



Faculty of Sciences
Department of Mathematics & Statistical Sciences

**The Marshall-Olkin-G Power Series family of
distributions with applications to complete
and censored data**

by

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A Thesis Submitted to the Faculty of Sciences in Partial Fulfilment of the Requirements
for the Award of a Masters Degree in Statistics at BIUST

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February, 2023



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PUBLICATIONS

Regarding the Work

Distribution	Journal	Status
The Marshall-Olkin-G Power Series Family of Distributions with Application to cancer data	Parkistan Journal of Statistics	Withdrawn

Table 1: **Submitted Paper**

We had submitted a manuscript to the Parkistan Journal of Statistics for consideration and possible publication title ,“The Marshall-Olkin-G Power Series Family of Distributions with Application to cancer data”, but we later withdraw it after learning about Cordeiro et. al.’s recent published paper, “A competitive family to the Beta and Kumaraswamy generators: Properties, Regressions and Applications”. In their article [14], they derived a model resembling ours.

Abstract

We reviewed the generalized family of distributions called the Marshall-Olkin-G Power Series (MO-GPS) family of distributions. Some special models of the MO-GPS family of distributions are reviewed in detail. Some structural statistical properties inclusive of moments, residual and reverse residual life function, stress - strength function, mean deviations, distribution of the order statistics and Rényi entropy are reviewed for the MO-GPS and its special models. Different estimation techniques are discussed, however the maximum likelihood estimation method is used to estimate the model parameters. A simulation study to examine the bias and mean square error of the maximum likelihood estimators was carried out. We model censored and complete data using the MO-GPS family of distributions.



Dedications

I dedicate this work to my brother, father and my late mother for their unwavering support and encouragement.



Acknowledgements

I thank the God Almighty the provider of knowledge and wisdom, for walking me through my studies and providing me with power and strength to undertake my research successfully. Without His grace and courage I would not have succeeded. I would like to extend my deep appreciation towards my supervisors Dr. Lesego Gabaitiri and Prof. Boikanyo Makubate for their time, guidance and advice they provided during my study period which made it possible for me to successfully complete this thesis. Special thanks goes to my father and brother for their unconditional love, sacrifice and support. I wish to express my sincere gratitude to my friends (W. Sengweni, S. Chamunorwa, P. J. Moyo and P. Sibanda) for their bravery and moral support. May our Lord the Almighty bless them all abundantly.



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1. Introduction

1.1 Introduction

The employment of appropriate statistical models that can realistically take into account the key aspects of the data and the processes being researched is now more vital than ever thanks to advancements in probability and statistics. In order to accommodate and provide more flexibility in the modeling of various real-world datasets, it has become necessary to generalize or modify the well-known classical distributions. There is an obvious demand for extended and generalized versions of distributions in many applied and practical areas, including lifetime analysis, finance, and insurance. Researchers in statistics have been working to improve classical models by adding some more shape, location, or size characteristics to these models in response to the growing demand for more flexible models. The statistical literature includes techniques for creating new families of distributions and has been researched in detail.

As mentioned, modeling and analyzing lifetime data are important aspects of statistical research in many applied sciences such as engineering, medicine, economics and so on. Various recent probability distributions discussed model some data by compounding the well-known lifetime distributions such as Marshall-Olkin, exponential, Weibull, generalized exponential, exponentiated Weibull, Burr-XII, with some discrete distributions such as zero-truncated power series distributions in general. The compounding approach gives new distributions that extend well-known families of distributions. Consequently, they offer more flexibility for modeling lifetime datasets. The flexibility of such compound distributions comes in terms of one or more failure rate shapes that may be decreasing, increasing, bathtub shaped or unimodal shaped.

By various techniques, new parameters can be introduced to expand families or classes of distributions for flexibility improvement or to construct covariate models. The introduction of a scale parameter results in accelerated life model, and taking powers of the survival function introduces a parameter that leads to the proportional hazards model.

One of the ways of introducing a parameter to expand a family of distributions was

introduced by Marshall and Olkin [34]. The Marshall-Olkin method was applied to yield a new two-parameter extension of the exponential distribution which serve as a competitor to commonly-used two-parameter families of life distributions as the Weibull, Burr XII, gamma and lognormal distributions. In addition, this method was applied to yield a new three-parameter Weibull distribution.

Nicy et al. [37] used Topp-Leone as the distribution generator and exponential distribution as the parent distribution to develop a Type 1 Topp-Leone q-Exponential distribution. The q-exponential distribution has two parameters which provide better flexibility with regards to its decay which is different from the general exponential distribution.

Gómez-Déniz et al. [22] introduced a two parameter model called the Log-Lindley distribution with bounded domain and this was achieved by a change of variable in the generalised Lindley distribution. Elgohari et al. [21] introduced the Odd-Burr Pareto type 11 model, its density was expressed as a mixture of the exponentiated Pareto type 11 model, properties were studied, bayesian and non-bayesian estimation methods were used.

We modelled censored and uncensored data using the Marshall-Olkin-G Power Series (MO-GPS) family of distributions and studied some statistical properties as well as some special models of the distribution due to the flexibility and adaptability of the Marshall-Olkin, Power Series, Burr XII, Log-Logistic, and Weibull distributions in modeling lifetime data as well as the need for modifications of the Marshall-Olkin model.

1.2 Some Statistical Properties of Probability Distributions

1.2.1 Probability Density Function

The probability density function (pdf) of a continuous random variable is a function which can be integrated to obtain the probability that the random variable takes in a given interval. If $g(x; \zeta)$ is a pdf of a continuous random variable then it must obey two conditions;

- i) the total probability of all possible values of X is 1; that is

$$\int_{-\infty}^{\infty} g(x; \zeta) dx = 1,$$

and

ii) the pdf can never be negative: $g(x; \zeta) \geq 0$ for $\forall x$.

1.2.2 Cumulative Distribution Function

Let X be a continuous random variable, a cumulative distribution function (cdf) is a function giving the probability that the random variable X is less than or equal to x , for every value x . The cdf $G(x; \zeta)$ is defined to be:

$$G(x; \zeta) = P(X \leq x), \quad -\infty < x < \infty. \quad (1.1)$$

For a continuous random variable, the cdf is the integral of its pdf; that is

$$G(x; \zeta) = \int_{-\infty}^x g(x; \zeta) dx.$$

1.2.3 Survival, Hazard and Reverse Hazard Functions

Let X be a continuous random variable with a cdf $G(x; \zeta)$ and pdf $g(x; \zeta)$ then the probability that a device, patient, or any other object of interest will survive longer than any given time x is provided by the survival function, which is defined by

$$S(x; \zeta) = \bar{G}(x; \zeta) = 1 - G(x; \zeta) = \int_x^{\infty} g(t; \zeta) dt. \quad (1.2)$$

The hazard rate function (hrf) is the likelihood that an object will survive to a certain point of time based on its survival to an earlier time (x). The hrf is the ratio of the pdf to the survival function and is given by

$$h(t) = \frac{g(x; \zeta)}{\bar{G}(x; \zeta)}. \quad (1.3)$$

The reverse hazard rate functions (rhrf) is defined as the ratio of the pdf to the cdf and is given by

$$\tau(x; \zeta) = \frac{g(x; \zeta)}{G(x; \zeta)}. \quad (1.4)$$

1.2.4 Quantile Function

Let X be a random variable with the cdf $G(x; \zeta)$, where $x \in \Re$. The quantile function is the inverse of the cdf given by $Q(u) = G^{-1}(x; \zeta)$ for $u \in (0, 1)$. This is equivalent

to solving

$$G(Q(u)) = u, u \in (0, 1). \quad (1.5)$$

1.2.5 Moments

Let $g(x; \zeta)$ be the pdf of a random variable X . The k^{th} moment about the origin is given by;

$$E(X^k) = \int_{-\infty}^{\infty} x^k g(x; \zeta) dx. \quad (1.6)$$

The special case of the k^{th} moment is the *mean* $E(X^1) = E(X)$. The k^{th} moment about the mean is given by;

$$E(X - \mu)^k = \int_{-\infty}^{\infty} (x - \mu)^k g(x; \zeta) dx. \quad (1.7)$$

Some special cases of the k^{th} moment about the mean include the

$$variance = E(X - \mu)^2,$$

$$skewness = E(x - \mu)^3 \text{ and}$$

$$kurtosis = E(X - \mu)^4.$$

1.2.6 Moment Generating Function

The Moment Generating Function (mgf) of a random variable x about point 'y' is denoted by $M_x(t)$ and is defined by;

$$M_x(t) = E(e^{t(x-y)}) = \int_{-\infty}^{\infty} e^{t(x-y)} g(x; \zeta) dx, \quad (1.8)$$

where $g(x; \zeta)$ is a pdf. The mgf about the origin is given by

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} g(x; \zeta) dx.$$

1.2.7 Probability Weighted Moment

The expectation of a function of a random variable given that the variable's mean exists is referred as the probability weighted moment (PWM). The $(i, j)^{th}$ PWM of

a random variable X , say $(\rho_{i,j})$ is given by

$$\rho_{i,j} = E(X^i F(X)^j) = \int_{-\infty}^{\infty} x^i g(x; \zeta) G(x; \zeta)^j dx, \quad (1.9)$$

where $g(x; \zeta)$ and $G(x; \zeta)$ are the pdf and cdf respectively.

1.2.8 Mean deviation

The mean deviation about the mean is a statistical measure that is used to calculate the average deviation from the mean value of the given data and is given by;

$$\alpha_1(x) = \int_0^{\infty} |x - \mu| g(x; \zeta) dx \quad (1.10)$$

where $\mu = E(X)$.

The mean deviation about the median measures the average deviation from the median value of a data set and is given by;

$$\alpha_2(x) = \int_0^{\infty} |x - M| g(x; \zeta) dx \quad (1.11)$$

where $M = \text{median}(X)$.

1.2.9 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves are defined by

$$B(u) = \frac{1}{u\mu} \int_0^v xg(x; \zeta) dx, \quad (1.12)$$

and

$$L(u) = \frac{1}{\mu} \int_0^v xg(x; \zeta) dx \quad (1.13)$$

where $\mu = E(X)$, $v = G^{-1}(u) = Q(u)$ can be easily calculated using equation (1.5) and $0 \leq u \leq 1$.

1.2.10 Residual and reverse residual life function

Residual life function is a function of t that calculates the expected additional lifetime given that a component has survived until time t . The n^{th} moment of the residual

life function is given by

$$a_n(t) = E((x-t)^n | x > t) = \frac{1}{\bar{G}(x; \zeta)} \int_t^\infty (x-t)^n g(x; \zeta) dx, \quad t \geq 0, \quad (1.14)$$

and the residual life function is obtained when $n = 1$ and is given by

$$a_1(t) = E((x-t) | x > t) = \frac{1}{\bar{G}(x; \zeta)} \int_t^\infty (x-t) g(x; \zeta) dx, \quad t \geq 0. \quad (1.15)$$

Reverse residual life function calculates the expected waiting time that a component was alive at time x given that it has failed at time t for $x < t$. The n^{th} moment of reverse residual life function is given by

$$b_n(t) = E((t-x)^n | x \leq t) = \frac{1}{G(x; \zeta)} \int_0^t (t-x)^n g(x; \zeta) dx, \quad t \geq 0, \quad (1.16)$$

and reverse residual life function is obtained when $n = 1$ and is given by

$$b_1(t) = E((t-x) | x \leq t) = \frac{1}{G(x; \zeta)} \int_0^t (t-x) g(x; \zeta) dx, \quad t \geq 0. \quad (1.17)$$

1.2.11 Stress - Strength Function

In the reliability study, the estimation of the stress-strength parameter suggested by Birnbaum [7] is crucial. The parameter R evaluates system performance and is widely employed in the context of mechanical dependability of a system in order to determine the probability of a system failure. Let X be the strength of a system that is subjected to stress Y . When the applied stress exceeds the system's strength, the system breaks down. The stress-strength reliability is then defined as the probability of not failing ($P[X > Y]$). When X and Y are independent $G(x; \zeta_1)$ and $G(y; \zeta_2)$ respectively, the reliability parameter for a distribution is given by

$$\begin{aligned} R = P(Y < X) &= \int_0^\infty \int_0^x g(x; \zeta_1) g(y; \zeta_2) dy dx \\ &= \int_{x=0}^\infty g(x; \zeta_1) G(x; \zeta_2) dx. \end{aligned}$$

1.2.12 Entropy

Shannon entropy and Rényi entropy are two popular measures of entropy. Rényi entropy (Rényi [31]) is an extension of Shannon entropy [40] for any continuous pdf

$(g(x; \zeta))$. Rényi entropy is defined to be

$$I_R(\nu) = \frac{1}{1-\nu} \log \left(\int_0^\infty (g(x; \zeta))^\nu dx \right), \nu \neq 1, \nu > 0. \quad (1.18)$$

Rényi entropy tends to Shannon entropy as $\nu \rightarrow 1$.

1.2.13 Order Statistics

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (*iid*) random variables from cdf and pdf $F(x)$ and $f(x)$ respectively. The i^{th} order statistic of n independent and identically distributed random variables is the i^{th} smallest element. The pdf of the i^{th} order statistic is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F^{i-1}(x) [1 - F(x)]^{n-i}.$$

1.2.14 Stochastic Ordering

The concept of stochastic ordering is regularly used to display the ordering mechanism in probability models see Shaked and Shanthikumar [39]. A random variable X_1 is said to be stochastically smaller than X_2 , ($X_1 <_{st} X_2$) in the

- Stochastic order ($X_1 <_{st} X_2$) if $G_{X_1}(x) \geq G_{X_2}(x)$, $\forall x$.
- Hazard rate order ($X_1 <_{hr} X_2$) if $h_{X_1}(x) \geq h_{X_2}(x)$, $\forall x$.
- Likelihood ratio order ($X_1 <_{lr} X_2$) if $\frac{g_{X_1}(x)}{g_{X_2}(x)}$ decreasing in x .

The stochastic orders given above are associated to each other and it holds that $X_1 <_{lr} X_2 \Rightarrow X_1 <_{hr} X_2 \Rightarrow X_1 <_{st} X_2$.

1.2.15 Maximum Likelihood Estimation

Let $X_1, X_2, \dots, X_n$ have a joint density denoted as

$$g_\theta(x_1, x_2, \dots, x_n) = g(x_1, x_2, \dots, x_n | \theta),$$

with respect to some parameter $\theta \in \mathbb{R}$, then the likelihood of θ is the function given by,

$$L(\theta) = g(x_1, x_2, \dots, x_n | \theta).$$

The MLE of the parameter θ is the value that maximizes $L(\theta)$ if available. If the X_i 's are independent and identically distributed (*iid*), then the likelihood simplifies

to

$$L(\theta) = \prod_{i=1}^n g(x_i|\theta). \quad (1.19)$$

Since the logarithm is an increasing function, maximizing $L(\theta)$ is corresponds to maximization of its log function, which is denoted by

$$\ell(\theta) = \log L(\theta) = \sum_{i=1}^n \log(g(x_i|\theta)). \quad (1.20)$$

1.2.16 Goodness-of-fit Statistics

Goodness-of-fit statistics (GOF) tests how well the model fits the observed data by testing the null hypothesis that H_0 : the model fits the dataset. The GOF, namely, Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), -2loglikelihood (-2 log L), Cramer-von Mises (W^*), Andersen-Darling (A^*) and the Kolmogorov-Smirnov (K-S) are all given in detail by Chen and Balakrishnan [16]. The goodness-of-fit statistics are respectively defined as follows;

$$\begin{aligned} AIC &= 2q - 2\log(L), \\ CAIC &= AIC + \frac{2q(q+1)}{n-q-1}, \\ BIC &= q \times \log(n) - 2\log(L), \\ K-S &= \sup_x |F_n(x) - F(x)|, \\ A^* &= -n - n^{-1} \sum_{i=1}^n (2i-1) [\log(F(x_{(i)})) + \log(1 - F(x_{(n-i+1)}))], \end{aligned}$$

and

$$W^* = (12n)^{-1} + \sum_{i=1}^n \left(\frac{2i-1}{2n} - F(x_{(i)}) \right)^2,$$

where $F_n(x) = n^{-1} \sum_{i=1}^n I_{x_i \leq x}$ is the empirical distribution, q is the number of parameters, n is the sample size, $F(x)$ is the cdf, $L(\hat{\theta})$ is the value of the likelihood function, $F(x_{(i)})$ is the cdf of the specified distribution and $x_{(i)}$ is the i^{th} sample of the chronologically arranged data. The model which fits the dataset best has the least value of these goodness-of-fit statistics.

1.3 Some Useful Distributions

In this section, we highlight the cdf and pdf of some useful continuous distributions.

1.3.1 Power Series Distribution

Let N be a discrete random variable following a power series distribution [25] assumed to be truncated at zero. The probability mass function (pmf) of N is given by

$$f_{\theta}(n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, 3, \dots \quad (1.21)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n \geq 1}$ is a sequence of positive real numbers. The power series family of distributions includes poisson, logarithmic, geometric and binomial distributions [25]. Table 1.1 displays the poisson, logarithmic, geometric and binomial distributions alongside their respective a_n , $C(\theta)$, $C'(\theta)$ and parameter space.

Distribution	$C(\theta)$	$C'(\theta)$	a_n	Parameter Space
Poisson	$e^{\theta} - 1$	e^{θ}	$(n!)^{-1}$	$(0, \infty)$
Logarithmic	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	n^{-1}	$(0, 1)$
Geometric	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	1	$(0, 1)$
Binomial	$(1 + \theta)^m - 1$	$m(1 + \theta)^{m-1}$	$\binom{m}{n}$	$(0, 1)$

Table 1.1: Some useful quantities for Power Series Distributions

1.3.2 Weibull Distribution

The cdf and pdf of a Weibull [35] distribution are given by

$$G_W(x; \eta, \nu) = 1 - e^{-\nu x^{\eta}}, \quad (1.22)$$

and

$$g_W(x; \eta, \nu) = \eta \nu x^{\eta-1} e^{-\nu x^{\eta}}, \quad (1.23)$$

respectively, where $x, \eta, \nu > 0$, and, ν and η are scale and shape parameters respectively.

1.3.3 Burr XII Distribution

The cdf and pdf of a Burr XII [9] distribution are given by

$$G_{BXII}(x; \alpha, \beta) = 1 - (1 + x^\alpha)^{-\beta}, \quad (1.24)$$

and

$$g_{BXII}(x; \alpha, \beta) = \alpha\beta x^{\alpha-1}(1 + x^\alpha)^{-\beta-1}, \quad (1.25)$$

respectively for $x, \alpha > 0$ and $\beta > 0$. The Log-Logistic distribution is a special case of the Burr XII distribution when $\beta = 1$.

1.3.4 Topp-Leone Distribution

The cdf and pdf of a Topp-Leone distribution are given by

$$G_{TL}(x; \theta) = (1 - (1 - x)^2)^\theta, \quad (1.26)$$

and

$$g_U(x; \theta) = 2\theta(1 - x)(1 - (1 - x)^2)^{\theta-1}, \quad (1.27)$$

respectively, for $0 < x < 1$ and $\theta > 0$.

1.4 Some methods for generating families of continuous distributions

In this section, we discuss in brief some of the generators used for generating families of distributions. Let $G(x; \zeta)$ and $g(x; \zeta)$ be the cdf and pdf of a continuous random variable X .

1.4.1 Exponentiated Generalized-G method

Let F be an absolutely continuous cdf with support on (a, b) , where the interval may be unbounded, and let α be a positive real number. The random variable X has an F^α - distribution if its cdf, denoted by $F_x(x)$ is given by $F_x(x) = F^\alpha(x) = [F(x)]^\alpha$ by Gholam [24]. The cdf and pdf of the Exponentiated Generalized-G (EGG) is

given by

$$F_{EGG}(x; \zeta) = [1 - (1 - G(x; \zeta))^a]^b, \quad (1.28)$$

and

$$f_{EGG}(x; \zeta) = abg(x; \zeta)(1 - G(x; \zeta))^{a-1}[1 - (1 - G(x; \zeta))^a]^{b-1}, \quad (1.29)$$

respectively, where $a > 0$, $b > 0$ and ζ is a parameter vector. More details regarding the EGG are given by Cordeiro et al. [12].

1.4.2 T-X family method

Alzaghal et al. [4] created the family of distributions known as T-X. The cdf and pdf of the T-X family of distribution are defined by,

$$F(x; \zeta) = \int_p^{H(G(x; \zeta))} v(u) du, \quad (1.30)$$

and

$$f(x; \zeta) = \left[\frac{d}{dx} H(G(x; \zeta)) \right] v[H(G(x; \zeta))] \quad (1.31)$$

where $v(u)$ is the pdf of a random variable $U \in [p, q]$, for $-\infty \leq p < q \leq \infty$ and $H(G(x; \zeta))$ is a function of the cdf $G(x; \zeta)$ of any continuous random variable X so that $H(G(x; \zeta))$ satisfies the following equations:

- i) $H(G(x; \zeta)) \in [p, q]$;
- ii) $H(G(x; \zeta))$ is differentiable and monotonically non-decreasing;
- iii) $H(G(x; \zeta)) \rightarrow p$ as $x \rightarrow -\infty$ and $H(G(x; \zeta)) \rightarrow q$ as $x \rightarrow \infty$.

Some Special cases of the T-X method include the Beta-G by Eugene, Lee and Famoye [20], Lomax generator by Cordeiro et al. [12], Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [13], Half-Logistic-G by Cordeiro, Alizadeh and Diniz [17] and the Gamma-G by Zografos et al. [42].

1.4.3 Marshall-Olkin-G Generator

The Marshall-Olkin (MO) extended family of distributions was developed by Marshall and Olkin [34]. The cdf of the Marshall-Olkin-G (MO-G) family of distributions

is given by

$$F_{MO-G}(x; \delta, \zeta) = 1 - \frac{\delta \bar{G}(x; \zeta)}{1 - \bar{\delta} \bar{G}(x; \zeta)}. \quad (1.32)$$

The corresponding pdf is given by

$$f_{MO-G}(x; \delta, \zeta) = \frac{\delta g(x; \zeta)}{[1 - \bar{\delta} \bar{G}(x; \zeta)]^2}, \quad (1.33)$$

where ζ is the parameter vector, $\bar{\delta} = 1 - \delta$.

1.4.4 Competing risk method

Let $S_{X_1}(x; \zeta)$ and $S_{X_2}(x; \zeta)$ be the survival function of the two continuous independent random variables say X_1 and X_2 respectively, let $X = \min(X_1, X_2)$, then the cdf of X is given by

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= 1 - [1 - G_1(x; \zeta)][1 - G_2(x; \zeta)] \\ &= 1 - (S_{X_1}(x; \zeta))(S_{X_2}(x; \zeta)). \end{aligned} \quad (1.34)$$

1.4.5 Compounding method

Consider a sequence of N independent and identically distributed (*iid*) random variables from $G(x; \zeta)$, where N is a discrete random variable following a power series distribution with cdf and pmf given by

$$F_\theta(n) = \frac{C_n(\theta)}{C(\theta)}, \quad n \in \mathbb{N} \quad (1.35)$$

and

$$f_\theta(n) = \frac{a_n \theta^n}{C(\theta)}, \quad n \in \mathbb{N} \quad (1.36)$$

where $C_n(\theta) = \sum_{k=1}^n a_k \theta^k$ is finite, $\theta > 0$. Let $X_{(1)} = \min(X_1, X_2, \dots, X_N)$ then the conditional distribution of $X_{(1)}$ given $N = n$ is given by

$$F(X_{(1)}|N = n) = 1 - \bar{G}^n(x; \zeta), \quad (1.37)$$

where $n \in \mathbb{N}_{>1}$. The joint cdf of $X_{(1)}$ and N is given by

$$F(X_{(1)}|N = n)f(N) = \frac{a_n\theta^n}{C(\theta)} - \frac{a_n\theta^n}{C(\theta)}\bar{G}^n(x; \zeta). \quad (1.38)$$

The marginal distribution of $X_{(1)}$ is given by

$$\begin{aligned} F_X(X_{(1)}) &= \sum_{n=1}^{\infty} \left[\frac{a_n\theta^n}{C(\theta)} - \frac{a_n\theta^n}{C(\theta)}\bar{G}^n(x; \zeta) \right] \\ &= 1 - \sum_{n=1}^{\infty} \frac{a_n\theta^n\bar{G}^n(x; \zeta)}{C(\theta)} \\ &= 1 - \frac{C(\theta\bar{G}(x; \zeta))}{C(\theta)}. \end{aligned} \quad (1.39)$$

The corresponding pdf is given by

$$f_X(X_{(1)}) = \frac{\partial F_X(X_{(1)})}{\partial x} = \frac{\theta g(x; \zeta) C'(\theta\bar{G}(x; \zeta))}{C(\theta)}, \quad (1.40)$$

where $C'(\theta) = \frac{\partial C(\theta)}{\partial \theta} = \sum_{n=1}^{\infty} a_n n \theta^{n-1}$.

