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Permutable Pairs of Quasi-Uniformities

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A Dissertation/Thesis Submitted to the Faculty of Sciences in Partial
Fulfilment of the Requirements for the Award of the Degree of Master of
Science in Pure and Applied Mathematics

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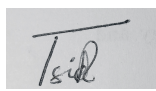
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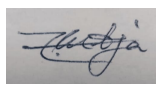


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I, **Zechariah Mushaandja**, hereby confirm that I have inspected the above titled thesis and, to the best of my knowledge, it is based on the original work of the candidate.

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Date: November 29, 2022

Abstract

In this project we study permutable pairs of quasi-uniformities. Our objective is to investigate the notion of permutability of quasi-uniformities in the context of conjugates, complements, adjacent and independent quasi-uniformities, following the work of authors like H. Weber, E.P de Jager and H.P.A. Künzi. The notion of quasi-uniformity which is a generalization of the concept of a uniformity is discussed in terms of surroundings. Two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X are said to be permutable provided $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{U}$. Some characterizations of permutability are presented. Among other results, we show that conjugates of permutable quasi-uniformities permute and conjugates of adjacent quasi-uniformities are adjacent. Complementary quasi-uniformities are discussed and a connection between complementary quasi-uniformities and permutability is shown. Furthermore, the notion of independent quasi-uniformities is discussed. In particular we present a well known result from ([3], corollary 4) in terms of independent quasi-uniformities. Some useful examples are also discussed.

Dedications

To my father.



Acknowledgements

I would like to thank my supervisor Dr Zechariah Mushaandja for his excellent supervision and support in helping me with the project. I would also like to thank SIMONS FOUNDATION for their financial support.

I am thankful to the Department of Mathematics and Statistical Sciences of Botswana International University of Science and Technology for giving me this opportunity.

I would also like to express my gratitude to my family, friends and colleagues for the courage they gave me throughout my entire studies. My parents deserve a particular note of thanks: your wise counsel and kind words have as always, served me well.



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Introduction

A lot of work has been done in the theory of quasi-uniformities by Eliza P. de Jager and Hans-Peter A. Künzi. In one of their papers ([1]), they described atoms of a complete lattice $(\mathcal{Q}(X), \subseteq)$ of all quasi-uniformities on a given non-empty set X . They also characterized those anti-atoms of the complete lattice $(\mathcal{Q}(X), \subseteq)$ that are not members of the quasi-proximity class of the largest quasi-uniformity, $\mathcal{D}$ on X . They further showed that non-discrete quasi-uniformity $\mathcal{U}$ including the discrete (quasi-)proximity has a complement in $(\mathcal{Q}(X), \subseteq)$ and each atom and each proximally non-discrete anti-atom of $(\mathcal{Q}(X), \subseteq)$ has a complement. In ([3]), they started considering permutable pairs of quasi-uniformities. Therein they showed that if X is a topological space and $\mathcal{P}$, is the Pervin quasi-uniformity of the space X then $\mathcal{P} \circ \mathcal{P}^{-1} = \mathcal{P}^{-1} \circ \mathcal{P}$ if and only if the space X is extremally disconnected and normal.

In their fairly recent paper,([7]) *More about complements of quasi-uniformities*, Eliza P. de Jager and Hans-Peter A. Künzi proved that if X is an infinite topological T_1 -space and $\mathcal{P}$, is the Pervin quasi-uniformity of X then $\mathcal{P}$ does not have a complement in $(\mathcal{Q}(X), \subseteq)$. They further presented that a hereditary precompact quasi-uniformity of X does not have a complement in $(\mathcal{Q}(X), \subseteq)$ provided that X is an infinite set. Maria Joao Ferreira, Mack Matlabyana and Jorge Picado(in [4]) extended some results from ([7] and [3]). They provided some enough conditions on the fixed class of subspaces that allowed them to combine some pair of observations under the same roof.

Eliza P. de Jager and Hans-Peter A. Künzi continued their work on quasi-uniformities in their paper, *Infima and complements on the lattice of quasi-uniformities*(see [2]) by showing that the smallest quasi uniformity of any family of proximally symmetric quasi-uniformity is proximally symmetric, while the largest quasi-uniformity of two proximally symmetric quasi-uniformity is proximally symmetric provided that the two quasi- uniformities are uniformities. On one hand they showed that the supremum of any family of transitive quasi-uniformities is transitive, while on the other side there are some transitive quasi-uniformities whose infimum with their conjugate quasi-uniformity is not transitive.

