

Magnetic field effect on a fractionalized blood flow model in the presence of magnetic particles and thermal radiations

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ARTICLE INFO

Article history:

Received 5 September 2019

Revised 9 November 2019

Accepted 18 November 2019

Available online 28 November 2019

Keywords:

Magnetohydrodynamics

Fractional derivatives

Magnetic field

Magnetic particles

ABSTRACT

The presence of magnetic particles is considered in a magneto-hydrodynamic blood flow through a circular cylinder. The fluid inside the tube is acted by an oscillating pressure gradient and an external constant magnetic field. The blood temperature is assumed to change with the blood and particle velocities, and the whole study is based on a mathematical model that includes Caputo fractional-order derivatives. Solutions for the particle and blood velocities, and blood temperature distribution, are obtained via the combination of the Laplace and Hankel transformation methods. Effects of the fractional-order parameter and magnetic field are addressed using numerical simulations. Results show that the applied magnetic field reduces the velocities of the fluid and particles, which remarkably affects the blood temperature. This is obvious for short and long time intervals. However, under long time intervals, particles seem to be accelerated, but their velocity is suitably controlled by the fractional-order parameter which also monitors the increase in blood temperature.

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1. Introduction

Biomagnetic fluid dynamics (BFD) is a new branch of science which is devoted to the study of fluids that flow under the influence of an external magnetic field. Recent advances in bioengineering and medical sciences have brought forth the importance of Biomagnetic fluids, with specific applications in targeted drug delivery [1], conception and development of magnetic devices for cell separation [2], attenuation of bleeding during surgeries and treatment of cancer tumors through magnetic hyperthermia [3]. Indeed, blood is a biomagnetic fluid, due to the strong presence of the erythrocytes playing the role of magnetic particles and the plasma as liquid carrier. Erythrocytes, or red blood cells (RBCs), are in fact negative charge carriers and this creates a magnetic field on the vessel wall, capable of influencing the pulsatile character of the blood flowing inside the vessel [4,5]. Moreover, the magnetization of blood can be augmented by adding artificially created nanoparticles to the flow as usually done in targeted drug delivery. More precisely, this implies that biofluids can behave like ferromagnetic fluids, reason why most of the magnetic particles that are frequently used in bio-medicine are manganese ferrite (CoFe_2O_4), iron oxides (Fe_2O_3 or Fe_3O_4), cobalt ferrite (CoFe_2O_4) [6], just to cite a few. One of the first models to address such aspect of BFD was proposed by Haik et al. [7], where they made a comparative study between mathematical models for BFD and ferrohydrodynamics (FHD). However, they did not consider the induced magnetic field, but rather assumed that the flow was under the effect of an applied magnetic field. This otherwise implies that biofluids should be considered electrically poor conductors, contrarily to what is reported in Refs. [8,9] and which supports that blood exhibits significantly high electrical conductivity, especially when additional magnetic particles are injected in blood to carry drugs. This may justify the use of the principles of magnetohydrodynamics (MHD) [10,11], which rely on the generation of Lorentz forces, in elaborating suitable mathematical models. Sharma et al. [12] studied for example the effect of external uniform magnetic field on flow parameters of both blood and magnetic particles based on a mathematical model using the MHD approach. Magnetic Drug Targeting (MDT) technique was used by Bose and Banerjee [13], where FHD and MHD principles were coupled to track the magnetic particles under the effect of a magnetic field. Mondal and Shit [6] studied the motion of magnetic nano-particles in a biofluid, in the presence of externally applied magnetic field, with strong applications to drug delivery for some therapeutic procedures in the treatment of tumor

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cells, infections and removing blood clots. Kefayati [14] studied the effect of a magnetic field on non-Newtonian blood flow between two-square concentric duct annuli, and further extended the study to non-Newtonian blood flow in a cavity driven by the motion of two facing lids [15]. In both cases, the results showed that the increment of Reynolds number augments the magnetic field effect on the flow of blood.

In most of the above contributions, biofluid-related problems were solved using classical models, i.e., with integer derivatives. Most of physical problems nowadays have found suitable formulation using the idea of fractional derivative, with applications in dynamical systems [16,17], neural systems [18,19], biophysics [20], mathematical biology [21], viscoelasticity [22,23] and fluid dynamics [24,25]. In that respect, Nehad et al. [26] and Ali et al. [27] studied blood flow with magnetic particles and magnetic field effect using the Caputo fractional derivatives concept in the fractal theory, starting from the classical model [28–32]. Generalized second grade fluids with Caputo-Fabrizio derivative were studied by Sheikh et al. [33], using the MHD formulation. Heat and mass transfer of differential-type fluids, with fractional Caputo derivatives were analyzed by Imran et al. [34]. Bansi et al. [5,35] proposed a fractionalized model to study the effect of heat transfer and magnetic field on the blood flowing inside oscillatory arteries and periodic pressure gradient. More recently, the effect of magnetic fields on blood flow was studied by Ali et al. [36] using the Caputo-Fabrizio derivative, with blood considered as a Casson fluid, with magnetic particles, flowing in the cylindrical domain.

The main objective of the present work is to study a mathematical model of blood flow in presence of magnetic particles and blood temperature, using Caputo fractional derivatives [37,38]. External magnetic effects are considered and the blood in the vessel is assumed to be acted by a periodic pressure gradient. The mathematical model, which includes Caputo fractional derivatives, is generalized from the model containing integer derivatives, and solutions for the blood velocity, particle velocity and blood temperature are obtained by conjointly making use of the finite Hankel and Laplace transforms. Since Laplace transformed expressions of velocities and temperature are functions of a modified Bessel function, the inverse Laplace transform is difficult to find traditionally. This is therefore done numerically, where we address the effect of some flow parameters such as the Reynolds and Hartmann numbers on the joint dynamics of blood and particles, along with the change in blood temperature, when the fractional-order parameter is varied.

2. Model equation and general analytical solution

2.1. Mathematical modeling of the problem

In the present model, blood, which is considered as a non-Newtonian fluid, is assumed to flow in a cylindrical vessel of radius R_0 , together with magnetic particles (see Fig. 1). This makes a two-phase mixture, with the solid phase consisting of magnetic particles and the liquid phase being the blood. We consider the magnetic particles to be uniformly distributed in the blood, while the ensemble flows in the axial direction x of the vessel. The magnetic field acts perpendicular to the vessel and the induced magnetic field is neglected because of the smallness of the magnetic Reynold's number [39]. The particles, the tube and the blood inside are supposed to be at rest at $t = 0$. Particles are subjected to an electromotive force resulting from the interaction of current with the magnetic field. Its expression can be found in Refs. [6,12,36] in the form

$$\vec{f}_{em} = -\sigma B_0^2 U(r, t) \vec{i}, \quad (1)$$

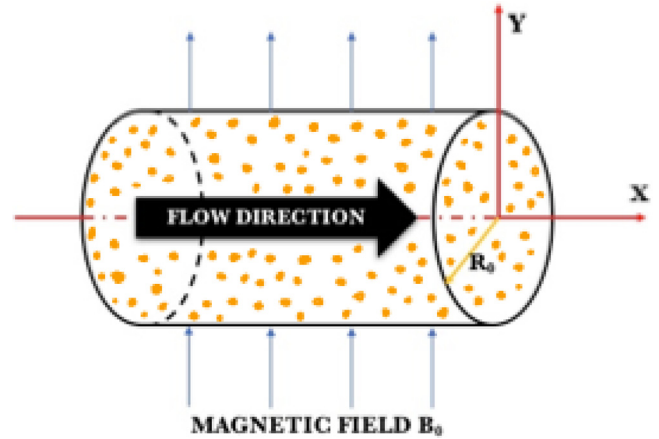


Fig. 1. Schematic representation of the model geometry with orange spots being the magnetic particles.

with $\vec{i}$ being the unit vector in the x -direction and $U(r, t)$, the axial velocity of the blood. B_0 is a uniform magnetic field and σ the electrical conductivity. The equation describing the motion of magnetic particles is governed by the Newton's second law and is given in the form [6,12,36]

$$m \frac{\partial v}{\partial t} = K(U - V), \quad (2)$$

where K is the Stokes constant, m is the average mass of the magnetic particles and V is the velocity of the particles.

