) = \frac{1 - \exp \left\{ 1 - \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^\beta \right\}}{1 + \exp \left\{ 1 - \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^\beta \right\}} = u,$$

for $0 \leq u \leq 1$ and

$$1 - u = (1 + u) \exp \left\{ 1 - \left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^\beta \right\},$$

such that

$$\left[1 + \left(\frac{(1 + (\frac{x}{b}))^{-a}}{1 - (1 + (\frac{x}{b}))^{-a}} \right)^\alpha \right]^\beta = [1 + \ln(1 + u) - \ln(1 - u)]^{\frac{1}{\beta}}.$$

Furthermore, we can write

$$\frac{\left(1 + \left(\frac{x}{b}\right)\right)^{-a}}{1 - \left(1 + \left(\frac{x}{b}\right)\right)^{-a}} = \left([1 + \ln(1 + u) - \ln(1 - u)]^{\frac{1}{\beta}} - 1\right)^{\frac{1}{\alpha}},$$

that is to say

$$\left(1 + \left(\frac{x}{b}\right)\right)^a = \left([1 + \ln(1 + u) - \ln(1 - u)]^{\frac{1}{\beta}} - 1\right)^{-\frac{1}{\alpha}} + 1.$$

Finally, the quantile values for the HLOPGW-Lx distribution are obtained by solving the equation

$$x(u) = b \left(\left(\left([1 + \ln(1 + u) - \ln(1 - u)]^{\frac{1}{\beta}} - 1 \right)^{-\frac{1}{\alpha}} + 1 \right)^{\frac{1}{a}} - 1 \right),$$

using iterative methods.

6.3.3 Half-Logistic Odd Power Generalized Weibull-Beta (HLOPGW-B) Distribution

Let the Beta distribution be the baseline distribution having shape parameters $a, b > 0$ having the cdf and pdf $G(x) = I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1 - t)^{b-1} dt$ and $g(x) = \frac{1}{B(a, b)} x^{a-1} (1 - x)^{b-1}$, for $0 < x < 1$, respectively. The cdf, pdf and hrf of HLOPGW-B distribution are given by

$$F(x; \alpha, \beta, a, b) = \frac{1 - \exp \left\{ 1 - \left[1 + \left(\frac{I_x(a, b)}{1 - I_x(a, b)} \right)^\alpha \right]^\beta \right\}}{1 + \exp \left\{ 1 - \left[1 + \left(\frac{I_x(a, b)}{1 - I_x(a, b)} \right)^\alpha \right]^\beta \right\}}, \quad (6.11)$$

$$f(x; \alpha, \beta, a, b) = \frac{2\alpha\beta [\mathbf{I}_x(a, b)]^{\alpha-1} \left[1 + \left(\frac{\mathbf{I}_x(a, b)}{1-\mathbf{I}_x(a, b)} \right)^\alpha \right]^{\beta-1} x^{a-1} (1-x)^{b-1}}{B(a, b) \left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{\mathbf{I}_x(a, b)}{1-\mathbf{I}_x(a, b)} \right)^\alpha \right]^\beta \right\} \right)^2} \\ \times \exp \left\{ 1 - \left[1 + \left(\frac{\mathbf{I}_x(a, b)}{1-\mathbf{I}_x(a, b)} \right)^\alpha \right]^\beta \right\} (1 - \mathbf{I}_x(a, b))^{-(\alpha+1)}$$

and

$$h(x; \alpha, \beta, a, b) = \frac{\alpha\beta [\mathbf{I}_x(a, b)]^{\alpha-1} \left[1 + \left(\frac{\mathbf{I}_x(a, b)}{1-\mathbf{I}_x(a, b)} \right)^\alpha \right]^{\beta-1} x^{a-1} (1-x)^{b-1}}{B(a, b) [1 - \mathbf{I}_x(a, b)]^{\alpha+1} \left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{\mathbf{I}_x(a, b)}{1-\mathbf{I}_x(a, b)} \right)^\alpha \right]^\beta \right\} \right)},$$

respectively for $\alpha, \beta, a, b > 0$. The pdf and hrf plots for the HLOPGW-B distribution are given in Figure 6.3 below.

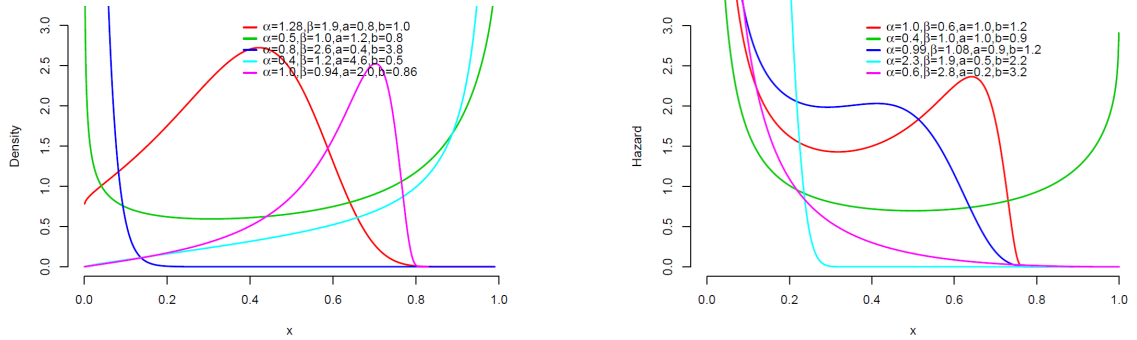


Figure 6.3: Plots of the pdf and hrf for the HLOPGW-B Distribution

Figure 6.3 above presents good illustration on the flexibility of the HLOPGW-B distribution for selected parameter values. The pdfs of the HLOPGW-B distribution exhibits various shapes that include reverse-J, uni-modal, left or right skewed shapes. The shapes of HLOPGW-B hrf are decreasing, reverse-J, bathtub, bathtub followed by upside down bathtub shape.

The quantile function for the HLOPGW-B distribution is derived by inverting equation 6.11, such that

$$F(x; \alpha, \beta, a, b) = u, \quad 0 \leq u \leq 1,$$

and therefore we can write

$$1 - u = (1 + u) \exp \left\{ 1 - \left[1 + \left(\frac{\mathbf{I}_x(a, b)}{1 - \mathbf{I}_x(a, b)} \right)^\alpha \right]^\beta \right\},$$

such that

$$\frac{\mathbf{I}_x(a, b)}{1 - \mathbf{I}_x(a, b)} = \left([1 + \ln(1 + u) - \ln(1 - u)]^{\frac{1}{\beta}} - 1 \right)^{\frac{1}{\alpha}},$$

which simplifies to

$$\mathbf{I}_x(a, b) = \left(\left([1 + \ln(1 + u) - \ln(1 - u)]^{\frac{1}{\beta}} - 1 \right)^{-\frac{1}{\alpha}} + 1 \right)^{-1}.$$

Finally, we can write the quantile function for the HLOPGW-B distribution as

$$x(u) = Q^{-1} \left(a, b \left(\left([1 + \ln(1 + u) - \ln(1 - u)]^{\frac{1}{\beta}} - 1 \right)^{-\frac{1}{\alpha}} + 1 \right)^{-1} \right),$$

which can be solved using iterative methods.

6.4 Mathematical and Statistical Properties

In this section, some of the statistical properties of the HLOPGW-G family of distributions, namely; order statistics, entropy, moments, incomplete moments, probability weighted moments (PWMs) and generating functions are derived.

6.4.1 Distribution of Order Statistics

The use of order statistics is critical in the study of probability and statistical inference. It has wider applications in various studies namely; hypothesis testing, theory of estimation, reliability, and statistical quality control. The pdf of the i^{th} order statistics can be obtained using equation (6.12),

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} F(x)^{j+i-1}, \quad (6.12)$$

where $B(.,.)$ is the beta function. Substituting equations (6.3) and (6.4) into equation (6.12), and applying the generalized binomial series expansion used under the density expansion, we can write the pdf of the i^{th} order statistic from the HLOPGW-G family of distributions as

$$\begin{aligned} f_{i:n}(x) &= \frac{2\alpha\beta}{B(i, n-i+1)} \binom{n-i}{j} \sum_{m,n,p,k,q,w=0}^{\infty} \binom{-(\alpha(q+1)+1)}{w} (-1)^w \\ &\times \binom{\beta(k+1)-1}{q} \frac{(m+n+1)^p (-1)^{k+n}}{p!} \binom{p}{k} \binom{j+i-1}{n} \binom{-(j+i+1)}{m} \\ &\times (G(x; \underline{\xi}))^{w+\alpha(q+1)-1} g(x; \underline{\xi}) \\ &= \sum_{w,q=0}^{\infty} v_{w,q}^* g_{w,q}(x; \underline{\xi}), \end{aligned} \quad (6.13)$$

after some algebra, where $g_{w,q}(x; \underline{\xi}) = (w + \alpha(q+1))g(x; \underline{\xi})[G(x; \underline{\xi})]^{w+\alpha(q+1)-1}$ is an Exp-G distribution with power parameter $w + \alpha(q+1)$ and

$$\begin{aligned} v_{w,q}^* &= \frac{2\alpha\beta(-1)^w}{B(i, n-i+1)} \binom{n-i}{j} \sum_{m,n,p,k=0}^{\infty} \binom{-(\alpha(q+1)+1)}{w} \binom{p}{k} \\ &\times \binom{\beta(k+1)-1}{q} \frac{(m+n+1)^p (-1)^{k+n}}{p!(w + \alpha(q+1))} \binom{j+i-1}{n} \binom{-(j+i+1)}{m}. \end{aligned}$$

Finally, the pdf of the i^{th} order statistics from the HLOPGW-G family of distributions can be expressed as an infinite linear combination of Exp-G densities.

