



FACULTY OF SCIENCES
DEPARTMENT OF PHYSICS AND ASTRONOMY

**Impact of Higher-Order Dispersions and Non-Kerr Nonlinearities on Soliton Pulse Trains
Induced by Modulational Instability in Doped and Undoped Optical Fibers**

BY

KARABO KEFILWE NDEBELE

STUDENT ID: 14000818

BSc (Physics) (BIUST)

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Supervisor: **Professor Conrad Bertrand Tabi**
Department of Physics and Astronomy
Faculty of Sciences, BIUST
Email: tabic@biust.ac.bw

Co-supervisor: **Professor Timoléon Crépin Kofané**
Department of Physics and Astronomy
Faculty of Sciences, BIUST
Email: kofanet@biust.ac.bw

MARCH, 2022

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Candidate (please write in caps or type) KARABO KEFILWE NDEBELE

Student ID: 14000818

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ON SOLITON PULSE TRAINS INDUCED BY MODULATIONAL INSTABILITY IN DOPED AND
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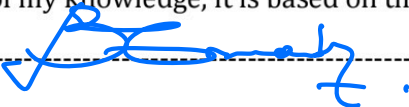
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List of Abbreviations

PCF: Photonic crystal fibers

NLS: Nonlinear Schrödinger

HNLS: Higher-order nonlinear Schrödinger

CGL: Complex Ginzburg-Landau

MI: Modulational instability

DSF: Dispersion-shifted optical fibers

dB: Decibels

Gb: Giga bites

WDM: Wavelength division multiplexing

GVD: Group velocity dispersion

SiO₂: Silica

Al₂O₃: Aluminium oxide

GeO₂: Germanium-dioxide

Er₂O₃: Erbium oxide

EFDA: Erbium-doped fiber amplifier

Nd⁺³: ionized neodymium

SPM: Self-phase modulation

XPM: Cross-phase modulation

FOC: Fiber optic cable

CD: Chromatic dispersion

PMD: Polarisation mode dispersion

SBS: Stimulated Brillouin scattering

SRS: Stimulated Raman scattering

FWM: Four-wave mixing

RZ: Return to zero

CRZ: Chirped return to zero

USPs: Ultrashort pulses

TOD: Third-order dispersion

FOD: Fourth-order dispersion

FFOD: Fifth-order dispersion

SOD: Sixth-order dispersion

SS: Self-steepening

SFS: Self-frequency shift

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Abstract

We start by considering the theory of electromagnetic waves, from the Maxwell's equations, the extended (1+1)D complex Ginzburg-Landau equation, higher-order nonlinear Schrödinger equation are derived with the third, fourth, fifth and sixth-order dispersions and the cubic, quintic, and septic nonlinear terms, describing the dynamics of extremely short pulses in nonlinear undoped and doped optical fibers. The linear stability analysis is employed to extract an expression for the modulational instability gain of a CW solution. The sensitivity of the system to higher-order dispersions and nonlinear terms is extensively shown the insistance on the balance of interactions between the sixth-order dispersion and nonlinarity, septic self-steepening and the septic self-frequency shift terms. The system is confronted to full numerical simulations, pointing out that the input CW gives rise to a broad range of behaviors, in relation to nonlinear patterns formations in the derived HNLS equation. To much excitement, under the motivation of modulational instability, a suitable balance between the sixth-order dispersion and the septic self-frequency shift term is achieved to bring forth some high influence on the propagation direction of the optical wave patterns. **Keywords:** Modulational Instability; Higher-order nonlinear Schrödinger equation; Complex Ginzburg-Landau equation; Sixth-order dispersion; Septic nonlinearity; Septic self- steepening; Septic Raman scattering.

Motivation

Communication is an essential necessity in life, the world keeps growing larger, civilization keeps changing. Today's civilization is an era of the internet. With the internet, people are able to communicate all over the world without any barriers hence the need to evolve data transmission forms. Optical fiber communication is a scientific innovation that is aimed at improving data communication. Nonlinear dynamics of optical fibers in data transmission were ignored until researchers discovered that they can exclude a rich set of dynamics that are innately sensitive, relatively if the focus is in ultrashort pulses. Optical fiber communication systems essentially use optical wave signals as carriers therefore, the improvement of bit rate distance product can be shown with improvement in various or significant orders of magnitude. After the first introduction of the lasers in 1960, attempts to use it for communication reasons, because of its coherent radiation proved to fail. Fortunate enough, in 1966, the optical fiber medium was introduced as the most suitable decision to transmit the optical signals due to its unique nature of confining an electromagnetic field in the plane perpendicular to the axis of an optical fiber core. Sadly, they had extremely high losses which were fairly solved in the 1970s by introducing multimode optical fibers which operated in the $0.8\mu\text{m}$ wavelength region. Remarkably, over the years generation of fiber-optic communication systems have extended their focus to the careful management of fiber dispersion which brings us to the approach that the concept of optical solitons is established on. Our main aim is to introduce wave propagation equations for ultrashort pulses preferably from subpicosecond to femtosecond regimes with minimum loss by

manipulation of higher-order dispersions and Kerr and non-Kerr nonlinearities. Thus this dissertation focuses on the nonlinear dynamics of optical fibers.

General Introduction

Solitons in optics is one of the key areas of research that are growing vastly in photonic sciences [1–6]. Optical solitons have become a true marvel in the field of telecommunication since the 1990s. There are several branches of soliton dynamics that are of interest. For example the study of a wide form of waveguides like optical fibers, crystals, PCFs as well as optical metamaterials and metasurfaces have become an important avenue of research. Optical solitons are the basic tie in telecommunication industry. It is well known that we have two main classes of solitons, that is spatial and temporal solitons. Generally, in optical fibers, we refer to temporal solitons in cases where electromagnetic field is seen to be spatially confined; consequently, temporal solitons were the first to be discovered, thus often referred to as just solitons in optics. They represent nonlinear pulses or wave packets with the sole ability to travel without change in shape and velocity across intercontinental distances [7–15]. One special characteristic of optical solitons is that they have the ability to interact like particles [12]. Optical solitons are also described as localized solutions to nonlinear dispersive wave equations that do not change upon propagation. In integrable systems, such waves have been proven to remain intact after collision events, particularly they behave as particle-like entities. Impeccably, these self-localized or self-trapped wave packets are said to owe their existence to the balance between nonlinearity and dispersion effects. Hasegawa and Tappert [16] reported these magnificent properties of solitons and they were proven in an experimental setup by Mollenauer et al. [17]. Since then, the model

equation for generation of a wave pulse in a media with Kerr nonlinearity is well known as the cubic nonlinear Schrödinger (NLS) equation, which can also be described as the propagation equation for picosecond pulses. Indeed, the Kerr effect, supported by the intensity effect, results in optical pulse spectrum asymmetric broadening fields [18], and propagation of light pulse. In relation to such fathomable effects, the pulse peak is observed to tilt to the flank of the pulse, hence named the self-steepening effect [19]. In real fibers, explicitly, the core media is not homogeneous. The fact is, there are some significant always non-uniformity contributed by many factors, mostly, the discrepancy in the lattice parameter of the fiber media and the differentiation of the fiber geometry. For an inhomogeneous fiber, a generalized NLS equation has been discovered to be the most important realistic and vast nonlinear model [20–23].

Over the past decades, studies on the light pulse generation through optical fibers [24–26] and optical fiber lasers [7] have fused enormous interest in the investigations of nonlinear effects which are described by the NLS, higher-order nonlinear Schrödinger (HNLS) equation [26, 28, 29] and higher-order complex Ginzburg-Landau (HCGL) equation [31–33]. From the theoretical point, the nonlinear response of a medium to a high power light beam is commonly explained by the expansion in a power series of the electric field wave amplitude, the coefficients of which are the so-called nonlinear susceptibilities. Such theoretical predictions of structures of this kind involve some saturation of the nonlinearity. Studies have profusely shown and affirmed that in optical ultrashort pulse beams, the value of group-velocity dispersion (GVD) is important, and its slope is governed by the third-order dispersion (TOD) and curvature brought forward by the fourth-order dispersion (FOD) become invariant assets [27, 34, 35]. Hence, prudent consideration should be taken into account to the influence of higher-order effects for optical pulses, such as third-, fourth-, fifth- and sixth-order dispersion, the self-steepening effect, and self-frequency shift. The self-steepening effect, occurs with an optical shock at the trailing edge, like the GVD and TOD give birth to the spectral

asymmetrical broadening of the pulse. The self-frequency shift from the stems of stimulated Raman scattering, because of the delayed reaction of the medium, consequently leading to the redshift increase in the pulse spectrum, the long wavelength components are said to exhibit the Raman gain at the will of short wavelength components. Undeniably, optical fiber communication systems are tremendously affected by these effects. For an extremely short optical pulse between 10^{-12} s and 10^{-15} s regimes, the gradient of the nonlinear terms are said to be related to the effect of the self-steepening effect and self-frequency shift, which coherently affect the stable transmission of femtosecond optical pulses. Therefore, the growing need of a model equation taking heed of the higher-order effects, such as the extended NLS equations

A way to describe those higher-order effects is to take into consideration the second term in the material response function to expand, that is represented by the quintic nonlinear susceptibility. Furthermore, solutions for HNLS equations have been intensively studied with the aid of certain distinct methods, by the balance effects of higher-order dispersions and nonlinear terms, from which more soliton solutions have been reported [36–39]. Even so, when the saturation is seen to be very strong, a self-focusing seventh-order or septic susceptibility is also needed. Various experiments have been reported in materials such as doped chalcogenide glasses that display not only third- and fifth-order nonlinearities but even seventh-order nonlinearity [40]. In other words, doped chalcogenide glasses can be classified as cubic-quintic-septic nonlinear materials. Hence, to study the optical soliton pulse of such a doped optical fiber material whose characteristics exhibit as high as the septic nonlinearity, focuses have shifted on deriving higher-order cubic-quintic-septic complex Ginzburg-Landau (CGL) equations [41, 42]. The CGL equation can be used to describe femtosecond pulse propagation in nonlinear optical fibers with gain and spectral filtering. In the physical essence, the spectral filtering in femtosecond systems appears in a natural manner as the bandwidth of the optical fiber amplifiers is restricted. Spectral filtering

is considered to play a positive role as it limits the deviations of the central frequency of the optical pulse. Solitons in optical fibers are analytic solutions for the family of the NLS-like equations [43–50].

It has been known that when the GVD in an optical fiber is below zero, with the nonlinearity of a planar waveguide which is self-focusing, normally a constant amplitude continuous wave has been seen not to be stable through the modulational instability (MI). Still, GVD and self-phase modulation exhibit symmetric broadening in the time and frequency domains, respectively, and therefore counterbalances to generate solitons. In certain conditions, bright and dark soliton solutions are observed to exist in the negative GVD and positive GVD regions. Inherently, there are more prominent likelihoods to having soliton pulse generation with all higher-order effects that inflict asymmetrical broadening. The generation mechanism of optical solitons has been generally believed to be the modulation instability. In various branches of physics, MI is a significant phenomenon [51–53], which can be viewed as a dynamical instability process. Whereby a small amplitude or phase perturbation is raised owing to the interaction among dispersive and nonlinear effects, forcibly in the time domain or diffraction in the spatial domain and as a role in the beginning stages of noise-seeded long pulse super-continuum generation [51, 54]. Moreover, when a small perturbation is performed on a continuous wave, it disintegrates into a periodic pulse train that finally generates short pulses. The acuity in the media of non-Kerr has been confirmed to be an immense and exciting study field in MI, for scrutinization of the optical pulse promulgation in NLS equations with higher-order effects [55–57]. The most important cause behind this is that the MI may be referred as, the ancestor of soliton construction.

Thesis Overview

- General objective:

- *Propose enhanced model equations for optical pulse propagation in doped and un-doped optical fibers based on Maxwell's equations for electromagnetic waves.

- Specific Objectives:

- *Find and investigate the instability criteria of continuous waves and study the effects of nonlinear and dispersion on MI growth rate for the newly derived HNLS equation for ultrashort pulses.

- *Study the combined effects of nonlinear, dispersion, amplification and loss on the newly derived (1+1)D higher-order cubic-quintic-septic CGL equation for ultrashort pulses in doped optical fibers.

The thesis is outlined as follows.

- Chapter 1 introduces fiber optic communication systems, giving a brief background overview. It also focuses on inaugurating the idea of solitons, optical solitons, doped optical fibers, modulational instability phenomenon, and lastly kicks off with the structure and applications of optical fibers.

- Chapter 2 studies the linear and nonlinear effects affecting the wave propagation in optical fibers, namely, modal, chromatic and polarization mode dispersions, self-phase modulation, cross-phase modulation, second-order and third-order nonlinear effects.

- Chapter 3 focuses on the fundamentals of ultrashort and nonlinear fiber optics based on understanding and having a quantitative description of the propagation of the pulse in an optical medium. With this technical note, this chapter offers a narrative definition of the general NLS equation for picosecond pulses which is derived from Maxwell's equations for electromagnetic waves. This is further accompanied by the general NLS MI process for optical pulses.

- Chapter 4 progresses to ultrashort pulse propagation equations for undoped optical fibers, which are governed by higher-order NLS equations. It is well known that when ultrashort pulses are taken into consideration, other physical phenomena such as TOD, FOD, FFOD, SOD, SS effect, and SFS should be accounted for by the HNLS equation. To start, we elude the MI phenomenon for a previously known HNLS equation and study the MI gain spectra in positive and negative GVD regimes with special cases for the SOD, septic nonlinearity, SS, and SFS. Thereafter, the focus is shifted on presenting the modeling process of the newly derived HNLS equation starting with Maxwell's equations for electromagnetic waves. For better comparisons, the MI process for the newly derived HNLS equation is also well eluded to study the MI spectra in both normal and anomalous regimes with special cases for the SOD, septic nonlinearity, SS and SFS.

- Chapter 5 derives the (1+1)D higher-order cubic-quintic-septic complex Ginzburg-Landau equation which describes the femtosecond pulse propagation equation in doped

optical fibers. The model is further analyzed for pulse train generation by MI phenomenon, the MI growth rate spectra in both normal and anomalous regimes with special cases for the SOD, septic nonlinearity, SS, and SFS is also well studied.

- The general conclusion is deduced lastly, offering a complete proposition and reasoning of the thesis.

Chapter 1

Optical Fiber

1.1 Fiber-Optic Communication Systems

Communication is an essential tool of human life, just as in the olden days, people used different forms to communicate such as, talking drums, smoke signaling, and cow horns, to name a few. But with civilization, modes of communication were also evolved. The history of optical fibers dates back to how electrical communication evolved. In this era of information technology, there is a high dependency on the way information is transmitted along with the need of fast, accurate, and high volume data communication. The age of electrical communication began with the emergence of telegraphy[58] from letters to modern-day social network services, with methods of communication also continuously evolving. The need to connect numerous telephone lines and computers, led to a network of topologies with routes that are capable of transmitting large amounts of data, thus, technology in transmission has continued to evolve to accommodate the high increase of capacity of social networks. The first transmission lines were wire pairs, which were replaced by coaxial cables in effort to improve the capacity. Still the number of telephones increased, coaxial cables were not sufficient enough. Hence leading to the development and deployment of microwave communication systems[59].

The fiber optics field has drastically grown in scientific research and developments. Initially, optical fibers were conceived as a light carrying medium and for medical endoscopic imaging purposes and applications. Thereafter, the emergence of Laser in 1960, optical fibers were discovered as information carrying medium for telecommunication applications[60]. This gave rise to the fact that the frequency operation has been perpetually shifted to higher frequencies in efforts to increase the bit rate distance product, whereas the highest bit rate is highly dependent on the magnitude of the carrier frequency. Moving further to metallic systems, attenuation commonly increases with frequency which in turn limits the bit rate distance product that may be achieved. In order to support the demand in bit rate distance product, studies have proven it very important that the carrier frequency has to be high enough to overpower the limitation[60]. Therefore, the next higher frequency band which is the main light region, is the frequency band which gave birth to optical fiber communication systems. Optical fiber communication systems often use optical waves as carriers, therefore, the bit rate distance product can be made better in as many different orders of magnitude in comparison to the microwaves and coaxial cable systems.

Optical fibers were first introduced in 1966, but they had profusively high loss exceeding 1000 dB/km. This was henceworth solved in the 1970s when optical fibers having moderate attenuation loss were produced, proposing the first generation of optical communication systems, that use multimode optical fibers operating in the 0.8 μ m wavelength region. The multimode optical fibers operated at a bit rate of 45 Mb/s which allowed repeater spacing of up to 10 km[63]. Their leading advantage was the core which was large enough that the coupling of the light from the source into the fiber was found to be easy. The disadvantage, since the core diameter was large, the optical waves travelled with different paths, leading to the arrival at the re-

ceiver having time different time delays, therefore causing intermodal dispersion (pulse spreading), intramodal dispersion (group-velocity dispersion which resulted from the different spectral components of signal travelling along an optical fiber with different speeds). The desire to eliminate intermodal dispersion gave rise to the second generation of optical fibers in 1978 which utilised single-mode optical fibers. In single-mode optical fibers, waves travel in one direction, thus only chromatic dispersion appears. The good part about using such kind of optical fibers is that they have less internal (Rayleigh) scattering, and attenuation was less than that of a multimode optical fiber. A common single-mode fiber displays the lowest dispersion in the $0.8\mu\text{m}$ region. Therefore, the multimode fiber becomes inactive in long distance communication systems with a large bit-rate distance product since single-mode fibers have added advantages of eliminating intermodal dispersion and reducing scattering loss. Although attenuation is at its lowest in a well-known single-mode fiber, in the wavelength window of $1.55\mu\text{m}$, chromatic dispersion is very large, which is intolerable for a very high bit-rate distance product. Narrow-linewidth lasers are required for the specific ability to make use of the lowest attenuation at this particular wavelength window.

Also, new kinds of fibers were employed, known as the dispersion-shifted optical fiber (DSF), together with lasers propagating in a single longitudinal mode, with core and cladding profiles that are tailored with respect that the chromatic dispersion is its lowest at the wavelength region of $1.55\mu\text{m}$. Both the approaches achieved the bit-rate distance product of about 500 Gb/skm, and this was in the third generation of optical fibers in the 1980s. The drawback brought by the third generation for the wavelength window of $1.55\mu\text{m}$ systems was the optical wave signal as it regenerated itself periodically using electronic repeaters spaced apart by 60-70 km. In the 1990s, optical fibers were employed with the development of optical amplifiers in which the bit rate distance product improvement became practically possible, which is the fourth generation of optical fibers, this was by using optical amplification in increment of

the repeater spacing and wavelength-division multiplexing for the bit rate increment. Hence, the bandwidth offered by the optical fiber was utilized at maximum with the help of the wavelength division multiplexing (WDM) technique. With such kind of optical fibers, amplification of all channels without optical-electrical conversion was achieved, leading to the spacing between electrical regenerate repeaters being largely extended, giving a bit-rate distance product larger than 10000 Gb/skm. One other approach to curb the dispersion effect is by making use of the fiber nonlinearity to try and balance the effect of dispersion resulting in the preservation of the shape of the optical signal whilst it travels along with the transmission link. This is a system that is soliton-based. The fifth generation of fiber-optic communication systems deals with the extension of the wavelength range over which the WDM system can operate frequently enough. It also attempts to increase the bit rate of each channel within the WDM signal. This system requires careful management of fiber dispersion hence the concept of optical solitons.

1.2 Optical Solitons

Solitons can be defined as localized waves that develop from the interplay of linear and nonlinear processes that would singularly cause wave decay. Generally, solitons are found in numerous physical settings, including liquids[64, 65] optical fibers, plasmas[66] and condensed matter physics[67, 68]. Optical solitons in fibers have long been an experimental reality, and they play an important role as information carriers in future high-speed communication systems. Since the mechanism for forming optical solitons is well known both theoretically and experimentally, current research is devoted to examining how solitons behave when perturbed. The interactions of solitons were investigated using the unperturbed nonlinear Schrödinger equation [69–71] together with various perturbations[72, 73]. The observations of the interaction forces between solitons in optical fibers was made known experimentally [74, 75]. At first, when nonlinear

optics were proposed, there were introduced with the knowledge that light beams are self-trapping, thus they create their waveguide with aid of the nonlinear Kerr effect [76]. This connection was introduced by Zakharov and Shabat [77], who proved the main types of optical solitons, as spatial (nonlinear effect balance with diffraction) and temporal (nonlinear effect balance with dispersion). Optical solitons (spatial, temporal, or spatiotemporal) are known as light beams or pulses with the ability to trap themselves with the support of the balance between diffraction or dispersion and various nonlinearities. They are ubiquitous objects in optical media [78]. Generally, light pulses in optical fibers remit as solitons and are usually discussed within the constitution of the cubic NLS equation that incorporates effects such as group velocity dispersion and self-phase modulation, depending on the anomalous and normal regimes. The main focus is to generate pulses shorter than 10fs hence higher-order dispersions play the key role, as such the wave dynamics can be described by the higher-order nonlinear Schrödinger equation.

1.3 Modulational instability phenomena

Modulational instability (MI) in optical fibers has been discussed for over 30 years now as a phenomena to generate a train of pulses with high repetition rate that can be identified with a train of solitons. Recently, researchers have taken an interest in studies of modulational instability in nonlinear optical fibers steered by basic important and applied aspects. . Significantly, MI is described as the ancestor of envelope soliton construction in optical fibers. MI is delineated as a process in which weak perturbations from the steady-state grow exponentially as a result of the interplay between the nonlinearity (self-phase modulation) and group velocity dispersion (GVD) [79, 80]. MI follows from nonlinear interaction that also abets for solitons. Therefore, the expeditious growth of MI changes leads to the wave breakup causing the formation of soliton trains. Most of the time, soliton trains are seen by the sideband evolution

having the carrier frequency that is directly proportional to the power input. The perturbations normally come from the noise, or are sometimes produced by the frequency shift in the signal. Consequently, the evolving instability could be defined as spontaneous or induced MI. Furthermore, induced MI takes place in the time domain and forms the break-up of the perturbed continuous wave, generating ultrashort train pulses, accompanied by the repetition rate which is caused by the frequency detuning of the signal from the carrier frequency. In an optical fiber system, the first observation was made in the temporal domain, whereby the generated ultrashort pulses took over a temporal with dependence towards the input power of the cw and the perturbation frequency. Significantly, in the frequency domain, the development of high-repetition rate pulse trains following MI are steered by the growth of rapid sidebands. Such pulse effects have impetus importance in fiber-optic telecommunications. Its important to note that MI in optical fibers was first introduced by Hasegawa and Brinkman [81] but proven experimentally by Tai et al [82]. Genuinely, the MI is a common physical phenomenon, which has been introduced in waves on deep water [80], electromagnetic waves in Kerr media[83], in plasma physics [25], and in other areas. In relation fiber optics, the MI significance in erbium-doped fiber amplifiers was recently shown in the case of negative GVD which will also be observed later on.

1.4 Doped Optical Fibers

Erbium-doped optical fibers are customarily made by a chemical-vapor-deposition process in which silica (SiO_2), aluminium oxide (Al_2O_3), germanium-dioxide (GeO_2) and erbium oxide (Er_2O_3) precursors are vaporized into carrier gas, mixed, decomposed at high temperature to the oxides and hen deposited as a fine soot glass[84]. The emergence of erbium-doped glass fibers have been making waves in research, particularly active-fiber technology in the $1.55\mu\text{m}$ wavelength region for long distance transmission systems. After the most prominent work on fiber lasers [85, 86], research on rare-earth

doped-fibers became almost inactive until the introduction of high-gain erbium-doped fiber amplifiers (EDFAs) [87, 88], and their functions in lightwave transmission systems [89, 90]. Active fibers are being found having various applications in optical amplifiers, lasers, switches, and a variety of nonlinear devices. They were first reported in rare-earth doped fiber amplifiers dating as far back as the 1960's [85, 91, 92], when Snitzer developed fiber amplifiers and lasers from triply ionized neodymium (Nd^{3+}) laser glasses. Such fiber amplifiers and laser structures demonstrated their significance in terms of increased power densities by pump power confinement in a waveguiding structure. The ability to use Er^{3+} doped fiber as amplifiers in the third-telecommunications was made aware of by Payne et al. [93] in 1987, with the use of fibers which are typically silica based and their fabrication varies to the inside tube deposition method, which was discovered in the late 1980s [94, 95]. In the early 1990s, investigations on EDFA's ability to perform ion concentration had been solely based on germanium co-doped fibers [96]. But the discovery that aluminum co-doped EDFAs offer a wider and flatter spectral gain and better solubility of erbium ions caused a shift to aluminum co-doped fibers which were introduced and examined with various fiber geometries and erbium confining factors in the late 1990s [97]. Recently, focuses have shifted to chalcogenide fiber doped glasses due to their characteristics to exhibit as high as the seventh-order nonlinearity [40]. In long distance optical fiber communication systems, compensation of attenuation of the optical pulse is of vital importance. Thus, the losses could be compensated either by the use of periodically spaced amplifiers or by erbium-doped fiber amplifiers. It is widely known and accepted that nonlinear models stem from several distinct physical nature involving nonlinearity and periodicity of medium properties can be seen to be almost the same mathematically. Hence, the analysis of such generic nonlinear models has been found to be quite profound in a range of physical bodies. More importantly, when the frequency and gain and loss which depend on intensity have to be considered, the governing optical pulse propagation equation is the cubic complex Ginzburg-Landau (CGL) equation [41, 98].

1.5 Structure, Properties and Applications of Optical Fibers

An optical fiber is described as a cylindrical dielectric waveguide from low-loss materials such as silica glass [99]. Normally, it composes of a central core with a guided light, an outer cladding in which the refractive index is relatively less than that of the core [100], as shown in Fig.1.5. Optical fibers can be divided into classes such as step-index fibers or graded index fibers, and multimode or single-mode fibers. Light in optical fibers propagates in the form of modes that is single-mode fibers and multimode fibers. When the core diameter of an optical fiber is small only a single mode is permitted and the fiber is said to be a single-mode and fibers with large core diameters are said to be multimode fibers [101], as shown in Fig.1.3. Optical fibers are used in different fields, such as medical, where they are used as light guides, imaging and also as lasers for surgeries, industrial uses for optical fibers are for imaging, sensory devices and as writing in automobiles and in industrial settings. Interestingly, in telecommunications, optical fibers are used for data transmission.

1.6 Summary

In this chapter, our key purpose was to instigate the idea fiber-optic communication system and also give a background view on optical fiber communication together with doped optical fibers. Solitons have also been formally elaborated as to the reason why the modulational instability phenomenon is so prominent in this study. Lastly, the structure of the optical fiber was addressed.

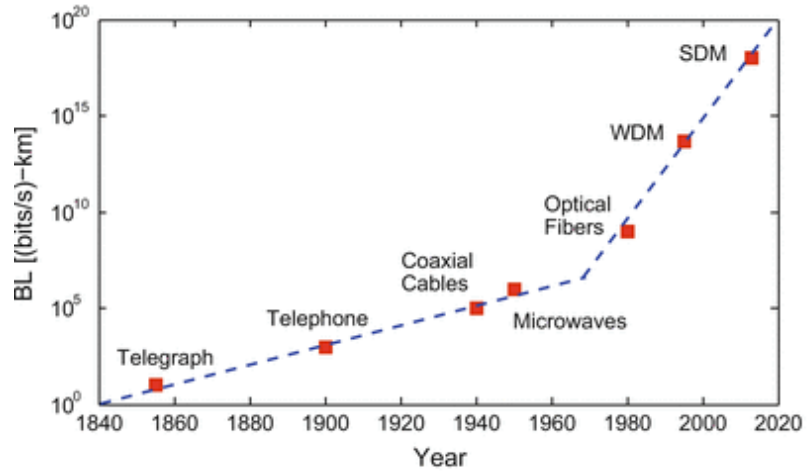


Figure 1.1: Graph representation of bit rate-distance product increment from 1850-2020 [61].

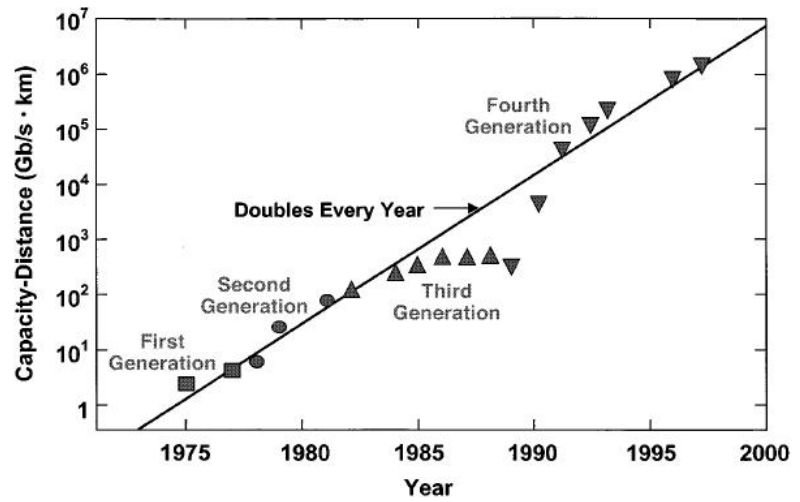


Figure 1.2: Graph representation of the progressive bit rate-distance product of lightwave communication systems[62].

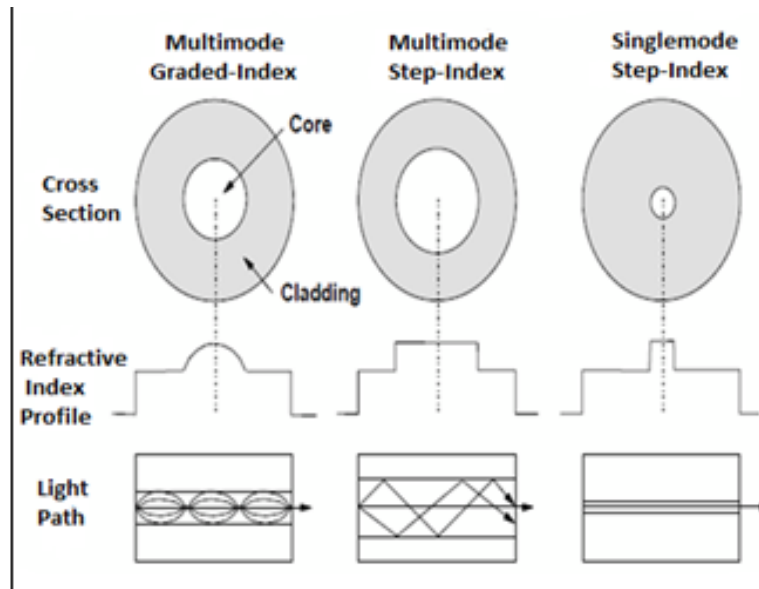


Figure 1.3: Fiber Optics modes displaying the profiles of the refractive index and the light path in the respective modes[117].

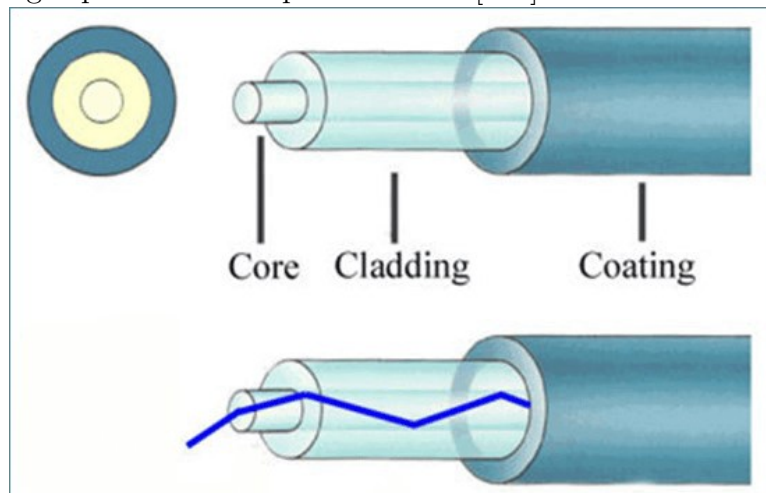


Figure 1.4: An overview of the optical fiber[117].

Chapter 2

Wave Propagation in optical fibers

2.1 Introduction

Light transmission in optical fibers does not rely only on the structure of the fiber, but also on the nature of the pulse. From a theoretical viewpoint, the response of a pulse propagation is induced by dispersions, Kerr and non-Kerr nonlinearities of the system. Generally the pulse propagation for optical solitons, there should be a balance between nonlinearity and dispersion in the system. Hence, the evolution of optical beam is has twoimportant effects: linear effects, caused by dispersion and or diffraction, and nonlinear effects, due to the processes such as self phase modulation (SPM) and cross-phase modulation (XPM) [26, 102, 103].

2.2 Linear effects of optical fibers

Problems brought by light generation in optical fibers is due to the varies in the group velocities brought by the modes. Hence resulting a set of different travel times and as such, causing the light pulses to be broad as they travel along the optical fiber. This kind of effect is well described as modal dispersion and it roots for the velocity at which ultrashort pulses can travel without overlapping, and the velocity

at which fiber-optic communication system can operate to be reduced[104, 105]. The dispersion effect could be limited by introducing fiber cores with graded refractive index from a really high value to a considerably low one at its center at the core cladding boundary[99, 100]. The resulting fiber is the Graded-index fiber whilst the convectional fibers with constant refractive indexes in the core and cladding are known as the Step-index fibers. Wave propagation in step-index fibers could be of guided rays or guided waves, there are also different modes that govern the wave propagation in step-index fibers. Optical rays are typically directed total internal reflections in the fiber core, provided the angle of incidence on the core-cladding bounding appears fairly larger than the critical angel $\theta_c = \sin^{-1}(\frac{n_2}{n_1})$ and stays the same as the rays bounce off [106–108]. Guided rays in step-index fibers could either be meridional rays, skewed rays and numerical arpeture [109]. Guided waves, in reference to monochromatic light in step-index fibers using electromagnetic waves, determine the electric and magnetic fields of guided waves that satisfy Maxwell's equations as well as the boundary conditions imposed by the cylindrical dielectric core and cladding. Each of the components of the electric and magnetic fields must obey the Helmholtz equation $\nabla^2 U + n^2 k_0^2 U = 0$ and $k_0 = \frac{2\pi}{\lambda_0}$, in both the regions, that is, the core and the cladding of the step-index fiber [100, 107, 110]. In cylindrical coordinates, the Helmholtz equation is written as

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} + n^2 k_0^2 U = 0, \quad (2.1)$$

where $U = U(r, \phi, z)$ is the complex amplitude which describes any of the Cartesian components of the electric and magnetic fields or the axial components E_z and H_z in cylindrical coordinates. The solution of the guided waves is traveling in the z -direction, with the propagation constant β . Thus, the z reliance of U is of the form $e^{-jl\beta z}$. This solution is periodic in the angle ϕ with period 2π , taking the form $e^{-jl\phi}$. If we substitute for U in Eq.(2.1) as $U = u(r)e^{-jl\phi}e^{-jl\beta z}$, with conditions that the wave is guided, the propagation constant is way less compared to the wave number

in the core, that is, $\beta < n_1^2 k_0^2$, and way much bigger compared to the wave number in the cladding, that is, $\beta > n_2^2 k_0^2$. To simplify what we just wrote, the guided wave is $k_T^2 = n_1^2 k_0^2 - \beta^2$ in the core, and $\gamma^2 = \beta^2 - n_2^2 k_0^2$ in the cladding, where k_T and γ are real and positive guided waves. Therefore, with all conditions and boundaries, the Helmholtz equation results as

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} \left(k_T^2 - \frac{l^2}{r^2} \right) u = 0 \quad r < a(\text{core}), \quad (2.2)$$

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} \left(\gamma^2 - \frac{l^2}{r^2} \right) u = 0 \quad r > a(\text{cladding}). \quad (2.3)$$

Eqs. (2.2) and (2.3) are common differential equations with solutions from the Bessel functions family of equations.

We have to take into consideration that in all waveguides there is a set of unique solutions known as modes, of which each has a very particular propagation constant, whose characterized field distribution are in the transverse plane and two independent polarization states. A mode is well defined by the azimuthal index l and radial distributions r . Moreover, to have a deeper understanding of modes, it is necessary to enforce boundary conditions. The conditions are that the centermost components of the electric and magnetic field complex amplitudes, E_z and H_z should constantly continue at the core-cladding boundary $r = a$, establishing a relation between the coefficients of proportionality, in such a way that we only have one unknown for both E_z and H_z [110]. As previously mentioned, optical fibers appear to be for now the most appealing and dominant transmission media in long-distance transmission. When compared to other media, they have better attenuation and dispersion effects than coaxial cables. Attenuation of an optical fiber could be described as a measure of the value of light lost amongst the input and the output. Particularly, the total attenuation is found

by adding all the losses of the medium[111]. It can also be described as the reduction of signal strength as shown in Fig. 2.1 and this attenuation happens in all kinds of signals, either analog or digital. In most instances, it is defined as attenuation loss, this is a common effect that occurs in long distance transmission of signals. In most cables, for example, conventional or fiber optic cables (FOCs), are expressed in terms decibels (dBs) for each foot, kilometer, or thousand foot, etc. This results in high cable efficiency when there is less attenuation in single distance units. [112]. Most

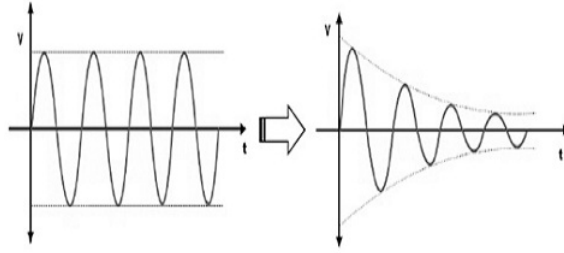


Figure 2.1: Attenuation in signals [112].

phenomena causing the reduction of light intensity, normally contribute to the optical attenuation. The optical transmission of an optical fiber can be valued through the input optical power, $P(0)$, and the output optical power, $P(z)$, seen after light oscillates a distance, z , through the fiber length. It is established as

$$P(z) = P(0) \exp(-\alpha_{total}z), \quad (2.4)$$

and

$$\%T = \frac{P(z)}{P(0)}, \quad (2.5)$$

where α_{total} is the total attenuation coefficient, $\%T$ is described as the percentage optical power transmission. The Eq.(2.5) is described as the Beer's law and depicts that the transmitted optical power exponentially reduces with the oscillation distance

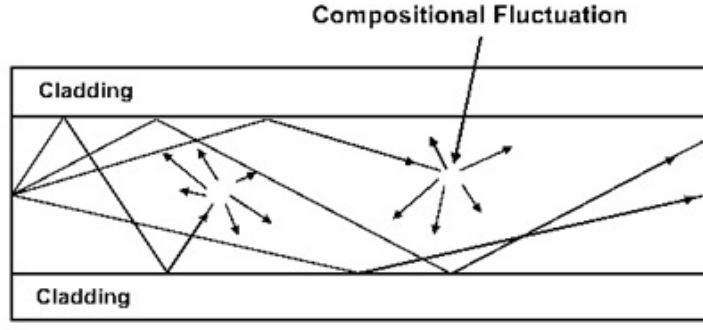


Figure 2.2: Attenuation in an optical fiber[112]

through the optical fiber [113]. The optical fiber's attenuation coefficient (which is sometimes referred to as the fiber loss) is expressed as follows

$$\alpha = \frac{10}{z(km)} \log \left(\frac{P(z)}{P(0)} \right). \quad (2.6)$$

Attenuation could happen in the form of absorption, scattering and other extrinsic effects. Absorption is uniform in order that, the identical amount of the material in question usually absorbs almost the identical fraction of light at similar wavelength. Let us for instance, take a scenario where one has three blocks of the same type of glass, with each of the blocks 1cm thick. Then, the three blocks are said to absorb the same fraction of the light passing through them, thus such action is known as absorption. It could also be formalized as the intrinsic material absorption which is due to the interaction of the pulse generation with one or more major components of the material or glass that constitute the optical fiber material composition. Another form of absorption is the extrinsic impurity ion absorption, which is mostly formed by the availability of a quantity of metallic ions from water dissolved in glass or material[100, 111]. Scattering losses occur due to the interaction of a light beam with a particle such that the energy losses in the direction of propagating wave and transferring it to other directions [111]. Rayleigh scattering is one of the pure effects that cause attenuation of light in glass. Whereby fiber fabrication processes are formed from small scale inhom-

geneities. We expect variations to be random and local such that the glass also creates random inhomogeneities in the refractive index that behave as really small scattering centers, also it accomodate approximately 96% of attenuation in optical fiber. Silica molecule in the core interact with the light as it travels[114, 115]. The direction of the propahation of light depends on the incident angle, that is whether the light will propagate forward, or escape the core, or will be reflected back toward the light source[99, 100, 111, 116]. Earlier on, it is mentioned that attenuation and dispersion are the main performance limiting factors of the optical fiber communication systems. But, what is dispersion and how does it affect the optical fiber? Dispersion could be described as the spreading of the pulse in an optical fiber. It is known that as light travels through the fiber, certain aspects like the numerical aperture, core diameter, refractive index profile, wavelength, and laser line width lead to the broadening of the pulse[117].

The effects of optical fibers such as Rayleigh scattering, absorption and attenuation are all linear effects of optical fibers. Dispersion contributes to both linear and nonlinear effects of optical fibers. The linear effects of dispersion are modal dispersion (which has been afore mentioned), chromatic dispersion and polarization mode dispersion. Chromatic Dispersion (CD) is well knowledgeable as form whereby the spreading of the pulse occurs, caused by various distinct wavelengths of light propagating with not much difference in velocity through the fiber, because of the glass fiber's refraction index which is a wavelength reliant quantity. A much simpler explanation is that it transpires when different wavelengths propagate at different velocities[117, 118]. Chromatic dispersion is said to cause performance limits of single mode fibers, hence more innovative fiber designs have been made aiming at reducing the effect with graded index cores, selected refractive index profiles, such that the wavelength at which the waveguide dispersion compensates material dispersion is shifted to the wavelength at which the fiber is to be used, leading to the dispersion-shifted fibers (which have been succesfully fabricated)[100]. Polarisation mode dispersion (PMD) usually happens

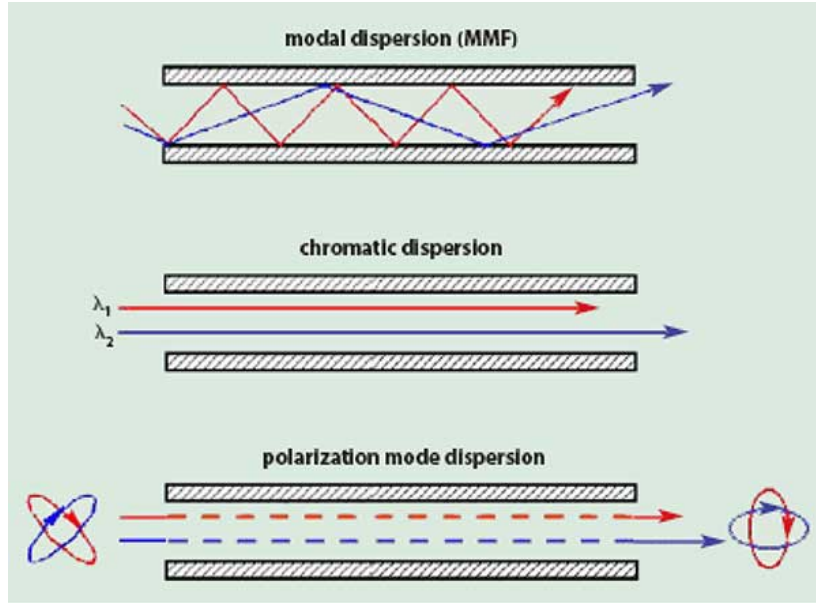


Figure 2.3: Different types of linear dispersion.[112]

when various planes of light inside a fiber travel at different speeds making it impossible to transmit data reliably at high speeds[119]. Figure 2.3 alludes more on the three types of linear dispersion we have explicitly mentioned.

2.3 Nonlinear effects of optical fibers

The nonlinear effects that have been regarded generally arise owing to the refractive index dependence of electric field on the intensity which is proportional to the square field amplitude[121, 122]. At considerably high optical light intensities, the non-linear refraction in the core (Kerr effect), which is described as the differences of the refraction index with light intensity. Therefore, nonlinear impairments become much more practical problem in optical networks since long-haul transmission commonly relies on high power lasers to transmit optical pulses over long spans to overcome attenuation[123, 124]. Nonlinear impairments rely mostly on the fiber type, length and could be categorized into two classes. Firstly, nonlinear effects that mainly affect the energy of an optical pulse known as; stimulated Brillouin scattering (SBS), stim-

ulated Raman scattering (SRS) and four-wave mixing (FWM). SBS is described as a narrow band effect that is related to data channels operating in the terahertz range, thus resulting in much short wavelengths causing amplifications of longer wavelengths but in fact depleting themselves. SBS is concerned with incident light interacting with acoustic waves in the silica glass erupting down-shifted scattered light in a manner which is similar to SRS[125, 126]. The SRS simply deals with two or more signals propagating at different wavelengths inside the fiber, This causes power transmissions from lower to higher wavelength channels[127]. The incident light is said to interact with the molecular vibrations in a fiber and scattered light is therefore produced as down-shifted by the Stokes frequency [128, 129].