The unsteady pulsatile flow of blood in an axisymmetric cylindrical blood vessel of radius R_0 , considered to be affected by a uniform transverse magnetic field B_0 , is governed by the following momentum equation in polar coordinates [5,35]

$$\rho \frac{\partial U}{\partial t} = -\frac{\partial P}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \mu \frac{\partial U}{\partial r} \right) + KN(V - U) - \sigma B_0^2 U, \quad (3)$$

where P is the blood pressure. μ , σ and ρ are the dynamic coefficient of viscosity, the electrical conductivity and the density of blood, respectively. N is the number of magnetic particles per unit volume. The first three terms on the right-hand side of Eq. (3) represent the pressure gradient, the viscosity and the force generated by the relative motion between the fluid and the particle. The last term stands for the Lorentz force brought by the magnetic field. The oscillating pressure gradient $\frac{\partial P}{\partial x}$ is given by

$$-\frac{\partial P}{\partial x} = b_0 + b_1 \cos(\omega t), \quad t > 0 \quad (4)$$

where b_0 and b_1 are the amplitude of the systolic and diastolic pressure gradient, respectively [12,35,40], with ω being the frequency. Eq. (3) can also be rewritten, by introducing the kinematic viscosity $\hat{\mu} = \mu/\rho$, as

$$\frac{\partial U}{\partial t} = \frac{1}{\rho} (b_0 + b_1 \cos(\omega t)) + \frac{1}{r} \frac{\partial}{\partial r} \left(r \hat{\mu} \frac{\partial U}{\partial r} \right) + \frac{KN}{\rho} (V - U) - \frac{\sigma B_0^2 U}{\rho}. \quad (5)$$

The energy equation, related to thermal radiation is given by [4,5]

$$\rho C_p \frac{\partial T}{\partial t} = \kappa \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) - \frac{\partial q_r}{\partial r} + \hat{\mu} \left(\frac{\partial U}{\partial r} \right)^2, \quad t > 0, \quad r \in [0, R_0], \quad (6)$$

where the term $\hat{\mu} \left(\frac{\partial U}{\partial r} \right)^2$ in (6) represents the viscous dissipation due to the assumption of the unidirectional flow. C_p and κ represent the specific heat at constant pressure and thermal conductivity. The radiative flux q_r is written assuming that blood is an

optically thin fluid with a relatively low density and heat absorption coefficient. q_r can then be simplified as proposed by Cogley et al. [41] in the form

$$\begin{aligned}\frac{\partial q_r}{\partial r} &= 4\alpha^2(T - T_\infty), \\ \frac{\partial U}{\partial x} &= 0, \quad T = T_\infty \quad \text{on} \quad r = 0 \\ U &= 0, \quad T = T_w \quad \text{on} \quad r = R_0,\end{aligned}\quad (7)$$

where $\alpha^2 = \int_0^\infty \phi \chi \frac{\partial B}{\partial T}$ is the linear Planck mean absorption coefficient, with B and ϕ being respectively the Planck's constant and radiation absorption coefficient, and χ being the frequency. The above formulated models equations correspond to the initial and boundary conditions

$$\begin{aligned}U(r, 0) &= 0, \quad V(r, 0) = 0, \quad T(r, 0) = T_\infty, \quad r \in [0, R_0]; \\ U(R_0, t) &= 0, \quad V(R_0, t) = 0, \quad T(R_0, t) = T_w, \quad t > 0.\end{aligned}\quad (8)$$

The set of model Eqs. (2), (3) and (6) can be generalized to their time-fractional versions by multiplying each of them by $\lambda = \sqrt{\frac{R_0 \rho}{b_0}}$. This leads to the fractional-order equations

$$\begin{aligned}\lambda^a D_t^a U &= \frac{\lambda}{\rho} [b_0 + b_1 \cos(\omega t)] + \lambda \hat{\mu} \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} \right) \\ &\quad - \frac{\lambda KN}{\rho} (V - U) - \frac{\lambda \sigma B_0^2}{\rho} U,\end{aligned}\quad (9a)$$

$$\lambda^a D_t^a V = \frac{\lambda K}{m} (U - V), \quad (9b)$$

$$-\lambda^a D_t^a T = -\frac{\kappa}{\rho C_p} \left(\frac{\partial^2 T}{\partial r^2} - \frac{1}{r} \frac{\partial T}{\partial r} \right) + \frac{4\alpha^2}{\rho C_p} (T - T_\infty) + \frac{\mu}{\rho C_p} \left(\frac{\partial U}{\partial r} \right)^2, \quad (9c)$$

where

$$D_t^a f(r, t) = \frac{1}{\Gamma(1-a)} \int_0^t \frac{1}{(t-\tau)^a} \frac{\partial f(r, \tau)}{\partial \tau} d\tau \quad (10)$$

is the Caputo fractional derivative of order a [18,19,37,38]. In the meantime, recent works have developed new directions in fractional derivative operators and using Mittag-Leffler functions. For example, a new definition of fractional derivative, with a smooth kernel which considers two different representations for the temporal and spatial variable, was proposed in Ref. [42]. Operators including the Atangana-Baleanu (AB) derivative and the Caputo-Fabrizio derivative, respectively related to the Mittag-Leffler law and the exponential law were applied to the non-linear Kaup-Kupersmidt equation [43]. Making use of the traditional Caputo derivative and Caputo-Fabrizio derivative with fractional order and no singular kernel, the nonlinear Kaup-Kupersmidt was extended to the span of fractional calculus by Atangana and Doungmo [44]. Moreover, further relationship and new results of Atangana-Baleanu derivative, with some integral transform operators, were obtained from a simple nonlinear system [45]. Another important feature of the non-local and non-singular kernel operator was pointed out and applied to chaotic models whose the bifurcation, period doubling dynamics and chaotic behaviors were discussed [46,47]. Additionally, the AB fractional derivative with non-singular kernel and non-local kernel was applied to chaotic processes with two-parameter derivative, including non-singular and non-local kernel based on the one-parameter Mittag-Leffler function [46,47]. To further proceed, the following dimensionless variables can be introduced

$$\begin{aligned}r^* &= \frac{r}{R_0}, \quad t^* = \frac{t}{\lambda}, \quad U^* = \frac{U}{U_0}, \quad V^* = \frac{V}{U_0}, \quad b_0^* = \frac{\lambda b_0}{\rho U_0}, \quad b_1^* = \frac{\lambda b_1}{\rho U_0}, \\ \omega^* &= \lambda \omega, \quad T^* = \frac{T - T_\infty}{T_w - T_\infty}.\end{aligned}\quad (11)$$

Considering all the above and dropping the (*) notation, we get the following fractionalized set of equations

$$D_t^a U = [b_0 + b_1 \cos(\omega t)] + \frac{1}{Re} \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} \right) - R(V - U) - \mathcal{H}^2 U, \quad (12a)$$

$$GD_t^a V = U - V, \quad (12b)$$

$$-D_t^a T + \frac{\kappa}{\rho C_p} \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) - \frac{4\alpha^2}{\rho C_p} T = -\frac{\hat{\mu}}{C_p} \left(\frac{\partial U}{\partial r} \right)^2, \quad (12c)$$

where $Re = \frac{R_0^2}{\lambda \mu}$ is a Reynold number, $R = \frac{KN\lambda}{\rho}$ is a particle concentration parameter, $\mathcal{H} = B_0 R_0 \sqrt{\frac{\sigma}{\mu}}$ is a magnetic parameter or Hartmann number, $G = \frac{m}{K\lambda}$ is the particle mass parameter. The initial and boundary conditions then become

$$\begin{aligned}U(r, 0) &= 0, \quad V(r, 0) = 0, \quad T(r, 0) = 0, \quad r \in [0, 1], \\ U(1, t) &= 0, \quad V(1, t) = 0, \quad T(1, t) = 0, \quad t > 0.\end{aligned}\quad (13)$$

3. Solution of the mathematical model

To solve the mathematical model (12), we make use of the Laplace transform with respect to time and the Hankel transform with respect to the radial variable. The temporal transformation then leads to the equations

$$\begin{aligned}s^a \bar{U}(r, s) &= \frac{1}{Re} \left[\frac{\partial^2 \bar{U}(r, s)}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{U}(r, s)}{\partial r} \right] + \frac{b_0}{s} + \frac{b_1 s}{s^2 + \omega^2} + R \bar{V}(y, s) \\ &\quad - (R + \mathcal{H}^2) \bar{U}(r, s),\end{aligned}\quad (14a)$$

$$\bar{V}(r, s) = \frac{\bar{U}(r, s)}{Gs^a + 1} \quad (14b)$$

$$\begin{aligned}s^a \bar{T}(r, s) &= \frac{\kappa}{\rho C_p} \left[\frac{\partial^2 \bar{T}(r, s)}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{T}(r, s)}{\partial r} \right] - \frac{4\alpha^2}{\rho C_p} \bar{T}(r, s) \\ &\quad + \frac{\hat{\mu}}{C_p} \left(\frac{\partial \bar{U}(r, s)}{\partial r} \right)^2 - \frac{\hat{\mu}}{C_p} \lim_{t \rightarrow 0} \left\{ \int_0^t \left(\frac{\partial U(r, \tau)}{\partial r} \right)^2 d\tau \right\},\end{aligned}\quad (14c)$$

with $\bar{U}(1, s) = 0$, $\bar{V}(1, s) = 0$, and $\bar{T}(1, s) = 0$. In the rest of study, it is considered that $\lim_{t \rightarrow 0} \left\{ \int_0^t \left(\frac{\partial U(r, \tau)}{\partial r} \right)^2 d\tau \right\} \rightarrow 0$. Applying the Hankel transform [48,49] of zero order to Eqs. (14a) and (14c), and using the boundary conditions given in (13), we obtain

