

OPTIMAL INVESTMENT-CONSUMPTION-INSURANCE STRATEGY WITH
INFLATION RISK AND STOCHASTIC INCOME IN AN ITÔ-LÉVY SETTING

by

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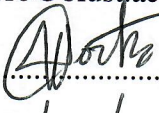
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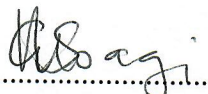
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Approval

This dissertation titled: “OPTIMAL INVESTMENT-CONSUMPTION-INSURANCE STRATEGY WITH INFLATION RISK AND STOCHASTIC INCOME IN AN ITÔ-LÉVY SETTING”, has been examined and approved as meeting the partial requirements for the fulfillment of Master of Science degree in Statistics at BIUST.

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Dedication

I dedicate this project to my late great grandmother Gaoganwe, my late grandfather Gabat-soswe and to my beautiful girls Wenzu and Zuri Moagi you are and will always be my motivation.

Abstract

In this study, we focus on finding the possible optimal strategy where there is consumption, stochastic income, purchasing of life insurance and investment in risky and risk-less investments are made. The stock follows a jump diffusion process and the bond is linked to inflation making the two risky and we have the investment position taken also on the money market account. The wealth process is determined by the different generations of the life of an investor, that is before the investor dies and after the investor dies. We also look into the optimal portfolio the beneficiaries benefit after the investor dies. We applied the concept of change of probability measures considering the Girsanov's and the Radon-Nikodym theorems, we found the generator of the Backward Stochastic differential equations defined and employed the Hamilton-Jacobi-Bellman dynamic programming (HJB) in finding the stochastic optimal controls of interest. The optimal strategies are explicitly obtained.

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Chapter 1

Introduction

1.1 Background Information

Over the past years, life insurance has been studied extensively incorporating the different ways of making it sound to the field of applied mathematics and statistics. In the financial context, life insurance has been analyzed to develop insights based on the life insurance demand, see Hakansson (1969) and Lewis (1989). They demonstrated how uncertain the nature of mortality is, the utility and strength of bequest desire, risk aversion, increasing household wealth and inter-temporal rates of substitution sway the choice and sort of life insurance to buy.

In 2015, Kronborg and Steffensen (2015) studied optimal consumption, investment and life insurance with surrender option guarantee where an investor's lifetime was considered in

investing with deterministic labour income and possibly willing to continue to invest in a Black-Scholes market and buy life insurance or annuities, they also considered that the savings were not supposed to be negative which is a non-negative reserve constraint in the context of life insurance and one constraint taken into account was that the savings did earn a minimum return as an interest rate guarantee. Guambe and Kufakunesu (2015) introduced optimal investment consumption and life insurance with capital constraints using the martingale approach and proved the existence of the optimal strategy, the optimal martingale measure and obtained the explicit solutions for the power utility functions when the investor is restricted to capital guarantee. Huang et al. (2008) believe that there is a blinding light between the way financial advisors advise clients about life insurance and how they promote investment products that they are presented as if they were a “separation theorem” that justified their relative invariance. Their aim was to analyze the decision of how much life insurance a family unit should have to protect against the loss of its breadwinner and how the family can distribute their financial resources between risky assets and risk free assets. Their findings were that the demand for life insurance should be dependent on the riskiness of the human capital and how it is correlated with financial returns. Further, they found that, highly anticipated levels of wage volatility or high levels of correlation induce reduction in the demand for life insurance.

Different scholars have in researched on the different approaches of studying optimal investment consumption. They further extended the study by including conditions and con-

straints considering the economic downturns e.g., (inflation), how the assets are to be allocated and how consumption is to occur to obtain maximal investment benefits. In 2019, Li and Xia (2019) studied the optimal investment strategy under the disordered return and random inflation. The optimal portfolio problem was vested in such a way that the inflation risk was taken into account, the risky asset price is changed by the impacts of major events which leads to the asset returns disorder at a random time and when investors have partial information in the financial market. Maslov and Zhang (1998) researched on optimal investment strategy for risky assets where investment in a portfolio of assets subject to multiplicative Brownian motion, the strategy providing the maximal typical long term growth rate of the investors capital, determination of the optimal fraction of capital an investor is supposed to keep in risky assets and the different assets weights in an optimal portfolio. The volatility and average return of an asset are considered to be indicators that determine its optimal weight. Mataramvura and Øksendal (2008) studied risk minimizing portfolios and HJBI equations for stochastic differential games that minimize the convex risk measure of the terminal wealth in a jump diffusion market. They extended the approach to solving a zero-sum stochastic differential game between an agent and a market as a min-max problem obtained in the process by proving the Hamilton–Jacobi–Bellman–Isaacs (HJBI).

Recently, Li and Xia (2019) studied the optimal investment strategy under the disordered return and random inflation on how to adjust the investment strategy in order to cope with the inflation risk as it is important to the investor. The Martingale approach has been pre-

viously used by some scholars to study the portfolio with inflation risk. Bo and Capponi (2018) used the martingale approach in studying optimal investment consumption problem in a model economy capturing market and credit risk dependencies. They numerically analyzed the sensitivity of the optimal investment strategies to risk aversion, default risk and volatility. Serrano (2021) also used the martingale method to determine the optimal threshold and investment strategy that maximizes the expected utility from dividend payouts that follows a consumption stock when studying portfolio allocation in a Lévy-type jump diffusion model with nonlife insurance risk. In 2014 López and Serrano (2014) studied martingale approach to optimal portfolio-consumption problem in Markov modulated pure-jump models. In the study they gave sufficient conditions for existence of optimal policies and did find the closed form expressions for the optimal value function for agents with logarithmic and fractional power utility in the case of two state Markov chains.

In 2011, Siu (2011) gave the optimal allocation strategy of the long term strategic assets with the inflation risk. Brennan and Xia (2002) studied the dynamic optimal portfolio with inflation, Wachter (2002) studied Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets. In 2015, Fei et al. (2015) studied the impact of inflation and the jumps on optimal asset allocations of an investor under asset prices with a jump environment. Two optimal consumption and portfolio problems with the financial markets of Markovian switching and inflation research was carried out by Fei (2013). In 2021, Doctor (2021) studied application of Generalized Geometric

Itô-Lévy Process to Investment-Consumption-Insurance Optimization Problem under Inflation Risk using the martingale approach to maximize the utility of an agent investing in risky and risk-less assets.

1.2 Problem Statement

The dissertation entirely establishes a strategy that maximizes the investor's wealth when an investor takes investment position in stock, inflation linked bond, money market account and purchases life insurance while consumption is made in a finite time horizon. The level of consumption is taken into account so that the investor does not collapse the strategy. Unique strategies are determined for both life generations that is, pre-death and post death of an investor. The choice of utility functions include the power and logarithmic functions in optimizing the wealth of the investor. In the study, the strategies in which beneficiaries should adopt when the investor dies is also studied and established.

1.3 Objectives

The aim of the study is to find the strategy that optimize the expected utility of the wealth of an investor while consuming and purchasing life insurance. This is done with consideration of the effect of inflation and the compensated diffusion jumps in the stock price. We aim to:

1. Find an investor's optimal strategy assuming the power and logarithmic utility functions.
2. Determine the wealth process in the different generations of an investor's life that is pre death and post death.
3. Establish an optimal strategy the beneficiaries adopt should the investor die under the power and logarithmic utility functions.
4. Indicate the value functions an investor should choose when longing for optimal returns with purchase of life insurance and consumption being done.
5. Establish value functions beneficiaries should consider in continuing with investment when the investor dies and the amount to consume not to collapse the strategy.

1.4 Significance of the study

The study's relevance is that it:

1. Establishes the optimal strategy an investor should inherit assuming the power and logarithmic utility functions. The wealth process composition is made up of risky, risk-less assets, purchase of life insurance and consumption made with caution.
2. Sheds light to investors on how to maximize their returns and diversify risk by investing across various financial instruments within a portfolio. Moreover, it also gives

guide to beneficiaries on how investment can be continued even when the investor dies.

1.5 Scope of the study

A comprehensive study to establish the optimal strategy that an investor would inherit should they invest in risk-less asset, risky asset, purchase life insurance and still consume. Investment strategies are obtained in an Itô-Lévy setting. The utility functions considered are the power and logarithmic functions. The investment strategies on both generations of the life of an investor are all presented.

1.6 Limitation

Some calculations led to highly nonlinear partial differential equations which were difficult to solve and present solutions explicitly.

Chapter 2

Literature Review

When studying optimal investment it is important to review utility theory and what portfolio theory encompasses.

2.1 Utility Theory

A utility function $U(x)$ is a function that shows the worthiness associated with goods and services. It measures the benefit consumers get from certain goods and services, that is, how much benefit investors get from the performance of portfolios. As all investors wish for high returns, it is important to understand that high returns come with a lot of risk. Therefore, a concrete understanding of an individual's utility with their money gives investors a clear insight on the grounds to take on investment strategies. The choice that investors make is dependent on the portfolios presented as to maximize their returns. This

was studied comprehensively by Markowitz (1952) on portfolio selection on the different stages of selecting a suitable portfolio with presentation of the expected utility rule and the curve that relates the utility to various return levels.

Utility functions were also discussed by Gerber and Pafum (1998) presenting the exponential, power utility functions of first kind e.g. quadratic utility functions and power utility functions of second kind e.g. logarithmic utility functions with the aim of giving a concise survey of utility functions as well as their applications which can be of interest to actuaries and insurance systems. They considered the utility functions with two basic properties; that the function be an increasing, concave and twice differentiable function. Utility theory was also studied intensively by Von Neumann and Morgenstern (1947) discussing the existence of a utility function which can be obtained from a set of axioms held by preference ordering. The study was extended by Borch and Pafumi (1974) where utility theory was utilized to form and solve problems relevant to insurance.

Markowitz (1952) assigned every utility function studied to a risk aversion function in which the exponential utility function yielded a constant risk aversion. Observations revealed that the risk aversion grew with increasing wealth and became limitless for certain parameters, suggesting that close to the immersion point little utility is acquired by a monetary gain. As for the power utility function of second kind, the risk aversion depicted a decreasing function of wealth which is believed to be typical for some investors. Using

the properties above we assume that the second derivative of $U(x)$, $U''(x)$ is maximal when $U''(x) < 0$. The exponential function is given by

$$U(x) = \frac{1}{n} \left(1 - \frac{1}{e^{nx}} \right), \quad n > 0, -\infty < x < \infty. \quad (2.1)$$

When x gets large $U(x) \rightarrow \frac{1}{n}$, which is finite.

The power utility function of first kind is given by

$$U(x) = \frac{d^{e+1} - (d-x)^{e+1}}{(e+1)d^e}, \quad (2.2)$$

for $d, e > 0$ and $d > x$. The model is non existent when $d = x$ but setting $U(x) = \frac{d}{e+1}$ for $x \geq d$ from (2.2). We consider the quadratic utility function

$$U(x) = x - \frac{x^2}{2d}, \quad e = 1, \quad d > x. \quad (2.3)$$

The Power utility function of the second kind is given by

$$U(x) = \frac{\frac{x}{x^a} - 1}{1 - a}, \quad (2.4)$$

for $a, x > 0$ and $a \neq 1$. But, for $a = 1$ we define $U(x)$ as;

$$U(x) = \ln(x), \quad x > 0, \quad (2.5)$$

which is the logarithmic utility function.

Risk Aversion Functions

The risk aversion function of any utility function is defined by;

$$r(x) = \frac{-U''(x)}{U'(x)}. \quad (2.6)$$

For the given functions above we define their risk aversion functions as follows;

1. For the exponential utility function its risk aversion is a constant $y > 0$, that is;

$$r(x) = y, \quad -\infty < x < \infty. \quad (2.7)$$

2. For the power utility function of the first kind we obtain the risk aversion to be increasing with wealth and grows without bound for x approaching d ;

$$r(x) = \frac{e}{d-x}, \quad d > x. \quad (2.8)$$

3. As for the power utility function of the second kind we obtain the risk aversion to be a decreasing function of the investor's wealth.

$$r(x) = \frac{a}{x}, \quad x > 0. \quad (2.9)$$

2.2 Portfolio selection theory

Portfolio selection is a continuously studied topic in the field of financial business. Pandey (2012) believes that portfolio selection is a process of having to describe and choose the most efficient portfolio in a set. This ability to select an optimal portfolio is dependent on the risk tolerance and risk aversion of an investor. The theory of portfolio selection was also extensively discussed by Zhang et al. (2004) on how Markowitz (1952) characterized the return and risk for combining more than two financial assets where costs or other restraints to transactions are non-existent by applying the mean and variance. Finite variances are assumed for the rate of return of assets since short selling is not limited. Portfolio theory encompasses four units which include the individual security valuation, performance measurement, asset allocation and portfolio optimization. Security valuation is a process of determining the worth of a security. It includes description of expected returns and risks of assets, the level of debt and data that can be quantified in an individual's assets. In order to decide on the portfolio it is essential for investors to select or consider looking at the price of a share to buy and the worth of an asset when selling. Valuation models include liquidating value, intrinsic value, book value and the replacement value compared to the market price.

Book value includes the accounting process which is the total value of an individual's assets short of the individual's outstanding liabilities. Studies on book value revealed it as an important variable that affects the equity share of a market value that is affected by in-

vestment and dividends decisions of an individual. Kane and Marcus (1986) studied the impact of book value on market value of an equity share and stated that assessing the effect of book value on the market value of equity needs to be analyzed carefully. The total value of an individual's tangible assets if they were to be sold and debts be paid is calculated by subtracting the value of all assets and liabilities from the financial report of an investor. Vieira (2020) studied liquidity constraints for portfolio selection based on financial volume and stated that applying liquidity constraints in a portfolio selection model was performed in an empirical test conducted in some emerging markets. Intrinsic value is the difference between asset's price and the option's strike price.

Asset allocation is the process of how an individual should invest into different assets. Asset allocation creates and balances a portfolio. Its best work depends on the ability to tolerate risk and the time horizon an investor has. Bekkers et al. (2009) showed how the mean-variance and market-neutral portfolio approaches can be combined to determine optimal portfolios. This shows how crucial portfolio selection is on an individual's investment. The expected time an investor has towards a particular goal in an investment is called the horizon time. The length of the horizon determines the level of risk a investor can have, the longer the horizon the more the riskier portfolio an investor can form and the shorter the horizon the less risky portfolio an investor may consider. Risk tolerance is the propensity and preparedness to lose your investments in exchange for probable better returns. Some investors prefer high risk investments in order to get better returns which also creates a

room for high likelihood of losing their original investments. Low risk takers in investment tend to preserve their investments due to their low risk tolerance. Thus, risk tolerance is an important feature to take into account when investing. This was also shown by Sulaiman (2012) that it is an important element to determine the optimal composition of assets in a portfolio in consideration to the risk and the relative return and a utility aspect for any investment decision to maximize the expected utility as it is a goal for any investment.

Portfolio optimization is the process of selecting the best portfolio with minimal risk and maximal return amongst others in a set of portfolios considered. Optimization techniques can be used in portfolio selection, they include the classical Markowitz mean-variance model that weighs risk expressed as variance against expected return. Portfolio optimization was also studied by Pardalos (1997) and stated how important portfolio selection is in investment decision making and studied the optimization algorithms on how they address computational complexity issues related to portfolio selection problems. Portfolio measurement is the measure of the rate of return of a portfolio in an investment. The rate of return is measured as the total sum of dividend and capital lost or gained divided by the purchase price. There are two methods to measure this rate which includes the Time Weighted Rate of Return and value Weighted rate of return. These were studied in detail by Pardalos (1997) where managed portfolios were discussed and how well the methods work and for which portfolios. The expected return of an investment portfolio is given by;

$$E(R_p) = E\left(\sum_{k=1}^n W_k \cdot R_k\right) = \sum_{k=1}^n W_k \cdot E(R_k), \quad (2.10)$$

where W_k are weights of individual shares and n is the number of securities. The variance of return is used to measure risk in an investment. Calculation of risk through the Markowitz formula in minimizing the risk is given by

$$\sigma_p = \sqrt{\left(\sum_{x=1}^N W_x^2 \sigma_x^2 + \sum_{x=1}^N \sum_{y=1}^N W_x W_y C_{xy}\right)} \quad (2.11)$$

where W_x is the amount invested in asset x , W_y is amount invested in asset y , C_{xy} is the co-variance between security x and y , N is the number of security and σ_x^2 is the variance.