6.4.2 Entropy

An entropy is a measure of variation for uncertainty on a random variable X with the probability distribution $f(x)$. There are two common types of entropy, namely, Rényi entropy [89] and Shannon entropy [94]. Shannon entropy is a special case of Rényi entropy. In this paper, we derive the Rényi entropy ($I_R(\nu)$) of the HLOPGW-G family of distributions as follows

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\int_0^\infty f^\nu(x) dx \right], \nu \neq 1, \nu > 0. \quad (6.14)$$

Using equation (6.4), $f^\nu(x)$, can be written as

$$\begin{aligned} f^\nu(x) &= (2\alpha\beta)^\nu \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^{\nu(\beta-1)} \exp \left\{ \nu \left(1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right) \right\} \\ &\times (\overline{G}(x; \underline{\xi}))^{-\nu(\alpha+1)} \left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^\alpha \right]^\beta \right\} \right)^{-2\nu} \\ &\times (G(x; \underline{\xi}))^{\nu(\alpha-1)} g(x; \underline{\xi})^\nu. \end{aligned}$$

Applying the generalized binomial expansion, used under sub-section 6.2.2, the Rényi entropy for the HLOPGW-G family of distributions can be written as

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\sum_{m=0}^{\infty} w_m e^{(1-\nu)I_{REG}} \right], \nu \neq 1, \nu > 0, \quad (6.15)$$

after some mathematical simplifications, where

$$\begin{aligned} w_m &= \sum_{j,i,k,p,s=0}^{\infty} \binom{-\alpha((\nu+z)+\nu)}{m} (-1)^m \frac{(\nu+j)^i}{i!} \binom{i}{k} \binom{-2\nu}{j} (2\alpha\beta)^\nu \\ &\times \binom{\nu(\alpha-1)+\alpha z+m}{p} \binom{p}{s} (-1)^{s+p+k} \left(\frac{s}{\nu} + 1 \right)^{-\nu}, \end{aligned} \quad (6.16)$$

and $I_{REG} = (1 - \nu)^{-1} \log \int_0^\infty \left[\left(\frac{s}{\nu} + 1 \right) g(x; \underline{\xi}) [G(x; \underline{\xi})]^{\frac{s}{\nu}} \right]^\nu dx$ is Rényi entropy of Exp-G distribution with parameter $\frac{s}{\nu} + 1$. Lastly, the Rényi entropy of the HLOPGW-G family of distributions can be obtained directly from Rényi entropy of Exp-G distribution.

6.4.3 Moments

The r^{th} ordinary moment can be derived using equation (6.6) as follows:

$$\mu'_r = E(X^r) = \sum_{p=0}^{\infty} v_p E(Y_p), \quad (6.17)$$

where Y_p follows an Exp-G distribution with power parameter p and v_p is given in equation (6.7). The s^{th} central moment of X is given by

$$\mu_s = \sum_{r=0}^s \binom{s}{r} (-\mu'_1)^{s-r} E(X^r) = \sum_{r=0}^s \sum_{p=0}^{\infty} v_p \binom{s}{r} (-\mu'_1)^{s-r} E(Y_p).$$

The cumulants of X follow recursively from

$$k_s = \mu'_s - \sum_{r=0}^{s-1} \binom{s-1}{r-1} k_r \mu'_{s-r},$$

where $k_1 = \mu'_1$, $k_2 = \mu'_2 - \mu'^2_1$, $k_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu'^3_1$, etc. Ordinary moments may also be used to calculate the measures of dispersion, namely, variance, skewness and kurtosis.

The r^{th} incomplete moment of X is given by

$$\phi_r(z) = \int_{-\infty}^z x^r f(x) dx = \sum_{p=0}^{\infty} v_p \int_{-\infty}^z x^r g_p(x; \underline{\xi}) dx. \quad (6.18)$$

The incomplete moment is very useful and can be used to estimate some important quantities such as Lorenz and Bonferroni curves. These quantities have a wide application in demography, economics, insurance, medicine and reliability. Mathematically, the Lorenz and Bonferroni curves for a given probability p are given by $L(p) = \phi_1(q)/\mu'_1$ and $B(p) = \phi_1(q)/(p\mu'_1)$, respectively, where μ'_1 is given by equation

(6.17), with $r = 1$ and $q = Q(p)$ is the quantile function of X at p . The incomplete moment (equation (6.18)) can be expressed as

$$\phi_r(z) = \sum_{p=0}^{\infty} v_p H_p(z), \quad (6.19)$$

where $H_p(z) = \int_{-\infty}^z x^r g_p(x; \underline{\xi}) dx$ is the r^{th} incomplete moment of the Exp-G distribution.

We present the first five moments with the standard deviation (SD or σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for the HLOPGW-U distribution for some parameter values (see Table 6.3, for details).

Table 6.3: Moments of the HLOPGW-U distribution for some parameter values

	(0.5,0.7,1.2)	(0.6,0.5,1.1)	(0.2,0.9,1.1)	(0.4,0.8,1.4)	(0.8,0.7,1.5)
$E(X)$	0.2897	0.3378	0.1941	0.1932	0.2698
$E(X^2)$	0.2125	0.2651	0.1401	0.1314	0.1963
$E(X^3)$	0.1715	0.2233	0.1137	0.1015	0.1560
$E(X^4)$	0.1451	0.1948	0.0971	0.0834	0.1301
$E(X^5)$	0.1263	0.1738	0.0853	0.0710	0.1118
SD	0.3585	0.3886	0.3200	0.3067	0.3514
CV	1.2372	1.1501	1.6486	1.5873	1.3021
CS	0.7689	0.5396	1.4263	1.3791	0.8385
CK	1.9504	1.5610	3.4527	3.4072	2.0670

6.4.4 Probability Weighted Moments

Probability Weighted Moments (PWMs) are very useful in estimating parameters of distributions which are not in closed form. The (j, i) th PWM, say $\eta_{j,i}$ of $X \sim \text{HLOPGW-G}(\alpha, \beta; \underline{\xi})$ distribution is defined by

$$\eta_{j,i} = E(X^j F(X)^i) = \int_{-\infty}^{\infty} x^j f(x) F(x)^i dx.$$

Using equation (6.13), we can write

$$\begin{aligned}
f(x)F(x)^i &= 2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p,k,q,w=0}^{\infty} \binom{-(\alpha(q+1)+1)}{w} (-1)^w \binom{\beta(k+1)-1}{q} \\
&\times \frac{(m+n+1)^p (-1)^{k+n}}{p!} \binom{p}{k} \binom{i-1}{n} \binom{-(i+1)}{m} \alpha\beta \\
&\times (G(x; \underline{\xi}))^{w+\alpha(q+1)-1} g(x; \underline{\xi}),
\end{aligned}$$

which simplifies to

$$f(x)F(x)^i = \sum_{w,q=0}^{\infty} h_{w,q}^* g_{w,q}(x; \underline{\xi}),$$

where

$$\begin{aligned}
h_{w,q}^* &= 2\alpha\beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p,k=0}^{\infty} \binom{-(\alpha(q+1)+1)}{w} (-1)^w \binom{\beta(k+1)-1}{q} \\
&\times \frac{(m+n+1)^p (-1)^{k+n}}{p!} \binom{p}{k} \binom{i}{n} \binom{-(i+2)}{m} \\
&\times (G(x; \underline{\xi}))^{w+\alpha(q+1)-1} g(x; \underline{\xi})
\end{aligned}$$

and $g_{w,q}(x; \underline{\xi}) = (w + \alpha(q + 1))g(x; \underline{\xi})[G(x; \underline{\xi})]^{w+\alpha(q+1)-1}$ is an **Exp-G** distribution with power parameter $w + \alpha(q + 1)$. The PWM is therefore given by

$$\begin{aligned}
\eta_{j,i} &= \sum_{w,q=0}^{\infty} h_{w,q}^* \int_{-\infty}^{\infty} x^j g_{w,q}(x; \underline{\xi}) dx \\
&= \sum_{w,q=0}^{\infty} h_{w,q}^* E(T_{w,q}^j),
\end{aligned}$$

where $T_{w,q}^j$ is j^{th} power of an **Exp-G** distributed random variable having power parameter $w + \alpha(q + 1)$.

6.4.5 Moment Generating Functions

The moment generating function (mgf) of HLOPGW-G family of distributions is given by

$$M_{\mathbf{x}}(t) = E(e^{tX}) = \sum_{p=0}^{\infty} v_p M_p(t),$$

where $M_p(t)$ is the mgf of Exp-G with power parameter p . The mgf of HLOPGW-G family of distributions can be derived directly from that of the Exp-G distribution.