$$\begin{aligned}-\frac{r_n^2 \bar{U}_H(y_n, s)}{Re} &= \left[\frac{Gs^{2a} + [1 + G(R + \mathcal{H}^2)]s^a + \mathcal{H}^2}{Gs^a + 1} \right] \bar{U}_H(r_n, s) \\ &\quad - \left(\frac{b_0}{s} + \frac{b_1 s}{s^2 + \omega^2} \right) \frac{J_1(r_n)}{r_n},\end{aligned}\quad (15)$$

where $\bar{U}_H(r_n, s) = \int_0^1 r \bar{U}(r, s) J_0(r r_n) dy$ is the finite Hankel transform of function $\bar{U}(r, s)$ and r_n ($n = 1, 2, \dots$) are positive roots of the equation $J_0(\chi) = 0$, J_0 being the Bessel function of the first kind of order zero. The inverse Laplace transform for Eq. (15) of fluid can be written in the equivalent form

$$\begin{aligned}\bar{U}_H(r_n, s) &= \frac{b_0 Re}{c_n} \left[(1 - Ga_{1n}) \frac{s^{-1}}{s^a + a_{1n}} - (1 - Ga_{2n}) \frac{s^{-1}}{s^a + a_{2n}} \right] \frac{J_1(r_n)}{r_n} \\ &\quad - \frac{b_1 Re}{c_n} \frac{s}{s^2 + \omega^2} \left[(1 - Ga_{1n}) \frac{s^{-1}}{s^a + a_{1n}} \right. \\ &\quad \left. - (1 - Ga_{2n}) \frac{s^{-1}}{s^a + a_{2n}} \right] \frac{J_1(r_n)}{r_n},\end{aligned}\quad (16)$$

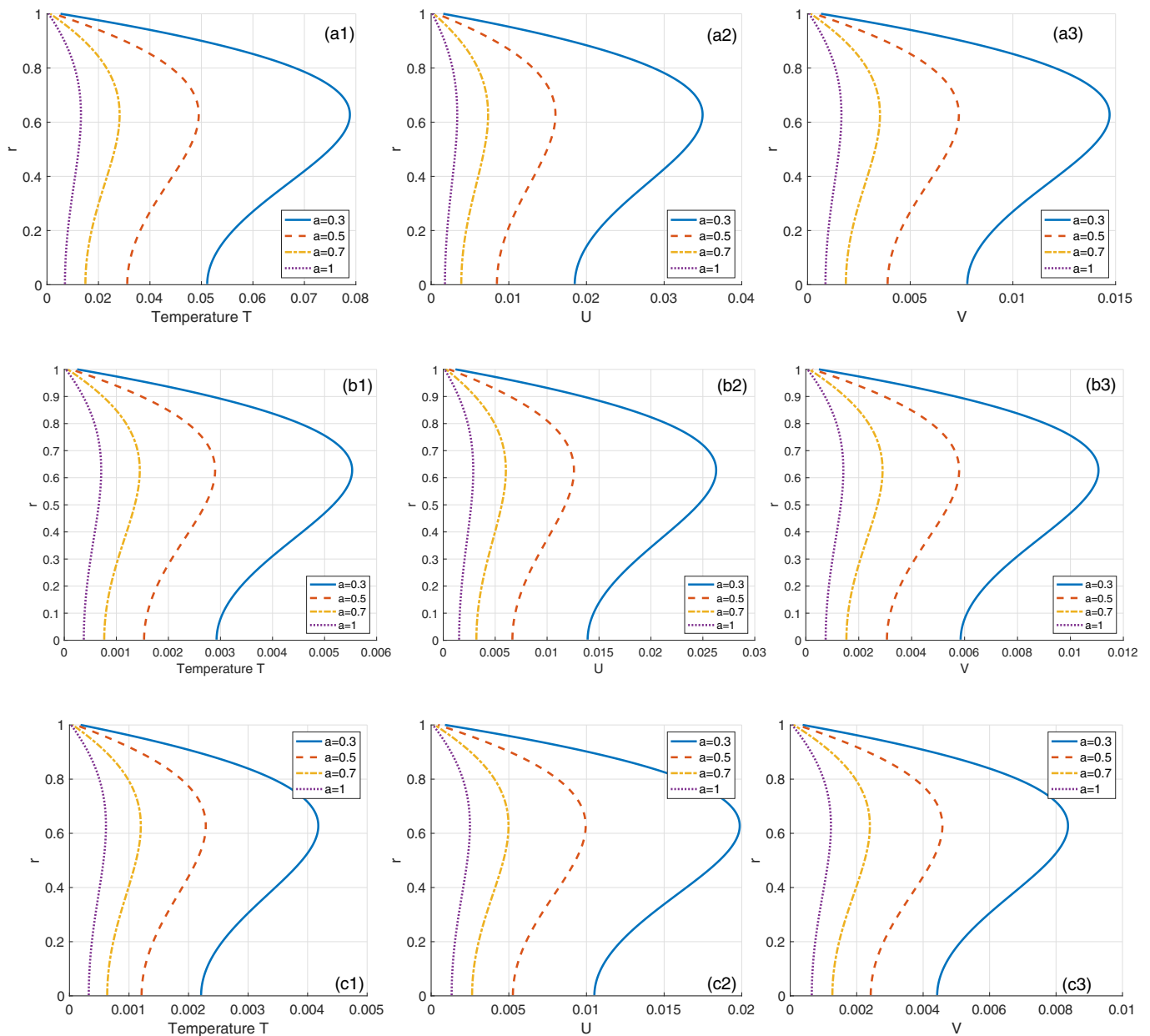


Fig. 2. Profiles of blood temperature $T(r, t)$, axial velocity $u(r, t)$, and particle velocity $v(r, t)$ for small values of time t , with $S = 0.01$, $A_0 = 10^{-7}$, $A_1 = 10^{-7}$, $\omega = \pi/4$, $\beta = 0.5$, $\lambda = 0.5$, $G = 1.5$, $R = 0.2$, $Re = 3$ and $\mathcal{H} = 1$. Panels (aj)_{j=1,2,3} correspond to $t = 0.05$, panels (bj)_{j=1,2,3} give results for $t = 0.2$ and panels (cj)_{j=1,2,3} correspond to $t = 0.3$. In all the panels, the fractional parameter takes the values $a = 0.3, 0.5, 0.7$ and 1 .

where

$$a_{1n} = \frac{b_n - c_n}{2ReG}, \quad a_{2n} = \frac{b_n + c_n}{2ReG}, \quad b_n = Re[1 + G(R + \mathcal{H}^2)] + Gr_n^2, \quad c_n = \sqrt{b_n^2 - 4ReG(Re\mathcal{H}^2 + r_n^2)}. \quad (17)$$

Inserting Eq. (16) into Eq. (14c) and applying the Hankel transform leads to

$$\bar{T}_H(r_n, s) = \frac{2\hat{\mu}Re\left(\frac{A_0}{s} + \frac{A_1s}{s^2 + w^2}\right)}{c_p\left(\beta r_n^2 + s + \frac{4\alpha^2}{c_p}\right)} \frac{\partial}{\partial r_n} \left[\left(\frac{1 - a_{1n}G}{s^a + a_{1n}} - \frac{1 - a_{2n}G}{s^a + a_{2n}} \right) \frac{J_1(r_n)}{r_n c_n} \right]. \quad (18)$$

The Bessel expansion form, given by

$$J_{n+1}(x) = \frac{nJ_n(x)}{x} - J'_n(x) \quad \text{and} \quad J_{n+1}(x) - J_{n-1}(x) = -2J'_n(x), \quad (19)$$

and the approximation

$$\frac{\partial}{\partial r_n} \left(\frac{J_1(r_n)}{r_n c_n} \right) = -\frac{J_2(r_n)}{r_n c_n} - \frac{c'_n}{c_n^2} (J_2(r_n) + J'_1(r_n)),$$

$$\frac{\partial}{\partial r_n} \left(\frac{1 - a_{1n}G}{s^a + a_{1n}} - \frac{1 - a_{2n}G}{s^a + a_{2n}} \right) = \frac{-a'_{1n}G(s^a + a_{1n}) - a'_{1n}(1 - a_{1n}G)}{(s^a + a_{1n})^2} + \frac{a'_{2n}G(s^a + a_{2n}) + a'_{2n}(1 - a_{2n}G)}{(s^a + a_{2n})^2} \quad (20)$$

can be employed to expand Eq. (18) so that

$$\begin{aligned} \bar{T}_H(r_n, s) = & \frac{\gamma b_0 c'_n}{c_n^2} \frac{J_1(r_n)}{r_n} \left[\frac{s^{-1}(1 - a_{2n}G)}{(s^a + l_{2n})(s^a + l_{1n})} - \frac{s^{-1}(1 - a_{1n}G)}{(s^a + k_{2n})(s^a + k_{1n})} \right] \\ & + \frac{s\gamma b_1 c'_n}{(s^2 + w^2)c_n^2} \frac{J_1(r_n)}{r_n} \left[\frac{(1 - a_{2n}G)}{(s^a + l_{2n})(s^a + l_{1n})} \right. \\ & - \left. \frac{(1 - a_{1n}G)}{(s^a + k_{1n})(s^a + k_{2n})} \right] + \frac{\gamma b_0 J_2}{c_n} \frac{J_2}{r_n} \left[\frac{(1 - a_{2n}G)s^{-1}}{(s^a + l_{2n})(s^a + l_{1n})} \right. \\ & - \left. \frac{(1 - a_{1n}G)s^{-1}}{(s^a + k_{2n})(s^a + k_{1n})} \right] + \frac{\gamma b_1 s}{c_n(s^2 + w^2)} \end{aligned}$$