Chapter 3

Preliminary Definitions and Concepts

3.1 Definitions

This chapter gives the most fundamental definitions and concepts which will be used on the oncoming chapters.

Definition 3.1.1 *If Ω is a given set then a σ -algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:*

1. $\emptyset \in \mathcal{F}$,
2. $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$,
3. $A_1, A_2, \dots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called measurable space.

Definition 3.1.2 A probability measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function,

$$\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$$

such that

$$1. \mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1,$$

2. if $A_1, A_2, \dots \in \mathcal{F}$ and $\{A_i\}_{i=1}^\infty$ is disjoint (that is, $A_i \cap A_j = \emptyset, i \neq j$) then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called the probability space. The space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete if $\mathcal{F}$ contains all subsets G on Ω with $\mathbb{P}$ -outer measure zero, i.e.,

$$\mathbb{P}(G) := \inf\{\mathbb{P}(F) : F \in \mathcal{F}, G \subset F\} = 0$$

see more from Øksendal (2003). The subsets F of Ω which belong to $\mathcal{F}$ are called $\mathcal{F}$ -measurable sets. These sets are said to be events and we use interpretation; $\mathbb{P}(F)$ is the probability that the event F occurs, that is $\mathbb{P}(F) = 1$ means the event F occurs almost surely (a.s.) or F occurs with probability 1.

Definition 3.1.3 Let $\mathbb{F} = (\mathcal{F}_t)_{t \in I}$ be an increasing family (in the sense of inclusion) of σ -algebra, I be an ordered index set with $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$, for $s, t \in I$. Then $\mathbb{F} = (\mathcal{F}_t)_{t \in I}$ is called a filtration.

In general, the information used by the decision makers will increase as time passes. Assuming that the decision marker never “forgets” past information, the data set must be

increasing over time:

$$\mathcal{F}_{t_0} \subseteq \mathcal{F}_{t_1} \subseteq \dots \subseteq \mathcal{F}_{t_k} \subseteq \mathcal{F}_{t_{k+1}} \subseteq \dots,$$

where $t_i, i = 0, 1, \dots$ are the times when the data set becomes available. In our context, an increasing family of σ -algebra signifies the family of information sets that become continuously available to the decision maker (an investor) as time passes.

The quadruplet $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a filtered probability space. The family $(\mathcal{F}_t)_{t \in I}$ is usually used to model events which can be observed up to time t . If the value of X_t is included in the information set $\mathcal{F}_t$ at each $t \geq 0$, then X_t is adapted to $\mathcal{F}_t$. That is, the value X_t will be known, given the information set $\mathcal{F}_t$ at time t .

Definition 3.1.4 (*Random variable*) A random variable X is an $\mathcal{F}$ -measurable function

$$X : \Omega \rightarrow \mathbb{R}^n, \quad \Omega \text{ is finite}$$

Every random variable induces a probability measure ψ_X on $\mathbb{R}^n$ by

$$\psi_X(\mathbb{B}) = \mathbb{P}(X^{-1}(\mathbb{B})), \quad \text{where } \mathbb{B} - \text{Borel set}$$

is called the distribution of X (Øksendal (2003)).

Throughout our discussion we define $L^1_{\mathcal{F}}(0, T; \mathbb{R})$ as a set of all $\mathcal{F}$ -measurable functions that are integrable with respect to t , and $L^2_{\mathcal{F}}(0, T; \mathbb{R})$ as a set of $\mathcal{F}$ -measurable functions that are square integrable with respect to t .

Definition 3.1.5 A stochastic process is a parametrized collection of random variables $(X_t)_{t \in T}$ which evolve with time, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assuming val-

ues in $\mathbb{R}^n$. The parameter T is usually the half line $[0, \infty)$, but can also be an interval $[a, b]$.

Definition 3.1.6 A stochastic process $(X_t)_{t \in [0, T]}$ is called a *supermartingale* with respect to a filtration $(\mathcal{F}_t)_{t \in [0, T]}$ (with probability measure $\mathbb{P}$) if,

1. $\forall t, X_t$ is $\mathcal{F}_t$ -measurable,
2. X_t is integrable for each $t \in [0, T]$,
3. $X_s \geq \mathbb{E}[X_t \mid \mathcal{F}_s]$ a.s. for all $0 \leq s \leq t \leq T$.

Definition 3.1.7 A stochastic process $(X_t)_{t \in [0, T]}$ is called a *submartingale* with respect to a filtration $(\mathcal{F}_t)_{t \in [0, T]}$ (with probability measure $\mathbb{P}$) if,

1. $\forall t, X_t$ is $\mathcal{F}_t$ -measurable,
2. X_t is integrable for each $t \in [0, T]$,
3. $X_s \leq \mathbb{E}[X_t \mid \mathcal{F}_s]$ a.s. for all $0 \leq s \leq t \leq T$.

Definition 3.1.8 A stochastic process $(X_t)_{t \in [0, T]}$ is called a *martingale* with respect to a filtration $(\mathcal{F}_t)_{t \in [0, T]}$ (with probability measure $\mathbb{P}$) if,

1. $\forall t, X_t$ is $\mathcal{F}_t$ -measurable,
2. X_t is integrable for each $t \in [0, T]$,
3. $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$ a.s. for all $0 \leq s \leq t \leq T$.

Definition 3.1.9 A standard Brownian motion is a stochastic process $(B_t)_{t \in [0, \infty)}$ with values in $\mathbb{R}$ such that

1. $B_0 = 0$ a.s.,
2. the sample paths $t \rightarrow B_t$ are a.s. continuous,
3. the increments are independent, i.e. $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_k} - B_{t_{k-1}}$ are independent $\forall 0 \leq t_1 < t_2 < \dots < t_k$,
4. $\forall t \geq 0$ and $h > 0$, the increments $B_{t+h} - B_t$ are normally distributed with expectation zero and variance h .

Definition 3.1.10 (Itô process) Let B_t be 1-Dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. A 1-D Itô process is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$X_t = x_0 + \int_0^t \alpha(s, w) ds + \int_0^t \beta(s, w) dB_s, \quad X_0 = x_0 \quad (3.1)$$

In shorter form

$$dX_t = \alpha(t, w)dt + \beta(t, w)dB_t, \quad X_0 = x_0 \quad (3.2)$$

where $\beta(t, w) \in \mathcal{W}_{\mathcal{H}}$ so that

$$\mathbb{P} \left[\int_0^t \beta(s, w)^2 ds < \infty, \quad \forall t \geq 0 \right] = 1$$

We also assume that $\alpha(t, w)$ is $\mathcal{H}_t$ -adapted and

$$\mathbb{P} \left[\int_0^t |\alpha(s, w)| ds < \infty, \quad \forall t \geq 0 \right] = 1$$

Theorem 3.1.11 *(The general Itô formula) Let $dX_t = \alpha dt + \beta dB_t$ be an n -Dimensional Itô process where α and β are two measurable functions. Let $g(t, x) = (g_1(t, x), \dots, g_n(t, x))$ be a C^2 map from $[0, \infty) \times \mathbb{R}^n$ into $\mathbb{R}^n$. Then the process*

$$Y_t = g(t, X(t))$$

is again an Itô process.

$$dY_t = \frac{\partial g(t, X_t)}{\partial t} dt + \frac{\partial g(t, X_t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g(t, X_t)}{\partial x^2} (dX_t)^2 \quad (3.3)$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dB_t = dB_t \cdot dt = dt \cdot dt = 0, \quad dB_t \cdot dB_t = dt.$$

Similarly, suppose $Y(t, x)$ is a real-valued function with continuous partial derivatives $Y_t(t, x)$, $D_x Y(t, x)$ and $D_{xx} Y(t, x) \quad \forall t \geq 0, x \in \mathbb{R}$ and in equation (3.2) where α and β are measurable functions. We also assume that the process $D_x Y(t, X_t)\beta(t)$ has an integrable square. Then $Y(t, X_t)$ can also be presented

$$\begin{aligned} Y(T, X_T) - Y(0, X_0) &= \int_0^T (Y_t(t, X_t) + D_x Y(t, X_t)\alpha(t) + \frac{1}{2} D_{xx} Y(t, X_t)\beta(t)^2) dt \\ &\quad + \int_0^T D_x Y(t, X_t)\beta(t) dB_t, \quad a.s. \end{aligned} \quad (3.4)$$

[For the proof refer to Brzezniak and Zastawniak (1999)]

Theorem 3.1.12 *(The Itô Representation Theorem) Let $F \in L^2(\mathcal{F}_T^{(n)})$. Then there exists a unique stochastic process $f(t, \omega) \in \mathcal{V}^n(t, 0)$ such that*

$$F(\omega) = \mathbb{E}[F] + \int_0^T f(t, \omega) dB_t. \quad (3.5)$$

Theorem 3.1.13 (*The Martingale Representation Theorem*) Let $B_t = (B_1(t), \dots, B_n(t))$ be n -dimensional Brownian motion. Suppose M_t is an $\mathcal{F}^{(n)}$ -martingale with respect to $\mathbb{P}$ and that

$$M_t \in L^2(\mathbb{P}), \forall t \geq 0.$$

Then there exist a unique stochastic process $g(t, \omega)$ such that

$$g \in \mathcal{V}^{(n)}, \forall t \geq 0$$

and

$$M_t = \mathbb{E}[M_0] + \int_0^t g(s, \omega) dB_s \text{ a.s. } \forall t \geq 0. \quad (3.6)$$

Theorem 3.1.14 (*Existence and Uniqueness Theorem for Stochastic Differential Equation(SDE)*) Let $T > 0$ and $b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ be measurable functions satisfying

$$|b(t, x)| + |\sigma(t, x)| \leq c(1 + |x|), \quad x \in \mathbb{R}^n, \quad t \in [0, T] \quad (3.7)$$

and

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq c|x - y|, \quad x, y \in \mathbb{R}^n, \quad t \in [0, T]$$

for some constant $c > 0$. Let Z be a random variable which is independent of the σ -algebra

$G_B = \sigma(B_s, 0 \leq s \leq T)$ and such that $\mathbb{E}[|Z|^2] < \infty$. The stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0 \quad (3.8)$$

has a unique solution that is continuous with respect to t , $X_t(\omega)$, with the property that $X_t(\omega)$

is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $(B_s)_{s \leq t}$ and we have $\mathbb{E}[\int_0^T |X_t|^2 dt] < \infty$.

3.2 Itô-diffusion and Generator

In this section we briefly give some properties of the Itô-diffusion that are relevant and important to the rest of the work. We take into consideration the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and the terminal time $T > 0$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$. Let $B = (B_t)_{t \in [0, T]}$ be an m -dimensional Brownian motion. Consider a stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \quad (3.9)$$

where $X_t \in \mathbb{R}^n$, $b(t, X_t) \in \mathbb{R}^n$ and $\sigma(t, X_t) \in \mathbb{R}^{n \times m}$. b is called the drift coefficient, σ is known as the diffusion coefficient or the volatility.

Definition 3.2.1 (*Time Homogeneous Itô Diffusion*) *A time homogeneous Itô diffusion is a stochastic process*

$$X_t(\omega) = X(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n \quad (3.10)$$

satisfying a stochastic differential equation of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geq s, \quad X_s = s, \quad (3.11)$$

where B_t is m -dimensional Brownian motion, $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ and also satisfying the following condition

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|, \quad x, y \in \mathbb{R}^n \quad (3.12)$$

where

$$|\sigma|^2 = \sum |\sigma_{i,j}|^2.$$

Definition 3.2.2 Suppose $X = (X_t)$ is an Itô diffusion and let D_L be a domain of L^2 that is, a class of all functions $[0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ with their limits existing for all s, x . A generator Lg of the process X is defined as

$$Lg(s, x) = \lim_{t \rightarrow s} \frac{\mathbb{E}_{s,x}[g(t, X_t)] - g(s, x)}{t - s} \quad (3.13)$$

$\forall s \geq 0, x \in \mathbb{R}^n$ and $g \in D_L$ of L^2 .

Definition 3.2.3 Let $\mathcal{L}$ be the differential operator defined on $C^{1,2}$ by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \quad (3.14)$$

with $a_{i,j} = (\sigma \sigma^T)_{i,j}$ and let $f \in C^{1,2}$ such that $\forall u \geq t \geq 0, x \in \mathbb{R}$

$$\mathbb{E}_{t,x} \left[\int_t^u |\mathcal{L}f(s, X_s)| ds \right] < \infty$$

and

$$\mathbb{E}_{t,x} \left[\int_t^u (D_x f(s, X_s)^T \sigma(s, X_s))^2 ds \right] < \infty$$

Then $f \in D_L$ and $Lf = \mathcal{L}f$.

Theorem 3.2.4 (Dynkin's formula) Let $f \in D_L$ and τ an almost surely finite stopping time with respect to $\mathbb{F}$. Then

$$\mathbb{E}_x[f(\tau, X_\tau)] = f(0, x) + \mathbb{E}_x \left[\int_0^\tau \mathcal{L}f(s, X_s) ds \right]. \quad (3.15)$$

Note that if τ is the first exit time of the bounded set, $\mathbb{E}^x[\tau] < \infty$, then the above Dynkin's formula holds for any function $f \in C^2$. [For the proof refer to Øksendal (2003)].

3.3 Stochastic Control of Jump Diffusion processes

In this section, we introduce the concept of stochastic control for jump diffusions as a motivation to our research (see Øksendal and Sulem (2005)).

3.3.1 Itô-Lévy processes

We give a brief survey of stochastic calculus for Itô-Lévy processes,

Definition 3.3.1 *The Itô Lévy process of a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is defined as a process $\Psi(t)$ having the properties:*

- i. $\Psi(0) = 0$
- ii. Ψ has independent stationary increments.
- iii. Ψ being stochastically continuous then the jump of Ψ at time t is

$$\Delta\Psi(t) = \Psi(t) - \Psi(t-1).$$

When finding the number of jumps of Ψ up to time t , the jump measure $J([0, t], U)$ is used as per given set $U \subset \mathbb{R}_0 \equiv \mathbb{R} \setminus \{0\}$. It can be proven that U contains infinitely many jumps in a given finite interval with the assumption that $\bar{U} \subset \mathbb{R}_0$. Given the Lévy measure $\nu(\cdot)$ of Ψ then it is defined by:

$$\nu(U) = E(J([0, t], U)).$$

The random measure $J([0, t], U)$ has the differential notation $J(dt, d\eta)$ where η is the generic jump size. We define $\tilde{J}(\cdot)$ as the compensated jump measure of Ψ given as

$$\tilde{J}(dt, d\eta) \equiv J(dt, d\eta) - \nu(d\eta)dt. \quad (3.16)$$

We consider the condition below on $\nu(\cdot)$

$$\int_{\mathbb{R}} \eta^2 \nu(d\eta) < \infty,$$

which we assume that it is a valid case $\forall t \geq 0$ such that

$$E[\Psi^2(t) < \infty]. \quad (3.17)$$

Considering all Lévy processes, the conditions allow that a Lévy process hold the following property

$$\int_{\mathbb{R}} (1 \wedge |\eta|) \nu d\eta = \infty, \quad (3.18)$$

which implies that there are infinitely many small jumps, but under the assumption that

$$\int_{\mathbb{R}} \eta^2 \nu(d\eta) < \infty$$

the Itô-Lévy decomposition theorem has it that any Lévy process takes the form:

$$\Psi(t) = at + bB(t) + \int_0^t \int_{\mathbb{R}} \eta \tilde{J}(ds, d\eta) \quad (3.19)$$

where $B(t)$ is the Brownian motion and a, b are real numbers.