6.5 Maximum Likelihood Estimation

Let $X_i \sim HLOPGW - G(\alpha, \beta; \underline{\xi})$, for $i = 1, \dots, n$, with parameter vector $\Delta = (\alpha, \beta; \underline{\xi})^T$. The log-likelihood for $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\begin{aligned} \ell = & n \ln(2\alpha\beta) + (\beta - 1) \sum_{i=1}^n \ln \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\overline{G}(x_i; \underline{\xi})} \right)^{\alpha} \right] \\ & + \sum_{i=1}^n \ln \left(1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\overline{G}(x_i; \underline{\xi})} \right)^{\alpha} \right]^{\beta} \right) \\ & + (\alpha - 1) \sum_{i=1}^n \ln (G(x_i; \underline{\xi})) - (\alpha + 1) \sum_{i=1}^n \ln (\overline{G}(x_i; \underline{\xi})) + \sum_{i=1}^n \ln(g(x_i; \underline{\xi})) \\ & - 2 \sum_{i=1}^n \ln \left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{G(x; \underline{\xi})}{\overline{G}(x; \underline{\xi})} \right)^{\alpha} \right]^{\beta} \right\} \right). \end{aligned}$$

The elements of the score vector are given in **appendix I.9**. These functions are not in closed form and can only be solved using iterative methods from applicable softwares. The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \underline{\xi}})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. To obtain confidence intervals for model parameters $(\alpha, \beta, \underline{\xi})$, and test hypothesis concerning these parameters, the observed information matrix is required and is given by

$$J(\Delta) = \begin{bmatrix} J_{\alpha\alpha}(\Delta) & J_{\alpha\beta}(\Delta) & J_{\alpha\xi}(\Delta) \\ J_{\beta\alpha}(\Delta) & J_{\beta\beta}(\Delta) & J_{\beta\xi}(\Delta) \\ J_{\xi\alpha}(\Delta) & J_{\xi\beta}(\Delta) & J_{\xi\xi}(\Delta) \end{bmatrix},$$

where $J_{rs} = -\frac{\partial^2 \ell(\Delta)}{\partial r \partial s}$, for $r, s = \alpha, \beta, \xi$. Under the usual regularity conditions, $\hat{\Delta}$ is asymptotically normal distributed, (see, [57]), that is $\hat{\Delta} \sim N_3(\mathbf{0}, I^{-1}(\Delta))$ as $n \rightarrow \infty$, where $I(\Delta)$ is the expected information matrix. The asymptotic behavior remains valid if $I(\Delta)$ is replaced by $J(\hat{\Delta})$, the observed information matrix evaluated at $\hat{\Delta}$.

6.6 Monte-Carlo Simulations

In this section, a Monte-Carlo simulation study is conducted using the R package to evaluate and examine consistency of the maximum likelihood estimators. We replicate N=1000 times for sample size $n=35, 50, 100, 200, 400, 800$ and 1000 from the HLOPGW-U distribution. The simulation results are given under Table 6.4. From these results, we deduce that as the sample size increases, the mean estimates of the parameters tends to be closer to the true parameter values, since RMSEs and average bias converges towards zero in all instances.

6.7 Applications

The HLOPGW-U model is applied to two real data examples to show the applicability of the proposed distribution when compared to other known non-nested distributions. The best performing model is examined using the goodness-of-fit statistics, namely, -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*) and Andersen-Darling (A^*) as given under [32]. The model that gives smaller values of these statistics is deemed to have the best fit.

The R software is used to estimate model parameters through the nlm func-

Table 6.4: Simulation Results for HLOPGW-U distribution; Mean, Average Bias and RMSE

Parameter	n	Set I: $\alpha=1.5, \beta=2.3, \lambda=0.4$			Set II: $\alpha=2.4, \beta=2.2, \lambda=1.0$			Set III: $\alpha=1.0, \beta=0.6, \lambda=0.2$		
		Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE
α	35	1.87951	0.37951	0.02292	3.25398	0.85398	0.13859	1.25608	0.25608	0.06165
	50	1.81423	0.31423	0.01466	2.88445	0.48445	0.09954	1.20708	0.20708	0.05434
	100	1.78820	0.28820	0.01322	2.82462	0.42462	0.08474	1.16660	0.16660	0.03770
	200	1.70913	0.20913	0.01107	2.70839	0.30839	0.07360	1.14674	0.14674	0.03230
	400	1.65626	0.15626	0.00993	2.58788	0.18788	0.02719	1.13883	0.13883	0.02596
	800	1.55146	0.05146	0.00657	2.50097	0.10097	0.01936	1.07845	0.07845	0.02160
	1000	1.52254	0.02254	0.00585	2.46558	0.06558	0.01267	1.03982	0.03982	0.01960
β	35	3.13086	0.83086	0.02740	3.25206	1.05206	0.04275	0.96949	0.36949	0.05887
	50	2.80014	0.50014	0.02550	3.03913	0.83913	0.03694	0.95911	0.35911	0.05629
	100	2.67201	0.37201	0.02343	2.88386	0.68386	0.03524	0.92805	0.32805	0.05271
	200	2.60188	0.30188	0.02150	2.62187	0.42187	0.03294	0.89822	0.29822	0.04222
	400	2.57350	0.27350	0.02080	2.43844	0.23844	0.03089	0.81605	0.21605	0.03924
	800	2.45784	0.15784	0.01751	2.36367	0.16367	0.02874	0.64635	0.04635	0.03639
	1000	2.32463	0.02463	0.01602	2.29299	0.09299	0.02519	0.64455	0.04455	0.01638
λ	35	0.65602	0.25602	0.02391	1.65237	0.65237	0.00428	0.42520	0.22520	0.01435
	50	0.57980	0.17980	0.02189	1.38159	0.38159	0.00404	0.40158	0.20158	0.01040
	100	0.56430	0.16430	0.01866	1.33753	0.33753	0.00384	0.37555	0.17555	0.00956
	200	0.52100	0.12100	0.01431	1.12633	0.12633	0.00347	0.35466	0.15466	0.00795
	400	0.50120	0.10120	0.01320	1.08837	0.08837	0.00319	0.34575	0.14575	0.00715
	800	0.47581	0.07581	0.00893	1.07426	0.07426	0.00295	0.31678	0.11678	0.00632
	1000	0.46915	0.06915	0.00716	1.07045	0.07045	0.00254	0.29994	0.09994	0.00326

tion. The results for these Model parameter estimates together with their standard errors (in parenthesis) and the goodness-of-fit-statistics are given under Tables 6.5 and 6.6. The fitted densities and observed probability plot (see [31]) that demonstrate how best these model fits the observed data sets are shown in Figures 6.4 and 6.5.

The HLOPGW-U distribution was compared to other competing three parameter non-nested models, namely the Marshall-Olkin Log-logistic (MO-LLoG) distribution by [106], exponentiated-Fréchet (EFr) distribution by [77], Marshall-Olkin extended inverse Weibull (IWMO) by [87], Marshall-Olkin extended Fréchet (MOEFr) by [21], exponentiated Weibull (EW) by [88] and Exponentiated Half-Logistic Exponential (EHLE) by [2]. The pdfs of these models are as follows;

$$f_{IWMO}(x; \alpha, \theta\lambda) = \frac{\alpha\lambda\theta^{-\lambda}x^{-\lambda-1}\mathbf{e}^{-(\theta x)^{-\lambda}}}{[\alpha - (\alpha - 1)\mathbf{e}^{-(\theta x)^{-\lambda}}]^2},$$

for $\alpha, \theta, \lambda > 0$,

$$f_{EFr}(x; \alpha, \lambda, \delta) = \alpha\lambda\delta^\lambda \left[1 - \mathbf{e}^{-(\delta/x)^\lambda}\right]^{\alpha-1} x^{-(1+\lambda)} \mathbf{e}^{-(\lambda+1)(\delta/x)^\lambda},$$

for $\alpha, \lambda, \delta > 0$,

$$f_{MOEFr}(x; \alpha, \lambda, \delta) = \frac{\alpha \lambda \delta^\lambda x^{-(\lambda+1)} e^{-(\delta/x)^\lambda}}{[1 - \bar{\alpha}(1 - e^{-(\delta/x)^\lambda})]^2},$$

for $\alpha, \lambda, \delta > 0$,

$$f_{EW}(x; \alpha, \beta, \delta) = \alpha \beta \delta x^{\beta-1} e^{-\alpha x^\beta} (1 - e^{-\alpha x^\beta})^\delta,$$

for $\alpha, \beta, \delta > 0$,

$$f_{MO-LLoG}(x; \alpha, \beta, \gamma) = \frac{\alpha^\beta \beta \gamma x^{\beta-1}}{(x^\beta + \alpha^\beta \gamma)^2},$$

for $\alpha, \beta, \gamma > 0$

and

$$f_{EHLE}(x; \lambda, \alpha, a) = \frac{2a\lambda\alpha \exp(-\alpha\lambda x)(1 - \exp(-\alpha\lambda x))^{a-1}}{(1 + \exp(-\alpha\lambda x))^{a+1}},$$

for $\lambda, \alpha, a > 0$.

6.7.1 Half-Way House (Failure times in days (49 Cases) Data

The first dataset presents failure times in days (49 cases) for half-way house parolees in the District of Columbia and it was first studied by [101]. The dataset is given by; 13, 16, 20, 22, 22, 25, 32, 45, 49, 59, 64, 70, 88, 8, 89, 93, 95, 10, 112, 116, 122, 147, 150, 151, 177, 179, 190, 204, 207, 221, 233, 240, 245, 247, 264, 267, 272, 283, 291, 301, 307, 320, 337, 343, 352, 362, 367, 396, 421.