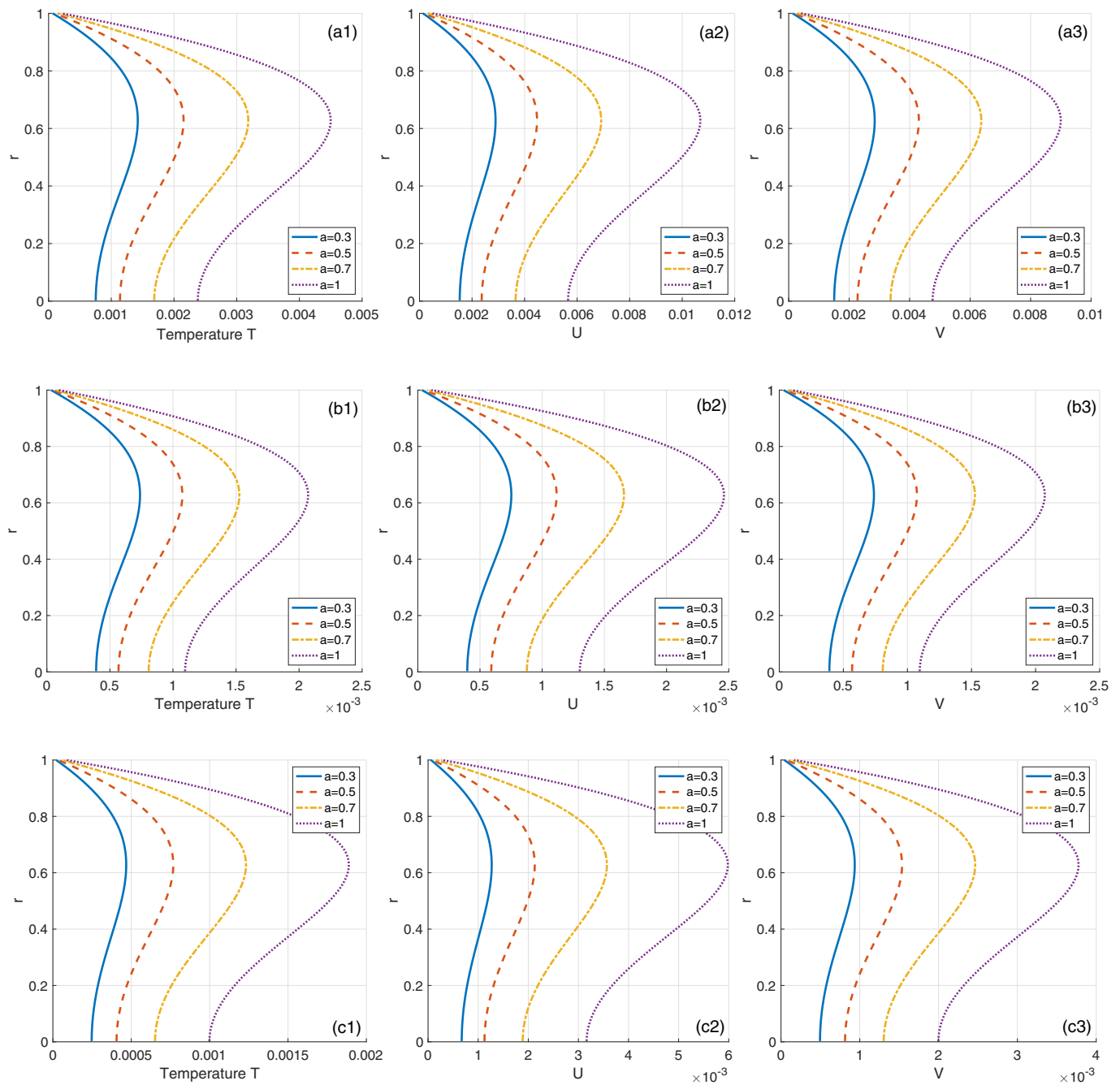


Fig. 3. Profiles of blood temperature $T(r, t)$, axial velocity $u(r, t)$, and particle velocity $v(r, t)$ for large values of time t , with $S = 0.01$, $A_0 = 10^{-7}$, $A_1 = 10^{-7}$, $\omega = \pi/4$, $\beta = 0.5$, $\lambda = 0.5$, $G = 1.5$, $R = 0.2$, $Re = 3$ and $\mathcal{H} = 1$. Panels (aj)_{j=1,2,3} correspond to $t = 1$, panels (bj)_{j=1,2,3} give results for $t = 3$ and panels (cj)_{j=1,2,3} correspond to $t = 5$. In all the panels, the fractional parameter takes the values $a = 0.3, 0.5, 0.7$ and 1 .

$$\times \frac{J_2}{r_n} \left[\frac{(1 - a_{2n}G)}{(s^q + l_{2n})(s^q + l_{1n})} - \frac{(1 - a_{1n}G)}{(s^q + k_{2n})(s^q + k_{1n})} \right], \quad (21a)$$

$$\begin{aligned} \tilde{T}_{H_2}(r_n, s) = & \frac{\gamma b_0 s^{-1}}{c_n} \frac{J_1(r_n)}{r_n} \left[\frac{(-a'_{1n}G)}{(s^q + k_{2n})(s^q + k_{1n})} - \frac{a'_{1n}(1 - a_{1n}G)}{(s^q + a_{1n})(s^q + k_{2n})(s^q + k_{1n})} \right] \\ & + \frac{a_{2n}G}{(s^q + l_{2n})(s^q + l_{1n})} + \frac{a'_{2n}(1 - a_{2n}G)}{(s^q + a_{2n})(s^q + l_{2n})(s^q + l_{1n})} \\ & + \frac{\gamma b_1 s J_1(r_n)}{c_n (s^2 + w^2) r_n} \left[\frac{(-a'_{1n}G)}{(s^q + k_{2n})(s^q + k_{1n})} - \frac{a'_{1n}(1 - a_{1n}G)}{(s^q + a_{1n})(s^q + k_{2n})(s^q + k_{1n})} \right] \\ & + \frac{a_{2n}G}{(s^q + l_{2n})(s^q + l_{1n})} + \frac{a'_{2n}(1 - a_{2n}G)}{(s^q + a_{2n})(s^q + l_{2n})(s^q + l_{1n})} \end{aligned} \quad (21b)$$

where

$$\begin{aligned} a'_{1n} = & \frac{(b'_n - c'_n)}{2ReG}, \quad a'_{2n} = \frac{(b'_n + c'_n)}{2ReG}, \quad b'_n = 2Gr_n, \quad b_{n1} = (\beta r_n^2 + p + a_{1n}), \\ b_{n2} = & (\beta r_n^2 + p + a_{2n}), \end{aligned}$$

$$c'_n = \frac{2Gr_n}{c_n} (-Re + ReG(R + \mathcal{H}^2) + Gr_n^2),$$

$$c_{n1} = \sqrt{(\beta r_n^2 + p + a_{1n})^2 - 4a_{1n}(p + \beta r_n^2)},$$

$$c_{n2} = \sqrt{(\beta r_n^2 + p + a_{2n})^2 - 4a_{2n}(p + \beta r_n^2)}, \quad k_{1n} = \frac{b_{n1} - c_{n1}}{2}, \quad k_{2n} = \frac{b_{n1} + c_{n1}}{2},$$

$$l_{1n} = \frac{b_{n2} - c_{n2}}{2}, \quad l_{2n} = \frac{b_{n2} + c_{n2}}{2}, \quad p = \frac{4\alpha^2}{C_p}, \quad \gamma = \frac{2\hat{\mu}Re}{C_p}. \quad (22)$$

In what follows, we make use of the special functions

$$\begin{aligned} L^{-1} \left\{ \frac{1}{s^a + m} \right\} = & F_a(-m, t) = \sum_{n=0}^{\infty} \frac{(-m)^n t^{(n+1)a-1}}{\Gamma(a(n+1))}, \quad a > 0 \\ L^{-1} \left\{ \frac{S^\gamma}{s^a + m} \right\} = & R_{a,\gamma} = \sum_{n=0}^{\infty} \frac{(-m)^n t^{(n+1)a-1-\gamma}}{\Gamma(a(n+1) - \gamma)}, \quad Re(a - \gamma) > 0, \end{aligned} \quad (23)$$