The Itô-Lévy processes to be studied are of the form

$$X(t) = x + \int_0^t \alpha(s, x)ds + \int_0^t B(s, x)dB(s) + \int_0^t \int_{\mathbb{R}} \gamma^2(s, \eta, x) \tilde{J}(ds, d\eta), \quad (3.20)$$

with $\int_0^t |\alpha(s)|ds + \int_0^t B^2(s)ds + \int_0^t \int_{\mathbb{R}} \gamma^2(s, \eta) \nu(d\eta)ds < \infty$ a.s and $\alpha(t), B(t)$ and $\gamma(t, \eta)$ being processes that can be predicted with respect to the filtration generated by $\Psi(s)$ for $s \leq t$.

In this thesis we assume that the Lévy measure $\nu(\cdot)$ of Ψ satisfies $\int_{\mathbb{R}} \min\{1, |\Psi|\} \nu(d\Psi) < \infty$, so the jumps have finite variation.

Theorem 3.3.2 (*The 1-dimensional Itô formula, Øksendal and Sulem (2005), Protter (1990), Applebaum (2004)*). Suppose $\Psi(t) \in \mathbb{R}$ is an Itô-Lévy process of the form

$$d\Psi(t) = \alpha(t, \omega)dt + \beta(t, \omega)dB(t) + \int_{\mathbb{R}} \gamma(t, \eta, \omega) \tilde{J}(dt, d\eta), \quad (3.21)$$

where

$$\tilde{J}(dt, d\eta) = \begin{cases} J(dt, d\eta) - \nu(d\eta)dt & \text{if } |\eta| < R; \\ J(dt, d\eta) & \text{if } |\eta| \geq R \end{cases} \quad (3.22)$$

for some $R \in [0, \infty]$, $\int_0^t \alpha(s)dt + \int_0^t \beta^2(s)dt + \int_0^t \gamma^2(s, \eta) \nu(d\eta)dt < \infty$ a.s., and $\alpha(t), \beta(t)$ and $\gamma(t, \eta)$ are predictable process (predictable w.r.t. the filtration $\mathcal{F}_t$ generated by $\Psi(s)$, for

$s \leq t$). Let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f(t, \Psi(t))$. Then $Y(t)$ is again an Itô-Lévy process and

$$\begin{aligned} dY(t) = & \frac{\partial f}{\partial t}(t, \Psi(t))dt + \frac{\partial f}{\partial \Psi}(t, \Psi(t))[\alpha(t, \omega)dt + \beta(t, \omega)dB(t)] \\ & + \frac{1}{2}\beta^2(t, \omega)\frac{\partial^2 f}{\partial \Psi^2}(t, \Psi(t))dt \\ & + \int_{|\eta| < R} \left\{ f(t, \Psi(t-) + \gamma(t, \eta)) - f(t, \Psi(t-)) - \frac{\partial f}{\partial \Psi}(t, \Psi(t-))\gamma(t, \eta) \right\} \nu(d\eta)dt \\ & + \int_{\mathbb{R}} \left\{ f(t, \Psi(t-) + \gamma(t, \eta)) - f(t, \Psi(t-)) \right\} \tilde{J}(dt, d\eta). \end{aligned} \quad (3.23)$$

Note 3.3.3

$$\left\{ \begin{array}{l} \text{If } R = 0, \text{ then } \tilde{J} = J \text{ everywhere;} \\ \text{If } R = \infty, \text{ then } \tilde{J} = J \text{ everywhere.} \end{array} \right. \quad (3.24)$$

Theorem 3.3.4 (*The Itô-Lévy isometry, Øksendal and Sulem (2005)*) Let $\Psi(t) \in \mathbb{R}$ be as in (3.21) but with $\Psi(0) = 0$ and $\alpha = 0$. Then

$$\mathbb{E}[\Psi^2(T)] = \mathbb{E}\left[\int_0^T \beta^2(t)dt + \int_0^T \gamma^2(t, \eta)dt\right]. \quad (3.25)$$

We also state martingale properties as follows: If the quantities in theorem (3.3.4) are finite, then

$$M(t) = \int_0^t \int_{\mathbb{R}} \gamma(t, \eta) \tilde{J}(ds, d\eta) \quad (3.26)$$

is a martingale for $t \leq T$.

3.3.2 Girsanov theorem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space. Let Q be another probability measure on $\mathcal{F}_T$. Q is equivalent to $\mathbb{P}|_{\mathcal{F}_T}$ if $\mathbb{P}|_{\mathcal{F}_T} \ll Q$ and $Q \ll \mathbb{P}|_{\mathcal{F}_T}$, or, equivalently, if $\mathbb{P}$ and Q have the same zero sets in $\mathcal{F}_T$. By Radon-Nikodym theorem this is the case if and only if we have

$$dQ(\omega) = Z(T)d\mathbb{P}(\omega) \text{ and } d\mathbb{P}(\omega) = Z^{-1}(T)dQ(\omega) \text{ on } \mathcal{F}_T$$

for some $\mathcal{F}_T$ -measurable random variable $Z(T) > 0$ a.s. $\mathbb{P}$.

Lemma 3.3.5 Suppose $Q \ll \mathbb{P}$ with $\frac{dQ}{d\mathbb{P}} = Z(T)$ on $\mathcal{F}_T$. Then for all $t \in [0, T]$,

$$Q|_{\mathcal{F}_t} \ll \mathbb{P}|_{\mathcal{F}_t}$$

and

$$Z(t) = \frac{d(Q|_{\mathcal{F}_t})}{d(\mathbb{P}|_{\mathcal{F}_t})} = \mathbb{E}_{\mathbb{P}}[Z(T)|\mathcal{F}_t]. \quad (3.27)$$

In particular, $Z(t)$ is a $\mathbb{P}$ -martingale.

Theorem 3.3.6 (Girsanov theorem I for jump processes) Let $\Psi(t)$ be a 1-dimensional Itô-Lévy process of the form

$$d\Psi(t) = \alpha(t, \omega)dt + \int_{\mathbb{R}} \gamma(t, \eta)\tilde{J}(dt, d\eta). \quad (3.28)$$

Assume that there exists a process $\theta(s, \eta) \leq 1$ such that

$$\int_{\mathbb{R}} \gamma(t, \eta)\theta(t, \eta)\nu(d\eta) = \alpha(t) \text{ for a.a. } (t, \omega) \quad (3.29)$$

and such that the process $Z(t)$ defined by

$$\begin{aligned} Z(t) &= \exp \left\{ \int_0^t \int_{\mathbb{R}} \ln(1 - \theta(s, \eta)) \tilde{J}(ds, d\eta) \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}} \{\ln(1 - \theta(s, \eta)) + \theta(s, \eta)\} \nu(d\eta) ds \right\} \end{aligned} \quad (3.30)$$

exists for $0 \leq t \leq T$. Define a measure Q on $\mathcal{F}_T$ by

$$dQ(\omega) = Z(T) d\mathbb{P}(\omega). \quad (3.31)$$

Assume that

$$\mathbb{E}_{\mathbb{P}}[Z(T)] = 1, \quad (3.32)$$

then Q is an equivalent local martingale measure for $\Psi(t)$.

3.3.3 Dynamic programming

Fix a domain $\mathcal{S} \subset \mathbb{R}$ (solvency region) and let $Y(t) = Y^u(t)$ be a stochastic process of the form

$$\begin{aligned} dY(t) &= b(Y(t), u(t))dt + \sigma(Y(t), u(t))dB(t) \\ &\quad + \int_{\mathbb{R}} \gamma(Y(t-), u(t-), \eta) \tilde{J}(dt, d\eta), \quad Y(0) = y \in \mathbb{R}, \end{aligned} \quad (3.33)$$

where $b : \mathbb{R} \times U \rightarrow \mathbb{R}$, $\sigma : \mathbb{R} \times U \rightarrow \mathbb{R}$ and $\gamma : \mathbb{R} \times U \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions, $U \subset \mathbb{R}$ is a given set. The process $u(t) = u(t, \omega) : [0, \infty) \times \Omega \rightarrow U$ is the control process, assumed to be cadlag and adapted. Thus $Y(t)$ is a controlled process. Define the performance criterion

$$J^u(y) = \mathbb{E}^y \left[\int_0^{\tau_S} f(Y(t), u(t))dt + g(Y(\tau_S)) \cdot \chi_{\{\tau_S < \infty\}} \right] \quad (3.34)$$

where $\tau_S = \inf\{t > 0; Y(t) \notin S\}$ is the bankruptcy time and $f : S \times U \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions. The control process u is said to be admissible ($u \in \mathcal{A}$) if (3.33) has a unique, strong solution $Y(t)$ for all $y \in S$ and

$$\mathbb{E}^y \left[\int_0^{\tau_S} f^-(Y(t), u(t)) dt + g^-(Y(\tau_S)) \cdot \chi_{\{\tau_S < \infty\}} \right] < \infty.$$

The stochastic control problem is to find the value function $\phi(y)$ and optimal control $u^* \in \mathcal{A}$ defined by

$$\phi(y) = \sup_{u \in \mathcal{A}} J^u(y) = J^{u^*}(y). \quad (3.35)$$

If $u = u(y)$ is a Markov control then $Y^u(t)$ is a Lévy diffusion with generator

$$\begin{aligned} \mathcal{L}\varphi(y) &= \mathcal{L}^u\varphi(y) = b(y, u(y)) \frac{\partial \varphi}{\partial y}(y) + \frac{1}{2} \sigma^2(y, u(y)) \cdot \frac{\partial^2 \varphi}{\partial y^2} \\ &+ \int_{\mathbb{R}} \left\{ \varphi(y + \gamma(y, u(y), \eta)) - \varphi(y) - \frac{\partial \varphi}{\partial y}(y) \gamma(y, u(y), \eta) \right\} \nu(d\eta). \end{aligned} \quad (3.36)$$

We now formulate Hamilton-Jacobi-Bellman (HJB) equation associated with (3.35).

Theorem 3.3.7 (*Hamilton-Jacobi-Bellman (HJB) equation, Øksendal and Sulem (2005)*)

a) Suppose $\varphi \in C^2(S) \cap C(S)$ satisfies the following:

i. $\mathcal{L}^v\varphi(y) + f(y, v) \leq 0 \quad \forall y \in S, v \in U$

ii. $Y(\tau_S) \in \partial S$ a.s. on $\{\tau_S < \infty\}$ and

$$\lim_{t \rightarrow \tau_S^-} \varphi(Y(t)) = g(Y(\tau_S)) \cdot \chi_{\{\tau_S < \infty\}} \quad \forall$$

$$u \in \mathcal{A}$$

iii.

$$\begin{aligned} \mathbb{E}^y \left[|\varphi(Y(\tau))| + \int_0^{\tau_S} \{ |\mathcal{L}\varphi(Y(t))| + |\sigma(Y(t)) \frac{\partial \varphi}{\partial y}(Y(t))|^2 \right. \\ \left. + \int_{\mathbb{R}} |\varphi(Y(t) + \gamma(Y(t), u(t), \eta)) - \varphi(Y(t))|^2 \nu(d\eta) \} dt \right] < \infty \end{aligned} \quad (3.37)$$

$\forall u \in \mathcal{A}$ and all $\tau \in \mathcal{T}$.

iv. $\{\varphi^-(Y(\tau))\}_{\tau \leq \tau_S}$ is uniformly integrable for all $u \in \mathcal{A}$ and $y \in \mathcal{S}$. Then

$$\varphi(y) \geq \phi(y), \quad \forall y \in \mathcal{S}. \quad (3.38)$$

b) Moreover, suppose that for each $y \in \mathcal{S}$ there exists $v = \hat{u}(y) \in U$ such that

v. $\mathcal{L}^{\hat{u}(y)}\varphi(y) + f(y, \hat{u}(y)) = 0$ and

vi. $\{\varphi(Y^{\hat{u}(y)}(\tau))\}_{\tau \leq \tau_S}$ is uniformly integrable.

Suppose $u^*(y) = u(Y(t-)) \in \mathcal{A}$. Then u^* is an optimal control and

$$\varphi(y) = \phi(y) = J^{u^*}(y), \quad \forall y \in \mathcal{S}. \quad (3.39)$$

Theorem 3.3.8 (Dominance Convergence Theorem) If $\{f_n : \mathbb{R} \rightarrow \mathbb{R}\}$ is a sequence of measurable functions which converge pointwise almost everywhere to f , and if there exists an integrable function g such that $|f_n(x)| \leq g(x)$ for all n and for all x then f is integrable and,

$$\int_{\mathbb{R}} f(x) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n(x) dx, \quad \text{then } f_n(x) \rightarrow 0 \text{ pointwise.} \quad (3.40)$$

Lemma 3.3.9 (Fatou's Lemma) Let (X, Σ, μ) be a measure space and $\{f_n : X \rightarrow [0, \infty]\}$ a sequence of nonnegative measurable functions. Then the function $\liminf_{n \rightarrow \infty} f_n$ is measurable

and

$$\int_X \liminf_{n \rightarrow \infty} f_n \, d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n \, d\mu. \quad (3.41)$$

[Refer to Royden (2018) for the proof].

Chapter 4

Model Description

We suppose that our model is built in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ which follows the characteristics of the Brownian motion $B_t = \{B_t^0, B_t^1, B_t^2, 0 \leq t \leq T\}$ in a finite time horizon, i.e., $0 \leq t \leq T$ to be considered. The Brownian motions $\langle B_t^0, B_t^1, B_t^2 \rangle$ are all correlated such that $dB_t^0 \cdot dB_t^1 = \rho_{0,1}dt$, $dB_t^0 \cdot dB_t^2 = \rho_{0,2}dt$, $dB_t^1 \cdot dB_t^2 = \rho_{1,2}dt$, are correlated with the correlation coefficient $-1 \leq \rho_{i,j} \leq 1$. For $\rho_{i,j} = 1$ then the Brownian motions are strongly correlated and for $\rho_{i,j} = -1$ then they are negatively correlated. We have the following for $i, j = 0, 1, 2$ and $i \neq j$;

$$\rho_{i,j}B_t^i = \sqrt{1 - \rho_{i,j}^2} B_t^j. \quad (4.1)$$

We take into account an important economic factor inflation, since the value for money in a long term investment has to be considered as inflation can lead to reduction in the value of investment returns of an investor. This was also discussed by Zhang and Zheng (2019) that

inflation could lead to diminution of financial wealth of the insurer when studying optimal investment Reinsurance policy with stochastic interest and inflation rates. We can measure inflation by the inflation rate, by considering the consumer price index (CPI) which includes the consumer price and retail price indices for any change in the index.

The investor inherits in three assets namely, a money market account, an inflation-linked index bond and stock. The wealth of an insurance policy holder is indicated by a portfolio distributing the wealth accordingly with π_1 being the fraction of the wealth at time t held by the stock, π_2 being held in the inflation-linked index bond which shall be defined indepth along the descriptions and $(1 - \pi_1 - \pi_2)$ denotes the fraction held in the money market account such that all the fractions held add up to 1. We denote the market value of the money market account R_t at a given time t as;

$$dR_t = rR_t dt, \quad (4.2)$$

where $r > 0$ denote the interest rate to be compounded over t .