In optical waves we expect the probe frequency wave to experience optical gain produced by the pump frequency wave, that is if the waves are separated by the Stokes frequency in a Raman active medium [130–133]. This gain phenomenon elucidates the SRS where the peak gain is inversely proportional to the pump wavelength. FWM, in general terms, is a photon-photon scattering phenomenon, the process involves scattering of photons with averagely high intensity pulses, called pump, the third-order nonlinearity, $\chi^{(3)}$ of the material (silica glass) as a result producing two daughter photons named the signal and idler photons, respectively[134, 135]. The angular frequencies $\omega_1, \omega_2, \dots, \omega_n$ and the intensity reliance on the refractive index does not only cultivate phase shifts within a channel but also generate signals at frequencies that are new such as $2\omega_i - \omega_j$ and $\omega_i + \omega_j - \omega_k$ [136, 137]. Such a phenomenon is known as the four-wave mixing. FWM effect is mostly dependent on the bit rate and channel spacing and fiber dispersion, therefore, causing a reduction in the channel-spacing causes an increase in the FWM effect, and thus decreasing the dispersion. Ordinarily, most signals propagated by FWM are said to have low powers since they lack exact phase matching and the attenuation of signals caused by fiber loss. The frequency elements are produced in this manner are referred to as FWM products.

Self-Phase Modulation (SPM) and Cross-Phase Modulation (XPM) are part of the nonlinear effects that mostly affect the shape of an optical signal. The main focus of this dissertation is mainly on single-mode optical fibers in which SPM, even though a single light pulse might be affected by this nonlinearity due to the fact that the phase is modulated by optical intensity fluctuations of a similar pulse. Such nonlinearity effect well defined as Self-Phase Modulation (SPM). The interaction of the SPM and the GVD effect produce quite unique spectral elements of optical pulse at which is said to propagate at different speeds. Such effect in a temporal system causes pulse broadening therefore it in turn leads to an overlap between adjacent bits hence bit error rate from such increase. While in XPM, the invention of WDM is the root to having multiple signals that could be transmitted on the same fiber. Particularly in a WDM system the optical intensity or power changes of a optical pulse generated in an optical fiber induces the phases of other propagating signals through the XPM process. XPM is known to develop from the same conditions as SPM, the difference that the XPM is more concerned with multi channel transmission whereas SPM effect focusses more on single channel transmission systems. XPM, is a nonlinearity effect that causes more limitations than SPM for WDM systems since it is presumed that it elludes for more channels to generate the phase shift and intensity changes[138].

The Kerr effect which is a nonlinear optical effect which happens when light oscillates in crystals, glasses as well other media and is caused by the refractive index's reliance on power. The Kerr nonlinearity expresses itself according to the reliance with the input signal in three distinct types. The physical nature of the nonlinear polarization usually causes modifications in the light pulse behaviour[139]. The stimulated effects such as SBS and SRS is modulated by the inelastic scattering at a much higher power level. Most nonlinear effects give rise to the gains of channels with the expense of depleting power from other channels excluding the SPM and the XPM which affect

phase signals that may bring about the spectral broadening hence increasing dispersion [121, 139–142]. Any dielectric medium expresses itself as a nonlinear system for the case of enormous electromagnetic fields. Originally the nonlinearity is found in the harmonic motion of bound electrons caused by an applied field. Polarization $\vec{P}$ in total is modulated by electric fields is demonstrated as follows

$$P = \epsilon_0 \chi^{(1)} E + \epsilon_0 \chi^{(2)} E^2 + \epsilon_0 \chi^{(3)} E^3 + \dots, \quad (2.7)$$

where ϵ_0 is the permittivity of vacuum and $\chi^{(k)}$ ($k = 1, 2, \dots$) is the k^{th} -order susceptibility. The polarizing P is governed by linear susceptibility $\chi^{(1)}$. The second order susceptibility $\chi^{(2)}$ eludes for the second harmonic generation and sum-frequency generation. Significantly, optical fibers do not hold for the second order nonlinear refractive effects. It is important to mention that, the electric quadrupole and magnetic-dipole moments normally brings about weak second-order nonlinear effects. It is worth mentioning, that the third-order susceptibility $\chi^{(3)}$ generates lowest-order nonlinear effects in fibers[26]. Optical fibers are considered to be an isotropic medium, therefore, polarization vector $\vec{P}$ in every situation is in the direction of the electric field vector $\vec{E}$. In an electric field,

$$\vec{E} = E_0 \cos(\omega t - kx), \quad (2.8)$$

resulting in the polarisation equation being,

$$\begin{aligned} \vec{P} = & \epsilon_0 \chi^{(1)} E_0 \cos(\omega t - kz) + \epsilon_0 \chi^{(2)} E_0^2 \cos^2(\omega t - kz) + \\ & \epsilon_0 \chi^{(3)} E_0^3 \cos^3(\omega t - kz) + \dots \end{aligned} \quad (2.9)$$

With the use of some trigonometric relations, the polarization is given by

$$\begin{aligned}\vec{P} = & \frac{1}{2}\varepsilon_0\chi^{(2)}E_0^2 + \varepsilon_0\chi^{(1)} + \frac{3}{4}\chi^{(3)}E_0^2E_0\cos(\omega t - kz) + \\ & \frac{1}{2}\varepsilon_0\chi^{(2)}E_0^2\cos 2(\omega t - kz) + \frac{1}{4}\varepsilon_0\chi^{(3)}E_0^3\cos 3(\omega t - kz) + \dots\end{aligned}\quad (2.10)$$

Most effects of first and second terms are not so important they are constant terms which produces DC field across the medium. The third term propagating at frequency ω is analyzed as first or most significant harmonic of polarization. The fourth term propagating with frequency 2ω is known as the second harmonic of polarization. With no exemption, the fifth term with frequency 3ω is eludes for the third harmonic of polarization. For optical fibers, $\chi^{(2)}$ is ignored or neglected, thus, the equation below

$$\vec{P} = \varepsilon_0\chi^{(1)} + \frac{3}{4}\chi^{(3)}E_0^2E_0\cos(\omega t - kz) + \frac{1}{4}\varepsilon_0\chi^{(3)}E_0^3\cos 3(\omega t - kz). \quad (2.11)$$

Due to the differences in the refractive index of the fiber, there is no lack of phase between frequencies ω and 3ω which creates a phase mismatch on the second term of the above equation resulting in

$$P = \varepsilon_0\chi^{(1)}E_0\cos(\omega t - kz) + \frac{3}{4}\varepsilon_0\chi^{(3)}E_0^3\cos(\omega t - kz), \quad (2.12)$$

which is an equation which contains both linear and nonlinear terms of the polarizations from the first and second terms respectively [26, 122]. In order to understand FMW effect [143][122], a WDM signal is considered as the sum of n monochromatic waves. An electric field of this kind of field can be expressed as

$$E = \sum_{p=1}^n E_p \cos(\omega_p t - k_p z), \quad (2.13)$$

with the nonlinear polarizaton taken as

$$P_{nl} = \epsilon_0 \chi^{(3)} E^3, \quad (2.14)$$

in which P_{nl} takes the form of

$$P_{nl} = \epsilon_0 \chi^{(3)} \sum_{p=1}^n \sum_{q=1}^n \sum_{r=1}^n E_p \cos(\omega_p t - k_p z) E_q \cos(\omega_q t - k_q z) E_r \cos(\omega_r t - k_r z), \quad (2.15)$$

its expansion is given as

$$\begin{aligned} P_{nl} = & \frac{3}{4} \epsilon_0 \chi^{(3)} \sum_{p=1}^n E_p^2 + 2 \sum_{q \neq p} E_p E_q E_p \cos(\omega_p t - k_p z) \\ & + \frac{1}{4} \epsilon_0 \chi^{(3)} \sum_{p=1}^n E_p^3 \cos(3\omega_p t - 3k_p z) + \\ & \frac{3}{4} \epsilon_0 \chi^{(3)} \sum_{p=1}^n \sum_{q \neq p} E_p^2 E_q \cos((2\omega_p - \omega_q)t - (2k_p - k_q)z) + \\ & \frac{3}{4} \epsilon_0 \chi^{(3)} \sum_{p=1}^n \sum_{q \neq 1} E_p^2 E_q \cos((2\omega_p - \omega_q)t - (2k_p - k_q)z) + \\ & \frac{6}{4} \epsilon_0 \chi^{(3)} \sum_{p=1}^n \sum_{q > p} \sum_{r > q} E_p E_q E_r \cos((2\omega_p + \omega_q + \omega_r)t - (k_p + k_q + k_r)z) + \\ & \cos((\omega_p + \omega_q + \omega_r)t - (k_p + k_q + k_r)z) + \\ & \cos((\omega_p - \omega_q + \omega_r)t - (k_p - k_q + k_r)z) + \\ & \cos((\omega_p - \omega_q - \omega_r)t - (k_p - k_q - k_r)z). \end{aligned} \quad (2.16)$$

The Eq. (2.16's first terms well represent the SPM and the XPM effects. The second, third and fourth terms would be ignored due to phase mismatch, for the main reason that in most cases real fibers $k(3\omega) \neq 3k(\omega)$ thus the differences $(3\omega - 3k)$ are defined as a phase mismatch. To have a deeper understanding of the phase mismatch as a mismatch of phases between distinct signals that travel through the fiber at distinct group velocities. The last terms represent the FMW phenomenon [148]. Those are the

terms tell that three EM waves generated in a fiber produces pulses with frequencies $(\omega_p \pm \omega_q \pm \omega_r)$ [149]. Four-wave mixing (FWM) is well stipulated analogous to inter-modulation alteration in electrical systems. Then lastly, the polarization quantities explain that FWM is from frequencies like $(\omega_p + \omega_q - \omega_r)$.

The intensity for a plane wave is taken as

$$I = \frac{1}{2} c \epsilon_0 n_l E_0^2. \quad (2.17)$$

The velocity of light is represented by c and n_l is the linear refractive index of the medium at two fields resulting in the polarization being

$$P = \epsilon_0 \chi^{(1)} + \frac{3}{2} \frac{\chi^{(3)}}{c \epsilon_0 n_l} I E_0 \cos(\omega t - k z). \quad (2.18)$$

In the following describe the effective susceptibility and refractive index in which the effective susceptibility of the medium is described as

$$\chi_{eff} = \frac{P}{\epsilon_0 E} = \chi^{(1)} + \frac{3}{2} \frac{\chi^{(3)}}{c \epsilon_0 n_l} I, \quad (2.19)$$

and the effective refractive index as

$$n_{eff} = (1 + \chi_{eff})^{\frac{1}{2}}. \quad (2.20)$$

If we introduce the effective susceptibility in the above equation with the notion that the linear refractive index n_l is $n_l = (1 + \chi^{(1)})^{\frac{1}{2}}$. The second term of the resulting equation $n_{nl} = \frac{3}{2} \frac{\chi^{(3)}}{c \epsilon_0 n_l}$, where n_{nl} is the nonlinear refractive index. For fused silica $n_l \approx 1.46$ and $n_{nl} \approx 3.2 \times 10^{-20} m^2/W$. Nonlinear terms account for soliton formation.

The nonlinear effects rely on transmission length. An optical pulse is known to oscillate along the link length, therefore, it reduces the power in response to the of fiber attenuation[143][122]. The optical power defined as $P(z) = P_{in} \exp(-\alpha z)$ is

described travelling at distance z through the fiber link, P_{in} is described as power input (when $z = 0$) and α as the coefficient of attenuation. The link length (L), it's effective length is expressed as,

$$P_{in}L_{eff} = \int_{z=0}^L P(z)dz. \quad (2.21)$$

Using Eq. (2.21) on optical power and effective length, the effective link length is obtained as,

$$L_{eff} = \frac{(1 - \exp(-\alpha z))}{\alpha}. \quad (2.22)$$

Due to the communication fibers being are long enough, we must have $L \gg 1/\alpha$. This follows in $L_{eff} \approx 1/\alpha$. The link length L in relation to amplifiers spaced l distance apart, the effective length is approximated by

$$L_{eff} = \frac{(1 - \exp(-\alpha z))}{\alpha} \frac{L}{l}. \quad (2.23)$$

Power is known to be evenly distributed through the cross-section of the fiber, it is without a doubt that effective cross-sectional area (A_{eff}) could be used. The A_{eff} in relation to the actual area (A) and the cross-sectional distribution of intensity $I(r, \theta)$ in described as follows [122, 143]

$$A_{eff} = \left[\frac{\int_r \int_\theta r dr d\theta I(r, \theta)}{\int_r \int_\theta r dr d\theta I^2(r, \theta)} \right] \quad (2.24)$$

The polar coordinates are denoted by r and θ , respectively. The optical phases change with time similar to the optical signal [139]. This nonlinear process is actively expressed in the SPM. Several parts of the pulse experience distinct phase shifts, due to the reliance of intensity on phase changes hence producing frequency chirping. The effect of SPM in broadening the spectrum of the signal [144], maintaining the temporal shape

undistorted. Due to the effect of chirping which is proportional to the transmitted signal power. The phase shift ϕ of field E through fiber length L is shown as

$$\phi = \frac{2\pi}{\lambda} nL, \quad (2.25)$$

where λ is the wavelength of the optical pulse propagating in fiber of refractive index n , and nL is known as the optical path length. The intensity could be time-reliant, that is when the light beam is temporarily modulated, then the phase ϕ , will rely on time [145], which brings about change in frequency spectrum

$$\omega = \frac{d\phi}{dt}. \quad (2.26)$$

As we have previously explained SPM is concerned with the intensity reliance of refractive index. Therefore, fluctuations in signal intensity cause changes to the signal phase. These changes in phase induce additional chirps, which lead to dispersion penalty that is considerably small if the input power is much less compared to a given threshold value. The accurately known chirping of the input pulses could be of significance in the reduction of the SPM effects. Hence, chirped RZ (return to zero) or CRZ (chirped return to zero) modulation can be employed. The power reliance on the nonlinear phase constant (ϕ_{NL}) is supposed to impact on communication systems [26][146]. As propounded beforehand, when two or more optical pulses oscillate simultaneously, the XPM usually in the company of the SPM and exists not only from the nonlinear refractive index but of also other co-propagating beams[147]. Most importantly, the effective refractive index of a nonlinear medium could be described by the input power (P) and the the effective core area (A_{eff}) as

$$n_{eff} = n_l + n_{nl} \frac{P}{A_{eff}}. \quad (2.27)$$

The nonlinear effects also rely on ratio of light power to the cross sectional area of

the fiber [122]. Furthermore, it has been well alluded that the mode shape does not fluctuate but the propagation constant becomes more power reliant,

$$k_{eff} = k_l + k_{nl}P, \quad (2.28)$$

where k_l is the linear portion of the propagation constant. Nonlinear propagation responsible for the phase shift travels at a distance L inside the fiber, is represented by

$$\phi_{NL} = \int_0^L (k_{eff} - k_l)dz, \quad (2.29)$$

leading to the nonlinear phase shift being

$$\phi_{NL} = k_{nl}P_{in}L_{eff}. \quad (2.30)$$

It was also found that when different optical pulses oscillate simultaneously, the nonlinear phase shift of the first channel relies on the power and the signal power of other channels. For the case of two channels, the phase shift of the first channel is given as

$$\phi_{NL}^i = k_{eff}L_{eff}(P_1 + 2P_2). \quad (2.31)$$

For N -channels transmission system, the shift for the i th channel can be given as[144]

$$\phi_{NL}^i = k_{nl}L_{eff}\left(P_i + 2\sum_{n \neq i}^N P_n\right). \quad (2.32)$$

The factor 2 in Eq.(2.32) has its origin in the form of nonlinear susceptibility [26] and is supposed to indicate that the XPM is twice as effective as the SPM for the amount of power. The first term represents the influence of SPM and the second term that of XPM. [122].

As shown in Table 2.1, we have only discussed nonlinear effects based on the Kerr-effect. The other important nonlinear effects that we should mention are the second-order and third-order nonlinear optical effects. Second-order nonlinear optical effects are not represented in the complementary metal-oxide semiconductor (CMOS) compatible materials (e.g. SiSiN, SiO₂ and Ge). This comes from the knowledge that these materials are centro-symmetric, hence, second-order nonlinear $\chi^{(2)}$ susceptibility, in a centro-symmetric media is found in the electric-dipole [150]. This becomes a problem for second-order electro-optic effects or second harmonic generation (SHG) in the CMOS compatibility area. $\chi^{(2)}$ in silicon is seen through material engineering. The nonlinear optics was established when SHG was discovered by Franken et. al. in 1961[151]. It was further realized that at high intensity, light-matter interaction through the microscopic properties (polarizability, absorption cross-section, and lifetime), led the changes in the macroscopic properties (polarization, absorption, and susceptibility) as a function of interacting intensity [152–155]. These intensity reliant parameters can be measured by several nonlinear techniques [154–158]. From the intensity-dependent macroscopic parameters, one can estimate the susceptibility. The susceptibility tells the material behavior as a function of wavelength and pulse width of the optical fiber. The oscillation of an electromagnetic wave in a nonlinear medium is defined as

$$\nabla^2 \vec{E}(r, t) = \frac{1}{c^2} \frac{\partial^2 \vec{E}(r, t)}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{P}(r, t)}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{P}_{NL}(r, t)}{\partial t^2}. \quad (2.33)$$

Because of symmetry, the odd-order susceptibilities are observed in any material, however the even-order ones only occur in non-centrosymmetric materials.

$$\vec{P}_{NL} = \chi^{(1)} E(t) + \chi^{(2)} E^2(t) + \chi^{(3)} E^3(t) \quad (2.34)$$

A graphical representation of the significant nonlinearities is shown in Figs 2.4 and 2.5. In general, nonlinear optical properties allow for various frequency conversion effects

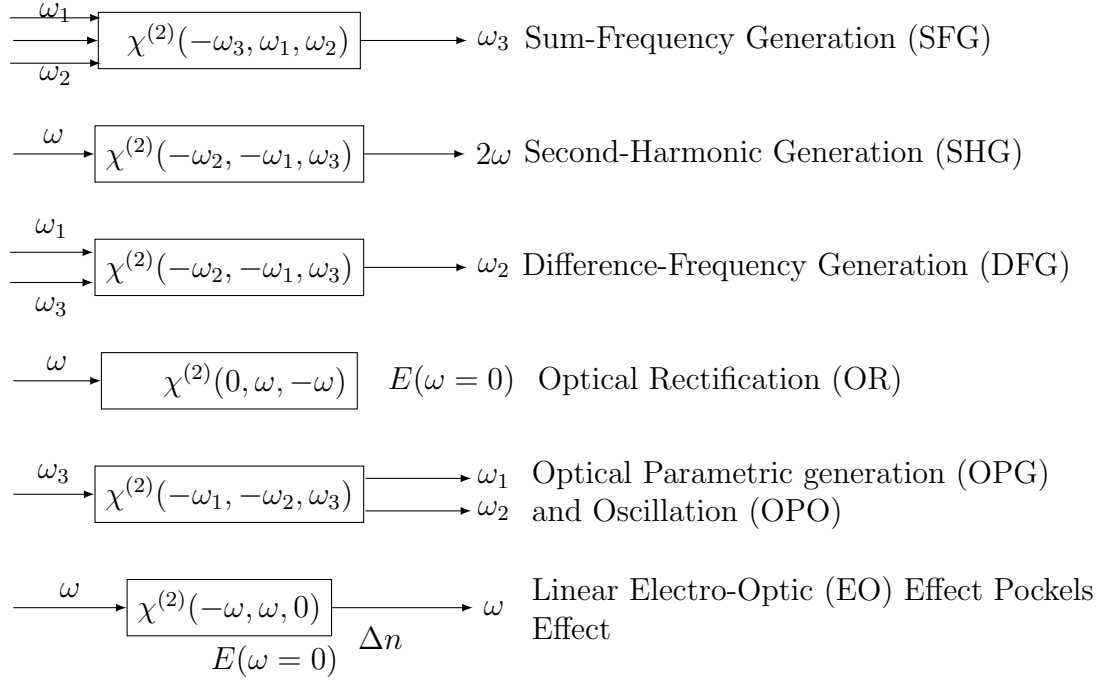


Figure 2.4: Schematic representation of important nonlinear effects of the second-order $\chi^{(2)}$.

(sum-frequency generation, difference-frequency generation, optical parametric generation, etc.), and generation of static or quasi-static electric fields (optical rectification, THz-wave generation), as well as for the field-induced change of the linear optical properties (electro-optic effect, optical Kerr effect), i.e., the field-induced change of the linear material susceptibility $\chi^{(1)}$ at optical frequencies. The electro-optic effect is a nonlinear optical effect, for which one of the applied fields is of low frequency (DC to THz) compared with the optical field. Sum-frequency is generated by the mixing of two incident light pulses of frequencies ω_1 and ω_2 creating a wave of $\omega_3 = \omega_1 + \omega_2$. This is in the form of nonlinear polarization

$$P_j^{\omega_3} = \varepsilon_0 \chi_{ijk}^{(2)}(-\omega_3, \omega_1, \omega_2) E_j^{\omega_1} E_k^{\omega_2}. \quad (2.35)$$

The nonlinear polarization for SHG can be shown by the use of the nonlinear optical coefficient d_{ijk} , d_{ijk} characterizes macroscopic samples that usually describes nonlinear

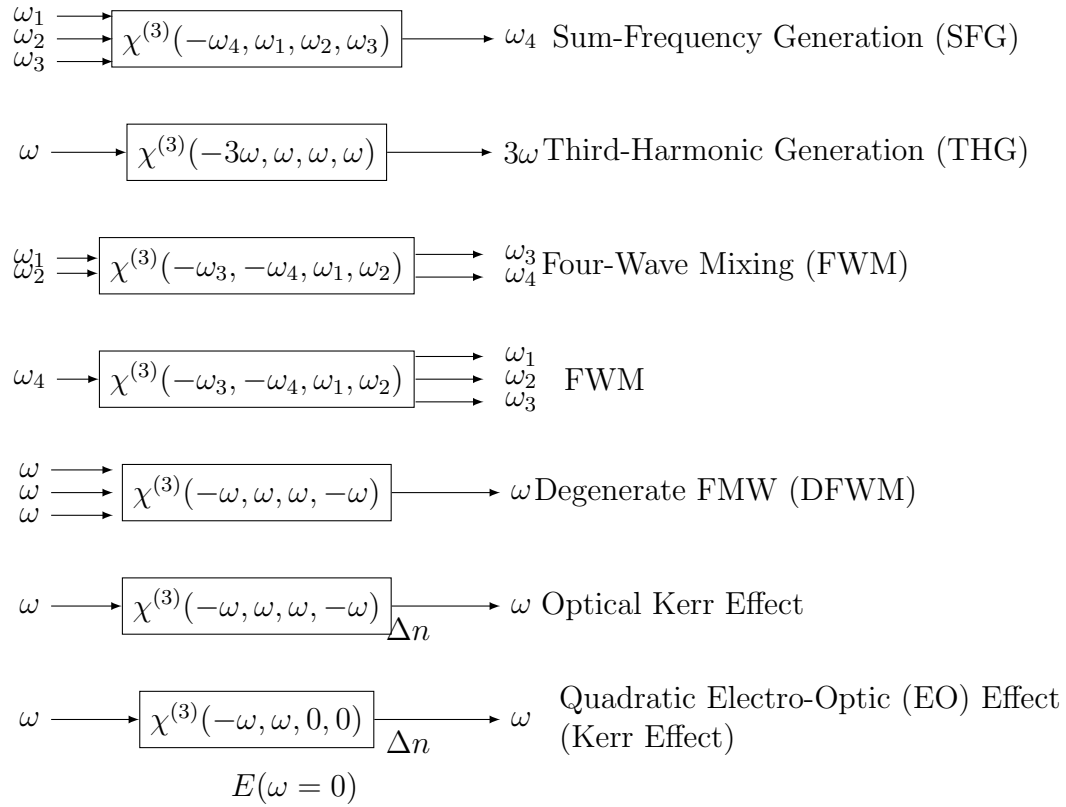


Figure 2.5: Schematic representation of important nonlinear effects of the third-order $\chi^{(3)}$.

optical characterization as follows

$$P_j^{2\omega} = \frac{1}{2}\varepsilon_0\chi_{ijk}^{(2)}(-2\omega, \omega, \omega)E_j^\omega E_k^\omega = \varepsilon_0d_{ijk}(-2\omega, \omega, \omega)E_j^\omega E_k^\omega. \quad (2.36)$$

Sum-frequency and second-harmonic generation are general ways used to make new rational output that occurs from laser systems with the mandate to get the short wavelength range towards the ultraviolet region. The efficiency for harmonic generation can be calculated using the nonlinear wave equation (2.33) with the nonlinear polarizations given above. The nonlinear polarization for the difference frequency generation can be written as

$$P_i^{\omega_2} = \varepsilon_0\chi_{ijk}^{(2)}(-\omega_2, \omega_3, -\omega_1)E_j^{\omega_3} E_k^{*\omega_1}. \quad (2.37)$$

The pulse with the frequency ω_3 is commonly the strongest and therefore called the pump wave. Optical parametric generation (OPG) is a unique case of difference frequency generation, pump pulse is incident on the nonlinear material, producing two waves at the frequencies ω_1 and ω_2 . In order to improve the efficiency of either phenomenon, the nonlinear medium can be put in a cavity with highly reflecting mirrors for the frequencies ω_1 and/or ω_2 ; for such, the phenomena is referred to as optical parametric oscillation (OPO). In contrast to sum-frequency generation, difference-frequency generation and optical parametric oscillation are well-suited to coherent light production at longer wavelengths, i.e., near and mid infrared region. Some other application is optical parametric amplification where the strong pump beam at frequency ω_3 transfers energy with the amplification of an optical signal at frequency ω_1 . In such a particular case, the output at frequency ω_2 remains as it is and thus often redefined as the idler beam. Optical parametric generation and oscillation are of unique significance due to their ability to permit single frequency laser to be used to create optical output that is tunable over a large number of wavelengths. Tuning is accomplished by modifying the phase-matching condition using, for example, angle or temperature changes.

Optical rectification (OR) is a unique form of difference-frequency generation. Similarly, as in the process of SHG, only one optical field of frequency ω is incident. In OR, this degenerate wave mixing generates a static polarization in the material according to

$$P_i^0 = \frac{1}{2}\varepsilon_0\chi_{ijk}^{(2)}(0, \omega, -\omega)E_j^\omega E_k^{*\omega}. \quad (2.38)$$

Optical rectification can be used, for example, to generate waves at THz frequencies using an ultrashort laser pulse with a large bandwidth as a pump source. The frequency components of such pulses are different from each other to produce nonlinear polarization from 0 to several THz (2.5). In centrosymmetric media, the second-order nonlinear term is absent. The dominant nonlinearity is of the third-order

$$P_i^{NL} = \varepsilon_0\chi_{ijkl}^{(3)}E_jE_kE_l, \quad (2.39)$$

and gives rise to the third-harmonic generation and related mixing phenomena. The third-order response of any medium to a monochromatic optical field of frequency ω consists of a nonlinear polarization containing ω and 3ω frequency components

$$P_i^{3\omega} = \frac{1}{4}\varepsilon_0\chi_{ijkl}^{(3)}(-3\omega, \omega, \omega, \omega)E_j^\omega E_k^\omega E_l^\omega, \quad (2.40a)$$

$$P_i^{3\omega} = \frac{3}{4}\varepsilon_0\chi_{ijkl}^{(3)}(-\omega, -\omega, \omega, \omega)E_j^{*\omega} E_k^\omega E_l^\omega. \quad (2.40b)$$

The first process is known as third-harmonic generation (THG). Similar to the process of second-harmonic generation, the efficiency of THG is negligible unless the phase matching conditions are satisfied. The second effect, the generation of the polarization component at frequency ω , is equivalent to changing the effective refractive index according to

$$\Delta n(\omega) \cong \frac{3\chi^{(3)}(-\omega, -\omega, \omega, \omega)}{8n_0(\omega)}|E^\omega|^2, \quad (2.41)$$

where $n_0(\omega)$ is the unperturbed index at frequency ω (the tensor notation is omitted

for simplicity). The relation (2.41) can be easily derived if we insert the nonlinear polarization (2.40b) into the wave equation for the production of an optical field inside a medium [159, 160]. The whole refractive index is proven to be a linear function of the optical intensity ($I \propto |E^\omega|^2$) given by

$$n(I) = n_0 + n_2 I. \quad (2.42)$$

This effect is known as the optical Kerr effect. It is a self-modulated effect, in which the phase velocity of the wave depends on the wave's intensity. As a result, a wide variety of important processes occur, such as self-phase modulation and self-focusing effects.

2.4 Summary

Table 2.1: Table showing comparison on the different nonlinear effects based on the Kerr-effect [122]

Nonlinear phenonon characteristics $\rightarrow$ $\downarrow$	SPM	XPM	FWM
1. Bit rate	Dependent	Dependent	Independent
2. Origin	Nonlinear Susceptibility $\chi^{(3)}$	Nonlinear Susceptibility $\chi^{(3)}$	Nonlinear Susceptibility $\chi^{(3)}$
3. Effects of $\chi^{(3)}$	Phase shift due to pulse itself only	Phase shift is alone due to co-propagating signals	New waves are generated
4> Shape of broadening	Symmetrical	Maybe symmetrical	-
5. Energy transfer between medium and optical pulse	No	No	No
6. Channel Spacing	No effect	Increases on decreasing the space	Increases on the decreasing the space

It is of utmost importance that before we move forward to the optical pulse propagation equations, the linear together with the nonlinear effects of dispersion be explicitly presented and their contribution is illustrated. Henceforth, in this chapter, the main linear dispersion effects have alluded as chromatic, modal, polarization mode dispersion, and attenuation which result in physical distortion of the optical fibers. Special attention to the nonlinear effects which have been carefully elaborated on, since, they are of important significance in the spectral components of the optical pulse, formation of optical solitons.

Chapter 3

Nonlinear Schrödinger (NLS) equation

3.1 Introduction

It has long been introduced, that nonlinear effects play a very significant part in the pulse propagation of optical fibers in the theoretical and experimental essence. Several effects associated with the fiber nonlinear index have been used for constructive purposes, e.g., the use of soliton pulses for undistorted or much accurate pulse propagation over long distances[16, 80] pulses and the creation of ultrashort pulses by means of nonlinear pulse-compression techniques. The structure of a fiber-optic communication model requires much deeper knowledge of the nonlinear propagation of optical pulses, with emphasis on fiber losses and fiber dispersion[161] which have been thoroughly introduced in Chapter 2. Following suit, the optical pulse propagation equation will be presented.

3.2 Presentation of the standard NLS equation

The analysis of the optical signal propagation for an optical fiber is mostly investigated through Maxwell's equations,[98],

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (3.1)$$

which is known as Maxwell-Faraday equation,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \quad (3.2)$$

Maxwell-Ampere equation and

$$\nabla \cdot \vec{D} = \rho_f, \quad (3.3a)$$

$$\nabla \cdot \vec{B} = 0, \quad (3.3b)$$

which are Maxwell-Gauss equations. Here, $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields vectors respectively, $\vec{D}$ and $\vec{B}$ are the corresponding electric and magnetic flux densities, $\vec{J}$ and ρ_f are the current density vector and the charge density. Optical fibers are a medium without free charges therefore $\vec{J} = 0$ and $\rho_f = 0$. This is accompanied by the constitutive relations

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}, \quad (3.4a)$$

$$\vec{B} = \mu_0 \vec{H} + \vec{M}, \quad (3.4b)$$

where ε_0 represents vacuum permittivity, μ_0 normally stands for the vacuum permeability and $\vec{P}$ and $\vec{M}$ are the induced electric and magnetic polarisations. Since optical fibers are a nonmagnetic medium, then $\vec{M} = 0$. Maxwell's equations are the main skeleton to attaining the wave equation used to define the propagation of light in optical fibers. The wave equation is derived by first taking the curl of the Maxwell-Faraday equation and using the right hand side of the second constitutive relation

results in

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}, \quad (3.5)$$

c is the speed of light in a vacuum defined as $c = 1/\sqrt{\varepsilon_0 \mu_0}$. $\vec{P}$ is the induced polarization as mentioned above and can be separated into linear and nonlinear parts [161, 162] as

$$\vec{P}(\vec{r}, t) = \vec{P}_L(\vec{r}, t) + \vec{P}_{NL}(\vec{r}, t), \quad (3.6)$$

with linear and nonlinear polarizations defined below

$$\vec{P}_L(\vec{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \chi^{(1)}(t - t') \cdot \vec{E}(\vec{r}, t) dt', \quad (3.7)$$

$$\vec{P}_{NL}(\vec{r}, t) = \varepsilon_0 \int \int \int_{-\infty}^{\infty} \chi^{(3)}(t - t_2, t - t_2, t - t_3) : \vec{E}(\vec{r}, t_1) \vec{E}(\vec{r}, t_2) \vec{E}(\vec{r}, t_3) dt_1 dt_2 dt_3, \quad (3.8)$$

As previously explained, $\chi^{(1)}$ and $\chi^{(3)}$ are the first and third order susceptibilities of the fiber, one would notice that the second order susceptibility $\chi^{(2)}$ is ignored and that is due to the fact that an optical fiber possesses inversion symmetry[26, 162]. From the above equations using the second derivatives of vector identities [163] we obtain

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{P}_{NL}}{\partial t^2}. \quad (3.9)$$

A propagation equation is developed from above but several assumptions are made regarding the nonlinearity of the system[26]. Such as;

1. The nonlinear-induced polarization is treated as perturbation since it is small.
2. The optical field can retain its polarization along the fiber length, with the aid of the scalar approach.

3. The optical field is defined as quasi-monochromatic therefore, the spectral width is $\delta\omega$ is small in comparison with its center frequency ω_0 , i.e. $\delta\omega/\omega_0 \ll 1$.

The approximation of the slowly varying envelope is used to divide the constantly changing side of the electric field by re-writing the field as

$$\vec{E}(\vec{r}, t) = \frac{1}{2}\hat{x}[E(\vec{r}, t)e^{-i\omega_0 t} + c.c.], \quad (3.10)$$

$$\vec{P}_L(\vec{r}, t) = \frac{1}{2}\hat{x}[P_L(\vec{r}, t)e^{-i\omega_0 t} + c.c.], \quad (3.11)$$

$$P_{NL}(\vec{r}, t) = \frac{1}{2}\hat{x}[P_{NL}(\vec{r}, t)e^{-i\omega_0 t} + c.c.], \quad (3.12)$$

c.c. is the complex conjugate, $\hat{x}$ is the polarization unit vector of the light with the assumption that is polarized linearly along the x-axis and $E(\vec{r}, t)$ is well defined as the slowly varying function with respect to optical carrier frequency ω_0 . It is not difficult to work within the Fourier domain, therefore, performing Fourier transform on the slowly varying function

$$\tilde{E}(\vec{r}, \omega - \omega_0) = \int_{-\infty}^{\infty} E(\vec{r}, t)e^{i(\omega - \omega_0)t} dt, \quad (3.13)$$

the linear polarization simply results to

$$\tilde{P}_L(\vec{r}, \omega) = \varepsilon_0 \chi^{(1)} \tilde{E}(\vec{r}, \omega). \quad (3.14)$$

The nonlinear-induced polarization is broken down by the assumption that the nonlinear response is instantaneous, hence the nonlinear polarization equation reduces to

a delta function response,

$$\vec{P}_{NL}(\vec{r}, t) = \varepsilon_0 \chi^{(3)} : \vec{E}(\vec{r}, t) \vec{E}(\vec{r}, t) \vec{E}(\vec{r}, t), \quad (3.15)$$

the vertical dots describe the tensor manner of the third-order susceptibility. The nonlinear polarization is established by treating the field as monochromatic waves[150]. All the fields in this section are treated as scalar variables

$$\vec{E} = E \cos(\omega_0 t). \quad (3.16)$$

With that taken into consideration

$$\begin{aligned} P_{NL} &= \varepsilon_0 \chi^{(3)} E^3 \cos^3(\omega_0 t) \\ &= \varepsilon_0 \chi^{(3)} E^3 \left[\frac{1}{4} \cos(3\omega_0 t) + \frac{3}{4} \cos(\omega_0 t) \right]. \end{aligned} \quad (3.17)$$

The nonlinear induced polarization Eq. (3.15) is discovered to be propagating with the frequency ω_0 as well as at the third harmonic $3\omega_0$. Since, the third harmonic is fairly small for optical fibers, it can be ignored, thus reducing the nonlinear polarization to

$$P_{NL} = \varepsilon_0 \varepsilon_{NL} E(\vec{r}, t), \quad (3.18)$$

where ε_{NL} is described as the nonlinear impact to the dielectric constant and can be found from Eq.(3.17) as

$$\varepsilon_{NL} = \frac{3}{4} \chi^{(3)} |E(\vec{r}, t)|^2. \quad (3.19)$$

The continuous wave equation is solved within the Fourier domain with the assumption that the nonlinear contribution of the dielectric is constant. This is, in turn, justified by the slowly varying approximation and the nonlinear polarization being treated as

a perturbation [26]. When using the Fourier transform of Eq.(3.10) and using the above equations of the nonlinear polarization the wave equation reduces to a Helmotz equation of the form

$$\nabla^2 \tilde{E} + \varepsilon(\omega) k_0 \tilde{E} = 0, \quad (3.20)$$

where $\tilde{E}$ is the electric field in the Fourier domain and $k_0 = \omega/c$ with the dielectric constant including both linear and nonlinear contributions from the induced polarization being

$$\varepsilon(\omega) = 1 + \chi^{(1)}(\omega) + \varepsilon_{NL}. \quad (3.21)$$

The dielectric constant denoted by the refractive index $\tilde{n}$ and the absorption coefficient $\tilde{\alpha}$. But both $\tilde{n}$ and $\tilde{\alpha}$ rely on the intensity due to the contribution of ε_{NL} [26], therefore the need to introduce

$$\tilde{n} = n + n_2 |E|^2, \quad (3.22)$$

$$\tilde{\alpha} = \alpha + \alpha_2 |E|^2, \quad (3.23)$$

using Eqs. (3.22) and (3.23) the nonlinear-index coefficient n_2 and the two-photon absorption coefficient α_2 are defined by

$$n_2 = \frac{3}{8n} \text{Re}(\chi^{(3)}), \quad (3.24)$$

$$\alpha_2 = \frac{3\omega_0}{4nc} \text{Im}(\chi^{(3)}). \quad (3.25)$$

α_2 is taken to be really small in silica fibers therefore it's contribution is taken to be

non-existent. The electromagnetic wave equation in Eq.(3.20) could be simplified by the technique known as separation of variables therefore the electric field is re-written as

$$\tilde{E}(\vec{r}, \omega - \omega_0) = F(x, y) \tilde{A}(z, \omega - \omega_0) e^{(i\beta_0 z)}, \quad (3.26)$$

in this case $\beta_0 = n\omega/c$ is the wave number. This results in

$$\frac{d^2 F}{dx^2} + \frac{d^2 F}{dy^2} + [\varepsilon(\omega)k_0^2 - \beta^2] F = 0, \quad (3.27)$$

$$2i\beta_0 \frac{d\tilde{A}(z)}{dz} + (\tilde{\beta}^2 - \beta_0^2) \tilde{A} = 0. \quad (3.28)$$

$\tilde{\beta}$ is the separation constant(eigenvalue). Eq. (3.28) is well known equation defining the propagation of an optical field in an optical fiber. In obtaining the equation, the second derivative $\partial^2 \tilde{A} / \partial z^2$, we therefore its contribution with the assumption that $\tilde{A}(z, \omega)$ is a slowly varying function of z . $\tilde{\beta}$ is normally found by simplifying the eigenvalue Eq.(3.27) for the fiber modes [26]. The dielectric constant $\varepsilon(\omega)$, normally given by the following equation (which is an approximation)[98]

$$\varepsilon = (n + \Delta n)^2 \approx n^2 + 2n\Delta n, \quad (3.29)$$

where Δn is a small perturbation given by

$$\Delta n = n_2 |E|^2 + \frac{i\tilde{\alpha}}{2k_0} \quad (3.30)$$

Eq.(3.27) could also be simplified by the first-order perturbation theory [164]. In single-mode fibers, $F(x, y)$ generally contributes to the modal distribution of the fundamental fiber mode. By the use of the first order perturbation theory, the modal distribution

function is seen to be not affected by Δn , rather, the eigenvalue $\tilde{\beta}$ becomes

$$\tilde{\beta}(\omega) = \beta(\omega) + \Delta\beta(\omega), \quad (3.31)$$

where

$$\Delta\beta(\omega) = \frac{n(\omega) \int \int_{-\infty}^{\infty} \Delta n |F(x, y)|^2 dx dy}{\beta(\omega) \int \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}. \quad (3.32)$$

The electric field can also be described as

$$\vec{E}(\vec{r}, t) = \frac{1}{2} \hat{x} \{ F(x, y) A(z, t) e^{i(\beta_0 z - \omega_0 t)} + c.c. \}, \quad (3.33)$$

where $A(z, t)$ is defined as the slowly varying pulse envelope that is satisfied by the Fourier transform $\tilde{A}(z, \omega - \omega_0)$. Thus Eq.(3.28) which can be defined as

$$\frac{\partial \tilde{A}}{\partial z} = i [\beta(\omega) + \Delta\beta(\omega) - \beta_0] \tilde{A} \quad (3.34)$$

The explanation on Eq.(3.34) is that as each spectral contribution in the pulse envelope oscillates down the fiber, it acquires a phase shift with a magnitude that is frequency and intensity reliant[26]. But, what is still uncertain is the functional form of $\beta(\omega)$. Thus, it is very important to expand it in a Taylor series around the frequency ω_0

$$\beta(\omega) = \beta_0 + (\omega - \omega_0)\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\beta_2 + \frac{1}{6}(\omega - \omega_0)^3\beta_3 + \dots, \quad (3.35)$$

where $\beta_0 \equiv \beta(\omega_0)$ and other parameters are defined as

$$\beta_m = \left(\frac{d^m \beta}{d\omega^m} \right)_{\omega=\omega_0}; (m = 1, 2, \dots). \quad (3.36)$$

Using Eqs. (3.34), (3.35), (3.36) and replacing $\omega - \omega_0$ with the differential operator

$i\partial/\partial t$ together with the Fourier transform of $A(z, t)$

$$A(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(z, \omega - \omega_0) e^{-i(\omega - \omega_0)t} d\omega, \quad (3.37)$$

The resulting equation for $A(z, t)$ becomes

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} = i\Delta\beta_0 A, \quad (3.38)$$

where $\Delta\beta_0$ comprises of the effects of both the fiber loss and nonlinearity. With the assumption that $F(x, y)$ in Eq.(3.38) varies a little over the pulse bandwidth and using $\beta(\omega) \approx n(\omega)\omega/c$, the above equation takes the form

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A = i\gamma(\omega_0) |A|^2 A. \quad (3.39)$$

Eq.(3.39) describes generation of picosecond pulses in single-mode fibers, is known as the nonlinear Schrodinger (NLS) equation, it also composes of the fiber loss effects by α . Effects of chromatic dispersion are expressed by β_1 and β_2 , the pulse envelope travels at the group velocity $v_g = 1/\beta_1$. The nonlinear value γ is denoted by

$$\gamma(\omega_0) = \frac{n_2(\omega_0)\omega_0}{cA_{eff}}. \quad (3.40)$$

The pulse amplitude A is said to be normalized in such a way that $|A|^2$ denotes the optical power, the parameter A_{eff} is particularly defined by the effective mode area and is described as

$$A_{eff} = \frac{(\int \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy)^2}{\int \int_{-\infty}^{\infty} |F(x, y)|^4 dx dy}. \quad (3.41)$$

The NLS equation describes the motion of picosecond pulses which gives account to the second-order dispersion and self-phase modulation but fails to give a correct analysis for pulse width which is way less than one picosecond.

3.3 Modulational instability process

To study the linear stability analysis, we allude the light pulse propagation as a continuous wave in an optical fiber[26]. If in Eq.(3.39) the fiber loss is ignored the equation is observed taking the form

$$\frac{\partial A}{\partial z} + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} = i\gamma(\omega_0) |A|^2 A \quad (3.42)$$

Taking assumptions of the following;

1. The amplitude A does not depend on the time at the input end of the fiber at $z = 0$.
2. With the assumption that A is not reliant on time during propagation, the steady state solution becomes

$$A = \sqrt{P} \exp(i\phi_{NL}), \quad (3.43)$$

where P is the input power and $\phi_{NL} = \gamma Pz$ is the nonlinear phase shift. What Eq.(3.43) implies is that the wave propagates through the fiber without any changes apart from the acquired power-dependent phase shift. Therefore, to study the stability of the steady state solution Eq. (3.43) we must perturb the solution, hence

$$A = \sqrt{P + a} \exp(i\phi_{NL}), \quad (3.44)$$

thus we obtain a linearized equation

$$i \frac{\partial a}{\partial z} = \frac{\beta_2}{2} \frac{\partial^2 a}{\partial t^2} - \gamma P(a + a^*), \quad (3.45)$$

Eq. (3.45) is solved in the frequency domain, a^* has Fourier elements at frequencies Ω and $-\Omega$, therefore, we introduce a solution in the following form

$$a(z, t) = u(z) \exp[i(Kz - \Omega t)] + v(z) \exp[-i(Kz - \Omega t)], \quad (3.46)$$

where K is known as the wave number and Ω is defined as the frequency of the perturbation. Eqs. (3.46) and (3.47) produce a set of two homogenous equations that have a nontrivial solution when K and Ω are interms of the dispersion relation

$$K = \pm \frac{1}{2} |\beta_2 \Omega| |\Omega^2 + \text{sgn}(\beta_2) \Omega_c^2|^{1/2}, \quad (3.47)$$

The MI gain spectrum is normally attained from the dispersion relation by setting $\text{sgn}(\beta_2) = -1$ and $g(\Omega) = 2\text{Im}(K)$.

$$g(\Omega) = |\beta_2 \Omega| (\Omega^2 - \Omega_c^2)^{1/2}. \quad (3.48)$$

The Eq.(3.48) is an equation that defines the MI growth rate, and with it we obtain the MI gain peaks in the anomalous regime shown in Fig. 3.1.