Table 6.5: The Models parameter estimates and goodness-of-fit statistics for half-way house data

Model	Estimates			Statistics							
	α	β	λ	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
HLOPGW-U	0.6323 (0.1124)	1.4380 (0.2898)	0.1825 (0.0553)	590.94	596.94	597.48	602.62	0.0338	0.2684	0.0703	0.97
OPGW-U	1.0465 (0.1296)	40.4810 (18.2100)	1.01×10^4 (0.0081)	594.48	600.48	601.01	606.15	0.0755	0.5129	0.0932	0.79
Weibull	-	0.0011 (0.0009)	1.3056 (0.1576)	600.53	604.53	604.79	608.32	0.1455	0.9488	0.1169	0.52
EW	α 1.2358 (0.7570)	β 0.2499 (0.0760)	δ 37.5554 (19.7592)	619.64	625.64	626.17	631.31	0.3787	2.2668	0.1706	0.12
MO-LLoG	α 18.7236 (4.5697)	β 1.6231 (0.1940)	γ 24.6109 (2.1419)	614.51	620.51	621.04	626.18	0.3040	1.8809	0.1360	0.33
EFr	α 5.8717 (7.7003)	δ 6.8262 (1.0339)	λ 0.0599 (0.1813)	673.89	679.89	680.42	685.56	0.6275	3.5043	0.3072	1.93×10^{-3}
MOEFr	116.2064 (9.8739)	6.8742 (5.6548)	1.6021 (0.2030)	614.31	620.31	620.84	625.99	16.6901	96.7639	0.9928	$< 2.2 \times 10^{-16}$
IWMO	α 1.16×10^2 (2.71×10^{-5})	λ 1.6021 (0.1818)	θ 0.1455 (5.52×10^{-2})	614.30	620.30	620.83	625.98	0.3129	1.9192	0.1375	0.31
EHLE	λ 3.0255 (3.32×10^{-7})	α 2.68×10^{-3} (3.82×10^{-4})	a 1.0677 (0.1907)	600.30	606.30	606.83	611.97	0.1396	0.9148	0.1098	0.60

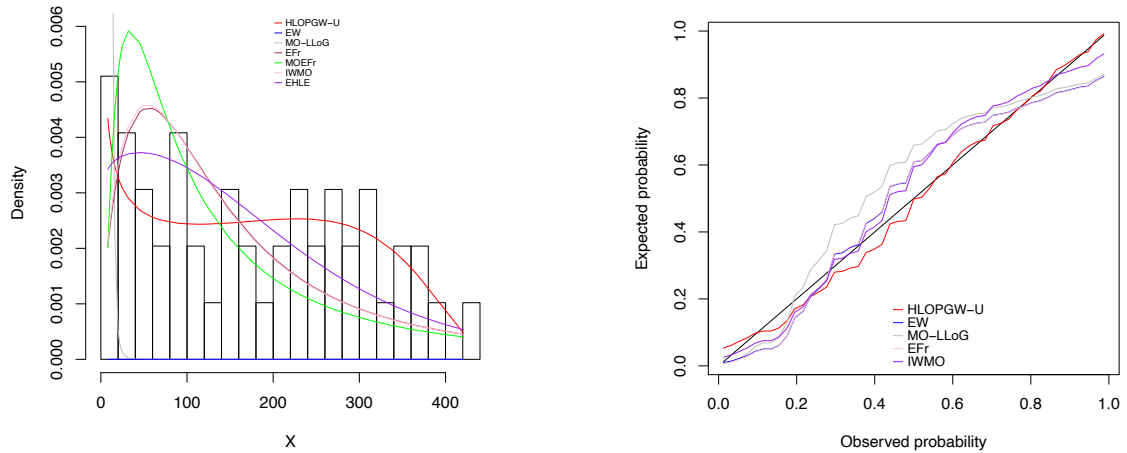


Figure 6.4: Fitted pdfs and observed probability plots for half-way house data

Note that the HLOPGW-U distribution gives a better fit to the half-way house data set compared to other non-nested models considered in the comparison. In addition, from the fitted density plots (see Figure 6.4), we confirm that the proposed model fits the heavy-tailed data well.

6.7.2 Leukemia Data

The second data set relates to lifetimes in days of fourty three (43) blood patients who had leukemia and was first collected from one hospital in Saudi Arabia and reported by [3]. The data is as follows; 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924 ,983 ,1025, 1062, 1063 ,1165, 1191, 1222 ,1222 ,1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1519, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852, 1899, 1925, 1965.

Table 6.6: The Model parameter estimates and goodness of fit statistics for leukemia data

Model	Estimates			Statistics							
	α	β	λ	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
HLOPGW-U	1.0049 (0.2293)	0.8746 (0.2289)	0.2267 (0.0493)	648.17	654.17	654.79	659.46	0.0199	0.1471	0.0650	0.99
OPGW-U	1.8505 (0.2370)	20.8620 (7.8754)	1.003×10^4 (0.0041)	651.10	657.10	657.72	662.39	0.0341	0.2557	0.0776	0.95
Weibull	- -	1.65×10^{-7} (1.23×10^{-9})	2.1767 (0.9613)	660.28	664.28	664.59	667.80	0.1482	0.9563	0.1456	0.32
EW	α 0.0667 (0.0287)	β 0.5572 (0.1595)	δ 15.5090 (11.3452)	685.74	691.74	692.36	697.03	0.4393	2.5642	0.1978	0.07
MO-LLoG	α 202.9888 (47.3798)	β 3.1368 (0.4129)	γ 228.1210 (13.4405)	673.59	679.59	680.21	684.88	0.3152	1.8977	0.1557	0.25
EFr	α 1.8668 (0.5507)	δ 167.4499 (45.1699)	λ 0.2905 (0.1253)	758.63	764.63	765.25	769.92	0.7989	4.4088	0.3942	3.14×10^{-6}
MOEFr	α 1.46×10^4 (0.0277)	δ 53.8340 (22.5960)	λ 3.1362 (0.0411)	673.59	679.59	680.21	684.88	13.5791	84.0707	0.9992	$< 2.200 \times 10^{-16}$
IWMO	α 1.52×10^4 (2.77×10^{-8})	λ 3.1362 (0.0412)	θ 0.0019 (7.93×10^{-3})	673.59	679.59	680.21	684.88	0.3158	1.9008	0.1557	0.25
EHLE	λ 1.1326 (4.11×10^{-7})	α 1.39×10^{-3} (1.72×10^{-4})	a 2.6078 (0.0583)	667.43	673.43	674.05	678.72	0.2459	1.5160	0.1494	0.29

Furthermore, from the results shown in Table 6.6 that the HLOPGW-U distribution fit the leukemia data set better than other non-nested models. The HLOPGW-U distribution gives smaller values of the goodness-of-fit statistics that is A^* , W^* and SS . Furthermore, according to the fitted density plots (see Figure 6.5), we note that the HLOPGW-U model fit the relief times data better than the selected non-nested.

6.8 Conclusions

A new family of generalized distribution namely the Half Logistic odd Power Generalized Weibull-G (HLOPGW-G) family of distributions. The new distribution is a

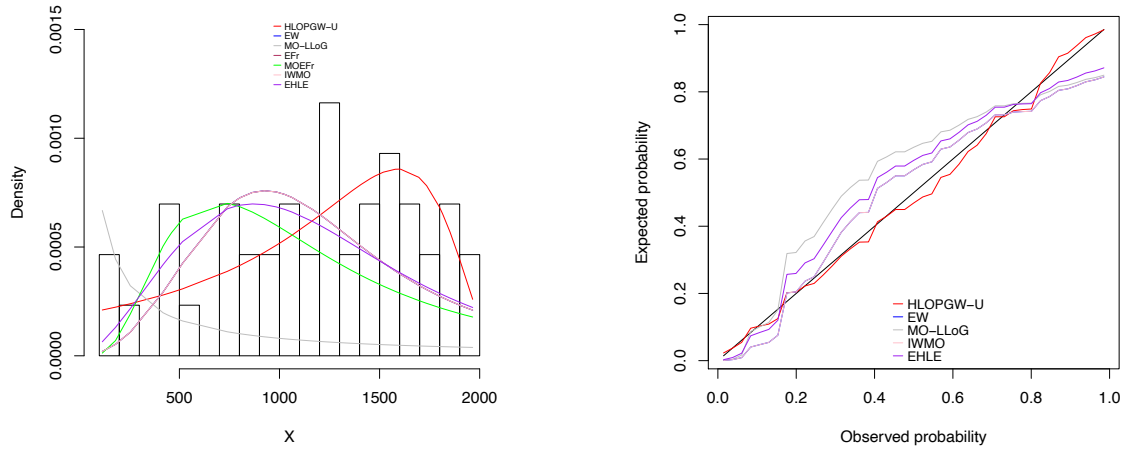


Figure 6.5: Fitted pdfs and probability plots for leukemia data

linear combination of the exponentiated-G distribution. The maximum likelihood estimation technique is used to estimate the model parameters. We applied the HLOPGW-U to two data sets that are heavily-skewed. Our new proposed model performs better than the selected equal-parameter non-nested models. Lastly, we expect that the new family of generalized distributions will find wider applicability in various disciplines such as finance, economics, engineering, reliability and survival analysis just to mention a few.

In addition, another new family of distributions belonging to the same family of distribution with the above model namely; the Half-Logistic Odd Power Generalized Weibull-G Power Series family, was also generated. The details on the development of this new class of distributions are also omitted in this thesis.

Chapter 7

Conclusions and Suggestions For Further Studies

7.1 Conclusions

In this work, we introduced and studied in detail new generalized families of distributions, namely, the Weibull Odd Burr III-G (WOBIII-G), Ristić-Balakrishnan Odd Burr III-G (RBOBIII-G), Ristić-Balakrishnan Marshall-Olkin-G (RBMO-G), Exponentiated Half-logistic Odd Burr III-G (EHL-OBIII-G) and the Half-Logistic Odd Power Generalized Weibull-G (HLOPGW-G) families of distributions. We extensively derived some of the mathematical and statistical properties for these distributions and further shown that these new family of distributions can be expressed as linear combination of the Exp-G distribution. The proposed distributions can model heavily tailed and skewed data. These distributions exhibit monotonic and non monotonic hazard rate functions. We conducted some simulation experiments to evaluate the effectiveness and accuracy of the maximum likelihood estimates. In addition, the usefulness and practicability of some of the special cases for these generalized models was demonstrated via applications to some real data sets and their performances were compared to some of the existing equal-parameter models. In conclusion, we have evidently demonstrated the usefulness of these newly proposed models by showing their better fits to real life datasets compared to other models considered in this study. Finally, we are optimistic that the proposed models will find wider applicability

in various fields such as actuarial science, economics, engineering, survival analysis and reliability, just to mention these few.

7.2 Suggestions for Reading and Further Studies

- Develop multivariate generalizations of these new models and stochastic dependence order via the parameters of the models.
- Consider different parameter estimation techniques on the proposed models.
- Relate the newly developed models in the context of whether they are among the extreme types such as exponential type, Cauchy type or the limited type.