Since our stock has a diffusion jump which is a L  vy process then the dynamics of the stock follows the following It  -L  vy process;

$$dS_t = S_{t-}[\alpha dt + \sigma dB_t^0 + \int_{\mathbb{R}} \gamma(t, \eta) \tilde{J}(d\eta, dt)], \quad (4.3)$$

where α , σ and $r > 0$ are the stock's drift and volatility rates respectively, $\gamma > -1$ and B_t^0 is the Brownian motion driving the stock. We assume the inequality $\alpha - r > 0$ is satisfied for

economic equilibrium purposes.

We consider the consumer good¹ an investor purchases in their time of life in the economy over a time interval satisfying a stochastic process:

$$dQ(t) = Q(t)[Z(t)dt + \varrho(t)dB^1(t)]_{\forall t \in [0, T]}, \quad Q(0) = Q_0, \quad (4.4)$$

with $B^1(t)$ being a Brownian motion driving the consumption good, $\varrho(t)$ is the volatility price index rate of the commodity good and $Z(t)$ is the stochastic drift rate representing the expected inflation rate over time governed by the time dependent Ornstein- Uhlenbeck (OU) process also applied by Chaiyapo and Phewchean (2017) and is given by:

$$dZ(t) = \varsigma(t)[\beta(t) - Z(t)]dt + \bar{\varrho}(t)dB^1(t), \quad (4.5)$$

where $\beta(t)$ is the long-run mean of the inflation rate, $\bar{\varrho}(t)$ is the volatility rate and $\varsigma(t)$ is the rate of mean reversion of the inflation rate. The quantity functions $\varrho(t)$, $\beta(t)$, $\varsigma(t)$ and $\bar{\varrho}(t)$ are continuous and deterministic with $t \in [0, T]$. We have the index bond $M(t)$ which is linked to inflation and has the price level process

$$\frac{dM(t)}{M(t)} = k(t)dt + \frac{dQ(t)}{Q(t)} = (k(t) + Z(t))dt + \varrho(t)dB^1(t), \quad (4.6)$$

where $k(t)$ is the interest rate at time t .

The insured have an incoming source of funds which comes in at different time intervals, therefore we consider it to be random over a given period of time. The income rate at time

¹Items that individuals and households buy, they include (packaged goods, clothing, beverages, automobiles, and electronics) for their own use and enjoyment.

t is given by $\psi(Y_t, t)$ where Y_t is referred to as the state variable which is an Itô process such that;

$$dY_t = \alpha_1(Y_t, t)dt + \sigma_1(Y_t, t)dX_t, \quad (4.7)$$

and follows the properties of Itô processes. With the assumption that $\alpha_1(Y_t, t)$ and $\sigma_1(Y_t, t)$ satisfies Lipschitz and growth conditions in Y_t and are continuous so we obtain a unique solution.

The policy holder purchases an insurance policy that pays premiums $P_t > 0$ over a unit time interval, supposedly on monthly basis which is an $\mathcal{F}_t$ adapted and measurable process with

$$\int_0^T P(s)ds < \infty. \quad (4.8)$$

We consider the lifetime of the policy holder at $t > 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The insurance is determined by the retirement and the time of death of the policy holder τ which is finite. The death benefit is paid to the beneficiaries at the time of death of the policy holder. The continuation of the policy can be renegotiated by the family members when the holder dies based on the predetermined terms and conditions of the insurance policy. We consider the probability mass function $f(t_l)$ to be the lifetime of the holder, then the cumulative mass function of t is given by;

$$F(t_l) = P(t < t_l) = \int_0^{t_l} f(s)ds, \quad (4.9)$$

then the probability that $t > t_l$ is the survival function given by,

$$G(t_l) = P(t > t_l) = 1 - F(t_l). \quad (4.10)$$

The instantaneous force of mortality $\mu(t_l)$ for a policy holder to be alive at time t_l is given by,

$$\begin{aligned}\mu(t_l) &= \lim_{\Delta t_l \rightarrow 0} \frac{P(t_l \leq t < t_l + \Delta t_l | t \geq t_l)}{\Delta t_l} \\ &= \lim_{\Delta t_l \rightarrow 0} \frac{P(t_l \leq t < t_l + \Delta t_l)}{\Delta t_l P(t \geq t_l)} \\ &= \frac{f(t_l)}{1 - F(t_l)} = -\frac{d}{dt_l}(\ln(1 - F(t_l))),\end{aligned}\tag{4.11}$$

which is the hazard rate function. The conditional probability of survival of the holder is defined as,

$$\bar{F}(t_l) = 1 - F(t_l) = P(t > t_l | \mathcal{F}_t) = \exp\left(-\int_0^{t_l} \mu(s) ds\right),\tag{4.12}$$

and the conditional survival probability density of the death of the holder is given by,

$$f(t_l) = \mu(t_l) \exp\left(-\int_0^{t_l} \mu(s) ds\right).\tag{4.13}$$

With the conditions above, the wealth process is given below in consideration with the following;

$$\pi_1 = \frac{\theta_t^1 S_t}{W_t}; \implies \pi_1 W_t = \theta_t^1 S_t,\tag{4.14}$$

$$\pi_2 = \frac{\theta_t^2 M_t}{W_t}; \implies \pi_2 W_t = \theta_t^2 M_t,\tag{4.15}$$

$$(1 - \pi_1 - \pi_2) = \frac{\theta_t^0 R_t}{W_t}; \implies (1 - \pi_1 - \pi_2) W_t = \theta_t^0 R_t,\tag{4.16}$$

where $\theta = (\theta_t^0, \theta_t^1, \theta_t^2)$ is the trading strategy that is (θ_t^0 being the number of shares held in the money market account, θ_t^1 being the number of shares held in stock, θ_t^2 being the number of shares held in the inflation linked bond), S_t is the price of the stock, R_t being the price

of the bond, W_t is the money held by the investor at time $t \in [0, T]$ and $\pi_1, \pi_2, (1 - \pi_1 - \pi_2)$ are as defined previously.

The total wealth process at time $t \in [0, T]$, where T is the time of retirement is given by; (further explained in appendix (7))

1. Pre-death case

For $t < \tau \wedge T$ that is $t \in [0, \tau \wedge T]$ we have,

$$\begin{aligned} dW_t = & \pi_1 W_t (\alpha dt + \sigma dB_t^0 + \int_{\mathbb{R}} \gamma(t, \eta) \tilde{J}(d\eta, dt)) - C(t)dt \\ & - \mu(t)(P(t) - W_t)dt + \pi_2 W(t)[(k(t) + Z(t))dt + \varrho(t)dB^1(t)] \\ & + (1 - \pi_1 - \pi_2)rW_t dt + \psi(Y_t, t)dt \end{aligned} \quad (4.17)$$

$$\begin{aligned} dW_t = & [W_t(\pi_1(\alpha - r) + \pi_2(k(t) + Z(t) - r) + r + \mu(t)) \\ & + \psi(Y_t, t) - \mu(t)P(t) - C(t)]dt + W_t[\pi_1 \sigma dB_t^0 - \pi_2 \varrho(t)dB_t^1] \\ & + \pi_1 W_t \int_{\mathbb{R}} \gamma(t, \eta) \tilde{J}(d\eta, dt), \end{aligned} \quad (4.18)$$

2. Post-death case

We have;

$$\begin{aligned} dW_t = & \pi_1 W_t (\alpha dt + \sigma dB_t^0 + \int_{\mathbb{R}} \gamma(t, \eta) \tilde{J}(dt, d\eta)) \\ & + (1 - \pi_1 - \pi_2)rW_t dt + \pi_2 W(t)[(k(t) + Z(t))dt + \varrho(t)dB^1(t)] \\ & - C(t)dt + \psi(Y_t, t)dt \end{aligned} \quad (4.19)$$

$$\begin{aligned}
dW_t = & (W_t[\pi_1(\alpha - r) + \pi_2(k(t) + Z(t) - r) + r] - C(t) \\
& + \psi(Y_t, t))dt + W_t[\sigma\pi_1dB_t^0 - \pi_2\varrho(t)dB_t^1] \\
& + \pi_1W_t \int_{\mathbb{R}} \gamma(t, \eta)\tilde{J}(dt, d\eta)
\end{aligned} \tag{4.20}$$

where $\tau \wedge T = \min \{\tau, T\}$, and $\mu(t)(P(t) - W_t)dt$ corresponds to the risk premium rate to pay for the life insurance at time t , $\psi(Y_t, t)$ is the stochastic income rate and the consumption rate $C(t)$ which are non negative and continuous.

4.1 Portfolio Selection

4.1.1 Pre-death case

We employ the Stochastic programming approach in finding the suitable investment strategy $(\pi_1, \pi_2, P(t), C(t))$ that maximizes the terminal wealth of an investor, the consumption and premiums to be paid during the minimum time $[\tau, T]$. Let the hurdle rate² be a positive $\mathcal{F}_t$ adapted process defined by $\Phi(t)$ and the utility function of the terminal wealth, consumption and premiums be given as U , we then select a strategy that maximize the performance function given by:

$$\begin{aligned} J(t, W(t), \pi_1, \pi_2, C, P) &:= \sup_{(\pi_1, \pi_2, C, P) \in \chi} \mathbb{E} \left[\int_0^{\tau \wedge T} e^{-\int_0^s \Phi(u) du} \right. \\ &\quad \times (U_1(C(s)) + U_2(P(s))) ds \\ &\quad \left. + e^{-\int_0^T \Phi(u) du} (U_3(W(T)))_{\{\tau < T\}} \right], \end{aligned} \quad (4.21)$$

where $U_i, i = 1, 2, 3$ is the utility function for the consumption, premiums and the terminal wealth of an investor and χ is a set of admissible strategies, that is,

$$\chi := \{\pi_1, \pi_2, C, P := (\pi_1(t), \pi_2(t), C(t), P(t))_{t \in [t, T]}\}. \quad (4.22)$$

At the terminal time t the set (π_1, π_2, C, P) is $\mathcal{F}_t$ adapted with P as defined in inequality (4.8)

and

$$\int_0^T \left\{ C(t) + \pi_1^2(t) + \pi_2^2(t) + \left[\int_{\mathbb{R}} |\gamma(t, \eta)|^2 \nu ds \right] \right\} dt < \infty. \quad (4.23)$$

²The minimum rate of return or target rate that investors are expecting to receive on an investment.

Using the conditional probability of survival in (4.12) we redefine the performance function as:

$$\begin{aligned}
 J(t, W(t), \pi_1, \pi_2, C, P) &:= \sup_{(\pi_1, \pi_2, C, P) \in \chi} \mathbb{E} \left[\int_0^T e^{-\int_0^s \Phi(u) du} \right. \\
 &\quad \times \bar{F}(s)(U_1(C(s)) + U_2(P(s))) ds \\
 &\quad \left. + e^{-\int_0^T \Phi(u) du} \bar{F}(T)(U_3(W(T))) \right], \tag{4.24}
 \end{aligned}$$

we have,

$$\begin{aligned}
 J(t, W(t), \pi_1, \pi_2, C, P) &:= \sup_{(\pi_1, \pi_2, C, P) \in \chi} \mathbb{E} \left[\int_0^T e^{-\int_0^s \Phi(u) du} \right. \\
 &\quad \times e^{-\int_0^s \mu(u) du} (U_1(C(s)) + U_2(P(s))) ds \\
 &\quad \left. + e^{-\int_0^T \Phi(u) du} \times e^{-\int_0^T \mu(u) du} (U_3(W(T))) \right], \tag{4.25}
 \end{aligned}$$

therefore we obtain,

$$\begin{aligned}
 J(t, W(t), \pi_1, \pi_2, C, P) &:= \sup_{(\pi_1, \pi_2, C, P) \in \chi} \mathbb{E} \left[\int_0^T e^{-\int_0^s (\Phi(u) + \mu(u)) du} (U_1(C(s)) \right. \\
 &\quad \left. + U_2(P(s))) ds + e^{-\int_0^T (\Phi(u) + \mu(u)) du} (U_3(W(T))) \right]. \tag{4.26}
 \end{aligned}$$

4.1.2 Conditions on the random parameters considered

We impose conditions since we are dealing with random parameters and diffusion jumps which will guarantee the unique and existent solutions, they are as follows:

1. $r \neq \alpha$
2. The interest rate r , the force of mortality $\mu(t)$ and the discount rate $\Phi(t)$ are bounded away from zero, thus there exist $\epsilon > 0$ such that for $\wp := r, \mu, \Phi$, we have $|\wp| \geq \epsilon$.

3. The exponential integrability conditions are satisfied by the random parameters, that

is, $\mathbb{E}\left[e^{(\Theta \int_0^T |\varphi(t)|dt)}\right] < \infty$, for large Θ as defined by Guambe and Kufakunesu (2015).

4.1.3 Value Function

The value function is the maximal payoff function, hence in the present problem our value function is an $\mathcal{F}_t$ measurable random process which we will use the Hamilton-Jacobi-Bellman (HJB) equation incorporating the backward stochastic differential equation (BSDE) because we cannot partially differentiate as always, thus it is given by:

$$V(t, W) = \sup_{(\pi_1, \pi_2, C, P) \in \chi} J(t, w, \pi_1, \pi_2, C, P) = J(t, w, \pi_1^*, \pi_2^*, C^*, P^*), \quad (4.27)$$

with $(\pi_1^*, \pi_2^*, C^*, P^*) \in \chi$ being the main strategy of interest to be established.

Chapter 5

The HJB equation and the BSDE with the diffusion jump

Incorporating the diffusion jump we consider the following conditions:

1. The terminal condition $\xi \in \mathbb{L}^2(\Omega, \{\mathcal{F}_t\}, \mathbb{P}, \mathbb{R})$, where $H(T) = \xi$.
2. A mapping f (generator) $f : \Omega \times [0, T] \times \mathbb{R} \times L_v^2(\mathbb{R}) \mapsto \mathbb{R}$ is predictable.
3. $\mathbb{E}[\int_0^T |f(t, 0, 0)|^2 dt] < \infty$; with $|f(t, y, v) - f(t, y, v')|^2 \leq h(|y - y'|^2 + \int_{\mathbb{R}} |v(\eta) - v'(\eta)|^2 \nu(d\eta))$,

thus the BSDE with a diffusion jump is given by:

$$f(t, H(t), \Upsilon(t))dt = \int_{\mathbb{R}} \Upsilon(t, \eta) \tilde{J}(d\eta, dt) - dH(t). \quad (5.1)$$

With the conditions above we are assured that there is a unique solution (H, Υ) to (5.1). The parameter Υ is a control process that controls the process H so it satisfies the first condition above.