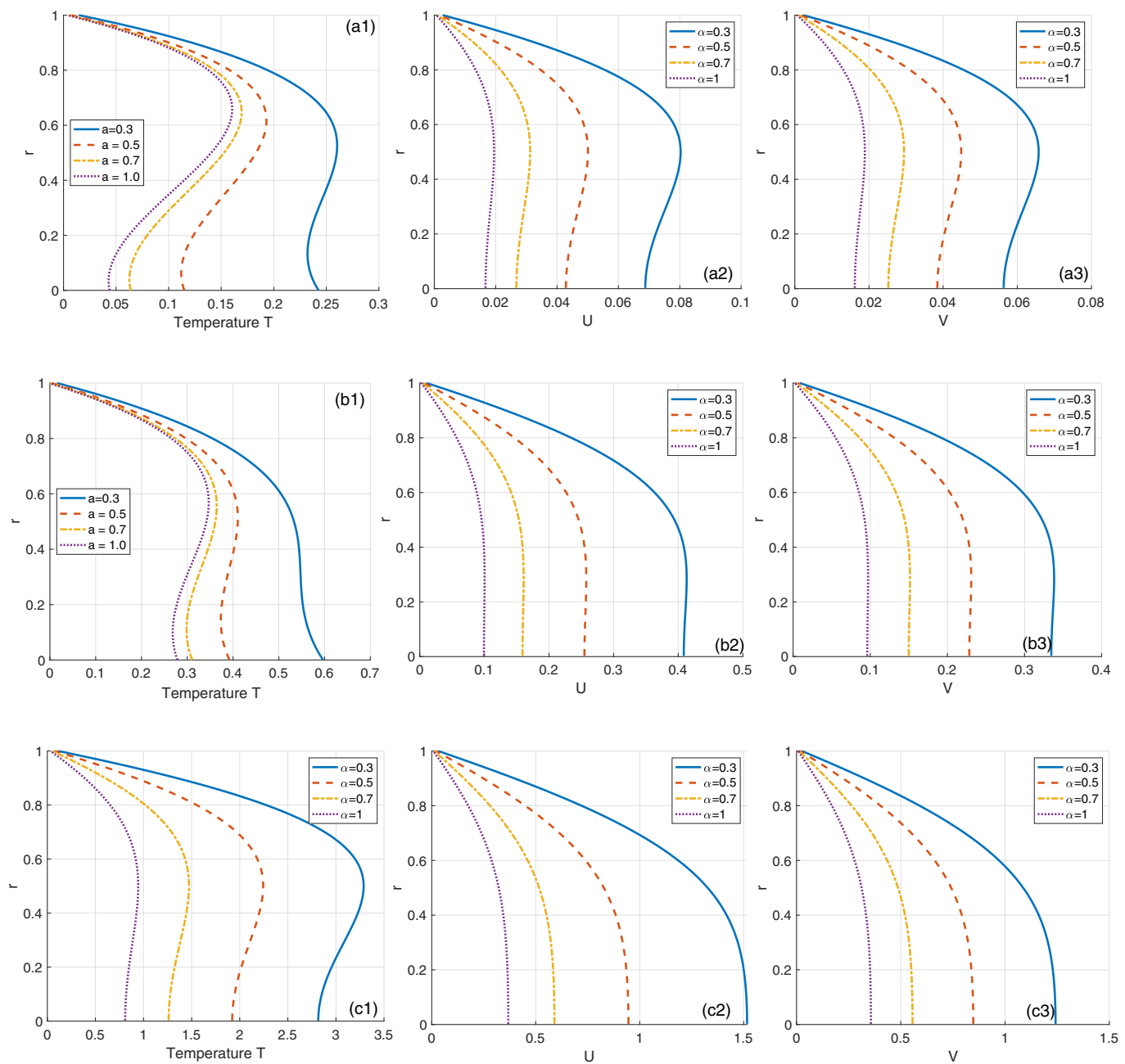


Fig. 4. Profiles of blood temperature $T(r, t)$, axial velocity $u(r, t)$, and particle velocity $v(r, t)$ for different values of Reynolds number Re , at $t = 0.2$ (small value of time), with $S = 0.01$, $A_0 = 10^{-7}$, $A_1 = 10^{-7}$, $\omega = \pi/4$, $\beta = 0.5$, $\lambda = 0.5$, $G = 1.5$, $R = 0.2$ and $\mathcal{H} = 1$. Panels (aj)_{j=1,2,3} correspond to $Re = 3$, panels (bj)_{j=1,2,3} give results for $Re = 4$ and panels (cj)_{j=1,2,3} correspond to $Re = 5$. In all the panels, the fractional parameter takes the values $a = 0.3, 0.5, 0.7$ and 1 .

where $F_a(., .)$ is the Robotnov-Hartley's function, and $R_{a,\gamma}(., .)$ is the Lorenzo-Hartley's function [50]. For the particular case $a = 1$, Eq. (23) become

$$F_1(-m, t) = e^{-mt} \quad \text{and} \quad R_{1,-1}(-m, t) = \frac{1 - e^{-mt}}{m}. \quad (24)$$

Based on the fact that $T(r, t) = T_1(r, t) + T_2(r, t)$, the inverse Hankel transform of Eqs. (16) and (21) leads to the solutions

$$\begin{aligned} U(r, t) &= 2 \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)} [F_{1n}(t) + F_{2n}(t)], \\ T_1(r, t) &= 2 \sum_{n=1}^{\infty} \left(\frac{J_0(rr_n)}{r_n J_1(r_n)} [F_{3n}(t) + F_{4n}(t)] + \frac{J_0(rr_n) J_2(r_n)}{r_n J_1^2(r_n)} [F_{5n}(t) + F_{6n}(t)] \right), \\ T_2(r, t) &= 2 \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)} [F_{7n}(t) + F_{8n}(t) + F_{9n}(t) + F_{10n}(t)], \quad \text{for } 0 < a < 1 \end{aligned}$$

$$U(r, t) = 2 \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)} [A_{1n}(t) + A_{2n}(t)], \quad (25a)$$

$$\begin{aligned} T_1(r, t) &= 2 \sum_{n=1}^{\infty} \left(\frac{J_0(rr_n)}{r_n J_1(r_n)} [A_{3n}(t) + A_{4n}(t)] + \frac{J_0(rr_n) J_2(r_n)}{r_n J_1^2(r_n)} [A_{5n}(t) + A_{6n}(t)] \right), \\ T_2(r, t) &= 2 \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)} [A_{7n}(t) + A_{8n}(t) + A_{9n}(t) + A_{10n}(t)], \quad \text{for } a = 1 \end{aligned} \quad (25b)$$

where F_{in} and A_{in} ($i = 1, \dots, 10$) are given in Appendices A and B, respectively.

4. Results and discussion

In this section, we collect the information due to the fractional-order parameter a and the other flow parameters, on temperature