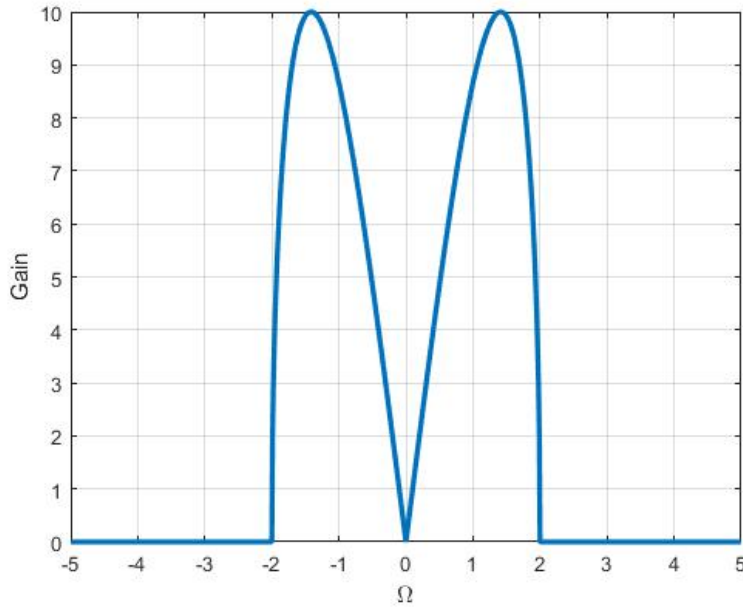


Figure 3.1: Shows the gain spectra peaks for an optical fiber with $\beta_2 = -5 \text{ ps}^2/\text{km}$.

3.4 Summary

The prevailing purpose of this chapter was to introduce and formulate the nonlinear Schrödinger equation as the pulse propagation equation for an optical fiber from Maxwell's equations. The input of the nonlinearity effect was observed as it came into play. The system's modulational instability was studied.

Chapter 4

Higher-order nonlinear Schrödinger (HNLS) equation

4.1 Introduction

The NLS equation in nonlinear evolution equations has been considered as the equation most suitable to illustrate some nonlinear phenomena in optical fibers [51, 52, 216–220]. It is important to shift attention on the ultrashort pulses (USPs)[221–225] with the aim of improving the quality of communication with high bit rate and long-distance (sub-picosecond and femtosecond regimes) and enhancement of information capacity. However, when the USPs are considered, corresponding physical characteristics, like the higher-order dispersion, self-steepening (also known as the Kerr dispersion which causes asymmetrical spectral broadening), self-frequency shift (caused by the stimulated Raman scattering), and cubic-quintic-septic nonlinearity, are expressed by HNLS equation to account for such combinations [16, 25, 226–234, 236]. The focus of this chapter, is to study the MI of HNLS equations discussing the dynamics of ultrashort pulses in nonlinear optical fibers, under with the combined effects of dispersion, loss, gain, cubic and cubic-quintic-septic nonlinearities, and cubic-quintic-septic self-steepening and self-frequency shift terms, respectively.

4.2 Derivation of the new HNLS equation for ultrashort pulses

In the previous example, we had adopted a model from a paper, the problem was that the model does not appear to show the origin for the coefficients, hence in the following subsection, we derive our own HNLS equation following Agrawal's methods shown in the previous chapter.

4.2.1 Model equation

The afore mentioned Maxwell's equations Eqs. (3.1-3.4b) result into the following equation which defines the complex dielectric constant, related to the pulse propagation in optical fibers

$$\varepsilon(\omega) = n_f^2(\omega) + 2in_f(\omega)\frac{\alpha_f}{k_0}, \quad (4.1)$$

where $\alpha_f = 0$ is the fiber loss and $n_f(\omega)$ represents the refractive index of the fiber given by

$$n_f(\omega) = n(\omega) + n_2 |E|^2 + n_4 |E|^4 + n_6 |E|^6, \quad (4.2)$$

Considering terms up to $|E|^6$, $(n_f(\omega))^2$ is written as

$$n_f(\omega)^2 = n(\omega)^2 + 2n(\omega)n_2 |E|^2 + 2n(\omega)n_4 |E|^4 + 2n(\omega)n_6 |E|^6 + n_2^2 |E|^4 + 2n_2n_4 |E|^6, \quad (4.3)$$

which we make use of, together with Eq.(4.3) into Eq.(4.2), to get

$$\varepsilon(\omega) = n(\omega)^2 + 2n(\omega)\Delta n, \quad (4.4)$$

with Δn being given by

$$\Delta n = [n_2] |E|^2 + \left[\frac{n^2(\omega)}{2} + n_4 \right] |E|^4 + \left[n_6 + \frac{n_2n_4}{n(\omega)} \right] |E|^6, \quad (4.5)$$

Following Agrawal [166], the propagation equation for $\tilde{\psi}(z, \omega - \omega_0)$, which is the Fourier transform of $\psi(z, t)$, is written in the form of Eq.(3.34). We should put into consideration that nonlinearities in relation to the self-steepening and self-frequency shift that are found through the investigation of the propagation constant. The Taylor's expansion in series, up to sixth-order, of $\beta(\omega)$, about the carrier frequency in Eq.(3.34) is shown in Appendix A. The final result for the pulse-propagation equation is the HNLS equation having up to the sixth order of dispersion is as follows

$$i\psi_z - \beta_2\psi_{tt} - i\beta_3\psi_{ttt} + \beta_4\psi_{tttt} + i\beta_5\psi_{ttttt} - \beta_6\psi_{ttttt} + \gamma_1|\psi|^2\psi + \gamma_2|\psi|^4\psi + \gamma_3|\psi|^6\psi + i\alpha_1(|\psi|^2\psi)_t + i\alpha_3(|\psi|^4\psi)_t + i\alpha_5(|\psi|^6\psi)_t + \alpha_2(|\psi|^2)_t\psi + \alpha_4(|\psi|^4)_t\psi + \alpha_6(|\psi|^6)_t\psi = 0, \quad (4.6)$$

where

$$\begin{aligned} \beta_2 &= \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega^2} \right)_0, \quad \beta_3 = \frac{1}{6} \left(\frac{\partial^3 \beta}{\partial \omega^3} \right)_0, \quad \beta_4 = \frac{1}{24} \left(\frac{\partial^4 \beta}{\partial \omega^4} \right)_0, \quad \beta_5 = \frac{1}{120} \left(\frac{\partial^5 \beta}{\partial \omega^5} \right)_0, \quad \beta_6 = \frac{1}{720} \left(\frac{\partial^6 \beta}{\partial \omega^6} \right)_0 \\ \beta_1 &= \left(\frac{\partial \beta}{\partial \omega} \right)_0, \quad \gamma_1 = \left(\frac{\partial \beta}{\partial |\psi|^2} \right)_0 + \frac{k_0 n_2}{4A_{eff}}, \quad \gamma_2 = \frac{1}{2} \left(\frac{\partial \beta}{\partial |\psi|^4} \right)_0 + \frac{k_0}{16A_{eff1}} \left(\frac{n_2^2}{2} + n_4 \right) \\ \gamma_3 &= \frac{1}{6} \left(\frac{\partial \beta}{\partial |\psi|^6} \right)_0 + \frac{k_0}{64A_{eff2}} \left(n_6 + \frac{n_4 n_2}{n(\omega)} \right), \quad \alpha_1 = \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^2)} \right)_0, \quad \alpha_3 = \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^4)} \right)_0 \\ \alpha_5 &= \frac{1}{6} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^6)} \right)_0, \quad \alpha_2 = \left(\frac{\partial \beta}{\partial (|\psi|^2)_t} \right)_0, \quad \alpha_4 = \frac{1}{2} \left(\frac{\partial \beta}{\partial (|\psi|^4)_t} \right)_0, \quad \alpha_6 = \frac{1}{6} \left(\frac{\partial \beta}{\partial (|\psi|^6)_t} \right)_0, \end{aligned}$$

with $A_{eff} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^4 dx dy}$ is the refractive core area of the fiber, $A_{eff1} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^6 dx dy}$ and $A_{eff3} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^8 dx dy}$. More precisely, $\psi(z, t)$ is the envelope amplitude of the electric field, t is the retarded time frame and z is the propagation distance originating from the transformation, $T = t - z/v_g = t - \beta_1$. All the coefficients are real constants. The above equation is a higher-order nonlinear Schrödinger equation with higher-order dispersion and cubic, quintic and septic non-Kerr nonlinearities where $\psi(z, t)$ represents the complex envelope of the electric field while z and t are the propagation distance and time, respectively. γ_1 , γ_2 and γ_3 are the cubic, quintic and septic nonlinearities, respectively while α_1 , α_3 and α_5 are the cubic, quintic and septic self

steepening terms, respectively. The terms α_2 , α_4 and α_6 are the cubic, quintic and septic intra-pulse Raman scattering terms, respectively. β_2 is the group velocity dispersion (GVD) term, β_3 is the third-order dispersion (TOD) term, β_4 is the fourth-order dispersion (FOD), β_5 is the fifth-order dispersion (FFD) term and β_6 is the sixth-order dispersion (SOD) term. The production of small-core PCFs have been made possible fabrication of the PCFs [167–170] since the nonlinear interactions which are strong happen to occur at presumably power peaks that are low and when the propagation distances are undeniably short. The four-wave mixing (FWM) theory has been developed, which does not only include the slowly varying envelope approximation of the NLS equation and is based on the full wave equation, allowing the rigorous study of the left-most instability band (the first FWM peak) and the far-detuned instability peak (the secondary FWM). In this approach, the perturbation wave numbers are the roots of the fourth-order algebraic equation [171], where the two extra roots are used to describe the excitation of the forward pulses and the backward pulses, while all previous findings on FWM in fibers with nontrivial dispersion characteristics resulted in a quadratic equation for the perturbation wave numbers [172–176]. These two extra roots have negative imaginary parts, generating instabilities [171]. In general, the first FWM peak, is known to have an analog in the standard NLS equation in the negative GVD regime, which coincides with a Stokes element a bit stronger than the anti-Stokes one, and the second peak, which occurs in the model from a corrected fiber dispersion, and is said not to be present in the general NLS, that produces a stronger anti-Stokes wave. In the meantime, a generalized NLS equation has been derived, including self-steepening effects, while the summation characterizing higher-order dispersion is taken up to order $N = 20$, which we assume to be high enough in such a manner that the dispersion profile is well approximated in the frequency domain [171]. It has been, however, pointed out that the generalized NLS equation is somehow not sufficient for better analysis between experimental and theoretical results since we approximate the reliance of the nonlinearity on the detuning parameter [171]. Indeed, the appearance

of the second FWM peak is made certain by positive higher-order dispersion terms greater or equal to 6, namely the positive dispersions of orders two, four, and six, respectively, and needs adequately large values of the detuning parameter due to the higher-order dispersions terms are very small. However, due to extra degrees of freedom, PCFs allow mode propagation to be easily manipulated. They are made of an array of microscopic air holes distributed throughout their whole length that surrounds a pure silica core. Therefore, nonlinear effects are enhanced due to light concentration into a very small area between silica and air, and their balance with higher-order dispersions can be used to further elaborate on the optical pulse communication in highly nonlinear photonic crystal fiber, which imposes to elude the order of nonlinear terms in the general NLS equation beyond Kerr nonlinearity.

4.2.2 Dimensionless HNLS equation

To simplify the problem at hand, it would be advisable to reduce the number of parameters. There are three most significant reasons as to why expressing complex equations in dimensionless or dimensionally reduced form.

1. Mathematical techniques are much easier to identify and familiarize with.
2. The values at which the equation's numerical solution is introduced may be made less.
3. Gives ideas on what elements of the equation may be neglected or approximated.

Therefore, Eq. (4.6) can be normalized by making use of the transformations $Z = z/L_D$ for space, $\tau = t/T_0$ for time and $u = \frac{\psi}{\sqrt{P_0}}N$ for amplitude, which lead to the following normalized HNLS equation :

$$\begin{aligned}
iu_Z - k_2 u_{\tau\tau} - ik_3 u_{\tau\tau\tau} + k_4 u_{\tau\tau\tau\tau} + ik_5 u_{\tau\tau\tau\tau\tau} - k_6 u_{\tau\tau\tau\tau\tau\tau} + n_{NL} |u|^2 u & \quad (4.7) \\
+ n_{NL4} |u|^4 u + n_{NL6} |u|^6 u + in_{SS}(|u|^2 u)_\tau + in_{SS4}(|u|^4 u)_\tau + \\
in_{SS6}(|u|^6 u)_\tau + n_{SFS}(|u|^2)_\tau u + n_{SFS4}(|u|^4)_\tau u + n_{SFS6}(|u|^6)_\tau u = 0,
\end{aligned}$$

where

$$\begin{aligned}
L_D &= \frac{T_0^2}{\beta_2}, \quad L_{Dm} = \sum_{m=3}^6 \frac{T_0^m}{\beta_m}, \quad k_2 = \text{sgn}(\beta_2), \quad k_m = \sum_{m=3}^6 \frac{L_D}{L_{Dm}}, \quad N^2 = \frac{L_D}{L_{NL}}, \quad L_{NL} = \frac{1}{\gamma_1 P_0}, \\
L_{NL4} &= \frac{1}{\gamma_2 P_0^2}, \quad L_{NL6} = \frac{1}{\gamma_3 P_0^3}, \quad L_{SS} = \frac{T_0}{\alpha_1 P_0}, \quad L_{SS4} = \frac{T_0}{\alpha_3 P_0^2}, \quad L_{SS6} = \frac{T_0}{\alpha_5 P_0^3}, \\
L_{SFS} &= \frac{T_0}{\alpha_2 P_0}, \quad L_{SFS4} = \frac{T_0}{\alpha_4 P_0^2}, \quad L_{SFS6} = \frac{T_0}{\alpha_6 P_0^3}, \quad n_{NL} = \frac{L_D}{L_{NL} N^2}, \quad n_{NL4} = \frac{L_D}{L_{NL4} N^4}, \\
n_{NL6} &= \frac{L_D}{L_{NL6} N^6}, \quad n_{SS} = \frac{L_D}{L_{SS} N^2}, \quad n_{SS4} = \frac{L_D}{L_{SS4} N^4}, \quad n_{SS6} = \frac{L_D}{L_{SS6} N^6}, \quad n_{SFS} = \frac{L_D}{L_{SFS} N^2}, \\
n_{SFS4} &= \frac{L_D}{L_{SFS4} N^4}, \quad n_{SFS6} = \frac{L_D}{L_{SFS6} N^6}.
\end{aligned}$$

The above quantities are dimensionless parameters depending on the typical values [41, 42, 177]: $P_0 = 10\text{W}$, $\beta_6 = 0.5\text{ps}^6\text{m}^{-1}$, $\beta_5 = 0.00123\text{ps}^5\text{m}^{-1}$, $\beta_4 = 10\text{ps}^4\text{m}^{-1}$, $\beta_3 = 0.009\text{ps}^3\text{m}^{-1}$, $\beta_2 = 50\text{ps}^2\text{m}^{-1}$, $\gamma_1 = 1\text{ kW}^{-1}\text{m}^{-1}$, $\gamma_2 = 1\text{ kW}^{-2}\text{m}^{-1}$, $\gamma_3 = 0.031\text{ kW}^{-1}\text{m}^{-3}$, $\alpha_1 = -0.0247\text{ kW}^{-1}/[(2\pi)\text{mTHz}]$, $\alpha_2 = 0.03705\text{ kW}^{-1}\text{psm}^{-1}$, $\alpha_3 = -0.0247\text{ kW}^{-2}/[(2\pi)\text{mTHz}]$, $\alpha_4 = -0.030875\text{ kW}^{-2}\text{psm}^{-1}$, $\alpha_5 = 0.004\text{ kW}^{-3}/[(2\pi)\text{mTHz}]$ and $\alpha_6 = 0.02\text{ kW}^{-3}\text{psm}^{-1}$.

4.2.3 Instability criteria of continuous waves on the dimensionless HNLS equation

Modulational instability analysis of Eq.(4.7) leads to the following gain,

$$G(\Omega) = 2 \sqrt{(-am + 2em)\Omega + ((a^2)/4 + 2k_2m - e^2 + b^2)\Omega^2 - k_2a\Omega^3 + (k_2^2 + 2k_4m)\Omega^4 - k_4a\Omega^5 + 2(k_2k_4 + k_6m)\Omega^6 - k_6a\Omega^7 + (ka_4^2 + 2k_2k_6)\Omega^8 + 2k_4k_6\Omega^{10} + ka_6^2\Omega^{12}} \quad (4.8)$$

where $a = 2in_{SFS}P_0 + 4in_{SFS4}P_0^2 + 6in_{SFS6}P_0^3$, $b = n_{SS}P_0 + 2n_{SS4}P_0^2 + 3n_{SS6}P_0^3$, $e = -in_{SFS}P_0 - 2in_{SFS4}P_0^2 - 3in_{SFS6}P_0^3$ and $m = n_{NL}P_0 + 2n_{NL4}P_0^2 + 3n_{NL6}P_0^3$.

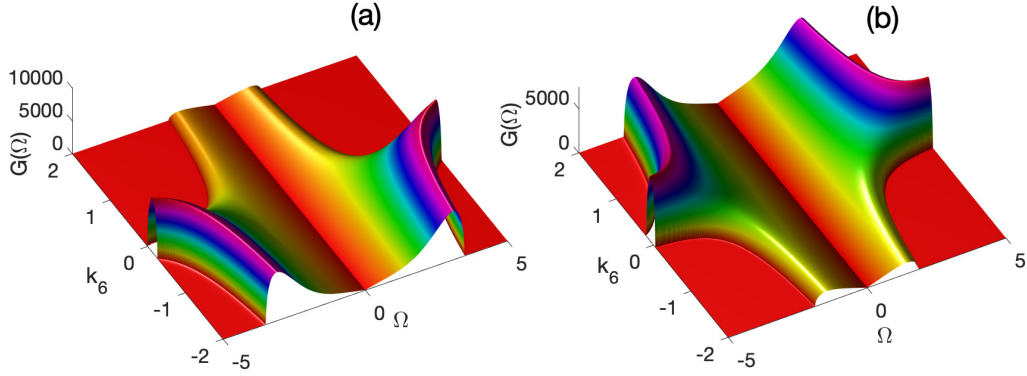


Figure 4.1: The panels show plot of the variation of maximum MI gain as a function of perturbation frequency Ω , and the SOD term k_6 for (a) the normal ($k_2 = 1$) and (b) the anomalous regimes ($k_2 = -1$), with the other parameters being: $P_0 = 8$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$ and $n_{SFS6} = 0.021$.

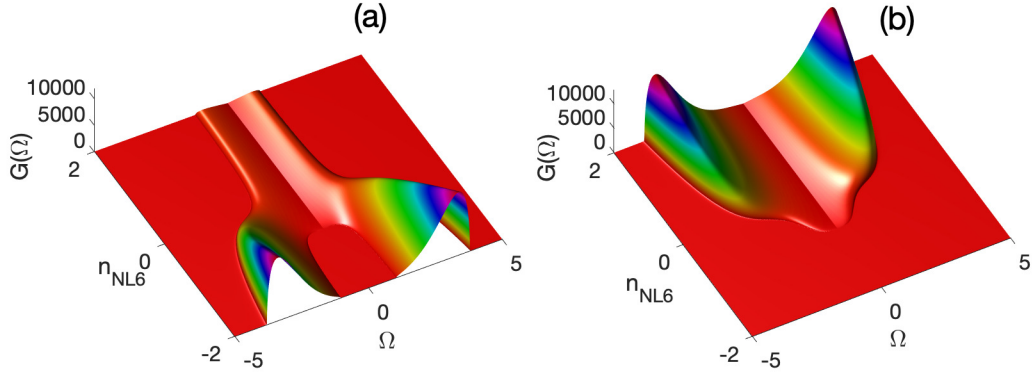


Figure 4.2: The variation of maximum MI gain as a function of perturbation frequency Ω , and the septic non-Kerr nonlinearity parameter n_{NL6} . Panel (a) show results for $k_6 = 0.02$ and panel (b) is plotted for $k_6 = -0.02$, with the other parameters being: $P_0 = 10$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$ and $n_{SFS6} = 0.021$.

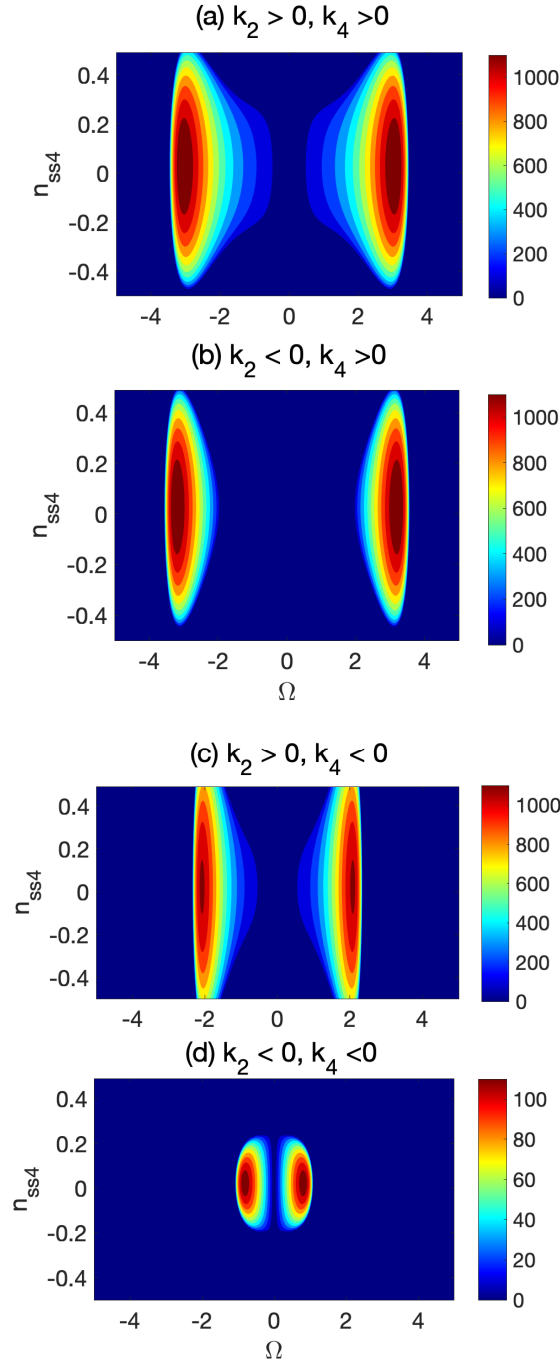


Figure 4.3: Variation of maximum MI gain versus the perturbation frequency Ω and the quintic self-steepening term n_{SS4} . The FOD is considered positive in Panels (a) and (b), respectively obtained for $k_2 > 0$ and $k_2 < 0$. Panels (c) and (d) are plotted for a negative FOD coefficient k_4 , with $k_2 > 0$ and $k_2 < 0$, respectively. The other parameters being: $P_0 = 10$, $k_6 = 0.02$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$ and $n_{SFS6} = 0.021$.

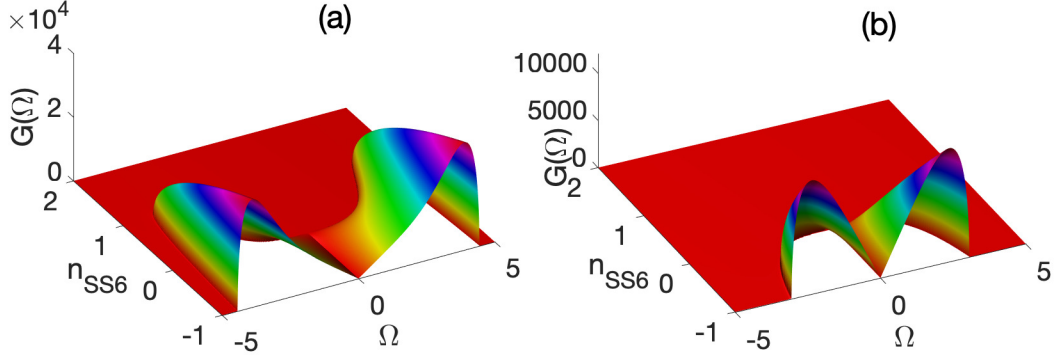


Figure 4.4: Panels show the variation of maximum MI gain versus the perturbation frequency Ω and the septic self-steepening term n_{SS6} and for (a) $k_6 = 0.02$, (b) $k_6 = -0.02$, with the other parameters being: $P_0 = 10$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$ and $n_{SFS6} = 0.021$.

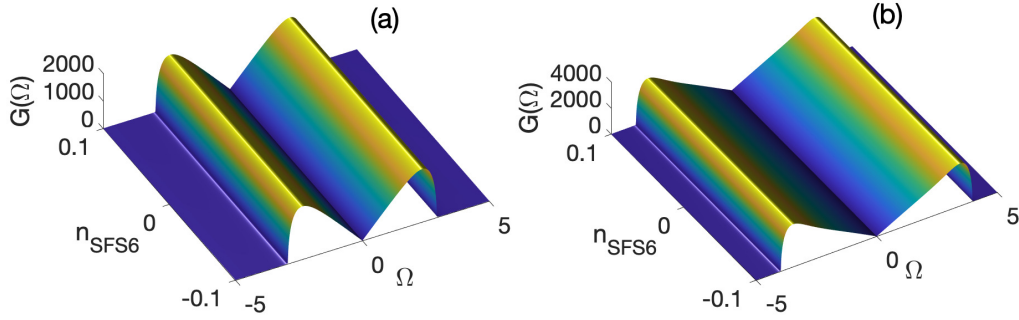


Figure 4.5: Variation of maximum MI gain versus the perturbation frequency Ω and the septic self-frequency shift term n_{SFS6} , and for (a) $k_6 = 0.02$, (b) $k_6 = -0.02$, with the other parameters being: $P_0 = 10$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$ and $n_{SFS4} = 0.0004$.

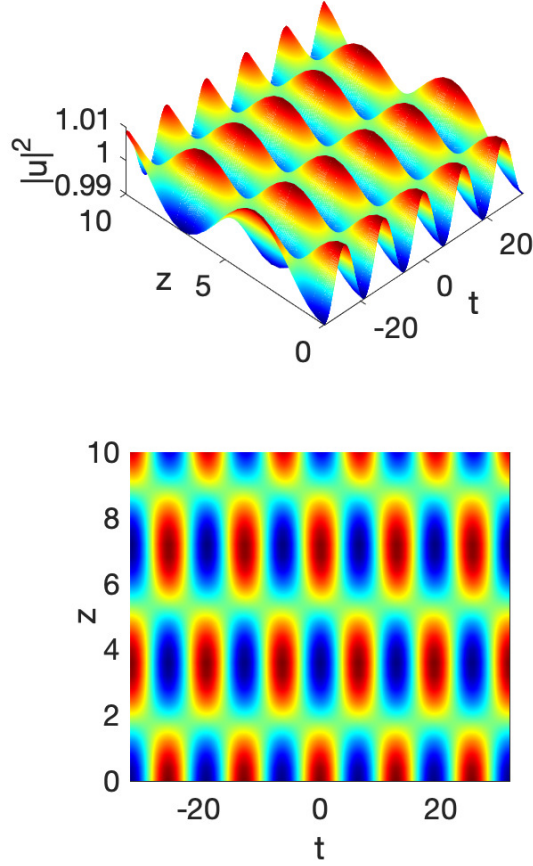


Figure 4.6: The panels show the evolution of the CW under condition (vii) of the soliton stability analysis which predict MI to be violated if $k_2/k_5 < 0$, with $k_2 = -1$ and $k_5 = 0.00123$. The rest of the parameters are: $k_6 = 0$, $P_0 = 10$, $k_3 = 0$, $k_4 = 0$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$ and $n_{SFS6} = 0.021$.

4.2.4 Effects of nonlinear and dispersion on modulational instability growth rate

To start, the gain spectrum is displayed in Fig. 4.1 versus the perturbation frequency Ω and the SOD parameter k_6 , both for the normal [see Fig. 4.1(a)] and anomalous [see Fig. 4.1(b)] GVD regimes. For $k_2 > 0$, one notices two gain peaks that divert from the center $\Omega = 0$ when k_6 approaches zero. However, two regimes are detected, i.e., when $k_6 > 0$ lobes of the frequency domain are large and reduced for $k_6 < 0$, expanding the stability zone. The reverse scenario is obtained for $k_2 < 0$, where the large frequency bandwidth is observed for $k_6 > 0$ and small bandwidth is observed for $k_6 < 0$. In the rest of the stability analysis, the effects of higher-order nonlinear terms will be confronted to the two SOD regimes, i.e., $k_6 > 0$ and $k_6 < 0$ under normal GVD. The first example to be addressed in this regard is the impact of the nonlinear septic term on MI, which is depicted in Fig. 4.2, where panel (a) is plotted for $k_6 > 0$ and panel (b) for $k_6 < 0$. For the case $k_6 > 0$, high-intensity side lobes appear for $n_{NL6} < 0$, but the maximum gain increases, together with the frequency bandwidth, in the interval $-2 \leq n_{NL6} < 0$. In this particular case, the bandwidth for $n_{NL6} < 0$ is large and completely disappears for $n_{NL6} \geq 0$, leaving two symmetric lobes of minor intensity, but still enough for the CW to disintegrate into nonlinear structures. On the other side, when $k_6 < 0$, the instability is expected to be more pronounced for $n_{NL6} > 0$, since the gain side lobes are of high intensity and tend to disappear when $n_{NL6} < 0$, although there is a low-intensity tongue of instability surrounded by a large region of stability [see Fig. 4.2(b)].

Fig. 4.3 displays the MI gain with respect to the quintic SS parameters and the frequency Ω for $k_4 > 0$ (panels (a) and (b)) and $k_4 < 0$ (panels (c) and (d)). Each of the cases presents lobes of instability for positive and negative GVD. In Fig. 4.3(a), for example, for which $k_2 > 0$, there appear two peaks of instability in the gain spectrum, which show instability to be possible for all values of n_{ss4} . The lobes being symmetric,

the gap between them increases for $k_2 < 0$, and the instability bandwidth shrinks [see Fig. 4.3(b)]. The same spectrum of behaviors appears in Fig. 4.3(c) and (d) for which $k_4 < 0$. For $k_2 > 0$, the two symmetric lobes appear and get shrank when $k_2 < 0$, for which there appears two breasts of instability. Also, the bandgap between gain peaks has reduced. We should stress that the case $k_4 < 0$ and k_2 being either negative or positive was recently explored by Tiam et al. [178, 179], who further found the existence of single-hump solitons numerically for positive values of $k_2 > 0$. Interestingly, considering a generalized NLS equation many multipulse solitons were found under a suitable balance between the quartic and quadratic dispersions with only the Kerr nonlinearity. Many other classes of solitons may exist when signs of k_4 and k_2 change, which encourages further investigations directed towards the activation of MI with non-Kerr nonlinearities.

The competition between the septic SS and SOD parameters is summarized in Fig. 4.4, where panels (a) and (b) correspond respectively to $k_6 > 0$ and $k_6 < 0$. In Fig. 5.1(a), when $k_6 > 0$, the MI gain is limited as n_{SS6} tends to zero and disappears at $n_{SS6} = 1$, leaving small intensity lobes separated by a large bandgap frequency compared to the large frequency bandwidth observed for $-1 \leq n_{SS6} \leq 0$. For the case $k_6 < 0$ [see Fig. 4.4(b)], no instability should be expected for $n_{SS6} > 0$. However, two symmetric lobes show the presence of instability under respective balances of nonlinear and dispersive effects. The MI gain with respect to the perturbation frequency Ω and the self-frequency shift (SFS) parameter is plotted in Fig. 4.5, with panels (a) and (b) corresponding respectively to $k_6 > 0$ and $k_6 < 0$. In general, two distinct and symmetric sidebands appear in the gain spectrum and remain the same even with increasing n_{SFS6} . However, the bandwidth is larger for $k_6 < 0$ than when k_6 is positive.

The above-discussed results are a good sign of the existence of unstable waves in the model under study, especially when the values of parameters are well-chosen. However, the model is very sensitive as many parameters are involved but satisfactorily, information about CW instability allows us to make the right choice of such parameter

values when numerical simulations are carried out. From the expression of the gain (4.8), the reader may notice the odd terms of coefficients k_3 and k_5 do not contribute to the dynamics of the MI growth rate. This is a confirmation of predictions from Shagalov [180] who proposed a separated technique to study the impact of such terms in the numerical occurrence of MI. In so doing, the relationship between soliton emergence and MI has been debated in several contributions. Especially, attention has been paid to the stability of solitons, as they share the same parametric conditions with the occurrence of MI. Before proceeding to the numerical analysis of our above predictions, it would be of interest to investigate the impact of individual parameters and their different combinations on the emergence and stability of solitons.

4.2.5 Stability of solitons and MI occurrence

When $k_2 \neq 0$, $n_{NL} \neq 0$, with the rest of the coefficients being equal to zero, we are in the presence of the cubic quintic NLS equation that admits stable soliton solutions. Such solutions imply the appearance of MI and the subsequent appearance of wave trains under the condition that dispersion and nonlinearity be well-balanced. However, when the high-dispersion terms are different from zero, the stability of soliton solution gets alternated with a substantial impact on the appearance of MI, primarily when the long-time evolution of solitary waves is investigated. Therefore, before proceeding to the numerical analysis of MI, it is necessary to examine the impact of each dispersion term and their combinations on the emergence of stable solitons in the model under study. To investigate the effects of such parameters on soliton stability, we adopt the method developed in Refs. [181, 182] and recently used by Shagalov [180].

A small-amplitude wave $u(Z, \tau) = u_0 e^{-ikZ + i\omega\tau}$ is adopted as a trial solution to Eq.(7), which leads to the dispersion relation

$$k(\omega) = -k_2\omega^2 + k_3\omega^3 - k_4\omega^4 + k_5\omega^5 - k_6\omega^6. \quad (4.9)$$

To proceed, we focus on the zero-dispersion regime, which implies the relation $\frac{k(\omega)}{\omega} = 0$ whose real solutions imply a resonance interaction between the soliton and small amplitude wave leading to a stable soliton solution. This depends on a number of combinations between the dispersion coefficients k_2 , k_3 , k_4 , k_5 and k_6 . With no pretension of being exhaustive, we focus mainly on the following cases:

(i) For $k_4 = k_5 = k_6 = 0$, $k_2 \neq 0$ and $k_3 \neq 0$, Eq.(4.9) admits the solution $\omega_r = k_2/k_3$. The soliton may be unstable and its decay will take place with emission of waves with frequency ω_r under the condition $k_2 < 0$ and $k_3 > 0$ or $k_2 > 0$ and $k_3 < 0$. This particular case is commonly known and several numerical simulations have confirmed the results.

(ii) For $k_3 = k_5 = k_6 = 0$, $k_2 \neq 0$ and $k_4 \neq 0$, the dispersion relation will admit the solutions $\omega_r = \pm \sqrt{-k_2/k_4}$, which implies that the soliton will be stable if $k_2/k_4 > 0$ and unstable if $k_2/k_4 < 0$.

(iii) For $k_3 \neq 0$, $k_5 = k_6 = 0$, $k_2 \neq 0$ and $k_4 \neq 0$, two roots are obtained for the dispersion relation: $\omega = \frac{-k_3 \pm \sqrt{k_3^2 - 4k_2k_4}}{2k_4}$. Solutions will be two real roots for $k_3^2 - 4k_2k_4 > 0$, which leads to the condition $k_2k_4 > k_3^2/4$ for the soliton to be stable.

(iv) For $k_2 \neq 0$, $k_3 \neq 0$, $k_4 \neq 0$, $k_5 \neq 0$, and $k_6 = 0$, the dispersion relation reduces to a third-order polynomial and the stability of the soliton can be studied through the Routh-Hurwitz (RH) criterion. In doing so, the soliton will be stable, so that MI takes place if the conditions $k_4/k_5 \leq 0$, $k_2/k_5 \leq 0$ and $k_4k_3 \geq k_2k_5$ are simultaneously satisfied.

(v) For $k_2 \neq 0$, $k_3 \neq 0$, $k_4 \neq 0$, $k_5 \neq 0$, and $k_6 \neq 0$, the RH stability criterion, from the obtained fourth-order polynomials, imposes $k_6 < 0$, $k_5 > 0$, $k_4 < 0$, $k_3 > 0$,

$k_2 < 0$, $k_4 k_5 > k_3 k_6$ and $k_3 k_4 k_5 > k_3^2 k_6 + k_2 k_5^2$ for the soliton to be stable.

(vi) For $k_2 \neq 0$, $k_3 \neq 0$, $k_4 \neq 0$, $k_5 = 0$, and $k_6 \neq 0$, from the RH stability criterion, we get $k_6 < 0$, $k_4 < 0$, $k_3 > 0$, $k_2 < 0$ and $k_3 k_6 > 0$ for the soliton to be stable and MI to take place.

(vii) For $k_2 \neq 0$, $k_3 = 0$, $k_4 = 0$, $k_5 = 0$, and $k_6 = 0$, the dispersion relation (4.9) has the soliton $\omega_r = \sqrt[3]{k_2/k_5}$ so that for $k_2/k_5 > 0$, the soliton will be stable and unstable otherwise. In this particular case, it is also obvious that the fifth-order nonlinearity stabilizes the soliton and gives more chances to the MI to take place.

The stability of solitons reveals more interesting and underlying phenomena that are mostly related to the appearance of wave trains via the activation of MI. This is true, as it has long been proven that solitons and MI share the same parameter region. This gets more complicated when several parameters are involved and require more argument input to justify parameter values' choice and their contribution to the studied MI process. For instance, from the above analysis, we have realized that higher-order dispersion terms contribute to stabilizing the soliton and, thus, play a highly significant key in the MI phenomenon. However, the above stability analysis being local, addressing a more complete version of the study via a long-time numerical simulation of the predicted entities is necessary. This is for the simple reason that the model contains nonlinear and dispersive nonlinear terms beyond the dispersion terms, which may violate some of the above stability criteria and lead to new phenomena and behaviors in the long-time evolution of modulationally unstable wave patterns. This will somehow justify our choice of parameter values in the study of the MI gain and give more credit to the proposed model.

4.3 Numerical analysis of unstable wave modulation

In efforts to check the accuracy of our linear stability analysis, direct numerical simulations have been performed on the model Eq. (4.7). The presence of noise or any other weak perturbations in the phase and/or the amplitude of a continuous wave CW or quasi-continuous wave oscillating in a nonlinear medium can evolve exponentially into amplitude w pulses at distinct modulation frequencies under definite conditions. This whole process is called MI and is expressed in many other physical systems and has been investigated extensively in several areas over time due to its theoretical and practical importance. First, the structure and behaviour of systems affected by MI which are described by the one-dimensional (1D) focusing NLS equation, which is an equation that models the evolving of nonlinear dispersive wave packets which are weak. Therefore, one can analyze the first (i.e., linear) stage of MI by first linearizing the NLS equation around the CW system. It is a well-known fact that the linear stage of the MI is composed of the exponential evolution of all the perturbations gravitating in the Fourier spectrum domain below a distinct threshold through the generation of sideband frequencies and is closely associated with the concept of self-localized waves, or solitons. MI can be said to be one of the main So, MI is one of the principal techniques that lead leads to the appearance of localized coherent nonlinear structures and the formation of a train of ultrashort pulses with Terahertz repetition rates both in the regimes of normal and anomalous dispersion. Secondly, the linearization stops being valuable when the amplitude of the evolving perturbation becomes comparable to the CW system, that is, at the nonlinear stage of MI, corresponding to the long-term spatiotemporal dynamics. In this respect, a certain structures of the MI development are known to be strongly dependent on the type of initial conditions considered, namely noisy perturbations of the plane wave, localized perturbations of the plane wave, and harmonic perturbations of the plane wave, respectively. For the first condition involving a CW field which is perturbed by a sine-modulation at the frequency

of maximum MI gain, the dynamics exhibits a recurrent behavior where Akhmediev breather-like structures are observed periodically. In essence, the Akhmediev breather is normally referenced as the exact solution of MI [183–187]. Interestingly, for noisy initial conditions, structures close to the Akhmediev breather, Kuznetsov-Ma solitons, and Peregrine solitons, which are breather solutions of the 1D NLS equation, evolve locally from the generation of a perturbed modulational unstable CW background. MI is considered to be also the basis on generation of rogue waves [188–191]. Moreover, the so-called superregular solitonic solutions, which is an intrinsic type of breather solutions of the 1D NLS equation, have been used to discuss the development of a distinct type of small perturbations of the plane wave which are local [192–194]. Recently, a different MI scenario which is inclusive of the evolution of a symmetrically expanding nonlinear wave like design analyzed by the modulated elliptic solution of the 1D NLS equation has been reported [195]. When a Gaussian noise perturbs the initial beam, it is found that the beam undergoes modulation instability that leads to filamentation into large-scale solitons coinciding with the lower stable branch and small-scale upper branch solitons as the beam intensifies, respectively. The general approach to spatiotemporal MI processes has been presented, showing that the MI gain bands strongly rely on the initial noise conditions. In particular, a general solution of the equation governing the evolution of the spatiotemporal of MI perturbation in terms of a combination of cosine and sine hyperbolic functions has been obtained. In this paper, with the ambition to generate bright solitons via MI, the right perturbation to be adopted is a weak sine perturbation, leading to the initial condition [172]

$$u(0, t) = \sqrt{P_0} [1 + a_m \sin(\Omega_m t)] e^{i\phi_{NL}}, \quad (4.10)$$

where $a_m = 0.01$ is the modulational amplitude and $\Omega_m = 0.5$ is the frequency of a weak sinusoidal modulation frequency imposed on the CW wave. Obviously, the value of Ω_m is within the bandwidth that is supposed to give rise to unstable waves.

Numerical results are obtained from Eq. (4.7) through the split-step Fourier method, with a particular interest in the emergence of MI as a result of the balanced competition between SOD and septic nonlinearity, self-steepening, and self-frequency shift effects.

To start, it is of importance to confirm the stability of soliton performed in the previous section through one of the cases, (vii), for example. It is predicted that for $k_2/k_5 > 0$, the soliton solution will be stable, and MI will be likely to occur. Otherwise, the soliton solution will be unstable, taking away all chances for MI to emerge in the system. We deliberately violate the stability condition by imposing $k_2/k_5 < 0$, which leads to the patterns obtained in Fig. 4.6, where the CW breaks into wave patterns that have the shape of solitonic objects. This shows such conditions might be true, but they are balanced by the other parameters that may violate the soliton solution's stability condition. Otherwise, such conditions are not enough to predict instability, which justifies the results from different gain spectra, where instability is possible negative and positive values of parameters. This explains why we insist on the competitive effect between higher-order nonlinearity and dispersion in the rest of the paper.