The following HJB equation is satisfied by the value function defined above. We let φ be defined as $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ such that $\varphi \in C^2_{\circ}(\mathbb{R}^n)$. We assume that partial derivatives $\varphi_t, \varphi_w, \varphi_z, \varphi_y, \varphi_h, \varphi_{zz}, \varphi_{yy}, \varphi_{wy}, \varphi_{wz}, \varphi_{yz}, \varphi_{ww}$ w.r.t t, w, z, y, h exist, where $\varphi_t = \frac{\partial \varphi}{\partial t}$, $\varphi_z = \frac{\partial \varphi}{\partial z}$, $\varphi_{zz} = \frac{\partial^2 \varphi}{\partial z^2}$, etc., therefore the generator is given by:

$$\begin{aligned}
\mathcal{L}^{\pi_1, \pi_2, c, p}[\varphi(t, w, z, y, h)] = & \varphi_t + \varphi_w(w[r + \mu] + \psi(y, t) - \mu P - C) \\
& + \varphi_y \alpha_1(y, t) + \frac{1}{2} \varphi_{yy} \sigma_1^2(y, t) + \varphi_z \varsigma[\beta - Z] \\
& + \frac{1}{2} \varphi_{zz} \bar{\varrho}^2 - \varphi_{wy} \rho \varrho \sigma_1(y, t) w - \varphi_{wz} \bar{\varrho} \varrho w \\
& + \varphi_{yz} \rho \sigma_1(y, t) \bar{\varrho} + \sup_{\pi_1 \in \mathbb{R}} \left\{ \frac{1}{2} \varphi_{ww} \sigma^2 \pi_1^2 w^2 \right. \\
& + \varphi_w w \pi_1 (\alpha - r) - \varphi_{wz} \sigma \varrho \rho w^2 \pi_1 \\
& + \left. \varphi_{wy} \sigma(t) \sigma_1(y, t) w \pi_1 + \varphi_{wz} \rho \bar{\varrho} \sigma w \pi_1 \right\} \\
& + \sup_{\pi_2 \in \mathbb{R}} \left\{ \frac{1}{2} \varphi_{ww} \varrho^2 \pi_2^2 w^2 + \varphi_w w \pi_2 (k + Z - r) \right\} \\
& - \varphi_h f(t, H, Y) + \int_{\mathbb{R}} [\varphi(t, w + \pi_1 w \gamma(t, \eta), z, y, h) \\
& - \varphi(t, w, z, y, h) - \pi_1 w \gamma(t, \eta) \varphi_w] \nu(d\eta) \\
& + \int_{\mathbb{R}} [\varphi(t, w, z, y, h + Y) - \varphi(t, w, z, y, h) \\
& - Y \varphi_w] \nu(d\eta),
\end{aligned} \tag{5.2}$$

in consideration with the following equations;

$$dY_t = \alpha_1(Y_t, t)dt + \sigma_1(Y_t, t)dX_t \tag{5.3}$$

$$dZ(t) = \varsigma(t)[\beta(t) - Z(t)]dt + \bar{\varrho}(t)dB^1(t) \tag{5.4}$$

$$\begin{aligned}
dW_t &= (W_t[\pi_1(\alpha - r) + \pi_2(k(t) + Z(t) - r) + r + \mu(t)] + \psi(Y_t, t) \\
&\quad - \mu(t)P(t) - C(t))dt + W_t[\pi_1\sigma dB_t^0 - \pi_2\varrho(t)dB_t^1] \\
&\quad + \pi_1 W_t \int_{\mathbb{R}} \gamma(t, \eta) \nu(d\eta, dt).
\end{aligned} \tag{5.5}$$

We define a function G as follows:

$$G(t, w, h, \pi_1, \pi_2, c, p) = \mathcal{L}^{\pi_1, \pi_2, c, p}[\varphi(t, w, z, y, h)] + U(c) + U(p). \tag{5.6}$$

$U(x)$ is a utility function. We present the verification theorem associated with the present problem for the combination of the HJB equation and the BSDE equation.

Theorem 5.0.1 (*Verification theorem Shen and Wei (2014)*). *Let the pair (ξ, f) be as defined in equation (5.1) and satisfies the conditions defined above. Let O be a solvency region with the closure $\bar{O}$. Let a function G be given such that $G \in C^\epsilon(O) \cup C(\bar{O})$ and an admissible control $(\pi_1^*, \pi_2^*, C^*, P^*) \in \chi$ such that;*

1. $G(t, w, H(t), \pi_1, \pi_2, C, P) \leq 0 \quad \forall (\pi_1, \pi_2, C, P) \in \chi$;
2. $G(t, w, H(t), \pi_1^*, \pi_2^*, C^*, P^*) = 0$;
3. For all $(\pi_1, \pi_2, C, P) \in \chi$, $\lim_{t \rightarrow T^-} \varphi(t, w, z, y, h) = U(w)$;
4. Let $\mathcal{K}$ be the set of stopping times $\mathcal{K} \leq T$. The family $\varphi(k, W(k), H(k))_{k \in \mathcal{K}}$ is uniformly integrable.

Then;

$\varphi(t, w, h) = \sup_{(\pi_1, \pi_2, C, P) \in \chi} J(t, w, \pi_1, \pi_2, C, P) = J(t, w, \pi_1^*, \pi_2^*, C^*, P^*)$ and $(\pi_1^*, \pi_2^*, C^*, P^*)$ is an optimal control.

Proof. We define the localizing sequence of stopping times as follows;

$$\begin{aligned} \mathcal{K}_{\bar{O}}^N &:= T \wedge \inf \{ t \geq 0 \mid (t, W(t), H(t)) \notin \bar{O} \} \\ &\wedge \inf \{ t > 0 \mid t \geq N \text{ or } |W(t)| \geq N \text{ or } \|H(t)\| \geq N \} \end{aligned}$$

Using the Dynkin formula we have;

$$\begin{aligned} \varphi(t, w, H(t)) &= \mathbb{E}_{t,w} \left[e^{-\int_t^{\mathcal{K}_{\bar{O}}^N} (\Phi(u) + \mu(u)) du} \varphi(\mathcal{K}_{\bar{O}}^N, W(\mathcal{K}_{\bar{O}}^N), H(\mathcal{K}_{\bar{O}}^N)) \right. \\ &\quad \left. - \int_t^{\mathcal{K}_{\bar{O}}^N} e^{-\int_t^s (\Phi(u) + \mu(u)) du} \mathcal{L}^{\pi_1, \pi_2, C, P} [\varphi(s, W(s), H(s)) ds] \right]. \end{aligned} \quad (5.7)$$

For all $(\pi_1, \pi_2, C, P) \in \chi$, using the conditions incorporated above (1) to equation (5.7) we obtain;

$$\begin{aligned} \varphi(t, w, H(t)) &= \mathbb{E}_{t,w} \left[\int_t^{\mathcal{K}_{\bar{O}}^N} e^{-\int_t^s (\Phi(u) + \mu(u)) du} [U_1(C(s)) + U_2(P(s))] ds \right. \\ &\quad \left. + e^{-\int_t^{\mathcal{K}_{\bar{O}}^N} (\Phi(s) + \mu(s)) ds} \varphi(\mathcal{K}_{\bar{O}}^N, W(\mathcal{K}_{\bar{O}}^N), H(\mathcal{K}_{\bar{O}}^N)) \right]. \end{aligned} \quad (5.8)$$

Using Fatou's lemma and the Dominated Convergence Theorem yields;

$$\begin{aligned} \varphi(t, w, H(t)) &\geq \lim_{N \rightarrow \infty} \inf \mathbb{E}_{t,w} \left[\int_t^{\mathcal{K}_{\bar{O}}^N} e^{-\int_t^s (\Phi(u) + \mu(u)) du} [U_1(C(s)) + U_2(P(s))] ds \right. \\ &\quad \left. + e^{-\int_t^{\mathcal{K}_{\bar{O}}^N} (\Phi(s) + \mu(s)) ds} \varphi(\mathcal{K}_{\bar{O}}^N, W(\mathcal{K}_{\bar{O}}^N), H(\mathcal{K}_{\bar{O}}^N)) \right] \\ &\geq \mathbb{E}_{t,w} \left[\lim_{N \rightarrow \infty} \inf \left\{ \int_t^{\mathcal{K}_{\bar{O}}^N} e^{-\int_t^s (\Phi(u) + \mu(u)) du} [U_1(C(s)) + U_2(P(s))] ds \right. \right. \\ &\quad \left. \left. + e^{-\int_t^{\mathcal{K}_{\bar{O}}^N} (\Phi(s) + \mu(s)) ds} \varphi(\mathcal{K}_{\bar{O}}^N, W(\mathcal{K}_{\bar{O}}^N), H(\mathcal{K}_{\bar{O}}^N)) \right\} \right] = J(\pi_1, \pi_2, C, P), \end{aligned} \quad (5.9)$$

which completes the proof.

The above theorem implies that the value function is the solution of the following system

of the HJB and BSDE;

$$\begin{cases} \sup_{(\pi_1, \pi_2, C, P) \in \mathcal{X}} \left\{ \mathcal{L}^{\pi_1, \pi_2, C, P}[\varphi(t, W, H(t))] + U(C) + U(P) \right\} = 0, & \varphi(T, w, \xi) = U(w) \\ f(t, H(t), Y(t))dt = \int_{\mathbb{R}} Y(t, \eta) \tilde{J}(d\eta, dt) - dH(t) \end{cases} \quad (5.10)$$

Therefore in general, differentiating equation (5.6) with respect to π_1, π_2, c, p and equating the derivatives to zero gives the following general optimal controls:

The proportion of wealth of an investor invested in stock π_1^* is given by;

$$\begin{aligned} \frac{\partial G(\cdot)}{\partial \pi_1} &= \varphi_{ww} \sigma^2 \pi_1 w^2 + \varphi_w w (\alpha - r) - \varphi_{wz} \sigma \varrho(t) \rho w^2 \\ &+ \varphi_{wy} \sigma(t) \sigma_1(Y_t, t) w + \varphi_{wz} \sigma \bar{\varrho}(t) \sigma(t) w \\ &+ \int_{\mathbb{R}} [\varphi_{\pi_1}(t, w + \pi_1 w \gamma(\eta, t), z, y, H(t)) - w \gamma(t, \eta) \varphi_w] \nu(d\eta) = 0 \end{aligned} \quad (5.11)$$

π_1^* is the solution of the above equation.

To obtain the proportion invested in the inflation linked bond π_2^* we have;

$$\begin{aligned} \frac{\partial G(\cdot)}{\partial \pi_2} &= \varphi_{ww} \varrho^2(t) \pi_2 w^2 + \varphi_w w (k(t) + Z(t) - r) = 0, \\ \pi_2^* &= -\frac{\varphi_w}{\varphi_{ww}} \left[\frac{(k(t) + Z(t) - r)}{w \varrho^2(t)} \right]. \end{aligned} \quad (5.12)$$

In general we have the optimal consumption process c^* and premiums p^* to be paid are given as;

$$\begin{aligned} \frac{\partial G(\cdot)}{\partial c} &= -w \varphi_w + U'(c) = 0 \\ U'(c) &= w \varphi_w. \end{aligned} \quad (5.13)$$

$$\frac{\partial G(\cdot)}{\partial p} = -\mu(t)w\varphi_w + U'(p) = 0 \quad (5.14)$$

$$U'(p) = \mu(t)w\varphi_w.$$

We consider the CRRA utility defined as follows:

$$U(x) = \begin{cases} \frac{x^\lambda}{\lambda}, & \text{if } \lambda \in \mathbb{R} \setminus \{0\} \\ \ln x, & \text{if } \lambda = 0. \end{cases} \quad (5.15)$$

5.1 The Power Utility

Suppose we have the type of a constant relative risk aversion (CRRA) utility function given by the power utility;

$$U(x) = \frac{x^\lambda}{\lambda}, \quad \lambda \in \mathbb{R} \setminus \{0\}. \quad (5.16)$$

We choose the value function as,

$$\varphi(w) = \frac{w^\lambda}{\lambda} e^{(1-\lambda)H(t)}, \quad \lambda \in \mathbb{R} \setminus \{0\}. \quad (5.17)$$

Consider the following differentials;

$$\varphi_{wz}(w) = 0 \quad (5.18)$$

$$\varphi_{wy}(w) = 0 \quad (5.19)$$

$$\varphi_w(w) = w^{\lambda-1} e^{-(\lambda-1)H(t)} \quad (5.20)$$

$$\varphi_{ww}(w) = (\lambda - 1)w^{\lambda-2} e^{-(\lambda-1)H(t)} \quad (5.21)$$

$$\varphi_{\pi_1}(w) = (w + \pi_1 w \gamma(\eta, t))^{\lambda-1} w \gamma(\eta, t) e^{-(\lambda-1)H(t)} \quad (5.22)$$

Substituting (5.18), (5.19), (5.20), (5.21) and (5.22) into (5.11), (5.12), (5.13) and (5.14)

we have the following optimal strategies.

The optimal proportion of wealth invested in stock $\pi_1^*(t)$ is the solution to the following equation:

$$\begin{aligned} - w^\lambda e^{-(\lambda-1)H(t)} (\alpha - r) &= (\lambda - 1) w^\lambda \sigma^2 \pi_1 e^{-(\lambda-1)H(t)} \\ + \int_{\mathbb{R}} &\left[w^\lambda e^{-(\lambda-1)H(t)} \gamma(\eta, t) \left((1 + \pi_1 \gamma(\eta, t))^{\lambda-1} - 1 \right) \right] \nu(d\eta). \end{aligned} \quad (5.23)$$

For the proportion of wealth invested in the inflation linked bond $\pi_2^*(t)$ we have;

$$\begin{aligned}
 \pi_2^* &= -\frac{\varphi_w}{\varphi_{ww}} \left[\frac{(k(t) + Z(t) - r)}{w\varrho^2(t)} \right] \\
 &= -\frac{w^{\lambda-1} e^{-(\lambda-1)H(t)}}{(\lambda-1)w^{\lambda-2} e^{-(\lambda-1)H(t)}} \left[\frac{(k(t) + Z(t) - r)}{w\varrho^2(t)} \right] \\
 &= \frac{(k(t) + Z(t) - r)}{\varrho^2(t)(\lambda-1)}.
 \end{aligned} \tag{5.24}$$

Then the optimal consumption process $c^*(t)$ is given;

$$\begin{aligned}
 U'(c) &= c^{\lambda-1} \\
 c^{\lambda-1} &= w\varphi_w \\
 c^*(t) &= \frac{1}{[w\varphi_w]^{(\lambda-1)}} \\
 &= \frac{\lambda}{w(\lambda-1)} e^{-H(t)}.
 \end{aligned} \tag{5.25}$$

The investor pays premiums $p^*(t)$ as given by;

$$\begin{aligned}
 U'(p) &= p^{(\lambda-1)} \\
 p^{(\lambda-1)} &= \mu(t)w\varphi_w \\
 p^*(t) &= \frac{1}{[\mu(t)w\varphi_w]^{(\lambda-1)}} \\
 &= \frac{1}{[\mu(t)w^\lambda]^{(\lambda-1)}} e^{-H(t)}.
 \end{aligned} \tag{5.26}$$

With the optimal strategies in section (5.1), substituting π_1^* , (5.24), (5.25) and (5.26) into

equation (5.6) we obtain the following;

$$\begin{aligned}
0 &= \frac{(1-\lambda)}{\lambda} e^{(1-\lambda)H(t)} \left\{ w^\lambda \left[\lambda \left(\frac{(r+\mu)}{(1-\lambda)} + \frac{1}{w(1-\lambda)} \right) \psi(Y_t, t) \right. \right. \\
&\quad - \frac{e^{-H(t)}}{(1-\lambda)} (\mu^\lambda w) \frac{1}{\lambda-1} - \frac{e^{-H(t)}}{(1-\lambda)} w \frac{1}{\lambda-1} - \frac{1}{2} \sigma^2 (\pi_1^*)^2 \\
&\quad + \pi_1^* \frac{(\alpha-r)}{(1-\lambda)} + \frac{3(k+Z-r)^2}{2(1-\lambda)^2 \varrho^2} \Big) - f(t, H(t), \Upsilon(t)) \\
&\quad + \frac{1}{(1-\lambda)} \left\{ \int_{\mathbb{R}} [(1 + \pi_1^* \gamma(t, \eta))^\lambda - 1 - \lambda \pi_1^* \gamma(t, \eta)] \nu(d\eta) \right. \\
&\quad + \left. \int_{\mathbb{R}} [e^{(\lambda-1)\gamma(t, \eta)} - 1 - \frac{\gamma(t, \eta)}{w}] \nu(d\eta) \Big\} \right\} \\
&\quad + \frac{\lambda^2}{w \lambda - 1} \frac{e^{-\lambda H(t)}}{(1-\lambda)} \left(1 + \mu \frac{\lambda}{\lambda-1} \right) \Big\}.
\end{aligned} \tag{5.27}$$