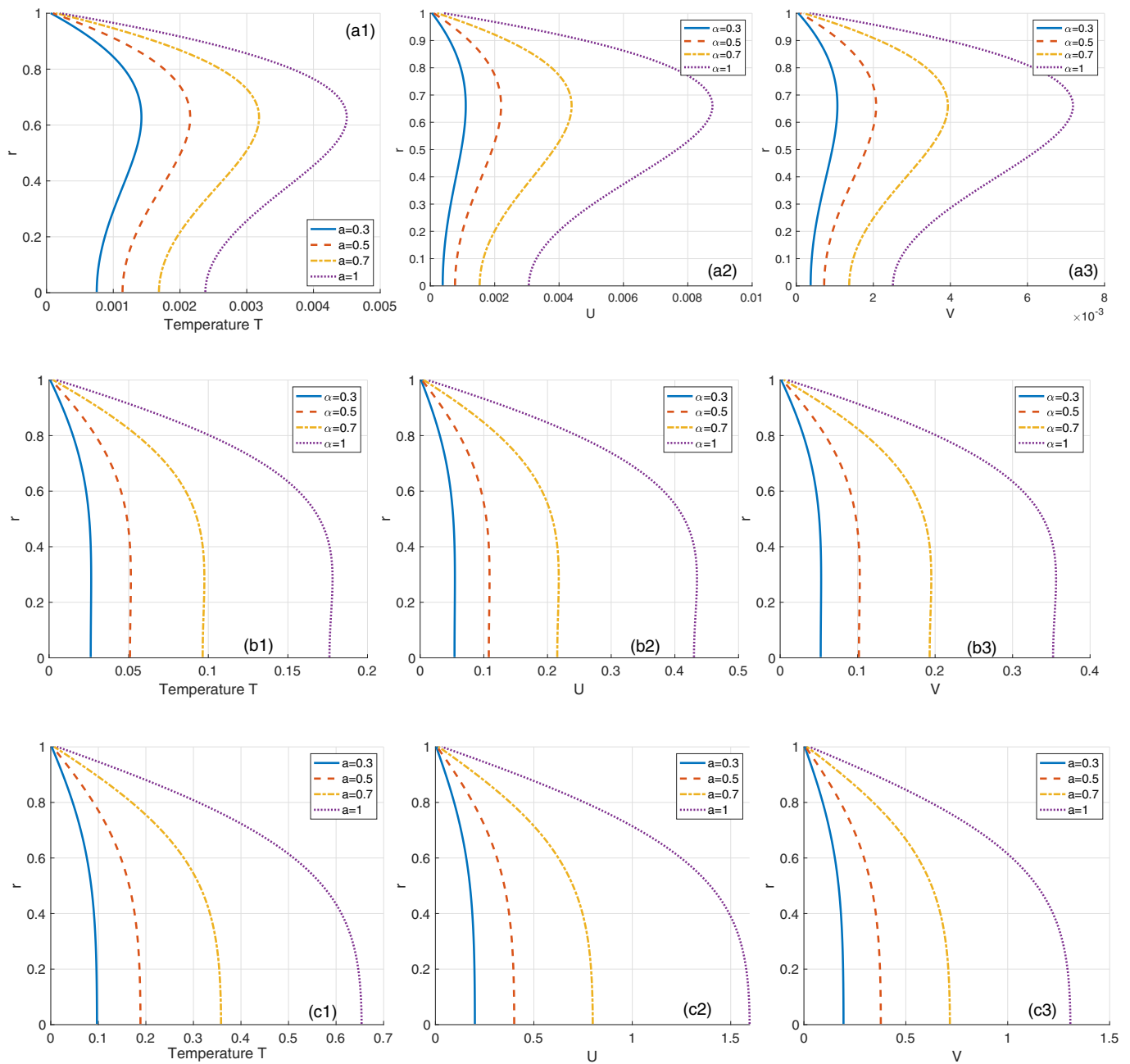


Fig. 5. Profiles of blood temperature $T(r, t)$, axial velocity $u(r, t)$, and particle velocity $v(r, t)$ for different values of Reynolds number Re , at $t = 3$ (large value of time), with $S = 0.01$, $A_0 = 10^{-7}$, $A_1 = 10^{-7}$, $\omega = \pi/4$, $\beta = 0.5$, $\lambda = 0.5$, $G = 1.5$, $R = 0.2$ and $\mathcal{H} = 1$. Panels (aj)_{j=1,2,3} correspond to $Re = 3$, panels (bj)_{j=1,2,3} give results for $Re = 4$ and panels (cj)_{j=1,2,3} correspond to $Re = 5$. In all the panels, the fractional parameter takes the values $a = 0.3, 0.5, 0.7$ and 1 .

T , fluid velocity U and particle velocity V using numerical simulations and solutions (25). In that respect, we need positive solutions of the equation $J_0(x) = 0$. The corresponding results are recorded in Figs. 2–7, where the effect of a is studied in different situations related to changes in the Reynolds number (Re) and the Hartmann number ($\mathcal{H}$). For the rest of the calculations, other parameter values have been considered as: $A_0 = 10^{-7}$, $A_1 = 10^{-7}$, $S = 0.01$, $\omega = \pi/4$, $R = 0.2$, $\lambda = 0.5$, $\beta = 0.5$ and $G = 0.8$.

In Figs. 2 and 3, we represent flow characteristics at small and high values of the time t , respectively. For the two cases, the temperature generally increases from the center of the vessel and drops near the wall. With the help of the fractional parameter, it is possible to control the increase of T , which is lower for $a = 1$. We should however stress that blood temperature is higher for larger

time. The same remark is made for the velocities, where the velocity of blood is higher than that of magnetic particles. This is different from the study proposed by Ali et al. [27], where the fractional parameter was controlling only the velocities, while it controls both the velocities and the temperature in our case.

Fig. 4 gives results for small values of time, i.e., $t = 0.2$, with changing the Reynolds number Re . For any of the values of the latter, the temperature, the blood velocity and particle velocity decrease with increasing the fractional parameter. Otherwise, the increase in velocity simply implies a decrease in the blood viscosity. Also, the velocity in general is an increasing function of the fractional parameter as supported by the calculations. On the contrary, temperature decreases with a , although changing the value of Re does not really have a significant effect on the blood temperature

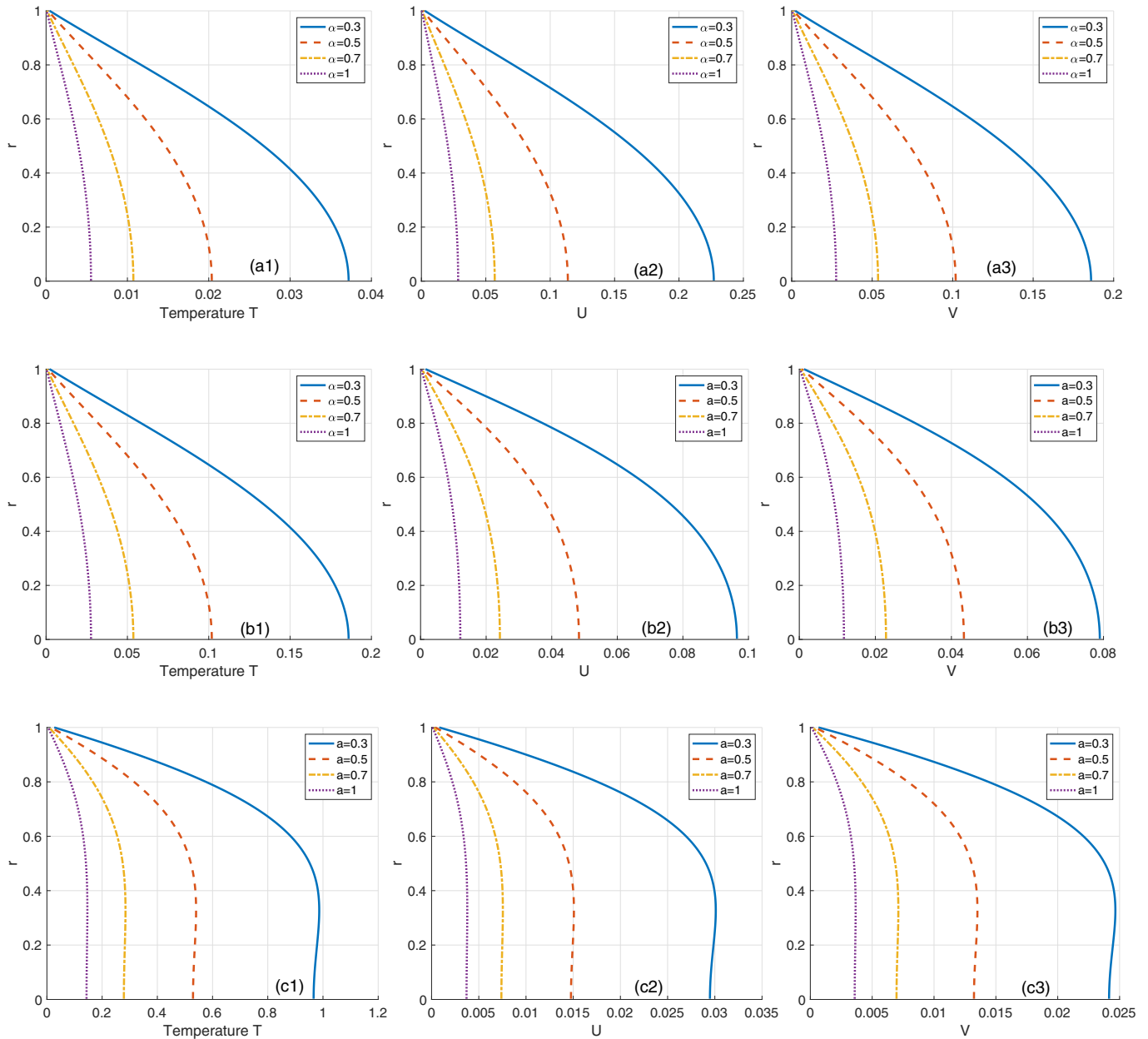


Fig. 6. Profiles of blood temperature $T(r, t)$, axial velocity $u(r, t)$, and particle velocity $v(r, t)$ for different values of Hartmann number H , at $t = 0.2$ (large value of time), with $S = 0.01$, $A_0 = 10^{-7}$, $A_1 = 10^{-7}$, $\omega = \pi/4$, $\beta = 0.5$, $\lambda = 0.5$, $G = 1.5$, $R = 0.2$ and $Re = 3$. Panels (aj)_{j=1,2,3} correspond to $H = 1$, panels (bj)_{j=1,2,3} give results for $H = 2$ and panels (cj)_{j=1,2,3} correspond to $H = 3$. In all the panels, the fractional parameter takes the values $a = 0.3, 0.5, 0.7$ and 1 .

inside the vessel. One also clearly notices that blood temperature is lower near the tissues and increases progressively near the center-line. However, the temperature profile is such that there is a peak inside the vessel, which is significantly important in keeping the tissues safe. For large values of time, i.e., $t = 3$ (see Fig. 5), Re also takes the same values as in Fig. 4, but one notices small temperatures even though they still decrease with increasing a . The velocities of blood and particles also increase when Re changes, except that they are very small compared to what was obtained in Fig. 4. This confirms the results obtained in Ref. [35], where particles were not included in the model. Moreover, such a behavior was described by Shah et al. [51], who also obtained that the Reynolds number, coupled to an increasing fractional parameter, can contribute to increase considerably the speed of blood and cause an acceleration of the magnetic particles. The influence of the magnetic field on the blood temperature, blood velocity and particles

velocity is depicted in Figs. 6 and 7. In Fig. 6, temperature and velocity profiles are recorded for a small value of time ($t = 0.5$) and columns from left to right correspond to $H = 1, 2$ and 3 (Hartmann number). One notices that the velocities of the blood and the particles are reduced with increasing H . This becomes more predominant over long exposure time $t = 3$ as depicted in Fig. 7. In this last case, the temperature rather increases, which to our opinion has an impact not only on the blood viscosity, but also on the velocity of blood and the conveyed particles. Such a result is not surprising because, when a moving electrically conducting fluid is exposed to a magnetic field, there is electric induction in the fluid, and this is pronounced when magnetic particles are present.