In Fig. 4.4, under normal GVD, results are presented for $k_6 > 0$ [see Fig. 4.7(a) _{$j=1,2,3$}] and $k_6 < 0$ [see Fig. 4.7(b) _{$j=1,2,3$}], with n_{NL6} changing in each of the cases. When $k_6 > 0$, the chosen values of parameters in gain spectra give rise to nonlinear patterns as predicted. Particularly, for $n_{NL6} = -0.8$, the instability takes place early, but increasing the septic nonlinear parameter reduces the recurrence of the patterns which develop after a long distance. The same phenomenon is obvious when $k_6 < 0$, except that patterns appear to be exotic pulses whose frequency and recurrence change with the propagation distance. Remarkably, for higher values of n_{NL6} , both for k_6 positive and negative, the perturbed CW breaks up into amplified pulses: the amplitude increases gradually during propagation. It is then ostensible that while controlling the occurrence of MI in the fiber through the septic nonlinearity, different scenarios such as wave trains, erratic patterns of high or low amplitude can appear, depending on the competition among the various intrinsic nonlinearities and dispersions. Indeed,

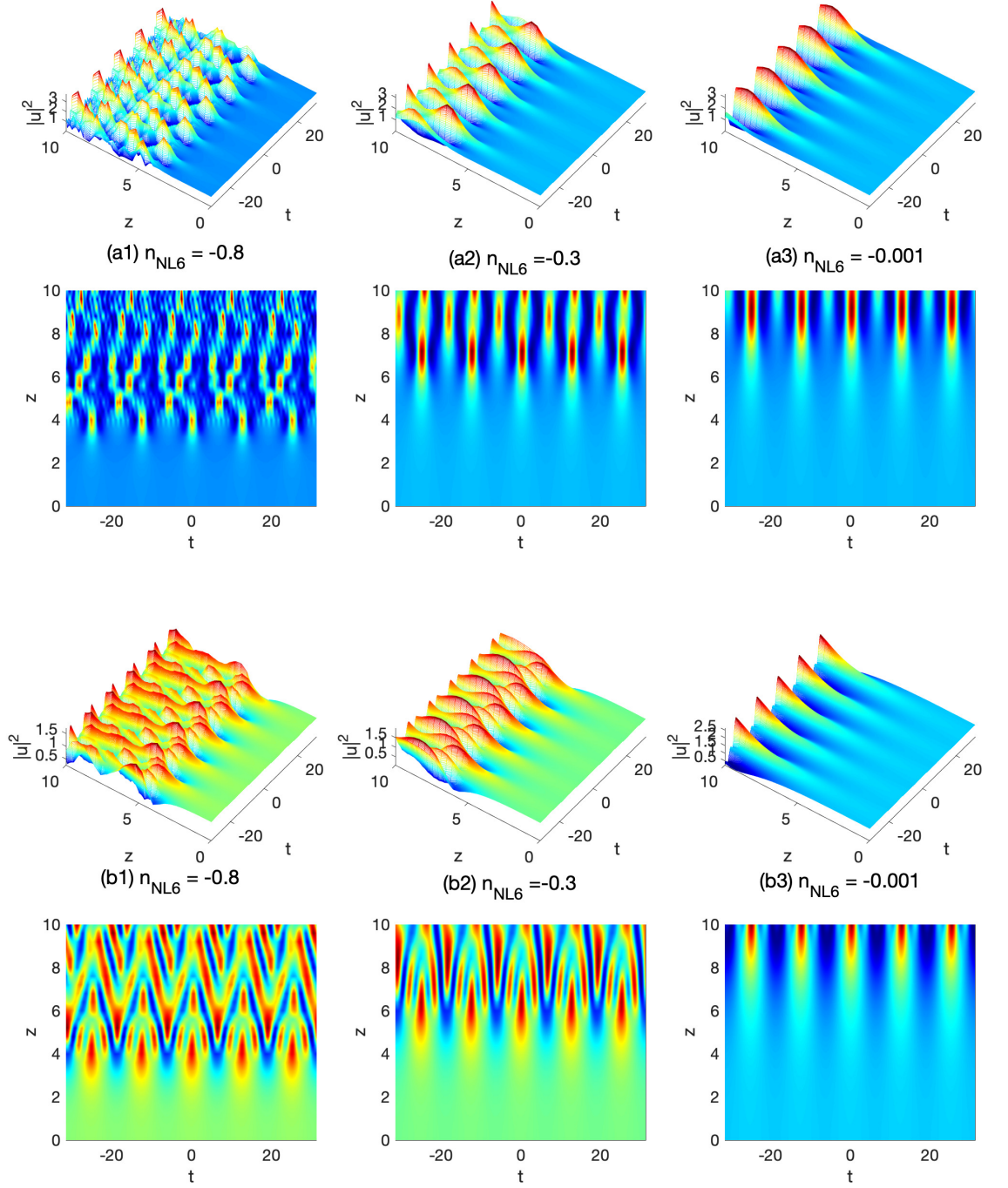


Figure 4.7: The panels show the evolution of the CW under the effect of the septic nonlinearity term for: (aj)_{j=1,2,3} $k_6 = 0.0008$, (bj)_{j=1,2,3} $k_6 = -0.0008$, and $P_0 = 10$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$ and $n_{SFS6} = 0.021$.

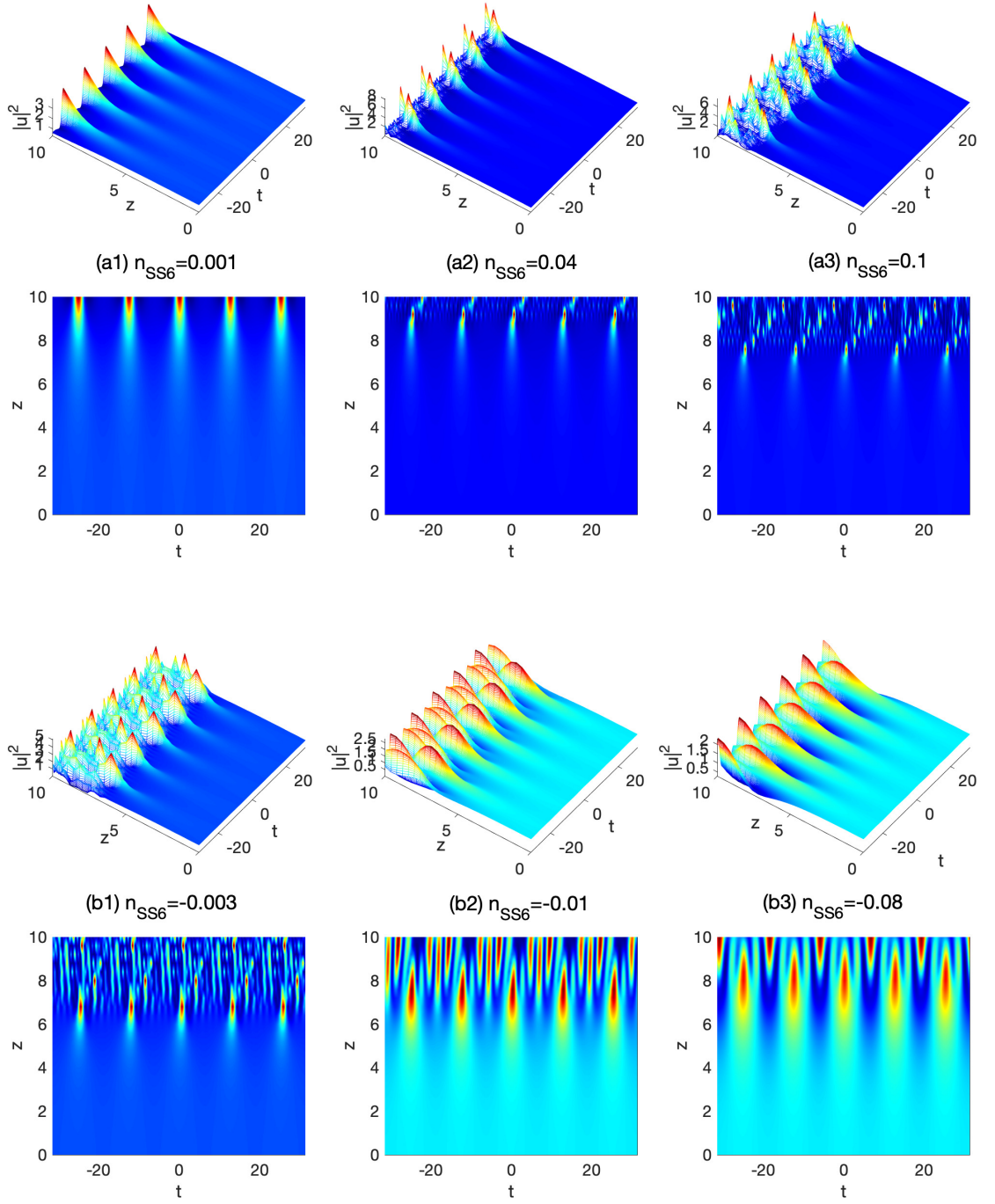


Figure 4.8: The panels show the evolution of the CW under the effect of the septic self-steepening term for: (aj) $_{j=1,2,3}$ $k_6 = 0.0003$, (bj) $_{j=1,2,3}$ $k_6 = -0.0003$, and $P_0 = 10$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$, and $n_{SFS6} = 0.021$.

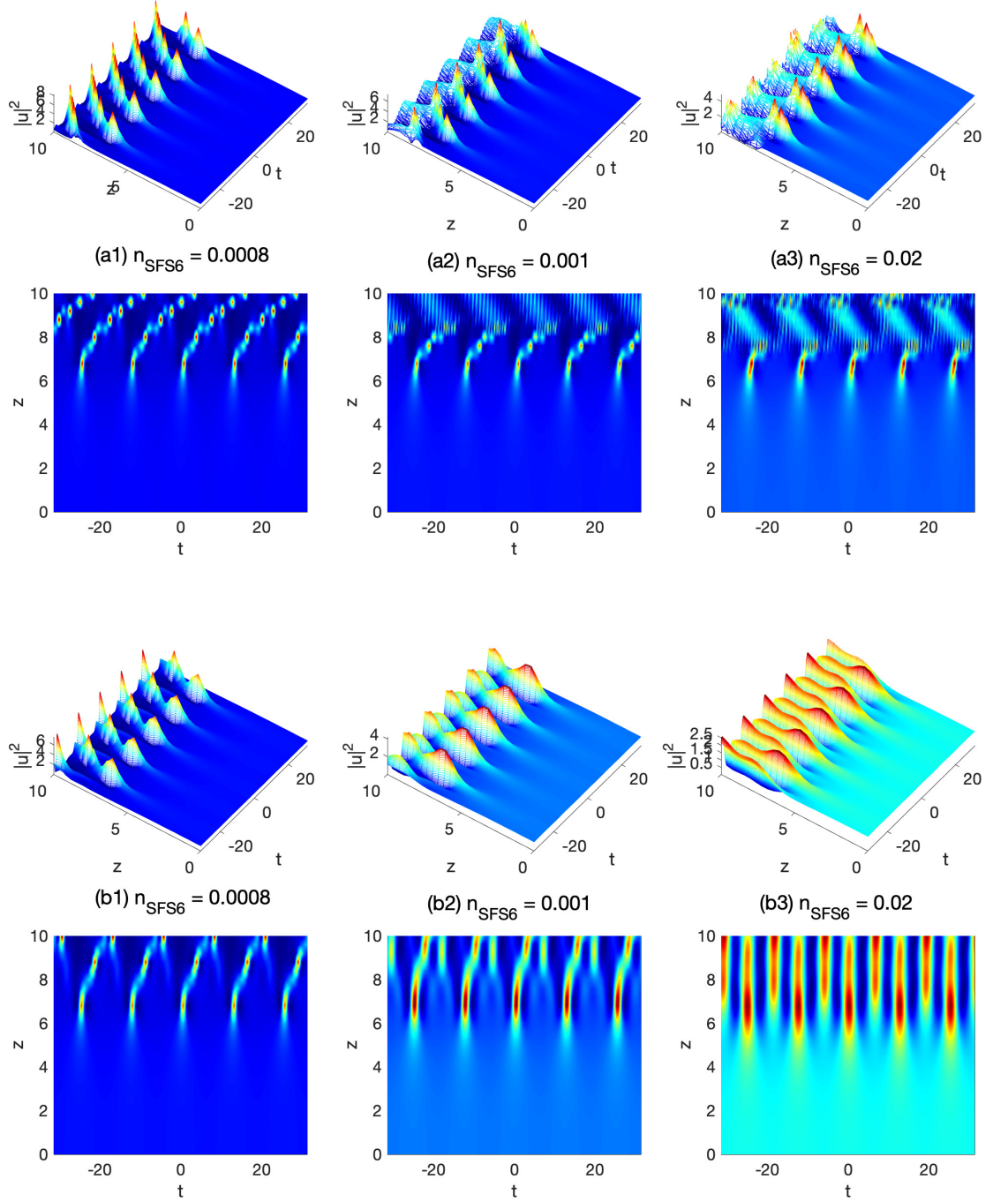


Figure 4.9: The panels show the evolution of the CW under the effect of the septic self-frequency shift term for: (aj) $_{j=1,2,3}$ $k_6 = 0.0008$, (bj) $_{j=1,2,3}$ $k_6 = -0.0008$, and $P_0 = 10$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{\text{NL}} = 1$, $n_{\text{NL4}} = 1$, $n_{\text{NL6}} = 0.03$, $n_{\text{SS}} = -0.0247$, $n_{\text{SS4}} = 0.03705$, $n_{\text{SS6}} = -0.0247$, $n_{\text{SFS}} = -0.030875$ and $n_{\text{SFS4}} = 0.0004$.

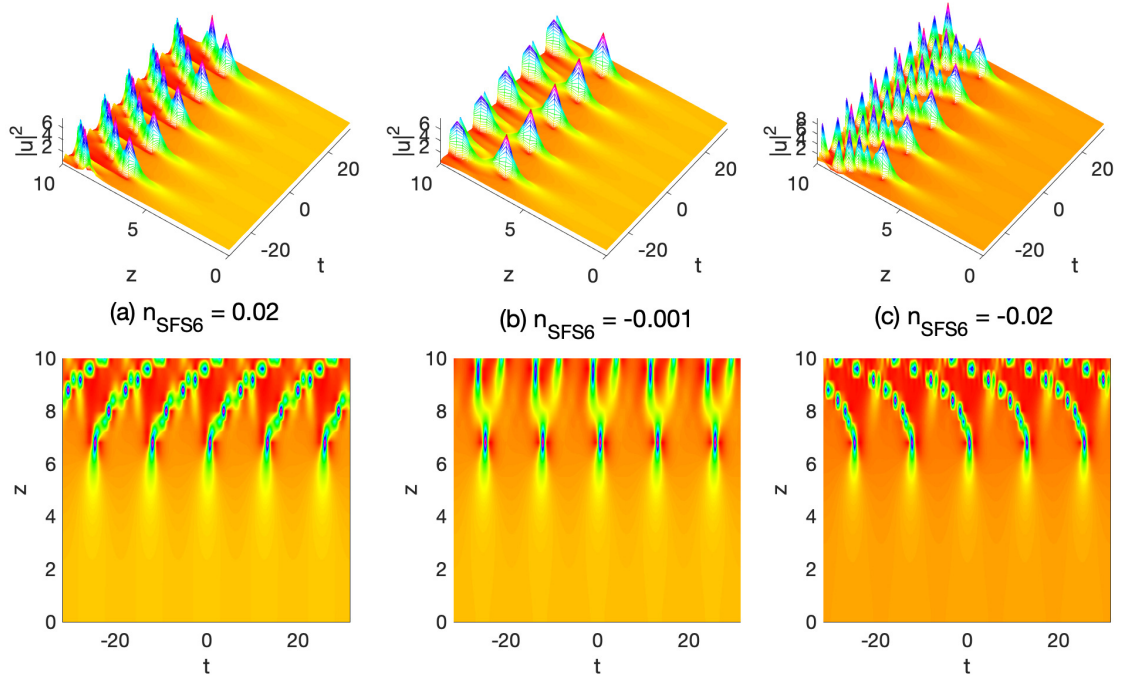


Figure 4.10: Effect of the septic self-frequency shift term on the propagation direction of the disintegrated CW for: (a) $n_{SFS6} = 0.02$, $k_6 = 0.0008$, (b) $n_{SFS6} = -0.001$, $k_6 = 0.0008$, (c) $n_{SFS6} = -0.02$, $k_6 = -0.0008$, and $P_0 = 10$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$ and $n_{SFS4} = 0.0004$.

this could not be foretold via the linear stability analysis, but the obtained results reinforce the efficiency of the MI mechanism through which optical wave patterns of various forms and characteristics can be generated. This is more reinforced by the instability features obtained in Fig. 4.5, which picture the effect of the SS term on the longtime evolution of the perturbed CW, both for $k_6 > 0$ [see Fig. 4.8(a j) $_{j=1,2,3}$] and $k_6 < 0$ [see Fig. 5.5(b j) $_{j=1,2,3}$]. We should however stress here that given the spectrum of behaviors obtained from the gain of Fig. 4.4, the maximum gain is located in the region $n_{SS6} < 0$, so that one expects the CW to disintegrate likely for such value. However, considering the case $k_6 > 0$, the panels (a j) $_{j=1,2,3}$ of Fig. 4.9 show that with increasing n_{SS6} , the MI patterns change from progressively amplified pulses to much erratic patterns, with a tendency to shift on the right. On the other side, when we consider $k_6 < 0$, given that the MI bandwidth is exclusively situated in the area where n_{SS6} is negative, decreasing its value produces the same effect, except that for $n_{SS6} = -0.01$ and $n_{SS6} = -0.08$, the MI takes place in the form of alternating multi-humped pulses. One fact that is however common in the theory of solitons is also obvious here, the space and time expansion of solitonic objects influences their characteristics, mainly the intensity: for $n_{SS6} = -0.003$, the light patterns are smaller with high intensity in comparison with the rest of the cases.

Fig. 4.8 shows the behaviors of the unstable CW when the SFS changes both for k_6 positive and negative. Under the condition $k_6 > 0$, one sees from Fig. 4.8(a1) that for $n_{SFS6} = 0.0008$, the longtime evolution of MI is characterized by right-shifted patterns. With increasing the effect of the SFS, the shifted coherent optical patterns travel over a short distance and turn into erratic structures shifted toward the left [see Fig. 4.8(a2)]. This later feature gets pronounced when $n_{SFS6} = 0.02$, where there is a mix-up of coherent optical patterns along with left-shifted erratic modulated structures as shown in Fig. 4.8(a3). Under the same conditions as in Fig. 4.8(a j) $_{j=1,2,3}$, but with $k_6 < 0$, the optical patterns generated by MI are of low intensity [see Fig. 4.8(b1)], shifted to the right. However, the action of the increasing SFS effect reduces the number of

wave trains, leaving patterns that are extended over the distance [see Fig. 4.8(b2)], which, for $n_{SFS6} = 0.02$ turn out to be trains of alternating and oscillating pulses [see Fig. 4.8(b3)]. Indubitably, the direction of the propagating solitons is importantly affected by the septic SFS term and such direction can change when parameters are well-balanced, with appropriate signs. In the numerical experiment shown in Fig. 4.9, one clearly confirms that when n_{SFS6} and k_6 are positive, the solitonic patterns are shifted to the right [see Fig. 4.9(a)], which is corrected for $n_{SFS6} = -0.001$ as seen in Fig. 4.9(b). Nevertheless, it is found that the direction can shift to the left for $n_{SFS6} = -0.02$ and $k_6 < 0$. As said earlier, model Eq. (4.7) contains many parameters that render it difficult to control the dynamics of any emerging pattern. In the meantime, we should stress that studies have been carried out on earlier models, in absence of the higher-order dispersive and nonlinear effects such as the septic nonlinearity, the septic self-steepening, and self-phase shift terms.

4.4 Summary

In this particular chapter, the model we have derived the higher-order cubic-quintic-septic nonlinear Schrödinger equation with the group-velocity dispersion (GVD), third-, fourth-, fifth- and sixth-order dispersions, and the cubic, quintic, and septic nonlinear terms, for ultrashort pulses in nonlinear optical fibers. The MI phenomenon has certainly been fully addressed, where we have considered the impact of SOD and septic nonlinearity, septic self-steepening, and the septic Raman scattering on MI under a normal dispersion regime. Additionally, the total input power has also been found to bring about some new effects to the manifestation of MI, this in competition with the septic self-steepening under normal frequency regime and positive SOD. Moreover, the model exhibits maximum MI spectral gain values much greater compared to the Higher-order NLS equation [54, 197] therefore, offering larger bandwidth with possible more ultra-short pulses (solitons)[198], hence the possible attractiveness in

long-distance optical fiber communication. However, it should be stressed that the HNLS only stands for undoped optical fibers therefore a new model is significant in describing the ultra-short pulses in doped optical fibers.

Chapter 5

(1+1)D Higher-order

cubic-quintic-septic complex

Ginzburg-Landau (CGL) equation

5.1 Introduction

The emergence of optical solitons in optical fiber has been well dedined and analyzed by the NLS equations [44, 199–204]. Still so, in doped optical fibers, from the quiddity of gain and output loss in the fiber, the light pulses in the fiber are constantly affected by gain effects such as amplification and the output loss, therefore the evolution of solitons induced by MI is finely displayed by the complex Ginzburg-Landau equation (CGLE) [24, 29]. Furthermore, with respect to the fiber loss and periodic amplification in long-distance soliton optical fiber communication systems, the nature of solitons in such a system cannot only be represented by HNLS equation, but also needs to be described by CGLE [205–210]. The main goal of this chapter is to investigate the MI gain of a newly derived extended (1+1) D higher-order cubic-quintic-septic CGL equation with the TOD, FOD, fifth-order dispersion (FFOD), and sixth-order dispersion (SOD) and the cubic, quintic, and septic nonlinear terms, defining the dynamics of extremely

short pulses in nonlinear doped optical fibers. In our discussion, we start with the Maxwell's equations which describe the behaviour of the nonlinear medium to an electromagnetic wave. Then, we report on the derivation of the new governing wave equation. Furthermore, we comprehensively discuss the theory of MI as well as the role played by higher-order dispersions and higher-order nonlinearities. Our analysis also gives rise to some concluding remarks.

5.2 Derivation of the (1+1)D higher-order cubic-quintic-septic CGL equation

Like the previous derived models for continuous wave for ultra-short pulses we start deriving the CGL model from Maxwell's equations the difference between the models is the resulting complex dielectric constant which here is defined as

$$\varepsilon(\omega) = n_f^2(\omega) + 2in_f(\omega)\frac{\alpha_f}{k_0} + \chi_d(\omega), \quad (5.1)$$

where α_f is the fiber loss and $n_f(\omega)$ represents the refractive index of the fiber which has already been defined in the previous NLS equations. Hence,

$$\begin{aligned} \Delta n = & \left[n_2 + \frac{i\alpha_f n_2}{n(\omega)k_0} \right] |E|^2 + \left[\frac{n(\omega)}{2} + n_4 + \frac{i\alpha_f n_4}{n(\omega)k_0} \right] |E|^4 + \\ & \left[n_6 + \frac{n_2 n_4}{n(\omega)} + \frac{i\alpha_f n_6}{n(\omega)k_0} \right] |E|^6 + \frac{i\alpha_f}{k_0} + \frac{\chi_d(\omega)}{2n(\omega)} \end{aligned} \quad (5.2)$$

The dopant which is represented by $\chi_d(\omega)$, the susceptibility of dopants is modeled as a two-level system whose dynamic response is described by Bloch equations[166] and the associated population relaxation time T_1 and dipole relaxation time T_2 [98], is taken to be

$$\chi_d(\vec{r}, \omega) = \frac{\sigma_s W(r) n_0 c / \omega_0}{(\omega - \omega_a) T_2 + i}, \quad (5.3)$$

σ_s is the transition cross section, $W(r) = N_2 - N_1$, is well defined as the population-inversion density, and ω_a is the atomic frequency. The rate equations for the atomic densities for the upper N_1 and lower N_2 energy levels of the two-level system normally used to obtain the following rate equation for the gain[98, 211],

$$\frac{dg_p}{dt} = \frac{g_0 - g_p}{T_1} - \left[\frac{2\sigma_s}{\hbar\omega} \right] g_p I, \quad (5.4)$$

where g_0 is the small-signal gain created by optical pumping and I is the optical intensity. In the steady state Eq.(4.7) is readily solved to obtain $g_p = g_0/(I + I/I_s)$, where $I_s = \hbar\omega/(2\sigma_s T_1)$ is the saturation intensity.

A Taylor expansion of χ_d up-to the sixth-order expansion is shown in the appendix. The propagation equation which is defined by Eq.(3.34) is given by the following equation

$$\frac{\partial \tilde{\psi}}{\partial z} = i(\beta(\omega) + \Delta\beta - \beta_0)\tilde{\psi}, \quad (5.5)$$

It is of vital importance that nonlinearities in relation to the self-steepening and self-frequency shift are derived or rather their input is shown through the propagation constant. With the Taylor's expansion in series, up to sixth-order, of $\beta(\omega)$, about the carrier frequency therefore becomes:

$$\begin{aligned} \beta(\omega) = & \beta_0 + \left(\frac{\partial \beta}{\partial \omega} \right)_0 (\omega - \omega_0) + \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega^2} \right)_0 (\omega - \omega_0)^2 + \frac{1}{6} \left(\frac{\partial^3 \beta}{\partial \omega^3} \right)_0 (\omega - \omega_0)^3 \\ & + \frac{1}{24} \left(\frac{\partial^4 \beta}{\partial \omega^4} \right)_0 (\omega - \omega_0)^4 + \frac{1}{120} \left(\frac{\partial^5 \beta}{\partial \omega^5} \right)_0 (\omega - \omega_0)^5 + \frac{1}{720} \left(\frac{\partial^6 \beta}{\partial \omega^6} \right)_0 (\omega - \omega_0)^6 \\ & + \left(\frac{\partial \beta}{\partial (|\psi|^2)} \right)_0 |\psi|^2 + \frac{1}{2} \left(\frac{\partial \beta}{\partial (|\psi|^4)} \right)_0 |\psi|^4 + \frac{1}{6} \left(\frac{\partial \beta}{\partial (|\psi|^6)} \right)_0 |\psi|^6 \\ & + \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^2)} \right)_0 (\omega - \omega_0) |\psi|^2 + \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^4)} \right)_0 (\omega - \omega_0) |\psi|^4 + \frac{1}{6} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^6)} \right)_0 (\omega - \omega_0) |\psi|^6 \\ & + \left(\frac{\partial \beta}{\partial (|\psi|^2)_t} \right)_0 (|\psi|^2)_t + \frac{1}{2} \left(\frac{\partial \beta}{\partial (|\psi|^4)_t} \right)_0 (|\psi|^4)_t + \frac{1}{6} \left(\frac{\partial \beta}{\partial (|\psi|^6)_t} \right)_0 (|\psi|^6)_t \end{aligned} \quad (5.6)$$

The final result for the pulse-propagation equation is the Higher order cubic-quintic-

septic Ginzburg-Landau equation having up to the sixth order of dispersion is as follows

$$\begin{aligned}
& i\psi_z + (p_r + ip_i)\psi_{tt} + (d_{3r} + id_{3i})\psi_{ttt} + (d_{4r} + id_{4i})\psi_{tttt} + (d_{5r} + id_{5i})\psi_{ttttt} + (d_{6r} + id_{6i})\psi_{ttttt} \\
& + (q_r + iq_i)|\psi|^2 + (C_r + iC_i)|\psi|^4\psi + (S_r + iS_i)|\psi|^6\psi + im_i(|\psi|^2\psi)_t + im_{1i}(|\psi|^4\psi)_t \\
& + im_{2i}(|\psi|^6\psi)_t + n_r(|\psi|^2)_t\psi + n_{1r}(|\psi|^4)_t\psi + n_{2r}(|\psi|^6)_t\psi + (\gamma_r + i\gamma_i)\psi = 0,
\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}
p_r &= \frac{-1}{2} \left(\frac{\partial^2 \beta}{\partial \omega^2} \right)_0 + \frac{g_p T_2^2}{2} \frac{(3\delta - \delta^3)}{(1 + \delta^2)^3}, \quad p_i = \frac{g_p T_2^2}{2} \frac{(3\delta^2 - 1)}{(1 + \delta^2)^3}, \quad d_{3r} = \frac{g_p T_2^3}{2} \frac{(-4\delta + 4\delta^3)}{(1 + \delta^2)^4} \\
d_{3i} &= \frac{-1}{6} \left(\frac{\partial^3 \beta}{\partial \omega^3} \right)_0 - \frac{g_p T_2^3}{2} \frac{(1 + \delta^4 - 6\delta^2)}{(1 + \delta^2)^4}, \quad d_{4r} = \frac{1}{24} \left(\frac{\partial^4 \beta}{\partial \omega^4} \right)_0 - \frac{g_p T_2^4}{2} \frac{(5\delta - 10\delta^3 + \delta^5)}{(1 + \delta^2)^5} \\
d_{4i} &= -\frac{g_p T_2^4}{2} \frac{(-1 + 10\delta^2 - 5\delta^4)}{(1 + \delta^2)^5}, \quad d_{5i} = \frac{1}{120} \left(\frac{\partial^5 \beta}{\partial \omega^5} \right)_0 + \frac{g_p T_2^5}{2} \frac{(1 - 15\delta^2 + 15\delta^4 - \delta^6)}{(1 + \delta^2)^6}, \\
d_{5r} &= \frac{g_p T_2^5}{2} \frac{(-6\delta - 6\delta^5 + 20\delta^3)}{(1 + \delta^2)^6}, \quad d_{6r} = \frac{-1}{720} \left(\frac{\partial^6 \beta}{\partial \omega^6} \right)_0 - \frac{g_p T_2^6}{2} \frac{(7\delta - 35\delta^3 + 21\delta^5 - \delta^7)}{(1 + \delta^2)^7}, \\
d_{6i} &= -\frac{g_p T_2^6}{2} \frac{(-1 + 21\delta^2 - 35\delta^4 + 7\delta^6)}{(1 + \delta^2)^7}, \quad q_r = \left(\frac{\partial \beta}{\partial |\psi|^2} \right)_0 + \frac{k_0 n_2}{4A_{eff}}, \quad q_i = \frac{\alpha f n_2}{n(\omega) k_0}, \\
C_r &= \frac{1}{2} \left(\frac{\partial \beta}{\partial |\psi|^4} \right)_0 + \frac{k_0}{16A_{eff1}} \left(\frac{n_2^2}{2} + n_4 \right), \quad C_i = \frac{\alpha f n_4}{16A_{eff1} n(\omega) k_0}, \\
S_r &= \frac{1}{6} \left(\frac{\partial \beta}{\partial |\psi|^6} \right)_0 + \frac{k_0}{64A_{eff2}} \left(n_6 + \frac{n_4 n_2}{n(\omega)} \right), \quad S_{i;} = \frac{\alpha f n_6}{64A_{eff2} n(\omega)}, \\
\gamma_r &= \frac{g_p}{2} \frac{\delta}{1 + \delta^2}, \quad \gamma_i = \alpha_f - \frac{g_p}{2} \frac{1}{1 + \delta^2}, \quad \beta_{1r} = \frac{g_p T_2}{2} \frac{(-2\delta)}{(1 + \delta^2)^2}, \quad \beta_{1i} = \left(\frac{\partial \beta}{\partial \omega} \right)_0 + \frac{g_p T_2}{2} \frac{(1 - \delta^2)}{(1 + \delta^2)^2}, \\
m_i &= \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^2)} \right)_0, \quad m_{1i} = \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^4)} \right)_0, \quad m_{2i} = \frac{1}{6} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^6)} \right)_0, \\
n_r &= \left(\frac{\partial \beta}{\partial (|\psi|^2)_t} \right)_0, \quad n_{1r} = \left(\frac{\partial \beta}{\partial (|\psi|^4)_t} \right)_0, \quad n_{1r} = \left(\frac{\partial \beta}{\partial (|\psi|^4)_t} \right)_0, \quad n_{2r} = \left(\frac{\partial \beta}{\partial (|\psi|^6)_t} \right)_0,
\end{aligned}$$

with $A_{eff} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^4 dx dy}$ is the refractive core area of the fiber, $A_{eff1} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^6 dx dy}$ and $A_{eff2} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^8 dx dy}$. A_{eff1} is normally presented as an experimental quantity as $\frac{3}{4}A_{eff}$ and A_{eff2} as $\frac{1}{2}A_{eff}$ [41]. More precisely, $\psi(z, t)$ is the envelope amplitude of the electric field, t taken as a the retarded time frame and z as the propagation dis-

tance originating from the transformation, $T = t - z/v_g = t - \beta_1$. All the coefficients are real constants. The parameter p_r is for measuring the wave dispersion, while p_i measures the spectral filtering, q_r and q_i stands for the Kerr nonlinearity coefficient and the nonlinear gain-absorption (if positive), respectively. The parameters C_r and C_i describe the nonlinear saturation of the Kerr nonlinearity (if negative) and the saturation of the nonlinear gain-absorption (if negative), respectively. The parameters S_r and S_i are higher-order correction terms to the nonlinear refractive and the nonlinear amplification-absorption gain, respectively. γ_r and γ_i stand for the coefficient of the linear gain (if positive) and frequency shift, respectively. The parameters $d_3 = (d_{3r} + id_{3i})$, $d_4 = (d_{4r} + id_{4i})$, $d_5 = (d_{5r} + id_{5i})$ and $d_6 = (d_{6r} + id_{6i})$ account for the TOD, FOD, fifth-order dispersion (FFOD) and the sixth-order dispersion (SOD), respectively. The terms m_i, m_{1i}, m_{2i} are the cubic, quintic and septic self-steepening pulse edge, respectively and the terms n_r, n_{1r}, n_{2r} represent the cubic, quintic and septic soliton self-frequency shift terms, respectively, which result from the induced Raman scattering processes.

5.2.1 Dimensionless (1+1)D higher-order cubic-quintic-septic CGL equation

In order to simplify the problem at hand, it would be advisable to reduce the number of parameters. Therefore, Eq. (5.1) can be normalized by making use of the transformations $s = \sigma_r z$ for space, $\tau = \sqrt{\frac{\sigma_r}{A_r}} t$ for time and $A = \sqrt{\frac{Q_r}{\sigma_r}} \psi$ for amplitude, which lead to the following dimensionless (1+1)D CGL equation:

$$\begin{aligned}
A_s = & (1 + ic_0)A + (1 + ic_2)A_{\tau\tau} + a_3(1 + ic_3)A_{\tau\tau\tau} + a_4(1 + ic_4)A_{\tau\tau\tau\tau} + a_5(1 + ic_5)A_{\tau\tau\tau\tau\tau} + \\
& a_6(1 + ic_6)A_{\tau\tau\tau\tau\tau\tau} + (1 + i\alpha_1)|A|^2 A + e_2(1 + i\alpha_2)|A|^4 A + e_3(1 + i\alpha_3)|A|^6 A + \\
& ie_4(|A|^2 A)_\tau + ie_5(|A|^4 A)_\tau + ie_6(|A|^6 A)_\tau + e_7(|A|^2)_\tau A + e_8(|A|^4)_\tau A + e_9(|A|^6)_\tau A,
\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}
c_0 &= \frac{\sigma_i}{T_0}, \quad c_2 = \frac{L_{D2}A_i}{T_0^2}, \quad c_3 = \frac{L_{D3}b_{2i}}{T_0^3}, \quad c_4 = \frac{L_{D4}b_{3i}}{T_0^4}, \quad c_5 = \frac{L_{D5}b_{4i}}{T_0^5}, \quad c_6 = \frac{L_{D6}b_{5i}}{T_0^6}, \\
a_3 &= \frac{L_{D2}\sqrt{L_{D2}^3\sigma_r}}{L_{D3}T_0^3}, \quad a_4 = \frac{L_{D2}^3\sigma_r}{L_{D4}T_0^4}, \quad a_5 = \frac{L_{D2}\sqrt{L_{D2}^5\sigma_r^3}}{L_{D5}T_0^5}, \quad a_6 = \frac{L_{D2}^4\sigma_r^2}{L_{D6}T_0^6}, \quad \alpha_1 = \frac{Q_i L_{NL1}}{L_{D2}}, \\
\alpha_2 &= \frac{R_i L_{NL2}}{L_{D2}}, \quad \alpha_3 = \frac{T_i L_{NL1}}{L_{D2}}, \quad e_2 = \frac{\sigma_r L_{NL1}^2}{L_{NL2} L_{D2}}, \quad e_3 = \frac{\sigma_r^2 L_{NL1}^3}{L_{NL3} L_{D2}^2}, \quad e_4 = \frac{L_{D2}^{\frac{3}{2}}}{L_{SS1} \sigma_r^{\frac{1}{2}} T_0}, \\
e_5 &= \frac{\sigma_r^{\frac{3}{2}}}{L_{SS2} L_{D2}^{\frac{1}{2}} T_0}, \quad e_6 = \frac{\sigma_r^{\frac{5}{2}}}{L_{SS3} L_{D2}^{\frac{3}{2}} T_0}, \quad e_7 = \frac{L_{D2}^{\frac{3}{2}}}{L_{SFS1} \sigma_r^{\frac{1}{2}} T_0}, \quad e_8 = \frac{\sigma_r^{\frac{3}{2}}}{L_{SFS2} L_{D2}^{\frac{1}{2}} T_0}, \quad e_9 = \frac{\sigma_r^{\frac{5}{2}}}{L_{SFS3} L_{D2}^{\frac{3}{2}} T_0},
\end{aligned}$$

where $\sigma_r = i\gamma_r = 1/T_0$ and the terms $\sigma_i = -\gamma_i$, $A_i = p_r$, $b_{2i} = d_{3i}$, $b_{3r} = d_{4r}$, $b_{4i} = d_{5r}$, $b_{5i} = d_{6r}$, $Q_i = q_r$, $R_i = C_r$, $T_i = S_r$, $M_i = -m_i$, $M_{1i} = -m_{1i}$, $M_{2i} = -m_{2i}$, $N_r = in_r$, $N_{1r} = in_{1r}$, and $N_{2r} = in_{2r}$. In efforts to obtain the dimensionless coefficients of Eq.(5.8) we have also considered the following inherent lengths: $L_{Dm} = \sum_{m=2}^6 \frac{T_0^m}{\beta_m}$ is the dispersion length (β_m are m-order parameters of dispersion), $L_{NLm} = \sum_{m=1}^3 \frac{T_0^m}{\gamma_m P_{0m}}$ is the nonlinear length (γ_m is the nonlinear amplification-absorption gain term), $L_{SSm} = \sum_{m=1}^3 \frac{T_0^m}{\alpha_m P_{0m}}$ is the self-steepening length (α_m is the self-steepening pulse edge term), and $L_{SFSm} = \sum_{m=1, i=4}^{3,6} \frac{T_0^m}{\alpha_i P_{0m}}$ is the self-frequency shift length (α_i is the self-frequency shift term [98, 215])

5.3 Linear stability analysis and modulational instability gain spectrum on the (1+1)D higher-order cubic-quintic-septic CGL equation

The MI is studied on model Eq. (5.8) via the linear stability analysis of a CW solution, which propagates inside the fiber with the initial input power P , i.e.,

$$A(s, \tau) = \sqrt{P} \exp[i\phi_{NL} + (1 + ic_0)s], \quad (5.9)$$

Table 5.1: Doped fiber parameters for a two-level system

Physical Parameters	Symbols	Values
Wavelength	λ_0	$1.55\mu\text{m}$ [213]
Atomic frequency	ω_a	$4 \times 10^{15}\text{Hz}$ [214]
Fiber loss	α	0.22dB/km [213]
Nonlinear parameter	γ	$2/\text{Wkm}$ [213]
Relaxation time	T_R	3fs [166]
Linear refractive index	n_0	1.45 [98]
Nonlinear refractive index which originate from $\chi^{(3)}$	n_2	$2.7 \times 10^{-13}\text{cm}^2/\text{W}$ [166]
Nonlinear refractive index which originate from $\chi^{(5)}$	n_4	$-7.8 \times 10^{-23}\text{cm}^4/\text{W}^4$ [41]
Nonlinear refractive index which originate from $\chi^{(7)}$	n_6	$7.2 \times 10^{-33}\text{cm}^6/\text{W}^3$ [41]
Dipole relaxation time	T_2	100fs [98]
Effective mode area	A_{eff}	$50\mu\text{m}^2$ [98]
Peak gain	g_p	$6.8/\text{Wkm}$ [166]
GVD coefficient	β_2	$-20\text{ps}^2/\text{km}$ [166]
TOD coefficient	β_3	$0.04\text{ps}^3/\text{km}$ [41]
FOD coefficient	β_4	$0.005\text{ps}^4/\text{km}$ [41]
FFOD coefficient	β_5	$0.06\text{ps}^5/\text{km}$ [41]
SOD coefficient	β_6	$0.07\text{ps}^6/\text{km}$ [41]

Table 5.2: Normalization of the higher-order CGL equation parameters

Unnormalized dimensions	Normalization quantity	Normalized dimensions
Time coordinate	$\tau = \sqrt{\frac{\sigma_r}{A_r}} t$	$\tau = t/T_0$
Space coordinate	$s = \sigma_r z$	$s = z/Z_0$
Wave amplitude	$A = \sqrt{\frac{Q_r}{\sigma_r}} \psi$	$A = \psi/\psi_0$
GVD coefficient	$c_2 = \frac{A_i}{A_r}$	$c_2 = \frac{L_{D2} A_i}{T_0^2}$
TOD coefficient	$c_3 = \frac{b_{2i}}{b_{2r}}$	$c_3 = \frac{L_{D3} b_{2i}}{T_0^3}$
FOD coefficient	$c_4 = \frac{b_{3i}}{b_{3r}}$	$c_4 = \frac{L_{D4} b_{3i}}{T_0^4}$
FFOD coefficient	$c_5 = \frac{b_{4i}}{b_{4r}}$	$c_5 = \frac{L_{D5} b_{4i}}{T_0^5}$
SOD coefficient	$c_6 = \frac{b_{5i}}{b_{5r}}$	$c_6 = \frac{L_{D6} b_{5i}}{T_0^6}$
cubic nonlinear coefficient	$\alpha_1 = \frac{Q_i}{Q_r}$	$\alpha_1 = \frac{Q_i L_{NL1}}{L_{D2}}$
quintic nonlinear coefficient	$\alpha_2 = \frac{R_i}{R_r}$	$\alpha_2 = \frac{R_i L_{NL2}}{L_{D2}}$
septic nonlinear coefficient	$\alpha_3 = \frac{T_i}{T_r}$	$\alpha_3 = \frac{T_i L_{NL3}}{L_{D2}^2}$
cubic self-steepening coefficient	$e_4 = \frac{M_i}{\sqrt{\sigma_r} A_r}$	$e_4 = \frac{L_{D2}^{\frac{3}{2}}}{L_{SS1} \sigma_r^{\frac{1}{2}} T_0}$
quintic self-steepening coefficient	$e_5 = \frac{M_{1i} \sigma_r^{3/2}}{Q_r^2 A_r^{1/2}}$	$e_5 = \frac{\sigma_r^{\frac{3}{2}}}{L_{SS2} L_{D2}^{\frac{1}{2}} T_0}$
septic self-steepening coefficient	$e_6 = \frac{M_{2i} \sigma_r^{5/2}}{Q_r^3 A_r^{1/2}}$	$e_6 = \frac{\sigma_r^{\frac{5}{2}}}{L_{SS3} L_{D2}^{\frac{3}{2}} T_0}$
Third-order self-frequency shift	$e_7 = \frac{N_r}{\sqrt{\sigma_r} A_r}$	$e_7 = \frac{L_{D2}^{\frac{3}{2}}}{L_{SFS1} \sigma_r^{\frac{1}{2}} T_0}$
Fifth-order self-frequency shift	$e_8 = \frac{N_{1r} \sigma_r^{3/2}}{A_r^{1/2} Q_r^2}$	$e_8 = \frac{\sigma_r^{\frac{3}{2}}}{L_{SFS2} L_{D2}^{\frac{1}{2}} T_0}$
Seventh-order self-frequency shift	$e_8 = \frac{N_{2r} \sigma_r^{5/2}}{A_r^{1/2} Q_r^3}$	$e_9 = \frac{\sigma_r^{\frac{5}{2}}}{L_{SFS3} L_{D2}^{\frac{3}{2}} T_0}$
Nonlinear gain parameter	$c_0 = \frac{\sigma_i}{\sigma_r}$	$c_0 = \frac{\sigma_i}{T_0}$

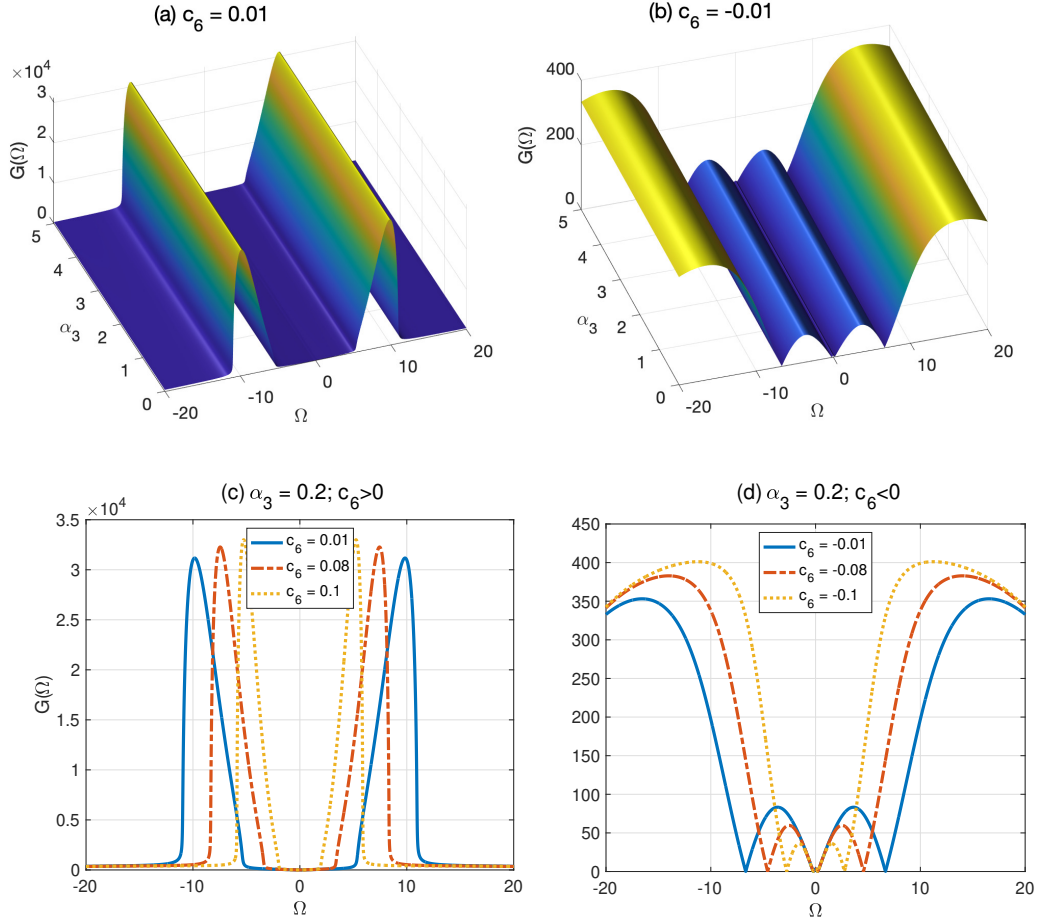


Figure 5.1: Panels (a) and (b) show the MI gain $G(\Omega)$ as a function of the septic nonlinear coefficient e_3 and the frequency Ω for positive and negative values of the SOD coefficient c_6 , respectively. Panels (c) and (d) display $G(\Omega)$ versus the frequency Ω for $e_3 = 0.2$ and for positive and negative values of c_6 . The other parameter values are: $P = 20$, $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$ and $e_9 = 6 \times 10^{-4}$.