Now suppose $X_{(n)} = \max(X_1, X_2, \dots, X_N)$ then the conditional distribution of $X_{(n)}$ given $N = n$ is given by

$$F(X_{(n)}|N = n) = G^n(x; \zeta), \quad (1.41)$$

where $n \in \mathbb{N}_{>1}$. The joint cdf of $X_{(n)}$ and N is given by

$$F(X_{(n)}|N = n)f(N) = \frac{a_n\theta^n}{C(\theta)}G^n(x; \zeta). \quad (1.42)$$

The marginal distribution of $X_{(n)}$ is given by

$$F_X(X_{(n)}) = \frac{C(\theta G(x; \zeta))}{C(\theta)} \quad (1.43)$$

The corresponding pdf is given by

$$f_X(X_{(n)}) = \frac{\partial F_X(X_{(n)})}{\partial x} = \frac{\theta g(x; \zeta) C'(\theta G(x; \zeta))}{C(\theta)}, \quad (1.44)$$

where $C'(\theta) = \frac{\partial C(\theta)}{\partial \theta} = \sum_{n=1}^{\infty} a_n n \theta^{n-1}$.

1.5 Objectives

The main objective is to critically review and apply the Marshall-Olkin-G Power Series (MO-GPS) distribution.

1.5.1 Specific Objectives

- i) To review the MO-GPS and its statistical properties.
- ii) To estimate parameters of the MO-GPS using the MLE technique.
- iii) To illustrate the performance of the reviewed model.
- iv) To apply the MO-GPS to complete and censored dataset.

1.6 Motivation

We were inspired to examine the MO-GPS family of distributions by the desired characteristics of the MO-G and the power series distributions, as well as the flexibility gained in modeling lifespan datasets exhibiting both monotonic and non-monotonic failure rates. The MO-GPS family of distributions is capable of producing desired density function forms, including left-skewed, right-skewed, symmetric, and reversed-J shapes. Since the distribution may be used with various degrees of skewed datasets and heavy-tailed datasets, we anticipate that academics would pay attention to it.

1.7 Outline of the thesis

This thesis is outlined as follows: In Chapter 2, the generalized reviewed distribution, namely the Marshall-Olkin-G Power Series (MO-GPS) distribution including the hrf and rhrf, quantile function and various sub-models and statistical properties are presented together with the maximum likelihood estimation for both censored and uncensored data. We discuss various estimation methods in Chapter 3. Chapter 4 has some special models of the MO-GPS family of distribution namely the Marshall-Olkin-G Logarithmic distribution and the Marshall-Olkin-G Binomial distribution together with their statistical properties. In Chapter 5, estimates of model parameters are compared to those of other models, a simulation study is conducted to evaluate the effectiveness of the maximum likelihood estimators of the MO-GPS

distribution, and applications of the suggested model to actual data are shown. A special case of the Marshall-Olkin-Weibull Logarithmic distribution is considered for simulations and applications to real life dataset. Summary and areas of further studies are presented in Chapter [6](#).



2. The Marshall-Olkin-G Power Series Family of Distributions

2.1 Introduction

In this chapter, we illustrate how the Marshall-Olkin-G Power Series (MO-GPS) family of distributions is obtained via the compounding method described in Section 1.4.5 where $G(x; \zeta)$ is the Marshall-Olkin-G (MO-G) distribution given in equation (1.32). Some of the important statistical properties of the MO-GPS family of distribution such as the cdf, pdf, survival function, hrf, rhf, quantile function, the linear representation of the pdf, moments, mgf, mean deviation, Bonferroni, Lorenz Curves, Rényi entropy, pdf of order statistic and stress strength model are presented.

2.2 Generating the MO-GPS family of distributions

Let N be a discrete random variable following a power series distribution with pmf given in equation (1.36). Given N , let $X_1, X_2, \dots, X_N$ be *iid* random variables from the Marshall Olkin-G distribution with pdf and cdf given in equation (1.33) and (1.32) respectively. Let $X_{(1)} = \min(X_1, X_2, \dots, X_N)$, then by equation (1.39) and (1.40), the cdf and pdf of the MO-GPS family of distributions is given by;

$$F_{MO-GPS}(x; \delta, \theta, \zeta) = 1 - \frac{C\left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)}\right]}{C(\theta)}, \quad (2.1)$$

and

$$f_{MO-GPS}(x; \delta, \theta, \zeta) = \frac{\theta \delta g(x; \zeta) C'\left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)}\right]}{[1 - \delta \bar{G}(x; \zeta)]^2 C(\theta)}, \quad (2.2)$$

respectively, for $\delta, \theta > 0$, $\bar{\delta} = 1 - \delta$, ζ is a vector of parameters and $C'(\theta)$ is the derivative of $C(\theta)$, that is

$$C'(\theta) = \sum_{n=0}^{\infty} (n+1) a_{n+1} \theta^n.$$

Thereafter, if a random variable X has the MO-GPS density, we write $X \sim MO - GPS(\delta, \theta, \zeta)$.

On the other hand, if we consider $X_{(n)} = \max(X_1, X_2, \dots, X_N)$, then by equation (1.43) and equation (1.44), the cdf and pdf of the MO-GPS family of distributions are given by

$$F_{MO-GPS}(x; \theta, \delta, \zeta) = \frac{C\left(\theta \left[1 - \frac{\delta \bar{G}(x; \zeta)}{1 - \delta G(x; \zeta)}\right]\right)}{C(\theta)}.$$

and

$$f_{MO-GPS}(x; \theta, \delta, \zeta) = \frac{\frac{\theta \delta g(x; \zeta)}{[1 - \delta \bar{G}(x; \zeta)]^2} C'\left(\theta \left[1 - \frac{\delta \bar{G}(x; \zeta)}{1 - \delta G(x; \zeta)}\right]\right)}{C'(\theta)}, \quad (2.3)$$

respectively for $\delta, \theta > 0$, $\bar{\delta} = 1 - \delta$, ζ is a vector of parameters and $C'(\theta)$ is the derivative of $C(\theta)$.

In this thesis we will consider the minimum statistic $X_{(1)}$ however, similar results can be obtained for the maximum statistic $X_{(n)}$.

Corollary 1: *The MO-G family of distributions defined in equation (1.32) is a limiting special case of the MO-GPS family of distributions when $\theta \rightarrow 0^+$.*

Proof:

$$\begin{aligned} \lim_{\theta \rightarrow 0^+} F_{MO-GPS}(x; \zeta) &= \lim_{\theta \rightarrow 0^+} \left(1 - \frac{\sum_{n=1}^{\infty} a_n \left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta G(x; \zeta)} \right]^n}{\sum_{n=1}^{\infty} a_n \theta^n} \right) \\ &= 1 - \lim_{\theta \rightarrow 0^+} \left(\frac{\frac{a_1 \theta \delta \bar{G}(x; \zeta)}{1 - \delta G(x; \zeta)} + \sum_{n=2}^{\infty} a_n \left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta G(x; \zeta)} \right]^n}{a_1 \theta + \sum_{n=2}^{\infty} a_n \theta^n} \right) \\ &= 1 - \lim_{\theta \rightarrow 0^+} \left(\frac{\frac{\delta \bar{G}(x; \zeta)}{1 - \delta G(x; \zeta)} + a_1^{-1} \sum_{n=2}^{\infty} a_n \theta^{n-1} \left[\frac{\delta \bar{G}(x; \zeta)}{1 - \delta G(x; \zeta)} \right]^n}{1 + a_1^{-1} \sum_{n=2}^{\infty} a_n \theta^{n-1}} \right) \\ &= 1 - \frac{\delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)}, \end{aligned}$$

since,

$$\lim_{\theta \rightarrow 0^+} a_1^{-1} \sum_{n=2}^{\infty} a_n \theta^{n-1} \left[\frac{\delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]^n = 0,$$

$$\lim_{\theta \rightarrow 0^+} a_1^{-1} \sum_{n=2}^{\infty} a_n \theta^{n-1} = 0$$

and

$$\lim_{\theta \rightarrow 0^+} \frac{\delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} = \frac{\delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)}.$$

2.3 The survival, hazard and reverse hazard rate function

The survival, hazard and reverse hazard rate function of the MO-GPS family of distributions are given by;

$$S_{MO-GPS}(x; \delta, \theta, \zeta) = \frac{C \left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]}{C(\theta)}, \quad (2.4)$$

$$h_{MO-GPS}(x; \delta, \theta, \zeta) = \frac{\theta \delta g(x; \zeta) C' \left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]}{[1 - \delta \bar{G}(x; \zeta)]^2 C \left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]}, \quad (2.5)$$

and

$$\tau_{MO-GPS}(x; \delta, \theta, \zeta) = \frac{\theta \delta g(x; \zeta) C' \left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]}{[1 - \delta \bar{G}(x; \zeta)]^2 \left(C(\theta) - C \left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right] \right)}, \quad (2.6)$$

respectively.

2.4 Quantile Function

The quantile function $Q(u)$ is obtained by solving the non-linear equation;

$$F_{MO-GPS}(Q(u)) = 1 - \frac{C \left[\frac{\theta \delta \bar{G}(Q(u))}{1 - \delta \bar{G}(Q(u))} \right]}{C(\theta)} = u,$$

for $0 \leq u < 1$. Therefore,

$$\begin{aligned} C\left[\frac{\theta\delta\bar{G}(Q(u))}{1-\delta\bar{G}(Q(u))}\right] &= (1-u)C(\theta) \\ \frac{\theta\delta\bar{G}(Q(u))}{1-\delta\bar{G}(Q(u))} &= C^{-1}((1-u)C(\theta)) \\ G(Q(u)) &= 1 - \frac{C^{-1}((1-u)C(\theta))}{\theta\delta + \delta\left[C^{-1}((1-u)C(\theta))\right]}. \end{aligned}$$

Thus, the quantile function of the MO-GPS family of distributions is given by;

$$Q(u) = G^{-1}\left\{1 - \frac{C^{-1}((1-u)C(\theta))}{\theta\delta + \delta\left[C^{-1}((1-u)C(\theta))\right]}\right\} \quad (2.7)$$

for $0 \leq u < 1$.

Consequently, we can generate random numbers from the MO-GPS family of distributions based on equation (2.7) for specific baseline cdf $G(\cdot)$.

2.5 Linear Representation of the Density Function

In this section, we present the pdf of the MO-GPS family of distributions as an infinite linear combination of exponentiated-G densities. We use the general results for series expansion of distributions to derive the density series expansion of the MO-GPS family of distributions. We can write equation (2.2) as

$$\begin{aligned} f_{MO-GPS}(x; \delta, \theta, \zeta) &= \theta\delta g(x; \zeta)[1 - \delta\bar{G}(x; \zeta)]^{-2} \frac{C'\left[\frac{\theta\delta\bar{G}(x; \zeta)}{1-\delta\bar{G}(x; \zeta)}\right]}{C(\theta)} \\ &= \theta\delta g(x; \zeta)[1 - \delta\bar{G}(x; \zeta)]^{-2} \sum_{n=1}^{\infty} \frac{na_n}{C(\theta)} \left[\frac{\theta\delta\bar{G}(x; \zeta)}{1-\delta\bar{G}(x; \zeta)}\right]^{n-1} \\ &= g(x; \zeta) \sum_{n=1}^{\infty} \frac{na_n\theta^n\delta^n}{C(\theta)} (1 - G(x; \zeta))^{n-1} [1 - \delta\bar{G}(x; \zeta)]^{-(n+1)} \\ &= g(x; \zeta) \sum_{n=0}^{\infty} \frac{(n+1)a_{n+1}\theta^{n+1}\delta^{n+1}}{C(\theta)} (1 - G(x; \zeta))^n \\ &\times [1 - \delta\bar{G}(x; \zeta)]^{-(n+2)} \end{aligned}$$

now, for $\delta \in (0, 1)$ using the following expansions;

$$\begin{aligned} [1 - \bar{\delta}\bar{G}(x; \zeta)]^{-(n+2)} &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k+1}{k} \bar{\delta}^k (1 - G(x; \zeta))^k \\ (1 - G(x; \zeta))^{n+k} &= \sum_{p=0}^{\infty} (-1)^p \binom{n+k}{p} G^p(x; \zeta) \end{aligned}$$

we have,

$$\begin{aligned} f_{MO-GPS}(x; \delta, \theta, \zeta) &= \sum_{n,k,p=0}^{\infty} \frac{(n+1)a_{n+1}\theta^{n+1}\delta^{n+1}(-1)^{k+p}\bar{\delta}^k}{C(\theta)} \binom{n+k+1}{k} \\ &\times \binom{n+k}{p} g(x; \zeta) G^p(x; \zeta) \\ &= \sum_{p=0}^{\infty} d_{p+1}^* g_{p+1}^*(x; \zeta) \end{aligned} \quad (2.8)$$

where

$$g_{p+1}^*(x; \zeta) = (p+1)g(x; \zeta)G^p(x; \zeta) \quad (2.9)$$

is an exponentiated-G (Exp-G) distribution with power parameter $(p+1)$ and

$$d_{p+1}^* = \sum_{n,k=0}^{\infty} \frac{(n+1)a_{n+1}(\theta\delta)^{n+1}(-1)^{k+p}\bar{\delta}^k}{(p+1)C(\theta)} \binom{n+k+1}{k} \binom{n+k}{p}. \quad (2.10)$$

For $\delta > 1$, using the following expansions;

$$\begin{aligned} \left[\delta \left(1 - \frac{\delta-1}{\delta} G(x; \zeta) \right) \right]^{-(n+2)} &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k+1}{k} \delta^{-(n+2)} \left(\frac{\delta-1}{\delta} \right)^k G^k(x; \zeta) \\ (1 - G(x; \zeta))^{n+k} &= \sum_{p=0}^{\infty} (-1)^p \binom{n+k}{p} G^p(x; \zeta) \end{aligned}$$

we have,

$$\begin{aligned}
f_{MO-GPS}(x; \delta, \theta, \zeta) &= \sum_{n,k,p=0}^{\infty} \frac{(n+1)a_{n+1}\theta^{n+1}\delta^{n+1}(-1)^{k+p}}{C(\theta)} \delta^{-(n+2)} \left(\frac{\delta-1}{\delta}\right)^k \\
&\times \binom{n+k+1}{k} \binom{n+k}{p} g(x; \zeta) G^{p+k}(x; \zeta) \\
&= \sum_{p,k=0}^{\infty} z_{p+k+1}^* g_{p+k+1}^*(x; \zeta)
\end{aligned} \tag{2.11}$$

where

$$g_{p+k+1}^*(x; \zeta) = (p+k+1)g(x; \zeta)G^{p+k}(x; \zeta) \tag{2.12}$$

is an exponentiated-G (Exp-G) distribution with power parameter $(p+k+1)$ and

$$z_{p+k+1}^* = \sum_{n=0}^{\infty} \frac{(n+1)a_{n+1}(\theta\delta)^{n+1}(-1)^{k+p}}{(p+k+1)C(\theta)} \delta^{-(n+2)} \left(\frac{\delta-1}{\delta}\right)^k \binom{n+k+1}{k} \binom{n+k}{p}. \tag{2.13}$$

Equation (2.8) and (2.11) are similar to equation (21) and (22) from Cordeiro et. al. [14] for $\delta \in (0, 1)$ and $\delta > 1$ respectively.

Moreover, for $\delta = 1$,

$$f_{MO-GPS}(x; \delta, \theta, \zeta) = g(x; \zeta) \sum_{n=0}^{\infty} \frac{(n+1)a_{n+1}\theta^{n+1}}{C(\theta)} (1 - G(x; \zeta))^n$$

since $[1 - \bar{\delta}G(x; \zeta)]^{-(n+2)} = 1$. now, using the following expansion;

$$(1 - G(x; \zeta))^n = \sum_{p=0}^{\infty} (-1)^p \binom{n}{p} G^p(x; \zeta),$$

equation (2.14) can be expressed as

$$f_{MO-GPS}(x; \delta, \theta, \zeta) = \sum_{p=0}^{\infty} w_{p+1}^* g_{p+1}^*(x; \zeta) \tag{2.14}$$

where

$$w_{p+1}^* = \sum_{n=0}^{\infty} \frac{(-1)^p (n+1)a_{n+1}\theta^{n+1}}{(p+1)C(\theta)} \binom{n}{p} \tag{2.15}$$

and $g_{p+1}^*(x; \zeta)$ is given in equation (2.9) and is an exponentiated-G (Exp-G) distribution with power parameter $(p + 1)$.

Thus, the statistical properties of the MO-GPS class of distributions including r^{th} moment, incomplete moment, the moment generating function, mean deviations, Bonferroni curves and Lorenz curves can be obtained directly from those of the Exp-G family of the distributions using equation (2.8), (2.11) and (2.14) for $\delta \in (0, 1)$, $\delta > 1$ and $\delta = 1$ respectively.

2.6 Moments and Generating Function

In this section, we present the r^{th} moment, incomplete moment PWM and mgf of the MO-GPS family of distributions.

2.6.1 The r^{th} Moment

Let Y_α , denote the Exp-G random variable with power parameter (α) . Then the r^{th} moment of Y , say μ'_r is given by

$$\mu'_r = E(Y^r) = \sum_{p=0}^{\infty} \beta_\alpha^* E(Y_\alpha^r), \quad (2.16)$$

where

$$\beta_\alpha^* = d_{p+1}^*, \quad \text{for } \delta \in (0, 1) \text{ and } \alpha = p + 1, \quad (2.17)$$

$$\beta_\alpha^* = z_{p+k+1}^*, \quad \text{for } \delta > 1 \text{ and } \alpha = p + k + 1 \quad (2.18)$$

and

$$\beta_\alpha^* = w_{p+1}^*, \quad \text{for } \delta = 1 \text{ and } \alpha = p + 1 \quad (2.19)$$

where d_{p+1}^* is as given in equation (2.10). The mean, variance, skewness and kurtosis can be easily derived from equation (2.16).

2.6.2 Incomplete Moment

The r^{th} incomplete moment can be obtained as follows;

$$I_X(t) = \int_0^t x^r f_{MO-GPS}(x; \delta, \theta, \zeta) dx = \sum_{p=0}^{\infty} \beta_{\alpha}^* I_{\alpha}(t), \quad (2.20)$$

where $I_{\alpha}(t) = \int_0^t x^r g_{\alpha}^*(x; \zeta) dx$ and β_{α}^* is as defined in equations (2.17), (2.18) and (2.19) for $\delta \in (0, 1)$, $\delta > 1$ and $\delta = 1$ respectively.

2.6.3 Probability Weighted Moments

The $(i, j)^{th}$ PWM of a random variable $X \sim MO - GPS(\delta, \theta, \zeta)$, say $(\varsigma_{i,j})$ is given by

$$\varsigma_{i,j} = \int_{-\infty}^{\infty} x^i f_{MO-GPS}(x; \delta, \theta, \zeta) F_{MO-GPS}(x; \delta, \theta, \zeta)^j dx. \quad (2.21)$$

By making use of expansions in Section 2.5, for $\delta \in (0, 1)$, it can be easily shown that;

$$f(x; \delta, \theta, \zeta) F^j(x; \delta, \theta, \zeta) = \sum_{p=0}^{\infty} h_{p+1}^* g_{p+1}^*(x; \zeta) \quad (2.22)$$

where $g_{p+1}^*(x; \zeta)$ is given in equation (2.9) and

$$\begin{aligned} h_{p+1}^* &= \sum_{q,i,m,n=0}^{\infty} (-1)^{i+p+q} \frac{(n+1)a_{n+1}(\delta\theta)^{n+i+1}\bar{\delta}^q}{(p+1)[C(\theta)]^{i+1}} d_{r,i} \\ &\times \binom{n+i+q+1}{q} \binom{n+i+q}{p} \binom{j}{i}, \end{aligned} \quad (2.23)$$

and $d_{r,i} = (rb_0)^{-1} \sum_{h=1}^r [i(h+1) - r] b_n d_{r-h,i}$ and $d_{0,i} = b_0^i$. As such, the PWM of X for MO-GPS for $\delta \in (0, 1)$ is given by

$$\varsigma_{i,j} = \sum_{p=0}^{\infty} h_{p+1}^* \int_{-\infty}^{\infty} x^i g_{p+1}^*(x; \zeta) dx = \sum_{p=0}^{\infty} h_{p+1}^* E[Y_{p+1}^i], \quad (2.24)$$

where $E[Y_{p+1}^i]$ is the i^{th} moment of Exp-G density with power parameter $(p+1)$ and $g_{p+1}^*(x; \zeta)$ is given in equation (2.9).

Now, for $\delta > 1$, the PWM of X for MO-GPS for $\delta > 1$ is given by

$$\varsigma_{i,j} = \sum_{p,k=0}^{\infty} j_{p+k+1}^* \int_{-\infty}^{\infty} x^i g_{p+k+1}^*(x; \zeta) dx = \sum_{p,k=0}^{\infty} j_{p+k+1}^* E[Y_{p+k+1}^i], \quad (2.25)$$

where $E[Y_{p+k+1}^i]$ is the i^{th} moment of Exp-G density with power parameter $(p+k+1)$, $g_{p+k+1}^*(x; \zeta)$ is given in equation (2.12),

$$\begin{aligned} j_{p+k+1}^* &= \sum_{n,i,r=0}^{\infty} (-1)^{i+p+k} \frac{(n+1)a_{n+1}(\delta\theta)^{n+i+1}\bar{\delta}^k}{(p+k+1)[C(\theta)]^{i+1}} d_{r,i} \\ &\times \binom{n+i+k+1}{k} \binom{n+i+k}{p} \binom{j}{i} \delta^{-(n+i+2)} \left(\frac{\delta-1}{\delta}\right)^k, \end{aligned}$$

and $d_{r,i} = (rb_0)^{-1} \sum_{h=1}^r [i(h+1) - r] b_n d_{r-h,i}$ and $d_{0,i} = b_0^i$.

Moreover, for $\delta = 1$, the PWM of X for MO-GPS for $\delta = 1$ is given by

$$\varsigma_{i,j} = \sum_{p=0}^{\infty} u_{p+1}^* \int_{-\infty}^{\infty} x^i g_{p+1}^*(x; \zeta) dx = \sum_{p=0}^{\infty} u_{p+1}^* E[Y_{p+1}^i], \quad (2.26)$$

where $E[Y_{p+1}^i]$ is the i^{th} moment of Exp-G density with power parameter $(p+1)$, $g_{p+1}^*(x; \zeta)$ is given in equation (2.9),

$$\begin{aligned} u_{p+1}^* &= \sum_{n,i,r=0}^{\infty} (-1)^{i+p} \frac{(n+1)a_{n+1}\theta^{n+i+1}}{(p+1)[C(\theta)]^{i+1}} d_{r,i} \\ &\times \binom{n+i}{p} \binom{j}{i}, \end{aligned}$$

and $d_{r,i} = (rb_0)^{-1} \sum_{h=1}^r [i(h+1) - r] b_n d_{r-h,i}$ and $d_{0,i} = b_0^i$.

2.6.4 Moment Generating Function

The moment generating function (mgf) $M_Y(t) = E(e^{tY})$ of Y is given by

$$M_Y(t) = \sum_{p=0}^{\infty} \beta_{\alpha}^* M_{\alpha}(t), \quad (2.27)$$

where $M_{\alpha}(t)$ is the mgf of Y_{α} . Hence, $M_Y(t)$ can be determined from the Exp-G generating function and β_{α}^* is as defined in equations (2.17), (2.18) and (2.19).

2.7 Mean deviation, Bonferroni and Lorenz Curves

The mean deviation about the mean and about the median are presented in this section together with the Bonferroni and Lorenz Curves for the MO-GPS.