In [5], C. H Yan generalized a theorem of Dubreil-Jacotin characterizing commuting equivalence relations. She defined the notion of commuting quasi-order relations on a set and obtained a structure theorem for pairs of commuting quasi-order relations. In 2016, B. Bazigaran and M. Pourattar published a paper on *Atoms in the lattice of quasi-uniformities*([16]).

In particular they showed that a quasi-uniformity is an atom in $(Q(X), \subseteq)$ if and only if it is generated by some special preorders and furthermore, they observed that in the $(Q(X), \subseteq)$ an atom need to be transitive and totally bounded while it is not a uniformity. *On lattices of uniformities* ([9]), a paper by Hans Weber published in 2007, provides the study of lattice theoretical properties of uniformities. He used the concept of permutable uniformities to study the distributivity laws in the set of all uniformities on a set.

John W. Carlson in his paper, *Nearness and Quasi-uniform structures* ([8]), introduced the idea of a locally symmetric quasi-uniform space. In one of his results, he showed that all symmetric topological spaces generate a compatible locally right symmetric quasi-uniformity. He further proved that the underlying topology of a quasi-uniform space $(X, \mathcal{U})$ is Lindelöf if and only if $\mathcal{U}$ is countably pre-compact, that is every weak Cauchy filter that has a countable intersection property clusters.

Objectives of the study

The objective of this project is to study permutable pairs of quasi-uniformities. Our aim is to investigate the notion of permutability of quasi-uniformities in the context of conjugates, complements, adjacent and independent quasi-uniformities. We review relevant results from literature on this topic. Moreover a connection between complementary, independent quasi-uniformities and permutability is shown. Among other things, we provide proofs for several results that appear in literature without proofs.

Problem Statement

In this project, we consider the following questions:

- 1). Let $\mathcal{U}$ and $\mathcal{V}$ be independent quasi-uniformities on a set X and let $\mathcal{A}$ and $\mathcal{B}$ be quasi-uniformities on a set X such that $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$. Our aim is to check whether $\mathcal{A}$ and $\mathcal{B}$ permute.
- 2). Let $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ be members of $(Q(X), \subseteq)$ such that $\mathcal{U}$ and $\mathcal{V}$ are complements. Our aim is to check whether $\mathcal{U} \vee \mathcal{V}$ and $\mathcal{W}$ permute.
- 3). Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ and $\mathcal{Z}$ be members of $(Q(X), \subseteq)$. If $\mathcal{U}$ is a complement of $\mathcal{V}$ and $\mathcal{W}$ is a complement of $\mathcal{Z}$. Then what can we say about:
 - i) $\mathcal{U} \vee \mathcal{V}$ and $\mathcal{W} \wedge \mathcal{Z}$,
 - ii) $\mathcal{U} \wedge \mathcal{V}$ and $\mathcal{W} \vee \mathcal{Z}$?
- 4) Why are conjugates of adjacent quasi-uniformities also adjacent?

Notations

X	:	Arbitrary set
Δ	:	Diagonal element of X
$\mathcal{U}, \mathcal{V}$	:	Quasi-uniformities on X
$\mathcal{U}^{-1}, \mathcal{V}^{-1}$	:	Conjugates of $\mathcal{U}$ and $\mathcal{V}$ respectively
$\mathcal{I}$	:	Indiscrete uniformity
$\mathcal{D}$	:	Discrete uniformity
$\mathcal{U} _A$	:	Subspace uniformity
$\mathcal{B}$	:	Base for a (quasi) uniformity
$\tau_{\mathcal{U}}$	:	Uniform topology
$\wedge$	:	Meet
$\vee$	:	Join

1. Introduction to uniform spaces

In this chapter, we begin by giving definitions and some notation for uniform spaces that we will use in this thesis. We continue by presenting important results on uniform spaces from literature (see [3], [6], [9]) and providing examples of uniformities.

1.1 Uniform Spaces

We shall first give some notation and then the definition of a uniform space $(X, \mathcal{U})$. We take that the reader is used to the basic theory of sets and relations, but every now and then we will repeat well known facts to fix our terminology and completeness of our work.