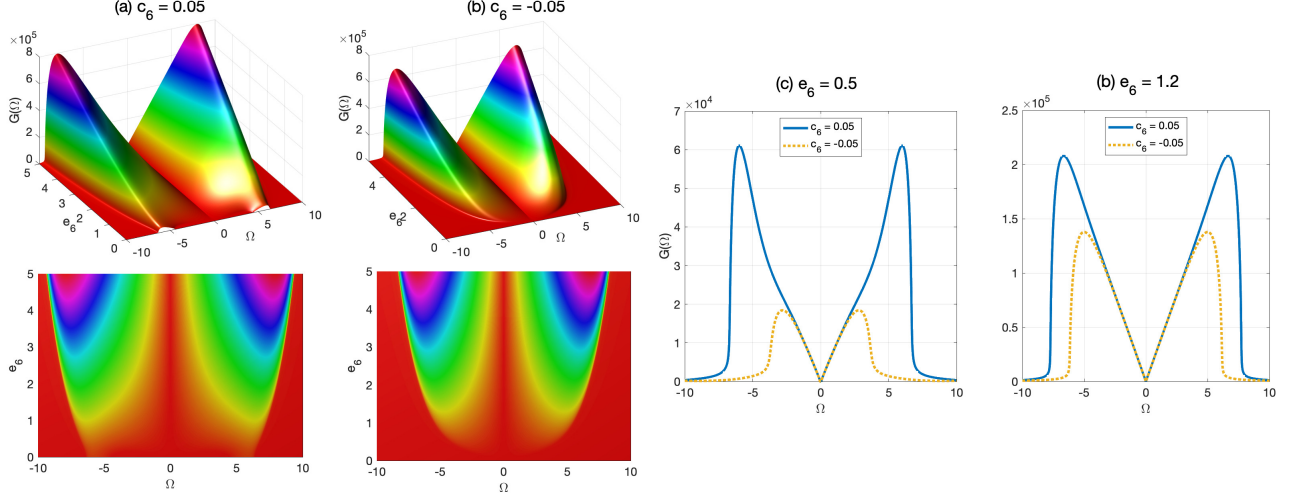


Figure 5.2: Panels (a) and (b) show the MI gain $G(\Omega)$ versus the sixth-order self-steepening parameter e_6 and Ω under the influence of SOD coefficient c_6 . Panels (c) and (d) show $G(\Omega)$ versus the frequency Ω when e_6 takes the respective values 0.5 and 1.2 for $c_6 > 0$ and $c_6 < 0$. The used parameter values are: $P = 20$, $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$ and $e_9 = 6 \times 10^{-4}$.

, where $\phi_{NL} = Pe_2 \int_0^s \exp(2s)ds + P^2e_3 \int_0^s \exp(4s)ds + P^3e_4 \int_0^s \exp(6s)ds$ is the non-linear phase shift due to the self-phase modulation (SPM) and higher-order nonlinear terms. The stability of solution Eq. (5.9) requires the introduction of slight perturbation so that

$$A(s, \tau) = \left[\sqrt{P} + a(s, \tau) \right] e^{i\phi_{NL}}, \quad (5.10)$$

where $a(s, \tau)$ is denoted as the small perturbation, with $|a(s, \tau)| \ll \sqrt{P}$ being the complex field. This perturbed solution means that if the perturbed field evolves exponentially, the steady state will become unstable. Substituting Eq. (5.10) in Eq. (5.8), and linearizing around the unperturbed solution, lead to the following equation for the

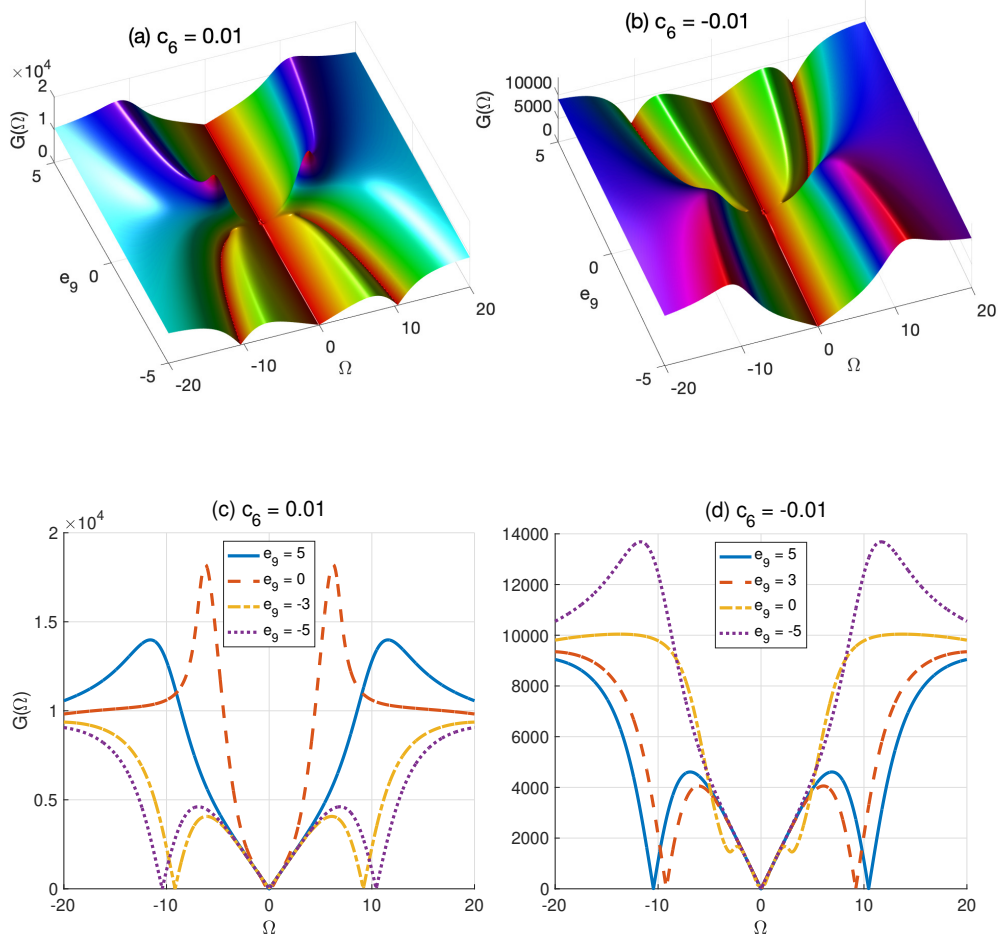


Figure 5.3: Panels (a) and (b) display plots of the MI gain $G(\Omega)$ versus the frequency Ω and the septic Raman scattering parameter e_9 . Each of the panels contains results for three increasing values of the septic Raman scattering parameter e_9 , where (a) corresponds to $c_6 = 0.01$, (b) to $c_6 = -0.01$. Panels (c) and (d) give the evolution of the gain with changing e_9 for $c_6 = 0.01$ and $c_6 = -0.01$, respectively.

perturbed field:

$$\begin{aligned}
a_s = & -[(1 + ic_0)(\sqrt{P} + a) + (1 + ic_2)a_{\tau\tau} + a_3(1 + ic_3)a_{\tau\tau\tau} + a_4(1 + ic_4)a_{\tau\tau\tau\tau} \\
& + a_5(1 + ic_5)a_{\tau\tau\tau\tau\tau} + a_6(1 + ic_6)a_{\tau\tau\tau\tau\tau\tau} + P(1 + i\alpha_1)(a + a^*) + 2P^2e_2(1 + i\alpha_2)(a + a^*) \\
& + 3P^3e_3(1 + i\alpha_3)(a + a^*) + ie_4P(2a_\tau + a_\tau^*) + ie_5P^2(3a_\tau + 2a_\tau^*) \\
& + ie_6P^3(4a_t + 3a_t^*) + e_7P(a_\tau + a_\tau^*) + 2e_8P^2(a_\tau + a_\tau^*) + 3e_9P^3(a_\tau + a_\tau^*)], \quad (5.11)
\end{aligned}$$

where a^* is the complex conjugate of the perturbed field. Further, solution for Eq. (5.11) can be adopted in the form

$$a(s, \tau) = u(s)e^{-i\Omega\tau} + v(s)e^{i\Omega\tau} \quad (5.12)$$

From the above equation $u(s)$ and $v(s)$ are the complex perturbation fields and Ω is the complex modulation frequency. Making use of the ansatz of the perturbed field of Eq. (5.12) in the linearized Eq. (5.11) leads to a 2×2 matrix for the perturbed fields shown in the Appendix:

$$i \frac{\partial}{\partial s} \begin{pmatrix} u(s) \\ v^*(s) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} u(s) \\ v^*(s) \end{pmatrix}, \quad (5.13)$$

$v^*(s)$ is described as the complex conjugate of the field $v(s)$, with the matrix elements being given by

$$\begin{aligned}
T_{11} &= -(1 + ic_2)\Omega^2 + ia_3(1 + ic_3)\Omega^3 + a_4(1 + ic_4)\Omega^4 - ia_5(1 + ic_5)\Omega^5 - a_6(1 + ic_6)\Omega^6 \\
&\quad + n_{11} - i\Omega P(2ie_4 + 3ie_5P + 4e_6P^2 + e_7 + 2e_8P + 3e_9P^2), \\
T_{12} &= n_{11} - i\Omega P(ie_4 + 2ie_5P + 3ie_6P^2 + e_7 + 2e_8P + 3e_9P^2), \\
T_{21} &= -[n_{11} - i\Omega P(ie_4 + 2ie_5P + 3ie_6P^2 + e_7 + 2e_8P + 3e_9P^2)], \\
T_{22} &= -[-(1 + ic_2)\Omega^2 + ia_3(1 + ic_3)\Omega^3 + a_4(1 + ic_4)\Omega^4 - ia_5(1 + ic_5)\Omega^5 - a_6(1 + ic_6)\Omega^6 \\
&\quad + n_{11} - i\Omega P(2ie_4 + 3ie_5P + 4e_6P^2 + e_7 + 2e_8P + 3e_9P^2)].
\end{aligned} \tag{5.14}$$

In order to find the MI gain, we should keep in mind that the wavenumber K of the perturbation can be found as the eigenvalue of the matrix $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, through the characteristic equation $\text{Det}|T - KI| = 0$, with I being a 2×2 identity matrix. This leads to the dispersion relation

$$K = \frac{1}{2} \left[T_{11} + T_{22} + \sqrt{(T_{11} - T_{22})^2 + 4T_{12}T_{21}} \right]. \tag{5.15}$$

For MI to take place, the wavenumber K should get a nonzero imaginary part that will be responsible for an exponential growth of the perturbed amplitude. In general, the occurrence of MI is predicted through the power gain, defined as $G(\Omega) = 2\text{Im}(K)$, which is obtained in this particular case in the form

$$G(\Omega) = 2 \sqrt{
\begin{aligned}
&2n_{11}^2 + (P^2a^2 - 2(1 + ic_2)n_{11})\Omega^2 + ((1 + ic_2)^2 + 2a_4(1 + ic_4)n_{11})\Omega^4 \\
&- 2((1 + ic_2)a_4(1 + ic_4) + a_6(1 + ic_6)n_{11})\Omega^6 + (a_4^2(1 + ic_4)^2 + 2(1 + ic_2)a_6(1 + ic_6))\Omega^8, \\
&- 2a_4(1 + ic_4)a_6(1 + ic_6)\Omega^{10} + a_6^2(1 + ic_6)^2\Omega^{12}
\end{aligned}
} \tag{5.16}$$

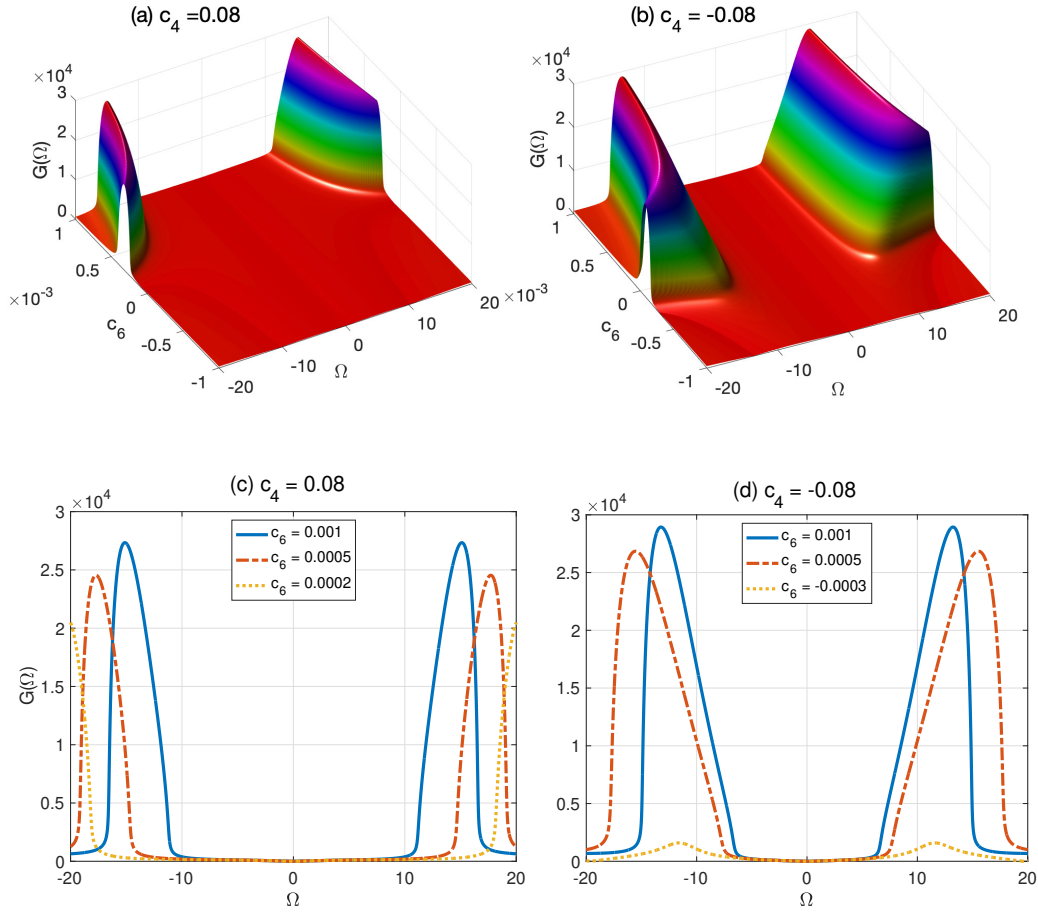


Figure 5.4: The panels show the MI gain $G(\Omega)$ versus the SOD parameter c_6 and Ω for different values of the FOD coefficient c_4 , under normal dispersion regime, i.e., $c_2 < 0$, with the other parameters being: $P = 20$, $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$ and $e_9 = 6 \times 10^{-4}$.

where $a = ie_4 + 2ie_5P + 3ie_6P^2 + e_7 + 2e_8P + 3e_9P^2$ and $n_{11} = (1 + ic_0)\sqrt{P} + (1 + i\alpha_1)P + e_2(1 + i\alpha_2)P^2 + e_3(1 + \alpha_3)P^3$.

5.3.1 Influence of nonlinear and dispersion effects on the modulational instability gain

In order to proceed we consider the parameter values of Table 1 and, using the transformations formulas of Table 2 we get the following dimensionless parameters to be used in the rest of this paper: $P = 20$, $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term

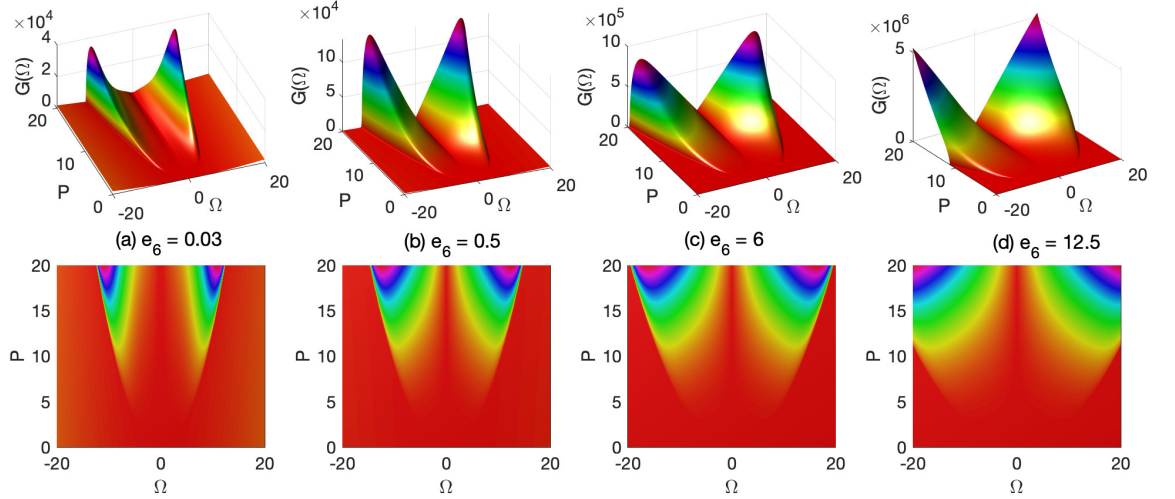


Figure 5.5: The panels show the MI gain $G(\Omega)$ versus the input power P and Ω for different values of e_6 , the septic self-steepening parameter e_6 . The rest of parameter values are: $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$ and $e_9 = 6 \times 10^{-4}$

$\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$ and $e_9 = 6 \times 10^{-4}$. The impact of the dispersion parameters c_2 and c_4 on the evolution of MI gain through the perturbation frequency Ω , with respect to the SOD parameter c_6 , is particularly addressed. The corresponding results, when $c_2 = 0.001$, are summarized in Fig. 5.4, where panels (a) and (b) correspond, respectively, to $c_4 = 0.08$ and $c_4 = -0.08$, and panels (c) and (d) are their corresponding representations versus the frequency Ω with changing c_6 . In general, under normal dispersion regime, i.e., $c_2 > 0$, it is obvious that the bandwidth and position of the MI lobes are functions of c_6 . However, for $c_4 > 0$, the instability tends to disappear for c_6 decreasing, which is the inverse for $c_4 < 0$. One should, however, notice that in both cases, the instability is characterized by the presence of lobes that are similar and symmetric, which at this stage justifies the suitability of the model under our study. But when we focus on the anomalous dispersion regime, i.e., $c_2 < 0$, two cases are also regarded, that is, c_4 positive and negative, as depicted in Fig. 5.4. As previously noticed, MI manifests itself by the presence of two symmetric

and identical sidebands, but, depending on the sign of c_4 , there is a threshold value of c_6 below which instability is not possible. This is clearly shown in Fig. 5.4(c) and (d), which respectively correspond to $c_4 > 0$ and $c_4 < 0$.

The evolution of nonlinear pulses in the optical fiber is highly dependent on the balance between nonlinear and dispersive effects. This implies the importance of nonlinear effects in the process of optical soliton characterization and propagation. In the chapter before, we have seen that the dispersion deeply influences the occurrence of MI, especially when FOD and SOD dispersions are involved. Here, we are mainly interested in how the balance between higher-order nonlinearities may combine with the FOD and SOD for MI to take place, this under normal frequency regime. To start, we study the features of the septic nonlinearity under the effect of positive and negative SOD. This is shown in Fig. 5.1, where panels (a) and (b) have been recorded, respectively, for $c_6 = 0.01$ and $c_6 = -0.01$, with their corresponding 2D representations versus the frequency Ω in Figs. 5.1(c) and (d). In the first case, i.e., $c_6 > 0$, the CW is unstable for any value of e_3 , although the frequency bandwidth is limited by two symmetric and identical lobes. According to Fig. 5.1(c), the distance between the lobes is very sensitive to the change in the SOD whose increase causes the decrease of the bandwidth as well as an increase of the intensity. When $c_6 < 0$, the MI frequency domain is also sensitive to the decrease of c_6 as noticed in Fig. 5.1(d). Instead of two lobes like in the previous case, here we rather have two symmetric wings separated by two minor lobes around $\Omega = 0$. The intensity increases when c_6 gets smaller, while the frequency bandwidth responsible for MI, supported by the large MI wings, gets expanded, therefore reducing the windows of instability materialized by the minor lobes.

Another illustration of how nonlinearity and dispersion can combine and monitor the emergence of modulated waves is given by Fig. 5.2, where the competition between the SOD and the septic self-steepening gives rise to lobes of instability whose intensity and expansion are very sensitive to the sign and value of c_6 . For $c_6 > 0$, the instability

takes place for all values of e_6 , but smaller values of the latter reduce the gain intensity and frequency bandwidth as supported by Figs. 5.2(a) and (c).

When $c_6 < 0$, we also notice the presence of two gain peaks which tend to disappear when $e_6 \rightarrow 0$ (see Fig. 5.2(b)). Fig. 5.3 shows results related to the competition between the SOD and the septic Raman scattering. As a whole, when $c_6 > 0$, the instability domain shows two different behaviors for $e_9 > 0$ and $e_9 < 0$. In Figs. 5.3(a) and (c), where $c_6 > 0$, the MI gain, for $e_9 > 0$, shows some wings of instability with frequency peaks at $e_9 = 0$, while for $e_9 < 0$, the instability is supported by two symmetric and identical frequency lobes of lower intensity, between lateral lobes of high intensity. Moreover, when $c_6 < 0$, as presented in Figs. 5.3(b) and (d), the inverse features of the case $c_6 < 0$ are obtained, except that the gain intensity is lower here and the peaks at $e_9 = 0$ are inexistent. We should also notice that some of the features ostensible in this case are shared with those obtained for the septic nonlinearity which, once more, supports the involvement of higher nonlinear effects in the process of MI on the one hand, and their mutual competition that may surely be regulated through the right choice of dispersion parameters, on the second hand. In the meantime, the frequency bandwidths detected are those expected to support the break-up of the CW into modulated waves and soliton-like objects. Nevertheless, one aspect that should not be ignored is how such a process can be monitored by the total input power P , combined with the modulational frequency Ω . This is clearly shown in Fig. 5.5, where in general the MI gain increases with P , while the frequency bandwidth gets enlarged with increasing the septic self-steepening parameter.

5.4 Summary

In summary, we have successfully derived an extended (1+1)D higher-order cubic-quintic-septic CGL equation with the third, fourth, fifth, and sixth-order dispersions and the cubic, quintic, and septic nonlinear terms, describing the dynamics of ex-

tremely short pulses in nonlinear doped optical fibers. Through the linear stability analysis of CW, the MI gain has explicitly been derived and used to comprehensively explore the features of MI under the effects of the SOD and the higher-order nonlinear parameters. The subsequent results can therefore be summarized as follows: (i) Under normal and anomalous regimes, the FOD parameter has been found to importantly affect the involvement of the SOD in the process of MI. Particularly, in the anomalous regime, the instability was quenched for negative SOD, while the intensity and expansion of the existing bandwidths were found to respond to increasing values of c_6 . (ii) The MI features of the septic nonlinearity under the effect of positive SOD have shown that the CW may be unstable for any value of the septic nonlinear coefficient. Under negative SOD, two symmetric wings separated by two minor lobes around $\Omega = 0$ were observed. In the same vein, the effect of the septic self-steepening and Raman scattering on the occurrence of MI displayed more interesting features, with a new gain profile and, more importantly, their sensitive response to the change in the SOD value and sign. Additionally, the total input power has also been found to bring about some new effects to the manifestation of MI, this in competition with the septic self-steepening under normal frequency regime and positive SOD. Moreover, the model exhibits a maximum MI spectral gain value much greater compared to the Higher-order NLS equation [165] and the CGL equation[212] therefore, offers a larger bandwidth with possible more ultra-short pulses (solitons)[198], therefore making it a more viable solution for long-distance optical fiber communication. We should however stress that one of the main difficulty in the derived model was the high number of parameters, the reason why we entirely work on a non-dimensional equivalence of the proposed CGL equation.

General Conclusion

The influence of higher-order dispersions and non-Kerr nonlinearities on soliton pulse, induced by modulational instability in doped and undoped optical fibers, was intensively studied. The first chapter focused on introducing the background overview on optical fibers, elucidating on solitons and soliton formation, modulational instability process, doped optical fibers, together with the structure and applications of optical fibers. Chapter 2 was based on explaining more on the linear (modal, chromatic, and polarization mode dispersions) and nonlinear effects associated with the wave propagations in optical fibers. In nonlinear effects, special attention was paid to the Kerr nonlinearities, namely; SPM, XPM, and FMW in WDM systems. Furthermore, the non-Kerr nonlinearities were also not forgotten, second-order (SFG, SHG, DFG, OP, OPG, oscillation, and Pockels effect) and third-order effects (THG, FMW, DFMW, optical Kerr effect, and quadratic electro-optic effect) were carefully enumerated. The third chapter introduced the modeling of the standard NLS equation. Subsequently, the MI growth rate of the equation was presented showing centrosymmetric side peaks.

The fourth and fifth chapters were the basis for our results and discussion of our findings. In chapter 4, we explicitly investigated the linear stability analysis of a newly derived HNLS equation, where we were interested in how the balance between higher-order nonlinearities combine with the FOD and SOD under the normal and anomalous regimes. MI growth rate displayed sensitivity towards the septic SFS and SS effect terms for special cases. In one of the special cases, the MI side lobes or peaks tend to disappear. It was clear, that the SOD produced symmetrical broadening effects

whereas the TOD and FFOD displayed asymmetrical broadening effects. It was also observed that septic non-Kerr nonlinearities proved to be crucial, as they decreased the MI gain peak values. Significant enough, the growth rate displayed some sensitivity to the septic SFS and SS effect parameters, an increase in the redshift spectra was also observed.

In chapter 5, we comprehensively discussed the theory of MI, as well as the role played by higher-order dispersions and nonlinearities on the newly derived (1+1)D higher-order CGL equation. Consequently, it was found that under the anomalous and normal regimes the FOD significantly affected the SOD in the MI process. The MI appeared to be very sensitive to any value of the septic nonlinearities, as it appeared to be unstable. Comparably, all the models beautifully exhibited high MI spectral gain values, in which the newly derived (1+1)D higher-order CGL spiked even higher MI spectral values, therefore offering a larger bandwidth with more possible ultrashort soliton pulses responsible for long-distance optical fiber communication. Moreover, our results indicated greater stability for the systems and stable soliton pulses of the equations. We should stress that the newly derived (1+1)D higher-order CGL equation is an even better and more interesting new feature in doped optical fibers.

Interesting extensions of this thesis may be:

- Investigation of higher-order MI dynamics in the higher-order NLSE systems.
- The analysis of breather dynamics in terms of the Darboux transformation will allow us to interpret the complex evolution at low instability modulation frequencies in terms of the superposition of multiple elementary instabilities governed by distinct dynamical parameter sets.
- Use the total soliton energy and perform many numerical simulations in order to observe a sequence of bifurcations such as period-doubling bifurcations or more complex dynamics such as bifurcation diagram showing the period tripling bifurcation with additional long period pulsations associated with the higher-order CGL equation.

-
- Focused on explicit analytic form for the two-breather solution of the higher-order NLSE systems with imaginary eigenvalues to describe various nonlinear combinations of Akhmediev breathers and Kuznetsov-Ma solitons.
 - Studied families of higher-order rational solutions of the higher-order NLSE systems with free real parameters, such as circular rogue wave clusters.
 - Spatiotemporal solitons in optical media have attracted much attention. A spatiotemporal soliton is referred to as a "light bullet" localized in all spatial dimensions and in the time dimension. So, the generation of light bullet in higher-order NLSE systems and in higher-order CGL media might be of importance in soliton-based communication systems.

Appendix

Appendix 1: HNLS equation

Example 1: Presentation of the previously known HNLS equation

2×2 matrix terms

$$T_{11} = [-\beta_2\Omega^2 - \beta_3\Omega^3 + \beta_4\Omega^4 + \beta_5\Omega^5 - \beta_6\Omega^6 + n_1 + \Omega P_0(2\alpha_1 + \alpha_2 + 3\alpha_3P_0 + 2\alpha_4P_0 + 4\alpha_5P_0^2 + 3\alpha_6P_0^2) - i\Gamma], \quad (\text{A.1a})$$

$$T_{12} = [n_1 + \Omega P_0(\alpha_1 + \alpha_2 + 2\alpha_3P_0 + 2\alpha_4P_0 + 3\alpha_5P_0^2 + 3\alpha_6P_0^2)], \quad (\text{A.1b})$$

$$T_{21} = -[n_1 - \Omega P_0(\alpha_1 + \alpha_2 + 2\alpha_3P_0 + 2\alpha_4P_0 + 3\alpha_5P_0^2 + 3\alpha_6P_0^2)], \quad (\text{A.1c})$$

$$T_{22} = -[-\beta_2\Omega^2 + \beta_3\Omega^3 + \beta_4\Omega^4 - \beta_5\Omega^5 - \beta_6\Omega^6 + n_1 - \Omega P_0(2\alpha_1 + \alpha_2 + 3\alpha_3P_0 + 2\alpha_4P_0 + 4\alpha_5P_0^2 + 3\alpha_6P_0^2) + i\Gamma]. \quad (\text{A.1d})$$

Example 2: Derivation of the new HNLS equation for ultrashort pulses

A Taylor expansion of the propagation constant $\beta(\omega)$ up-to the sixth-order expansion where $\beta(\omega)$ is expressed as

$$\begin{aligned}
\beta(\omega) = & \beta_0 + \left(\frac{\partial \beta}{\partial \omega} \right)_0 (\omega - \omega_0) + \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega^2} \right)_0 (\omega - \omega_0)^2 + \frac{1}{6} \left(\frac{\partial^3 \beta}{\partial \omega^3} \right)_0 (\omega - \omega_0)^3 \\
& + \frac{1}{24} \left(\frac{\partial^4 \beta}{\partial \omega^4} \right)_0 (\omega - \omega_0)^4 + \frac{1}{120} \left(\frac{\partial^5 \beta}{\partial \omega^5} \right)_0 (\omega - \omega_0)^5 + \frac{1}{720} \left(\frac{\partial^6 \beta}{\partial \omega^6} \right)_0 (\omega - \omega_0)^6 \\
& + \left(\frac{\partial \beta}{\partial (|\psi|^2)} \right)_0 |\psi|^2 + \frac{1}{2} \left(\frac{\partial \beta}{\partial (|\psi|^4)} \right)_0 |\psi|^4 + \frac{1}{6} \left(\frac{\partial \beta}{\partial (|\psi|^6)} \right)_0 |\psi|^6 \\
& + \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^2)} \right)_0 (\omega - \omega_0) |\psi|^2 + \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^4)} \right)_0 (\omega - \omega_0) |\psi|^4 + \frac{1}{6} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^6)} \right)_0 (\omega - \omega_0) |\psi|^6 \\
& + \left(\frac{\partial \beta}{\partial (|\psi|^2)_t} \right)_0 (|\psi|^2)_t + \frac{1}{2} \left(\frac{\partial \beta}{\partial (|\psi|^4)_t} \right)_0 (|\psi|^4)_t + \frac{1}{6} \left(\frac{\partial \beta}{\partial (|\psi|^6)_t} \right)_0 (|\psi|^6)_t.
\end{aligned} \tag{A.2a}$$

(1+1)D Higher-order cubic-quintic-septic CGL equation for ultrashort pulses

A Taylor expansion of χ_d up-to the sixth-order expansion where $\chi_d(\omega)$ [41] is found to be;

$$\begin{aligned}
\chi_d(\omega) = & \frac{g_p n(\omega)}{k_0} \left[- \frac{(-\delta + i)}{1 + \delta^2} + \frac{(1 - \delta^2 + 2i\delta)(\omega - \omega_0)T_2}{(1 + \delta^2)^2} - \right. \\
& \frac{(3\delta - \delta^3 - i + 3i\delta^2)(\omega - \omega_0)^2 T_2^2}{(1 + \delta^2)^3} \\
& + \frac{(-1 - \delta^4 - 4i\delta + 4i\delta^3 + 6\delta^2)(\omega - \omega_0)^3 T_2^3}{(1 + \delta^2)^4} - \\
& \frac{(-5\delta + 10\delta^3 - \delta^5 + i_1 0i\delta^2 + 5i\delta^4)(\omega - \omega_0)^4 T_2^4}{(1 + \delta^2)^5} \\
& - \frac{(1 - 15\delta^2 + 15\delta^4 - \delta^6 + 6i\delta + 6i\delta^5 - 20i\delta^3)(\omega - \omega_0)^5 T_2^5}{(1 + \delta^2)^6} \\
& \left. - \frac{(7\delta - 35\delta^3 + 21\delta^5 - \delta^7 - i + 21i\delta^2 - 35i\delta^4 + 7i\delta^6)(\omega - \omega_0)^6 T_2^6}{(1 + \delta^2)^7} \right],
\end{aligned} \tag{B.1}$$

where k_0 is the wave number and $g_p = \sigma_s W(r)$ is the peak gain.

$\Delta\beta$ is defined by eq.(3.32) where β is defined as the propagation constant hence, the following equation is obtained.

$$\begin{aligned}
\Delta\beta = & k_0 \left(n_2 + \frac{i\alpha_f n_2}{n(\omega)k_0} \right) \frac{|\psi|^2}{4A_{eff}} + k_0 \left(\frac{n(\omega)}{2} + n_4 + \frac{i\alpha_f n_4}{n(\omega)k_0} \right) \frac{|\psi|^4}{16A_{eff}} + \\
& k_0 \left(n_6 + \frac{n_4 n_2}{n(\omega)} + \frac{i\alpha_f n_6}{n(\omega)k_0} \right) \frac{|\psi|^6}{64A_{eff}} \\
& + i\alpha_f + \frac{g_p}{2n} \left[-\frac{(-\delta + i)}{1 + \delta^2} + \frac{(1 - \delta^2 + 2i\delta)(\omega - \omega_0)T_2}{(1 + \delta^2)^2} \right. \\
& - \frac{(3\delta - \delta^3 - i + 3i\delta^2)(\omega - \omega_0)^2 T_2^2}{(1 + \delta^2)^3} \\
& + \frac{(-1 - \delta^4 - 4i\delta + 4i\delta^3 + 6\delta^2)(\omega - \omega_0)^3 T_2^3}{(1 + \delta^2)^4} \\
& - \frac{(-5\delta + 10\delta^3 - \delta^5 + i_1 0i\delta^2 + 5i\delta^4)(\omega - \omega_0)^4 T_2^4}{(1 + \delta^2)^5} \\
& - \frac{(1 - 15\delta^2 + 15\delta^4 - \delta^6 + 6i\delta + 6i\delta^5 - 20i\delta^3)(\omega - \omega_0)^5 T_2^5}{(1 + \delta^2)^6} \\
& \left. - \frac{(7\delta - 35\delta^3 + 21\delta^5 - \delta^7 - i + 21i\delta^2 - 35i\delta^4 + 7i\delta^6)(\omega - \omega_0)^6 T_2^6}{(1 + \delta^2)^7} \right].
\end{aligned} \tag{B.2}$$

The equation below is adopted from eq.(3.34) by substituting $\Delta\beta(\omega)$ (eq.(B.2)) and

$\beta(\omega)$ (eq.(5.1)) into eq.(3.34)

$$\begin{aligned}
\frac{\partial \tilde{\psi}}{\partial z} = & i \left[i \left(\frac{\partial \beta}{\partial \omega} \right)_0 \frac{\partial \psi}{\partial t} - \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega^2} \right)_0 \frac{\partial^2 \psi}{\partial t^2} - \frac{i}{6} \left(\frac{\partial^3 \beta}{\partial \omega^3} \right)_0 \frac{\partial^3 \psi}{\partial t^3} + \right. \\
& \frac{1}{24} \left(\frac{\partial^4 \beta}{\partial \omega^4} \right)_0 \frac{\partial^4 \psi}{\partial t^4} + \frac{i}{120} \left(\frac{\partial^5 \beta}{\partial \omega^5} \right)_0 \frac{\partial^5 \psi}{\partial t^5} - \frac{1}{720} \left(\frac{\partial^6 \beta}{\partial \omega^6} \right)_0 \frac{\partial^6 \psi}{\partial t^6} \\
& + \left(\frac{\partial \beta}{\partial(|\psi|^2)} \right)_0 |\psi|^2 \psi + \frac{1}{2} \left(\frac{\partial \beta}{\partial(|\psi|^4)} \right)_0 |\psi|^4 \psi + \\
& \frac{1}{6} \left(\frac{\partial \beta}{\partial(|\psi|^6)} \right)_0 |\psi|^6 \psi + i \left(\frac{\partial^2 \beta}{\partial \omega \partial(|\psi|^2)} \right)_0 \frac{\partial(|\psi|^2 \psi)}{\partial t} + \\
& \frac{i}{2} \left(\frac{\partial^2 \beta}{\partial \omega \partial(|\psi|^4)} \right)_0 \frac{\partial(|\psi|^4 \psi)}{\partial t} + \frac{i}{6} \left(\frac{\partial^2 \beta}{\partial \omega \partial(|\psi|^6)} \right)_0 \frac{\partial(|\psi|^6 \psi)}{\partial t} + \\
& k_0 \left(n_2 + \frac{i \alpha_f n_2}{n(\omega) k_0} \right) \frac{|\psi|^2}{4 A_{eff}} + k_0 (n(\omega) + n_4 + \frac{i \alpha_f n_4}{n(\omega) k_0}) \frac{|\psi|^4}{4 A_{eff}} + \\
& k_0 \left(n_6 + \frac{n_4 n_2}{n(\omega)} + \frac{i \alpha_f n_6}{n(\omega) k_0} \right) \frac{|\psi|^6}{4 A_{eff}} \\
& + i \alpha_f \psi + \frac{g_p}{2 n(\omega)} \left[- \frac{(-\delta + i)}{1 + \delta^2} \psi + \frac{(1 - \delta^2 + 2i\delta)(\omega - \omega_0) T_2}{(1 + \delta^2)^2} \right. \\
& - \frac{(3\delta - \delta^3 - i + 3i\delta^2)(\omega - \omega_0)^2 T_2^2}{(1 + \delta^2)^3} \\
& + \frac{(-1 - \delta^4 - 4i\delta + 4i\delta^3 + 6\delta^2)(\omega - \omega_0)^3 T_2^3}{(1 + \delta^2)^4} \\
& - \frac{(-5\delta + 10\delta^3 - \delta^5 + i - 10i\delta^2 + 5i\delta^4)(\omega - \omega_0)^4 T_2^4}{(1 + \delta^2)^5} \\
& - \frac{(1 - 15\delta^2 + 15\delta^4 - \delta^6 + 6i\delta + 6i\delta^5 - 20i\delta^3)(\omega - \omega_0)^5 T_2^5}{(1 + \delta^2)^6} \\
& \left. - \frac{(7\delta - 35\delta^3 + 21\delta^5 - \delta^7 - i + 21i\delta^2 - 35i\delta^4 + 7i\delta^6)(\omega - \omega_0)^6 T_2^6}{(1 + \delta^2)^7} \right].
\end{aligned} \tag{B.3}$$

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List of Publications

1. **K. K. Ndebele**, C. B. Tabi, and T. C. Kofané, “*Modulational instability in non-linear doped optical fiber induced by the cubic-quintic-septic complex Ginzburg-Landau equation with higher-order dispersions*,” JOSA B **37**, A214-A227 (2020).
2. **K. K. Ndebele**, C. B. Tabi, C. G. Tiofack Latchio, and T. C. Kofané, “*Higher-order dispersion and nonlinear effects of optical fibers under septic self-steepening and self-frequency shift*”, Physical Review E **104**, 044208 (2021).

Modulational instability in nonlinear doped optical fiber induced by the cubic–quintic–septic complex Ginzburg–Landau equation with higher-order dispersions

KARABO KEFILWE NDEBELE,^{1,3} CONRAD BERTRAND TABI,^{1,*}  AND TIMOLÉON CRÉPIN KOFANÉ^{1,2,4}

¹Department of Physics and Astronomy, Botswana International University of Science and Technology, Private Bag 16 Palapye, Botswana

²Laboratoire de Mécanique, Département de Physique, Faculté des Sciences, Université de Yaoundé I, B.P. 812 Yaoundé, Cameroon

³e-mail: karabo.ndebele@studentmail.biust.ac.bw

⁴e-mail: tckofane@yahoo.com

*Corresponding author: tabic@biust.ac.bw

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Considering the theory of electromagnetic waves, from the Maxwell's equations, we introduce an extended $(1 + 1)$ D complex Ginzburg–Landau equation with third-, fourth-, fifth-, and sixth-order dispersions and the cubic, quintic, and septic nonlinear terms, describing the dynamics of extremely short pulses in nonlinear doped optical fibers. Furthermore, we study the modulational instability of the plane wave theoretically using the linear stability analysis, via the gain spectrum. The linear theory predicts instability for any amplitude of the primary wave, where not only the higher-order dispersion terms highly influence the occurrence of modulational instability but also the combined effect between dispersion and higher-order nonlinear terms modifies the gain spectrum. Mainly, importance is paid to the competition between sixth-order dispersion and septic non-Kerr nonlinearities. This gives rise to new spectra of instability and therefore confirms the validity of both the proposed model and subsequent modulational instability manifestation. © 2020 Optical Society of America

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1. INTRODUCTION

The generation of solitons in optical fibers, predicted by Hasegawa and Tappert [1,2], has enabled the generation of stable picosecond and subpicosecond pulses in the near infrared. Verification of many of the predicted soliton pulse characteristics was carried out in a series of experiments by Mollenauer *et al.* [3–5]. Furthermore, the formation of spatial optical solitons was demonstrated first in planar waveguides with the cubic (Kerr) self-focusing nonlinearity [5–8]. In fact, the confinement of light in short intervals of time and small regions of space is a well-known problem in nonlinear fiber optics, where temporal optical solitons are due to the confining action of the nonlinear material that prevents the spreading of the pulse caused by frequency dispersion. Moreover, an extensive literature on spatial optical solitons also exists [9–15], where the confinement of light in the localized region of the transverse plane is based on beam propagation in a nonlinear medium: in this case, the nonlinearity counterbalances the spreading effect of diffraction. Because of these defining features, the optical soliton is being considered as the ideal and natural data bit for long- and

short-distance information transmission because, unlike pulses in a linear dispersive fiber, optical solitons are self-confined, propagating for a long distance without changing shape.