2.7.1 Mean deviation about the mean

Let $X \sim MO - GPS(\delta, \theta, \zeta)$, the mean deviation about the mean is given by

$$\begin{aligned}\alpha_1(x) &= 2\mu F_{MO-GPS}(\mu) - 2 \int_0^\mu x f_{MO-GPS}(x; \delta, \theta, \zeta) dx \\ &= 2\mu F_{MO-GPS}(\mu) - 2 \int_0^\mu x \beta_\alpha^* g_\alpha^*(x; \zeta) dx,\end{aligned}\quad (2.28)$$

where $\mu = E(X)$, g_α^* is as defined in equation (2.9) for $\delta \in (0, 1)$ and $\delta = 1$, g_α^* is as defined in equation (2.12) for $\delta > 1$ and β_α^* is as defined in equations (2.17), (2.18) and (2.19).

2.7.2 Mean deviation about the median

Let $X \sim MO - GPS(\delta, \theta, \zeta)$, the mean deviation about the median is given by

$$\begin{aligned}\alpha_2(x) &= \mu - 2 \int_0^m x f_{MO-GPS}(x; \delta, \theta, \zeta) dx \\ &= \mu - 2 \int_0^m x \beta_\alpha^* g_\alpha^*(x; \zeta) dx,\end{aligned}\quad (2.29)$$

where $\mu = E(X)$, g_α^* is the same as equation (2.9) for $\delta \in (0, 1)$ and $\delta = 1$, g_α^* is the same as equation (2.12) for $\delta > 1$ and β_α^* is as defined in equations (2.17), (2.18) and (2.19).

2.7.3 Bonferroni curve

Using equation (1.12), the Bonferroni curve of the MO-GPS family of distributions is defined by

$$\begin{aligned}B(u) &= \frac{1}{u\mu} \int_0^v x f_{MO-GPS}(x; \delta, \theta, \zeta) dx \\ &= \frac{1}{u\mu} \int_0^v \sum_{p=0}^{\infty} x \beta_\alpha^* g_\alpha^*(x; \zeta) dx\end{aligned}\quad (2.30)$$

where $\mu = E(X)$, $v = Q(u)$, g_α^* is the same as equation (2.9) for $\delta \in (0, 1)$ and $\delta = 1$, g_α^* is the same as equation (2.12) for $\delta > 1$ and β_α^* is as defined in equations (2.17),

(2.18) and (2.19).

2.7.4 Lorenz curve

Using equation (1.13), the Lorenz curve of the MO-GPS family of distributions is defined by

$$\begin{aligned} L(u) &= \frac{1}{\mu} \int_0^v x f_{MO-GPS}(x; \delta, \theta, \zeta) dx \\ &= \frac{1}{\mu} \int_0^v \sum_{p=0}^{\infty} x \beta_{\alpha}^* g_{\alpha}^*(x; \zeta) dx \end{aligned} \quad (2.31)$$

where $\mu = E(X)$, $v = Q(u)$, g_{α}^* is the same as equation (2.9) for $\delta \in (0, 1)$ and $\delta = 1$, g_{α}^* is the same as equation (2.12) for $\delta > 1$ and β_{α}^* is as defined in equations (2.17), (2.18) and (2.19).

2.8 Residual and reverse residual life functions

In this Section, we present the residual and reverse residual life functions of the MO-GPS family of distribution.

2.8.1 Residual life function

Corollary 2: *If n is an integer value greater than 1 ($n > 1$) and $x > t$, then the n^{th} moment of residual life is given by;*

$$a_n^*(t) = \frac{1}{\bar{G}(x; \zeta)} \sum_{p=0}^{\infty} \beta_{\alpha}^* \sum_{m=0}^{\infty} \binom{n}{m} (-t)^m \int_t^{\infty} x^{n-m} g_{\alpha}^*(x; \zeta) dx. \quad (2.32)$$

where g_{α}^* is the same as equation (2.9) for $\delta \in (0, 1)$ and $\delta = 1$ with power parameter (α), g_{α}^* is the same as equation (2.12) for $\delta > 1$ with power parameter (α), β_{α}^* is as defined in equations (2.17), (2.18) and (2.19) and $\bar{G}(x; \zeta)$ is the survival function.

Proof: The n^{th} moment of residual life of X is given by;

$$a_n^*(t) = E\{(x - t)^n | x > t\} = \frac{1}{\bar{G}(x; \zeta)} \int_t^{\infty} (x - t)^n f_{MO-GPS}(x; \delta, \theta, \zeta) dx. \quad (2.33)$$

Since n is an integer value, we have the following general Binomial series expansion as

$$(a - b)^n = \sum_{m=0}^{\infty} \binom{n}{m} a^{n-m} (-b)^m. \quad (2.34)$$

Using the series expansion in equation (2.34) and infinite mixture representation in equation (2.8) and changing the order of summation and integration, we obtain

$$a_n^*(t) = \frac{1}{\bar{G}(x; \zeta)} \sum_{p=0}^{\infty} \beta_{\alpha}^* \sum_{m=0}^{\infty} \binom{n}{m} (-t)^m \int_t^{\infty} x^{n-m} g_{\alpha}^*(x; \zeta) dx. \quad (2.35)$$

The mean residual life or the life expectation at age “t” defined by

$$a_1^*(t) = E((X - t)/X > t) = \frac{1}{\bar{G}(x; \zeta)} \sum_{p=0}^{\infty} \beta_{\alpha}^* \sum_{m=0}^{\infty} \binom{1}{m} (-t)^m \int_t^{\infty} x^{1-m} g_{\alpha}^*(x; \zeta) dx$$

represents the specified additional life length for a unit which is alive at age “t”.

2.8.2 Reverse residual life function

Corollary 3: If n is an integer value greater than 1 ($n > 1$) and $x > t$, then the n^{th} moment of reverse residual life is given by;

$$b_n^*(t) = \frac{1}{G(x; \zeta)} \sum_{p=0}^{\infty} \beta_{\alpha}^* \sum_{m=0}^{\infty} \binom{n}{m} (t)^{n-m} (-1)^m \int_0^t x^m g_{\alpha}^*(x; \zeta) dx, \quad (2.36)$$

where g_{α}^* is the same as equation (2.9) for $\delta \in (0, 1)$ and $\delta = 1$ with power parameter (α), g_{α}^* is the same as equation (2.12) for $\delta > 1$ with power parameter (α) and β_{α}^* is as defined in equations (2.17), (2.18) and (2.19).

Proof: The n^{th} moment of reverse residual life is given as;

$$b_n^*(t) = E((t - x)^n / x \leq t) = \frac{1}{G(x; \zeta)} \int_0^t (t - x)^n f_{MO-GPS}(x; \delta, \theta, \zeta) dx. \quad (2.37)$$

Now, using the series expansion given in equation (2.34) and infinite mixture representation in equation (2.8) and changing the order of summation and integration, the results follow,

$$b_n^*(t) = \frac{1}{G(x; \zeta)} \sum_{p=0}^{\infty} \beta_{\alpha}^* \sum_{m=0}^{\infty} \binom{n}{m} (t)^{n-m} (-1)^m \int_0^t x^m g_{\alpha}^*(x; \zeta) dx. \quad (2.38)$$

Mean inactivity time (MIT) or mean waiting time (MWT), also called the mean

reversed residual life function is given by

$$b_1^*(t) = E((X - t)/X \leq t) = \frac{1}{G(x; \zeta)} \sum_{p=0}^{\infty} \beta_{\alpha}^* \sum_{m=0}^{\infty} \binom{1}{m} (t)^{1-m} (-1)^m \int_0^t x^m g_{\alpha}^*(x; \zeta) dx,$$

and it represents the waiting elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.

2.9 Stress-Strength Model

Let X and Y be independent random variables such that $X \sim MO-GPS(\delta_1, \theta_1, \zeta_1)$ and $Y \sim MO-GPS(\delta_2, \theta_2, \zeta_2)$ respectively, the stress-strength model of the MO-GPS family of distributions is given by

$$\begin{aligned} R = P(Y < X) &= \int_0^{\infty} \int_0^x f_{MO-GPS}(x; \delta_1, \theta_1, \zeta_1) f_{MO-GPS}(y; \delta_2, \theta_2, \zeta_2) dy dx \\ &= \int_{x=0}^{\infty} f_{MO-GPS}(x; \delta_1, \theta_1, \zeta_1) F_{MO-GPS}(x; \delta_2, \theta_2, \zeta_2) dx. \end{aligned}$$

The reliability parameter for MO-GPS distribution can be expressed as

$$R = \int_{x=0}^{\infty} \frac{\theta_1 \delta_1 g(x; \zeta_1) C' \left[\frac{\theta_1 \delta_1 \bar{G}(x; \zeta_1)}{1 - \delta_1 \bar{G}(x; \zeta_1)} \right]}{[1 - \delta_1 \bar{G}(x; \zeta_1)]^2 C(\theta_1)} \left(1 - \frac{C \left[\frac{\theta_2 \delta_2 \bar{G}(x; \zeta_2)}{1 - \delta_2 \bar{G}(x; \zeta_2)} \right]}{C(\theta_2)} \right) dx. \quad (2.39)$$

2.10 Rényi Entropy

The Rényi entropy of the MO-GPS family of distributions is given by

$$\begin{aligned} I_R(\nu) &= \frac{1}{1 - \nu} \log \left(\int_0^{\infty} \left[\theta \delta g(x; \zeta) [1 - \delta \bar{G}(x; \zeta)]^{-2} \frac{C' \left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]}{C(\theta)} \right]^{\nu} dx \right) \\ &= \frac{1}{1 - \nu} \log \left(\int_0^{\infty} \left(\frac{\theta \delta g(x; \zeta)}{C(\theta)} \right)^{\nu} [1 - \delta \bar{G}(x; \zeta)]^{-2\nu} \right. \\ &\quad \times \left. \left(\left[\sum_{n=0}^{\infty} (n+1) a_{n+1} \left[\frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]^n \right]^{\nu} dx \right) \right), \nu \neq 1, \nu > 0. \end{aligned} \quad (2.40)$$

We then apply the results of a power series raised to a positive integer, with $a_n = (n+1)a_{n+1}\theta^n$, see Gradshteyn & Ryzhik [23] and the following expansions for $\delta \in$

$(0, 1)$;

$$\begin{aligned}(1 - \bar{\delta} \bar{G}(x; \zeta))^{-(2\nu+ns)} &= \sum_{q=0}^{\infty} (-1)^q \binom{2\nu+ns+q-1}{q} \bar{\delta}^q \bar{G}^q(x; \zeta) \\ (1 - G(x; \zeta))^{ns+q} &= \sum_{p=0}^{\infty} (-1)^p \binom{ns+q}{p} G^p(x; \zeta)\end{aligned}$$

we have the Rényi Entropy for the MO-GPS distribution for $\delta \in (0, 1)$ given by

$$\begin{aligned}I_R(\nu) &= \frac{1}{1-\nu} \log \left[\sum_{n,s,k,p,q=0}^{\infty} b_{s,k} \left(\frac{\theta \delta}{C(\theta)} \right)^{\nu} (\delta \theta)^{ns} \bar{\delta}^q (-1)^{p+q} \binom{ns+q}{p} \right. \\ &\quad \times \left. \binom{2\nu+ns+q-1}{q} \int_0^{\infty} g^{\nu}(x; \zeta) G^p(x; \zeta) dx \right] \quad (2.41)\end{aligned}$$

where $b_{s,k} = (sa_0)^{-1} \sum_{l=1}^s [k(l+1) - s] a_l b_{s-l,k}$, $b_{0,k} = a_0^k$, $\nu \neq 1$ and $\nu > 0$.

Now, for $\delta > 1$, using the following expansions;

$$\begin{aligned}\left[\delta \left(1 - \frac{\delta-1}{\delta} G(x; \zeta) \right) \right]^{-(2\nu+ns)} &= \sum_{q=0}^{\infty} (-1)^q \binom{2\nu+ns+q-1}{q} \delta^{-(2\nu+ns)} \left(\frac{\delta-1}{\delta} \right)^q G^q(x; \zeta) \\ (1 - G(x; \zeta))^{ns+q} &= \sum_{p=0}^{\infty} (-1)^p \binom{ns+q}{p} G^p(x; \zeta)\end{aligned}$$

we have the Rényi Entropy for the MO-GPS distribution for $\delta > 1$ given by

$$\begin{aligned}I_R(\nu) &= \frac{1}{1-\nu} \log \left[\sum_{n,s,k,p,q=0}^{\infty} b_{s,k} \left(\frac{\theta \delta}{C(\theta)} \right)^{\nu} (\delta \theta)^{ns} \bar{\delta}^q (-1)^{p+q} \binom{ns+q}{p} \right. \\ &\quad \times \left. \binom{2\nu+ns+q-1}{q} \delta^{-(2\nu+ns)} \left(\frac{\delta-1}{\delta} \right)^q \int_0^{\infty} g^{\nu}(x; \zeta) G^{p+q}(x; \zeta) dx \right] \quad (2.42)\end{aligned}$$

where $b_{s,k} = (sa_0)^{-1} \sum_{l=1}^s [k(l+1) - s] a_l b_{s-l,k}$, $b_{0,k} = a_0^k$, $\nu \neq 1$ and $\nu > 0$.

Moreover, for $\delta = 1$, using the following expansion;

$$(1 - G(x; \zeta))^{ns} = \sum_{p=0}^{\infty} (-1)^p \binom{ns}{p} G^p(x; \zeta)$$

the Rényi Entropy for the MO-GPS distribution for $\delta = 1$ given by

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{n,s,k,p=0}^{\infty} b_{s,k} \left(\frac{\theta}{C(\theta)} \right)^{\nu} \theta^{ns} (-1)^p \binom{ns}{p} \right. \\ \left. \times \int_0^{\infty} g^{\nu}(x; \zeta) G^p(x; \zeta) dx \right] \quad (2.43)$$

where $b_{s,k} = (sa_0)^{-1} \sum_{l=1}^s [k(l+1) - s] a_l b_{s-l,k}$, $b_{0,k} = a_0^k$, $\nu \neq 1$ and $\nu > 0$.

2.11 Stochastic Ordering

The concept of stochastic ordering is regularly used to display the ordering mechanism in probability models. For more details on stochastic ordering see Shaked and Shanthikumar [39]. A random variable X_1 is said to be stochastically smaller than X_2 , ($X_1 <_{st} X_2$) in the

- Stochastic order ($X_1 <_{st} X_2$) if $F_{X_1}(x) \geq F_{X_2}(x)$ for all x .
- Hazard rate order ($X_1 <_{hr} X_2$) if $h_{X_1}(x) \geq h_{X_2}(x)$ for all x .
- Likelihood ratio order ($X_1 <_{lr} X_2$) if $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ decreasing in x .

The stochastic orders given above are associated to each other and it holds that $X_1 <_{lr} X_2 \Rightarrow X_1 <_{hr} X_2 \Rightarrow X_1 <_{st} X_2$. Now, let $X_1 \sim MO - GPS(\delta_1, \theta, \zeta)$ and $X_2 \sim MO - GPS(\delta_2, \theta, \zeta)$. If $\delta_1 < \delta_2$, then the ratio $\frac{f(x; \delta_1, \theta, \zeta)}{g(x; \delta_2, \theta, \zeta)}$ is decreasing in x .

Then according to the definition of likelihood ratio ordering, is given by

$$\frac{f(x; \delta_1, \theta, \zeta)}{g(x; \delta_2, \theta, \zeta)} = \frac{\delta_1(1 - \bar{\delta}_2 \bar{G}(x; \zeta))^2 C' \left[\theta \frac{\delta_1 \bar{G}(x; \zeta)}{1 - \bar{\delta}_1 \bar{G}(x; \zeta)} \right]}{\delta_2(1 - \bar{\delta}_1 \bar{G}(x; \zeta))^2 C' \left[\theta \frac{\delta_2 \bar{G}(x; \zeta)}{1 - \bar{\delta}_2 \bar{G}(x; \zeta)} \right]}.$$

Taking derivative with respect to x , we obtain

$$\frac{\partial}{\partial x} \left(\frac{f(x; \delta_1, \theta, \zeta)}{g(x; \delta_2, \theta, \zeta)} \right) = \frac{\delta_1}{\delta_2} (\delta_2 - \delta_1) \frac{(1 - \bar{\delta}_2 W)}{(1 - \bar{\delta}_1 W)^3} W' C'' \left[\frac{(1 - \delta_2 W)}{(1 - \delta_1 W)^3} W' \right],$$

where $W = \bar{G}(x; \zeta)$ and $W' = \frac{dW}{dx}$.

If $\delta_1 < \delta_2$, then $\frac{\partial}{\partial x} \left(\frac{f(x; \delta_1, \theta, \zeta)}{g(x; \delta_2, \theta, \zeta)} \right) < 0$, which implies that $X_1 <_{lr} X_2$. Thus, X_1 and X_2 are stochastically ordered.

2.12 Order Statistics

Let $X_1, X_2, \dots, X_m$ be independent and identically distributed MO-GPS random variables. Then, the pdf of the k^{th} order statistics ($X_{k:m}$) can be written as

$$\begin{aligned} f_{k:m}(x) &= \frac{m!}{(k-1)!(m-k)!} \sum_{l=0}^{\infty} (-1)^l \binom{m-k}{l} f_{MO-GPS}(x; \zeta) F_{MO-GPS}^{k+l-1}(x; \zeta) \\ &= \frac{m!}{(k-1)!(m-k)!} \sum_{l,n=0}^{\infty} (-1)^l \binom{m-k}{l} g(x; \zeta) \frac{(n+1)a_{n+1}\theta^{n+1}\delta^{n+1}}{C(\theta)} \\ &\quad \times (\bar{G}(x; \zeta))^n [1 - \bar{\delta}\bar{G}(x; \zeta)]^{-(n+2)} \left[1 - \frac{C\left[\theta \frac{\delta\bar{G}(x; \zeta)}{1-\bar{\delta}\bar{G}(x; \zeta)}\right]}{C(\theta)} \right]^{k+l-1} \end{aligned} \quad (2.44)$$

Using the Binomial Series Expansion

$$\left(1 - \frac{C(\theta D)}{C(\theta)} \right)^{k+l-1} = \sum_{i=0}^{\infty} (-1)^i \binom{k+l-1}{i} \left[\frac{C(\theta D)}{C(\theta)} \right]^i$$

where $D = \frac{\delta\bar{G}(x; \zeta)}{1-\bar{\delta}\bar{G}(x; \zeta)}$ and power series raised to a positive integer result (see Gradshteyn and Ryzhik [23]) for more details, we have

$$\left[\frac{C(\theta D)}{C(\theta)} \right]^i = \sum_{r=0}^{\infty} \frac{d_{r,i}\theta^i}{C^i(\theta)} D^i$$

where $d_{r,i} = (rb_0)^{-1} \sum_{h=1}^r [i(h+1) - r] b_n d_{r-h,i}$ and $d_{0,i} = b_0^i$ we get

$$\begin{aligned} f_{k:m}(x) &= \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r=0}^{\infty} (-1)^{l+i} \binom{m-k}{l} g(x; \zeta) d_{r,i} \\ &\quad \times \frac{(n+1)a_{n+1}(\delta\theta)^{n+i+1}}{[C(\theta)]^{i+1}} \binom{k+l-1}{i} (\bar{G}(x; \zeta))^{n+i} \\ &\quad \times (1 - \bar{\delta}\bar{G}(x; \zeta))^{-(n+i+2)}. \end{aligned}$$

For $\delta \in (0, 1)$, using the following expansions;

$$\begin{aligned} (1 - \bar{\delta}\bar{G}(x; \zeta))^{-(n+i+2)} &= \sum_{q=0}^{\infty} (-1)^q \binom{n+i+q+1}{q} \bar{\delta}^q \bar{G}^q(x; \zeta) \\ (\bar{G}(x; \zeta))^{n+i+q} &= \sum_{p=0}^{\infty} (-1)^p \binom{n+i+q}{p} G^p(x; \zeta). \end{aligned}$$

Thus the distribution of the order statistics of the MO-GPS family of distributions for $\delta \in (0, 1)$ can be written as

$$f_{k:m}(x) = \sum_{p=0}^{\infty} Q_{p+1}^* g_{p+1}^*(x; \zeta) \quad (2.45)$$

where $g_{p+1}^*(x; \zeta)$ is given in equation (2.9) and

$$\begin{aligned} Q_{p+1}^* &= \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r,q=0}^{\infty} (-1)^{l+i+p+q} \frac{(n+1)a_{n+1}(\delta\theta)^{n+i+1}\bar{\delta}^q}{(p+1)[C(\theta)]^{i+1}} d_{r,i} \\ &\times \binom{n+i+q+1}{q} \binom{m-k}{l} \binom{n+i+q}{p} \binom{k+l-1}{i}. \end{aligned}$$

Thus, the pdf of the order statistic of the MO-GPS can be written as a linear combination of exponentiated-G (Exp-G) with power parameter $(p+1)$.

Now for $\delta > 1$, using the following expansions;

$$\begin{aligned} \left[\delta \left(1 - \frac{\delta-1}{\delta} G(x; \zeta) \right) \right]^{-(n+i+2)} &= \sum_{q=0}^{\infty} (-1)^q \binom{n+i+q+1}{q} \delta^{-(n+i+2)} \left(\frac{\delta-1}{\delta} \right)^q G^q(x; \zeta) \\ (\bar{G}(x; \zeta))^{n+i+q} &= \sum_{p=0}^{\infty} (-1)^p \binom{n+i+q}{p} G^p(x; \zeta). \end{aligned}$$

Thus the distribution of the order statistics of the MO-GPS for $\delta > 1$ can be written as

$$f_{k:m}(x) = \sum_{p=0}^{\infty} P_{p+q+1}^* g_{p+q+1}^*(x; \zeta) \quad (2.46)$$

where $g_{p+q+1}^*(x; \zeta)$ is given in equation (2.12) and

$$\begin{aligned} P_{p+q+1}^* &= \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r=0}^{\infty} (-1)^{l+i+p+q} \frac{(n+1)a_{n+1}(\delta\theta)^{n+i+1}\bar{\delta}^q}{(p+q+1)[C(\theta)]^{i+1}} d_{r,i} \\ &\times \binom{n+i+q+1}{q} \binom{m-k}{l} \binom{n+i+q}{p} \binom{k+l-1}{i} \delta^{-(n+i+2)} \left(\frac{\delta-1}{\delta}\right)^q. \end{aligned}$$

Thus, the pdf of the order statistic of the MO-GPS can be written as a linear combination of Exp-G with power parameter $(p+q+1)$.

Moreover for $\delta = 1$, using the following expansion;

$$(\bar{G}(x; \zeta))^{n+i} = \sum_{p=0}^{\infty} (-1)^p \binom{n+i}{p} G^p(x; \zeta).$$

Thus the distribution of the order statistics of the MO-GPS family of distributions for $\delta = 1$ can be written as

$$f_{k:m}(x) = \sum_{p=0}^{\infty} M_{p+1}^* g_{p+1}^*(x; \zeta) \quad (2.47)$$

where $g_{p+1}^*(x; \zeta)$ is given in equation (2.9) and

$$\begin{aligned} M_{p+1}^* &= \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r=0}^{\infty} (-1)^{l+i+p} \frac{(n+1)a_{n+1}\theta^{n+i+1}}{(p+1)[C(\theta)]^{i+1}} d_{r,i} \\ &\times \binom{m-k}{l} \binom{n+i}{p} \binom{k+l-1}{i}. \end{aligned}$$

Thus, the pdf of the order statistic of the MO-GPS can be written as a linear combination of Exp-G with power parameter $(p+1)$.

3. Estimation Techniques

In this section, we discuss various techniques of obtaining point estimators namely, the Maximum Likelihood Estimation, Method of Moments, Ordinary Least Squares Estimation, Anderson-Darling Estimation, Right-Tail Anderson-Darling Estimation, Cramer-von Mises Minimum Distance Estimation and the Maximum Product Spacing Estimation technique.

3.1 Maximum Likelihood Estimation

In this section, we present the estimates of the unknown parameters of the MO-GPS family of distribution for uncensored and censored data. We discuss parameter estimation for uncensored and censored data respectively, alternatives of censored data namely the type I and type II censoring are also discussed. Let $x_1, x_2, \dots, x_n$ be a random sample of size n from the MO-GPS family of distribution given in equation (2.2) such that parameters are estimated from a complete sample.