Notation 1.1.1. *Let X be a set. Subsets of the Cartesian product $X \times X$ are called relations on X . For $U, V \subseteq X \times X$;*

- i). $U^{-1} = \{(y, x) \mid (x, y) \in U\}$.
- ii). If $U = U^{-1}$ then we say U is symmetric.
- iii). $U \circ V = \{(x, y) \mid \text{there exists } z \in X \text{ with } (x, z) \in V \text{ and } (z, y) \in U\}$ and this is just the composition of relations.
- iv). $\Delta = \{(x, x) \mid x \in X\}$ is called the **diagonal** of X , sometimes denoted as Δ_X when confusion may arise.
- v). For $x \in X$, define $U[x] = \{y \in X \mid (x, y) \in U\}$.
- vi). For $A \subseteq X$, define $U[A] = \bigcup_{a \in A} U[a] = \{y \in X \mid \exists a \in A \text{ with } (a, y) \in U\}$.

Definition 1.1.2. [14] A uniformity on a set X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$, called entourages or surroundings such that;

- u1). $U \in \mathcal{U} \implies \Delta \subseteq U$,
- u2). $U \in \mathcal{U} \implies U^{-1} \in \mathcal{U}$,
- u3). $U \in \mathcal{U} \implies V \circ V \subseteq U$ for some $V \in \mathcal{U}$,
- u4). $U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U}$,
- u5). $U \in \mathcal{U}$ and $U \subseteq V \implies V \in \mathcal{U}$.

The tuple $(X, \mathcal{U})$ is called a **uniform space** and we say x and y are U -close if $(x, y) \in U$ for some $U \in \mathcal{U}$.

Definition 1.1.3. Let $(X, \mathcal{U})$ be a uniform space. The inverse of $\mathcal{U}$ is the set $\mathcal{U}^{-1} = \{U^{-1} \mid U \in \mathcal{U}\}$.

Remark 1.1.4. It is easy to see that $(X, \mathcal{U}^{-1})$ is a uniform space.

The proof follows directly from the fact that for any $U \in \mathcal{U}$ we have $U^{-1} \in \mathcal{U}$.

Before we provide the reader with examples of uniformities, we define a base of a uniformity.

Definition 1.1.5. A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is a base or basis for $\mathcal{U}$ if $\forall U \in \mathcal{U}, \exists B \in \mathcal{B}$ such that $B \subseteq U$. A subfamily $\mathcal{V}$ of $\mathcal{U}$ is a subbase or subbasis for $\mathcal{U}$ if the set of all finite intersections of elements of $\mathcal{V}$ gives a base for $\mathcal{U}$.

The following proposition provides examples of uniformities and their basis.

Proposition 1.1.6. a). Let X be a set. Then $\mathcal{D} = \{U \subseteq X \times X \mid \Delta \subseteq U\}$ is a uniformity on X called the **discrete** uniformity and it has $\{\Delta\}$ as a base. Note that $\mathcal{V} \subseteq \mathcal{D}$ for any uniformity $\mathcal{V}$ on X , thus $\mathcal{D}$ is the largest uniformity on X .

Proof: We show that $\mathcal{D}$ is a uniformity on X .

- u1): Obviously for every $U \in \mathcal{D}, \Delta \subseteq U$ by construction.
- u2): Let $U \in \mathcal{D}$. Then $U \subseteq X \times X$ and $\Delta \subseteq U$. Note that $U^{-1} \subseteq X \times X$ and $\Delta^{-1} = \Delta$. Hence $\Delta = \Delta^{-1} \subseteq U^{-1}$. So $U^{-1} \in \mathcal{D}$.

u3) Let $U \in \mathcal{D}$. Then there is $V = \Delta \in \mathcal{D}$, such that $V \circ V \subseteq U$.

u4): Let $U, V \in \mathcal{D}$. Then $U, V \subseteq X \times X$ and $\Delta \subseteq U, V$. So $U \cap V \subseteq X \times X$ and $\Delta \subseteq U \cap V$. So $U \cap V \in \mathcal{D}$.

u5); Let $U \in \mathcal{D}$ such that $U \subseteq V$. Then $\Delta \subseteq U \subseteq V$. So $\Delta \subseteq V$, hence $V \in \mathcal{D}$. $\square$

b). The **in-discrete/trivial** uniformity $\mathcal{I}$ on X is a uniformity with $X \times X$ as the only surrounding and it is the smallest uniformity on X .