Note that $\frac{1-\lambda}{\lambda} e^{(1-\lambda)H(t)} \neq 0$ for $\lambda \notin \{0, 1\}$ we have ;

$$\begin{aligned}
0 &= w^\lambda \left[\lambda \left(\frac{(r+\mu)}{(1-\lambda)} + \frac{1}{w(1-\lambda)} \right) \psi(Y_t, t) - \frac{e^{-H(t)}}{(1-\lambda)} (\mu^\lambda w) \frac{1}{\lambda-1} \right. \\
&\quad - \frac{e^{-H(t)}}{(1-\lambda)} w \frac{1}{\lambda-1} - \frac{1}{2} \sigma^2 (\pi_1^*)^2 + \pi_1^* \frac{(\alpha-r)}{(1-\lambda)} + \frac{3(k+Z-r)^2}{2(1-\lambda)^2 \varrho^2} \Big) \\
&\quad - f(t, H(t), \Upsilon(t)) + \frac{1}{(1-\lambda)} \left\{ \int_{\mathbb{R}} [(1 + \pi_1^* \gamma(t, \eta))^\lambda \right. \\
&\quad - 1 - \lambda \pi_1^* \gamma(t, \eta)] \nu(d\eta) + \int_{\mathbb{R}} [e^{(\lambda-1)\gamma(t, \eta)} - 1 - \frac{\gamma(t, \eta)}{w}] \nu(d\eta) \Big\} \Big] \\
&\quad + \frac{\lambda^2}{w \lambda - 1} \frac{e^{-\lambda H(t)}}{(1-\lambda)} \left(1 + \mu \frac{\lambda}{\lambda-1} \right),
\end{aligned} \tag{5.28}$$

then we have;

$$\begin{aligned}
f(t, H(t), Y(t)) = & \left[\lambda \left(\frac{(r + \mu)}{(1 - \lambda)} + \frac{1}{w(1 - \lambda)} \psi(Y_t, t) - \frac{e^{-H(t)}}{(1 - \lambda)} (\mu^\lambda w)^{\frac{1}{\lambda - 1}} \right. \right. \\
& - \frac{e^{-H(t)}}{(1 - \lambda)} w^{\frac{1}{\lambda - 1}} - \frac{1}{2} \sigma^2 (\pi_1^*)^2 + \pi_1^* \frac{(\alpha - r)}{(1 - \lambda)} + \frac{3(k + Z - r)^2}{2(1 - \lambda)^2 \varrho^2} \Big) \\
& + \frac{1}{(1 - \lambda)} \left\{ \int_{\mathbb{R}} [(1 + \pi_1^* \gamma(t, \eta))^\lambda - 1 - \lambda \pi_1^* \gamma(t, \eta)] \nu(d\eta) \right. \\
& + \left. \int_{\mathbb{R}} [e^{(\lambda - 1)\gamma(t, \eta)} - 1 - \frac{\gamma(t, \eta)}{w}] \nu(d\eta) \right\} \Big] \\
& - \frac{\lambda}{w^{\lambda - 1}} \frac{e^{-\lambda H(t)}}{\lambda(1 - \lambda)} \left(1 + \mu^{\frac{\lambda}{\lambda - 1}} \right),
\end{aligned} \tag{5.29}$$

for $\lambda \in (0, 1)$.

We observe that equation (5.29) is the generator of the equation (5.1), and consider the conditions stated by Guambe and Kufakunesu (2015), conditions "(C1 to C3)". In finding the unique solution (H, Y) , we let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F}_t)$ such that we have a function $L(\omega)$ that can define a new measure through the relation $d\mathbb{Q}(\omega) = L(\omega)d\mathbb{P}$. The probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, written $\mathbb{P} \sim \mathbb{Q}$ whenever $\forall \omega \in \Omega \mathbb{P}(\omega) > 0$ iff $\mathbb{Q}(\omega) > 0$. $\mathbb{Q}$ is considered a probability measure on $(\Omega, \mathcal{F}_t)$ if the probability measure $\mathbb{P}$ is defined and if the function L have the following properties;

$$\mathcal{K}_1 : L(t) \geq 0 \text{ a.s w.r.t } \mathbb{P}.$$

$$\mathcal{K}_2 : \int_{\Omega} L(t) d\mathbb{P}(t) = 1.$$

Given the equivalent probability measures above on the σ -algebra $\mathcal{F}$ we obtain

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L. \quad (5.30)$$

L is referred to as the Radon-Nikodym derivative and is given by;

$$\frac{dL(t)}{L(t)} = \int_{\mathbb{R}} \left[\frac{w(e^{(1-\lambda)\Upsilon(s,\eta)} - 1)}{(1-\lambda)\Upsilon(s,\eta)} - 1 \right] \tilde{J}(ds, d\eta) \quad (5.31)$$

The Girsanov's theorem explains the variability of the stochastic process under the new equivalent measure to the original probability measure. In this case we consider the change from probability measure $\mathbb{P}$ to $\mathbb{Q}$. We let

$$\nu(t, \eta) = \frac{w(e^{(1-\lambda)\Upsilon(t,\eta)} - 1)}{(1-\lambda)\Upsilon(t,\eta)} - 1. \quad (5.32)$$

Using the Girsanov's theorem we define the $(\mathbb{Q}, \mathcal{F}_t)$ compensated jump measure $\tilde{J}^{\mathbb{Q}}(dt, d\eta)$ as;

$$\tilde{J}^{\mathbb{Q}}(dt, d\eta) = J(dt, d\eta) - (\nu(t, \eta) + 1)\nu(d\eta)dt, \quad (5.33)$$

Therefore in the probability measure $\mathbb{Q}$ we have equation (5.1) with the generator (5.29) is

given as;

$$\begin{aligned}
 dH(t) = & -\left\{ \frac{\lambda(r+\mu)}{(1-\lambda)} + \frac{\lambda\psi(Y_t, t)}{w(1-\lambda)} - \frac{\lambda}{2}\sigma^2(\pi_1^*)^2 + \pi_1^* \frac{\lambda(\alpha-r)}{(1-\lambda)} \right. \\
 & + \frac{3\lambda(k+Z-r)^2}{2(1-\lambda)^2\varrho^2} + \frac{\lambda}{(1-\lambda)} \left\{ \int_{\mathbb{R}} [(1+\pi_1^*\gamma(t, \eta))^\lambda \right. \\
 & - 1 - \lambda\pi_1^*\gamma(t, \eta)]\nu(d\eta) \Big\} - \left(\frac{\lambda(\mu^\lambda w)^{\lambda-1}}{(1-\lambda)} + \frac{\lambda w^{\lambda-1}}{(1-\lambda)} \right) e^{-H(t)} \\
 & + \frac{\lambda}{w^{\lambda-1}} \frac{(1+\mu^{\lambda-1})e^{-\lambda H(t)}}{\lambda(1-\lambda)} \Big\} dt + \int_{\mathbb{R}} \Upsilon(t, \eta) \tilde{J}^Q(d\eta, dt)
 \end{aligned} \tag{5.34}$$

$$H(T) = 0.$$

The solution of the above equation (5.34) is represented below as:

$$\begin{aligned}
 \int_t^T dH(s) = & - \int_t^T \left\{ \frac{\lambda(r+\mu)}{(1-\lambda)} + \frac{\lambda\psi(Y_s, s)}{w(1-\lambda)} - \frac{\lambda}{2}\sigma^2(\pi_1^*)^2 \right. \\
 & + \pi_1^* \frac{\lambda(\alpha-r)}{(1-\lambda)} + \frac{3\lambda(k+Z-r)^2}{2(1-\lambda)^2\varrho^2} + \frac{\lambda}{(1-\lambda)} \\
 & \times \left\{ \int_{\mathbb{R}} [(1+\pi_1^*\gamma(s, \eta))^\lambda - 1 - \lambda\pi_1^*\gamma(s, \eta)]\nu(d\eta) \right\} \Big\} ds \\
 & + \int_t^T \int_{\mathbb{R}} \Upsilon(s, \eta) \tilde{J}^Q(d\eta, ds),
 \end{aligned} \tag{5.35}$$

The solution to the above equation is;

$$\begin{aligned}
 H(t) = & - \int_t^T \left\{ \frac{\lambda(r+\mu)}{(1-\lambda)} + \frac{\lambda\psi(Y_s, s)}{w(1-\lambda)} - \frac{\lambda}{2}\sigma^2(\pi_1^*)^2 \right. \\
 & + \pi_1^* \frac{\lambda(\alpha-r)}{(1-\lambda)} + \frac{3\lambda(k+Z-r)^2}{2(1-\lambda)^2\varrho^2} + \frac{\lambda}{(1-\lambda)} \\
 & \times \left\{ \int_{\mathbb{R}} [(1+\pi_1^*\gamma(s, \eta))^\lambda - 1 - \lambda\pi_1^*\gamma(s, \eta)]\nu(d\eta) \right\} \Big\} ds \\
 & + \int_t^T \int_{\mathbb{R}} \Upsilon(s, \eta) \tilde{J}^Q(d\eta, ds)
 \end{aligned} \tag{5.36}$$

defined by Łukasz Delong (2013) in consideration to the condition that

$\frac{\lambda}{w\lambda-1} \frac{\left(1 + \mu \frac{\lambda}{\lambda-1}\right) e^{-\lambda H(t)}}{\lambda(1-\lambda)} = 0$. Note that, $\frac{\lambda}{w\lambda-1} e^{-\lambda H(t)} \neq 0$ implies that $1 + \mu \frac{\lambda}{\lambda-1} = 0$. Thus,

$$\mu(t) = (-1) \frac{\lambda-1}{\lambda} = \mu^*, \quad \mu^* \text{ a constant} \quad (5.37)$$

for some λ . The above equation (5.37) is the hazard rate function of a policy holder to be alive at a certain time t . Therefore the policy holder's conditional probability of survival is given by;

$$\bar{F}(t_l) = e^{-\mu^* t_l}. \quad (5.38)$$

The conditional survival probability density of the death of the policy holder is given by;

$$f(t_l) = \mu^* e^{-\mu^* t_l}. \quad (5.39)$$

Theorem 5.1.1 *Given the optimal investment-consumption-insurance problem with inflation risk and stochastic income in an Itô Lévy setting, and value function*

$$\varphi(t, w, z, y, h) = \frac{w^\lambda}{\lambda} e^{(1-\lambda)H(t)} \quad \text{then,}$$

$\pi_1^*(t)$ solves:

$$\left(\sigma^2 \pi_1 (\lambda - 1) + \int_{\mathbb{R}} [(1 + \pi_1 \gamma(\eta, t))^{\lambda-1} \gamma(\eta, t) - \gamma(\eta, t)] \nu(d\eta) + (\alpha - r) \right) = 0; \quad \text{for} \quad (5.40)$$

$$w^\lambda e^{-(\lambda-1)H(t)} \neq 0$$

$$\pi_2^* = \left[\frac{(k(t) + Z(t) - r)}{\varrho^2(t)(\lambda - 1)} \right], \quad c^*(t) = w \frac{\lambda}{(\lambda - 1)} e^{-H(t)} \quad \text{and}$$

$$p^*(t) = [\mu(t) w^\lambda] \frac{1}{(\lambda - 1)} e^{-H(t)}.$$

5.2 The Logarithmic Utility

We consider the exponential utility function defined by equation (5.15) where $\lambda = 0$ therefore;

$$U(x) = \ln x, \quad 0 < x < \infty$$

We define the value function $\varphi(w)$ as;

$$\varphi(w) = e^{H(t)} \ln w. \quad (5.41)$$

Consider the following differentials;

$$\varphi_{wz}(w) = 0 \quad (5.42)$$

$$\varphi_{wy}(w) = 0 \quad (5.43)$$

$$\varphi_w(w) = \frac{e^{H(t)}}{w} \quad (5.44)$$

$$\varphi_{ww}(w) = -\frac{e^{H(t)}}{w^2} \quad (5.45)$$

$$\varphi_{\pi_1}(w) = \frac{e^{H(t)} \gamma(\eta, t)}{(1 + \pi_1 \gamma(\eta, t))}. \quad (5.46)$$

With the above differentials we obtain the controls as follows;

π_1^* is the proportion of wealth invested in stock and lies in $(0, 1)$, is the solution of the following equation;

$$e^{H(t)} [\sigma^2 \pi_1 - \int_{\mathbb{R}} [\gamma(\frac{-\pi_1 \gamma}{(1 + \pi_1 \gamma)})] \nu(d\eta)] = e^{H(t)} (\alpha - r), \quad (5.47)$$

since $e^{H(t)} \neq 0$, we have;

$$[\sigma^2 \pi_1 - \int_{\mathbb{R}} \left[\gamma \left(\frac{-\pi_1 \gamma}{(1 + \pi_1 \gamma)} \right) \right] \nu(d\eta)] = (\alpha - r). \quad (5.48)$$

We have the optimal proportion $\pi_2^*(t)$ given by;

$$\begin{aligned} \pi_2^*(t) &= -\frac{\varphi_w}{\varphi_{ww}} \left[\frac{(k(t) + Z(t) - r)}{w\varrho^2(t)} \right] \\ &= -\frac{e^{H(t)}}{w} \times -\frac{w^2}{e^{H(t)}} \left[\frac{(k(t) + Z(t) - r)}{w\varrho^2(t)} \right] \\ &= \left[\frac{(k(t) + Z(t) - r)}{\varrho^2(t)} \right]. \end{aligned} \quad (5.49)$$

The optimal consumption process $c^*(t)$ and the premium $p^*(t)$ are given by:

$$\begin{aligned} U'(c) &= \frac{1}{c} \\ \frac{1}{c} &= w\varphi_w \\ c &= \frac{1}{w\varphi_w}. \end{aligned}$$

Thus,

$$\begin{aligned} c^* &= \frac{1}{e^{H(t)} w \frac{w}{w}} \\ &= e^{-H(t)}. \end{aligned} \quad (5.50)$$

For p^* we have

$$\begin{aligned} U'(p) &= \frac{1}{p} \\ \frac{1}{p} &= \mu(t) w \varphi_w \\ p &= \frac{1}{\mu(t) w \varphi_w}. \end{aligned}$$

Simplifying further we obtain;

$$\begin{aligned} p^* &= \frac{1}{\frac{\mu(t)w e^{H(t)}}{w}} \\ &= \frac{e^{-H(t)}}{\mu(t)}. \end{aligned} \quad (5.51)$$

Taking the controls in the logarithmic utility, substituting π_1^* , (5.49), (5.50), (5.51) into (5.6)

we obtain the following;

$$\begin{aligned} 0 &= e^{H(t)} \ln w \left\{ \frac{1}{\ln w} \left((r + \mu) + \frac{\Psi(Y_t, t)}{w} - \frac{2e^{-H(t)}}{w} - \frac{\sigma^2(\pi_1^*)^2}{2} \right. \right. \\ &\quad + \pi_1^*(\alpha - r) + \frac{(k(t) + Z(t) - r)^2}{2Q^2} - f(t, H(t), \Upsilon(t)) \\ &\quad + \int_{\mathbb{R}} [\ln(1 + \pi_1^* w \gamma(t, \eta)) - \pi_1^* \gamma(t, \eta)] \nu(d\eta) \\ &\quad \left. \left. + \int_{\mathbb{R}} [\ln w (e^{\gamma(t, \eta)} - 1) - \frac{\gamma(t, \eta)}{w}] \nu(d\eta) - e^{-H(t)} (2H(t) - \ln \mu) \right) \right\}. \end{aligned} \quad (5.52)$$

Taking into consideration that $e^{H(t)} \ln w \neq 0$, we have ;

$$\begin{aligned} 0 &= \frac{1}{\ln w} \left((r + \mu) + \frac{\Psi(Y_t, t)}{w} - \frac{2e^{-H(t)}}{w} - \frac{\sigma^2(\pi_1^*)^2}{2} + \pi_1^*(\alpha - r) \right) \\ &\quad + \frac{(k(t) + Z(t) - r)^2}{2Q^2 \ln w} - f(t, H(t), \Upsilon(t)) + \frac{1}{\ln w} \left(\int_{\mathbb{R}} [\ln(1 + \pi_1^* w \gamma(t, \eta)) \right. \\ &\quad - \pi_1^* \gamma(t, \eta)] \nu(d\eta) + \int_{\mathbb{R}} [\ln w (e^{\gamma(t, \eta)} - 1) - \frac{\gamma(t, \eta)}{w}] \nu(d\eta) \Big) \\ &\quad - \frac{e^{-H(t)}}{\ln w} (2H(t) - \ln \mu). \end{aligned} \quad (5.53)$$

Therefore we obtain;

$$\begin{aligned}
f(t, H(t), \Upsilon(t)) &= \frac{1}{\ln w} \left((r + \mu) + \frac{\Psi(Y_t, t)}{w} - \frac{2e^{-H(t)}}{w} - \frac{\sigma^2(\pi_1^*)^2}{2} \right. \\
&\quad \left. + \pi_1^*(\alpha - r) \right) + \frac{(k(t) + Z(t) - r)^2}{2\varrho^2 \ln w} \\
&\quad + \frac{1}{\ln w} \left(\int_{\mathbb{R}} [\ln(1 + \pi_1^* w \gamma(t, \eta)) - \pi_1^* \gamma(t, \eta)] \nu(d\eta) \right. \\
&\quad \left. + \int_{\mathbb{R}} [\ln w(e^{\gamma(t, \eta)} - 1) - \frac{\gamma(t, \eta)}{w}] \nu(d\eta) \right) \\
&\quad - \frac{e^{-H(t)}}{\ln w} (2H(t) - \ln \mu).
\end{aligned} \tag{5.54}$$

We have the generator above (5.54) for the BDSE with diffusion jump in equation (5.1).