3.1.1 Estimation of parameters for uncensored data

Let $X \sim \text{MO-GPS}(\delta, \theta, \zeta)$ and $\Delta = (\delta, \theta, \zeta)^T$ the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the MO-GPS family of distributions is given by

$$\begin{aligned} \ell_n(\Delta) &= n \log(\theta\delta) + \sum_{i=1}^n \log(g(x_i; \zeta)) - 2 \sum_{i=1}^n \log(1 - \bar{\delta}\bar{G}(x_i; \zeta)) \\ &\quad + \sum_{i=1}^n \log \left[C' \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right] - n \log(C(\theta)), \\ U_\theta(\Delta) &= \frac{n}{\theta} - \frac{nC'(\theta)}{C(\theta)} + \sum_{i=1}^n \frac{C'' \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)}{C' \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)} \left(\frac{\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right), \\ U_\delta(\Delta) &= \frac{n}{\delta} - 2 \sum_{i=1}^n \frac{\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} + \sum_{i=1}^n \frac{C'' \left[\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]}{C' \left[\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]} \\ &\quad \times \left(\frac{\theta\bar{G}(x_i; \zeta)}{(1 - \bar{\delta}\bar{G}(x_i; \zeta))} \right) \end{aligned}$$

and

$$\begin{aligned}
U_{\zeta_k}(\Delta) &= \sum_{i=1}^n \partial[\log(g(x_i; \zeta))]/\partial\zeta_k - 2 \sum_{i=1}^n \partial[\log(1 - \bar{\delta}\bar{G}(x_i; \zeta))]/\partial\zeta_k \\
&+ \sum_{i=1}^n \partial \left[\log \left[C' \left(\frac{\theta \bar{\delta}\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right] \right] / \partial\zeta_k.
\end{aligned}$$

These equations are non-linear systems of equation: $\hat{\Delta} = (\frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \zeta_k})^T = \mathbf{0}$, and they cannot be solved analytically but can be solved numerically using statistical softwares such as MATHEMATICA, MAPLE, Ox and R.

3.1.2 Estimation of parameters for censored data

Due to constraints like time and money, a researcher cannot wait for all of the subjects under study to fail in real-world research. As a result, the researcher can watch for a set amount of time and wait until a predetermined number of observations produce data. Type II and type I censoring, respectively, are the following options: In type I censoring, the time is fixed before sample selection and the number of survivors is a random variable. Prior to sample selection, the number of survivors is fixed, and time is a random variable in type II censoring. Let the lifetime of the first r failed items be $x_1, x_2, \dots, x_r$, the censoring be $x_{r+1}, x_{r+2}, \dots, x_n$ and $\Delta = [\delta, \theta, \zeta]^T$ be the vector of parameters, the likelihood function of type I and II censoring is given by;

$$\ell(x_i; \Delta) = A \left[\prod_{i=1}^r f(x_i; \Delta) \times \prod_{i=r+1}^n (1 - F(x_i; \Delta)) \right] \quad (3.1)$$

where $f(\cdot)$ and $1 - F(\cdot)$ are the pdf and survival function respectively and $A = \text{Constant}$. Inserting equations (2.2) and (2.1) into (3.1) we have the following log-

likelihood function;

$$\begin{aligned}
\ell_n(\Delta) &= \log(A) + n \log(\theta\delta) + \sum_{i=1}^r \log(g(x_i; \zeta)) - 2 \sum_{i=1}^r \log(1 - \bar{\delta}\bar{G}(x_i; \zeta)) \\
&+ \sum_{i=1}^r \log \left[C' \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right] - 2n \log(C(\theta)) \\
&+ \sum_{i=r+1}^n \log \left[C \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right], \\
U_\theta(\Delta) &= \frac{n}{\theta} - \frac{2nC'(\theta)}{C(\theta)} + \sum_{i=1}^r \frac{C'' \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)}{C' \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)} \left(\frac{\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \\
&+ \sum_{i=r+1}^n \left(\frac{C' \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)}{C \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)} \left(\frac{\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right), \\
U_\delta(\Delta) &= \frac{n}{\delta} - 2 \sum_{i=1}^r \frac{\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} + \sum_{i=1}^r \frac{C'' \left[\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]}{C' \left[\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]} \\
&\times \left(\frac{\theta G(x_i; \zeta) \bar{G}(x_i; \zeta)}{(1 - \bar{\delta}\bar{G}(x_i; \zeta))} \right) + \sum_{i=r+1}^n \frac{C' \left[\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]}{C \left[\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]} \\
&\times \left(\frac{\theta G(x_i; \zeta) \bar{G}(x_i; \zeta)}{(1 - \bar{\delta}\bar{G}(x_i; \zeta))} \right)
\end{aligned}$$

and

$$\begin{aligned}
U_{\zeta_k}(\Delta) &= \sum_{i=1}^r \partial[\log(g(x_i; \zeta))]/\partial\zeta_k - 2 \sum_{i=1}^r \partial[\log(1 - \bar{\delta}\bar{G}(x_i; \zeta))]/\partial\zeta_k \\
&+ \sum_{i=1}^r \partial \left[\log \left[C' \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right] \right] / \partial\zeta_k \\
&+ \sum_{i=r+1}^n \left(\partial \left[\log \left[C \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right] \right] / \partial\zeta_k \right).
\end{aligned}$$

Solving these systems of non-linear equations yield the maximum likelihood estimates of censored data.

3.2 Method of Moments

Method of moment estimate can be obtained by equating the first theoretical moment with the sample moment

$$E(x^k) = M_k, \quad k = 1, 2, \dots$$

for

$$M_k = \frac{1}{n} \sum_{i=1}^n X_i^k,$$

$E(x^k)$ is the k^{th} moment of the distribution and $x_i = x_1, x_2, \dots, x_n$.

3.3 Ordinary Least Squares Estimation

The classical estimation method called the Ordinary Least Squares Estimation (OLSE) was introduced by Swain et al. [41]. Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be an ordered sample from a MO-GPS distribution with cdf

$$F_{MO-GPS}(x; \underline{\theta}), \quad \underline{\theta} \in \Theta.$$

The OLSE of $\underline{\theta}$ that minimises the square of the difference between the theoretical cdf and ecdf:

$$L(\underline{\theta}) = \sum_{i=1}^n \left\{ F_{MO-GPS}(x_{(i)}) - \frac{i}{n+1} \right\}^2.$$

3.4 Anderson-Darling Estimation

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be an ordered sample from a MO-GPS distribution with cdf

$$F_{MO-GPS}(x; \underline{\theta}), \quad \underline{\theta} \in \Theta.$$

The Anderson-Darling Estimation (ADE) is another type of minimum distance estimator obtained by minimizing

$$L(\underline{\theta}) = -n - n^{-1} \sum_{i=1}^n (2i-1) \left[\ln(F_{MO-GPS}(x_{(i)})) + \ln(S_{MO-GPS}(x_{(n+1-i)})) \right]^2.$$

3.5 Right-Tail Anderson-Darling Estimation

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be an ordered sample from a MO-GPS distribution with cdf

$$F_{MO-GPS}(x; \underline{\theta}), \quad \underline{\theta} \in \Theta.$$

The Right-Tailed Anderson-Darling Estimation (RTADE) is also another type of minimum distance estimator obtained by minimizing

$$L(\underline{\theta}) = \frac{n}{2} - 2 \sum_{i=1}^n \ln(F_{MO-GPS}(x_{(i)})) - n^{-1} \sum_{i=1}^n (2i-1) \ln(S_{MO-GPS}(x_{(n+1-i)})).$$

3.6 Cramer-von Mises Minimum Distance Estimation

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be an ordered sample from a MO-GPS distribution with cdf

$$F_{MO-GPS}(x; \underline{\theta}), \quad \underline{\theta} \in \Theta.$$

The Cramer-von Mises Minimum Distance Estimation (CMDE) by MacDonald [33] of $\underline{\theta}$ is defined as the value of $\underline{\theta}$ that minimizes the square of the difference of the theoretical cdf and modified ecdf:

$$L(\underline{\theta}) = (12n)^{-1} + \sum_{i=1}^n \left\{ F_{MO-GPS}(x_{(i)}) - \frac{2i-1}{2n} \right\}^2.$$

3.7 Maximum Product Spacing Estimation

The Maximum Product Spacing Estimation (MPSE) is also a conventional estimator to obtain the estimate of the parameter and was discussed by Cheng and Amin [10] and has been proved by Coolen and Newby [11]. Let $x_1, x_2, \dots, x_n$ be an (*iid*) random sample of size n from a distribution with cdf

$$F_{MO-GPS}(x; \underline{\theta}), \quad \underline{\theta} \in \Theta.$$

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the corresponding ordered sample. Also, let $x_{(0)} = 0$ and $x_{(n+1)} = \infty$. Now, we define the spacings as the “gaps” between the values of the cdf at adjacent ordered points :

$$D_i(\underline{\theta}) = F_{MO-GPS}(x_{(i)}; \underline{\theta}) - F_{MO-GPS}(x_{(i-1)}; \underline{\theta}), \quad i = 1, 2, \dots, i+1.$$

The MPSE of $\underline{\theta}$ is the value that maximizes the logarithm of the geometric mean of sample spacings:

$$\hat{\underline{\theta}} = \operatorname{argmax}(S_n(\underline{\theta})),$$

where

$$S_n(\underline{\theta}) = (n+1)^{-1} \sum_{i=1}^{n+1} \ln(D_i(\underline{\theta})).$$



4. Some classes of MO-GPS Family of Distributions and Properties

Some classes of the MO-GPS family of distributions are presented in this chapter, other classes of the MO-GPS can be seen at Cordeiro et al. [14]. Cases of the Logarithmic and Binomial Power Series with $G(x; \zeta)$ and $g(x; \zeta)$ being the cdf and pdf of the baseline distribution respectively. Plots of the pdf and hrf are also presented.

4.1 Marshall-Olkin-G Logarithmic distribution

We present the Marshall-Olkin-G Logarithmic (MO-GLog) distribution with different baseline distribution. We make use of results from Table 1.1 together with $C''(\theta) = (1 - \theta)^{-2}$ and $C^{-1}(\theta) = 1 - e^{-\theta}$. The cdf and pdf of the MO-GLog distribution are given by

$$F_{MO-GLog}(x; \delta, \theta, \zeta) = 1 - \frac{\log \left[1 - \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]}{\log(1 - \theta)}, \quad (4.1)$$

and

$$f_{MO-GLog}(x; \delta, \theta, \zeta) = \frac{\theta \delta g(x; \zeta) \left[1 - \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]^{-1}}{(1 - \delta \bar{G}(x; \zeta))^2 (-\log(1 - \theta))} \quad (4.2)$$

respectively for $\delta, \theta > 0$, $\bar{\delta} = 1 - \delta$ and $\bar{G}(x; \zeta) = 1 - G(x; \zeta)$.

4.1.1 The Survival, Hazard and Reverse Hazard Rate Function

The survival, hazard and reverse hazard rate function of the MO-GLog family of distributions are given by

$$S_{MO-GLog}(x; \delta, \theta, \zeta) = \frac{\log \left[1 - \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]}{\log(1 - \theta)}, \quad (4.3)$$

$$h_{MO-GLog}(x; \delta, \theta, \zeta) = \frac{\theta \delta g(x; \zeta) \left[1 - \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]^{-1}}{(1 - \delta \bar{G}(x; \zeta))^2 \left(-\log \left[1 - \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right] \right)}, \quad (4.4)$$

and

$$\begin{aligned} \tau_{MO-GLog}(x; \delta, \theta, \zeta) &= \frac{\theta \delta g(x; \zeta) \left[1 - \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right]^{-1} \log(1 - \theta)}{(1 - \delta \bar{G}(x; \zeta))^2 (-\log(1 - \theta))} \\ &\times \left\{ \log(1 - \theta) - \log \left[1 - \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right] \right\}^{-1}, \end{aligned} \quad (4.5)$$

respectively, for $\delta, \theta > 0$, $\bar{\delta} = 1 - \delta$, $\bar{G}(x; \psi) = 1 - G(x; \psi)$ and parameter vector ζ .

4.1.2 Quantile Function and Density Expansion

The quantile function of the MO-GLog distributions can be obtained using results from equation (2.7) and can be expressed as;

$$Q(u) = G^{-1} \left\{ 1 - \frac{1 - (1 - \theta)^{1-u}}{\theta \delta + \bar{\delta} (1 - \theta)^{1-u}} \right\} \quad (4.6)$$

for $0 \leq u < 1$.

Using general results for series expansion of MO-GPS from equations (2.8), (2.11) and (2.14) the density series expansion of the MO-GLog distributions can be expressed as

$$f_{MO-GLog}(x; \delta, \theta, \zeta) = \sum_{p=0}^n e_{\alpha}^* g_{\alpha}^*(x; \zeta) \quad (4.7)$$

where $g_{\alpha}^*(x; \zeta)$ is an Exp-G distribution with power parameter (α),

$$e_{\alpha}^* = \sum_{n,k=0}^{\infty} \frac{(\theta \delta)^{n+1} (-1)^{k+p} \bar{\delta}^k}{(p+1)(-\log(1-\theta))} \binom{n+k+1}{k} \binom{n+k}{p} \quad (4.8)$$

for $\delta \in (0, 1)$ with $\alpha = p + 1$,

$$e_{\alpha}^* = \sum_{n=0}^{\infty} \frac{(\theta \delta)^{n+1} (-1)^{k+p}}{(p+k+1)(-\log(1-\theta))} \delta^{-(n+2)} \left(\frac{\delta-1}{\delta} \right)^k \binom{n+k+1}{k} \binom{n+k}{p} \quad (4.9)$$

for $\delta > 1$ with $\alpha = p + k + 1$ and

$$e_{\alpha}^* = \sum_{n=0}^{\infty} \frac{(-1)^p \theta^{n+1}}{(p+1)(-\log(1-\theta))} \binom{n}{p} \quad (4.10)$$

for $\delta = 1$ with $\alpha = p + 1$.

4.1.3 Moments and Moment Generating Function

The r^{th} moment of Y , say a'_r and the r^{th} incomplete moment of the MO-GLog can be obtained using equation (2.16) and (2.20) respectively and they are given by

$$a'_r = E(Y^r) = \sum_{p=0}^{\infty} e_{\alpha}^* E(Y_{\alpha}^r), \quad (4.11)$$

and

$$a_X(t) = \int_0^t x^r f_{MO-GLog}(x; \delta, \theta, \zeta) dx = \sum_{p=0}^{\infty} e_{\alpha}^* I_{\alpha}(t), \quad (4.12)$$

respectively, where $I_{\alpha}(t) = \int_0^t x^r g_{\alpha}^*(x; \zeta) dx$ and e_{α}^* is as defined in equations (4.8), (4.9) and (4.10) for $\delta \in (0, 1)$, $\delta > 1$ and $\delta = 1$ respectively.

The $(i, j)^{th}$ probability weighted moment (PWM) of a random variable $X \sim MO - GLog(\delta, \theta, \zeta)$, say $(\omega_{i,j})$ is obtained using equation (2.22) and is given by

$$\omega_{i,j} = \sum_{p=0}^{\infty} H_{\alpha}^* \int_{-\infty}^{\infty} x^i g_{\alpha}^*(x; \zeta) dx = \sum_{p=0}^{\infty} H_{\alpha}^* E[Y_{\alpha}^i], \quad (4.13)$$

where $E[Y_{\alpha}^i]$ is the i^{th} moment of Exponentiated-G (Exp-G) density with power parameter (α) ,

$$\begin{aligned} H_{\alpha}^* &= \sum_{q,i,m,n=0}^{\infty} (-1)^{i+p+q} \frac{(\delta\theta)^{n+i+1} \bar{\delta}^q}{(p+1)(-\log(1-\theta))^{i+1}} d_{r,i} \\ &\times \binom{n+i+q+1}{q} \binom{n+i+q}{p} \binom{j}{i} \end{aligned} \quad (4.14)$$

for $\delta \in (0, 1)$, $\alpha = p + 1$ and $g_{\alpha}^*(x; \zeta)$ given in equation (2.9),

$$\begin{aligned} H_{\alpha}^* &= \sum_{n,i,r=0}^{\infty} (-1)^{i+p+k} \frac{(\delta\theta)^{n+i+1} \bar{\delta}^k}{(p+k+1)(-\log(1-\theta))^{i+1}} d_{r,i} \\ &\times \binom{n+i+k+1}{k} \binom{n+i+k}{p} \binom{j}{i} \delta^{-(n+i+2)} \left(\frac{\delta-1}{\delta} \right)^k \end{aligned}$$

for $\delta > 1$, $\alpha = p + k + 1$ with $g_\alpha^*(x; \zeta)$ given in equation (2.12),

$$H_\alpha^* = \sum_{n,i,r=0}^{\infty} (-1)^{i+p} \frac{\theta^{n+i+1}}{(p+1)(-\log(1-\theta))^{i+1}} d_{r,i} \times \binom{n+i}{p} \binom{j}{i},$$

for $\delta > 1$, $\alpha = p+1$ with $g_\alpha^*(x; \zeta)$ given in equation (2.9) and $d_{r,i} = (rb_0)^{-1} \sum_{h=1}^r [i(h+1) - r] b_n d_{r-h,i}$ and $d_{0,i} = b_0^i$.

The moment generating function (mgf) of the MO-GLog say $M_Y(t) = E(e^{tY})$ of Y can be directly derived from equation (4.7) and is given by

$$M_Y(t) = \sum_{p=0}^{\infty} e_\alpha^* M_p(t), \quad (4.15)$$

where $M_p(t)$ is the mgf of Y_α , e_α^* is given in equations (4.8), (4.9) and (4.10). Hence, $M_Y(t)$ can be determined from the Exp-G generating function.

4.1.4 Mean deviation, Bonferroni and Lorenz Curves

Let $X \sim MO - GLog(\delta, \theta, \zeta)$, the mean deviation about the mean and about the median using equation (2.28) and (2.29) are given by

$$\begin{aligned} \alpha_{MO-GLog}^1(x) &= 2\mu F_{MO-GLog}(\mu) - 2 \int_0^\infty x f_{MO-GLog}(x; \delta, \theta, \zeta) dx \\ &= 2\mu F_{MO-GLog}(\mu) - 2 \int_0^\infty \sum_{p=0}^{\infty} x e_\alpha^* g_\alpha^*(x; \zeta) dx \end{aligned} \quad (4.16)$$

and

$$\begin{aligned} \alpha_{MO-GLog}^2(x) &= \mu - 2 \int_0^\infty x f_{MO-GLog}(x; \delta, \theta, \zeta) dx \\ &= \mu - 2 \int_0^\infty \sum_{p=0}^{\infty} x e_\alpha^* g_\alpha^*(x; \zeta) dx \end{aligned} \quad (4.17)$$

respectively where $\mu = E(X)$, e_α^* is given in equations (4.8), (4.9) and (4.10) and $g_\alpha^*(x; \zeta)$ is given in equation (2.9) (for $\delta \in (0, 1)$ and $\delta = 1$) and (2.12) (for $\delta > 1$).

The Bonferroni and Lorenz curves for the MO-GLog distribution are defined by

$$\begin{aligned} B_{MO-GLog}(u) &= \frac{1}{u\mu} \int_0^v x f_{MO-GLog}(x; \delta, \theta, \zeta) dx \\ &= \frac{1}{u\mu} \int_0^v \sum_{p=0}^{\infty} x e_{\alpha}^* g_{\alpha}^*(x; \zeta) dx \end{aligned} \quad (4.18)$$

and

$$\begin{aligned} L_{MO-GLog}(u) &= \frac{1}{\mu} \int_0^v x f_{MO-GLog}(x; \delta, \theta, \zeta) dx \\ &= \frac{1}{\mu} \int_0^v \sum_{p=0}^{\infty} x e_{\alpha}^* g_{\alpha}^*(x; \zeta) dx \end{aligned} \quad (4.19)$$

where $\mu = E(X)$, $v = Q(u)$, e_{α}^* is given in equations (4.8), (4.9) and (4.10) and $g_{\alpha}^*(x; \zeta)$ is given in equation (2.9) (for $\delta \in (0, 1)$ and $\delta = 1$) and (2.12) (for $\delta > 1$).

4.1.5 Residual and reverse residual life function

Using results from equation (2.32) and (2.36), the n^{th} moment of residual life of a MO-GLog is given by;

$$a_{(MO-GLog;n)}^*(t) = \frac{1}{\bar{G}(x; \zeta)} \sum_{p=0}^{\infty} e_{\alpha}^* \sum_{m=0}^{\infty} \binom{n}{m} (-t)^m \int_t^{\infty} x^{n-m} g_{\alpha}^*(x; \zeta) dx. \quad (4.20)$$

The n^{th} moment of reverse residual life of a MO-GLog is given by;

$$b_{(MO-GLog;n)}^*(t) = \frac{1}{G(x; \zeta)} \sum_{p=0}^{\infty} e_{\alpha}^* \sum_{m=0}^{\infty} \binom{n}{m} (t)^{n-m} (-1)^m \int_0^t x^m g_{\alpha}^*(x; \zeta) dx, \quad (4.21)$$

where e_{α}^* is given in equations (4.8), (4.9) and (4.10) and $g_{\alpha}^*(x; \zeta)$ is given in equation (2.9) (for $\delta \in (0, 1)$ and $\delta = 1$) and (2.12) (for $\delta > 1$), $\bar{G}(x; \zeta)$ is the survival function.

4.1.6 Stress-Strength Model

For X and Y being independent $F_{MO-GLog}(x; \delta_1, \theta_1, \zeta_1)$ and $F_{MO-GLog}(y; \delta_2, \theta_2, \zeta_2)$ respectively and for the stress-strength parameter $R_{MO-GLog} = P(Y < X)$, then, using equation (2.39), the reliability parameter for MO-GLog distribution is given

by

$$R_{MO-GLog} = \int_{x=0}^{\infty} \frac{\theta_1 \delta_1 g(x; \zeta_1) \left[1 - \frac{\theta_1 \delta_1 \bar{G}(x; \zeta_1)}{1 - \delta_1 \bar{G}(x; \zeta_1)} \right]^{-1}}{(1 - \delta_1 \bar{G}(x; \zeta_1))^2 (-\log(1 - \theta_1))} \left(1 - \frac{\log \left[1 - \frac{\theta_2 \delta_2 \bar{G}(x; \zeta_2)}{1 - \delta_2 \bar{G}(x; \zeta_2)} \right]}{\log(1 - \theta_2)} \right) dx. \quad (4.22)$$

4.1.7 Order Statistics and Rényi Entropy

Let $X_1, X_2, \dots, X_m$ be *iid* MO-GLog random variables. Then, using equation (2.47), the pdf of the k^{th} order statistics ($X_{k:m}$) of a MO-GLog can be written as

$$f_{k:m}(x) = \sum_{p=0}^{\infty} f_{\alpha}^* g_{\alpha}^*(x; \zeta) \quad (4.23)$$

where $g_{\alpha}^*(x; \zeta)$ is an Exp-G with power parameter (α),

$$\begin{aligned} f_{\alpha}^* &= \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r,q=0}^{\infty} (-1)^{l+i+p+q} \frac{(\delta\theta)^{n+i+1} \bar{\delta}^q}{(p+1)(-\log(1-\theta))^{i+1}} d_{r,i} \\ &\times \binom{n+i+q+1}{q} \binom{m-k}{l} \binom{n+i+q}{p} \binom{k+l-1}{i} \end{aligned}$$

for $\delta \in (0, 1)$ with $\alpha = p + 1$,

$$\begin{aligned} f_{p+q+1}^* &= \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r=0}^{\infty} (-1)^{l+i+p+q} \frac{(\delta\theta)^{n+i+1} \bar{\delta}^q}{(p+q+1)(-\log(1-\theta))^{i+1}} d_{r,i} \\ &\times \binom{n+i+q+1}{q} \binom{m-k}{l} \binom{n+i+q}{p} \binom{k+l-1}{i} \delta^{-(n+i+2)} \left(\frac{\delta-1}{\delta} \right)^q \end{aligned}$$

for $\delta > 1$ with $\alpha = p + k + 1$,

$$\begin{aligned} f_{p+1}^* &= \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r=0}^{\infty} (-1)^{l+i+p} \frac{\theta^{n+i+1}}{(p+1)(-\log(1-\theta))^{i+1}} d_{r,i} \\ &\times \binom{m-k}{l} \binom{n+i}{p} \binom{k+l-1}{i} \end{aligned}$$

for $\delta = 1$ with $\alpha = p + 1$ and $d_{r,i} = (rb_0)^{-1} \sum_{h=1}^r [i(h+1) - r] b_n d_{r-h,i}$ with $d_{0,i} = b_0^i$.