For example, numerical and experimental works by Sharma et al. [12,52,53] revealed that Fe_3O_4 Magnetic particles can be efficiently captured when the applied magnetic field intensity is increased, with important applications in targeted drug delivery.

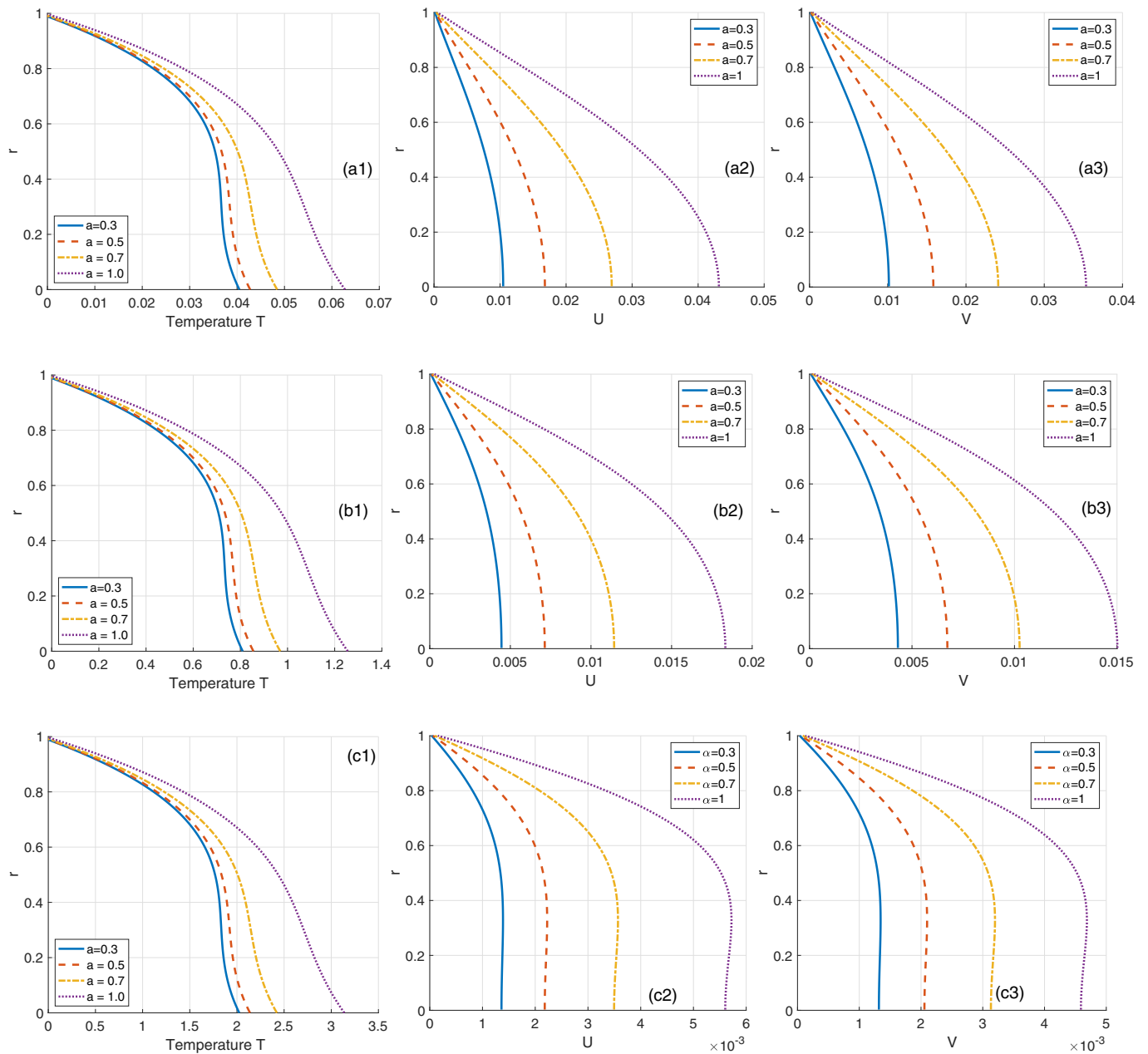


Fig. 7. Profiles of blood temperature $T(r, t)$, axial velocity $u(r, t)$, and particle velocity $v(r, t)$ for different values of Hartmann number $\mathcal{H}$, at $t = 3$ (large value of time), with $S = 0.01$, $A_0 = 10^{-7}$, $A_1 = 10^{-7}$, $\omega = \pi/4$, $\beta = 0.5$, $\lambda = 0.5$, $G = 1.5$, $R = 0.2$ and $Re = 3$. Panels (aj)_{j=1,2,3} correspond to $\mathcal{H} = 1$, panels (bj)_{j=1,2,3} give results for $\mathcal{H} = 2$ and panels (cj)_{j=1,2,3} correspond to $\mathcal{H} = 3$. In all the panels, the fractional parameter takes the values $a = 0.3, 0.5, 0.7$ and 1 .

Sheikholeslami and Shehzad [54] showed that in such conditions, an augmentation of the nanoparticles may deeply affect the thermal conduction of blood, and the whole process may be favorable in slowing down the blood flow as a result of a Lorentz force that is created and retards the flow of blood. At that stage, the temperature of blood may be enhanced as observed by Akbar et al. [55,56], while the opposite trend is observed in the case of velocities of fluid and particles. In the meantime, the fractional parameter that varies also plays a role in controlling the flow characteristics. One can for example notice that temperature and both velocities are lower for the classical value of the fractional parameter, i.e., $a = 1$. The various values that can be attributed to a make the model under our study flexible in the sense that it can be applied to a broad range of problems, each having specific applications either in technology or biomedicine.

5. Conclusion

The paper was devoted to the theoretical investigation of the motion of magnetic particles and blood flow in the presence of temperature and Caputo's fractional derivative. Considering the blood vessel as a cylindrical elastic tube, analytical solutions for the nonlinear fractional equations have been obtained through the Laplace transform, with respect to the time variable, and finite Hankel transform. Numerical calculations have been conducted, where information have been recorded for the blood temperature and both the blood and magnetic particle velocities. In general, for short intervals of time the blood flows faster than the particles, while the inverse is observed for large values of time, with magnetic particles being accelerated. The temperature increases from the center of the vessel and drops near the wall and all

the flow characteristics remain very sensitive to the change in the fractional-order parameter. The same results are obtained when the Reynolds number changes, since the temperature, the blood velocity and particle velocity decrease with increasing the fractional parameter. One also notices that the velocities of blood and particles are reduced with increasing $\mathcal{H}$, while the temperature remains an increasing function for both short and long time periods.

Declaration of Competing Interest

None.

Acknowledgments

CBT has received research grants from the [Botswana International University of Science and Technology](#), under the grant [DVC/RDI/2/1/16I \(25\)](#). CBT also thanks the Kavli Institute for Theoretical Physics ([KITP](#)), University of California Santa Barbara (USA), where this work was supported in part by the [National Science Foundation](#), Grant no. [NSF PHY-1748958](#), and the [Gordon and Betty Moore Foundation](#) Grant no. [2919.01](#).

Appendix A. Coefficients $F_{1n}(t)$ of Eq. (25a)