Proof: Note that here $\mathcal{U} = \mathcal{I} = \{X \times X\}$. So firstly, $\Delta \subseteq X \times X$, $(X \times X)^{-1} = X \times X$, $(X \times X) \circ (X \times X) = X \times X$, $(X \times X) \cap (X \times X) = X \times X$ and lastly we cannot find any super set of $X \times X$ which is not in $\mathcal{I}$. So $\mathcal{I}$ is a uniformity on X .

c). Let $\epsilon \in \mathbb{R}$, $\epsilon > 0$ and $D_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid |x - y| < \epsilon\}$. Then $\mathcal{B} = \{D_\epsilon \mid \epsilon > 0\}$ is a base for a uniformity on $\mathbb{R}$, called the **usual** uniformity.

d). Any metric d on a set X generates a metric uniformity $\mathcal{U}_d$ on X . $\mathcal{U}_d$ has $\{D_\epsilon \mid \epsilon > 0\}$ as the base, where $D_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. If d is a metric on X , the uniformities generated by d and $2d$ coincide, that is different metrics may generate the same uniformity ([12], definition 2.1.2).

Proposition 1.1.7. *Let $(X, \mathcal{U})$ be a uniform space and A a subset of X . Then $\mathcal{U}|_A = \{U \cap (A \times A) \mid U \in \mathcal{U}\}$ is a uniformity on A .*

Proof: We show that $\mathcal{U}|_A$ satisfies u1)-u5).

u1): Let $B \in \mathcal{U}|_A$. Then $B = U \cap (A \times A)$ for some $U \in \mathcal{U}$. So $\Delta_X \subseteq U$ as $\mathcal{U}$ is a uniformity on X . Since $A \subseteq X$ then $\Delta_A \subseteq \Delta_X \subseteq U$. Now we get that $\Delta_A \subseteq U \cap (A \times A) = B$ as already $\Delta_A \subseteq (A \times A)$. Hence $\Delta_A \subseteq B$.

u2): Let $B \in \mathcal{U}|_A$. Then $B = U \cap (A \times A)$ for some $U \in \mathcal{U}$. So $U^{-1} \in \mathcal{U}$. Now $B^{-1} = (U \cap (A \times A))^{-1} = U^{-1} \cap (A \times A) \in \mathcal{U}|_A$. Hence $B^{-1} \in \mathcal{U}|_A$.

u3): Let $B \in \mathcal{U}|_A$. Then obviously there is some $U \in \mathcal{U}$ such that $B = U \cap (A \times A)$. Since $\mathcal{U}$ is a uniformity, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. So $V \cap (A \times A) \in \mathcal{U}|_A$. Now $V \cap (A \times A) \circ V \cap (A \times A) = (V \circ V) \cap (A \times A) \subseteq U \cap (A \times A) = B$.

u4): Let $B, C \in \mathcal{U}_{|A}$. Then there is some $U, V \in \mathcal{U}$ such that $B = U \cap (A \times A)$ and $C = V \cap (A \times A)$. So $U \cap V \in \mathcal{U}$ since $\mathcal{U}$ is a uniformity. Now we have $B \cap C = U \cap (A \times A) \cap V \cap (A \times A) = (U \cap V) \cap (A \times A) \in \mathcal{U}_{|A}$. That is $B \cap C \in \mathcal{U}_{|A}$.

u5): Let $B \in \mathcal{U}_{|A}$ such that $B \subseteq C$ where $C = V \cap (A \times A)$ with $V \subseteq X \times X$. If we can show that $V \in \mathcal{U}$ then we are done. So there is some $U \in \mathcal{U}$ such that $B = U \cap (A \times A)$. Since $U \cap (A \times A) \subseteq V \cap (A \times A)$ then $U \subseteq V$. So $V \in \mathcal{U}$ by definition of $\mathcal{U}$, hence $C \in \mathcal{U}_{|A}$.

□

Definition 1.1.8. The uniformity $\mathcal{U}_{|A}$ on A defined in the above proposition is called the subspace (relative) uniformity. We say $(A, \mathcal{U}_{|A})$ is a uniform subspace of $(X, \mathcal{U})$.

Next we give a characterization of a base of a uniformity.