Using equations (2.41), (2.42) and (2.43), the Rényi entropy of the MO-GLog for

$\delta \in (0, 1)$, $\delta > 1$ and $\delta = 1$ is defined to be

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{n,s,k,p,q=0}^{\infty} b_{s,k} \left(\frac{\theta\delta}{(-\log(1-\theta))} \right)^{\nu} (\delta\theta)^{ns} \delta^q (-1)^{p+q} \binom{ns+q}{p} \right. \\ \left. \times \binom{2\nu+ns+q-1}{q} \int_0^{\infty} g^{\nu}(x; \zeta) G^p(x; \zeta) dx \right], \quad (4.24)$$

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{n,s,k,p,q=0}^{\infty} b_{s,k} \left(\frac{\theta\delta}{(-\log(1-\theta))} \right)^{\nu} (\delta\theta)^{ns} \delta^q (-1)^{p+q} \binom{ns+q}{p} \right. \\ \left. \times \binom{2\nu+ns+q-1}{q} \delta^{-(2\nu+ns)} \left(\frac{\delta-1}{\delta} \right)^q \int_0^{\infty} g^{\nu}(x; \zeta) G^{p+q}(x; \zeta) dx \right] \quad (4.25)$$

and

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{n,s,k,p=0}^{\infty} b_{s,k} \left(\frac{\theta}{(-\log(1-\theta))} \right)^{\nu} \theta^{ns} (-1)^p \binom{ns}{p} \right. \\ \left. \times \int_0^{\infty} g^{\nu}(x; \zeta) G^p(x; \zeta) dx \right] \quad (4.26)$$

respectively, where $b_{s,k} = (sa_0)^{-1} \sum_{l=1}^s [k(l+1) - s] a_l b_{s-l,k}$, $b_{0,k} = a_0^k$, $\nu \neq 1$ and $\nu > 0$.

4.1.8 Maximum Likelihood Estimation

We consider section 3.1 to obtain the maximum likelihood estimates of the parameters of the MO-GLog distribution and the Fisher information matrix can be applied to create confidence zones and intervals for specific model parameters.

4.1.8.1 Estimation of parameters for uncensored data

Let $X \sim \text{MO-GLog}(\delta, \theta, \zeta)$ and $\Delta = (\delta, \theta, \zeta)^T$ the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the

MO-GLog distributions is given by

$$\begin{aligned}
\ell_n(\Delta) &= n \log(\theta\delta) + \sum_{i=1}^n \log(g(x_i; \zeta)) - 2 \sum_{i=1}^n \log(1 - \bar{\delta}\bar{G}(x_i; \zeta)) \\
&\quad + \sum_{i=1}^n \log \left[\left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1} \right] - n \log(-\log(1 - \theta)), \\
U_\theta(\Delta) &= \frac{n}{\theta} + \frac{n(1 - \theta)^{-1}}{\log(1 - \theta)} + \sum_{i=1}^n \left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1} \left(\frac{\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right), \\
U_\delta(\Delta) &= \frac{n}{\delta} - 2 \sum_{i=1}^n \frac{\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} + \sum_{i=1}^n \left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1} \\
&\quad \times \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{(1 - \bar{\delta}\bar{G}(x_i; \zeta))} \right)
\end{aligned}$$

and

$$\begin{aligned}
U_{\zeta_k}(\Delta) &= \sum_{i=1}^n \partial[\log(g(x_i; \zeta))]/\partial\zeta_k - 2 \sum_{i=1}^n \partial[\log(1 - \bar{\delta}\bar{G}(x_i; \zeta))]/\partial\zeta_k \\
&\quad + \sum_{i=1}^n \partial \left[\log \left[\left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1} \right] \right] / \partial\zeta_k.
\end{aligned}$$

4.1.8.2 Estimation of parameters for censored data

Let $X \sim \text{MO-GLog}(\delta, \theta, \zeta)$ and $\Delta = (\delta, \theta, \zeta)^T$ the vector of model parameters, then, using equation (3.1), the log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random

sample of size n from the MO-GLog distributions is given by

$$\begin{aligned}
\ell_n(\Delta) &= \log(A) + n \log(\theta\delta) + \sum_{i=1}^r \log(g(x_i; \zeta)) - 2 \sum_{i=1}^r \log(1 - \bar{\delta}\bar{G}(x_i; \zeta)) \\
&+ \sum_{i=1}^r \log \left[\left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1} \right] - n \log(-\log(1 - \theta)) \\
&+ \sum_{i=r+1}^n \log \left[-\log \left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right], \\
U_\theta(\Delta) &= \frac{n}{\theta} + \frac{2n(1 - \theta)^{-1}}{\log(1 - \theta)} + \sum_{i=1}^r \left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1} \left(\frac{\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \\
&- \sum_{i=r+1}^n \left(\frac{\left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1}}{\log \left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)} \left(\frac{\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right), \\
U_\delta(\Delta) &= \frac{n}{\delta} - 2 \sum_{i=1}^r \frac{\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} + \sum_{i=1}^r \left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1} \\
&\times \left(\frac{\theta G(x_i; \zeta) \bar{G}(x_i; \zeta)}{(1 - \bar{\delta}\bar{G}(x_i; \zeta))} \right) - \sum_{i=r+1}^n \frac{\left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1}}{\log \left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)} \\
&\times \left(\frac{\theta G(x_i; \zeta) \bar{G}(x_i; \zeta)}{(1 - \bar{\delta}\bar{G}(x_i; \zeta))} \right)
\end{aligned}$$

and

$$\begin{aligned}
U_{\zeta_k}(\Delta) &= \sum_{i=1}^r \partial[\log(g(x_i; \zeta))]/\partial\zeta_k - 2 \sum_{i=1}^r \partial[\log(1 - \bar{\delta}\bar{G}(x_i; \zeta))]/\partial\zeta_k \\
&+ \sum_{i=1}^r \partial \left[\log \left[\left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{-1} \right] \right] / \partial\zeta_k \\
&+ \sum_{i=r+1}^n \left(\partial \left[\log \left[-\log \left(1 - \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right] \right] / \partial\zeta_k \right).
\end{aligned}$$

4.1.9 Special Cases of MO-GLog distribution

In this Section, we present the special cases of the MO-GLog distribution for different baseline distribution. The pdf and hrf plots are also presented.

4.1.9.1 Marshall-Olkin Weibull Logarithmic distribution

Suppose the baseline distribution is the Weibull distribution with cdf and pdf given by $G(x; a, b) = 1 - e^{-bx^a}$ and $g(x; a, b) = abx^{a-1}e^{-bx^a}$, for $a, b > 0$ then, the Marshall-

Olkin Weibull Logarithmic (MO-WLog) distribution has cdf and pdf given by;

$$F_{MO-WLog}(x; \delta, \theta, a, b) = 1 - \frac{\log \left[1 - \frac{\theta \delta e^{-bx^a}}{1 - \bar{\delta} e^{-bx^a}} \right]}{\log(1 - \theta)}, \quad (4.27)$$

and

$$\begin{aligned} f_{MO-WLog}(x; \delta, \theta, a, b) &= \frac{\theta \delta a b x^{a-1} e^{-bx^a} \left[1 - \frac{\theta \delta e^{-bx^a}}{1 - \bar{\delta} e^{-bx^a}} \right]^{-1}}{[1 - \bar{\delta} e^{-bx^a}]^2 (-\log(1 - \theta))} \\ &= \sum_{n,k=0}^{\infty} e_{\alpha}^* g_{\alpha}^*(x; \zeta), \end{aligned} \quad (4.28)$$

for $\delta, \theta, a, b > 0$, $\bar{\delta} = 1 - \delta$, e_{α}^* is given in equations (4.8), (4.9) and (4.10) and $g_{\alpha}^*(x; \zeta)$ is given in equations (2.9) and (2.12).

The corresponding survival and hazard rate function are given by

$$S_{MO-WLog}(t) = \frac{\log \left[1 - \frac{\theta \delta e^{-bt^a}}{1 - \bar{\delta} e^{-bt^a}} \right]}{\log(1 - \theta)}, \quad (4.29)$$

and

$$h_{MO-WLog}(t) = \frac{\theta \delta a b t^{a-1} e^{-bt^a} \left[1 - \frac{\theta \delta e^{-bt^a}}{1 - \bar{\delta} e^{-bt^a}} \right]^{-1}}{(1 - \bar{\delta} e^{-bt^a})^2 \left(-\log \left[1 - \frac{\theta \delta e^{-bt^a}}{1 - \bar{\delta} e^{-bt^a}} \right] \right)}, \quad (4.30)$$

respectively, for $\delta, a, b > 0$, $0 < \theta < 1$ and $\bar{\delta} = 1 - \delta$.

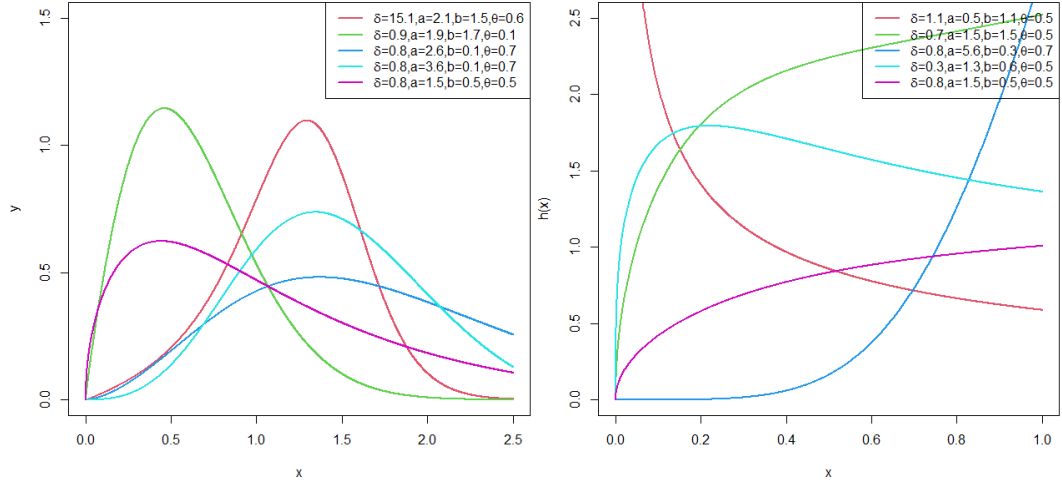


Figure 4.1: Plots of the pdf and hrf for the MO-WLog distribution

The pdf of the MO-WLog can handle data set that is right or left skewed, symmetric and almost symmetric, the hrf of the this distribution is showing increasing, decreasing and upside down bathtub shapes.

4.1.9.2 Marshall-Olkin Burr-XII Logarithmic distribution

Suppose the baseline distribution is the Burr-XII distribution with cdf and pdf given by $G(x; c, k) = 1 - (1 + x^c)^{-k}$ and $g(x; c, k) = ckx^{c-1}(1 + x^c)^{-k-1}$, for $a, b > 0$ and $x > 0$, then, the Marshall-Olkin Burr-XII Logarithmic (MO-BXIILog) distribution has cdf and pdf given by;

$$F_{MO-BXIILog}(x; \delta, \theta, c, k) = 1 - \frac{\log \left[1 - \frac{\theta \delta (1+x^c)^{-k}}{1 - \bar{\delta} (1+x^c)^{-k}} \right]}{\log(1 - \theta)}, \quad (4.31)$$

and

$$\begin{aligned} f_{MO-BXIILog}(x; \delta, \theta, c, k) &= \frac{\theta \delta c k x^{c-1} (1 + x^c)^{-k-1} \left[1 - \frac{\theta \delta (1+x^c)^{-k}}{1 - \bar{\delta} (1+x^c)^{-k}} \right]^{-1}}{[1 - \bar{\delta} (1 + x^c)^{-k}]^2 (-\log(1 - \theta))} \\ &= \sum_{n,k=0}^{\infty} e_{\alpha}^* g_{\alpha}^*(x; \zeta), \end{aligned} \quad (4.32)$$

for $\delta, \theta, c, k > 0$, $\bar{\delta} = 1 - \delta$, e_{α}^* is given in equations (4.8), (4.9) and (4.10) and $g_{\alpha}^*(x; \zeta)$ is given in equations (2.9) and (2.12).

The corresponding survival and hazard rate function are given by

$$S_{MO-BXII\text{Log}}(t) = \frac{\log \left[1 - \frac{\theta \delta (1+x^c)^{-k}}{1 - \bar{\delta} (1+x^c)^{-k}} \right]}{\log(1 - \theta)}, \quad (4.33)$$

and

$$h_{MO-BXII\text{Log}}(t) = \frac{\theta \delta c k x^{c-1} (1+x^c)^{-k-1} \left[1 - \frac{\theta \delta (1+x^c)^{-k}}{1 - \bar{\delta} (1+x^c)^{-k}} \right]^{-1}}{(1 - \bar{\delta} (1+x^c)^{-k})^2 \left(-\log \left[1 - \frac{\theta \delta (1+x^c)^{-k}}{1 - \bar{\delta} (1+x^c)^{-k}} \right] \right)}, \quad (4.34)$$

respectively, for $\delta, \theta, c, k > 0$ and $\bar{\delta} = 1 - \delta$.

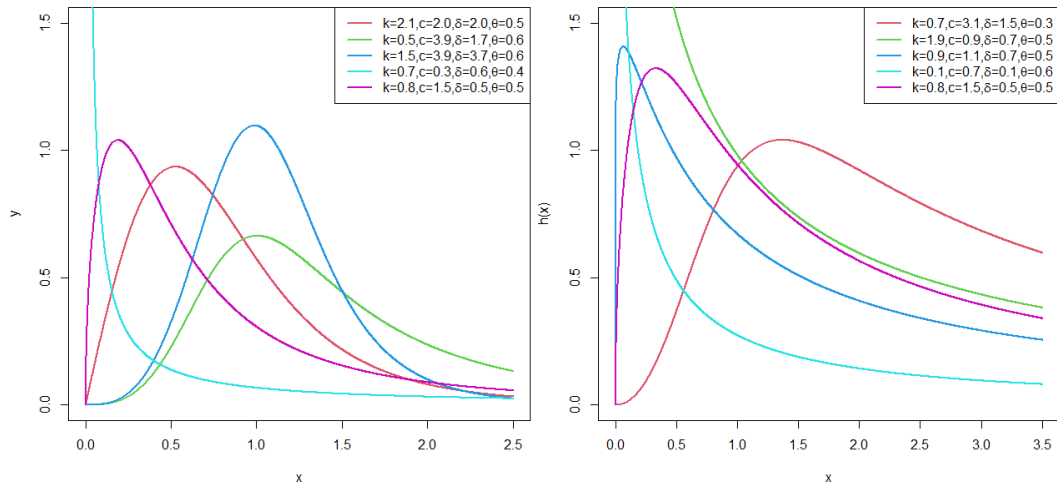


Figure 4.2: Plots of the pdf and hrf for the MO-BXIILog distribution

The pdf of the MO-BXIILog can handle data set that is right or left skewed, symmetric and almost symmetric, the hrf of the this distribution is exhibiting decreasing and upside down bathtub shapes.

4.1.9.3 Marshall-Olkin Topp-Leone Logarithmic distribution

Suppose the baseline distribution is the Topp-Leone distribution with cdf and pdf given by $G(x; a) = (1 - (1 - x)^2)^a$ and $g(x; a) = 2a(1 - x)(1 - (1 - x)^2)^{a-1}$, for $a > 0$ and $x > 0$, then, the Marshall-Olkin-Topp-Leone Logarithmic (MO-TLLog)

distribution has cdf and pdf given by;

$$F_{MO-TLLog}(x; \delta, \theta, a) = 1 - \frac{\log \left[1 - \frac{\theta \delta (1 - (1 - (1 - x)^2)^a)}{1 - \bar{\delta} (1 - (1 - (1 - x)^2)^a)} \right]}{\log(1 - \theta)}, \quad (4.35)$$

and

$$\begin{aligned} f_{MO-TLLog}(x; \delta, \theta, a) &= \frac{2\theta \delta a (1 - x) (1 - (1 - x)^2)^{a-1} \left[1 - \frac{\theta \delta (1 - (1 - (1 - x)^2)^a)}{1 - \bar{\delta} (1 - (1 - (1 - x)^2)^a)} \right]^{-1}}{(1 - \bar{\delta} (1 - (1 - (1 - x)^2)^a))^2 (-\log(1 - \theta))} \\ &= \sum_{n,k=0}^{\infty} e_{\alpha}^* g_{\alpha}^*(x; \zeta), \end{aligned} \quad (4.36)$$

for $\delta, \theta, a > 0$, $\bar{\delta} = 1 - \delta$, e_{α}^* is given in equations (4.8), (4.9) and (4.10) and $g_{\alpha}^*(x; \zeta)$ is given in equations (2.9) and (2.12).

The corresponding survival and hazard rate function are given by

$$S_{MO-TLLog}(t) = \frac{\log \left[1 - \frac{\theta \delta (1 - (1 - (1 - x)^2)^a)}{1 - \bar{\delta} (1 - (1 - (1 - x)^2)^a)} \right]}{\log(1 - \theta)}, \quad (4.37)$$

and

$$h_{MO-TLLog}(t) = \frac{\theta \delta a (1 - x) (1 - (1 - x)^2)^{a-1} \left[1 - \frac{\theta \delta (1 - (1 - (1 - x)^2)^a)}{1 - \bar{\delta} (1 - (1 - (1 - x)^2)^a)} \right]^{-1}}{(1 - \bar{\delta} (1 - (1 - (1 - x)^2)^a))^2 \left(-\log \left[1 - \frac{\theta \delta (1 - (1 - (1 - x)^2)^a)}{1 - \bar{\delta} (1 - (1 - (1 - x)^2)^a)} \right] \right)}, \quad (4.38)$$

respectively, for $\delta, \theta, a > 0$ and $\bar{\delta} = 1 - \delta$.

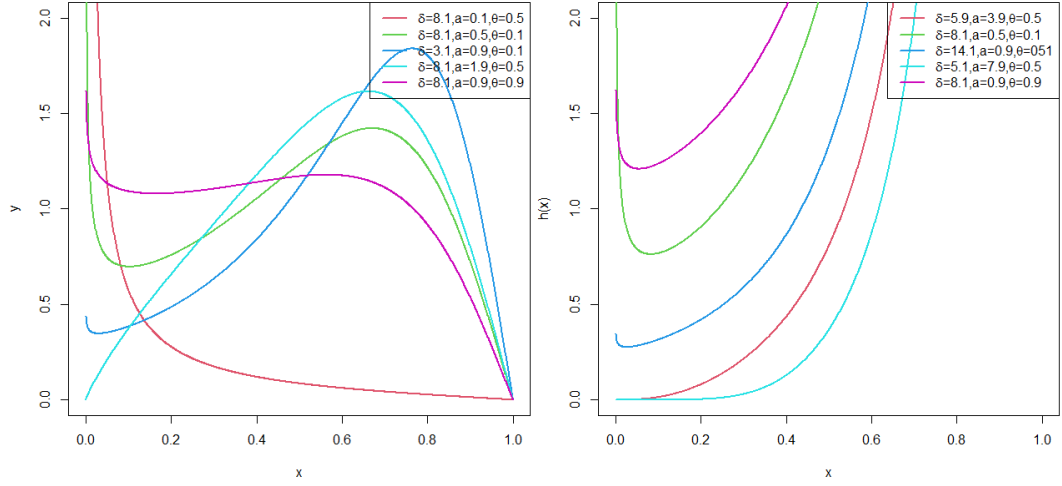


Figure 4.3: Plots of the pdf and hrf for the MO-TLLog distribution

The pdf of the MO-TLLog can handle data set of different shapes, the hrf of the this distribution is exhibiting increasing and bathtub shapes.

4.2 Marshall-Olkin-G Binomial distribution

We present the Marshall-Olkin-G Binomial (MO-GBin) distribution with different baseline distribution together with applications and simulations. By making use of results from Table 1.1 together with $C''(\theta) = m(m-1)(1+\theta)^{m-2}$ and $C^{-1}(\theta) = (1+\theta)^{\frac{1}{m}} - 1$, the cdf and pdf of the MO-GBin distribution are given by

$$F_{MO-GBin}(x; \delta, \theta, m, \zeta) = 1 - \frac{\left[1 + \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta G(x; \zeta)}\right]^m - 1}{(1 + \theta)^m - 1}, \quad (4.39)$$

and

$$f_{MO-GBin}(x; \delta, \theta, m, \zeta) = \frac{\theta \delta m g(x; \zeta) \left[1 + \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta G(x; \zeta)}\right]^{m-1}}{(1 - \delta \bar{G}(x; \zeta))^2 [(1 + \theta)^m - 1]} \quad (4.40)$$

respectively for $\delta, \theta, m > 0$, $\bar{\delta} = 1 - \delta$ and $\bar{G}(x; \zeta) = 1 - G(x; \zeta)$.

4.2.1 The Survival, Hazard and Reverse Hazard Rate Function

The survival, hazard and reverse hazard rate function of the MO-GBin family of distributions are given by

$$S_{MO-GBin}(x; \delta, \theta, \zeta) = \frac{\left[1 + \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)}\right]^m - 1}{(1 + \theta)^m - 1}, \quad (4.41)$$

$$\begin{aligned} h_{MO-GBin}(x; \delta, \theta, \zeta) &= \frac{\theta \delta m g(x; \zeta) \left[1 + \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)}\right]^{m-1}}{(1 - \delta \bar{G}(x; \zeta))^2} \\ &\times \left(\left[1 + \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)}\right]^m - 1\right)^{-1}, \end{aligned} \quad (4.42)$$

and

$$\begin{aligned} \tau_{MO-GBin}(x; \delta, \theta, \zeta) &= \frac{\theta \delta m g(x; \zeta) \left[1 + \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)}\right]^{m-1}}{(1 - \delta \bar{G}(x; \zeta))^2} \\ &\times \left(\left[(1 + \theta)^m - 1\right] - \left[1 + \frac{\theta \delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)}\right]^m - 1\right)^{-1}, \end{aligned} \quad (4.43)$$

respectively, for $\delta, \theta, m > 0$, $\bar{\delta} = 1 - \delta$, $\bar{G}(x; \psi) = 1 - G(x; \psi)$ and parameter vector ζ .

4.2.2 Quantile Function and Density Expansion

The quantile function $Q(u)$ and density expansion of the MO-GBin distribution can be obtained from equation (2.7) and (2.8) respectively and they are given by;

$$Q(u) = G^{-1} \left\{ 1 - \frac{(A - uA - 1)^{1/m} - 1}{\theta \delta + \delta[(A - uA - 1)^{1/m} - 1]} \right\} \quad (4.44)$$

and

$$f_{MO-GBin}(x; \delta, \theta, \zeta) = \sum_{p=0}^n l_{\alpha}^* g_{\alpha}^*(x; \zeta) \quad (4.45)$$

where $A = (1 + \theta)^m - 1$ and $0 \leq u < 1$, $g_\alpha^*(x; \zeta)$ is an exponentiated-G (Exp-G) distribution with power parameter (α) ,

$$l_\alpha^* = \sum_{n,k=0}^{\infty} \frac{(n+1)(\theta\delta)^{n+1}(-1)^{k+p}\bar{\delta}^k}{(p+1)((1+\theta)^m-1)} \binom{m}{n+1} \binom{n+k+1}{k} \binom{n+k}{p} \quad (4.46)$$

for $\delta \in (0, 1)$ with $\alpha = p + 1$ and $g_\alpha^*(x; \zeta)$ being the same as equation (2.9),

$$l_\alpha^* = \sum_{n=0}^{\infty} \frac{(n+1)(\theta\delta)^{n+1}(-1)^{k+p}}{(p+k+1)((1+\theta)^m-1)} \delta^{-(n+2)} \left(\frac{\delta-1}{\delta}\right)^k \binom{m}{n+1} \binom{n+k+1}{k} \times \binom{n+k}{p} \quad (4.47)$$

for $\delta > 1$ with $\alpha = p + k + 1$ and $g_\alpha^*(x; \zeta)$ being the same as equation (2.12) and

$$l_\alpha^* = \sum_{n=0}^{\infty} \frac{(-1)^p(n+1)\theta^{n+1}}{(p+1)((1+\theta)^m-1)} \binom{m}{n+1} \binom{n}{p} \quad (4.48)$$

for $\delta = 1$ with $\alpha = p + 1$ and $g_\alpha^*(x; \zeta)$ being the same as equation (2.9).