$$\begin{aligned} F_{1n}(t) &= \frac{b_0 Re}{c_n} [(1 - Ga_{1n})R_{a,-1}(-a_{1n}, t) - (1 - Ga_{2n})R_{a,-1}(-a_{2n}, t)], \\ F_{2n}(t) &= \frac{b_1 Re \cos(\omega t)}{c_n} [(1 - Ga_{1n})F_a(-a_{1n}, t) - (1 - Ga_{2n})F_a(-a_{2n}, t)], \\ F_{3n}(t) &= \frac{\gamma b_0 c'_n}{c_n^2} [R_{\alpha,-1}(-l_{2n}, t)F_{\alpha}(-l_{1n}, t)(1 - a_{2n}G) \\ &\quad - R_{\alpha,-1}(-k_{2n}, t)F_{\alpha}(-k_{1n}, t)(1 - a_{1n}G)] \\ F_{4n}(t) &= \frac{\gamma b_1 c'_n \cos(\omega t)}{c_n^2} [F_{\alpha}(-l_{2n}, t)F_{\alpha}(-l_{1n}, t)(1 - a_{2n}G) \\ &\quad - F_{\alpha}(-k_{2n}, t)F_{\alpha}(-k_{1n}, t)(1 - a_{1n}G)] \\ F_{5n}(t) &= \frac{\gamma b_0}{c_n} [R_{\alpha,-1}(-l_{2n}, t)F_{\alpha}(-l_{1n}, t)(1 - a_{2n}G) \\ &\quad - R_{\alpha,-1}(-k_{2n}, t)F_{\alpha}(-k_{1n}, t)(1 - a_{1n}G)] \\ F_{6n}(t) &= \frac{\gamma b_1 \cos(\omega t)}{c_n} [F_{\alpha}(-l_{2n}, t)F_{\alpha}(-l_{1n}, t)(1 - a_{2n}G) \\ &\quad - F_{\alpha}(-k_{2n}, t)F_{\alpha}(-k_{1n}, t)(1 - a_{1n}G)] \\ F_{7n}(t) &= \frac{\gamma b_0 G}{c_n} [a'_{2n}R_{\alpha,-1}(-l_{2n}, t)F_{\alpha}(-l_{1n}, t) \\ &\quad - a'_{1n}R_{\alpha,-1}(-k_{2n}, t)F_{\alpha}(-k_{1n}, t)] \\ F_{8n}(t) &= \frac{\gamma b_0}{c_n} [a'_{2n}(1 - a_{2n}G)F_{\alpha}(-a_{2n}, t)(R_{\alpha,-1}(-l_{2n}, t)F_{\alpha}(-l_{1n}, t)) \\ &\quad - a'_{1n}(1 - a_{1n}G)F_{\alpha}(-a_{1n}, t)(R_{\alpha,-1}(-k_{2n}, t)F_{\alpha}(-k_{1n}, t))] \\ F_{9n}(t) &= \frac{\gamma b_1 G \cos(\omega t)}{c_n} [a'_{2n}F_{\alpha}(-l_{2n}, t)F_{\alpha}(-l_{1n}, t) \\ &\quad - a'_{1n}F_{\alpha}(-k_{2n}, t)F_{\alpha}(-k_{1n}, t)] \\ F_{10n}(t) &= \frac{\gamma b_1 \cos(\omega t)}{c_n} [a'_{2n}(1 - a_{2n}G)F_{\alpha}(-a_{2n}, t)(F_{\alpha}(-l_{2n}, t)F_{\alpha}(-l_{1n}, t)) \\ &\quad - a'_{1n}(1 - a_{1n}G)F_{\alpha}(-a_{1n}, t)(F_{\alpha}(-k_{2n}, t)F_{\alpha}(-k_{1n}, t))] \end{aligned}$$

Appendix B. Coefficients $A_{1n}(t)$ of Eq. (25b)

$$\begin{aligned} A_{1n}(t) &= \frac{b_0 Re}{c_n} \left[\frac{c_n}{r_n^2 - Re\mathcal{H}^2} - \frac{(1 - Ga_{1n})e^{-a_{1n}t}}{a_{1n}} + \frac{(1 - Ga_{2n})e^{-a_{2n}t}}{a_{2n}} \right] \\ A_{2n}(t) &= \frac{b_1 Re}{c_n} \left[\frac{(1 - Ga_{1n})}{a_{1n}^2 + \omega^2} (a_{1n} \cos(\omega t) + \omega \sin(\omega t) - a_{1n}e^{-a_{1n}t}) \right. \\ &\quad \left. + \frac{(1 - Ga_{2n})}{a_{2n}^2 + \omega^2} (a_{2n} \cos(\omega t) + \omega \sin(\omega t) - a_{2n}e^{-a_{2n}t}) \right] \end{aligned}$$

$$\begin{aligned} A_{3n}(t) &= \frac{\gamma b_0 c'_n}{c_n^2} \left[(1 - a_{2n}G) \left(\frac{1 - e^{-l_{1n}t}}{l_{2n}l_{1n}} + \frac{e^{-l_{1n}t} - e^{-l_{2n}t}}{l_{2n}(l_{1n} - l_{2n})} \right) \right. \\ &\quad \left. - (1 - a_{1n}G) \left(\frac{1 - e^{-k_{1n}t}}{k_{2n}k_{1n}} + \frac{e^{-k_{1n}t} - e^{-k_{2n}t}}{k_{2n}(k_{1n} - k_{2n})} \right) \right] \\ A_{4n}(t) &= \frac{\gamma b_1 c'_n \cos \omega t}{c_n^2} \left[\frac{(1 - a_{2n}G)(e^{-l_{2n}t} - e^{-l_{1n}t})}{(l_{1n} - l_{2n})} \right. \\ &\quad \left. - \frac{(1 - a_{1n}G)(e^{-k_{2n}t} - e^{-k_{1n}t})}{(k_{1n} - k_{2n})} \right] \\ A_{5n}(t) &= \frac{\gamma b_0}{c_n} \left[(1 - a_{2n}G) \left(\frac{1 - e^{-l_{1n}t}}{l_{2n}l_{1n}} + \frac{e^{-l_{1n}t} - e^{-l_{2n}t}}{l_{2n}(l_{1n} - l_{2n})} \right) \right. \\ &\quad \left. - (1 - a_{1n}G) \left(\frac{1 - e^{-k_{1n}t}}{k_{2n}k_{1n}} + \frac{e^{-k_{1n}t} - e^{-k_{2n}t}}{k_{2n}(k_{1n} - k_{2n})} \right) \right] \\ A_{6n}(t) &= \frac{\gamma b_1 \cos \omega t}{c_n} \left[\frac{(1 - a_{2n}G)(e^{-l_{2n}t} - e^{-l_{1n}t})}{(l_{1n} - l_{2n})} \right. \\ &\quad \left. - \frac{(1 - a_{1n}G)(e^{-k_{2n}t} - e^{-k_{1n}t})}{(k_{1n} - k_{2n})} \right] \\ A_{7n}(t) &= \frac{\gamma b_0 G}{c_n} \left[a'_{2n} \left(\frac{1 - e^{-l_{1n}t}}{l_{2n}l_{1n}} + \frac{e^{-l_{1n}t} - e^{-l_{2n}t}}{l_{2n}(l_{1n} - l_{2n})} \right) \right. \\ &\quad \left. - a'_{1n} \left(\frac{1 - e^{-k_{1n}t}}{k_{2n}k_{1n}} + \frac{e^{-k_{1n}t} - e^{-k_{2n}t}}{k_{2n}(k_{1n} - k_{2n})} \right) \right] \\ A_{8n}(t) &= \frac{\gamma b_0}{c_n} \left[a'_{2n}(1 - a_{2n}G) \left(\frac{1}{l_{2n}l_{1n}} \left(\frac{1 - e^{-a_{2n}t}}{a_{2n}l_{1n}} + \frac{e^{-a_{2n}t} - e^{-l_{2n}t}}{l_{1n}(a_{2n} - l_{1n})} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{l_{2n}(l_{1n} - l_{2n})} \left(\frac{e^{-a_{2n}t} - e^{-l_{1n}t}}{l_{1n} - a_{2n}} \right) - \frac{1}{l_{2n}(l_{1n} - l_{2n})} \left(\frac{e^{-a_{2n}t} - e^{-l_{2n}t}}{l_{2n} - a_{2n}} \right) \right) \right. \\ &\quad \left. - a'_{1n}(1 - a_{1n}G) \left(\frac{1}{k_{2n}k_{1n}} \left(\frac{1 - e^{-a_{1n}t}}{a_{1n}k_{1n}} + \frac{e^{-a_{1n}t} - e^{-k_{1n}t}}{k_{1n}(a_{1n} - k_{1n})} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{k_{2n}(k_{1n} - k_{2n})} \left(\frac{e^{-a_{1n}t} - e^{-k_{1n}t}}{k_{1n} - a_{1n}} \right) - \frac{1}{k_{2n}(k_{1n} - k_{2n})} \left(\frac{e^{-a_{1n}t} - e^{-k_{2n}t}}{k_{2n} - a_{1n}} \right) \right) \right] \\ A_{9n}(t) &= \frac{\gamma b_1 G \cos \omega t}{c_n} \left[a'_{2n} \frac{e^{-l_{2n}t} - e^{-l_{1n}t}}{(l_{1n} - l_{2n})} - a'_{1n} \frac{e^{-k_{2n}t} - e^{-k_{1n}t}}{(k_{1n} - k_{2n})} \right] \\ A_{10n}(t) &= \frac{\gamma b_1 \cos \omega t}{c_n} \left[\frac{a'_{2n}(1 - a_{2n}G)}{l_{1n} - l_{2n}} \left(\frac{e^{-a_{2n}t} - e^{-l_{2n}t}}{l_{2n} - a_{2n}} - \frac{e^{-a_{2n}t} - e^{-l_{1n}t}}{l_{1n} - a_{2n}} \right) \right. \\ &\quad \left. - \frac{a'_{1n}(1 - a_{1n}G)}{k_{1n} - k_{2n}} \left(\frac{e^{-a_{1n}t} - e^{-k_{2n}t}}{k_{2n} - a_{1n}} - \frac{e^{-a_{1n}t} - e^{-k_{1n}t}}{k_{1n} - a_{1n}} \right) \right] \end{aligned}$$

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