Solitons arise in many fundamental areas of physics and technology: from nonlinear optics and ultrafast photonics, condensed matter and plasma physics to biophysics, elementary particle physics, cosmology and monster (rogue) waves in oceans, Bose–Einstein condensates, and electrical transmission lines [16–26]. A well-known example of an equation that admits pulse-like soliton solutions is the nonlinear Schrödinger (NLS) equation [27]. The NLS equation describes the evolution of an envelope function that is assumed to vary slowly over an optical cycle. Inverse scattering theory has been applied with great success to the NLS equation. The solutions are dark and bright solitons with tanh and sech profiles, respectively. The NLS equation has exact soliton solutions that correspond to a balance between nonlinearity and dispersion in the case of temporal solitons or between nonlinearity and diffraction in the case of spatial solitons. Another class of equations of particular interest is the cubic complex Ginzburg–Landau (CGL)

equation and its variants [28]. Such equations adequately model the dissipative soliton generation and propagation in systems including nanophotonics, nanoplasmonics, plasmas, and fluids, as well as superconductivity, superfluidity, elementary particles, electrical transmission line, doped optical fiber, laser physics, and biological systems [28], to name just a few. Recent investigations on the propagation of light pulses through optical fibers and optical fiber lasers have aroused considerable interest in the study of nonlinear effects described by the higher-order NLS (HONLS) equation [29–37] and higher-order CGL (HCGL) equation [38–43] that model several new effects such as third-order dispersion (TOD) and fourth-order dispersion (FOD), which become significant in ultrashort pulses. Moreover, focusing on the use of a weaker nonlinearity and the collapse stabilization, it appears that as the intensity of the incident light field becomes stronger, non-Kerr nonlinear effects become important, and the dynamics of pulses should be described by the NLS family of equations with higher-order nonlinear terms. In other words, the NLS equation has a plane wave solution of constant amplitude, and the modulational instability (MI) of the plane wave is determined by the sign of the product of the coefficient of the nonlinear and dispersion terms of the NLS equation. Therefore, if one of the two terms vanishes as in the case of a marginal state, one must introduce new ordering in order to balance the nonlinearity and the bandwidth [44–47]. Accordingly, the higher-order NLS equation, with group-velocity dispersion, TOD, FOD, self-phase modulation (SPM), self-steepening, and self-frequency shift due to stimulated Raman scattering, was studied [48–50], which describes the propagation of femtosecond pulses in optical fibers. The nonlinearity arising due to fifth-order susceptibility can be obtained in many optical materials, which include some semiconductors, where diffusion and recombination effects, etc., are important, and semiconductor-doped glasses, such as AlGaAs [51–53] and $\text{CdS}_x\text{Se}_{1-x}$, respectively, the polydiacetylene para-toluene sulfonate (PTS) μ -conjugated polymer [54–56], chalcogenide glasses (e.g., Ag-As-Se systems) [57–59], and some transparent organic materials [60]. However, when the saturation is very strong, a self-focusing seventh-order susceptibility $\chi^{(7)}$ is also needed. An experiment has been reported in material such as chalcogenide glasses that exhibit not only third- and fifth-order nonlinearities but even seventh-order nonlinearity [61]. In other words, chalcogenide glasses can be classified as cubic–quintic–septic nonlinear materials.

The development of erbium-doped glass fibers has been a major impetus to research on active-fiber technology in the 1.55 μm wavelength region. After the pioneering work on fiber lasers [62,63], research on rare-earth-doped fibers remained nearly dormant until the demonstration of high-gain erbium-doped fiber amplifiers (EDFAs) [64,65], and their use in lightwave transmission systems [66,67]. Now these active fibers are finding diverse applications in optical amplifiers, lasers, switches, and a variety of nonlinear devices. The first reports of rare-earth-doped fiber amplifiers date back to the 1960s [62,68,69] when Snitzer fabricated fiber amplifiers and lasers from Nd^{3+} laser glasses. These structures demonstrated the advantages to be gained in terms of increased power densities

by confining the pump power in a waveguiding structure. The possibility of using Er^{3+} -doped fiber as an amplifier in the third-telecoms window was reported by Payne *et al.* [70] in 1987, using fiber that was essentially silica based and fabricated by a variation to the inside tube deposition method, which was first attempted in the late 1980s [71,72]. In the early 1990s, studies on EDFA performance on ion concentration was focused on germanium co-doped fibers [73]. Aluminum co-doped EDFA's offer a wider and flatter spectral gain and better solubility of erbium ions; hence, aluminum co-doped fibers were introduced and examined with various fiber geometries and erbium confinement factors in the late 1990s [74]. Recently, focuses have shifted to chalcogenide-fiber-doped glasses due to their characteristics to exhibit as high as the seventh-order nonlinearity [61]; therefore, this work studies an extended $(1 + 1)D$ CGL equation with seventh-order nonlinearities.

One of the key issues of solitary wave theory is the universal MI phenomenon, which is one of the most fundamental processes in nonlinear wave systems in nature. MI, which is an indispensable mechanism for understanding pattern formation in a uniform medium, is a process in which the amplitude and phase modulations of a wave grow under the combined effects of nonlinearity and diffraction or dispersion in a spatially nonlinear field. It refers to the exponential growth of weak perturbations through the amplification of sideband frequencies. In optics, MI has constituted a wide field of intense theoretical and experimental research including ultrafast pulse generation [75], supercontinuum generation [76,77], four-wave mixing [78,79], Bragg gratings [80], parametric oscillators [81,82], and optical fiber systems [83–86].

The main objective of the present work is to investigate the MI gain of an extended $(1 + 1)D$ CGL equation with the TOD, FOD, fifth-order dispersion (FFOD), and sixth-order dispersion (SOD) and the cubic, quintic, and septic nonlinear terms, describing the dynamics of extremely short pulses in nonlinear doped optical fibers. In our contribution, we start with the Maxwell's equations describing the response of the nonlinear medium to an electromagnetic wave. Then, we report on the derivation of the new governing wave equation. Furthermore, we comprehensively discuss the theory of MI as well as the role played by higher-order dispersions and higher-order nonlinearities. Our analysis also gives rise to some concluding remarks.

2. DERIVATION OF THE $(1 + 1)D$ HIGHER-ORDER CUBIC–QUINTIC–SEPTIC CGL EQUATION

The optical signal propagation through an optical fiber is governed by Maxwell's equations. Under the assumption that the optical fiber is a non-magnetic medium without any free surface charges, such equations write

$$\begin{aligned}\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, & \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t}, \\ \nabla \cdot \vec{D} &= \rho_f, & \nabla \cdot \vec{B} &= 0,\end{aligned}\quad (1)$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic field vectors, respectively, with $\vec{D}$ and $\vec{B}$ being their corresponding electric and magnetic flux densities. $\vec{J}$ and ρ_f are the current density vector and the charge density. The optical fiber is a medium without free charges, which implies that $\vec{J} = 0$ and $\rho_f = 0$. The constitutive equations, relating the electric and magnetic fields, are given by

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad \text{and} \quad \vec{B} = \mu_0 \vec{H} + \vec{M}, \quad (2)$$

where ϵ_0 is the vacuum permittivity, and μ_0 is the vacuum permeability. $\vec{P}$ and $\vec{M}$ are the induced electric and magnetic polarizations, respectively. The optical fiber is a non-magnetic medium; therefore, the magnetic polarization is inexistent, i.e., $\vec{M} = \vec{0}$. The above Maxwell's equations result into the following equation that defines the complex dielectric constant, related to the pulse propagation in doped optical fibers [87]:

$$\epsilon(\omega) = n_f^2(\omega) + 2in_f(\omega)\frac{\alpha_f}{k_0} + \chi_d(\omega), \quad (3)$$

where α_f is the fiber loss, and $n_f(\omega)$ represents the refractive index of the fiber given by

$$n_f(\omega) = n(\omega) + n_2|E|^2 + n_4|E|^4 + n_6|E|^6. \quad (4)$$

Considering terms up to $|E|^6$, $(n_f(\omega))^2$ is written as

$$\begin{aligned} n_f(\omega)^2 &= n(\omega)^2 + 2n(\omega)n_2|E|^2 + 2n(\omega)n_4|E|^4 \\ &\quad + 2n(\omega)n_6|E|^6 + n_2^2|E|^4 + 2n_2n_4|E|^6, \end{aligned} \quad (5)$$

which we make use of, together with Eqs. (4) into (3), to get

$$\epsilon(\omega) = n(\omega)^2 + 2n(\omega)\Delta n, \quad (6)$$

with Δn being given by

$$\begin{aligned} \Delta n &= \left[n_2 + \frac{i\alpha_f n_2}{n(\omega)k_0} \right] |E|^2 + \left[\frac{n(\omega)}{2} + n_4 + \frac{i\alpha_f n_4}{n(\omega)k_0} \right] |E|^4 \\ &\quad + \left[n_6 + \frac{n_2 n_4}{n(\omega)} + \frac{i\alpha_f n_6}{n(\omega)k_0} \right] |E|^6 + \frac{i\alpha_f}{k_0} + \frac{\chi_d(\omega)}{2n(\omega)}. \end{aligned} \quad (7)$$

The dopant is represented by $\chi_d(\omega)$, the susceptibility of dopants is modeled as a two-level system whose dynamic response is governed by Bloch equations [21], and the associated population relaxation time T_1 and dipole relaxation time T_2 [87] are taken to be

$$\chi_d(\vec{r}, \omega) = \frac{\sigma_s W(r) n_0 c / \omega_0}{(\omega - \omega_a) T_2 + i}. \quad (8)$$

In Eq. (8), σ_s is the transition cross section, $W(r) = N_2 - N_1$ is the population-inversion density, and ω_a is the atomic frequency. The rate equations for the atomic densities for the upper N_1 and lower N_2 energy levels of the two-level system can be used to obtain the following rate equation for the gain [87,88]:

$$\frac{dg_p}{dt} = \frac{g_0 - g_p}{T_1} - \left[\frac{2\sigma_s}{\hbar\omega} \right] g_p I, \quad (9)$$

where g_0 is the small-signal gain created by optical pumping, and I is the optical intensity. In the steady state, Eq. (9) is readily solved to obtain $g_p = g_0 / (1 + I/I_s)$, where $I_s = \hbar\omega / (2\sigma_s T_1)$ is the saturation intensity.

In the rest of this work, the atomic susceptibility $\chi_d(\omega)$, which describes the response of the dopant, is extended in Taylor's series, up to the sixth order, as follows:

$$\chi_d^{(m)}(\omega) = a \sum_{m=0}^6 \frac{F_m}{m!} (\omega - \omega_0)^m, \quad (10)$$

where $F_m = \frac{\partial^m \chi_d(\omega)}{\partial \omega^m} |_{\omega=\omega_0}$, and $a = g_p n(\omega) / k_0$, with

$$\begin{aligned} F_0 &= -\frac{(-\delta + i)}{1 + \delta^2}, \quad F_1 = -\frac{(1 - \delta^2 + 2i\delta)T_2}{(1 + \delta^2)^2}, \\ F_2 &= -\frac{(3\delta - \delta^3 - i + 3i\delta^2)T_2^2}{(1 + \delta^2)^3}, \\ F_3 &= \frac{(-1 - \delta^4 - 4i\delta + 4i\delta^3 + 6\delta^2)T_2^3}{(1 + \delta^2)^4}, \\ F_4 &= -\frac{(-5\delta + 10\delta^3 - \delta^5 + i - 10i\delta^2 + 5i\delta^4)T_2^4}{(1 + \delta^2)^5}, \\ F_5 &= -\frac{(1 - 15\delta^2 + 15\delta^4 - \delta^6 + 6i\delta + 6i\delta^5 - 20i\delta^3)T_2^5}{(1 + \delta^2)^6}, \\ F_6 &= -\frac{(7\delta - 35\delta^3 + 21\delta^5 - \delta^7 - i + 21i\delta^2 - 35i\delta^4 + 7i\delta^6)T_2^6}{(1 + \delta^2)^7}. \end{aligned} \quad (11)$$

k_0 is the wavenumber, and δ is the detuning parameter given by $\delta = (\omega_0 - \omega_a)T$, in which $(\omega - \omega_a) = (\omega - \omega_0) + (\omega_0 - \omega_a)$.

Following Agrawal [21], the propagation equation for $\tilde{\psi}(z, \omega - \omega_0)$, which is the Fourier transform of $\psi(z, t)$, is written in the form

$$\frac{\partial \tilde{\psi}}{\partial z} = i(\beta(\omega) + \Delta\beta - \beta_0)\tilde{\psi}, \quad (12)$$

where the quantity $\Delta\beta$ is defined as

$$\Delta\beta = k_0 \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta n |f(x, y)|^2 dx, dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 dx, dy}, \quad (13)$$

with $f(x, y)$ being the modal distribution of the fundamental mode of the optical fiber. We should note that nonlinearities relative to the self-steepening and self-frequency shift are modeled through the propagation constant. The Taylor's expansion in series, up to sixth order, of $\beta(\omega)$, about the carrier frequency therefore becomes

$$\begin{aligned}
\beta(\omega) = & \beta_0 + \left(\frac{\partial \beta}{\partial \omega} \right)_0 (\omega - \omega_0) + \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega^2} \right)_0 (\omega - \omega_0)^2 + \frac{1}{6} \left(\frac{\partial^3 \beta}{\partial \omega^3} \right)_0 (\omega - \omega_0)^3 \\
& + \frac{1}{24} \left(\frac{\partial^4 \beta}{\partial \omega^4} \right)_0 (\omega - \omega_0)^4 + \frac{1}{120} \left(\frac{\partial^5 \beta}{\partial \omega^5} \right)_0 (\omega - \omega_0)^5 + \frac{1}{720} \left(\frac{\partial^6 \beta}{\partial \omega^6} \right)_0 (\omega - \omega_0)^6 \\
& + \left(\frac{\partial \beta}{\partial (|\psi|^2)} \right)_0 |\psi|^2 + \frac{1}{2} \left(\frac{\partial \beta}{\partial (|\psi|^4)} \right)_0 |\psi|^4 + \frac{1}{6} \left(\frac{\partial \beta}{\partial (|\psi|^6)} \right)_0 |\psi|^6 \\
& + \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^2)} \right)_0 (\omega - \omega_0) |\psi|^2 + \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^4)} \right)_0 (\omega - \omega_0) |\psi|^4 + \frac{1}{6} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^6)} \right)_0 (\omega - \omega_0) |\psi|^6 \\
& + \left(\frac{\partial \beta}{\partial (|\psi|^2)_t} \right)_0 (|\psi|^2)_t + \frac{1}{2} \left(\frac{\partial \beta}{\partial (|\psi|^4)_t} \right)_0 (|\psi|^4)_t + \frac{1}{6} \left(\frac{\partial \beta}{\partial (|\psi|^6)_t} \right)_0 (|\psi|^6)_t.
\end{aligned} \tag{14}$$

The final result for the pulse-propagation equation is the higher-order cubic–quintic–septic Ginzburg–Landau equation having up to the sixth order of dispersion as follows:

$$\begin{aligned}
& i\psi_z + (p_r + ip_i)\psi_{tt} + (d_{3r} + id_{3i})\psi_{ttt} + (d_{4r} + id_{4i})\psi_{tttt} + (d_{5r} + id_{5i})\psi_{ttttt} + (d_{6r} + id_{6i})\psi_{ttttt} \\
& + (q_r + iq_i)|\psi|^2 + (C_r + iC_i)|\psi|^4\psi + (S_r + iS_i)|\psi|^6\psi + im_i(|\psi|^2\psi)_t + im_{1i}(|\psi|^4\psi)_t \\
& + im_{2i}(|\psi|^6\psi)_t + n_r(|\psi|^2)_t\psi + n_{1r}(|\psi|^4)_t\psi + n_{2r}(|\psi|^6)_t\psi + (\gamma_r + i\gamma_i)\psi = 0,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
p_r &= \frac{-1}{2} \left(\frac{\partial^2 \beta}{\partial \omega^2} \right)_0 + \frac{g_p T_2^2 (3\delta - \delta^3)}{2(1 + \delta^2)^3}, \quad p_i = \frac{g_p T_2^2 (3\delta^2 - 1)}{2(1 + \delta^2)^3}, \quad d_{3r} = \frac{g_p T_2^3 (-4\delta + 4\delta^3)}{2(1 + \delta^2)^4}, \\
d_{3i} &= \frac{-1}{6} \left(\frac{\partial^3 \beta}{\partial \omega^3} \right)_0 - \frac{g_p T_2^3 (1 + \delta^4 - 6\delta^2)}{2(1 + \delta^2)^4}, \quad d_{4r} = \frac{1}{24} \left(\frac{\partial^4 \beta}{\partial \omega^4} \right)_0 - \frac{g_p T_2^4 (5\delta - 10\delta^3 + \delta^5)}{2(1 + \delta^2)^5}, \\
d_{4i} &= -\frac{g_p T_2^4 (-1 + 10\delta^2 - 5\delta^4)}{2(1 + \delta^2)^5}, \quad d_{5i} = \frac{1}{120} \left(\frac{\partial^5 \beta}{\partial \omega^5} \right)_0 + \frac{g_p T_2^5 (1 - 15\delta^2 + 15\delta^4 - \delta^6)}{2(1 + \delta^2)^6}, \\
d_{5r} &= \frac{g_p T_2^5 (-6\delta - 6\delta^5 + 20\delta^3)}{2(1 + \delta^2)^6}, \quad d_{6r} = \frac{-1}{720} \left(\frac{\partial^6 \beta}{\partial \omega^6} \right)_0 - \frac{g_p T_2^6 (7\delta - 35\delta^3 + 21\delta^5 - \delta^7)}{2(1 + \delta^2)^7}, \\
d_{6i} &= -\frac{g_p T_2^6 (-1 + 21\delta^2 - 35\delta^4 + 7\delta^6)}{2(1 + \delta^2)^7}, \quad q_r = \left(\frac{\partial \beta}{\partial |\psi|^2} \right)_0 + \frac{k_0 n_2}{4A_{\text{eff}}}, \quad q_i = \frac{\alpha_f n_2}{n(\omega)k_0}, \\
C_r &= \frac{1}{2} \left(\frac{\partial \beta}{\partial |\psi|^4} \right)_0 + \frac{k_0}{16A_{\text{eff1}}} \left(\frac{n_2^2}{2} + n_4 \right), \quad C_i = \frac{\alpha_f n_4}{16A_{\text{eff1}} n(\omega)k_0}, \\
S_r &= \frac{1}{6} \left(\frac{\partial \beta}{\partial |\psi|^6} \right)_0 + \frac{k_0}{64A_{\text{eff2}}} \left(n_6 + \frac{n_4 n_2}{n(\omega)} \right), \quad S_i = \frac{\alpha_f n_6}{64A_{\text{eff2}} n(\omega)}, \\
\gamma_r &= \frac{g_p}{2} \frac{\delta}{1 + \delta^2}, \quad \gamma_i = \alpha_f - \frac{g_p}{2} \frac{1}{1 + \delta^2}, \quad \beta_{1r} = \frac{g_p T_2}{2} \frac{(-2\delta)}{(1 + \delta^2)^2}, \quad \beta_{1i} = \left(\frac{\partial \beta}{\partial \omega} \right)_0 + \frac{g_p T_2}{2} \frac{(1 - \delta^2)}{(1 + \delta^2)^2}, \\
m_i &= \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^2)} \right)_0, \quad m_{1i} = \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^4)} \right)_0, \quad m_{2i} = \frac{1}{6} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^6)} \right)_0, \\
n_r &= \left(\frac{\partial \beta}{\partial (|\psi|^2)_t} \right)_0, \quad n_{1r} = \left(\frac{\partial \beta}{\partial (|\psi|^4)_t} \right)_0, \quad n_{1r} = \left(\frac{\partial \beta}{\partial (|\psi|^4)_t} \right)_0, \quad n_{2r} = \left(\frac{\partial \beta}{\partial (|\psi|^6)_t} \right)_0,
\end{aligned} \tag{16}$$

with $A_{\text{eff}} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^4 dx dy}$ the refractive core area of the fiber, $A_{\text{eff1}} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^6 dx dy}$, and $A_{\text{eff2}} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^8 dx dy}$. Experimentally, A_{eff1} is presented as $\frac{3}{4}A_{\text{eff}}$ and A_{eff2} as $\frac{1}{2}A_{\text{eff}}$ [89]. More precisely, $\psi(z, t)$ is the envelope amplitude of the electric field, t is the retarded time frame, and z is the propagation distance originating from the transformation $T = t - z/v_g = t - \beta_1$. All the coefficients are real constants. The parameter p_r measures the wave dispersion, while p_i measures the spectral filtering, and q_r and q_i represent the Kerr nonlinearity coefficient and the nonlinear gain absorption (if positive), respectively. The parameters C_r and C_i stand for the nonlinear saturation of the Kerr nonlinearity (if negative) and the saturation of the nonlinear gain-absorption (if negative), respectively. The parameters S_r and S_i are higher-order correction terms to the nonlinear refractive and nonlinear amplification-absorption gain, respectively. γ_r and γ_i stand for the coefficient of the linear gain (if positive) and frequency shift, respectively. The parameters $d_3 = (d_{3r} + id_{3i})$, $d_4 = (d_{4r} + id_{4i})$, $d_5 = (d_{5r} + id_{5i})$, and $d_6 = (d_{6r} + id_{6i})$ account for the TOD, FOD, FFOD, and SOD, respectively. The terms m_i , m_{1i} , m_{2i} are the cubic, quintic, and septic self-steepening pulse edges, respectively, and the terms n_r , n_{1r} , n_{2r} represent the cubic, quintic, and septic soliton self-frequency shift terms, respectively, which result from the induced Raman scattering processes.

In order to simplify the problem at hand, it would be advisable to reduce the number of parameters. Therefore, Eq. (15) can be normalized by making use of the transformations $s = \sigma_r z$ for space, $\tau = \sqrt{\frac{\sigma_r}{A_r}} t$ for time, and $A = \sqrt{\frac{Q_r}{\sigma_r}} \psi$ for amplitude, which lead to the following dimensionless (1 + 1)D CGL equation:

$$\begin{aligned} A_s = & (1 + ic_0)A + (1 + ic_2)A_{\tau\tau} + a_3(1 + ic_3)A_{\tau\tau\tau} \\ & + a_4(1 + ic_4)A_{\tau\tau\tau\tau} + a_5(1 + ic_5)A_{\tau\tau\tau\tau\tau} \\ & + a_6(1 + ic_6)A_{\tau\tau\tau\tau\tau\tau} + (1 + i\alpha_1)|A|^2 A \\ & + e_2(1 + i\alpha_2)|A|^4 A + e_3(1 + i\alpha_3)|A|^6 A \\ & + ie_4(|A|^2 A)_{\tau} + ie_5(|A|^4 A)_{\tau} + ie_6(|A|^6 A)_{\tau} \\ & + e_7(|A|^2)_{\tau} A + e_8(|A|^4)_{\tau} A + e_9(|A|^6)_{\tau} A, \end{aligned} \quad (17)$$

where

$$\begin{aligned} c_0 = \frac{\sigma_i}{T_0}, \quad c_2 = \frac{L_{D2}A_i}{T_0^2}, \quad c_3 = \frac{L_{D3}b_{2i}}{T_0^3}, \quad c_4 = \frac{L_{D4}b_{3i}}{T_0^4}, \quad c_5 = \frac{L_{D5}b_{4i}}{T_0^5}, \quad c_6 = \frac{L_{D6}b_{5i}}{T_0^6}, \\ a_3 = \frac{L_{D2}\sqrt{L_{D2}^3\sigma_r}}{L_{D3}T_0^3}, \quad a_4 = \frac{L_{D2}^3\sigma_r}{L_{D4}T_0^4}, \quad a_5 = \frac{L_{D2}\sqrt{L_{D2}^5\sigma_r^3}}{L_{D5}T_0^5}, \quad a_6 = \frac{L_{D2}^4\sigma_r^2}{L_{D6}T_0^6}, \quad \alpha_1 = \frac{Q_i L_{NL1}}{L_{D2}}, \\ \alpha_2 = \frac{R_i L_{NL2}}{L_{D2}}, \quad \alpha_3 = \frac{T_i L_{NL1}}{L_{D2}}, \quad e_2 = \frac{\sigma_r L_{NL1}^2}{L_{NL2} L_{D2}}, \quad e_3 = \frac{\sigma_r^2 L_{NL1}^3}{L_{NL3} L_{D2}^2}, \quad e_4 = \frac{L_{D2}^{\frac{3}{2}}}{L_{SS1} \sigma_r^{\frac{1}{2}} T_0}, \\ e_5 = \frac{\sigma_r^{\frac{3}{2}}}{L_{SS2} L_{D2}^{\frac{1}{2}} T_0}, \quad e_6 = \frac{\sigma_r^{\frac{5}{2}}}{L_{SS3} L_{D2}^{\frac{3}{2}} T_0}, \quad e_7 = \frac{L_{D2}^{\frac{3}{2}}}{L_{SFS1} \sigma_r^{\frac{1}{2}} T_0}, \quad e_8 = \frac{\sigma_r^{\frac{3}{2}}}{L_{SFS2} L_{D2}^{\frac{1}{2}} T_0}, \quad e_9 = \frac{\sigma_r^{\frac{5}{2}}}{L_{SFS3} L_{D2}^{\frac{3}{2}} T_0}, \end{aligned} \quad (18)$$

where $\sigma_r = i\gamma_r = 1/T_0$ and the terms $\sigma_i = -\gamma_i$, $A_i = p_r$, $b_{2i} = d_{3i}$, $b_{3r} = d_{4r}$, $b_{4i} = d_{5r}$, $b_{5i} = d_{6r}$, $Q_i = q_r$, $R_i = C_r$, $T_i = S_r$, $M_i = -m_i$, $M_{1i} = -m_{1i}$, $M_{2i} = -m_{2i}$, $N_r = in_r$, $N_{1r} = in_{1r}$, and $N_{2r} = in_{2r}$. In order to obtain the dimensionless coefficients of Eq. (17), we have also considered the following inherent lengths: $L_{Dm} = \sum_{m=2}^6 \frac{T_0^m}{\beta_m}$ is the dispersion length (β_m are m-order parameters of dispersion), $L_{NLm} = \sum_{m=1}^3 \frac{T_0^m}{\gamma_m P_{0m}}$ is the nonlinear length (γ_m is the nonlinear amplification-absorption gain term), $L_{SSm} = \sum_{m=1}^3 \frac{T_0^m}{\alpha_m P_{0m}}$ is the self-steepening length (α_m is the self-steepening pulse edge term), and $L_{SFSm} = \sum_{m=1, i=4}^{3,6} \frac{T_0^m}{\alpha_i P_{0m}}$ is the self-frequency shift length (α_i is the self-frequency shift term [42,87]).

We should further stress that Eq. (17) is the generalization of many cases that have been studied in the literature, depending on the values of parameters. For example, when we set $a_5 = a_6 = 0$, $e_3 = e_5 = 0$, and $e_6 = e_8 = e_9 = 0$, we recover the CGL equation that was used by Tiofack *et al.* [84] to show the impact of the FOD on the appearance of solitons in optical fibers via the activation of MI. Moreover, a modified HONLS equation was also considered in the same context, which can be recovered from Eq. (17) by imposing $a_5 = a_6 = 0$, $e_3 = e_6 = e_9 = 0$ and switching off the imaginary parts of the other coefficients. A NLS equation was extended, by Mohamadou *et al.* [82], to include the dopant effect, which resulted in a CGL equation with cubic and quintic nonlinearities, along with cubic Raman scattering and self-steepening effects. More recently, Djoko and Kofané [90] proposed a cubic–quintic–septic CGL equation in the absence of sixth-order self-steepening and Raman scattering effects, respectively materialized here by e_6 and e_9 . However, in the latter, the MI was not investigated, but the propagation of subpicosecond and femtosecond pulses in nonlinear fibers was studied via the variational method. It is relevant to mention that due to the complexity of Eq. (17), which involves several terms, many contributions have rather addressed the MI of some extended versions of the NLS equation [91–93]. This has been regarded by Nair *et al.* [91], for example, but in the absence of sixth-order self-steepening and Raman scattering effects. In some other contributions along the same line, the order on nonlinear terms has been extended in the NLS equation beyond Kerr nonlinearities, where Porsezian and co-workers [92] were

interested mainly in the effect of quintic non-Kerr nonlinearities. Cubic–quintic–septic nonlinear effects were addressed in the same context in the presence of SOD, TOD, and FOD, coupled to cubic Raman scattering and self-steepening [93].

Although several works have been dedicated to a homogeneous fiber core medium, a non-uniform or inhomogeneous core medium is, in general, adopted in an actual fiber in order to overcome the attenuation due to optical losses that are inherent to the optical fiber. In such conditions, the optical pulse is amplified and reshaped while propagating in an active zone doped with resonant atoms such as erbium, with the resonant interactions being capable of neutralizing the optical losses in the fiber. In this context, the more appropriate model to describe such effects is the CGL equation, in which, based on the dopant,

$$\begin{aligned} a_i = & -(1 + ic_0)(\sqrt{P} + a) + (1 + ic_2)a_{\tau\tau} + a_3(1 + ic_3)a_{\tau\tau\tau} + a_4(1 + ic_4)a_{\tau\tau\tau\tau} \\ & + a_5(1 + ic_5)a_{\tau\tau\tau\tau\tau} + a_6(1 + ic_6)a_{\tau\tau\tau\tau\tau\tau} + P(1 + i\alpha_1)(a + a^*) + 2P^2e_2(1 + i\alpha_2)(a + a^*) \\ & + 3P^3e_3(1 + i\alpha_3)(a + a^*) + ie_4P(2a_\tau + a_\tau^*) + ie_5P^2(3a_\tau + 2a_\tau^*) \\ & + ie_6P^3(4a_\tau + 3a_\tau^*) + e_7P(a_\tau + a_\tau^*) + 2e_8P^2(a_\tau + a_\tau^*) + 3e_9P^3(a_\tau + a_\tau^*), \end{aligned} \quad (21)$$

cubic–quintic–septic higher-order non-slowly-varying-envelope approximation (SVEA) terms may contribute to the SPM of shorter optical pulses, thus suggesting additional possibilities for experiments on new chalcogenide glasses capable of bringing about new effects related to higher-order self-steepening and self-frequency shift. Indubitably, the derivation of Eq. (17) is significant from the viewpoint of the fundamental principles of nonlinear optics, because: (i) it brings about the existence of a further physical channel whereby cubic–quintic–septic higher-order non-SVEA terms may contribute to the SPM of shorter optical pulses, and (ii) allows not only to unmask the physical origins of such corrections but also to deeply understand the physics involved in the model.

where $\phi_{NL} = Pe_2 \int_0^s \exp(2s)ds + P^2e_3 \int_0^s \exp(4s)ds + P^3e_4 \int_0^s \exp(6s)ds$ is the nonlinear phase shift due to the SPM and higher-order nonlinear terms. The stability of the solution in Eq. (19) requires the introduction of a slight perturbation so that

$$A(s, \tau) = \left[\sqrt{P} + a(s, \tau) \right] e^{i\phi_{NL}}, \quad (20)$$

where $a(s, \tau)$ is the small perturbation, with $|a(s, \tau)| \ll \sqrt{P}$ being the complex field. This perturbed solution means that if the perturbed field grows exponentially, the steady state will become unstable. Substituting Eqs. (20) into (19) and linearizing around the unperturbed solution lead to the following equation for the perturbed field:

where a^* is the complex conjugate of the perturbed field. Further, the solution to Eq. (21) can be adopted in the form

$$a(s, \tau) = u(s)e^{-i\Omega\tau} + v(s)e^{i\Omega\tau}, \quad (22)$$

where $u(s)$ and $v(s)$ are the complex perturbation fields, and Ω is the complex modulation frequency. Making use of the ansatz of the perturbed field of Eq. (22) in the linearized Eq. (21) leads to a 2×2 matrix of the following form for the perturbed fields:

$$i \frac{\partial}{\partial s} \begin{pmatrix} u(s) \\ v^*(s) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} u(s) \\ v^*(s) \end{pmatrix}, \quad (23)$$

where $v^*(s)$ is the complex conjugate of the field $v(s)$, with the matrix elements being given by

$$\begin{aligned} T_{11} = & -(1 + ic_2)\Omega^2 + ia_3(1 + ic_3)\Omega^3 + a_4(1 + ic_4)\Omega^4 - ia_5(1 + ic_5)\Omega^5 - a_6(1 + ic_6)\Omega^6 \\ & + n_{11} - i\Omega P(2ie_4 + 3ie_5P + 4e_6P^2 + e_7 + 2e_8P + 3e_9P^2), \\ T_{12} = & n_{11} - i\Omega P(ie_4 + 2ie_5P + 3ie_6P^2 + e_7 + 2e_8P + 3e_9P^2), \\ T_{21} = & -[n_{11} - i\Omega P(ie_4 + 2ie_5P + 3ie_6P^2 + e_7 + 2e_8P + 3e_9P^2)], \\ T_{22} = & -[-(1 + ic_2)\Omega^2 + ia_3(1 + ic_3)\Omega^3 + a_4(1 + ic_4)\Omega^4 - ia_5(1 + ic_5)\Omega^5 - a_6(1 + ic_6)\Omega^6 \\ & + n_{11} - i\Omega P(2ie_4 + 3ie_5P + 4e_6P^2 + e_7 + 2e_8P + 3e_9P^2)]. \end{aligned} \quad (24)$$

3. MODULATIONAL INSTABILITY: LINEAR STABILITY ANALYSIS AND GAIN SPECTRUM

The MI is studied on model Eq. (17) via the linear stability analysis of a CW solution, which propagates inside the fiber with the initial input power P , i.e.,

$$A(s, \tau) = \sqrt{P} \exp[i\phi_{NL} + (1 + ic_0)s], \quad (19)$$

In order to find the MI gain, we should keep in mind that the wavenumber K of the perturbation can be found as the eigenvalue of the matrix $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, through the characteristic equation $\text{Det}[T - KI] = 0$, with I being a 2×2 identity matrix. This leads to the dispersion relation

$$K = \frac{1}{2} \left[T_{11} + T_{22} + \sqrt{(T_{11} - T_{22})^2 + 4T_{12}T_{21}} \right]. \quad (25)$$

For MI to take place, the wavenumber K should get a nonzero imaginary part that will be responsible for an exponential growth of the perturbed amplitude. In general, the occurrence of MI is predicted through the power gain, defined as $G(\Omega) = 2\text{Im}(K)$, which is obtained in this particular case in the form

$$G(\Omega) = 2 \sqrt{2n_{11}^2 + (P^2 a^2 - 2(1 + ic_2)n_{11})\Omega^2 + ((1 + ic_2)^2 + 2a_4(1 + ic_4)n_{11})\Omega^4 - 2((1 + ic_2)a_4(1 + ic_4) + a_6(1 + ic_6)n_{11})\Omega^6 + (a_4^2(1 + ic_4)^2 + 2(1 + ic_2)a_6(1 + ic_6))\Omega^8 - 2a_4(1 + ic_4)a_6(1 + ic_6)\Omega^{10} + a_6^2(1 + ic_6)^2\Omega^{12}}, \quad (26)$$

where $a = ie_4 + 2ie_5P + 3ie_6P^2 + e_7 + 2e_8P + 3e_9P^2$, and $n_{11} = (1 + ic_0)\sqrt{P} + (1 + i\alpha_1)P + e_2(1 + i\alpha_2)P^2 + e_3(1 + \alpha_3)P^3$.

A. Dispersion Effects on MI

In order to proceed, we consider the parameter values in Table 1, and, using the transformation formulas in Table 2, we get the following dimensionless parameters to be used in the rest of this paper: $P = 20$, $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$, and $e_9 = 6 \times 10^{-4}$. The impact of the dispersion parameters c_2 and c_4 on the appearance of MI gain along the perturbation frequency Ω , with respect to the SOD parameter c_6 , is particularly addressed. The corresponding results, when $c_2 = 0.001$, are summarized in Fig. 1, where (a) and (b) correspond, respectively, to $c_4 = 0.08$ and $c_4 = -0.08$, and (c) and (d) are their corresponding representations versus the frequency Ω with changing c_6 . In general, under a normal dispersion regime, i.e., $c_2 > 0$, it is obvious that the bandwidth and position of the MI lobes are functions of c_6 . However, for $c_4 > 0$, the instability tends to disappear for c_6 decreasing, which is the

inverse for $c_4 < 0$. One should, however, notice that in both cases, the instability is characterized by the presence of lobes that are similar and symmetric, which at this stage justifies the suitability of the model under our study. On the other hand, in

the anomalous dispersion regime, i.e., $c_2 < 0$, two cases are also regarded, i.e., c_4 positive and negative, as depicted in Fig. 2. As previously noticed, MI manifests itself by the presence of two symmetric and identical sidebands, but, depending on the sign of c_4 , there is a threshold value of c_6 below which instability is

Table 2. Normalized Parameters for the Higher-Order CGL Equation

Unnormalized Dimensions	Normalization Quantity	Normalized Dimensions
Time coordinate	$\tau = \sqrt{\frac{\sigma_r}{A_r}} t$	$\tau = t/T_0$
Space coordinate	$s = \sigma_r z$	$s = z/Z_0$
Wave amplitude	$A = \sqrt{\frac{Q_r}{\sigma_r}} \psi$	$A = \psi/\psi_0$
GVD coefficient	$c_2 = \frac{A_i}{A_r}$	$c_2 = \frac{L_{D2} A_i}{T_0^2}$
TOD coefficient	$c_3 = \frac{b_{2i}}{b_{2r}}$	$c_3 = \frac{L_{D3} b_{2i}}{T_0^3}$
FOD coefficient	$c_4 = \frac{b_{3i}}{b_{3r}}$	$c_4 = \frac{L_{D4} b_{3i}}{T_0^4}$
FFOD coefficient	$c_5 = \frac{b_{4i}}{b_{4r}}$	$c_5 = \frac{L_{D5} b_{4i}}{T_0^5}$
SOD coefficient	$c_6 = \frac{b_{5i}}{b_{5r}}$	$c_6 = \frac{L_{D6} b_{5i}}{T_0^6}$
Cubic nonlinear coefficient	$\alpha_1 = \frac{Q_i}{Q_r}$	$\alpha_1 = \frac{Q_i L_{NL1}}{L_{D2}}$
Quintic nonlinear coefficient	$\alpha_2 = \frac{R_i}{R_r}$	$\alpha_2 = \frac{R_i L_{NL2}}{L_{D2}}$
Septic nonlinear coefficient	$\alpha_3 = \frac{T_i}{T_r}$	$\alpha_3 = \frac{T_i L_{NL3}}{L_{D2}}$
Cubic self-steepening coefficient	$e_4 = \frac{M_i}{\sqrt{\sigma_r} A_r}$	$e_4 = \frac{L_{D2}^{\frac{3}{2}}}{L_{SS1} \sigma_r^{\frac{1}{2}} T_0}$
Quintic self-steepening coefficient	$e_5 = \frac{M_{1i} \sigma_r^{3/2}}{Q_r^2 A_r^{1/2}}$	$e_5 = \frac{\sigma_r^{\frac{3}{2}}}{L_{SS2} L_{D2}^{\frac{1}{2}} T_0}$
Septic self-steepening coefficient	$e_6 = \frac{M_{2i} \sigma_r^{5/2}}{Q_r^3 A_r^{1/2}}$	$e_6 = \frac{\sigma_r^{\frac{5}{2}}}{L_{SS3} L_{D2}^{\frac{3}{2}} T_0}$
Third-order self-frequency shift	$e_7 = \frac{N_r}{\sqrt{\sigma_r} A_r}$	$e_7 = \frac{L_{D2}^{\frac{3}{2}}}{L_{SFS1} \sigma_r^{\frac{1}{2}} T_0}$
Fifth-order self-frequency shift	$e_8 = \frac{N_{1r} \sigma_r^{3/2}}{A_r^{1/2} Q_r^2}$	$e_8 = \frac{\sigma_r^{\frac{3}{2}}}{L_{SFS2} L_{D2}^{\frac{1}{2}} T_0}$
Seventh-order self-frequency shift	$e_9 = \frac{N_{2r} \sigma_r^{5/2}}{A_r^{1/2} Q_r^3}$	$e_9 = \frac{\sigma_r^{\frac{5}{2}}}{L_{SFS3} L_{D2}^{\frac{3}{2}} T_0}$
Nonlinear gain parameter	$c_0 = \frac{\sigma_i}{\sigma_r}$	$c_0 = \frac{\sigma_i}{T_0}$

Table 1. Doped Fiber Parameters for a Two-Level System

Physical Parameters	Symbols	Values
Wavelength	λ_0	1.55 μm [94]
Atomic frequency	ω_a	$4 \times 10^{15} \text{Hz}$ [95]
Fiber loss	α	0.22 dB/km [94]
Nonlinear parameter	γ	2/Wkm [94]
Relaxation time	T_R	3 fs [21]
Linear refractive index	n_0	1.45 [87]
Nonlinear refractive index originating from $\chi^{(3)}$	n_2	$2.7 \times 10^{-13} \text{cm}^2/\text{W}$ [21]
Nonlinear refractive index originating from $\chi^{(5)}$	n_4	$-7.8 \times 10^{-23} \text{cm}^4/\text{W}^4$ [89]
Nonlinear refractive index originating from $\chi^{(7)}$	n_6	$7.2 \times 10^{-33} \text{cm}^6/\text{W}^3$ [89]
Dipole relaxation time	T_2	100 fs [87]
Effective mode area	A_{eff}	$50 \mu\text{m}^2$ [87]
Peak gain	g_p	6.8/Wkm [21]
GVD coefficient	β_2	$-20 \text{ps}^2/\text{km}$ [21]
TOD coefficient	β_3	$0.04 \text{ps}^3/\text{km}$ [89]
FOD coefficient	β_4	$0.005 \text{ps}^4/\text{km}$ [89]
FFOD coefficient	β_5	$0.06 \text{ps}^5/\text{km}$ [89]
SOD coefficient	β_6	$0.07 \text{ps}^6/\text{km}$ [89]

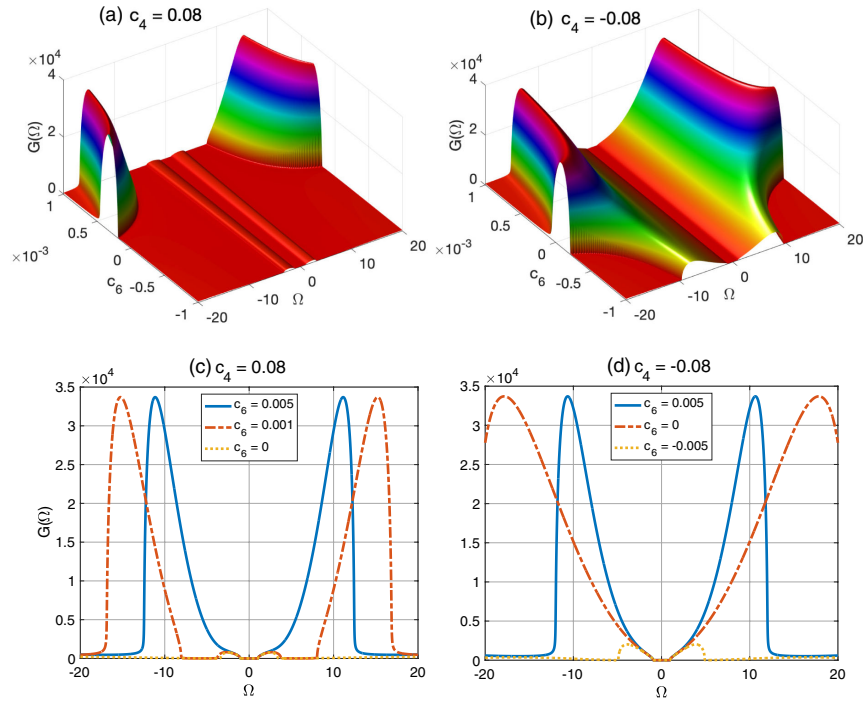


Fig. 1. Panels show the MI gain $G(\Omega)$ versus the SOD parameter c_6 and Ω for different values of the FOD coefficient c_4 , under normal dispersion regime, i.e., $c_2 > 0$, with the other parameters being: $P = 20$, $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$, and $e_9 = 6 \times 10^{-4}$.

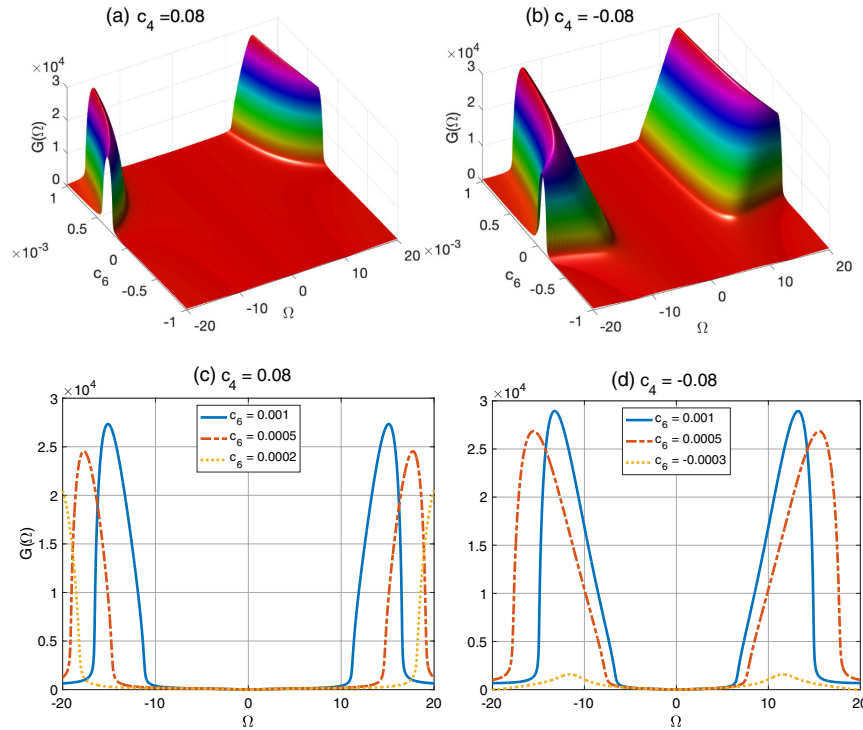


Fig. 2. Panels show the MI gain $G(\Omega)$ versus the SOD parameter c_6 and Ω for different values of the FOD coefficient c_4 , under normal dispersion regime, i.e., $c_2 < 0$, with the other parameters being: $P = 20$, $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$, and $e_9 = 6 \times 10^{-4}$.

not possible. This is clearly shown in Figs. 2(c) and 2(d), which respectively correspond to $c_4 > 0$ and $c_4 < 0$.