4.2.3 Moments and Moment Generating Function

The r^{th} moment of Y , say c'_r and the r^{th} incomplete moment of the MO-GBin can be obtained using equation (2.16) and (2.20) respectively and they are given by

$$c'_r = E(Y^r) = \sum_{p=0}^{\infty} l_\alpha^* E(Y_\alpha^r), \quad (4.49)$$

and

$$c_X(t) = \int_0^t x^r f_{MO-GBin}(x; \delta, \theta, \zeta) dx = \sum_{p=0}^{\infty} l_\alpha^* I_\alpha(t), \quad (4.50)$$

respectively, where $I_\alpha(t) = \int_0^t x^r g_\alpha^*(x; \zeta) dx$ and l_α^* is as defined in equations (4.46), (4.47) and (4.48) for $\delta \in (0, 1)$, $\delta > 1$ and $\delta = 1$ respectively.

Making use of equation (2.22), the probability weighted moment (PWM) of X for

MO-GBin is given by

$$\Xi_{i,j} = \sum_{p=0}^{\infty} n_{\alpha}^* \int_{-\infty}^{\infty} x^i g_{\alpha}^*(x; \zeta) dx = \sum_{p=0}^{\infty} n_{\alpha}^* E[Y_{\alpha}^i], \quad (4.51)$$

where $E[Y_{\alpha}^i]$ is the i^{th} moment of Exp-G density with power parameter (α) ,

$$\begin{aligned} n_{\alpha}^* &= \sum_{q,i,m,n=0}^{\infty} (-1)^{i+p+q} \frac{(n+1)(\delta\theta)^{n+i+1}\bar{\delta}^q}{(p+1)((1+\theta)^m - 1)^{i+1}} d_{r,i} \\ &\times \binom{m}{n+1} \binom{n+i+q+1}{q} \binom{n+i+q}{p} \binom{j}{i} \end{aligned} \quad (4.52)$$

for $\delta \in (0, 1)$ with $\alpha = p + 1$,

$$\begin{aligned} n_{\alpha}^* &= \sum_{n,i,r=0}^{\infty} (-1)^{i+p+k} \frac{(n+1)(\delta\theta)^{n+i+1}\bar{\delta}^k}{(p+k+1)((1+\theta)^m - 1)^{i+1}} d_{r,i} \\ &\times \binom{m}{n+1} \binom{n+i+k+1}{k} \binom{n+i+k}{p} \binom{j}{i} \delta^{-(n+i+2)} \left(\frac{\delta-1}{\delta}\right)^k \end{aligned} \quad (4.53)$$

for $\delta > 1$ with $\alpha = p + k + 1$,

$$\begin{aligned} n_{\alpha}^* &= \sum_{n,i,r=0}^{\infty} (-1)^{i+p} \frac{(n+1)\theta^{n+i+1}}{(p+1)((1+\theta)^m - 1)^{i+1}} d_{r,i} \\ &\times \binom{m}{n+1} \binom{n+i}{p} \binom{j}{i} \end{aligned} \quad (4.54)$$

for $\delta = 1$ with $\alpha = p + 1$, $d_{r,i} = (rb_0)^{-1} \sum_{h=1}^r [i(h+1) - r] b_n d_{r-h,i}$ and $d_{0,i} = b_0^i$. The moment generating function (mgf) of the MO-GBin say $M_Y(t) = E(e^{tY})$ of Y can be directly derived from equation (4.45) and is given by

$$M_Y(t) = \sum_{p=0}^{\infty} l_{\alpha}^* M_p(t), \quad (4.55)$$

where $M_p(t)$ is the mgf of Y_{α} and l_{α}^* is given in equations (4.46), (4.47) and (4.48) for $\delta \in (0, 1)$, $\delta > 1$ and $\delta = 1$ respectively. Hence, $M_Y(t)$ can be determined from the Exp-G generating function.

4.2.4 Mean deviation, Bonferroni and Lorenz Curves

Let $X \sim MO - GBin(\delta, \theta, \zeta)$, the mean deviation about the mean and about the median of the MO-GBin using equation (2.28) and (2.29) are given by

$$\begin{aligned}\alpha_{MO-GBin}^1(x) &= 2\mu F_{MO-GBin}(\mu) - 2 \int_0^\infty x f_{MO-GBin}(x; \delta, \theta, \zeta) dx \\ &= 2\mu F_{MO-GBin}(\mu) - 2 \int_0^\infty \sum_{p=0}^\infty x l_\alpha^* g_\alpha^*(x; \zeta) dx\end{aligned}\quad (4.56)$$

and

$$\begin{aligned}\alpha_{MO-GBin}^2(x) &= \mu - 2 \int_0^\infty x f_{MO-GBin}(x; \delta, \theta, \zeta) dx \\ &= \mu - 2 \int_0^\infty \sum_{p=0}^\infty x l_\alpha^* g_\alpha^*(x; \zeta) dx\end{aligned}\quad (4.57)$$

respectively where $\mu = E(X)$, l_α^* is given in equations (4.46), (4.47) and (4.48) and $g_\alpha^*(x; \zeta)$ is given in equation (2.9) (for $\delta \in (0, 1)$ and $\delta = 1$) and (2.12) (for $\delta > 1$). The Bonferroni and Lorenz curves for the MO-GBin distribution are defined by

$$\begin{aligned}B_{MO-GBin}(u) &= \frac{1}{u\mu} \int_0^v x f_{MO-GBin}(x; \delta, \theta, \zeta) dx \\ &= \frac{1}{u\mu} \int_0^v \sum_{p=0}^\infty x l_\alpha^* g_\alpha^*(x; \zeta) dx\end{aligned}\quad (4.58)$$

and

$$\begin{aligned}L_{MO-GBin}(u) &= \frac{1}{\mu} \int_0^v x f_{MO-GBin}(x; \delta, \theta, \zeta) dx \\ &= \frac{1}{\mu} \int_0^v \sum_{p=0}^\infty x l_\alpha^* g_\alpha^*(x; \zeta) dx\end{aligned}\quad (4.59)$$

where $\mu = E(X)$, $v = Q(u)$, l_α^* is given in equations (4.46), (4.47) and (4.48) and $g_\alpha^*(x; \zeta)$ is given in equation (2.9) (for $\delta \in (0, 1)$ and $\delta = 1$) and (2.12) (for $\delta > 1$).

4.2.5 Residual and reverse residual life function

Using results from equation (2.32) and (2.36), the n^{th} moment of residual life of a MO-GBin is given by;

$$a_{(MO-GBin;n)}^*(t) = \frac{1}{\bar{G}(x; \zeta)} \sum_{p=0}^{\infty} l_{\alpha}^* \sum_{m=0}^{\infty} \binom{n}{m} (-t)^m \int_t^{\infty} x^{n-m} g_{\alpha}^*(x; \zeta) dx. \quad (4.60)$$

The n^{th} moment of reverse residual life of a MO-GBin is given by;

$$b_{(MO-GBin;n)}^*(t) = \frac{1}{G(x; \zeta)} \sum_{p=0}^{\infty} l_{\alpha}^* \sum_{m=0}^{\infty} \binom{n}{m} (t)^{n-m} (-1)^m \int_0^t x^m g_{\alpha}^*(x; \zeta) dx, \quad (4.61)$$

where $g_{\alpha}^*(x; \zeta)$ is an Exp-G distribution with power parameter (α), l_{α}^* is given in equations (4.46), (4.47) and (4.48) and $g_{\alpha}^*(x; \zeta)$ is given in equation (2.9) (for $\delta \in (0, 1)$ and $\delta = 1$) and (2.12) (for $\delta > 1$) and $\bar{G}(x; \zeta)$ is the survival function.

4.2.6 Stress-Strength Model

For X and Y being independent $F_{MO-GBin}(x; \delta_1, \theta_1, \zeta_1)$ and $F_{MO-GBin}(y; \delta_2, \theta_2, \zeta_2)$ respectively and for the stress-strength parameter $R_{MO-GBin} = P(Y < X)$, then, using equation (2.39), the reliability parameter for MO-GBin distribution is given by

$$\begin{aligned} R_{MO-GBin} &= \int_{x=0}^{\infty} \frac{\theta_1 \delta_1 m_1 g(x; \zeta_1) \left[1 + \frac{\theta_1 \delta_1 \bar{G}(x; \zeta_1)}{1 - \delta_1 \bar{G}(x; \zeta_1)} \right]^{m_1 - 1}}{(1 - \delta_1 \bar{G}(x; \zeta_1))^2 [(1 + \theta_1)^{m_1} - 1]} \\ &\times \left(1 - \frac{\left[1 + \frac{\theta_2 \delta_2 \bar{G}(x; \zeta_2)}{1 - \delta_2 \bar{G}(x; \zeta_2)} \right]^{m_2} - 1}{(1 + \theta_2)^{m_2} - 1} \right) dx. \end{aligned} \quad (4.62)$$

4.2.7 Order Statistics and Rényi Entropy

Let $X_1, X_2, \dots, X_m$ be independent and identically distributed MO-GBin random variables. Then, by making use of equation (2.47), the pdf of the k^{th} order statistics ($X_{k:m}$) of the MO-GBin can be written as

$$f_{k:m}(x) = \sum_{p=0}^{\infty} m_{\alpha}^* g_{\alpha}^*(x; \zeta) \quad (4.63)$$

where

$$m_\alpha^* = \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r,q=0}^{\infty} (-1)^{l+i+p+q} \frac{(n+1)(\delta\theta)^{n+i+1}\bar{\delta}^q}{(p+1)((1+\theta)^m-1)^{i+1}} d_{r,i} \\ \times \binom{m}{n+1} \binom{n+i+q+1}{q} \binom{m-k}{l} \binom{n+i+q}{p} \binom{k+l-1}{i}.$$

for $\delta \in (0, 1)$ with $\alpha = p + 1$,

$$m_\alpha^* = \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r=0}^{\infty} (-1)^{l+i+p+q} \frac{(n+1)(\delta\theta)^{n+i+1}\bar{\delta}^q}{(p+q+1)((1+\theta)^m-1)^{i+1}} d_{r,i} \\ \times \binom{m}{n+1} \binom{n+i+q+1}{q} \binom{m-k}{l} \binom{n+i+q}{p} \binom{k+l-1}{i} \delta^{-(n+i+2)} \left(\frac{\delta-1}{\delta}\right)^q.$$

for $\delta > 1$ with $\alpha = p + q + 1$,

$$m_\alpha^* = \frac{m!}{(k-1)!(m-k)!} \sum_{l,n,i,r=0}^{\infty} (-1)^{l+i+p} \frac{(n+1)\theta^{n+i+1}}{(p+1)((1+\theta)^m-1)^{i+1}} d_{r,i} \\ \times \binom{m}{n+1} \binom{m-k}{l} \binom{n+i}{p} \binom{k+l-1}{i}.$$

for $\delta = 1$ with $\alpha = p + 1$, $d_{r,i} = (rb_0)^{-1} \sum_{h=1}^r [i(h+1) - r] b_n d_{r-h,i}$ and $d_{0,i} = b_0^i$.

Equations (2.41), (2.42) and (2.43) are used to derive the Rényi entropy of the MO-GBin. The Rényi Entropy for the MO-GBin distribution for $\delta \in (0, 1)$, $\delta > 1$ and $\delta = 1$ are given by

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{n,s,k,p,q=0}^{\infty} b_{s,k} \left(\frac{\theta\delta}{((1+\theta)^m-1)} \right)^\nu (\delta\theta)^{ns} \bar{\delta}^q (-1)^{p+q} \binom{ns+q}{p} \right. \\ \left. \times \binom{2\nu+ns+q-1}{q} \int_0^\infty g^\nu(x; \zeta) G^p(x; \zeta) dx \right], \quad (4.64)$$

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{n,s,k,p,q=0}^{\infty} b_{s,k} \left(\frac{\theta\delta}{((1+\theta)^m-1)} \right)^\nu (\delta\theta)^{ns} \bar{\delta}^q (-1)^{p+q} \binom{ns+q}{p} \right. \\ \left. \times \binom{2\nu+ns+q-1}{q} \delta^{-(2\nu+ns)} \left(\frac{\delta-1}{\delta} \right)^q \int_0^\infty g^\nu(x; \zeta) G^{p+q}(x; \zeta) dx \right], \quad (4.65)$$

and

$$\begin{aligned}
I_R(\nu) &= \frac{1}{1-\nu} \log \left[\sum_{n,s,k,p=0}^{\infty} b_{s,k} \left(\frac{\theta}{((1+\theta)^m - 1)} \right)^{\nu} \theta^{ns} (-1)^p \binom{ns}{p} \right. \\
&\quad \times \left. \int_0^{\infty} g^{\nu}(x; \zeta) G^p(x; \zeta) dx \right]
\end{aligned} \tag{4.66}$$

respectively, where $b_{s,k} = (sa_0)^{-1} \sum_{l=1}^s [k(l+1) - s] a_l b_{s-l,k}$, $b_{0,k} = a_0^k$, $\nu \neq 1$ and $\nu > 0$.

4.2.8 Maximum Likelihood Estimation

We consider section 3.1 to obtain the maximum likelihood estimates of the parameters of the MO-GBin distribution and the Fisher information matrix can be applied to create confidence zones and intervals for specific model parameters.

4.2.8.1 Estimation of parameters for uncensored data

Let $X \sim \text{MO-GBin}(\delta, \theta, \zeta)$ and $\Delta = (\delta, \theta, \zeta)^T$ the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the MO-GBin distributions is given by

$$\begin{aligned}
\ell_n(\Delta) &= n \log(\theta\delta) + \sum_{i=1}^n \log(g(x_i; \zeta)) - 2 \sum_{i=1}^n \log(1 - \bar{\delta}\bar{G}(x_i; \zeta)) \\
&\quad + \sum_{i=1}^n \log \left[m \left(1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{m-1} \right] - n \log((1+\theta)^m - 1), \\
U_{\theta}(\Delta) &= \frac{n}{\theta} - \frac{nm(1+\theta)^{m-1}}{(1+\theta)^m - 1} + \sum_{i=1}^n (m-1) \left[1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]^{-1} \left(\frac{\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right), \\
U_{\delta}(\Delta) &= \frac{n}{\delta} - 2 \sum_{i=1}^n \frac{\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} + \sum_{i=1}^n (m-1) \left[1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]^{-1} \\
&\quad \times \left(\frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)
\end{aligned}$$

and

$$\begin{aligned}
U_{\zeta_k}(\Delta) &= \sum_{i=1}^n \partial[\log(g(x_i; \zeta))]/\partial\zeta_k - 2 \sum_{i=1}^n \partial[\log(1 - \bar{\delta}\bar{G}(x_i; \zeta))]/\partial\zeta_k \\
&\quad + \sum_{i=1}^n \partial \left[\log \left[m \left(1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{m-1} \right] \right] / \partial\zeta_k.
\end{aligned}$$

4.2.8.2 Estimation of parameters for censored data

Let $X \sim \text{MO-GBin}(\delta, \theta, \zeta)$ and $\Delta = (\delta, \theta, \zeta)^T$ the vector of model parameters, then, using equation (3.1), the log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the MO-GBin distributions is given by

$$\begin{aligned}
\ell_n(\Delta) &= \log(A) + n \log(\theta\delta) + \sum_{i=1}^n \log(g(x_i; \zeta)) - 2 \sum_{i=1}^n \log(1 - \bar{\delta}\bar{G}(x_i; \zeta)) \\
&+ \sum_{i=1}^n \log \left[m \left(1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{m-1} \right] - n \log((1 + \theta)^m - 1) \\
&+ \sum_{i=r+1}^n \log \left[\left(1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^m - 1 \right], \\
U_\theta(\Delta) &= \frac{n}{\theta} - \frac{nm(1 + \theta)^{m-1}}{(1 + \theta)^m - 1} + \sum_{i=1}^r (m-1) \left[1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]^{-1} \left(\frac{\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \\
&+ \sum_{i=r+1}^n \left(\frac{m \left(1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{m-1}}{\left(1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^m - 1} \left(\frac{\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right) \right), \\
U_\delta(\Delta) &= \frac{n}{\delta} - 2 \sum_{i=1}^r \frac{\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} + \sum_{i=1}^r (m-1) \left[1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]^{-1} \\
&\times \left(\frac{\theta\bar{G}(x_i; \zeta)}{(1 - \bar{\delta}\bar{G}(x_i; \zeta))} \right) + \sum_{i=r+1}^n \frac{m \left[1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]^{m-1}}{\left[1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right]^m - 1} \\
&\times \left(\frac{\theta\bar{G}(x_i; \zeta)}{(1 - \bar{\delta}\bar{G}(x_i; \zeta))} \right)
\end{aligned}$$

and

$$\begin{aligned}
U_{\zeta_k}(\Delta) &= \sum_{i=1}^r \partial[\log(g(x_i; \zeta))]/\partial\zeta_k - 2 \sum_{i=1}^r \partial[\log(1 - \bar{\delta}\bar{G}(x_i; \zeta))]/\partial\zeta_k \\
&+ \sum_{i=1}^r \partial \left[\log \left[m \left(1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^{m-1} \right] \right] / \partial\zeta_k \\
&+ \sum_{i=r+1}^n \left(\partial \left[\log \left[\left(1 + \frac{\theta\delta\bar{G}(x_i; \zeta)}{1 - \bar{\delta}\bar{G}(x_i; \zeta)} \right)^m - 1 \right] \right] / \partial\zeta_k \right).
\end{aligned}$$

4.2.9 Special Cases of MO-GBin distribution

In this Section, we present the special cases of the MO-GBin distribution for different baseline distribution. The pdf and hrf plots are also presented.

4.2.9.1 Marshall-Olkin Weibull Binomial distribution

Suppose the baseline distribution is the Weibull distribution with cdf and pdf given by $G(x; a, b) = 1 - e^{-bx^a}$ and $g(x; a, b) = abx^{a-1}e^{-bx^a}$, for $a, b > 0$ then, the Marshall-Olkin Weibull Binomial (MO-WBin) distribution has cdf and pdf given by;

$$F_{MO-WBin}(x; \delta, \theta, a, b, m) = 1 - \frac{\left[1 + \frac{\theta \delta e^{-bx^a}}{1 - \delta e^{-bx^a}}\right]^m - 1}{(1 + \theta)^m - 1}, \quad (4.67)$$

and

$$\begin{aligned} f_{MO-WBin}(x; \delta, \theta, a, b, m) &= \frac{\theta \delta m a b x^{a-1} e^{-bx^a} \left[1 + \frac{\theta \delta e^{-bx^a}}{1 - \delta e^{-bx^a}}\right]^{m-1}}{[1 - \bar{\delta} e^{-bx^a}]^2 (1 + \theta)^m - 1} \\ &= \sum_{n,k=0}^{\infty} l_{\alpha}^* g_{\alpha}^*(x; \zeta), \end{aligned} \quad (4.68)$$

for $\delta, \theta, a, b > 0$, $\bar{\delta} = 1 - \delta$, l_{α}^* is given in equations (4.46), (4.47) and (4.48) and $g_{\alpha}^*(x; \zeta)$ is given in equations (2.9) and (2.12).

The corresponding survival and hazard rate function are given by

$$S_{MO-WBin}(t) = \frac{\left[1 + \frac{\theta \delta e^{-bt^a}}{1 - \delta e^{-bt^a}}\right]^m - 1}{(1 + \theta)^m - 1}, \quad (4.69)$$

and

$$\begin{aligned} h_{MO-WBin}(t) &= \frac{\theta \delta m a b t^{a-1} e^{-bt^a} \left[1 + \frac{\theta \delta e^{-bt^a}}{1 - \delta e^{-bt^a}}\right]^{m-1}}{[1 - \bar{\delta} e^{-bt^a}]^2} \\ &\times \left(\left[1 + \frac{\theta \delta e^{-bt^a}}{1 - \bar{\delta} e^{-bt^a}}\right]^m - 1 \right)^{-1} \end{aligned} \quad (4.70)$$

respectively, for $\delta, \theta, m, a, b > 0$, and $\bar{\delta} = 1 - \delta$.

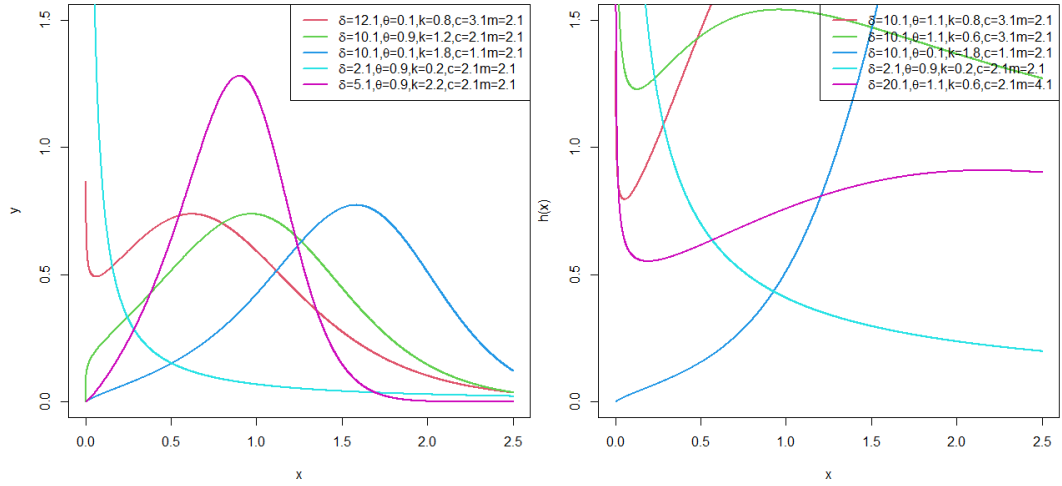


Figure 4.4: Plots of the pdf and hrf for the MO-WBin distribution

The pdf of the MO-WBin can handle different levels of skewed data. The hrf of the this distribution is exhibiting increasing, decreasing, bathtub and bathtub followed by upside down bathtub shapes.

4.2.9.2 Marshall-Olkin Burr-XII Binomial distribution

Suppose the baseline distribution is the Burr-XII distribution with cdf and pdf given by $G(x; c, k) = 1 - (1 + x^c)^{-k}$ and $g(x; c, k) = ckx^{c-1}(1 + x^c)^{-k-1}$, for $a, b > 0$ and $x > 0$, then, the Marshall-Olkin Burr-XII Binomial (MO-BXII Bin) distribution has cdf and pdf given by;

$$F_{MO-BXII Bin}(x; \delta, \theta, c, k, m) = 1 - \frac{\left[1 + \frac{\theta\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right]^m - 1}{(1 + \theta)^m - 1}, \quad (4.71)$$

and

$$\begin{aligned} f_{MO-BXII Bin}(x; \delta, \theta, c, k, m) &= \frac{\theta\delta mckx^{c-1}(1+x^c)^{-k-1} \left[1 + \frac{\theta\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right]^{m-1}}{[1 - \bar{\delta}(1+x^c)^{-k}]^2(1 + \theta)^m - 1} \\ &= \sum_{n,k=0}^{\infty} l_{\alpha}^* g_{\alpha}^*(x; \zeta), \end{aligned} \quad (4.72)$$

for $\delta, \theta, c, k > 0$, $\bar{\delta} = 1 - \delta$, l_{α}^* is given in equations (4.46), (4.47) and (4.48) and $g_{\alpha}^*(x; \zeta)$ is given in equations (2.9) and (2.12).

The corresponding survival and hazard rate function are given by

$$S_{MO-BXIIBin}(t) = \frac{\left[1 + \frac{\theta\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right]^m - 1}{(1+\theta)^m - 1}, \quad (4.73)$$

and

$$\begin{aligned} h_{MO-BXIIBin}(t) &= \frac{\theta\delta mckx^{c-1}(1+x^c)^{-k-1} \left[1 + \frac{\theta\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right]^{m-1}}{[1 - \bar{\delta}(1+x^c)^{-k}]^2} \\ &\times \left(\left[1 + \frac{\theta\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right]^m - 1 \right)^{-1} \end{aligned} \quad (4.74)$$

respectively, for $\delta, c, k, m, \theta > 0$ and $\bar{\delta} = 1 - \delta$.