Proposition 1.1.9. Let X be a set. A family $\emptyset \neq \mathcal{B}$ of subsets of $X \times X$ form a base for a uniformity $\mathcal{U}$ on X iff

b1). $U \in \mathcal{B} \implies \Delta \subseteq U$,

b2). $U \in \mathcal{B} \implies \exists B \in \mathcal{B}$ such that $B \subseteq U^{-1}$,

b3). $U \in \mathcal{B} \implies V \circ V \subseteq U$ for some $V \in \mathcal{B}$,

b4). $U, V \in \mathcal{B} \implies \exists B \in \mathcal{B}$ such that $B \subseteq U \cap V$.

Proof: ($\implies$). Suppose $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on X . We show that (b1)-(b4) hold.

(b1): Let $U \in \mathcal{B}$. Since $\mathcal{B}$ is a base for $\mathcal{U}$, then by definition of a base, $\mathcal{B} \subseteq \mathcal{U}$. Hence $U \in \mathcal{U}$. By definition of $\mathcal{U}$, $\Delta \subseteq U$. So for $U \in \mathcal{B}$, $\Delta \subseteq U$.

(b2): Let $U \in \mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{U}$ then $U \in \mathcal{U}$. By (u2), $U^{-1} \in \mathcal{U}$. By definition of a base, there exist some $B \in \mathcal{B}$ such that $B \subseteq U^{-1}$. So for $U \in \mathcal{B}$, there is some $B \in \mathcal{B}$ such that $B \subseteq U^{-1}$.

(b3): Let $U \in \mathcal{B}$. Then $U \in \mathcal{U}$. By definition of a uniformity $V \circ V \subseteq U$ for some $V \in \mathcal{U}$. Since $\mathcal{B}$ is a base for $\mathcal{U}$, $\exists V' \in \mathcal{B}$ such that $V' \subseteq V$. So $V' \circ V' \subseteq V \circ V \subseteq U$. That's for $U \in \mathcal{B} \exists V' \in \mathcal{B}$ such that $V' \circ V' \subseteq U$ as required.

(b4): Let $U, V \in \mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{U}$, $U, V \in \mathcal{U}$. By definition of $\mathcal{U}$, $U \cap V \in \mathcal{U}$. By definition of $\mathcal{B}$, $\exists B \in \mathcal{B}$ such that $B \subseteq U \cap V$. So for $U, V \in \mathcal{B}$ there exist $B \in \mathcal{B}$ such that $B \subseteq U \cap V$.

($\Leftarrow$). Suppose a non-empty family $\mathcal{B}$ of subsets of $X \times X$ satisfies b1)-b4). We show that there exists some $\mathcal{U}$ such that $\mathcal{B}$ is its base.

Let $\mathcal{U} := \{V \mid V \supseteq B \text{ for some } B \in \mathcal{B}\}$. Then for each $V \in \mathcal{U}$, there exists $B \in \mathcal{B}$ such that $B \subseteq V$. So $\mathcal{B}$ is a base for $\mathcal{U}$. We only need to show that $\mathcal{U}$ is a uniformity on X .

u1): Let $U \in \mathcal{U}$. Then by definition of $\mathcal{U}$, there exists $B \in \mathcal{B}$ such that $B \subseteq U$. So $\Delta \subseteq B$ by (b1). Hence $\Delta \subseteq U$.

u2): Let $U \in \mathcal{U}$. Then there is $B \in \mathcal{B}$ such that $B \subseteq U$. By (b2) there is some $A \in \mathcal{B}$ such that $A \subseteq B^{-1}$. Since $B \subseteq U$ then $B^{-1} \subseteq U^{-1}$. Hence $A \subseteq U^{-1}$. Thus $U^{-1} \in \mathcal{U}$.

u3): Let $U \in \mathcal{U}$. Then there is some $B \in \mathcal{B}$ such that $B \subseteq U$. By (b3), there exist $V \in \mathcal{B}$ with $V \circ V \subseteq B$. So $V \circ V \subseteq U$. Since $\mathcal{B} \subseteq \mathcal{U}$, $V \in \mathcal{U}$.

u4): Let $U, V \in \mathcal{U}$. Then there exist $A, B \in \mathcal{B}$ such that $A \subseteq U$ and $B \subseteq V$. By (b4), there is $C \in \mathcal{B}$ such that $C \subseteq A \cap B$. But $A \cap B \subseteq U \cap V$, so $C \subseteq U \cap V$. Hence $U \cap V \in \mathcal{U}$.

u5): Let $U \in \mathcal{U}$ and $U \subseteq V$. Then there is $B \in \mathcal{B}$ such that $B \subseteq U$. So $B \subseteq V$ since $U \subseteq V$. Hence $V \in \mathcal{U}$ by definition of a uniformity $\mathcal{U}$. $\square$

Remark 1.1.10. The set $\mathcal{B} = \{U \in \mathcal{U} \mid U \text{ is symmetric}\}$ is a basis for a uniformity $\mathcal{U}$ on a set X . Since, if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$. So $U \cap U^{-1} \in \mathcal{U}$ and is symmetric, and $U \cap U^{-1} \subseteq U$.