B. Nonlinear Effects on MI

The emergence of nonlinear waves in the optical fiber is highly dependent on the balance between nonlinear and dispersive effects. This implies the importance of nonlinear effects in the process of optical soliton characterization and propagation. In the previous section, we have seen that the dispersion deeply influences the occurrence of MI, especially when FOD and SOD dispersions are involved. Here, we are interested mainly in how the balance between higher-order nonlinearities may combine with the FOD and SOD for MI to take place, this under a normal frequency regime. To start, we study the features of the septic nonlinearity under the effect of positive and negative SOD. This is shown in Fig. 3, where (a) and (b) have been recorded, respectively, for $c_6 = 0.01$ and $c_6 = -0.01$, with their corresponding 2D representations versus the frequency Ω in Figs. 3(c) and 3(d). In the first case, i.e., $c_6 > 0$, the CW is unstable for any value of e_3 , although the frequency bandwidth is limited by two symmetric and identical lobes. According to Fig. 3(c), the distance between the lobes is very sensitive to the

change in the SOD whose increase causes the decrease of the bandwidth as well as an increase in intensity. When $c_6 < 0$, the MI frequency domain is also sensitive to the decrease in c_6 , as noticed in Fig. 3(d). Instead of two lobes as in the previous case, here we rather have two symmetric wings separated by two minor lobes around $\Omega = 0$. The intensity increases when c_6 gets smaller, while the frequency bandwidth responsible for MI, supported by the large MI wings, gets expanded, therefore reducing the windows of instability materialized by the minor lobes.

Another illustration of how nonlinearity and dispersion can combine and monitor the emergence of modulated waves is given by Fig. 4, where the competition between the SOD and the septic self-steepening gives rise to lobes of instability whose intensity and expansion are very sensitive to the sign and value of c_6 . For $c_6 > 0$, the instability takes place for all values of e_6 , but smaller values of the latter reduce the gain intensity and frequency bandwidth, as supported by Figs. 4(a) and 4(c). When $c_6 < 0$, we also notice the presence of two gain peaks that tend to disappear when $e_6 \rightarrow 0$ [see Fig. 4(b)]. Figure 5 shows results related to the competition between the SOD and the septic Raman scattering. As a whole, when $c_6 > 0$, the instability domain shows two different behaviors for $e_9 > 0$ and $e_9 < 0$.

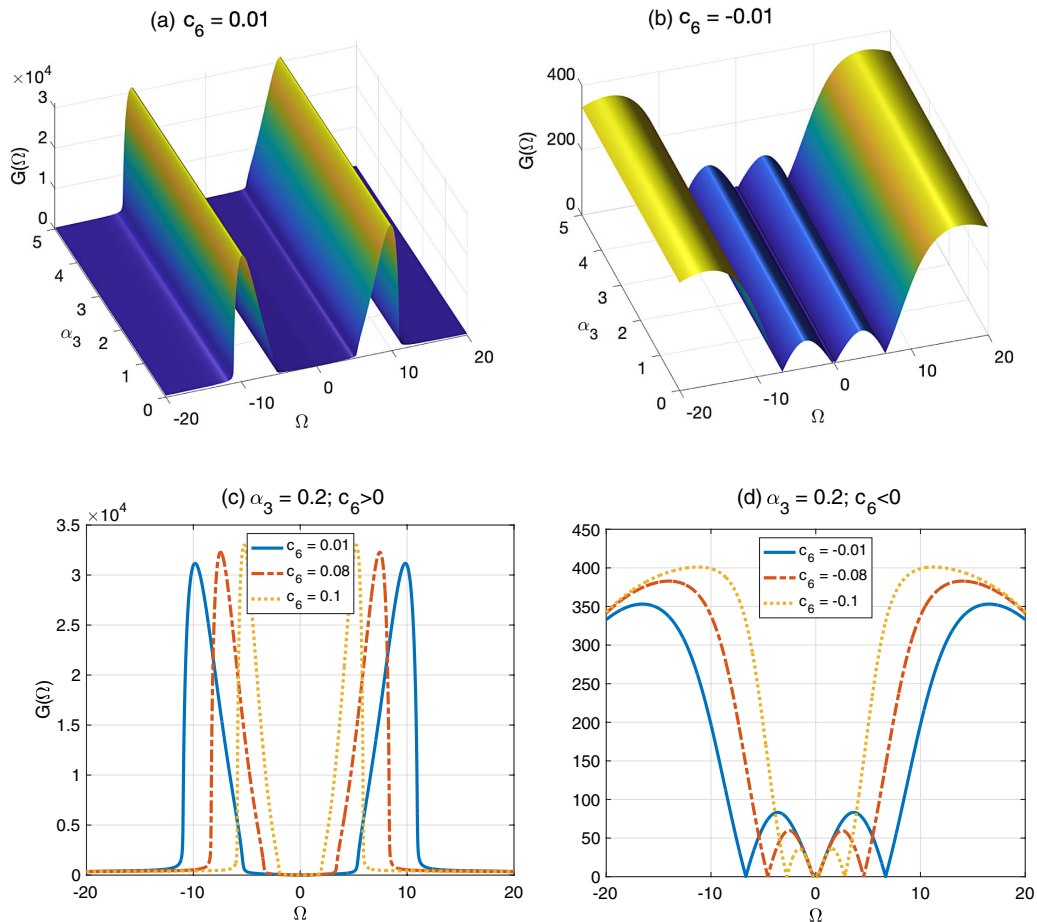


Fig. 3. (a), (b) MI gain $G(\Omega)$ as a function of the septic nonlinear coefficient e_3 and the frequency Ω for positive and negative values of the SOD coefficient c_6 , respectively. (c), (d) $G(\Omega)$ versus the frequency Ω for $e_3 = 0.2$ and for positive and negative values of c_6 . The other parameter values are: $P = 20$, $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$, and $e_9 = 6 \times 10^{-4}$.

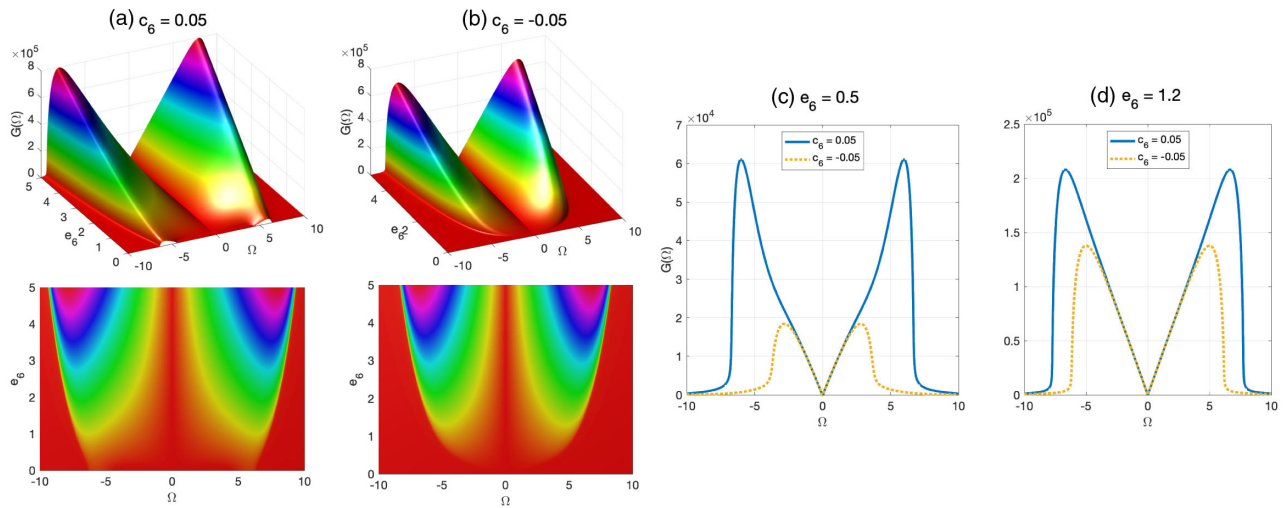


Fig. 4. (a), (b) MI gain $G(\Omega)$ versus the sixth-order self-steepening parameter e_6 and Ω under the influence of SOD coefficient c_6 . (c), (d) $G(\Omega)$ versus the frequency Ω when e_6 takes the respective values 0.5 and 1.2 for $c_6 > 0$ and $c_6 < 0$. The used parameter values are: $P = 20$, $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$, and $e_9 = 6 \times 10^{-4}$.

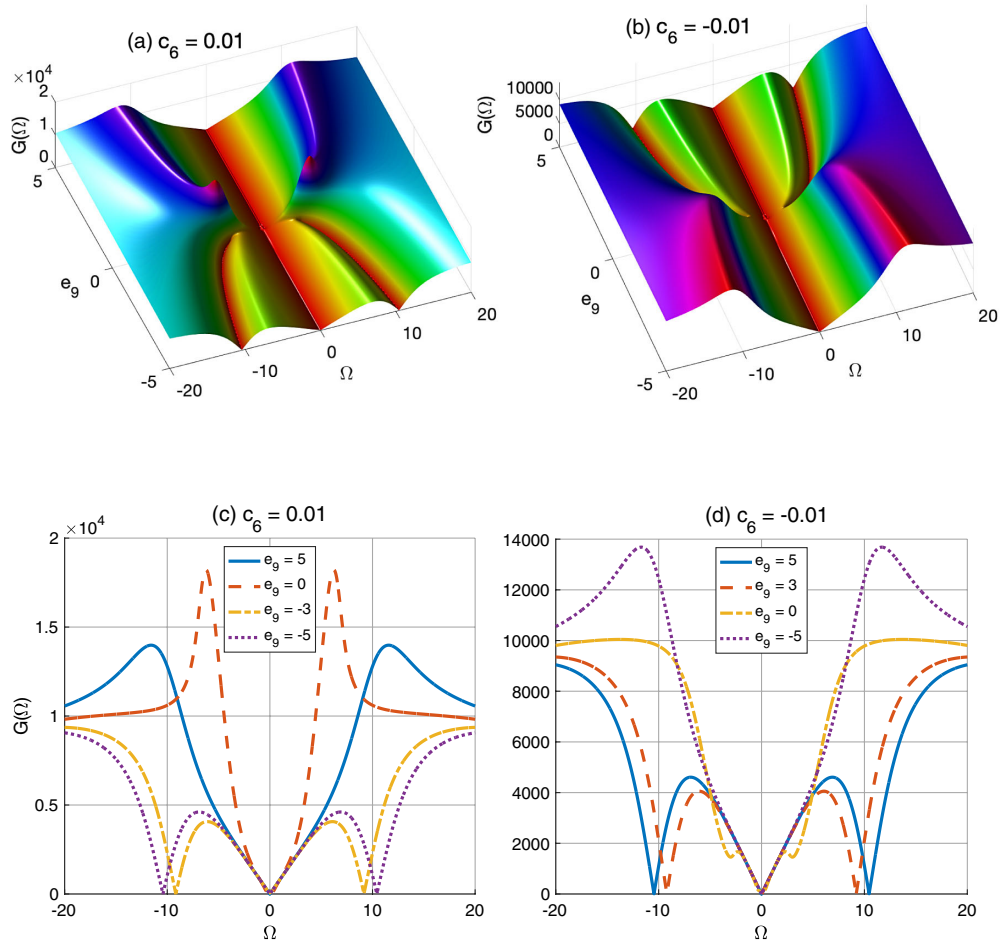


Fig. 5. (a), (b) Plots of the MI gain $G(\Omega)$ versus the frequency Ω and the septic Raman scattering parameter e_9 . Each of the panels contains results for three increasing values of the septic Raman scattering parameter e_9 , where (a) corresponds to $c_6 = 0.01$ and (b) to $c_6 = -0.01$. (c), (d) Evolution of the gain with changing e_9 for $c_6 = 0.01$ and $c_6 = -0.01$, respectively.

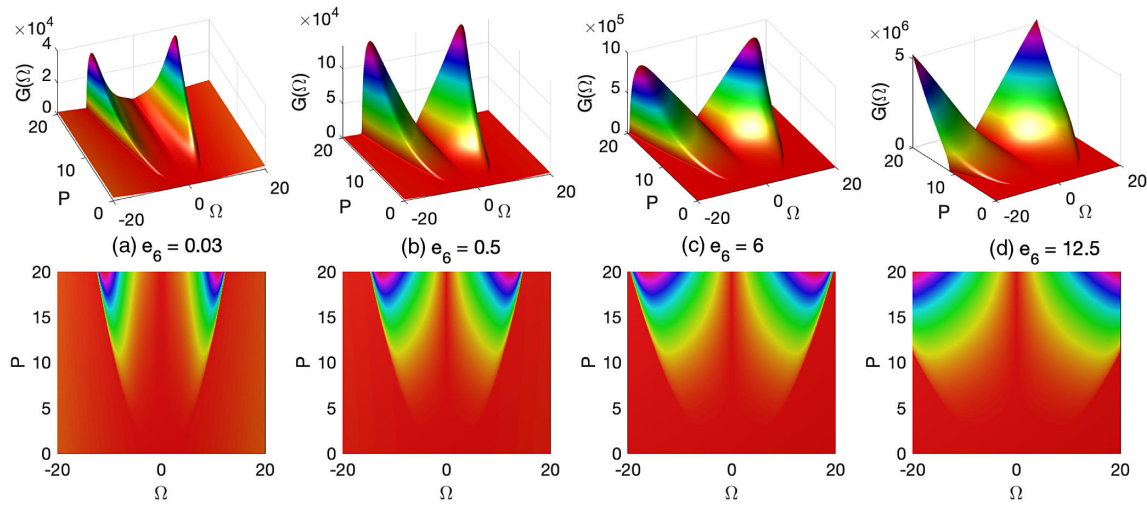


Fig. 6. Panels show the MI gain $G(\Omega)$ versus the input power P and Ω for different values of e_6 , the septic self-steepening parameter e_6 . The remaining parameter values are: $c_0 = -0.1$, $c_3 = 0.01$, $c_5 = 0.002$, with Kerr term $\alpha_1 = 1$ and non-Kerr terms $\alpha_2 = 0.01$, $\alpha_3 = 0.05$, $a_3 = 0.01$, $a_4 = 0.005$, $a_5 = 0.002$, $a_6 = 0.001$, $e_7 = 0.00375$, $e_8 = 0.00156$, and $e_9 = 6 \times 10^{-4}$.

In Figs. 5(a) and 5(c), where $c_6 > 0$, the MI gain, for $e_9 > 0$, shows some wings of instability with frequency peaks at $e_9 = 0$, while for $e_9 < 0$, the instability is supported by two symmetric and identical frequency lobes of lower intensity, between lateral lobes of high intensity. Moreover, when $c_6 < 0$, as illustrated in Figs. 5(b) and 5(d), the inverse features of the case $c_6 < 0$ are obtained, except that the gain intensity is lower here and the peaks at $e_9 = 0$ are inexistent. We should also notice that some of the features ostensible in this case are shared with those obtained for the septic nonlinearity, which, once more, supports the involvement of higher nonlinear effects in the process of MI on one hand, and their mutual competition that may surely be regulated through the right choice of dispersion parameters, on the other hand. In the meantime, the frequency bandwidths detected are those expected to support the break-up of the CW into modulated waves and soliton-like objects. Nevertheless, one aspect that should not be ignored is how such a process can be monitored by the total input power P , combined with the modulational frequency Ω . This is illustrated in Fig. 6, where in general, the MI gain increases with P , while the frequency bandwidth gets enlarged with increasing the septic self-steepening parameter.

4. CONCLUSION

In summary, we have successfully derived an extended $(1 + 1)D$ higher-order cubic–quintic–septic CGL equation with the TOD, FOD, FFOD, and SOD and the cubic, quintic, and septic nonlinear terms, describing the dynamics of extremely short pulses in nonlinear doped optical fibers. The MI phenomenon has been addressed exclusively on the proposed model, where we have considered the impact of SOD and septic nonlinearity, septic self-steepening, and the septic Raman scattering on MI under a normal dispersion regime. In fact, through the linear stability analysis of CW, the MI gain has been derived explicitly and used to comprehensively explore the features of MI under the effects of the SOD and the higher-order nonlinear parameters. The

subsequent results can therefore be summarized as follows: (i) under normal and anomalous regimes, the FOD parameter has been found to importantly affect the involvement of the SOD in the process of MI. Particularly, in the anomalous regime, the instability was quenched for negative SOD, while the intensity and expansion of the existing bandwidths were found to respond to increasing values of c_6 . (ii) The MI features of the septic nonlinearity under the effect of positive SOD have shown that the CW may be unstable for any value of the septic nonlinear coefficient. Under negative SOD, two symmetric wings separated by two minor lobes around $\Omega = 0$ were observed. In the same vain, the effect of the septic self-steepening and Raman scattering on the occurrence of MI displayed more interesting features, with a new gain profile and, more importantly, their sensitive response to the change in the SOD value and sign. Additionally, the total input power has also been found to bring about some new effects to the manifestation of MI, this in competition with the septic self-steepening under a normal frequency regime and positive SOD. Moreover, the model exhibits a maximum MI spectral gain value much greater compared to the HONLS equation [91,92] and the CGL equation [84], therefore offering a larger bandwidth with possibly more ultra-short pulses (solitons) [96], and hence its possible attractiveness in long distance optical fiber communication. We should, however, stress that one of the main difficulties in the derived model was the high number of parameters—the reason that we work entirely on a non-dimensional equivalence of the proposed CGL equation. Our next step is to generalize such a model to high dimensions and investigate the propagation and stability of light bullets in the presence of defects.

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Higher-order dispersion and nonlinear effects of optical fibers under septic self-steepening and self-frequency shift

Karabo K. Ndebele^{1,*}, Conrad B. Tabi^{1,†}, Camus G. Latchio Tiofack^{2,‡} and Timoléon C. Kofané^{1,2,3,§}

¹*Department of Physics and Astronomy, Botswana International University of Science and Technology,
Private Mail Bag 16 Palapye, Botswana*

²*Department of Physics, Laboratory of Mechanics, Faculty of Science, University of Yaoundé I, P.O. Box 812, Yaoundé, Cameroon*

³*Centre d'Excellence Africain en Technologies de l'Information et de la Communication,
University of Yaoundé I, P.O. Box 812, Yaoundé, Cameroon*



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We investigate the modulational instability (MI) of a continuous wave (cw) under the combined effects of higher-order dispersions, self steepening and self-frequency shift, cubic, quintic, and septic nonlinearities. Using Maxwell's theory, an extended nonlinear Schrödinger equation is derived. The linear stability analysis of the cw solution is employed to extract an expression for the MI gain, and we point out its sensitivity to both higher-order dispersions and nonlinear terms. In particular, we insist on the balance between the sixth-order dispersion and nonlinearity, septic self-steepening, and the septic self-frequency shift terms. Additionally, the linear stability analysis of cw is confronted with the stability conditions for solitons. Different combinations of the dispersion parameters are proposed that support the stability of solitons and the occurrence of MI. This is confronted with full numerical simulations where the input cw gives rise to a broad range of behaviors, mainly related to nonlinear patterns formation. Interestingly, under the activation of MI, a suitable balance between the sixth-order dispersion and the septic self-frequency shift term is found to highly influence the propagation direction of the optical wave patterns.

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I. INTRODUCTION

The study of exact traveling wave solutions plays a vital role to understand the nonlinear physical phenomena of nonlinear evolution equations, which exist in many areas of science, such as plasma physics, fluid dynamics, nonlinear optics, elastic media, optical fibers, hydrodynamics, biology, and many more [1–5]. Optical solitons have been an active topic of research during the past decades because of their potential applications in long-distance communication [6,7]. The pioneering works of Hasegawa and Tappert [8,9], who predicted solitons theoretically, and Mollenauer *et al.* [10,11], who observed them experimentally, made solitons a practical tool for data transmission. There has been much interest in recent years in the supercontinuum generation in optical fibers because of their high potential for applications in the fields of optical communications [12], generation of ultrashort pulses [13] in highly nonlinear dispersion-shifted fibers [14] using picosecond or femtosecond laser pulses with high peak power. It is well known that the standard nonlinear Schrödinger (NLS) equation can be used to describe the propagation of a picosecond optical pulse. However, higher-order effects, such as the third-order dispersion (TOD), fourth-order dispersion

(FOD), fifth-order dispersion (FFOD), sixth-order dispersion (SOD), which should be considered when the pulse width is below 10 fs [15–17], higher-order nonlinearities (should not be neglected when the optical field frequency approaches a resonant frequency of the optical fiber materials [17,18]), self-steepening (SS), and self-frequency shift (SFS) induced by Raman stimulated scattering are indispensable for describing the propagation of ultrashort pulses [19–22]. In the picosecond systems, the NLS equation is the central governing pulse envelope equation in which the self-phase modulation and the group-velocity dispersion (GVD) effects are just taken into description [23]. It is well known that as the intensity of the incident light field becomes stronger, non-Kerr nonlinear effects become important [18,24], and the NLS family of equations should describe the dynamics of the pulses with higher-order nonlinear terms [25–31] from which can emerge a new class of soliton solutions. One of the key issues of soliton theory is the universal modulational instability (MI) phenomenon, which leads to the emergence of localized coherent nonlinear structures and the formation of trains of soliton pulses.

MI is a fundamental and ubiquitous nonlinear phenomenon that pertains to a large variety of subfields of physics, such as fluid dynamics [2], plasma physics [32], atomic physics (atomic Bose-Einstein condensates), and nonlinear optics [33]. MI implies that as a result of the combined action of the group velocity dispersion (GVD) or the second-order chromatic dispersion and self-phase modulation (SPM) or the self-focusing Kerr nonlinearity of the medium, amplitude, or phase perturbations added to a continuous or quasicontinuous

*karabo.ndebele@studentmail.biust.ac.bw

†Corresponding author: tabic@biust.ac.bw

‡glatchio@yahoo.fr

§tckofane@yahoo.com

wave (cw) exhibit exponential growth. This is accompanied by sideband evolution with a frequency separation from the carrier, proportional to its power. In the anomalous dispersion regime, that is, when we use optical fiber with negative GVD, the dominant unstable MI frequencies can be shown to be proportional to the cw amplitude and inversely proportional to the square root of the dispersion [15,17]. Thus, the modulation that grows on the cw field increases frequency as the dispersion goes to zero. Even in the normal dispersion regime, MI can occur due to third- and fourth-order dispersion corrections [34]. In addition, various higher-order linear and nonlinear effects, such as higher dispersions, self-steepening, and time-delayed Raman effects have also been considered, and these effects are found to influence MI in optical fibers strongly [35,36]. In optics, MI has constituted a wide field of intense theoretical and experimental research including ultrafast pulse generation [37], supercontinuum generation [38,39], four-wave mixing (FWM) [40,41], Bragg gratings [42], parametric oscillators [43,44], and optical fiber systems [45–48]. In the meantime, many research activities have been carried to understand and characterize optical soliton under: (i) strong nonlinearity versus weak dispersion [49], (ii) strong dispersion versus weak nonlinearity [34,50], and (iii) a subtle balance between dispersion and nonlinearity with the main objective being the generation of solitary waves. Our paper relies on the third stream. We hope that a sixth-order linear dispersion and sixth-order nonlinear dispersions can suitably balance septic nonlinearity and lead to the formation of nonlinear structures. In our contribution, we start with Maxwell's equations describing the response of the nonlinear medium to electromagnetic waves. Then we report on the derivation of the one plus one dimensional [(1 + 1)D] cubic-quintic-septic NLS equation with third-, fourth-, fifth-, and sixth-order dispersions. Furthermore, we examine theoretically and numerically MI of few-cycle pulses using linear stability analysis of linearized equations for small perturbations and using direct numerical simulation to support our analytical predictions.

The rest of the paper is organized as follows. In Sec. II, we derive a (1 + 1)D cubic-quintic-septic NLS equation describing the dynamics of ultrashort pulses in nonlinear optical fibers under the combined effects of dispersion, loss, gain, cubic, and cubic-quintic-septic nonlinearities, and cubic-quintic-septic self-steepening and self-frequency shift terms, respectively. In Sec. III, the linear stability analysis of the MI is addressed, and instability zones, as well as the analytical expressions of the gain of MI, are obtained. Based on analytical findings, numerical simulations are carried out on the (1 + 1)D cubic-quintic-septic NLS equation using the split-step Fourier method. Particular attention is paid to the combined effects of the sixth-order dispersion cubic, quintic, septic self-steepening, and self-frequency shift parameters. Some concluding remarks are given in Sec. V.

II. DERIVATION OF THE HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION

We consider the cubic-quintic-septic nonlinearities of nonlinear medium, which arise from the expansion of the refractive index in the power series of intensity I of the light

pulse,

$$n_f(\omega) = n_0 + n_2 I + n_4 I^2 + n_6 I^3. \quad (1)$$

Here, n_0 is the linear refractive index coefficient, and n_2 , n_4 , and n_6 are the nonlinear refractive index coefficients, which originate from third-, fifth-, and seventh-order susceptibility, respectively, $I = |E|^2$, E being the electric field vector. The complex dielectric constant, related to optical signal propagation through an optical fiber is as follows:

$$\varepsilon(\omega) = n_f^2(\omega) = n_0^2 + 2n(\omega)\Delta n, \quad (2)$$

with Δn being given by

$$\Delta n = n_2 |E|^2 + \left[\frac{n_0^2}{2} + n_4 \right] |E|^4 + \left[n_6 + \frac{n_2 n_4}{n_0} \right] |E|^6. \quad (3)$$

Following Agrawal [17], the propagation equation for $\tilde{\psi}(z, \omega - \omega_0)$, which is the Fourier transform of $\psi(z, t)$, is written in the form

$$\frac{\partial \tilde{\psi}}{\partial z} = i[\beta(\omega) + \Delta\beta - \beta_0]\tilde{\psi}, \quad (4)$$

where $\beta(\omega)$ is Taylor's expansion in series, up to sixth-order, about the carrier frequency and the quantity $\Delta\beta$ is defined as

$$\Delta\beta = k_0 \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta n |f(x, y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 dx dy}, \quad (5)$$

with k_0 as the wave number and $f(x, y)$ being the modal distribution of the fundamental mode of the optical fiber. We should note that nonlinearities relative to the self-steepening and self-frequency shift are modeled through the propagation constant. Taking the inverse Fourier transform of Eq. (4), we obtain the following envelope equation for ultrashort pulses in optical fiber:

$$\begin{aligned} i\psi_z + \sum_{m=2}^6 (i)^m \partial_t^m \psi + \left(\sum_{m=1}^3 \gamma_m |\psi|^{2m} \right) \psi \\ + i \sum_{m=1,3,5} \alpha_m (|\psi|^{m+1} \psi)_t \\ + \sum_{m=2,4,6} \alpha_m (|\psi|^m)_t \psi = 0, \end{aligned} \quad (6)$$

where $\partial_t^m = \frac{\partial^m}{\partial t^m}$. More precisely, $\psi(z, t)$ is the envelope amplitude of the electric field, t is the retarded time frame, and z is the propagation distance originating from the transformation, $T = t - z/v_g = t - \beta_1$. All the coefficients are real constants. The above equation is a higher-order NLS equation with higher-order dispersion and cubic, quintic, and septic non-Kerr nonlinearities. γ_1 , γ_2 , and γ_3 are the cubic, quintic, and septic nonlinearities, respectively, whereas α_1 , α_3 , and α_5 are the cubic, quintic, and septic self-steepening terms, respectively. The terms α_2 , α_4 , and α_6 are the cubic, quintic, and septic self-frequency shift induced by stimulated Raman scattering, respectively. β_2 is the GVD term, β_3 is the TOD term, β_4 is the FOD, β_5 is the FFOD term, and β_6 is the SOD term. The expressions of parameters in Eq. (6) are given in the Appendix. Recent advances in the fabrication of photonic-crystal

fibers (PCFs) [51–54] have made possible the production of small-core PCFs where strong nonlinear interactions occur at relatively low peak powers and over short propagation distances. The FWM theory has been developed, which goes beyond the slowly varying envelope approximation of the NLS equation and is based on the full wave equation, allowing the rigorous study of the left-most instability band (the first FWM peak) and the far-detuned instability peak (the secondary FWM). In this approach, the perturbation wave numbers are the roots of the fourth-order algebraic equation [55] where the two extra roots can describe excitation of both forward and backward waves. At the same time, all previous results on FWM in fibers with nontrivial dispersion characteristics resulted in a quadratic equation for the perturbation wave numbers [17,56–59]. These two extra roots have negative imaginary parts, generating instabilities [55]. In general, the first FWM peak, which has a direct analog in the idealized NLS equation in the anomalous dispersion regime, corresponds to a wave with a Stokes component slightly stronger than the anti-Stokes one, and the second peak, which appears in the model due to modified fiber dispersion, and which does not exist in the idealized NLS, generates a stronger anti-Stokes wave. In the meantime, a generalized NLS equation has been derived, including self-steepening effects, whereas the summation characterizing higher-order dispersion is taken up to order $N = 20$, high enough to ensure that the dispersion profile is adequately approximated in the frequency domain under consideration [55]. It has been, however, pointed out that the generalized NLS equation is often insufficient for quantitative comparisons between experimental and theoretical results since the dependence of the nonlinearity on the detuning parameter is taken into only approximately [55]. Indeed, the existence of the second FWM peak is ensured by positive higher-order dispersion terms greater or equal to 6, namely, the positive dispersions of orders two, four, and six, respectively, and requires sufficiently large values of the detuning parameter because higher-order dispersions terms are very small. However, due to extra degrees of freedom, PCFs allow mode propagation to be easily manipulated. They are made of an array of microscopic air holes distributed throughout their entire length that surrounds a pure silica core. Therefore, nonlinear effects are enhanced due to light concentration into a very small area between silica and air, and their balance with higher-order dispersions can be used to explain optical pulse communication in highly nonlinear photonic crystal fiber, which imposes to extend the order of nonlinear terms in the standard NLS equation beyond Kerr nonlinearity.

In order to simplify the problem at hand, it would be advisable to reduce the number of parameters. Therefore, Eq. (6) can be normalized by making use of the transformations $Z = z/L_D$ for space, $\tau = t/T_0$ for time and $u = \frac{\psi}{\sqrt{P_0}}N$ for amplitude, which leads to the following normalized HNLS equation:

$$iu_Z + \sum_{m=2}^6 (i)^m k_m \partial_\tau^m u + n_{NL}|u|^2 u + n_{NL4}|u|^4 u + n_{NL6}|u|^6 u + in_{SS}(|u|^2 u)_\tau + in_{SS4}(|u|^4 u)_\tau$$

$$+ in_{SS6}(|u|^6 u)_\tau + n_{SFS}(|u|^2)_\tau u + n_{SFS4}(|u|^4)_\tau u + n_{SFS6}(|u|^6)_\tau u = 0, \quad (7)$$

where

$$k_2 = \text{sgn}(\beta_2), \quad k_m = \sum_{m=3}^6 \frac{L_D}{L_{Dm}}, \quad n_{NL} = \frac{L_D}{L_{NL}N^2}, \\ n_{NL4} = \frac{L_D}{L_{NL4}N^4}, \quad n_{NL6} = \frac{L_D}{L_{NL6}N^6}, \quad n_{SS} = \frac{L_D}{L_{SS}N^2}, \\ n_{SS4} = \frac{L_D}{L_{SS4}N^4}, \quad n_{SS6} = \frac{L_D}{L_{SS6}N^6}, \quad n_{SFS} = \frac{L_D}{L_{SFS}N^2}, \\ n_{SFS4} = \frac{L_D}{L_{SFS4}N^4}, \quad n_{SFS6} = \frac{L_D}{L_{SFS6}N^6}, \quad (8)$$

with $L_D = \frac{T_0^2}{\beta_2}$, $L_{Dm} = \sum_{m=3}^6 \frac{T_0^m}{\beta_m}$, $N^2 = \frac{L_D}{L_{NL}}$, $L_{NL} = \frac{1}{\gamma_1 P_0}$, $L_{NL4} = \frac{1}{\gamma_2 P_0^2}$, $L_{NL6} = \frac{1}{\gamma_3 P_0^3}$, $L_{SS} = \frac{T_0}{\alpha_1 P_0}$, $L_{SS4} = \frac{T_0}{\alpha_3 P_0^2}$, $L_{SS6} = \frac{T_0}{\alpha_5 P_0^3}$, $L_{SFS} = \frac{T_0}{\alpha_2 P_0}$, $L_{SFS4} = \frac{T_0}{\alpha_4 P_0^2}$, $L_{SFS6} = \frac{T_0}{\alpha_6 P_0^3}$. The above quantities are dimensionless parameters depending on the typical values [60–62]: $P_0 = 10$ W, $\beta_6 = 0.5$ ps⁶ m⁻¹, $\beta_5 = 0.00123$ ps⁵ m⁻¹, $\beta_4 = 10$ ps⁴ m⁻¹, $\beta_3 = 0.009$ ps³ m⁻¹, $\beta_2 = 50$ ps² m⁻¹, $\gamma_1 = 1$ kW⁻¹ m⁻¹, $\gamma_2 = 1$ kW⁻² m⁻¹, $\gamma_3 = 0.031$ kW⁻³ m⁻¹, $\alpha_1 = -0.0247$ kW⁻¹/[(2 π) mTHz], $\alpha_2 = 0.03705$ kW⁻¹ psm⁻¹, $\alpha_3 = -0.0247$ kW⁻²/[(2 π) mTHz], $\alpha_4 = -0.030875$ kW⁻² psm⁻¹, $\alpha_5 = 0.004$ kW⁻³/[(2 π) mTHz] and $\alpha_6 = 0.02$ kW⁻³ psm⁻¹.

The advantage of deriving the above higher-order nonlinear Schrödinger equation (HNLS) equation is that we see the mathematical input of the aforementioned respective nonlinearities that most researchers do not show. The model we have presented has a set of new parameters compared to the previous papers [31,63–65]. Before ending, let us mention that the self-steepening and self-frequency shift terms have been taken into account in optical fibers whose pulse dynamics are well described by the complex Ginzburg-Landau equation [46,66,67]. Our HNLS equation governs the dynamics of ultrashort pulses where septic self-steepening and self-frequency shift terms come into action. The existence of solitons in an optical fiber system described by the HNLS equation further requires the balance among self-steepening and self-frequency shift effects also called the derivative Kerr nonlinear terms. The balance among the derivative quintic non-Kerr nonlinear terms and the balance between the derivative septic non-Kerr nonlinear terms are all accounted for in Eq. (7). It is remarked that such balance aspects are also required for the formation of ultrashort soliton pulses [68].

III. MODULATIONAL INSTABILITY

A. Linear stability analysis and gain spectrum

The MI is studied on model Eq. (7) via the linear stability analysis of a CW solution, which propagates inside the optical fiber with the initial input power P_0 , i.e.,

$$u(Z, \tau) = \sqrt{P_0} e^{i\phi_{NL}}, \quad (9)$$

where $\phi_{NL} = n_{NL}P_0 + n_{NL4}P_0^2 + n_{NL6}P_0^3$ is the nonlinear phase shift due to the self-phase modulation and higher-order

nonlinear terms. The stability of solution (9) requires the introduction of slight perturbation so that

$$u(Z, \tau) = [\sqrt{P_0} + a(Z, \tau)]e^{i\phi_{NL}}, \quad (10)$$

where $a(Z, \tau)$ is the small perturbation with $|a(Z, \tau)| \ll \sqrt{P}$ being the complex field. This perturbed solution means that if the perturbed field grows exponentially, the steady state will become unstable. Substituting Eq. (10) in Eq. (7), and linearizing around the unperturbed solution, leads to the following equation for the perturbed field:

$$\begin{aligned} ia_Z - k_2 a_{\tau\tau} - ik_3 a_{\tau\tau\tau} + k_4 a_{\tau\tau\tau\tau} \\ + ik_5 a_{\tau\tau\tau\tau\tau} - k_6 a_{\tau\tau\tau\tau\tau\tau} + n_{NL}[P_0(a + a^*)] \\ + n_{NL4}[2P_0^2(a + a^*)] + n_{NL6}[3P_0^3(a + a^*)] \\ + in_{SS}[2P_0 a_\tau + P_0 a_\tau^*] + in_{SS4}P_0^2[3a_\tau + 2a_\tau^*] \\ + in_{SS6}P_0^3[4a_\tau + 3a_\tau^*] + n_{SFS}P_0[a_\tau + a_\tau^*] \\ + 2n_{SFS4}P_0^2[a_\tau + a_\tau^*] + 3n_{SFS6}P_0^3[a_\tau + a_\tau^*] = 0, \end{aligned} \quad (11)$$

where a^* is the complex conjugate of the perturbed field. Furthermore, solution for Eq. (11) can be adopted in the form

$$a(Z, \tau) = u(Z)e^{-i\Omega\tau} + v(Z)e^{i\Omega\tau}, \quad (12)$$

where $u(Z)$ and $v(Z)$ are the complex perturbation fields and Ω is the complex modulation frequency. Making use of the ansatz of the perturbed field of Eq. (12) in the linearized Eq. (11) leads to a 2×2 matrix of the following form for the perturbed fields:

$$i \frac{\partial}{\partial s} \begin{pmatrix} u(Z) \\ v^*(Z) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} u(Z) \\ v^*(Z) \end{pmatrix}, \quad (13)$$

where $v^*(Z)$ is the complex conjugate of the field $v(Z)$ with the matrix elements being given by

$$\begin{aligned} T_{11} &= -[k_2\Omega^2 + k_3\Omega^3 + ka_4\Omega^4 + k_5\Omega^5 + k_6\Omega^6 + m \\ &\quad + \Omega P_0(2n_{SS} + 3n_{SS4}P_0 + 4n_{SS6}P_0^2 \\ &\quad - in_{SFS} - 2in_{SFS4}P_0 - 3in_{SFS6}P_0^2)], \\ T_{12} &= -[m + \Omega P_0(n_{SS} + 2n_{SS4}P_0 + 3n_{SS6}P_0^2 \\ &\quad - in_{SFS} - 2in_{SFS4}P_0 - 3in_{SFS6}P_0^2)], \\ T_{21} &= [m - \Omega P_0(n_{SS} + 2n_{SS4}P_0 + 3n_{SS6}P_0^2 + in_{SFS} \\ &\quad + 2in_{SFS4}P_0 + 3in_{SFS6}P_0^2)], \\ T_{22} &= [k_2\Omega^2 - k_3\Omega^3 + ka_4\Omega^4 - k_5\Omega^5 + k_6\Omega^6 + m \\ &\quad - \Omega P_0(2n_{SS} + 3n_{SS4}P_0 + 4n_{SS6}P_0^2 \\ &\quad + in_{SFS} + 2in_{SFS4}P_0 + 3in_{SFS6}P_0^2)]. \end{aligned} \quad (14)$$

In order to find the MI gain, we should keep in mind that the wave number K of the perturbation can be found as the eigenvalue of the matrix $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, through the characteristic equation $\text{Det}[T - KI] = 0$ with I being a 2×2 identity matrix. This leads to the dispersion relation

$$K = \frac{1}{2}[T_{11} + T_{22} + \sqrt{(T_{11} - T_{22})^2 + 4T_{12}T_{21}}]. \quad (15)$$

For MI to take place, the wave number K should get a nonzero imaginary part that will be responsible for an exponential

growth of the perturbed amplitude. In general, the occurrence of MI is predicted through the power gain, defined as $G(\Omega) = 2 \text{Im}(K)$, which is obtained in this particular case in the form

$$G(\Omega) = \sqrt{c_2\Omega^2 - c_3\Omega^3 + c_4\Omega^4 - c_5\Omega^5 + c_6\Omega^6 - c_7\Omega^7 + c_8\Omega^8 + c_{10}\Omega^{10} + c_{12}\Omega^{12}}, \quad (16)$$

where

$$\begin{aligned} c_2 &= (8n_{NL}P_0 + 16n_{NL4}P_0^2 + 24n_{NL6}P_0^3)k_2 \\ &\quad + 4n_{SS}^2P_0^2 + 16n_{SS4}n_{SS}P_0^3 + 24n_{SS}n_{SS6}P_0^4 \\ &\quad + 16n_{SS4}^2P_0^4 + 48n_{SS4}n_{SS6}P_0^5 + 36n_{SS6}^2P_0^6 \\ &\quad + (8in_{SFS}P_0 + 16in_{SFS4}P_0^2 + 24in_{SFS6}P_0^3 + 4) \\ &\quad \times (n_{SS}P_0 + 2n_{SS4}P_0^2 + 3n_{SS6}P_0^3), \\ c_3 &= 8ik_2(n_{SFS}P_0 + 2n_{SFS4}P_0^2 + 3n_{SFS6}P_0^3), \\ c_4 &= k_4(8n_{NL}P_0 + 16n_{NL4}P_0^2 + 24n_{NL6}P_0^3) + 4k_2^2, \\ c_5 &= 8ik_4(n_{SFS}P_0 + 2n_{SFS4}P_0^2 + 3n_{SFS6}P_0^3), \\ c_6 &= k_6(8n_{NL}P_0 + 16n_{NL4}P_0^2 + 24n_{NL6}P_0^3) + 8k_2k_4, \\ c_7 &= 8ik_6(n_{SFS}P_0 + 2n_{SFS4}P_0^2 + 3n_{SFS6}P_0^3), \\ c_8 &= 4k_4^2 + 8k_2k_6, \quad c_{10} = 8k_4k_6, \quad c_{12} = 4k_6^2. \end{aligned} \quad (17)$$

As it can be noted, the expression of the MI gain depends on many physical parameters. Considering all such parameters will make our contribution interesting but too heavy. Therefore, a suitable choice of parameters can be made based on previous works relative to the higher-order effect on soliton stability and the MI process. Originally, the terminology soliton was reserved for a particular set of integrable solutions existing due to the suitable balance between dispersion (or diffraction) and nonlinearity. Optical solitons are used for long- and short-distance information transmission. Unlike pulses in a linear dispersive fiber, solitons are self-confined, propagating long distances without changing shape. A well-known example of an equation that admits pulselike soliton solutions is the NLS equation. It is now well known that the power is directly proportional to the dispersion in the experimental investigation of optical solitons. Because of the large dispersion of the conventional single-mode fiber at the wavelength of minimal losses ($\lambda = 1.5 \mu\text{m}$), a clear idea for power reduction was a shift of the operational wavelength towards the zero-dispersion wavelength [69–71] which is $\lambda = 1.31 \mu\text{m}$. In doing so, an additional term in expanding the wave number of the fiber mode around the central frequency ω_0 , namely, the TOD term, needs to be taken into account when extending the applicability of the NLS equation. Hence, there was a certain interest in understanding the effect of TOD on the existence and stability of the soliton [72–77] based on the Sasa-Satsuma [78] and Hirota [79] equations, which are the NLS equation with some TOD terms that cause great variation in MI properties which would induce some new nonlinear excitations. The linearized stability analysis for the Sasa-Satsuma equation [80] suggested that there are both MI and stability regimes for low perturbation frequencies on the continuous wave background. Indeed, it is well known that odd-order dispersion coefficients, such as TOD, do not affect

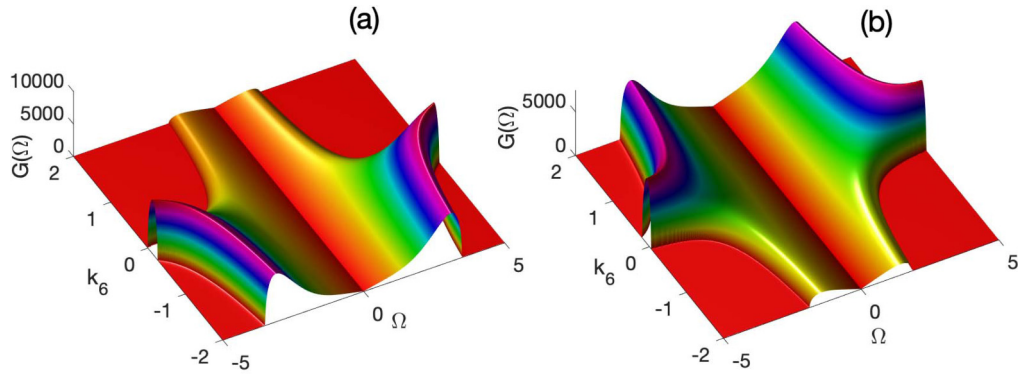


FIG. 1. The panels show plot of the variation of maximum MI gain as a function of perturbation frequency Ω and the SOD term k_6 for (a) the normal ($k_2 = 1$) and (b) the anomalous regimes ($k_2 = -1$) with the other parameters being: $P_0 = 8$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$ and $n_{SFS6} = 0.021$.

the MI condition [56]. The MI process has been revisited theoretically using a global stability analysis reformulated as an initial value problem. It has been demonstrated that MI gain of time-localized signals (i.e., pulsed signals) depends strongly on the TOD, leading to a dramatic reduction of the MI gain, contrary to the well-known case of time-extended signals (cw signals) [81]. The MI has been investigated in the region of minimum group-velocity dispersion through an extended NLS equation, taking into account FOD effects, and this term dominates the critical modulation-instability frequency when second-order dispersion approaches its minimum value at the so-called zero-dispersion wavelength [82]. The effect of TOD and FOD terms on MI has been investigated. It has been shown that the modulationally unstable waves evolve to the soliton-type characterized by the periodic generation of moving or stationary solitons and to a turbulent state, depending on the strength of the third- and fourth-order dispersion terms, which are related to the stability conditions for solitons [83]. It has been demonstrated numerically that in the presence of the quintic nonlinear term, which balances with the higher-order dispersion terms, the turbulent states can be controlled, leading to the periodic occurrence of the stable solitonlike pulses produced by MI of an extended NLS equation with the third- and fourth-order dispersion and the cubic-quintic nonlinear terms. It appears that even though the TOD term

does not directly contribute to the MI gain spectrum, it influences the evolution of MI and the dynamics of the soliton [66]. Also, for a fiber with the zero group-velocity dispersion wavelength in the visible region of the spectrum and a pump wavelength close to the zero group-velocity dispersion wavelength, the importance of the FOD term in the expansion of the propagation constant is considerable, leading to a strong parametric gain of the sidebands even when technically the fiber is being pumped in the normal dispersion regime [34]. The FOD-induced MI, which can be used for broadband wavelength conversion, has been investigated experimentally in single-mode optical fiber, and evidence of a new MI spectral window due to the FOD effect in the normal-dispersion regime has been demonstrated [84]. MI of the HNLS equation with non-Kerr nonlinearities in an optical context has been studied, and an analytical expression for MI gain presented to show the effects of non-Kerr nonlinearities and higher-order dispersions on MI gain spectra [65]. The HNLS equation contains real parameters related to GVD, TOD, and FOD, SPM due to the cubic nonlinearity (Kerr effect), self-steepening, and self-frequency shift due to stimulated Raman scattering, and quintic non-Kerr nonlinearity, respectively. In particular, MI gain spectra for the cubic-quintic NLS equation with FOD and stimulated Raman scattering terms have been plotted. Indeed, the expression of the MI gain has revealed that the TOD

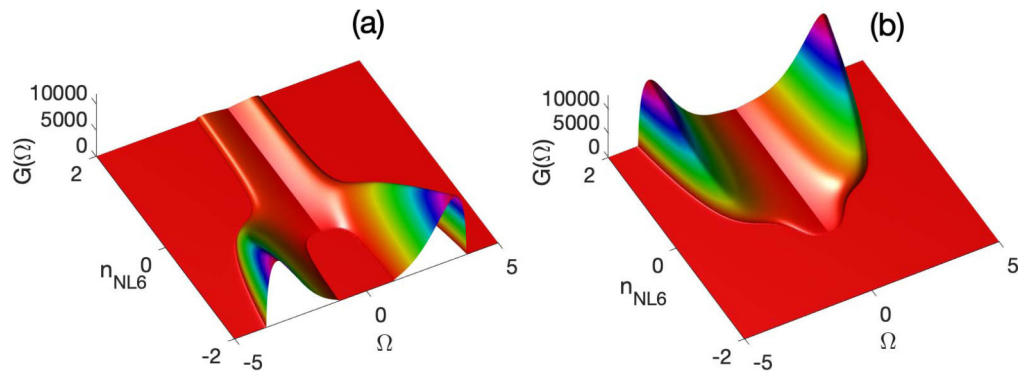


FIG. 2. The variation of maximum MI gain as a function of perturbation frequency Ω and the septic non-Kerr nonlinearity parameter n_{NL6} . Panel (a) shows results for $k_6 = 0.02$, and panel (b) is plotted for $k_6 = -0.02$ with the other parameters being: $P_0 = 10$, $k_5 = 0.00123$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$, and $n_{SFS6} = 0.021$.

term has no effect on the MI condition, but GVD and FOD terms influence the MI condition. The study also shows that the non-Kerr quintic nonlinear effect reduces the maximum value of the gain and bandwidth [65]. The MI process in glass fibers with a local saturable nonlinear refractive index, in operating conditions of anomalous dispersion but with a positive sign of the FOD, has been analyzed. It has been found that two types of MI processes become highly sensitive to the magnitude of the FOD. The first type of MI generates a single pair of sidebands, whereas the second type generates two pairs of sidebands [85]. However, among all the studies of the MI process in nonlinear optics, the question of MI analysis under the combined effects of higher-order dispersions, self-steepening, and self-frequency shift, cubic, quintic, and septic nonlinearities is investigated, here. That is why, in this paper, in particular, we insist on the balance between the sixth-order dispersion and nonlinearity terms on one hand, and between septic self-steepening and the septic self-frequency shift terms, on the other hand. To start, the gain spectrum is displayed in Fig. 1 versus the perturbation frequency Ω and the SOD parameter k_6 both for the normal [see Fig. 1(a)] and anomalous [see Fig. 1(b)] GVD regimes. For $k_2 > 0$, one notes two gain peaks that divert from the center $\Omega = 0$ when k_6 approaches zero. However, two regimes are detected, i.e., when $k_6 > 0$ lobes of the frequency domain are large and reduced for $k_6 < 0$, expanding the stability zone. The reverse scenario is obtained for $k_2 < 0$ where the large frequency bandwidth is observed for $k_6 > 0$ and small bandwidth is observed for $k_6 < 0$. In the rest of the stability analysis, the effects of higher-order nonlinear terms will be confronted to the two SOD regimes, i.e., $k_6 > 0$ and $k_6 < 0$ under normal GVD. The first example to be addressed in this regard is the impact of the nonlinear septic term on MI, which is depicted in Fig. 2 where panel (a) is plotted for $k_6 > 0$ and panel (b) for $k_6 < 0$. For the case $k_6 > 0$, high-intensity sidelobes appear for $n_{NL6} < 0$, but the maximum gain increases, along with the frequency bandwidth, in the interval $-2 \leq n_{NL6} < 0$. In this particular case, the bandwidth for $n_{NL6} < 0$ is large and completely disappears for $n_{NL6} \geq 0$, leaving two symmetric lobes of minor intensity but still enough for the cw to disintegrate into nonlinear structures. On the other side, when $k_6 < 0$, the instability is expected to be more pronounced for $n_{NL6} > 0$ since the gain sidelobes are of high intensity and tend to disappear when $n_{NL6} < 0$, although there is a low-intensity tongue of instability surrounded by a large region of stability [see Fig. 2(b)].