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Appendix

Some Useful Expansions

A.1 Series Expansion of W-OBIII-G Density Function

The WOBIII-G pdf can be written as:

$$\begin{aligned}
 f(x; a, b, \beta, \underline{\varphi}) &= ab\beta \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} b_{s,m} \binom{\beta(k+1)-1}{m} \\
 &\times \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b(\beta(k+1)+m+s)-1} \\
 &\times \frac{\left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a-1}}{\left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right)} \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}.
 \end{aligned}$$

Also, by applying the generalized binomial series expansion

$$\left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right)^{-1} = \sum_{z=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1)z!} \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-bz},$$

we get

$$\begin{aligned}
 f(x; a, b, \beta, \underline{\varphi}) &= ab\beta \sum_{k=0}^{\infty} (-1)^k \sum_{z=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1)z!} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} b_{s,m} \binom{\beta(k+1)-1}{m} \\
 &\times \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b(\beta(k+1)+m+s+z)-1} \\
 &\times \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a-1} \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}.
 \end{aligned}$$

Now, considering

$$\begin{aligned}
\left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-a}\right)^{-b(\beta(k+1)+m+s+z)-1} &= \sum_{i=0}^{\infty} \binom{-b(\beta(k+1)+m+s+z)-1}{i} \\
&\times \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-ai}, \tag{7.1}
\end{aligned}$$

the pdf then reduces to

$$\begin{aligned}
f(x; a, b, \beta, \underline{\varphi}) &= ab\beta \sum_{k,z,i,m,s=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m} (-1)^k \binom{\beta(k+1)-1}{m} \\
&\times \binom{-b(\beta(k+1)+m+s+z)-1}{i} (G(x; \underline{\varphi}))^{-a(i+1)-1} (\overline{G}(x; \underline{\varphi}))^{a(i+1)+1} \\
&\times \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}.
\end{aligned}$$

Using the generalized binomial series expansion

$$(G(x; \underline{\varphi}))^{-a(i+1)-1} = (1 - \overline{G}(x; \underline{\varphi}))^{-a(i+1)-1} = \sum_{p=0}^{\infty} \binom{-a(i+1)-1}{p} (-1)^p \overline{G}^p(x; \underline{\varphi}),$$

we have

$$\begin{aligned}
f(x; a, b, \beta, \underline{\varphi}) &= ab\beta \sum_{k,z,i,p,m,s=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m} (-1)^{k+p} \binom{\beta(k+1)-1}{m} \\
&\times \binom{-a(i+1)-1}{p} \binom{-b(\beta(k+1)+m+s+z)-1}{i} (\overline{G}(x; \underline{\varphi}))^{a(i+1)+p-1} g(x; \underline{\varphi}) \\
&= ab\beta \sum_{k,z,i,p,w,m,s=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m} (-1)^{k+p+w} \binom{\beta(k+1)-1}{m} \\
&\times \binom{a(i+1)+p-1}{w} \binom{-b(\beta(k+1)+m+s+z)-1}{i} \\
&\times \binom{-a(i+1)-1}{p} \frac{w+1}{w+1} (G(x; \underline{\varphi}))^w g(x; \underline{\varphi}) \\
&= \sum_{w=0}^{\infty} c_{w+1} g_{w+1}(x; \underline{\varphi}), \tag{7.2}
\end{aligned}$$

where $g_{w+1}(x; \underline{\varphi}) = (w+1) (G(x; \underline{\varphi}))^w g(x; \underline{\varphi})$ is the exponentiated-G (Exp-G) distribu-

tion with power parameter $w + 1$ and

$$\begin{aligned}
 c_{w+1} &= ab\beta \sum_{k=0}^{\infty} \sum_{z=0}^{\infty} \sum_{i,p=0}^{\infty} \frac{\Gamma(z+1)}{(w+1)\Gamma(1)z!} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} b_{s,m} (-1)^{k+p+w} \binom{\beta(k+1)-1}{m} \\
 &\times \binom{-a(i+1)-1}{p} \binom{a(i+1)+p-1}{w} \binom{-b(\beta(k+1)+m+s+z)-1}{i}.
 \end{aligned} \tag{7.3}$$

B.2 PWMs Series Expansion for W-OBIII-G Distribution

Consider the series expansion given by

$$\begin{aligned}
 \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-a}\right)^{-b(\beta(k+1)+m+s+z)-1} &= \sum_{i=0}^{\infty} \binom{-b(\beta(k+1)+m+s+z)-1}{i} \\
 &\times \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-ai},
 \end{aligned} \tag{7.4}$$

we have

$$\begin{aligned}
 f(x)(F(x))^m &= ab\beta \sum_{q,k=0}^{\infty} \sum_{z,i=0}^{\infty} \sum_{m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k}}{k!} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m} \binom{\beta(k+1)-1}{m} \\
 &\times \binom{-b(\beta(k+1)+m+s+z)-1}{i} (G(x; \underline{\varphi}))^{-a(i+1)-1} (\overline{G}(x; \underline{\varphi}))^{a(i+1)+1} \\
 &\times \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}.
 \end{aligned}$$

Applying the generalized series expansion

$$(G(x; \underline{\varphi}))^{-a(i+1)-1} = (1 - \overline{G}(x; \underline{\varphi}))^{-a(i+1)-1} = \sum_{p=0}^{\infty} \binom{-a(i+1)-1}{p} (-1)^p \overline{G}^p(x; \underline{\varphi}),$$

we have

$$\begin{aligned}
 f(x)(F(x))^m &= ab\beta \sum_{q,k,z,i,p,m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k+p}}{k!} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m} \\
 &\times \binom{\beta(k+1)-1}{m} \binom{-b(\beta(k+1)+m+s+z)-1}{i} \binom{-a(i+1)-1}{p} \\
 &\times (\overline{G}(x; \underline{\varphi}))^{a(i+1)+p-1} g(x; \underline{\varphi})
 \end{aligned}$$

$$\begin{aligned}
&= ab\beta \sum_{q,k,z,i,p,w,m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k+p+w}}{k!} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m} \\
&\times \binom{\beta(k+1)-1}{m} \binom{a(i+1)+p-1}{w} \binom{-b(\beta(k+1)+m+s+z)-1}{i} \\
&\times \binom{-a(i+1)-1}{p} \left(\frac{w+1}{w+1} \right) (G(x; \underline{\varphi}))^w g(x; \underline{\varphi}) \\
&= \sum_{w=0}^{\infty} \phi_{w+1} g_{w+1}(x; \underline{\varphi}), \tag{7.5}
\end{aligned}$$

where $g_{w+1}(x; \underline{\varphi}) = (w+1) (G(x; \underline{\varphi}))^w g(x; \underline{\varphi})$ is the exponentiated-G (Exp-G) distribution with power parameter $w+1$ and

$$\begin{aligned}
\phi_w &= ab\beta \sum_{q,k,z,i,p,m,s=0}^{\infty} \binom{m}{q} \frac{(-1)^{q+k+p+w}}{k!} (q+1)^k \frac{\Gamma(z+1)}{(w+1)\Gamma(1)z!} b_{s,m} \\
&\times \binom{-a(i+1)-1}{p} \binom{\beta(k+1)-1}{m} \binom{a(i+1)+p-1}{w} \binom{-b(\beta(k+1)+m+s+z)-1}{i}. \tag{7.6}
\end{aligned}$$

C.3 Distribution of Order Statistics Expansion for W-OBIII-G Distribution

Using equations (2.1) and (2.2) we write

$$\begin{aligned}
f(x) [F(x)]^{k+l-1} &= ab\beta \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta} \right) \\
&\times \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\
&\times \frac{\left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b-1}}{\left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right)} \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a-1} \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2} \\
&\times \left(1 - \exp \left(- \left[-\log \left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta} \right) \right)^{k+l-1}.
\end{aligned}$$

Applying the series expansion

$$\left(1 - \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-a}\right)^{-b}\right)^{-1} = \sum_{z=0}^{\infty} \frac{\Gamma(z+1)}{\Gamma(1)z!} \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-a}\right)^{-bz},$$

we get

$$\begin{aligned} f(x)(F(x))^{k+l-1} &= ab\beta \sum_{q,k=0}^{\infty} \sum_{z,m=0}^{\infty} \sum_{s=0}^{\infty} \binom{k+l-1}{q} (-1)^{q+k} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m} \\ &\times \binom{\beta(k+1)-1}{m} \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-a}\right)^{-b(\beta(k+1)-2+m+s+z)} \\ &\times \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-a-1} \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}. \end{aligned}$$

Now, consider

$$\begin{aligned} \left(1 + \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-a}\right)^{-b(\beta(k+1)-2+m+s+z)} &= \sum_{i=0}^{\infty} \binom{-b(\beta(k+1)-2+m+s+z)}{i} \\ &\times \left(\frac{G(x; \underline{\varphi})}{\overline{G}(x; \underline{\varphi})}\right)^{-ai} \end{aligned} \quad (7.7)$$

we have

$$\begin{aligned} f(x)(F(x))^{k+l-1} &= ab\beta \sum_{q,k=0}^{\infty} \sum_{z,i=0}^{\infty} \sum_{m,s=0}^{\infty} \binom{k+l-1}{q} (-1)^{q+k} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m} \\ &\times \binom{\beta(k+1)-1}{m} \binom{-b(\beta(k+1)-2+m+s+z)}{i} (G(x; \underline{\varphi}))^{-a(i+1)-1} \\ &\times (\overline{G}(x; \underline{\varphi}))^{a(i+1)+1} \frac{g(x; \underline{\varphi})}{(\overline{G}(x; \underline{\varphi}))^2}. \\ &= ab\beta \sum_{q,k=0}^{\infty} \sum_{z,i=0}^{\infty} \sum_{p,m,s=0}^{\infty} \binom{k+l-1}{q} (-1)^{q+k+p} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1)z!} b_{s,m} \\ &\times \binom{\beta(k+1)-1}{m} \binom{-b(\beta(k+1)+m+s+z)-1}{i} \binom{-a(i+1)-1}{p} \\ &\times (\overline{G}(x; \underline{\varphi}))^{a(i+1)+p-1} g(x; \underline{\varphi}) \\ &= ab\beta \sum_{q,k=0}^{\infty} \sum_{z,i=0}^{\infty} \sum_{p,w=0}^{\infty} \sum_{m,s=0}^{\infty} \binom{k+l-1}{q} (-1)^{q+k+p+w} (q+1)^k \frac{\Gamma(z+1)}{\Gamma(1)z!} \end{aligned}$$

$$\begin{aligned}
& \times b_{s,m} \binom{\beta(k+1)-1}{m} \binom{a(i+1)+p-1}{w} \binom{-b(\beta(k+1)+m+s+z)-1}{i} \\
& \times \binom{-a(i+1)-1}{p} \left(\frac{w+1}{w+1} \right) (G(x; \underline{\varphi}))^w g(x; \underline{\varphi}) \\
& = \sum_{w=0}^{\infty} b_{w+1} g_{w+1}(x; \underline{\varphi}),
\end{aligned} \tag{7.8}$$

where b_{w+1} and $g_{w+1}(x; \underline{\varphi})$ are as defined in equation (2.11).