We use the definition of equivalent measures defined in section (5.1) in finding the unique solution (H, γ) . We define the Radon-Nikodym derivative given as;

$$\frac{dL(t)}{L(t)} = \int_{\mathbb{R}} \left[\frac{w \ln w(e^\gamma - 1)}{\gamma(t, \eta)} - 1 \right] \tilde{J}(ds, d\eta). \tag{5.55}$$

In consideration to the Girsanov's theorem we consider the change of probability measure $\mathbb{P}$ to $\mathbb{Q}$. We suppose that,

$$\nu(\eta, t) := \frac{w \ln w(e^\gamma - 1)}{\gamma(t, \eta)} - 1 \tag{5.56}$$

then we have the compensated random measure defined under the probability measure $\mathbb{Q}$,

we then have the BSDE in equation (5.1) given by;

$$\begin{aligned}
dH(t) &= \left\{ \frac{1}{\ln w} \left((r + \mu) + \frac{\Psi(Y_t, t)}{w} - \frac{\sigma^2(\pi_1^*)^2}{2} + \pi_1^*(\alpha - r) \right) \right. \\
&\quad \left. + \frac{(k(t) + Z(t) - r)^2}{2\varrho^2} + \int_{\mathbb{R}} [\ln(1 + \pi_1^* w \gamma(t, \eta)) - \pi_1^* \gamma(t, \eta)] \nu(d\eta) \right\} \\
&\quad + e^{-H(t)} \left[\frac{-2}{w \ln w} - \frac{(2H(t) + \ln \mu)}{\ln w} \right] dt + \int_{\mathbb{R}} \Upsilon(t, \eta) \tilde{J}^{\mathbb{Q}}(d\eta, dt).
\end{aligned} \tag{5.57}$$

The solution of the above equation (5.57) is given by:

$$\begin{aligned}
 H(t) = & \mathbb{E} \left\{ \int_t^T \ln \left[1 + \int_t^s \left(\frac{-2}{w \ln w} \right) d\tilde{u} \right] \times \frac{1}{\ln w} (r + \mu) \right. \\
 & + \frac{\Psi(Y_t, t)}{w} - \frac{\sigma^2(\pi_1^*)^2}{2} + \pi_1^*(\alpha - r) + \frac{(k(t) + Z(t) - r)^2}{2\sigma^2} \\
 & \left. + \int_{\mathbb{R}} [\ln(1 + \pi_1^* w \gamma(t, \eta)) - \pi_1^* \gamma(t, \eta)] \nu(d\eta) \right\}
 \end{aligned} \tag{5.58}$$

whenever

1. $\mu(t) = e^{-2H(t)}$.
2. $1 + \pi_1^* w \gamma > 0$; for $\pi_1^* > \frac{-1}{w\gamma}$ and $w\gamma \neq 0$.

Note that the hazard rate function $\mu(t)$ above is dependent on $H(t)$. The policy holder's conditional probability of survival is given by;

$$\begin{aligned}
 \bar{F}(t_l) &= e^{-\mu t_l} \\
 &= \exp\{-t_l e^{-2H(t_l)}\},
 \end{aligned} \tag{5.59}$$

and their conditional survival probability density is given by;

$$\begin{aligned}
 f(t_l) &= \mu e^{-\mu t_l} \\
 &= \exp\{-(2H(t_l) + t_l e^{-2H(t_l)})\}
 \end{aligned} \tag{5.60}$$

Theorem 5.2.1 *Given the optimal investment-consumption-insurance problem with inflation risk and stochastic income in an Itô-Lévy setting, and value function*

$\varphi(t, w, z, y, h) = \ln w e^{H(t)}$ *then,*

$$p^* = \frac{e^{-H(t)}}{\mu(t)}, \quad (5.61)$$

$$c^* = e^{-H(t)}, \quad (5.62)$$

$$\pi_2^* = \left[\frac{(k(t) + Z(t) - r)}{\varrho^2(t)} \right] \quad (5.63)$$

and, $\pi_1^*(t)$ solves:

$$\sigma^2 \pi_1 - \int_{\mathbb{R}} \left[\gamma(\eta, t) \left(\frac{w}{(w + \pi_1 w \gamma(\eta, t))} - 1 \right) \right] \nu(d\eta) - (\alpha - r) = 0. \quad (5.64)$$

5.3 Post-death case

In this section we present the case when the investor dies at time τ after the retirement time T , that is $\tau > T$. Note that here the investor does not pay premiums thus $p = 0$. We have the wealth process as given by equation (4.20) and the beneficiaries obtain a share as given by;

$$W(\tau) + \frac{p(\tau)}{\tilde{\pi}(\tau)}, \quad (5.65)$$

where $\tilde{\pi}(\tau)$ is the premium insurance ratio, $\frac{p(\tau)}{\tilde{\pi}(\tau)}$ is the insurance benefit and $W(\tau)$ is the wealth of the investor. The investment-consumption strategy for the beneficiaries is given by

$$\chi := (\bar{\pi}_1, \bar{\pi}_2, c_a) \quad (5.66)$$

where $\bar{\pi}_1, \bar{\pi}_2$ are the proportions of wealth held in the stock and the inflation linked bond respectively, and c_a is the consumption amount consumed by the beneficiaries. It satisfies;

$$\int_0^T c_a(s) ds < \infty \quad (5.67)$$

The performance function is then given by:

$$\begin{aligned} J(t, W(t), \bar{\pi}_1, \bar{\pi}_2, c_a) &:= (\bar{\pi}_1, \bar{\pi}_2, c_a) \in \chi \mathbb{E}_{t,w} \left[\int_t^T e^{-\int_0^{\tau \wedge T} \Phi(u) du} \right. \\ &\quad \times \left. (U_1(c_a(s))) ds + e^{-\int_t^T \Phi(u) du} (U_2(W(T))_{\{\tau > T\}}) \right], \end{aligned} \quad (5.68)$$

where U_1 and U_2 are the utility functions for the consumption of the beneficiaries and the terminal wealth of an investor, respectively. The set $(\bar{\pi}_1, \bar{\pi}_2, c_a)$ is $\mathcal{F}_t$ adapted. The performance function is then redefined using the probability of survival as:

$$\begin{aligned} J(t, W(t), \bar{\pi}_1, \bar{\pi}_2, c_a) &:= \sup_{(\bar{\pi}_1, \bar{\pi}_2, c_a) \in \chi} \mathbb{E}_{t,w} \left[\int_t^T e^{-\int_0^{\tau \wedge T} \Phi(u) du} \right. \\ &\quad \times e^{\int_0^{\tau \wedge T} \mu(u) du} (U_1(c_a(s))) ds \\ &\quad \left. + e^{\int_t^T \mu(u) du} \times e^{-\int_t^T \Phi(u) du} (U_2(W(T))) \right]. \end{aligned} \quad (5.69)$$

Then we have

$$\begin{aligned} J(t, W(t), \bar{\pi}_1, \bar{\pi}_2, c_a) &:= \sup_{(\bar{\pi}_1, \bar{\pi}_2, c_a) \in \chi} \mathbb{E}_{t,w} \left[\int_t^T e^{-\int_0^{\tau \wedge T} (\Phi(u) + \mu(u)) du} \right. \\ &\quad \times (U_1(c_a(s))) ds \\ &\quad \left. + e^{-\int_t^T (\Phi(u) + \mu(u)) du} (U_2(W(T))) \right]. \end{aligned} \quad (5.70)$$

We consider the conditions imposed on the parameters in section (4.1.2). The defined function below is given as:

$$\bar{G} = \sup_{\bar{\pi}_1^*, \bar{\pi}_2^*, c_a^*} \{ \mathcal{L}^{\bar{\pi}_1, \bar{\pi}_2, c_a} [\varphi(t, w, z, y, h)] + U(c_a) \}. \quad (5.71)$$

We assume that the partial derivatives $\varphi_t, \varphi_w, \varphi_z, \varphi_y, \varphi_h, \varphi_{zz}, \varphi_{yy}, \varphi_{wy}, \varphi_{wz}, \varphi_{yz}, \varphi_{ww}$ w.r.t t, w, z, y, h exists, where $\varphi_t = \frac{\partial \varphi}{\partial t}$, $\varphi_z = \frac{\partial \varphi}{\partial z}$, $\varphi_{zz} = \frac{\partial^2 \varphi}{\partial z^2}$, etc. exists, then we have the generator of the wealth process as:

$$\begin{aligned}
\mathcal{L}^{\bar{\pi}_1, \bar{\pi}_2, c_a}[\varphi(t, w, z, y, h)] &= \varphi_t + \varphi_w(w[r + \mu] + \psi - c_a) + \varphi_y \alpha_1 \\
&+ \frac{1}{2} \varphi_{yy} \sigma_1^2 + \varphi_z S[\beta - Z] + \frac{1}{2} \varphi_{zz} \bar{\varrho}^2 \\
&- \varphi_{wy} \rho \varrho(t) \sigma_1 w - \varphi_{wz} \bar{\varrho} \varrho w + \varphi_{yz} \rho \sigma_1 \bar{\varrho} \\
&+ \sup_{\bar{\pi}_1 \in \mathbb{R}} \left\{ \frac{1}{2} \varphi_{ww} \sigma^2 \bar{\pi}_1^2 w^2 + \varphi_w w \bar{\pi}_1 (\alpha - r) \right. \\
&- \varphi_{wz} \sigma \varrho \rho w^2 \bar{\pi}_1 + \varphi_{wy} \sigma \sigma_1 w \bar{\pi}_1 \\
&+ \left. \varphi_{wz} \rho \bar{\varrho} \sigma w \bar{\pi}_1 \right\} + \sup_{\bar{\pi}_2 \in \mathbb{R}} \left\{ \frac{1}{2} \varphi_{ww} \varrho^2 \bar{\pi}_2^2 w^2 \right. \\
&+ \left. \varphi_w w \bar{\pi}_2 (k + Z - r) \right\} - \varphi_h f(t, H, Y) \\
&+ \int_{\mathbb{R}} [\varphi(t, w + \bar{\pi}_1 w \gamma(t, \eta), z, y, h) - \varphi(t, w, z, y, h) \\
&- \bar{\pi}_1 w \gamma(t, \eta) \varphi_w] \nu(d\eta) + \int_{\mathbb{R}} [\varphi(t, w, z, y, h + Y(t, \eta)) \\
&- \varphi(t, w, z, y, h) - Y(t, \eta) \varphi_w] \nu(d\eta),
\end{aligned} \tag{5.72}$$

considering equations (5.3), (5.4) and (4.20). We obtain the optimal controls $\bar{\pi}_1^*$, $\bar{\pi}_2^*$, c_a^* by differentiating (5.71) with respect to $\bar{\pi}_1$, $\bar{\pi}_2$ and c_a and equating them to zero.

The proportion $\bar{\pi}_1^*$ of wealth held from the invested stock is given by;

$$\begin{aligned}
\frac{\partial G(\cdot)}{\partial \bar{\pi}_1} &= \varphi_{ww} \sigma^2 \bar{\pi}_1 w^2 + \varphi_w w (\alpha - r) - \varphi_{wz} \sigma \varrho \rho w^2 \\
&+ \varphi_{wy} \sigma \sigma_1 w + \varphi_{wz} \rho \bar{\varrho} \sigma w \\
&+ \int_{\mathbb{R}} [\varphi_{\bar{\pi}_1}(t, w + \bar{\pi}_1 w \gamma(\eta, t), z, y, H(t)) - w \gamma(t, \eta) \varphi_w] \nu(d\eta) = 0.
\end{aligned} \tag{5.73}$$

$\bar{\pi}_1^*$ is the solution of the above equation.

We get the proportion of wealth in the inflation linked bond $\bar{\pi}_2^*$ as;

$$\frac{\partial G(\cdot)}{\partial \bar{\pi}_2} = \varphi_{ww} \varrho^2 \bar{\pi}_2 w^2 + \varphi_w w(k + Z - r) = 0,$$

$$\bar{\pi}_2^* = -\frac{\varphi_w}{\varphi_{ww}} \left[\frac{(k + Z - r)}{w \varrho^2} \right]. \quad (5.74)$$

We have the optimal consumption process c_a^* to be consumed as;

$$\begin{aligned} \frac{\partial G(\cdot)}{\partial c_a} &= -w \varphi_w + U'(c_a) = 0 \\ U'(c_a) &= w \varphi_w, \end{aligned} \quad (5.75)$$

given that $U'(c_a)$ is the differential of the utility function *w.r.t* c_a .

The following theorems state the solutions above given a specific utility function $U(w)$ and a specific value function $\varphi(t, w, z, y, h)$. The steps are omitted as they are obtained exactly like those in the pre-death case.

Theorem 5.3.1 *Given the optimal investment-consumption-insurance problem with inflation risk and stochastic income in an Itô-Lévy setting, and the value function $\varphi(t, w, z, y, h) = \frac{w^\lambda}{\lambda} e^{-(\lambda-1)H(t)}$, then,*

$$\begin{aligned} \bar{\pi}_2^*(t) &= \frac{(\lambda - 1)}{\varrho(t)} \left[\frac{(k(t) + Z(t) - r)}{\varrho(t)} \right], \\ c_a^*(t) &= w \frac{\lambda}{(\lambda - 1)} e^{-H(t)} \quad \text{and} \end{aligned} \quad (5.76)$$

$\bar{\pi}_1^*(t)$ solves;

$$\left(\frac{\sigma^2 \bar{\pi}_1}{\lambda - 1} + \int_{\mathbb{R}} [(1 + \bar{\pi}_1 \gamma(\eta, t))^{\lambda-1} \gamma(\eta, t) - \gamma(\eta, t)] \nu(d\eta) + (\alpha - r) \right) = 0; \quad \text{for} \quad (5.77)$$

$$w^\lambda e^{-(\lambda-1)H(t)} \neq 0.$$

Theorem 5.3.2 *Given the optimal investment-consumption-insurance problem with inflation risk and stochastic income in an Itô-Lévy setting, and the value function*

$\varphi(t, w, z, y, h) = \ln w e^{H(t)}$, *then,*

$$c_a^* = e^{-H(t)}, \quad (5.78)$$

$$\bar{\pi}_2^* = \left\lceil \frac{(k(t) + Z(t) - r)}{\varrho^2(t)} \right\rceil, \quad (5.79)$$

and $\bar{\pi}_1^*(t)$ solves:

$$\sigma^2 \bar{\pi}_1 - \int_{\mathbb{R}} \left[\gamma(\eta, t) \left(\frac{1}{(1 + \bar{\pi}_1 \gamma(\eta, t))} - 1 \right) \right] \nu(d\eta) - (\alpha - r) = 0. \quad (5.80)$$

The theorems above are obtained similarly as the theorems 5.1.1 and 5.2.1, respectively.