Figure 3 portrays the MI gain as a function of the quintic SS parameters and the frequency Ω for $k_4 > 0$ [panels (a) and (b)] and $k_4 < 0$ [panels (c) and (d)]. Each of the cases presents lobes of instability for normal and anomalous dispersion. In Fig. 3(a), for example, for which $k_2 > 0$, there appear two peaks of instability in the gain spectrum, which show instability to be possible for all values of n_{ss4} . The lobes being symmetric, the gap between them increases for $k_2 < 0$, and the instability bandwidth shrinks [see Fig. 3(b)]. The same spectrum of behaviors appears in Figs. 3(c) and 3(d) for which $k_4 < 0$. For $k_2 > 0$, the two symmetric lobes appear and get shrunk when $k_2 < 0$ for which there appears two breasts of instability. Also, the band gap between gain peaks has reduced. We should stress that the case $k_4 < 0$ and k_2 being

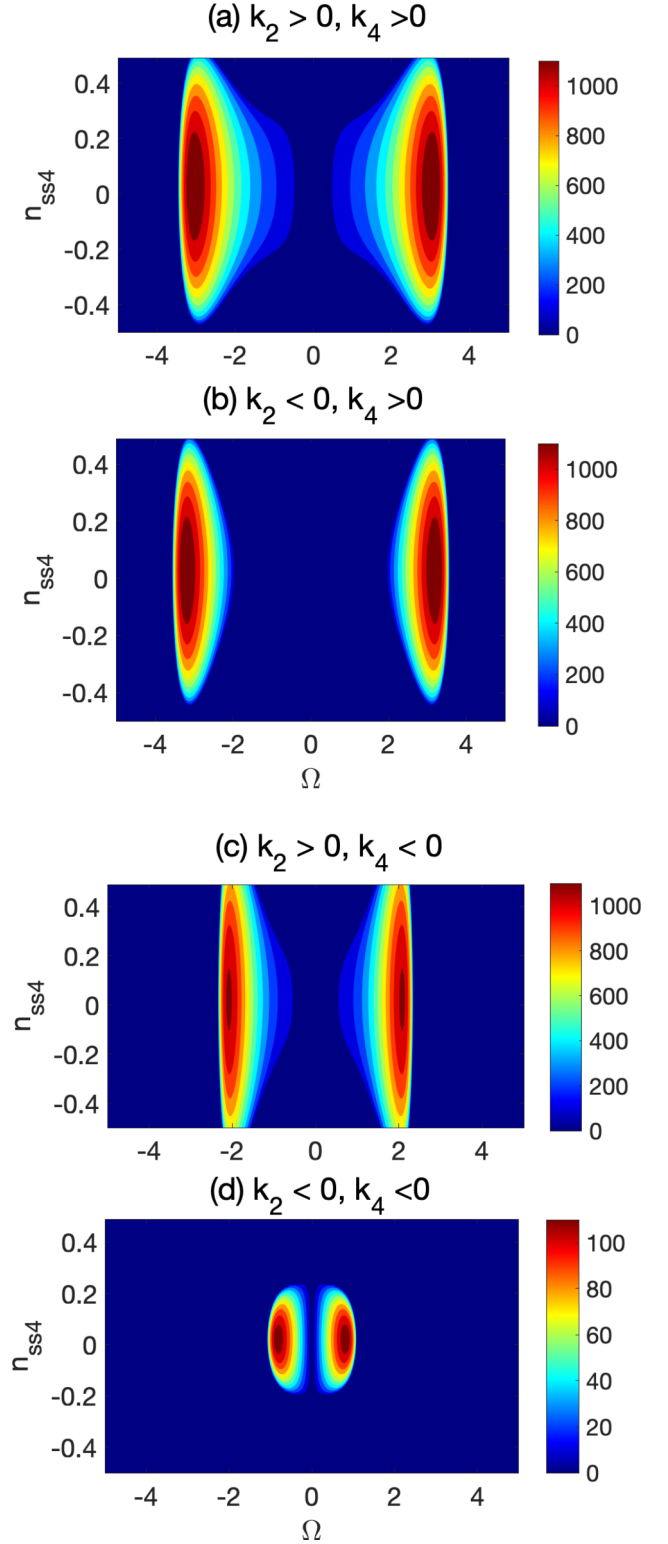


FIG. 3. Variation of maximum MI gain versus the perturbation frequency Ω and the quintic self-steepening term n_{ss4} . The FOD is considered positive in panels (a) and (b), respectively, obtained for $k_2 > 0$ and $k_2 < 0$. Panels (c) and (d) are plotted for a negative FOD coefficient k_4 with $k_2 > 0$ and $k_2 < 0$, respectively. The other parameters being: $P_0 = 10$, $k_6 = 0.02$, $k_5 = 0.001\,23$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{ss4} = 0.037\,05$, $n_{SFS} = -0.030\,875$, $n_{SFS4} = 0.0004$, and $n_{SFS6} = 0.021$.

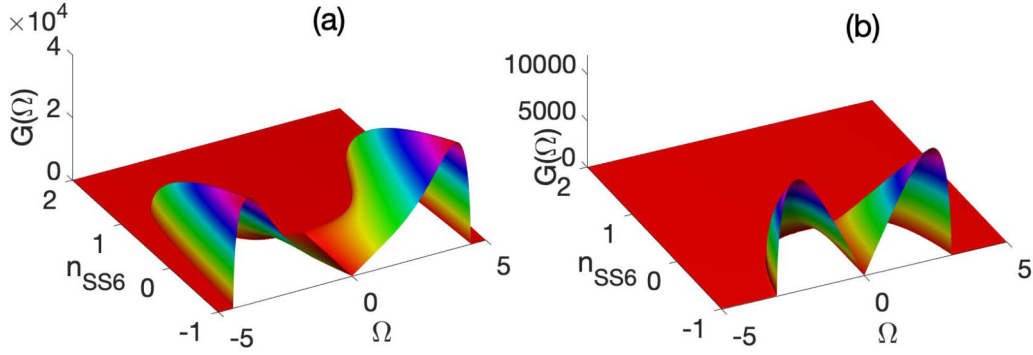


FIG. 4. Panels show the variation of maximum MI gain versus the perturbation frequency Ω and the septic self-steepening term n_{SS6} and for (a) $k_6 = 0.02$, (b) $k_6 = -0.02$ with the other parameters being: $P_0 = 10$, $k_5 = 0.001\,23$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.037\,05$, $n_{SFS} = -0.030\,875$, $n_{SFS4} = 0.0004$, and $n_{SFS6} = 0.021$.

either negative or positive were recently explored by Tiam *et al.* [86], who further found the existence of single-hump solitons numerically for positive values of $k_2 > 0$.

The competition between the septic SS and the SOD parameters is summarized in Fig. 4 where panels (a) and (b) correspond, respectively, to $k_6 > 0$ and $k_6 < 0$. In Fig. 4(a), when $k_6 > 0$, the MI gain is limited as n_{SS6} tends to zero and disappears at $n_{SS6} = 1$, leaving small intensity lobes separated by a large band gap frequency compared to the large frequency bandwidth observed for $-1 \leq n_{SS6} \leq 0$. For the case $k_6 < 0$ [see Fig. 4(b)], no instability should be expected for $n_{SS6} > 0$. However, two symmetric lobes show the presence of instability under a suitable balance between nonlinear and dispersive effects. The MI gain as a function of the perturbation frequency Ω and the SFS parameter is plotted in Fig. 5 with panels (a) and (b), respectively, corresponding to $k_6 > 0$ and $k_6 < 0$. In general, two distinct and symmetric sidebands appear in the gain spectrum and remain the same even with increasing n_{SFS6} . However, the bandwidth is larger for $k_6 < 0$ than when k_6 is positive. From the expression of the gain (16), the reader may note that the odd terms of coefficients k_3 and k_5 do not contribute to the dynamics of the MI growth rate. This is a confirmation of predictions from Shagalov [83] who proposed a separated technique to study the impact of such terms in the numerical occurrence of MI. In so doing, the relationship between soliton emergence and MI has been debated

in several contributions. Especially, attention has been paid to the stability of solitons as they share the same parameter conditions with the occurrence of MI. Before proceeding to the numerical analysis of our above predictions, it would be of interest to investigate the impact of each of the parameters and their various combinations for the emergence and stability of solitons.

B. Stability of solitons and MI occurrence

When $k_2 \neq 0$, $n_{NL} \neq 0$ with the rest of the coefficients being equal to zero, we are in the presence of the cubic quintic NLS equation that admits stable soliton solutions. Such solutions imply the appearance of MI and the subsequent appearance of wave trains under the condition that dispersion and nonlinearity be well balanced. However, when the high-dispersion terms are different from zero, the stability of soliton solution gets alternated with a substantial impact on the appearance of MI, primarily when the long-time evolution of solitary waves is investigated. Therefore, before proceeding to the numerical analysis of MI, it is necessary to examine the impact of each dispersion term and their combinations on the emergence of stable solitons in the model under study. To investigate the effects of such parameters on soliton stability, we adopt the method developed in Ref. [76] and recently used by Shagalov [83].

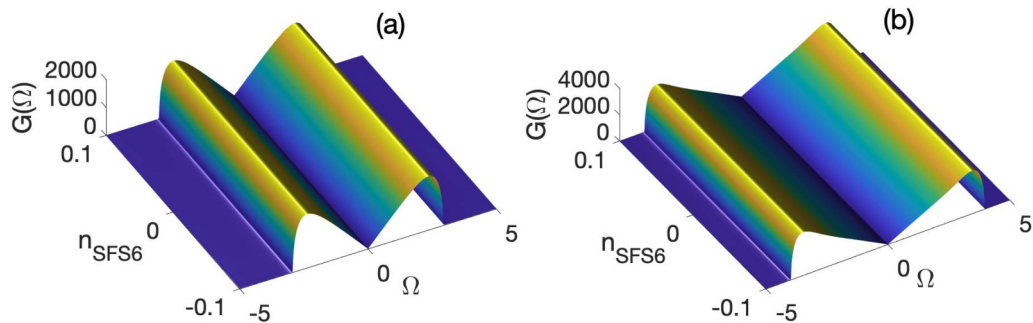


FIG. 5. Variation of maximum MI gain versus the perturbation frequency Ω and the septic self-frequency shift term n_{SFS6} and for (a) $k_6 = 0.02$, (b) $k_6 = -0.02$ with the other parameters being: $P_0 = 10$, $k_5 = 0.001\,23$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.037\,05$, $n_{vSS6} = -0.0247$, $n_{SFS} = -0.030\,875$, and $n_{SFS4} = 0.0004$.

A small-amplitude wave $u(Z, \tau) = u_0 e^{-ikZ + i\omega\tau}$ is adopted as a trial solution to Eq. (7), which leads to the dispersion relation,

$$k(\omega) = -k_2\omega^2 + k_3\omega^3 - k_4\omega^4 + k^5\omega^5 - k_6\omega^6. \quad (18)$$

To proceed, we focus on the zero-dispersion regime, which implies the relation $\frac{k(\omega)}{\omega} = 0$ whose real solutions imply a resonance interaction between the soliton and the small-amplitude wave leading to a stable soliton solution. This depends on a number of combinations between the dispersion coefficients k_2, k_3, k_4, k_5 , and k_6 . With no pretension of being exhaustive, we focus mainly on the following cases:

(i) For $k_4 = k_5 = k_6 = 0$, $k_2 \neq 0$, and $k_3 \neq 0$, Eq. (18) admits the solution $\omega_r = k_2/k_3$. The soliton may be unstable and its decay will take place with emission of waves with frequency ω_r under the condition $k_2 < 0$ and $k_3 > 0$ or $k_2 > 0$ and $k_3 < 0$. This particular case is commonly known and several numerical simulations have confirmed the results.

(ii) For $k_3 = k_5 = k_6 = 0$, $k_2 \neq 0$, and $k_4 \neq 0$, the dispersion relation will admit the solutions $\omega_r = \pm\sqrt{-k_2/k_4}$, which implies that the soliton will be stable if $k_2/k_4 > 0$ and unstable if $k_2/k_4 < 0$.

(iii) For $k_3 \neq 0$, $k_5 = k_6 = 0$, $k_2 \neq 0$, and $k_4 \neq 0$, two roots are obtained for the dispersion relation: $\omega = \frac{-k_3 \pm \sqrt{k_3^2 - 4k_2k_4}}{2k_4}$. Solutions will be two real roots for $k_3^2 - 4k_2k_4 > 0$, which leads to the condition $k_2k_4 > k_3^2/4$ for the soliton to be stable.

(iv) For $k_2 \neq 0$, $k_3 \neq 0$, $k_4 \neq 0$, $k_5 \neq 0$, and $k_6 = 0$, the dispersion relation reduces to a third-order polynomials and the stability of the soliton can be studied through the Routh-Hurwitz (RH) criterion. In doing so, the soliton will be stable so that MI takes place if the conditions $k_4/k_5 \leq 0$, $k_2/k_5 \leq 0$, and $k_4k_3 \geq k_2k_5$ are simultaneously satisfied.

(v) For $k_2 \neq 0$, $k_3 \neq 0$, $k_4 \neq 0$, $k_5 \neq 0$, and $k_6 \neq 0$, the RH stability criterion, from the obtained fourth-order polynomials, imposes $k_6 < 0$, $k_5 > 0$, $k_4 < 0$, $k_3 > 0$, $k_2 < 0$, $k_4k_5 > k_3k_6$, and $k_3k_4k_5 > k_3^2k_6 + k_2k_5^2$ for the soliton to be stable.

(vi) For $k_2 \neq 0$, $k_3 \neq 0$, $k_4 \neq 0$, $k_5 = 0$, and $k_6 \neq 0$, from the RH stability criterion, we get $k_6 < 0$, $k_4 < 0$, $k_3 > 0$, $k_2 < 0$, and $k_3k_6 > 0$ for the soliton to be stable and MI to take place.

(vii) For $k_2 \neq 0$, $k_3 = 0$, $k_4 = 0$, $k_5 = 0$, and $k_6 = 0$, the dispersion relation (18) has the soliton $\omega_r = \sqrt[3]{k_2/k_5}$ so that for $k_2/k_5 > 0$, the soliton will be stable and unstable otherwise. In this particular case, it is also obvious that the fifth-order nonlinearity stabilizes the soliton and gives more chances to the MI to take place.

The stability of solitons reveals more interesting and underlying phenomena that are mostly related to the appearance of wave trains via the activation of MI. This is true as it has long been proven that solitons and MI share the same parameter region. This gets more complicated when several parameters are involved and require more argument input to justify parameter values' choice and their contribution to the studied MI process. For instance, from the above analysis, we have realized that higher-order dispersion terms contribute to stabilizing the soliton and, therefore, play a vital role in the process of MI. However, the above stability analysis being

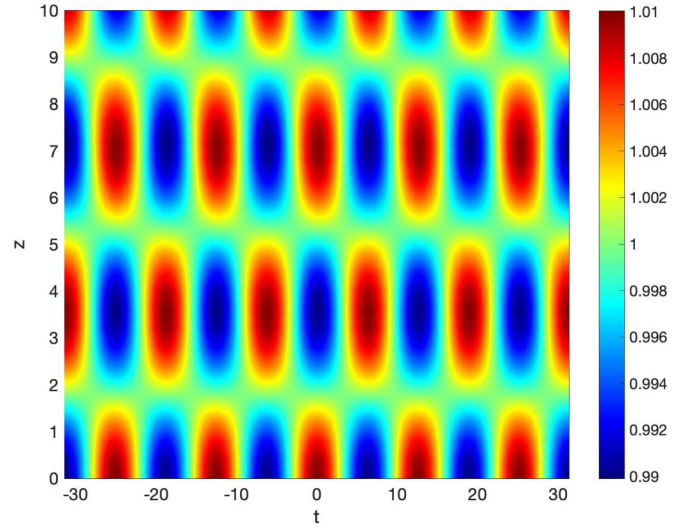


FIG. 6. The panels show the evolution of the cw under condition (vii) of the soliton stability analysis which predict MI to be violated if $k_2/k_5 < 0$ with $k_2 = -1$ and $k_5 = 0.00123$. The rest of the parameters are: $k_6 = 0$, $P_0 = 10$, $k_3 = 0$, $k_4 = 0$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$, and $n_{SFS6} = 0.021$.

local, addressing a more complete version of the study via a long-time numerical simulation of the predicted entities is necessary. This is for the simple reason that the model contains nonlinear and dispersive nonlinear terms beyond the dispersion terms, which may violate some of the above stability criteria and lead to new phenomena and behaviors in the long-time evolution of modulationally unstable wave patterns. This will somehow justify our choice of parameter values in the study of the MI gain and give more credit to the proposed model.

IV. NUMERICAL ANALYSIS OF UNSTABLE WAVE MODULATION

To check the accuracy of our linear stability analysis, direct numerical simulations have been performed on the model Eq. (7). Of course, a particular scenario of the MI development strongly depends on the type of initial conditions considered, namely, noisy perturbations of the plane wave, localized perturbations of the plane wave, and harmonic perturbations of the plane wave, respectively. In particular, a general solution of the equation governing the evolution of the spatiotemporal of MI perturbation in terms of a combination of cosine and sine hyperbolic functions has been obtained. In this paper, with the ambition to generate bright solitons via MI, the right perturbation to be adopted is a weak sine perturbation so that the initial condition be of the form [17]

$$u(0, t) = \sqrt{P_0} [1 + a_m \sin(\Omega_m t)] e^{i\phi_{NL}}, \quad (19)$$

where $a_m = 0.01$ is the modulational amplitude and $\Omega_m = 0.5$ is the frequency of a weak sinusoidal modulation frequency imposed on the cw wave. The value of Ω_m is within the bandwidth that is supposed to give rise to unstable waves. Numerical results are obtained from Eq. (7) through the

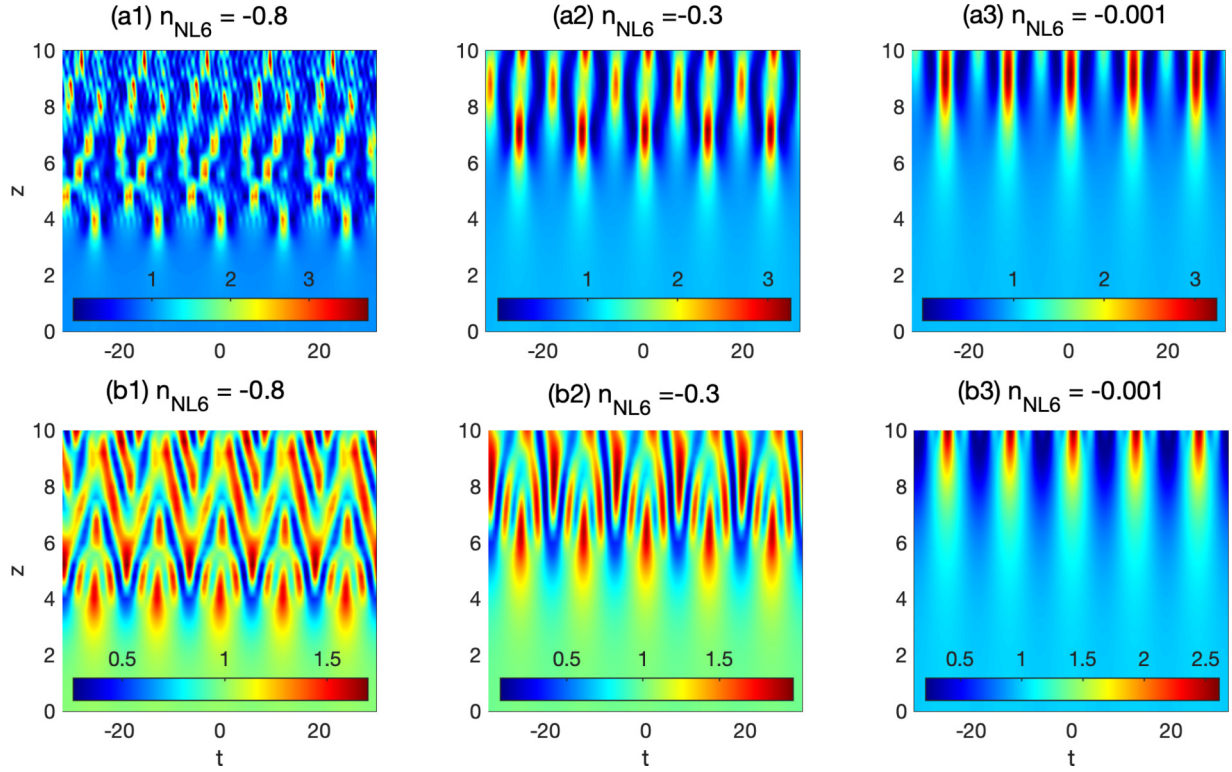


FIG. 7. The panels show the evolution of the cw under the effect of the septic nonlinearity term for (aj) $_{j=1-3}$ $k_6 = 0.0008$, (bj) $_{j=1-3}$ $k_6 = -0.0008$, and $P_0 = 10$, $k_5 = 0.001\,23$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.03705$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030875$, $n_{SFS4} = 0.0004$, and $n_{SFS6} = 0.021$.

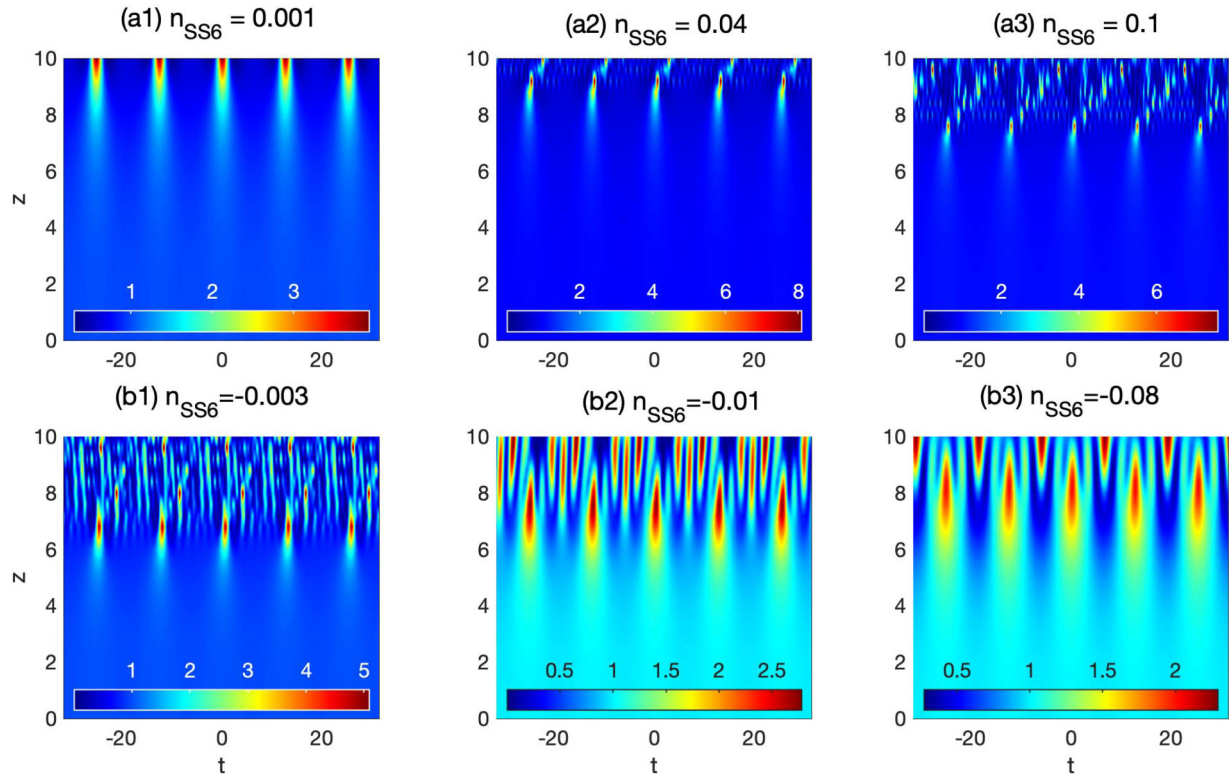


FIG. 8. The panels show the evolution of the cw under the effect of the septic self-steepening term for: (aj) $_{j=1-3}$ $k_6 = 0.0003$, (bj) $_{j=1-3}$ $k_6 = -0.0003$, and $P_0 = 10$, $k_5 = 0.001\,23$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{vSS} = -0.0247$, $n_{SS4} = 0.037\,05$, $n_{SFS} = -0.030\,875$, $n_{SFS4} = 0.0004$, and $n_{SFS6} = 0.021$.

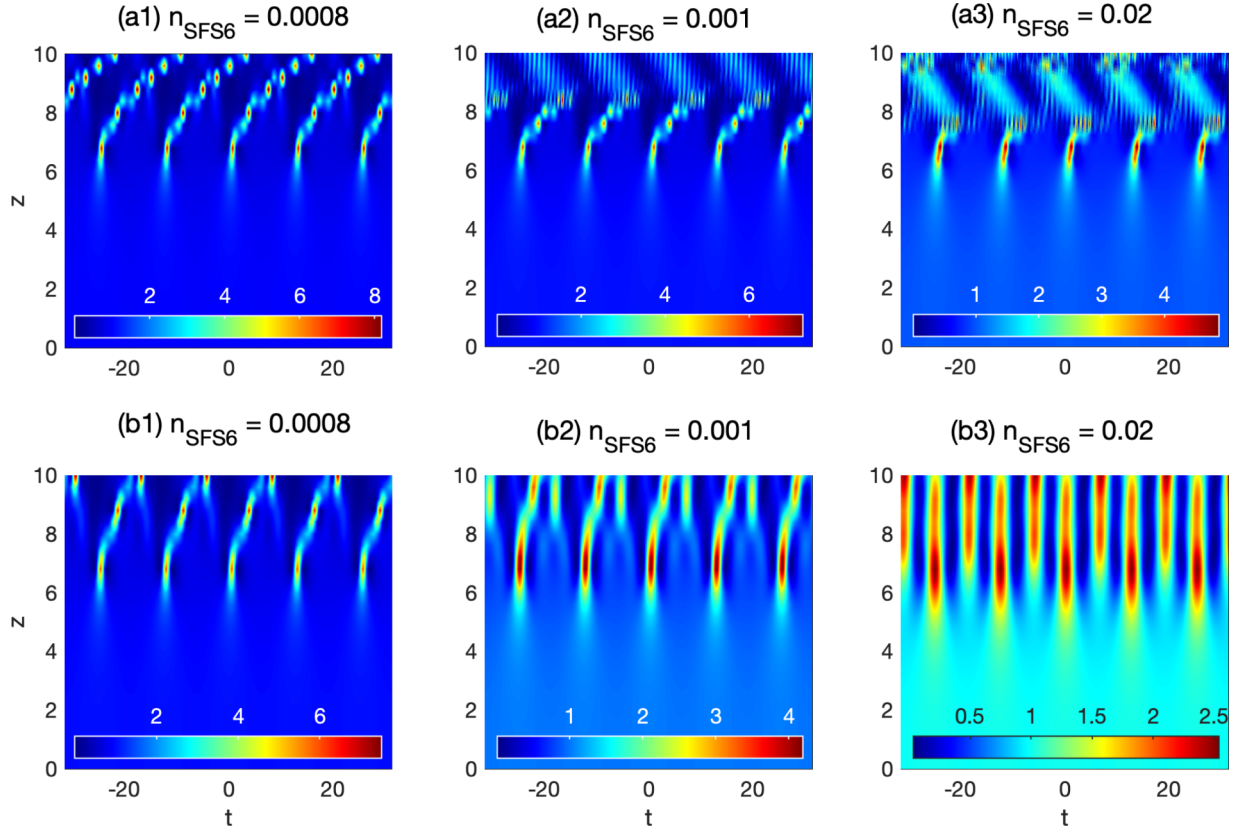


FIG. 9. The panels show the evolution of the cw under the effect of the septic self-frequency shift term for: (aj)_{j=1-3} $k_6 = 0.0008$, (bj)_{j=1-3} $k_6 = -0.0008$, and $P_0 = 10$, $k_5 = 0.001\,23$, $k_3 = 0.009$, $n_{NL} = 1$, $n_{NL4} = 1$, $n_{NL6} = 0.03$, $n_{SS} = -0.0247$, $n_{SS4} = 0.037\,05$, $n_{SS6} = -0.0247$, $n_{SFS} = -0.030\,875$, and $n_{SFS4} = 0.0004$.

split-step Fourier method with a particular interest in the emergence of MI as a result of the balanced competition among SOD and septic nonlinearity, self-steepening, and self-frequency shift effects. To start, it is of importance to confirm the stability of soliton performed in the previous section through one of the cases, (vii), for example. It is predicted that for $k_2/k_5 > 0$, the soliton solution will be stable, and MI will be likely to occur. Otherwise, the soliton solution will be unstable, taking away all chances for MI to emerge in the system. We deliberately violate the stability condition by imposing $k_2/k_5 < 0$, which leads to the patterns obtained in Fig. 6 where the cw breaks into wave patterns that have the shape of solitonic objects. This shows such conditions might be true, but they are balanced by the other parameters that may violate the soliton solution's stability condition. Otherwise, such conditions are not enough to predict instability, which justifies the results from different gain spectra where instability is possible negative and positive values of parameters. This explains why we insist on the competitive effect between higher-order nonlinearity and dispersion in the rest of the paper.

In Fig. 7, under normal GVD, results are presented for $k_6 > 0$ [see Fig. 7(aj)_{j=1-3}] and $k_6 < 0$ [see Fig. 7(bj)_{j=1-3}] with n_{NL6} changing in each of the cases. When $k_6 > 0$, the chosen values of parameters in gain spectra give rise to nonlinear patterns as predicted. Particularly, for $n_{NL6} = -0.8$, the instability takes place early, but increasing the nonlinear septic parameter reduces the recurrence of the patterns

which develop after a long distance. The same phenomenon is obvious when $k_6 < 0$, except that patterns appear to be exotic pulses whose frequency and recurrence change with the propagation distance. Remarkably, for higher values of n_{NL6} both for k_6 positive and negative, the perturbed cw breaks up into amplified pulses: The amplitude increases gradually during propagation. It is then ostensible that whereas controlling the occurrence of MI in the fiber through the septic nonlinearity, different scenarios, such as wave trains, erratic patterns of high or low amplitude can appear, depending on the competition among the various intrinsic nonlinearities and dispersions. Indeed, this could not be predicted via the linear stability analysis. Still, the obtained results reinforce the efficiency of the MI mechanism through which optical wave patterns of various forms and characteristics can be generated. This is more reinforced by the instability features obtained in Fig. 8, which picture the effect of the SS term on the long time evolution of the perturbed cw both for $k_6 > 0$ [see Fig. 8(aj)_{j=1-3}] and $k_6 < 0$ [see Fig. 8(bj)_{j=1-3}]. We should, however, stress here that given the spectrum of behaviors obtained from the gain of Fig. 4, the maximum gain is located in the region $n_{SS6} < 0$ so that one expects the cw to disintegrate likely for such value. However, considering the case $k_6 > 0$, the panels (aj)_{j=1-3} of Fig. 8 show that with increasing n_{SS6} , the MI patterns change from progressively amplified pulses to many erratic patterns with a tendency to shift on the right. On the other side, when we consider $k_6 < 0$, given that

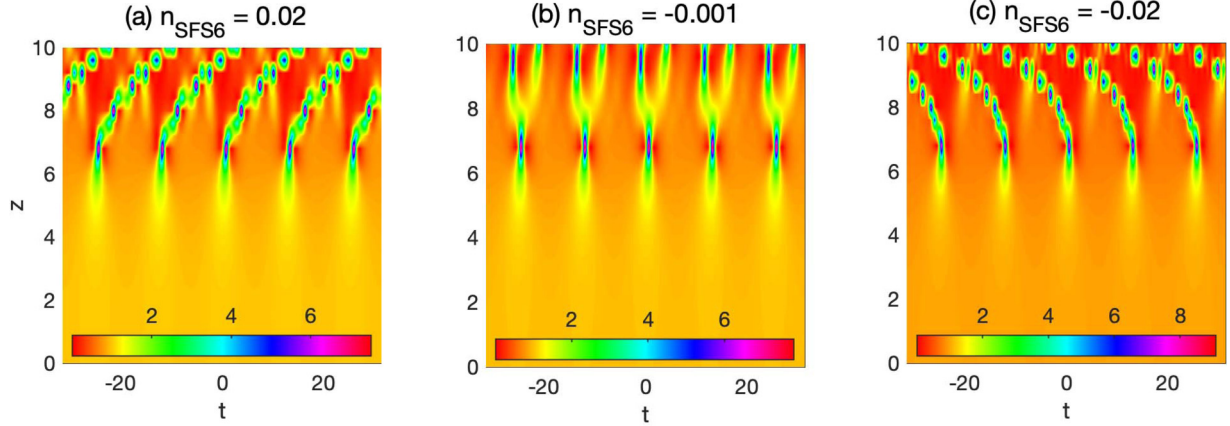


FIG. 10. Effect of the septic self-frequency shift term on the propagation direction of the disintegrated cw for (a) $n_{\text{SFS6}} = 0.02$, $k_6 = 0.0008$, (b) $n_{\text{SFS6}} = -0.001$, $k_6 = 0.0008$, (c) $n_{\text{SFS6}} = -0.02$, $k_6 = -0.0008$, and $P_0 = 10$, $k_5 = 0.001\,23$, $k_3 = 0.009$, $n_{\text{NL}} = 1$, $n_{\text{NL4}} = 1$, $n_{\text{NL6}} = 0.03$, $n_{\text{SS}} = -0.0247$, $n_{\text{SS4}} = 0.037\,05$, $n_{\text{SS6}} = -0.0247$, $n_{\text{SFS}} = -0.030\,875$, and $n_{\text{SFS4}} = 0.0004$.

the MI bandwidth is exclusively situated in the area where $n_{\text{SS6}} < 0$ is negative, decreasing its value produces the same effect except that for $n_{\text{SS6}} = -0.01$ and $n_{\text{SS6}} = -0.08$, the MI takes place in the form of alternating multihumped pulses. One fact that is, however, common in the theory of solitons is also obvious here, the space and time expansions of solitonic objects influences their characteristics, mainly the intensity for $n_{\text{SS6}} = -0.003$, the light patterns are smaller with high intensity compared to the rest of the cases.

Figure 9 shows the behaviors of the unstable cw when the SFS changes both for k_6 positive and negative. Under the condition $k_6 > 0$, one sees from Fig. 9(a1) that for $n_{\text{SFS6}} = 0.0008$, the long time evolution of MI is characterized by right-shifted patterns. With increasing the effect of the SFS, the shifted coherent optical patterns travel over a short distance and turn into erratic structures shifted toward the left [see Fig. 9(a2)]. This later feature gets pronounced when $n_{\text{SFS6}} = 0.02$ where there is a mix up of coherent optical patterns along with left-shifted erratic modulated structures as shown in Fig. 9(a3). Under the same conditions as in Fig. 9(aj)_{j=1-3} but with $k_6 < 0$, the optical patterns generated by MI are of low intensity [see Fig. 9(b1)], shifted to the right. However, the action of the increasing SFS effect reduces the number of wave trains, leaving patterns that are extended over the distance [see Fig. 9(b2)], which, for $n_{\text{SFS6}} = 0.02$ turn out to be trains of alternating and oscillating pulses [see Fig. 9(b3)]. Indubitably, the direction of the propagating solitons is importantly affected by the septic SFS term, and such a direction can change when parameters are well balanced with appropriate signs. In the numerical experiment shown in Fig. 10, one clearly confirms that when n_{SFS6} and k_6 are positive, the solitonic patterns are shifted to the right [see Fig. 10(a)], which is corrected for $n_{\text{SFS6}} = -0.001$ as seen in Fig. 10(b). Nevertheless, it is found that the direction can shift to the left for $n_{\text{SFS6}} = -0.02$ and $k_6 < 0$. As said earlier, model Eq. (7) contains many parameters that render it difficult to control the dynamics of any emerging pattern. In the meantime, we should stress that studies have been carried out on earlier models in the absence of the higher-order dispersive and nonlinear effects, such as the septic nonlinearity, the septic self-steepening, and the self-phase shift terms.

V. CONCLUSION

In this paper, we have successfully derived a generalized higher-order NLS equation with cubic, quintic, and septic nonlinearity with self-steepening and self-frequency shift terms of the septic order. Under the linear stability analysis of a cw solution, an analytical expression for the MI gain has been derived, and its sensitivity to the competitive effects among the SOD, the septic nonlinearity, the self-steepening, and self-frequency shift terms have been addressed. Different combinations of the dispersion coefficients have been explored that give rise to soliton stability and the emergence of MI. It was revealed that the instability condition for the soliton could be violated due to the presence of higher-order nonlinear terms. This has been confirmed via direct numerical simulations where under various dispersive and nonlinear effects, the cw has been found to disintegrate into trains of structures with solitonic features. Interestingly, during numerical experiments, we have discovered that the propagation direction of the trains of waves can be controlled by a suitable balance between the SOD and the septic self-frequency shift strength. This may be a valuable tool with undeniable and straightforward applications in nonlinear optic communications.

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APPENDIX: PARAMETERS FOR EQ. (6)

Parameters for Eq. (6) are defined as follows:

$$\beta_m = \frac{1}{m!} \left(\frac{\partial^m \beta}{\partial \omega^m} \right)_0, \quad (m = 2-6), \quad \gamma_1 = \left(\frac{\partial \beta}{\partial |\psi|^2} \right)_0 + \frac{k_0 n_2}{4A_{\text{eff}}},$$

$$\gamma_2 = \frac{1}{2} \left(\frac{\partial \beta}{\partial |\psi|^4} \right)_0 + \frac{k_0}{16A_{\text{eff}}} \left(\frac{n_2^2}{2} + n_4 \right),$$

$$\gamma_3 = \frac{1}{6} \left(\frac{\partial \beta}{\partial |\psi|^6} \right)_0 + \frac{k_0}{64A_{\text{eff}2}} \left(n_6 + \frac{n_4 n_2}{n(\omega)} \right), \quad (\text{A1})$$

$$\alpha_1 = \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^2)} \right)_0, \quad \alpha_3 = \frac{1}{2} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^4)} \right)_0,$$

$$\alpha_5 = \frac{1}{6} \left(\frac{\partial^2 \beta}{\partial \omega \partial (|\psi|^6)} \right)_0, \quad \alpha_2 = \left(\frac{\partial \beta}{\partial (|\psi|^2)} \right)_0,$$

$$\alpha_4 = \frac{1}{2} \left(\frac{\partial \beta}{\partial (|\psi|^4)} \right)_0, \quad \alpha_6 = \frac{1}{6} \left(\frac{\partial \beta}{\partial (|\psi|^6)} \right)_0,$$

where $A_{\text{eff}} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^4 dx dy}$, $A_{\text{eff}1} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^6 dx dy}$, $A_{\text{eff}2} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^8 dx dy}$ with A_{eff} as the refractive core area of the fiber. $A_{\text{eff}1} = (3/4)A_{\text{eff}}$ and $A_{\text{eff}2} = (1/2)A_{\text{eff}}$.

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