D.4 Entropy Series Expansion for W-OBIII-G Distribution

Using the series expansion

$$(G(x; \underline{\varphi}))^{-a(v+i)-v} = (1 - \overline{G}(x; \underline{\varphi}))^{-a(v+i)-v} = \sum_{p=0}^{\infty} (-1)^p \binom{-a(v+i)-v}{p} \overline{G}^p(x; \underline{\varphi}),$$

then the Rényi entropy for the W-OBIII-G family of distributions can be written as

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v} \log \left((ab\beta)^v \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s,z,i,p,w=0}^{\infty} b_{s,m} \binom{\beta(k+v)-v}{m} (-1)^{k+p+w} v^k \right. \\
&\quad \times \binom{-a(v+i)-v}{p} \binom{-b(\beta(k+v)+m+s+z)-v}{i} \binom{a(v+i)-v+p}{w} \frac{\Gamma(z+v)}{\Gamma(v)z!} \\
&\quad \times \left. \frac{1}{\left[1 + \frac{w}{v}\right]^v} \int_0^{\infty} \left(\left[\frac{w}{v} + 1 \right] (G(x; \underline{\varphi}))^{\frac{w}{v}} g(x; \underline{\varphi}) \right)^v dx \right) \\
&= \frac{1}{1-v} \log \left[\sum_{w=0}^{\infty} \tau_w \exp((1-v)I_{REG}) \right],
\end{aligned}$$

for $v > 0$, $v \neq 1$, where $I_{REG} = \frac{1}{1-v} \log \left[\int_0^{\infty} \left(\left[\frac{w}{v} + 1 \right] (G(x; \underline{\varphi}))^{\frac{w}{v}} g(x; \underline{\varphi}) \right)^v dx \right]$ is the Rényi entropy of Exp-G distribution with power parameter $\frac{w}{v} + 1$, and

$$\begin{aligned}
\tau_w &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s,z,i,p=0}^{\infty} (ab\beta)^v b_{s,m} \binom{\beta(k+v)-v}{m} (-1)^{k+p+w} v^k \\
&\quad \times \binom{-a(v+i)-v}{p} \binom{-b(\beta(k+v)+m+s+z)-v}{i} \binom{a(v+i)-v+p}{w} \frac{\Gamma(z+v)}{\Gamma(v)z!} \\
&\quad \times \frac{1}{\left[1 + \frac{w}{v}\right]^v}.
\end{aligned}$$

E.5 Elements of Score Vector for W-OBIII-G Distribution

The elements of a score vector, $U(\Delta)$ are given by

$$\begin{aligned}
\frac{\partial \ell}{\partial a} &= \frac{n}{a} + \beta b \sum_{i=1}^n \frac{\left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b-1} \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a} \ln \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)}{\left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right)} \\
&\times \left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right) \right]^{\beta-1} \\
&- (\beta - 1) \sum_{i=1}^n \frac{b \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b-1} \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a} \ln \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)}{\left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right) \right] \left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right)} \\
&+ (-b - 1) \sum_{i=1}^n \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a} \ln \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right) \\
&- \sum_{i=1}^n \ln \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right) - \sum_{i=1}^n \frac{b \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b-1} \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a} \ln \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)}{\left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial b} &= \frac{n}{b} - \beta b \sum_{i=1}^n \frac{\left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b} \ln \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)}{\left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right)} \\
&\times \left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right) \right]^{\beta-1} \\
&+ (\beta - 1) \sum_{i=1}^n \frac{\left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b} \ln \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)}{\left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right) \right] \left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right)} \\
&- \sum_{i=1}^n \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right) + \sum_{i=1}^n \frac{\left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b} \ln \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)}{\left(1 - \left(1 + \left(\frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}\right)^{-a}\right)^{-b}\right)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^n \left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta} \\
&\quad \times \ln \left(-\log \left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right) \\
&\quad + \sum_{i=1}^n \ln \left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right],
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell}{\partial \varphi_k} &= -\beta b a \sum_{i=1}^n \frac{\left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b-1} \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a-1} \frac{\partial G(x_i; \underline{\varphi})}{\partial \varphi_k}}{\left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \overline{G}^2(x_i; \underline{\varphi})} \\
&\quad \times \left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right]^{\beta-1} \\
&\quad + (\beta - 1) \sum_{i=1}^n \frac{b a \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b-1} \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a-1} \frac{\partial G(x_i; \underline{\varphi})}{\partial \varphi_k}}{\left[-\log \left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \right] \left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \overline{G}^2(x_i; \underline{\varphi})} \\
&\quad - a(-b-1) \sum_{i=1}^n \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a-1} \frac{\frac{\partial G(x_i; \underline{\varphi})}{\partial \varphi_k}}{\overline{G}^2(x_i; \underline{\varphi})} \\
&\quad + (-a-1) \sum_{i=1}^n \frac{\overline{G}(x_i; \underline{\varphi})}{G(x_i; \underline{\varphi})} \frac{\frac{\partial G(x_i; \underline{\varphi})}{\partial \varphi_k}}{\overline{G}^2(x_i; \underline{\varphi})} + \sum_{i=1}^n \frac{a b \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b-1} \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a-1} \frac{\partial G(x_i; \underline{\varphi})}{\partial \varphi_k}}{\left(1 - \left(1 + \left(\frac{G(x_i; \underline{\varphi})}{\overline{G}(x_i; \underline{\varphi})} \right)^{-a} \right)^{-b} \right) \overline{G}^2(x_i; \underline{\varphi})} \\
&\quad + \sum_{i=1}^n \frac{\frac{\partial g(x_i; \underline{\varphi})}{\partial \varphi_k}}{(g(x_i; \underline{\varphi}))} - 2 \sum_{i=1}^n \frac{\frac{\partial \overline{G}(x_i; \underline{\varphi})}{\partial \varphi_k}}{(\overline{G}(x_i; \underline{\varphi}))}.
\end{aligned}$$

F.6 Elements of Score Vector for RBOBIII-G Distribution

The elements of the score vector $U(\Theta)$ for RBOBIII-G family of distributions are given by

$$\begin{aligned}
 \frac{\partial \ell_n}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \ln[1 - G(x_i; \psi)] - \sum_{i=1}^n \ln[G(x_i; \psi)] \\
 &+ (\delta - 1) \sum_{i=1}^n \left\{ \frac{\beta \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{\beta-1} \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \ln \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)}{\left[-\log \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{-\beta} \right] \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{\beta}} \right\} \\
 &+ (\beta - 1) \sum_{i=1}^n \left\{ \frac{\left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \ln \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)}{1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha}} \right\}, \\
 \\
 \frac{\partial \ell_n}{\partial \beta} &= \frac{n}{\beta} + (\delta - 1) \sum_{i=1}^n \left\{ \frac{\left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{\beta} \ln \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{\beta}}{\left[-\log \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{-\beta} \right] \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{\beta}} \right\} \\
 &- \sum_{i=1}^n \ln \left[1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right], \\
 \\
 \frac{\partial \ell_n}{\partial \delta} &= \frac{n \Gamma'(\delta)}{\Gamma(\delta)} + \sum_{i=1}^n \ln \left[-\log \left(\left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{-\beta} \right) \right]
 \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \psi_k} = & -(\alpha - 1) \sum_{i=1}^n \frac{\frac{\partial G(x_i; \psi_k)}{\partial \psi_k}}{1 - G(x_i; \psi)} - (\alpha + 1) \sum_{i=1}^n \frac{\frac{\partial G(x_i; \psi_k)}{\partial \psi_k}}{G(x_i; \psi)} + \sum_{i=1}^n \frac{\frac{\partial g(x_i; \psi_k)}{\partial \psi_k}}{g(x_i; \psi)} \\
& - (\delta - 1) \sum_{i=1}^n \frac{\alpha \beta \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha-1} \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{\beta-1}}{\left[-\log \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{-\beta} \right] \left(1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha} \right)^{\beta}} \\
& \times \left[\frac{\frac{\partial G(x_i; \psi_k)}{\partial \psi_k} (1 - G(x_i; \psi)) + \frac{\partial G(x_i; \psi_k)}{\partial \psi_k} G(x_i; \psi)}{(1 - G(x_i; \psi))^2} \right] \\
& - (-\beta - 1) \sum_{i=1}^n \frac{\alpha \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha-1} \left[\frac{\frac{\partial G(x_i; \psi_k)}{\partial \psi_k} (1 - G(x_i; \psi)) + \frac{\partial G(x_i; \psi_k)}{\partial \psi_k} G(x_i; \psi)}{(1 - G(x_i; \psi))^2} \right]}{1 + \left(\frac{G(x_i; \psi)}{1 - G(x_i; \psi)} \right)^{-\alpha}}.
\end{aligned}$$