Substituting $\bar{\pi}_1^*$ and the strategies above into (5.71) we obtain the generators of the BSDE equation (5.1) with the considered utility functions and value functions.

$$\begin{aligned}
f(t, H(t), \Upsilon(t)) &= \left\{ \lambda \left[\frac{1}{1-\lambda} ((r+\mu) + \frac{\psi(y, t)}{w} + \pi_1^*(\alpha - r)) - \frac{\sigma^2(\pi_1^*)^2}{2} \right. \right. \\
&\quad \left. \left. - \frac{3}{2} \left(\frac{k+z-r}{\varrho} \right)^2 \right] \right\} - \frac{1}{1-\lambda} \left(\lambda e^{-H(t)} - \frac{1}{w\lambda-1} e^{-H(2\lambda-1)} \right) \\
&\quad + \frac{1}{(1-\lambda)} \left\{ \int_{\mathbb{R}} [(1 + \pi_1^* \gamma(t, \eta))^\lambda - 1 - \lambda \pi_1^* \gamma(t, \eta)] \nu(d\eta) \right. \\
&\quad \left. + \int_{\mathbb{R}} [e^{(\lambda-1)\gamma(t, \eta)} - 1 - \frac{\gamma(t, \eta)}{w}] \nu(d\eta) \right\},
\end{aligned} \tag{5.81}$$

and,

$$\begin{aligned}
f(t, H(t), \Upsilon(t)) &= \frac{1}{\ln w} \left((r+\mu) + \frac{\Psi(Y_t, t)}{w} - \frac{2e^{-H(t)}}{w} - \frac{\sigma^2(\pi_1^*)^2}{2} \right. \\
&\quad \left. + \pi_1^*(\alpha - r) \right) + \frac{(k(t) + Z(t) - r)^2}{2\varrho^2 \ln w} \\
&\quad + \frac{1}{\ln w} \left(\int_{\mathbb{R}} [\ln(1 + \pi_1^* w \gamma(t, \eta)) - \pi_1^* \gamma(t, \eta)] \nu(d\eta) \right. \\
&\quad \left. + \int_{\mathbb{R}} [\ln w (e^{\gamma(t, \eta)} - 1) - \frac{\gamma(t, \eta)}{w}] \nu(d\eta) \right) \\
&\quad - \frac{1}{\ln w} (H(t) + \frac{1}{w}) e^{-H(t)},
\end{aligned} \tag{5.82}$$

respectively.

Chapter 6

Discussion of Results

6.1 Results Analysis

The first model where an investor's utility function is the power utility with the value function chosen we observe that the interest rate r and drift parameter α have a notable impact in the fraction invested in the stock as well as the variance σ . The proportion invested in the inflation linked bond π_2^* is inversely proportional to the volatility of the price index rate of the commodity good, thus the proportion to be invested becomes smaller when the variance ϱ gets large. Therefore, in general when the risk is high in the market we invest a smaller fraction. We observe that the risk aversion increases with increasing investment fraction or wealth in the inflation linked bond, provided that the interest rate k and drift parameter z are all greater than the money market account interest rate that is, the bank's interest rate r , $k+z > r$. It is important to note that when the parameter $\lambda = 1$ the investment in π_2 vanishes

and investment position is only taken on the money market account, thus we have maximal wealth when $\varrho < (k + z - r)$ and $\lambda \neq 1$. Our admissible portfolio is given by theorem (5.1.1).

We also observe that the investor is allowed to consume during the process of investment. Note that the power utility has the consumption fraction given as in equation (5.25), thus $e^{-H(t)} \neq 0$, therefore consumption is made on the wealth process and consumption until $c^* = w$ will deplete all the wealth accumulated then there will be no use of the investment strategy. Caution and limit has to be made on the power utility when consuming so that the investment strategy does not collapse and lead to no wealth accumulated. The premiums to be paid are dependent on the time of death or retirement T , of an investor. If the investor dies the benefits are given to the beneficiary. We note that when the investor dies the beneficiary obtains a portion $W + \frac{P(\tau)}{\tilde{\pi}(\tau)}$ with a predetermined premium insurance benefit ratio $\tilde{\pi}(\tau)$, and when the investor survives until retirement time then the investor stops paying premiums, but continues to invest his/her fractions in the available asset positions determined in the paper.

In the model where we have the logarithmic utility function the fraction π_1^* is a function of the diffusion jump, the interest rate r , drift parameter α and the volatility σ . The diffusion jumps in the model are due to external changing asset price behaviour throughout the investment period. In essence, diffusion jumps can never be ignored in pricing assets, therefore we observe that volatility dynamics of the stock is also due to the jumps in the stock

price, thus the risk of maximizing the returns of the portfolio π_1^* depends on the movement of the diffusion jumps as the jump $\tilde{J}(d\eta, dt)$ and the Brownian motion B_t^0 are independent. Hence, at any given point it is more risky investing a large proportion in the stock as the risk is brought by independent factors but also gives an investor a chance to benefit from the high return on the investment. The proportion held in the inflation linked bond π_2^* is also determined by the level of risk in the market, if the risk is less we have maximal investment in the bond. The stock's risk is determined by the jump diffusion and the volatility σ therefore, it is best to invest in a risk-less asset while Zhang and Zheng (2019) has that the stochastic interest rate may lead to a reduction in the amount invested in the bond. We observe that the interest rates k , r and the drift parameter z has a notable impact on the investment made. We have maximal wealth also when $k + z > r$ and less risk ϱ^2 .

In essence, the change in stock prices as a result of sudden stock market jumps can either increase or decrease the wealth of an investor. These jumps are usually a result of government policy changes e.g., tax policies, monetary policy and central banking, trade and exchange rate policies and foreign stock markets. These gaps determine the level of volatility of the stock price which can either decrease or increase the risk of an asset. The jumps give rise to the demand and supply of stock shares, increased demand to own shares π_1^* is met by other investors selling their shares. Therefore the investor's decision to buy and sell is determined by the stock price jump at that given point in time, hence government policies can affect the wealth of an investor positively and negatively in the long run.

Looking into the fraction of consumption in the logarithmic utility we note that the pro-

portion of consumption is independent of the wealth accumulated or to be accumulated. For $t \in [0, T]$ we have the fraction given as in equation (5.50), thus at any given time t the investor will always consume, therefore the market allows arbitrage because the fraction of consumption is not determined by the wealth w an investor has. The admissible portfolio is as given in theorem (5.2.1). Inflation affects investments. In this study it is not guaranteed that diversification and asset allocation protects against losses. In the the study we realise that the bond with the inflation linked index does not have a stable stream of income even though interest rates remain the same until maturity.

It is evident that with the portfolio π_2^* , its purchasing power declines with increase in the inflation rate thus resulting in the bond price falling when inflation increases. This is also the case with the money market account as it has fixed interest rate. The savings purchasing power will be eroded with increasing rate of inflation which will be advisable for investors to also consider investing in assets that outperform markets when inflation hits. These include real estate investments, commodities and gold. Zhang and Zheng (2019) considered the reinsurance policy in their model to diversify the risk not just investing in different assets to reduce risk. We can clearly say that in both cases of the two utilities we invest more in a market with less risk.

6.1.1 Future Work

In the future, it would be interesting to consider the problem with extensive ways of solving jump diffusion equations and other techniques in finding the optimal portfolio. Also by

using more realistic market features such as stochastic investment opportunities, portfolio constraints and ways of managing risks.

Chapter 7

Conclusion

The dissertation entirely focused on the study of optimal investment-consumption-insurance strategy in an Itô-Lévy setting with diffusion jumps. The wealth process is determined at different generations of the life of an investor, where investment position is taken on both risky and risk-less assets. The assets comprise of stock with a diffusion jump, bond, and a money market account. Inflation as the most important economic factor is taken into account as it affects the wealth of an investor at any given time, thus the inflation is linked to the bond which makes the bond risky. The investor's incoming source of funds is taken to be a stochastic process that also contributes to his/her's wealth. The investor's consumption is finite and does not exceed the limit of what they have. We managed to employ both the BSDE with jumps and the HJB equation in finding the optimal investment strategies π_1^*, π_2^*, c^* , and p^* with the different conditions imposed on the parameters throughout the dissertation. When the investor dies the proportions invested are then the proportions that

beneficiaries can decide whether to sell or buy looking at the market the investment has been made on, and the insurance pays out the benefit they have agreed on with the holder in their time of life. The beneficiaries can either continue with the investment by increasing or decreasing the fractions looking at the level of risk in each one of them and continue to consume the proportion that will allow the investment to go on.

With the use and understanding of the theorems employed, that is the Radon-Nikodym theorem and the Girsanov's theorem, the consideration of the utility functions and value functions chosen, the power and logarithmic functions gave us the optimal strategies given by theorems (5.1.1) and (5.2.1). With the utility functions and value functions assessed there was a possibility to review how the instantaneous force of mortality, the conditional probability and the cumulative densities of the investor in their time of life are redefined. The solutions to the BDSE was made possible by the conditions given to the hazard rate function on both utilities considered. In the logarithmic utility the hazard function is dependent on the solution of the BSDE, which then implies that the conditional probability and the cumulative densities of the investor also depend on the solution of the BSDE.

The study can further be improved by using other methods or techniques in finding the optimal strategies, with emphasis made on diffusion jumps. The study on diffusion jumps could help in giving a function on the solution of π_1^* to improve the solution obtained. Different assumptions can be made on the values of the parameters and the utility functions

chosen and compare the solutions obtained. One can consider the different times of life of an investor and choose a more specific insurance an investor buys.

Appendix

Appendix 1

Note that we have the following equations:

$$\pi_1 W_t = \theta_t^1 S_t \quad (7.1)$$

$$\pi_2 W_t = \theta_t^2 M_t, \quad (7.2)$$

$$(1 - \pi_1 - \pi_2)W_t = \theta_t^0 R_t, \quad (7.3)$$

then we have the following;

$$dW_t = \theta_t^0 dR_t + \theta_t^1 dS_t + \theta_t^2 dM_t + \psi(Y_t, t)dt - C(t)dt - \mu(t)(P(t) - W_t)dt, \quad (7.4)$$

Substituting dR_t, dS_t, dM_t into equation (7.4) we obtain the following;

$$\begin{aligned} dW_t = & \theta_t^0 r R_t dt + \theta_t^1 S_t (\alpha dt + \sigma dB_t^0 + \int_{\mathbb{R}} \gamma(t, \eta) \tilde{J}(dt, d\eta)) \\ & + \theta_t^2 M(t)(k(t) + Z(t))dt + \varrho(t)dB_t^1 + \psi(Y_t, t) - C(Q_t, t)dt \\ & - \mu(t)(P(t) - W_t)dt, \end{aligned} \quad (7.5)$$

substituting (7.1), (7.2) and (7.3) then simplifying we obtain the wealth process of an investor.

Appendix 2

Simplifying the HJB equation

Let G in (5.6) be given in the form below;

$$\begin{aligned}
 \mathcal{B}_1. &= \varphi_t + \varphi_w(w[r + \mu(t)] + \psi(Y_t, t) - \mu(t)P^* - C^*) \\
 &+ \varphi_y\alpha_1(y_t, t) + \frac{1}{2}\varphi_{yy}\sigma_1^2(Y_t, t) + \varphi_zS(t)[\beta(t) - Z(t)] \\
 &+ \frac{1}{2}\varphi_{zz}\bar{\varrho}^2(t) - \varphi_{wy}\rho\varrho(t)\sigma_1(Y_t, t)w - \varphi_{wz}\bar{\varrho}(t)\varrho(t)w \\
 &+ \varphi_{yz}\rho\sigma_1(Y_t, t)\bar{\varrho}(t) \\
 &= w^\lambda e^{(1-\lambda)H(t)}[(r + \mu) + \frac{1}{w}\psi(Y_t, t) - (\mu^\lambda w)\frac{1}{\lambda - 1}e^{-H(t)} \\
 &- \frac{1}{w\lambda - 1}e^{-H(t)}],
 \end{aligned} \tag{7.6}$$

since $\varphi_t = \varphi_y = \varphi_z = \varphi_{yy} = \varphi_{zz} = \varphi_{wy} = \varphi_{wz} = \varphi_{yz} = 0$ explicitly.

$$\begin{aligned}
 \mathcal{B}_2. &= \frac{1}{2}\varphi_{ww}\sigma^2(\pi_1^*)^2w^2 + \varphi_w w\pi_1^*(\alpha - r) - \varphi_{wz}\sigma\varrho(t)\rho w^2\pi_1^* \\
 &+ \varphi_{wy}\sigma(t)\sigma_1(Y_t, t)w\pi_1^* + \varphi_{wz}\rho\bar{\varrho}(t)\sigma(t)w\pi_1^* \\
 &= w^\lambda e^{(1-\lambda)H(t)}[\frac{1}{2}(\lambda - 1)\sigma^2(\pi_1^*)^2 + \pi_1^*(\alpha - r)]
 \end{aligned} \tag{7.7}$$

$$\begin{aligned}
\mathcal{B}_3. &= \frac{1}{2}\varphi_{ww}\varrho^2(t)(\pi_2^*)^2w^2 + \varphi_w w \pi_2^*(k(t) + Z(t) - r) \\
&= w^\lambda e^{(1-\lambda)H(t)}(k(t) + Z(t) - r) \frac{(\lambda-1)^3}{\varrho^2} \left(\frac{1}{2} + \frac{1}{(\lambda-1)^2} \right)
\end{aligned} \tag{7.8}$$

$$\begin{aligned}
\mathcal{B}_4. &= -\varphi_h f(t, H(t), \Upsilon(t)) + \int_{\mathbb{R}} [\varphi(t, w + \pi_1^* w \gamma(t, \eta), z, y, h) \\
&- \varphi(t, w, z, y, h) - \pi_1^* w \gamma(t, \eta) \varphi_w] \nu(d\eta) + \int_{\mathbb{R}} [\varphi(t, w, z, y, h + \Upsilon(t, \eta)) \\
&- \varphi(t, w, z, y, h) - \Upsilon(t, \eta) \varphi_w] \nu(d\eta) \\
&= -\frac{w^\lambda}{\lambda} (1-\lambda) e^{(1-\lambda)H(t)} \left[f(t, H(t), \Upsilon(t)) + \frac{1}{(1-\lambda)} \left\{ \int_{\mathbb{R}} [(1 + \pi_1^* \gamma(t, \eta))^\lambda \right. \right. \\
&- \left. \left. 1 - \lambda \pi_1^* \gamma(t, \eta)] \nu(d\eta) + \int_{\mathbb{R}} [e^{(1-\lambda)\gamma(t, \eta)} - 1 - \frac{\gamma(t, \eta)}{w}] \nu(d\eta) \right\} \right]
\end{aligned} \tag{7.9}$$

$$\begin{aligned}
\mathcal{B}_5. &= U(c^*) + U(p^*) \\
&= \frac{\lambda^2}{w\lambda-1} e^{(1-\lambda)H(t)} e^{-\lambda H(t)} \left(1 + \mu \frac{\lambda}{\lambda-1} \right)
\end{aligned} \tag{7.10}$$

Combining the above forms of the HJB equation we finally have equation (5.27).

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