This chapter is mainly focused on introducing uniformities. The notion of uniform spaces was discussed. The subspace uniformity and the generalization of a base of a uniformity were discussed. We concluded the chapter by providing examples of uniformities and their basis.

2. Uniform Topology and Uniformly Continuous Functions

In this chapter we show that every uniformity on a set gives rise to a topology on that set called a uniform topology. Whenever we use a topological term to refer to a uniform space, it will be understood to refer to its uniform topology.

In the last part of this chapter, we define uniformly continuous functions on a uniform space $(X, \mathcal{U})$ and show that if a function is uniformly continuous then it is also continuous with respect to the underlying topology.

2.1 Uniform Topology

We begin by recalling definitions of a topology $\mathcal{T}$ on a set X and a neighbourhood base $\mathcal{N}_x$ of a point x in a set X .

Definition 2.1.1. [6] *Let X be a set. A topology on X is a set $\mathcal{T}$ containing subsets of X , called open sets, satisfying the following conditions:*

- i) $\emptyset, X \in \mathcal{T}$,*
- ii) If $O, P \in \mathcal{T}$, then $O \cap P \in \mathcal{T}$,*
- iii) If $(O_i)_{i \in I}$ is a family in $\mathcal{T}$, then $\bigcup_{i \in I} O_i \in \mathcal{T}$.*

We say $(X, \mathcal{T})$ is a topological space, sometimes we say, X is a topological space when no confusion can result about $\mathcal{T}$.

Example 2.1.2. a) Let $X \neq \emptyset$ be a set. Then $\mathcal{T} = \{O \mid O \subseteq X\}$ is a topology on X called the discrete topology.

b) For any non-empty set X , $\mathcal{T} = \{\emptyset, X\}$ is a topology on X called the indiscrete topology.

c) Let $X = \{x, y\}$ and $\mathcal{T} = \{\emptyset, \{x\}, X\}$. Then $\mathcal{T}$ is a topology on X called the Sierpinski topology and X is called the Sierpinski space.

d) Let $X \neq \emptyset$ be a set and $x \in X$. Then the set $\mathcal{T}_x = \{O \subseteq X \mid x \in O \text{ or } O = \emptyset\}$ is a topology called the particular point topology where x is the particular point. (more examples in [10])

Remark 2.1.3. a) The discrete topology is the largest topology and the indiscrete topology is the smallest topology on any given set.

b) Sierpinski topology is the smallest example of a topology which is neither trivial nor discrete.

c) Sierpinski topology is a special case of a particular point topology. In (d) above, if $X = \{x, y\}$ then the particular point topology where x is the particular point is the Sierpinski topology.

Definition 2.1.4. [6] The neighbourhood base $\mathcal{N}_x$ of x is a collection of all neighborhoods of x in a topological space X that satisfy the following properties:

N1) If $U \in \mathcal{N}_x$, then $x \in U$,

N2) If $U, V \in \mathcal{N}_x$ then $U \cap V \in \mathcal{N}_x$,

N3) If $U \in \mathcal{N}_x$ then there is a $V \in \mathcal{N}_x$ such that $U \in \mathcal{N}_y$ for every $y \in V$,

N4) If $U \in \mathcal{N}_x$ and $U \subseteq V$, then $V \in \mathcal{N}_x$.

Theorem 2.1.5. [12] *Let $(X, \mathcal{U})$ be a uniform space. The collection $\mathcal{N}_x = \{U[x] \mid U \in \mathcal{U}\}$ forms a neighbourhood base at x for all $x \in X$.*

Proof: We show that $\mathcal{N}_x$ is a neighbourhood base of x .

N1) Let $U[x] \in \mathcal{N}_x$. Then obviously $x \in U[x]$ for all $x \in X$.