G.7 Elements of Score Vector for RBMO-G Distribution

The elements of a score vector $U = (\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \psi_k})$ are given by

$$\frac{\partial \ell_n}{\partial \alpha} = -n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^n \ln \left[-\log \left(1 - \frac{\delta \bar{G}(x_i; \psi)}{1 - \delta \bar{G}(x_i; \psi)} \right) \right],$$

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \delta} = & (\alpha - 1) \sum_{i=1}^n \frac{\frac{\bar{G}(x_i; \psi) [1 - \delta \bar{G}(x_i; \psi)] - \delta [\bar{G}(x_i; \psi)]^2}{[1 - \delta \bar{G}(x_i; \psi)]^2}}{\left[-\log \left(1 - \frac{\delta \bar{G}(x_i; \psi)}{1 - \delta \bar{G}(x_i; \psi)} \right) \right] \left(1 - \frac{\delta \bar{G}(x_i; \psi)}{1 - \delta \bar{G}(x_i; \psi)} \right)} + \frac{n}{\delta} \\
& - 2 \sum_{i=1}^n \frac{\bar{G}(x_i; \psi)}{[1 - \delta \bar{G}(x_i; \psi)]}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \psi_k} = & (\alpha - 1) \sum_{i=1}^n \frac{\delta \frac{\partial \bar{G}(x_i; \psi_k)}{\partial \psi_k} [1 - \delta \bar{G}(x_i; \psi)] + \delta \bar{G}(x_i; \psi) \frac{\partial \bar{G}(x_i; \psi_k)}{\partial \psi_k}}{\left[-\log \left(1 - \frac{\delta \bar{G}(x_i; \psi)}{1 - \delta \bar{G}(x_i; \psi)} \right) \right] \left(1 - \frac{\delta \bar{G}(x_i; \psi)}{1 - \delta \bar{G}(x_i; \psi)} \right)} \\
& + \sum_{i=1}^n \frac{\frac{\partial g(x_i; \psi_k)}{\partial \psi_k}}{[g(x_i; \psi)]} + 2 \sum_{i=1}^n \frac{\bar{\delta} \frac{\partial \bar{G}(x_i; \psi_k)}{\partial \psi_k}}{[1 - \delta \bar{G}(x_i; \psi)]},
\end{aligned}$$

respectively.

H.8 Elements of Score Vector for EHL-OBIII-G Distribution

The elements of the score vector, $U(\Delta)$ are given by

$$\begin{aligned}
\frac{\partial \ell}{\partial a} &= \frac{n}{a} - (\alpha - 1) \sum_{i=1}^n \frac{b \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b-1} \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \ln \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)}{\left(\left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right)} \\
&+ (-b - 1) \sum_{i=1}^n \frac{\left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \ln \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)}{\left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)} - \sum_{i=1}^n \ln \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right) \\
&- (\alpha + 1) \sum_{i=1}^n \frac{b \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b-1} \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \ln \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)}{\left(1 + \left[1 - \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)}, \\
\\
\frac{\partial \ell}{\partial b} &= \frac{n}{b} + (\alpha - 1) \sum_{i=1}^n \frac{\left(\left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right) \ln \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)}{\left(\left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right)} \\
&- \sum_{i=1}^n \ln \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right) \\
&+ (\alpha + 1) \sum_{i=1}^n \frac{\left(\left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right) \ln \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)}{\left(1 + \left[1 - \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)}, \\
\\
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \ln \left(\left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right) \\
&- \sum_{i=1}^n \ln \left(1 + \left[1 - \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\overline{G}(x_i; \underline{\psi})} \right)^{-a} \right)^{-b} \right] \right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell}{\partial \underline{\psi}_k} &= (\alpha - 1)ba \sum_{i=1}^n \frac{\left(1 + \left(\frac{G(x_i; \underline{\psi})}{\bar{G}(x_i; \underline{\psi})}\right)^{-a}\right)^{-b-1} \left(\frac{G(x_i; \underline{\psi})}{\bar{G}(x_i; \underline{\psi})}\right)^{-a-1} \frac{\partial G(x_i; \underline{\psi})}{\partial \underline{\psi}_k}}{\left(\left(1 + \left(\frac{G(x_i; \underline{\psi})}{\bar{G}(x_i; \underline{\psi})}\right)^{-a}\right)^{-b}\right) \bar{G}^2(x_i; \underline{\psi})} \\
&- a(-b-1) \sum_{i=1}^n \left(\frac{G(x_i; \underline{\psi})}{\bar{G}(x_i; \underline{\psi})}\right)^{-a-1} \frac{\partial G(x_i; \underline{\psi})}{\partial \underline{\psi}_k} \frac{1}{\bar{G}^2(x_i; \underline{\psi})} + (-a-1) \sum_{i=1}^n \frac{\bar{G}(x_i; \underline{\psi})}{G(x_i; \underline{\psi})} \frac{\partial G(x_i; \underline{\psi})}{\partial \underline{\psi}_k} \frac{1}{\bar{G}^2(x_i; \underline{\psi})} \\
&+ \sum_{i=1}^n \frac{ab \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\bar{G}(x_i; \underline{\psi})}\right)^{-a}\right)^{-b-1} \left(\frac{G(x_i; \underline{\psi})}{\bar{G}(x_i; \underline{\psi})}\right)^{-a-1} \frac{\partial G(x_i; \underline{\psi})}{\partial \underline{\psi}_k}}{\left(1 + \left[1 - \left(1 + \left(\frac{G(x_i; \underline{\psi})}{\bar{G}(x_i; \underline{\psi})}\right)^{-a}\right)^{-b}\right]\right) \bar{G}^2(x_i; \underline{\psi})} \\
&+ \sum_{i=1}^n \frac{\frac{\partial g(x_i; \underline{\psi})}{\partial \underline{\psi}_k}}{(g(x_i; \underline{\psi}))} - 2 \sum_{i=1}^n \frac{\frac{\partial \bar{G}(x_i; \underline{\psi})}{\partial \underline{\psi}_k}}{(\bar{G}(x_i; \underline{\psi}))}.
\end{aligned}$$

I.9 Elements of Score Vector for HLOPGW-G Distribution

The elements of score vector $U = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \xi_k}\right)$ are given by:

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + (\beta - 1) \sum_{i=1}^n \frac{\left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)^\alpha \ln \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)}{\left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)^\alpha\right]} \\
&- \sum_{i=1}^n \frac{\beta \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)^\alpha \ln \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right) \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)^\alpha\right]^{\beta-1}}{\left(1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)^\alpha\right]^\beta\right)} \\
&+ \sum_{i=1}^n \ln(G(x_i; \underline{\xi})) - \sum_{i=1}^n \ln(\bar{G}(x_i; \underline{\xi})) \\
&+ 2 \sum_{i=1}^n \frac{\beta \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)^\alpha \ln \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right) \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)^\alpha\right]^{\beta-1} \exp \left\{1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)^\alpha\right]^\beta\right\}}{\left(1 + \exp \left\{1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})}\right)^\alpha\right]^\beta\right\}\right)},
\end{aligned}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n \ln \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right] - \sum_{i=1}^n \frac{\left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^\beta \ln \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]}{\left(1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^\beta \right)} \\ &+ 2 \sum_{i=1}^n \frac{\left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^\beta \ln \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right] \exp \left\{ 1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^\beta \right\}}{\left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^\beta \right\} \right)} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \xi_k} &= (\beta - 1) \sum_{i=1}^n \frac{\alpha \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^{\alpha-1} \left[\frac{\partial G(x_i; \underline{\xi})}{\partial \xi_k} \bar{G}(x_i; \underline{\xi}) - \frac{\partial \bar{G}(x_i; \underline{\xi})}{\partial \xi_k} G(x_i; \underline{\xi}) \right]}{\left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right] \bar{G}^2(x_i; \underline{\xi})} + \sum_{i=1}^n \frac{1}{g(x_i; \underline{\xi})} \frac{\partial g(x_i; \underline{\xi})}{\partial \xi_k} \\ &- \sum_{i=1}^n \frac{\alpha \beta \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^{\beta-1} \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^{\alpha-1} \left[\frac{\partial G(x_i; \underline{\xi})}{\partial \xi_k} \bar{G}(x_i; \underline{\xi}) - \frac{\partial \bar{G}(x_i; \underline{\xi})}{\partial \xi_k} G(x_i; \underline{\xi}) \right]}{\left(1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^\beta \right) \bar{G}^2(x_i; \underline{\xi})} \\ &+ (\alpha - 1) \sum_{i=1}^n \frac{\frac{\partial G(x_i; \underline{\xi})}{\partial \xi_k}}{(G(x_i; \underline{\xi}))} - (\alpha + 1) \sum_{i=1}^n \frac{\frac{\partial \bar{G}(x_i; \underline{\xi})}{\partial \xi_k}}{(\bar{G}(x_i; \underline{\xi}))} \\ &+ 2 \sum_{i=1}^n \frac{\alpha \beta \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^{\beta-1} \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^{\alpha-1} \left[\frac{\partial G(x_i; \underline{\xi})}{\partial \xi_k} \bar{G}(x_i; \underline{\xi}) - \frac{\partial \bar{G}(x_i; \underline{\xi})}{\partial \xi_k} G(x_i; \underline{\xi}) \right]}{\left(1 + \exp \left\{ 1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^\beta \right\} \right) \bar{G}^2(x_i; \underline{\xi})} \\ &\times \exp \left\{ 1 - \left[1 + \left(\frac{G(x_i; \underline{\xi})}{\bar{G}(x_i; \underline{\xi})} \right)^\alpha \right]^\beta \right\}, \end{aligned}$$

respectively.