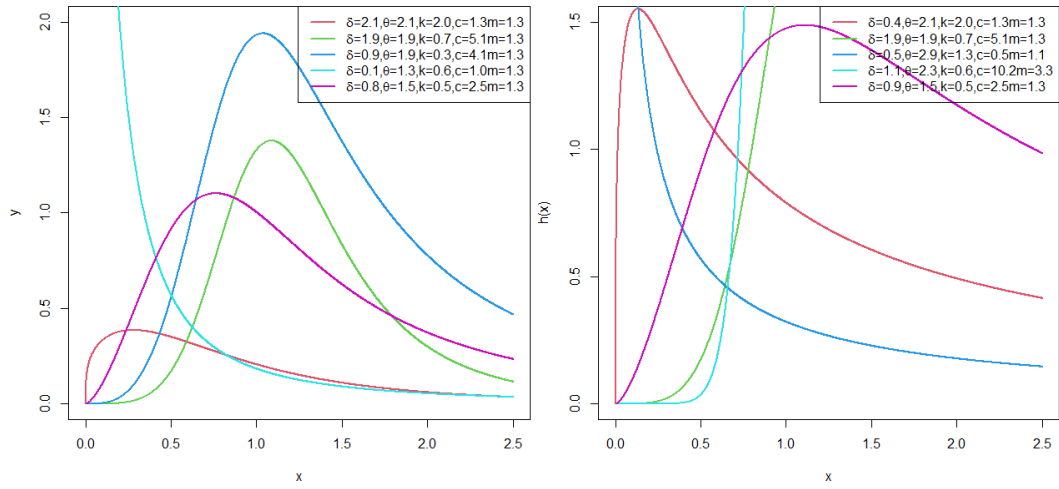


Figure 4.5: Plots of the pdf and hrf for the MO-BXIIBin distribution

The pdf of the MO-BXIIBin can handle data set that is right skewed and almost symmetric, the hrf of the this distribution is showing increasing, decreasing and upside down bathtub shapes.

4.2.9.3 Marshall-Olkin Topp-Leone Binomial distribution

Suppose the baseline distribution is the Topp-Leone distribution with cdf and pdf given by $G(x; a) = (1 - (1 - x)^2)^a$ and $g(x; a) = 2a(1 - x)(1 - (1 - x)^2)^{a-1}$, for $a > 0$ and $x > 0$, then, the Marshall-Olkin-Topp-Leone Binomial (MO-TLBin) distribution

has cdf and pdf given by;

$$F_{MO-TLBin}(x) = 1 - \frac{\left[1 + \frac{\theta\delta(1-(1-(1-x)^2)^a)}{1-\bar{\delta}(1-(1-(1-x)^2)^a)}\right]^m - 1}{(1+\theta)^m - 1}, \quad (4.75)$$

and

$$\begin{aligned} f_{MO-TLBin}(x) &= \frac{2\theta\delta ma(1-x)(1-(1-x)^2)^{a-1} \left[1 + \frac{\theta\delta(1-(1-(1-x)^2)^a)}{1-\bar{\delta}(1-(1-(1-x)^2)^a)}\right]^{m-1}}{(1-\bar{\delta}(1-(1-(1-x)^2)^a))^2 [(1+\theta)^m - 1]} \\ &= \sum_{n,k=0}^{\infty} l_{\alpha}^* g_{\alpha}^*(x; \zeta), \end{aligned} \quad (4.76)$$

for $\delta, \theta, a > 0$, $\bar{\delta} = 1 - \delta$, l_{α}^* is given in equations (4.46), (4.47) and (4.48) and $g_{\alpha}^*(x; \zeta)$ is given in equations (2.9) and (2.12).

The corresponding survival and hazard rate function are given by

$$S_{MO-TLBin}(t) = \frac{\left[1 + \frac{\theta\delta(1-(1-(1-x)^2)^a)}{1-\bar{\delta}(1-(1-(1-x)^2)^a)}\right]^m - 1}{(1+\theta)^m - 1}, \quad (4.77)$$

and

$$\begin{aligned} h_{MO-TLBin}(t) &= \frac{2\theta\delta ma(1-x)(1-(1-x)^2)^{a-1} \left[1 + \frac{\theta\delta(1-(1-(1-x)^2)^a)}{1-\bar{\delta}(1-(1-(1-x)^2)^a)}\right]^{m-1}}{(1-\bar{\delta}(1-(1-(1-x)^2)^a))^2} \\ &\times \left(\left[1 + \frac{\theta\delta(1-(1-(1-x)^2)^a)}{1-\bar{\delta}(1-(1-(1-x)^2)^a)}\right]^m - 1 \right)^{-1} \end{aligned} \quad (4.78)$$

respectively, for $\delta, \theta, m, a > 0$ and $\bar{\delta} = 1 - \delta$.

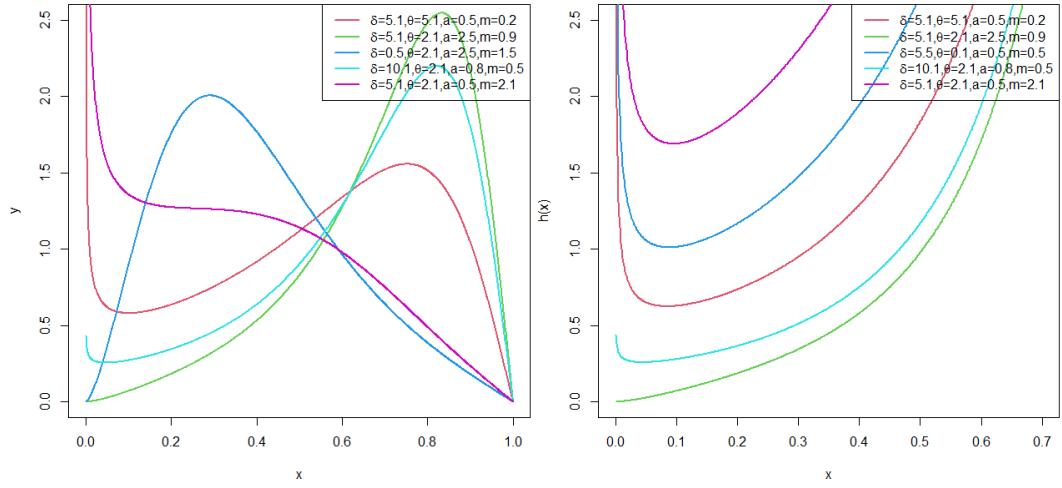


Figure 4.6: Plots of the pdf and hrf for the MO-TLBin distribution

The pdf of the MO-TLBin can handle data set with different levels of skewness and the hrf of the this distribution is exhibiting increasing and bathtub shapes.

5. Simulations and Applications

5.1 Simulation Study

We conducted a simulation study to efficiently and consistently estimate the parameters of the MO-GPS distribution using the special case of MO-WLoG distribution at different sample sizes. We used the MLE technique since MLEs are efficient, consistent, asymptotically normal and invariant under transformation. Let θ be the population parameter and $\hat{\theta}$ be the estimate of the population parameter. Average Bias (ABIAS) calculates the deviation of $\hat{\theta}$ from the true population parameter value of θ . Now we assess the behaviour of ABias as we increase the population size n . For $\hat{\theta}$ to be a good estimator of θ , ABias has to decrease as we increase n . Root Mean Square Error (RMSE) is the standard deviation of the residuals (prediction errors). Residuals are the difference between values predicted by the model and observed values. As such, if RMSE decreases as n increases then $\hat{\theta}$ is said to be a good estimator of θ . We simulated for complete data but however for censored data we can simulate using the maximization function MaxBFGS in Ox Console [19]. Tables 5.1 and Table 5.2 show the simulation results for selected initial parameter values. The ABIAS and RMSE for the estimated parameter ($\hat{\theta}$) are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

5.1.1 MO-WLoG Simulations

parameter	Sample Size	(0.5, 0.5, 2.5, 1.0)			(0.5, 0.02, 0.8, 1.0)		
		Mean	RMSE	ABias	Mean	RMSE	A.Bias
δ	25	1.2072	1.7670	0.7072	0.7498	1.0450	0.2498
	50	1.3134	1.7907	0.8134	0.6390	0.7342	0.1390
	100	1.1786	1.2113	0.6786	0.6040	0.5438	0.1040
	200	0.9874	0.9488	0.4874	0.5898	0.4013	0.0898
	400	0.8081	0.5831	0.3081	0.5831	0.3561	0.0831
	800	0.7085	0.4012	0.2085	0.5457	0.2230	0.0457
	1000	0.6820	0.3535	0.1820	0.5591	0.2444	0.0591
θ	25	0.6692	0.4054	0.1692	0.3911	0.5643	0.3711
	50	0.6804	0.4089	0.1804	0.2847	0.4641	0.2647
	100	0.7040	0.3964	0.2040	0.2355	0.4246	0.2155
	200	0.6689	0.3814	0.1689	0.1936	0.3756	0.1736
	400	0.6376	0.3686	0.1376	0.1769	0.3601	0.1569
	800	0.6010	0.3559	0.1010	0.1086	0.2609	0.0886
	1000	0.5929	0.3416	0.0929	0.1319	0.2969	0.1119
a	25	2.9784	0.8097	0.4784	0.9596	0.2618	0.1596
	50	2.9112	0.6973	0.4112	0.9215	0.1902	0.1215
	100	2.7748	0.5085	0.2748	0.8821	0.1450	0.0821
	200	2.7083	0.4159	0.2083	0.8440	0.1022	0.0440
	400	2.6589	0.2951	0.1589	0.8368	0.0829	0.0368
	800	2.6154	0.2197	0.1154	0.8162	0.0547	0.0162
	1000	2.6021	0.1979	0.1021	0.8196	0.0536	0.0196
b	25	0.9856	0.5902	-0.0144	0.8393	0.4301	-0.1607
	50	1.0203	0.5476	0.0203	0.8203	0.3881	-0.1797
	100	1.0250	0.4705	0.0250	0.8638	0.3427	-0.1362
	200	1.0087	0.4075	0.0087	0.9063	0.2750	-0.0937
	400	0.9757	0.2730	-0.0243	0.9197	0.2176	-0.0803
	800	0.9816	0.2000	-0.0184	0.9682	0.1613	-0.0318
	1000	0.9822	0.1816	-0.0178	0.9632	0.1493	-0.0368

Table 5.1: MO-WLoG Simulation Results 1

parameter	Sample Size	(0.5, 0.5, 1.8, 1.0)			(0.5, 0.5, 1.4, 1.0)		
		Mean	RMSE	Bias	Mean	RMSE	A.Bias
δ	25	1.1207	1.5045	0.6207	1.1635	1.7627	0.6635
	50	1.3014	1.7150	0.8014	1.1661	1.4356	0.6661
	100	1.1192	1.1835	0.6192	1.1948	1.4551	0.6948
	200	0.9354	0.8618	0.4354	0.9583	0.8890	0.4583
	400	0.7948	0.5612	0.2948	0.8130	0.5803	0.3130
	800	0.7008	0.3902	0.2008	0.6993	0.3827	0.1993
	1000	0.6852	0.3606	0.1852	0.6715	0.3494	0.1715
θ	25	0.6308	0.4142	0.1308	0.7049	0.4118	0.2049
	50	0.6880	0.4053	0.1880	0.6415	0.4007	0.1415
	100	0.6710	0.3931	0.1710	0.6703	0.3990	0.1703
	200	0.6624	0.3855	0.1624	0.6610	0.3823	0.1610
	400	0.6324	0.3706	0.1324	0.6542	0.3719	0.1542
	800	0.5945	0.3571	0.0945	0.6020	0.3553	0.1020
	1000	0.5937	0.3385	0.0937	0.5872	0.3400	0.0872
a	25	2.1994	0.5973	0.3994	1.7354	0.5033	0.3354
	50	2.0915	0.5178	0.2915	1.6198	0.3688	0.2198
	100	1.9898	0.3645	0.1898	1.5577	0.2943	0.1577
	200	1.9649	0.3104	0.1649	1.5228	0.2475	0.1228
	400	1.9163	0.2131	0.1163	1.4976	0.1744	0.0976
	800	1.8832	0.1600	0.0832	1.4678	0.1247	0.0678
	1000	1.8692	0.1369	0.0692	1.4573	0.1090	0.0573
b	25	0.9623	0.5856	-0.0377	0.9165	0.6223	-0.0835
	50	1.0194	0.5912	0.0194	0.9887	0.5444	-0.0113
	100	1.0261	0.4863	0.0261	1.0173	0.4876	0.0173
	200	0.9742	0.3735	-0.0258	0.9906	0.4025	-0.0094
	400	0.9741	0.2689	-0.0259	0.9676	0.2772	-0.0324
	800	0.9809	0.1993	-0.0191	0.9749	0.1966	-0.0251
	1000	0.9859	0.1811	-0.0141	0.9785	0.1826	-0.0215

Table 5.2: MO-WLoG Simulation Results 2

It can be clearly seen that as sample size n increases, the values of the root mean square error (RMSE) decays and average bias (ABIAS) decreases which shows that the maximum likelihood estimates are consistent. The simulation results presented shows that the special cases of the MO-GPS produces reliable results when estimating parameters for the model.

5.2 Applications

Using real data set, we illustrate the performance of the MO-GPS family of distributions by fitting the special case of the MO-WLoG distribution. The goodness-of-fit statistics mentioned in Section 1.2.16 are presented. The model with the smallest values of these statistics and a higher p-value for the K-S statistic, is identified as the best fitting model.

We used R software to estimate the model parameters via the **nlm** function. Model parameter estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the selected data sets are shown in Tables 5.3 and 5.4. We also present plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [15]) to show how well our model fits the observed data sets.

We compared the special cases distribution with other competing four parameter non-nested models, namely the Exponentiated Generalized Logarithmic (EGEL) distribution by Oluyede et al.[29], Odd Weibull-Topp-Leone-Log Logistic Logarithmic (OW-TL-LLoGL) distribution by Oluyede et al. [28], Exponentiated Power Lindley Poisson (EPLP) distribution by Pararai et al. [38], Exponentiated Log-Logistic Poisson (ELoLP) by da Silva et al. [18], Topp-Leone- Gompertz-Weibull (TLGW) by Oluyede et al. [30] and Generalized Gompertz-Poisson (GGP) by Saeed and Ali [32] . The pdfs of these distributions are given by:

$$f_{EGEL}(x; p, \alpha, \beta, \lambda) = \frac{p\alpha\beta\lambda e^{-\alpha\lambda x}[1 - e^{-\alpha\lambda x}]^{\beta-1}}{\ln(1-p)[p(1 - e^{-\alpha\lambda x})^{\beta} - 1]},$$

for $p, \alpha, \beta, \lambda > 0$,

$$\begin{aligned} f_{OWTLLLoGL}(x; \alpha, \lambda, \gamma, \theta) &= \frac{2\theta\gamma\alpha\lambda x^{\lambda-1}(1+x^\lambda)^{-3}[1-(1+x^\lambda)^{-2}]^{\gamma\alpha-1}}{[1-(1-(1+x^\lambda)^{-2})^\gamma]^{\alpha+1}} \\ &\times \exp\left(-\left[\frac{[1-(1+x^\lambda)^{-2}]^\gamma}{[1-(1-(1+x^\lambda)^{-2})^\gamma]}\right]^\alpha\right) \\ &\times \frac{\left(1-\theta\left(\exp\left(-\left[\frac{[1-(1+x^\lambda)^{-2}]^\gamma}{[1-(1-(1+x^\lambda)^{-2})^\gamma]}\right]^\alpha\right)\right)\right)^{-1}}{-\log(1-\theta)}, \end{aligned}$$

for $\alpha, \lambda, \gamma, \theta > 0$,

$$\begin{aligned} f_{EPLP}(x; \alpha, \beta, \theta, \omega) &= \frac{\alpha\beta^2\omega\theta(1+y^\alpha)y^{\alpha-1}e^{-\beta y^\alpha}}{(1+\beta)(e^\theta-1)} \left[1-\left(1+\frac{\beta y^\alpha}{\beta+1}\right)e^{-\beta y^\alpha}\right]^{\omega-1} \\ &\times \exp\left(\theta\left[1-\left(1+\frac{\beta y^\alpha}{\beta+1}\right)e^{-\beta y^\alpha}\right]^\omega\right), \end{aligned}$$

for $\alpha, \beta, \theta, \omega > 0$,

$$\begin{aligned} f_{ELoLP}(x; c, s, \alpha, \lambda) &= \frac{\alpha\lambda cs^{-c}x^{c-1}}{1-e^{-\lambda}} \left(1+\left(\frac{x}{s}\right)^c\right)^{-2} \left(1-\left(1+\left(\frac{x}{s}\right)^c\right)^{-1}\right)^{\alpha-1} \\ &\times \exp\left[-\lambda\left(1-\left(1+\left(\frac{x}{s}\right)^c\right)^{-1}\right)^\alpha\right], \end{aligned}$$

for $c, s, \alpha, \lambda > 0$,

$$\begin{aligned} f_{TL-Gom-W}(x; b, \gamma, \theta, \lambda) &= 2b\theta\beta\gamma x^{\beta-1}e^{-x^\beta}(e^{-x^\beta})^{-\gamma-1} \exp\left[\frac{\theta}{\gamma}(1-(e^{-x^\beta})^{-\gamma})\right] \\ &\times \left[1-\left(\exp\left[\frac{\theta}{\gamma}(1-(e^{-x^\beta})^{-\gamma})\right]\right)^2\right]^{b-1}, \end{aligned}$$

for $b, \gamma, c, \lambda > 0$ and

$$f_{GPP}(x; \theta, \gamma, \beta, \lambda) = \theta\lambda\beta e^{\gamma x-\theta}(1-t)t^{\lambda-1}e^{\theta t^\lambda}$$

for $\theta, \gamma, \beta, \lambda > 0$ and $t = 1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}$.

5.2.1 Complete Data

5.2.1.1 Remission Times of 128 Bladder Cancer Patients

The data consists of the remission times of 128 bladder cancer patients (see Lee and Wang [27] for details). The data set is as follows:

0.08 4.98 25.74 3.7 10.06 2.69 7.62 1.26 7.87 4.4 2.02 21.73 2.09 6.97 0.5 5.17 14.77
4.18 10.75 2.83 11.64 5.85 3.31 2.07 3.48 9.02 2.46 7.28 32.15 5.34 16.62 4.33 17.36
8.26 4.51 3.36 4.87 13.29 3.64 9.74 2.64 7.59 43.01 5.49 1.4 11.98 6.54 6.93 6.94 0.4
5.09 14.76 3.88 10.66 1.19 7.66 3.02 19.13 8.53 8.65 8.66 2.26 7.26 26.31 5.32 15.96
2.75 11.25 4.34 1.76 12.03 12.63 13.11 3.57 9.47 0.81 7.39 36.66 4.26 17.14 5.71 3.25
20.28 22.69 23.63 5.06 14.24 2.62 10.34 1.05 5.41 79.05 7.93 4.5 2.02 0.2 7.09 25.82
3.82 14.83 2.69 7.63 1.35 11.79 6.25 3.36 2.23 9.22 0.51 5.32 34.26 4.23 17.12 2.87
18.1 8.37 6.76 3.52 13.8 2.54 7.32 0.9 5.41 46.12 5.62 1.46 12.02 12.07

Model	Estimates				Statistics							
	δ	θ	a	b	$-2\log L$	AIC	AICC	BIC	W*	A*	K-S	p-value
MO-WLog	0.0638 (0.0642)	6.8469×10^{-6} (0.5025)	1.6042 (0.1629)	0.0033 (0.0038)	820.1842	828.1842	828.5094	839.5923	0.0284	0.2041	0.0324	0.9993
OW-TL-LLoGL	α	λ	γ	θ	1118.988	1126.988	1127.314	1138.396	5.0262	24.9088	0.9166	2.2×10^{-16}
	0.1801 (0.0305)	0.9741 (0.1372)	0.9771 (0.2614)	4.59×10^{-9} (0.0069)								
EGEL	p	α	β	λ	826.1552	834.1552	834.4804	845.5633	0.1122	0.6741	0.0725	0.5111
	1.4532×10^{-7} (0.0391)	0.0870 (0.0097)	1.2180 (0.1490)	1.3912 (0.0006)								
EPLP	θ	α	β	ω	820.867	828.867	829.1922	840.2751	0.0392	0.2568	0.0429	0.9725
	2.2382×10^{-7} (0.0305)	0.5666 (0.1016)	0.8182 (0.3112)	2.7647 (1.2868)								
ELoLP	c	s	α	λ	982.851	990.8544	991.1796	1002.263	0.1863	1.2345	0.4709	2.2×10^{-16}
	0.7517 (0.0721)	1.0984 (0.1809)	1.1102 (0.1021)	3.0195×10^{-9} (0.0051)								
TLGW	β	λ	θ	γ	821.1583	829.1583	829.4836	840.5665	0.0417	0.2747	0.0438	0.9662
	0.0235 (0.0167)	4.5143 (1.1940)	5.7081 (1.8968)	0.0913 (0.0296)								
GPP	β	λ	θ	γ	934.7457	942.7488	943.074	954.1569	0.2243	1.3329	0.3406	2.518×10^{-13}
	1.1012e (0.6681)	0.5063 (0.0575)	0.2062 (0.0600)	2.2494×10^{-10} (0.0072)								

Table 5.3: **Parameter estimates and goodness-of-fit statistics for various models fitted for the remission times of 128 bladder cancer patients data set**

Table 5.3 shows that the MO-WLog distribution has the highest P-values with the lowest goodness-of-fit statistic values and thus providing the overall best fit on the remission times of 128 bladder cancer patients data set as compared to non-nested models presented in this paper.

Here are some analysis based on the MO-WLog distribution since its providing the overall best fit. The variance-covariance matrix for MO-WLoG model the remission

times of 128 bladder cancer patients data set is given by

$$\begin{bmatrix} 0.0041 & 0.0081 & -0.0078 & 2.3634 \times 10^{-4} \\ 0.0081 & 0.2525 & 5.6918 \times 10^{-8} & -3.1323 \times 10^{-10} \\ -0.0078 & 5.6918 \times 10^{-8} & 0.0265 & -5.5639 \times 10^{-4} \\ 2.3634 \times 10^{-4} & -3.1323 \times 10^{-10} & -5.5639 \times 10^{-4} & 1.5173 \times 10^{-5} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$\delta \in [0.0638 \pm 0.1259]$, $\theta \in [6.8469 \times 10^{-6} \pm 0.9850]$, $a \in [1.6042 \pm 0.3193]$ and $b \in [0.0033 \pm 0.0076]$.

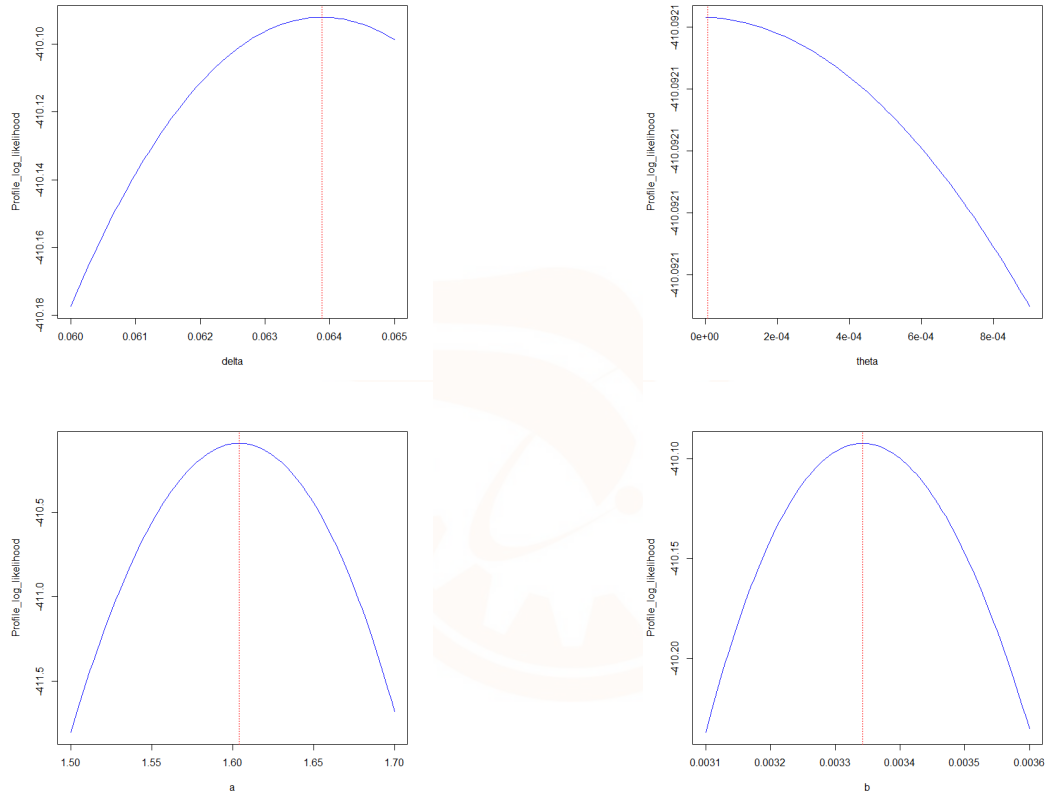


Figure 5.1: MO-WLoG distribution profile plots for the remission times of 128 bladder cancer patients data set.