Density Functions for Some non-nested Models

J.10 KW Distribution

$$\begin{aligned} f(x; a, b, c, \lambda) &= abc \lambda^c x^{c-1} \exp\{-(\lambda x)^c\} [1 - \exp\{-(\lambda x)^c\}]^{a-1} \\ &\times \{1 - [1 - \exp\{-(\lambda x)^c\}]^a\}^{b-1}, \end{aligned}$$

for $a, b, c, \lambda > 0$ and $x > 0$.

K.11 GWLLoG Distribution

$$f(x; \alpha, \beta, \gamma, a) = \frac{\alpha\beta\gamma x^{\gamma-1}}{a^\gamma} \left(1 + \left(\frac{x}{a}\right)^\gamma\right)^{-1} \left[\log\left(1 + \left(\frac{x}{a}\right)^\gamma\right)\right]^{\beta-1} \\ \times \exp\left\{-\alpha \left[\log\left(1 + \left(\frac{x}{a}\right)^\gamma\right)\right]^\beta\right\},$$

for $\alpha, \beta, \gamma, a > 0$ and $x > 0$.

L.12 NMW Distribution

$$f(x; \alpha, \gamma, \lambda, \beta) = \left(\alpha\gamma x^{\gamma-1} e^{\alpha x^\gamma} + \lambda\beta x^{\lambda-1} e^{-\beta x^\lambda}\right) e^{-e^{\alpha x^\gamma} + e^{-\beta x^\lambda}}, \quad x > 0,$$

for $\alpha, \beta, c, k > 0$ and $x > 0$.

M.13 BGL Distribution

$$f(x; \alpha, \lambda, a, b) = \frac{\alpha\lambda^2}{B(a, b)(1+\lambda)} (1+x)e^{-\lambda x} \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x}\right]^{\alpha\alpha-1} \\ \times \left[1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x}\right)^a\right]^{b-1},$$

for $\alpha, \lambda, a, b > 0$ and $x > 0$.

N.14 KOLLLoG Distribution

$$f(x; a, b, \lambda, c) = ab \left[\frac{\lambda^2}{(1+\lambda)} \frac{cx^{c-1}}{(1+x^c)^{-1}} \exp\left\{-\lambda \frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right\} \right] \\ \times \left[1 - \frac{\lambda + (1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}} \exp\left\{-\lambda \frac{(1-(1+x^c)^{-1})}{(1+x^c)^{-1}}\right\} \right]^{a-1} \\ \times \left(1 - \left[1 - \frac{\lambda + (1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}} \exp\left\{-\lambda \frac{(1-(1+x^c)^{-1})}{(1+x^c)^{-1}}\right\} \right]^a \right)^{b-1},$$

for $a, b, \lambda, c > 0$ and $x > 0$.

O.15 MO-K Distribution

$$f(x; \alpha, \beta, \theta, \delta) = \frac{\frac{\delta\alpha\theta}{\beta} \left(\frac{x}{\beta}\right)^{\theta-1} \left(\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}\right)^{\frac{-(\alpha+1)}{\alpha}}}{\left(\delta + (1-\delta) \left(\frac{\left(\frac{x}{\beta}\right)^{\alpha\theta}}{\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}}\right)^{\frac{1}{\alpha}}\right)^2},$$

for $\alpha, \beta, \theta, \delta > 0$ and $x > 0$.

R-Codes for Model Applications, Density and hrf Plots

```
#define HLOPGW_Uniform pdf

HLOPGW_U_pdf <- function(alpha,beta,lambda,x){
  ((2*alpha*beta*(1+(x/(lambda-x))^alpha)^(beta-1))*
  (exp(1-(1+(x/(lambda-x))^alpha)^beta))*(x/lambda)^(alpha-1))*
  (1/lambda))/(((1-(x/lambda))^(alpha+1))*
  (1+exp(1-(1+(x/(lambda-x))^alpha)^beta))^2)
}

## define HLPWG_LLoG_hazard
HLOPGW_U_hazard <-function(alpha,beta,lambda,x){
HLOPGW_U_pdf(alpha,beta,lambda,x)
}
x=seq(0,1,by=0.001)
par(mar = c(5,5,3,1), xpd = TRUE) # Specify par parameters
z1=HLOPGW_U_hazard(0.9,4.9,1.8,x)
par(mar = c())
plot(x,z1,ylim=range(c(0,4)),col=2,'l', frame.plot = FALSE,
lwd=2,xlab="x",ylab="Hazard")
z2=HLOPGW_U_hazard(0.35,1.2,0.5,x)
lines(x,z2,col=3,lwd=2)
z3=HLOPGW_U_hazard(1.5,5.8,2.8, x)
lines(x,z3,col=4,lwd=2)
z4=HLOPGW_U_hazard(6.9,8.8,2.3 ,x)
lines(x,z4,col=5,lwd=2)
z5=HLOPGW_U_hazard(0.4,2.3,8.4,x)
lines(x,z5,col=6,lwd=2)

###APPLICATIONS

HLOPGWU_LL<-function(alpha,beta,lambda){-sum(log(((2*alpha
*beta*(1+(x/(lambda-x))^alpha)^(beta-1))*(exp(1-(1+(x/
(lambda-x))^alpha)^beta))*(x/lambda)^(alpha-1))*(1/lambda))/
(((1-(x/lambda))^(alpha+1))*(1+exp(1-(1+(x/(lambda-x))^alpha)
^beta))^2))))}
mle.result<-mle2(HLOPGWU_LL,hessian = NULL,start=list
(alpha=2.201, beta=0.2281,lambda=500),optimizer="nlminb",
lower=0)
summary(mle.result)

##QUANTILES AND MOMENTS

HLOPGWU_quantile=function(parameter,u){
alpha=parameter[1]
```

```

beta=parameter[2]
lambda=parameter[3]
f=function(x) {
HLOPWGU_cdf(alpha,beta,lambda,x)-u
}
x=min(uniroot.all(f,lower=0,upper=1,tol=0.0001))
return(x)
}
HLOPWGU_QuantileTable=function(parameter_matrix) {
u=seq(0.1,0.9,0.1)
size=dim(parameter_matrix)[1]
Table_Quantile=matrix(NA,nrow=length(u),ncol=size)
row.names(Table_Quantile)=u
colnames(Table_Quantile)=apply(parameter_matrix,1,function(x)
{paste0('(',paste0(x,collapse=','),')')})
Table_Quantile
for(iter in 1:size){
parameter=parameter_matrix[iter,]
for(i in 1:length(u)){
Table_Quantile[i,iter]=HLOPWGU_quantile(parameter,u[i])}
}
return(Table_Quantile)
}
#-----
## table of quantile
parameter_matrix=as.matrix(rbind(
par1=c(0.4,0.6,1.0),
par2=c(0.5,0.6,1.0),
par3=c(0.2,0.8,1.0),
par4=c(0.5,0.5,1.0),
par5=c(0.9,0.6,1.0)
))
HLOPWGU_QuantileTable(parameter_matrix)
sink("C:/Users/ppeter/Desktop/SIMU/HLOPWGU_quantile_table.txt")
print(parameter_matrix)
print(HLOPWGU_QuantileTable(parameter_matrix))
sink()

#####MOMENTS#####

HLOPWGU_moments=function(parameter,k) {
alpha=parameter[1]
beta=parameter[2]
lambda=parameter[3]

f=function(x) {
(x^k) * (HLOPWGU_pdf(alpha,beta,lambda,x))
}
y=integrate(f,lower=0,upper=1,subdivisions = 1000000)
return(y$value)

```

```

}
HLOPWGU_MomentsTable=function(parameter_matrix,K){
list=c(paste0('EX',1:K),'SD','CV','CS','CK')
Table_Moments=matrix(NA,nrow=length(list),
ncol=dim(parameter_matrix)[1])
row.names(Table_Moments)=list
colnames(Table_Moments)=apply(parameter_matrix,1,function(x)
{paste0('(',paste0(x,collapse=','),')')})
Table_Moments
for(iter in 1:dim(parameter_matrix)[1]){
parameter=parameter_matrix[iter,]
for(i in 1:K){
Table_Moments[i,iter]=HLOPWGU_moments(parameter,i)
}
Table_Moments['SD',iter]=sqrt(Table_Moments['EX2',iter]-
Table_Moments['EX1',iter]^2)
Table_Moments['CV',iter]=sqrt(Table_Moments['EX2',iter]/
Table_Moments['EX1',iter]^2-1)
Table_Moments['CS',iter]=(Table_Moments['EX3',iter]-3*
Table_Moments['EX1',iter]*Table_Moments['EX2',iter]+2*
Table_Moments['EX1',iter]^3)/((Table_Moments['EX2',iter]-
Table_Moments['EX1',iter]^2)^(3/2))
Table_Moments['CK',iter]=(Table_Moments['EX4',iter]-4*Table_
Moments['EX1',iter]*Table_Moments['EX3',iter]+6*Table_Moments
['EX1',iter]^2*Table_Moments['EX2',iter]-3*Table_Moments
['EX1',iter]^4)/((Table_Moments['EX2',iter]-Table_Moments
['EX1',iter]^2)^2)
}
return(Table_Moments)
}
#-----
## Moments Table
parameter_matrix=as.matrix(rbind(
par1=c(0.5,0.7,1.2),
par2=c(0.6,0.5,1.1),
par3=c(0.2,0.9,1.1),
par4=c(0.4,0.8,1.4),
par5=c(0.8,0.7,1.5)
))
K=5
HLOPWGU_MomentsTable(parameter_matrix,K)

sink("C:/Users/ppeter/Desktop/SIMU/HLOPWGU_moment_table.txt")
print(parameter_matrix)
print(HLOPWGU_MomentsTable(parameter_matrix,K))
sink()
#-----

```