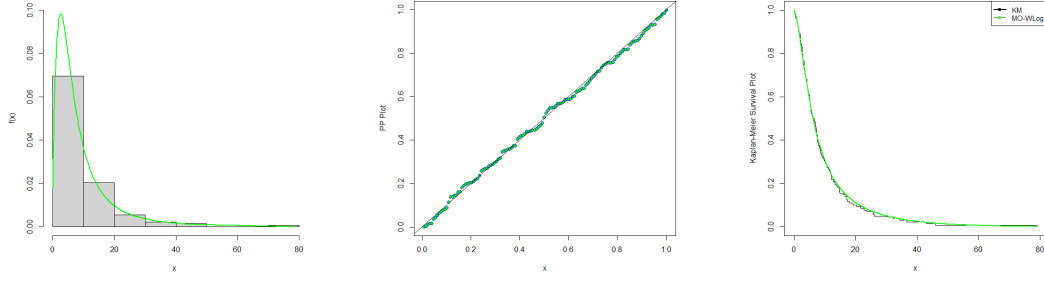


Figure 5.2: Fitted densities, probability plots and Kaplan-Meier survival plots of the MO-WLoG distribution for the remission times of 128 bladder cancer patients data.

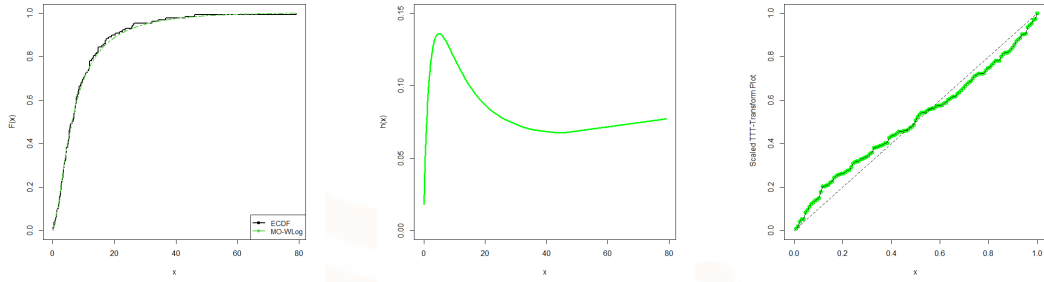


Figure 5.3: Estimated cdf, estimated hazard rate plot and Total-Time-on-Test of the MO-WLoG distribution for the remission times of 128 bladder cancer patients data.

Figure 5.1 shows the profile plots of the remission times of 128 bladder cancer patients datasets and it can be seen that our estimates are attaining the global maximum thus, our estimates maximizes the log-likelihood.

Figures 5.2 and 5.3 displays clearly that the MO-WLog distribution provide better fits and flexibility in fitting the remission times of 128 bladder cancer patients data set. The histogram shows that the MO-WLog distribution can accommodate the extremely tailed data sets. We also observe that fitted Kaplan-Meier and empirical cumulative distribution function (ECDF) curves are close to each other exhibiting better performance of our model also, the fitted hrf exhibits the upside-down bathtub shape. The Total-Time-on-Test (TTT) plot gives a hazard rate based on the 128 bladder cancer patients dataset as such our model's hazard rate plot is coinciding with the TTT as given in Figure 5.3 which shows a good fit.

5.2.1.2 Kevlar 49/Epoxy Strands

We have comprehensive data with precise periods of failure since the data represents

the stress-rupture life of kevlar 49/epoxy strands that were continuously subjected to sustained pressure at a 90 stress level until all had failed. The failure times in hours are shown below

0.01 0.01 0.02 0.02 0.02 0.03 0.03 0.04 0.05 0.06 0.07 0.07 0.08 0.09 0.09 0.10 0.10
0.11 0.11 0.12 0.13 0.18 0.19 0.20 0.23 0.24 0.24 0.29 0.34 0.35 0.36 0.38 0.40 0.42
0.43 0.52 0.54 0.56 0.60 0.60 0.63 0.65 0.67 0.68 0.72 0.72 0.72 0.73 0.79 0.79 0.80
0.80 0.83 0.85 0.90 0.92 0.95 0.99 1.00 1.01 1.02 1.03 1.05 1.10 1.10 1.11 1.15 1.18
1.20 1.29 1.31 1.33 1.34 1.40 1.43 1.45 1.50 1.51 1.52 1.53 1.54 1.54 1.55 1.58 1.60
1.63 1.64 1.80 1.80 1.81 2.02 2.05 2.14 2.17 2.33 3.03 3.03 3.34 4.20 4.69 7.89

Model	Estimates				Statistics							
	δ	θ	a	b	$-2 \log L$	AIC	AICC	BIC	W*	A*	K-S	p-value
MO-WLoG	3.9166 (3.8594)	8.2214×10^{-5} (0.0766)	0.6931 (0.1700)	1.9955 (0.8383)	203.6843	211.6843	212.101	222.1448	0.1189	0.7368	0.0704	0.6985
OW-TL-LLoGL	α 0.4286 (0.1199)	λ 1.2941 (0.3750)	γ 1.3794 (0.2288)	θ 1.1844×10^{-8} (0.0154)	218.9	226.9	227.4	237.4	0.2593	2.2359	0.3024	1.892×10^{-8}
EGEL	p 3.7308×10^{-8} (0.0237)	α 0.6013 (0.2142)	β 0.8575 (0.1106)	λ 1.3988 (0.4982)	205.8	213.8	214.3	224.3	0.1815	1.0312	0.0815	0.5129
EPLP	θ 1.1683 (1.2586)	α 0.7894 (0.2025)	β 1.7951 (0.6111)	ω 0.9385 (0.3828)	204.4	212.4	212.9	222.9	0.1348	0.8140	0.0708	0.6921
EBXHP	c 0.7150 (0.0682)	s 0.2935 (0.0661)	α 1.0195 (0.1232)	λ 2.6980×10^{-9} (0.0059)	271.0	279.0	279.5	289.5	0.6774	3.6501	0.2483	7.7780×10^{-6}
TLGW	β 0.3735 (0.4298)	λ 1.5051 (2.0039)	θ 2.4139 (0.9797)	γ 0.3145 (0.2598)	205.9	213.9	214.3	224.4	0.1681	0.9760	0.0842	0.4706
GGP	β 1.0515 (1.3346)	λ 0.6645 (0.2262)	θ 0.9400 (0.1716)	γ 5.1908×10^{-8} (0.01848)	205.3	213.3	213.7	223.8	0.1237	0.7737	0.0828	0.4933

Table 5.4: **Parameter estimates and goodness-of-fit statistics for various models fitted for the stress-rupture life of kevlar 49/epoxy strands data set**

Table 5.4 shows that the MO-WLog distributions dominate the highest P-value with lowest goodness-of-fit statistic values and provides the overall best fit on kevlar 49/epoxy strands data set as compared to non-nested models presented in this paper.

Here are some more statistical results based on the MO-WLog distribution since its providing the overall best fit.

The variance-covariance matrix for MO-WLog model on epoxy strands failure at

90% stress level data set is given by

$$\begin{bmatrix} 14.8949 & 0.0115 & -0.613 & 53.1821 \\ 0.0115 & 0.0058 & 5.6183 \times 10^{-8} & -2.4730 \times 10^{-7} \\ -0.6135 & 5.6183 \times 10^{-8} & 0.0289 & -0.1345 \\ 3.1821 & -2.4730 \times 10^{-7} & -0.1345 & 0.7028 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$\delta \in [3.9166 \pm 7.5644]$, $\theta \in [8.2214 \times 10^{-5} \pm 0.1502]$, $a \in [0.6931 \pm 0.3332]$ and $b \in [1.9955 \pm 1.6432]$.

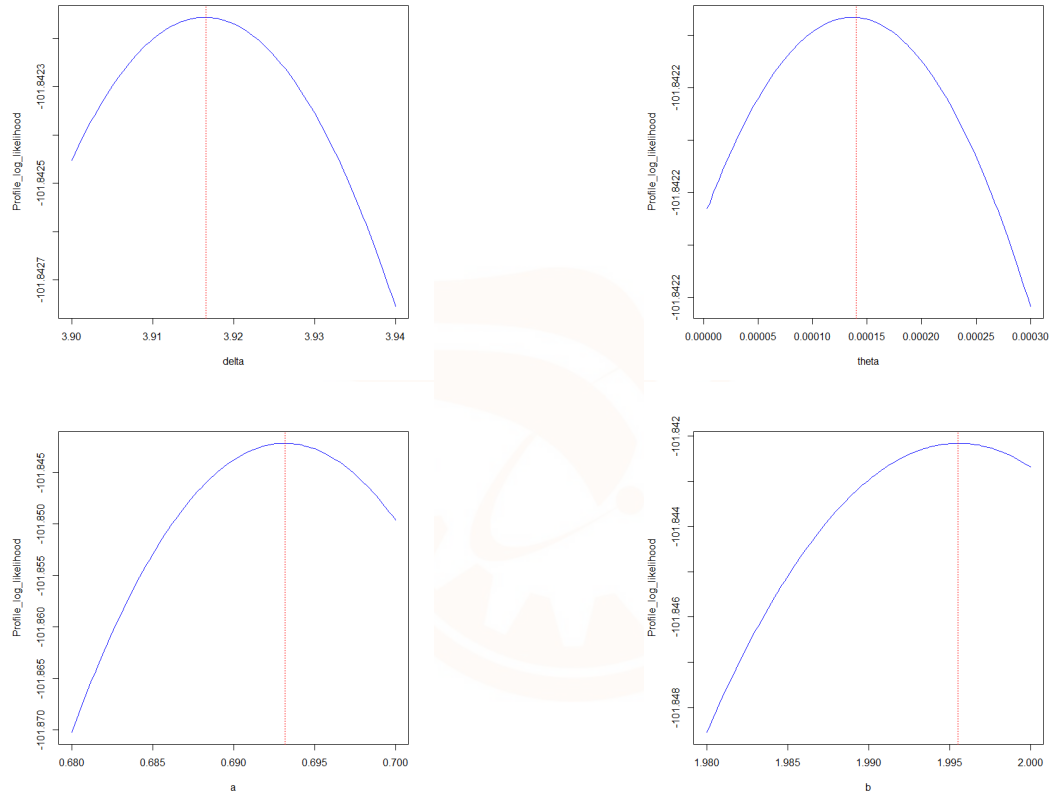


Figure 5.4: MO-WLoG distribution profile plots for the Kevlar data set.

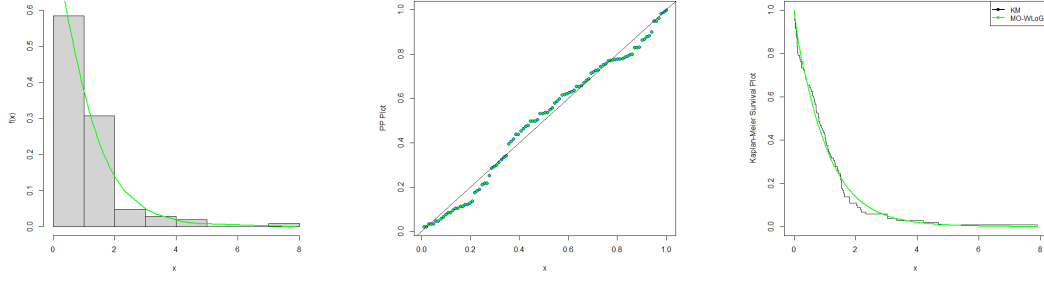


Figure 5.5: Fitted densities, probability plots and Kaplan-Meier survival plots of the MO-WLoG distribution for the Kevlar data.

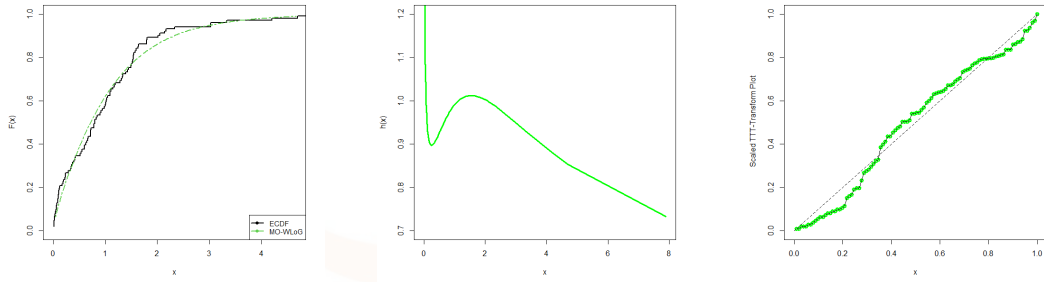


Figure 5.6: Estimated cdf, estimated hazard rate plot and Total-Time-on-Test of the MO-WLoG distribution for the Kevlar data.

To examine if our estimates maximizes the log-likelihood, Figure 5.4 shows the profile plots of the Kevlar datasets and it can be clearly seen that our estimates are attaining the global maximum.

Figures 5.5 and 5.6 displays clearly that the MO-WLog distribution provides better fit and flexibility in fitting Kevlar 49/epoxy strands data set. The histogram shows that the MO-WLoG distribution can accommodate the extremely tailed data sets. We also observe that fitted Kaplan-Meier and ECDF curves are close to each other exhibiting better performance of our model also, the fitted hrf exhibits bathtub followed by upside-down bathtub shape. The Total-Time-on-Test (TTT) plot gives a hazard rate based on the dataset as such our model's hazard rate plot has to coincide with the TTT as given in Figure 5.6.

5.2.2 Censored Dataset

5.2.2.1 Leukaemia Patients Dataset

The censored data of consideration represent the times of remission, in weeks, for a group of 30 leukaemia patients who underwent similar treatment (see, Lawless [26], pp. 139). The dataset is as follows;

1, 1, 2, 4, 4, 6, 6, 6, 7, 8, 9, 9, 10, 12, 13, 14, 18, 19, 24, 26, 29, 31, 42, 45, 50, 57, 60, 71, 85, 91.

Model	Estimates				Statistics							
	δ	θ	a	b	$-2 \log L$	AIC	AICC	BIC	W^*	A^*	K-S	p-value
MO-WLoG	0.3812 (0.4841)	3.8855×10^{-5} (0.2874)	1.1699 (0.2874)	0.0127 (0.0220)	253.3391	261.3391	262.9391	266.9439	0.0382	0.2530	0.0877	0.9751
EGEL	p	α	β	λ	257.6304	265.6303	267.2303	271.2351	0.0522	0.3203	0.1821	0.2729
	7.0818×10^{-9} (0.0160)	0.0326 (0.1588)	1.5154 (0.4153)	1.5746 (7.6617)								
EBXIIP	c	s	α	λ	331.4113	339.4113	341.0113	345.0161	0.0421	0.39336	0.6987	3.7770×10^{-13}
	0.7067 (0.1508)	0.8829 (0.3027)	0.7603 (0.1399)	7.8400×10^{-10} (0.0054)								
TLGW	β	λ	θ	γ	253.4068	261.4068	263.0068	267.0116	0.0396	0.2600	0.0857	0.9082
	0.1760 (0.4126)	0.0986 (0.4939)	2.9278 (5.6767)	0.4662 (0.6266)								
GGP	β	λ	θ	γ	438.8484	446.8484	448.4484	452.4532	0.0953	0.5619	0.6167	2.459×10^{-10}
	1.1001 (0.6541)	0.5007 (0.0963)	0.2223 (0.0282)	1.2093×10^{-11} (8.0252×10^{-4})								

Table 5.5: Parameter estimates and goodness-of-fit statistics for various models fitted for the censored data set

Table 5.5 shows that the MO-WLog distributions has the highest P-value with lowest goodness-of-fit statistic values and provides the overall best fit on the censored data set as compared to non-nested models presented in the table.

Here are some results of the MO-WLog distribution since it provides the best fit.

The variance-covariance matrix for MO-WLog model on the censored data set is given by

$$\begin{bmatrix} 0.2343 & -0.0047 & -0.1161 & 0.0100 \\ -0.0047 & 2.4803 \times 10^{-2} & -1.2529 \times 10^{-6} & 1.0680 \times 10^{-7} \\ -0.1161 & -1.2529 \times 10^{-6} & 8.2651 \times 10^{-2} & -6.0580 \times 10^{-3} \\ 0.0100 & 1.0680 \times 10^{-7} & -6.0580 \times 10^{-3} & 4.8411 \times 10^{-4} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$\delta \in [0.3812 \pm 0.9488]$, $\theta \in [3.8855 \times 10^{-5} \pm 0.3087]$, $a \in [1.1699 \pm 0.5634]$ and $b \in [0.0127 \pm 0.0431]$.

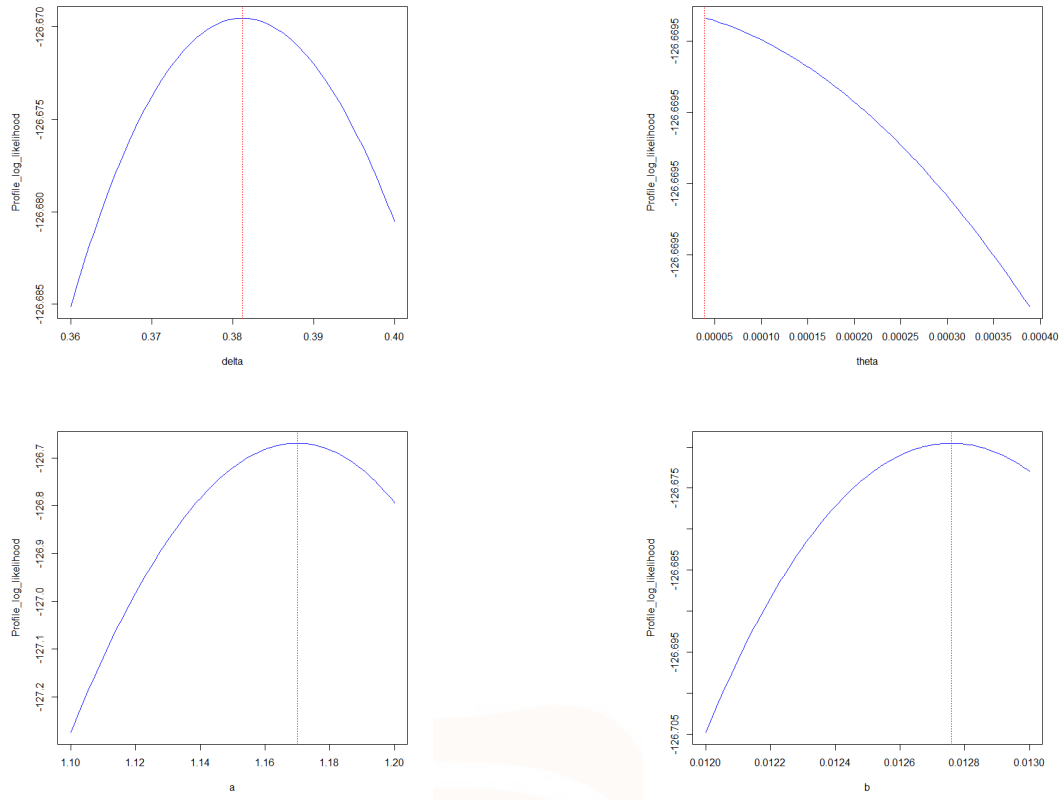


Figure 5.7: MO-WLoG distribution profile plots for the censored data set.

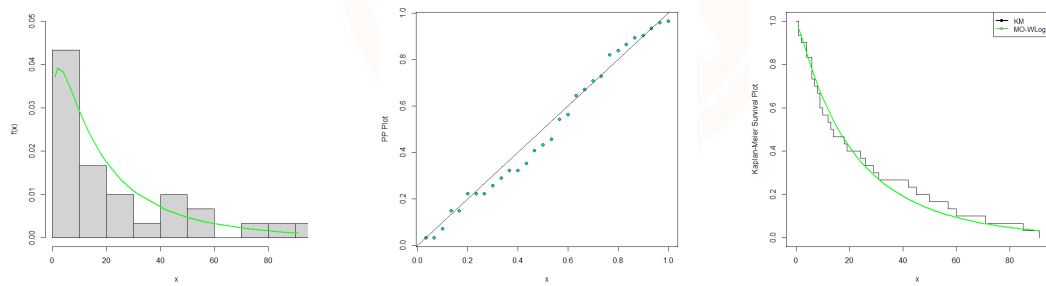


Figure 5.8: Fitted densities, probability plots and Kaplan-Meier survival plots of the MO-WLoG distribution for the censored data.

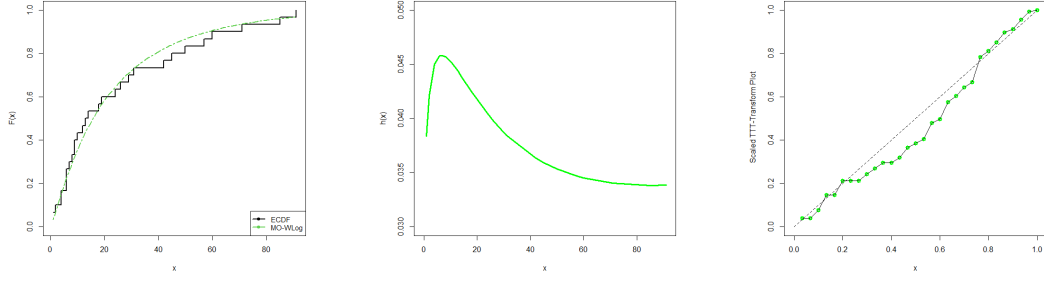


Figure 5.9: Estimated cdf, estimated hazard rate plot and Total-Time-on-Test of the MO-WLoG distribution for the censored data.

To examine if our estimates maximizes the log-likelihood, Figure 5.7 shows the profile plots of the remission times for a group of 30 leukaemia patients who underwent similar treatment dataset (censored data) and it can be clearly seen that our estimates are attaining the global maximum.

Figures 5.8 and 5.9 displays clearly that the MO-WLog distribution provides better fit and flexibility in fitting this censored data set. The histogram shows that the MO-WLoG distribution can accommodate the extremely tailed data sets. We also observe that fitted Kaplan-Meier and ECDF curves are close to each other exhibiting better performance of our model also, the fitted hrf exhibits an upside-down bathtub shape. The Total-Time-on-Test (TTT) plot is giving a hazard rate based on the censored dataset as such our model's hazard rate plot coincide with the TTT as given in Figure 5.9 which implies a good fit.

6. Conclusion and Areas of further study

6.1 Conclusion

A family of distributions, referred to as the Marshall-Olkin-G Power Series (MO-GPS) family of distributions was reviewed in detail and presented. Some statistical properties such as the quantile function, moments, mean deviations, Bonferroni and Lorenz curves, residual and reverse residual life function, stress-strength function, entropy, order statistics and maximum likelihood estimates for censored and uncensored data of the MO-GPS and some of its special models are also discussed. The performances of the estimators are examined by Monte Carlo simulation. Some estimators such as the Maximum Likelihood Estimator, Moments Estimator, Ordinary Least Squares Estimator, Anderson-Darling Estimator, Right-Tail Anderson-Darling Estimator, Cramer-von Misses Minimum Distance Estimator and Maximum Product Spacing Estimator are investigated for unknown parameters of MO-WLoG model. The efficiency of the estimators is checked via Monte Carlo simulation based on the biased on the bias and mean square error criteria. Application of a special cases, namely, the Marshall-Olkin-Weibull Logarithmic (MO-WLoG) distribution model to real life data sets namely, the remission times of 128 Bladder Cancer Patients data, Kevlar 49/Epoxy Strands and censored data are presented to show the usefulness, flexibility and applicability of our distribution. From Tables 5.3 and 5.4 it can be clearly seen that the MO-WLog has the highest P-value with lowest goodness-of-fit statistic values and provides the overall best fit thus, the distribution has a better fit as compared to presented non-nested models in this paper for specified datasets. Thus the MO-GPS family of distributions could play a reasonable role as a candidate for modelling real life dataset.

6.2 Areas of further study

We offer some ideas in this section for potential follow-up research projects. Other potential research areas could involve using the proposed model for regression analysis, applying Bayesian and Bootstrap techniques to the model, and estimating the model parameters under different censoring schemes since we reviewed and presented

the generalized model (MO-GPS), its properties, simulations, and applications. On the examined model, some researchers ought to take into account various sub-models and baseline distributions.



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