N2) Let $U[x], V[x] \in \mathcal{N}_x$. Then $x \in U[x] \cap V[x] = (U \cap V)[x]$. Therefore $U[x] \cap V[x] \in \mathcal{N}_x$.

N3) Let $U[x] \in \mathcal{N}_x$. Then $U \in \mathcal{U}$. Hence by definition of a uniformity, there is $V \in \mathcal{U}$ such that there is $V \in \mathcal{U}$ with $V \circ V \subseteq U$. We verify that if $y \in V[x]$ then $V[y] \subseteq U[x]$, for some $x, y \in X$. Let $z \in V[y]$. Then $(y, z) \in V$ and $(x, y) \in V$, so $(x, z) \in V \circ V$. Thus $(x, z) \in U$ since $V \circ V \subseteq U$. Hence $z \in U[x]$.

N4) Let $U[x] \in \mathcal{N}_x$ and $U[x] \subseteq V[y]$ for some $y \in X$. Then $x \in V[y]$, so $V[y] \in \mathcal{N}_x$. $\square$

The topology described in the theorem above is called the uniform topology or underlying topology $\mathcal{T}_\mathcal{U}$ of $(X, \mathcal{U})$. It is defined as

$$\mathcal{T}_\mathcal{U} = \{G \subseteq X \mid \forall x \in G \text{ there is } U \in \mathcal{U} \text{ such that } U[x] \subseteq G\}.$$

Proposition 2.1.6. $\mathcal{T}_\mathcal{U}$ as defined above is a topology on X .

Proof: Note that $\mathcal{T}_\mathcal{U} \subseteq \mathcal{P}(X)$ by construction.

i). $\forall x \in \emptyset \exists U \in \mathcal{U}$ such that $U[x] \subseteq \emptyset$ and $\forall x \in X \exists U \in \mathcal{U}$ such that $U[x] \subseteq X$. So $\emptyset, X \in \mathcal{T}_\mathcal{U}$.

ii) Let $G, F \in \mathcal{T}_\mathcal{U}$ and $x \in G \cap F$. Then $x \in G$ and $x \in F$. So $\exists U, V \in \mathcal{U}$ such that $U[x] \subseteq G$ and $V[x] \subseteq F$. Now we have $\forall x \in G \cap F, U[x] \cap V[x] = (U \cap V)[x] \subseteq G \cap F$. Hence $G \cap F \in \mathcal{T}_\mathcal{U}$.

iii) Suppose $(O_i)_{i \in I}$ is a family in $\mathcal{T}_\mathcal{U}$. That is $O_i \in \mathcal{T}_\mathcal{U}$ for every $i \in I$. Take $x \in \cup_{i \in I} O_i$. Then $x \in O_i$ for some $i \in I$. So there is $U \in \mathcal{U}$ such that $U[x] \subseteq O_i$. But $O_i \subseteq \cup_{i \in I} O_i$. Thus $U[x] \subseteq O_i \subseteq \cup_{i \in I} O_i$. So $U[x] \subseteq \cup_{i \in I} O_i$. Then $\forall x \in \cup_{i \in I} O_i$ for some $i \in I$, $\exists U \in \mathcal{U}$ such that $U[x] \subseteq \cup_{i \in I} O_i$. So $\forall x \in \cup_{i \in I} O_i$, $\exists U \in \mathcal{U}$ such that $U[x] \subseteq \cup_{i \in I} O_i$. Thus $\cup_{i \in I} O_i \in \mathcal{T}_\mathcal{U}$. $\square$

Remark 2.1.7. a) Any topological space that can be obtained as the uniform topology of a uniform space is said to be a uniformizable topological space.

b) The topology of a uniform subspace is the subspace topology. That's if $\mathcal{U}_A$ is the subspace uniformity of $(X, \mathcal{U})$ and $\mathcal{T}$ is a topology on a set X , then the topology induced by $\mathcal{U}_A$ is the same as the subspace topology of $(X, \mathcal{T})$.

Example 2.1.8. a) The indiscrete uniformity on a set X generates the indiscrete topology on X .

Proof: By definition $\mathcal{T}_{\mathcal{I}} = \{G \subseteq X \mid \forall x \in G \text{ there is } U \in \mathcal{I} \text{ such that } U[x] \subseteq G\}$. But $\mathcal{I} = \{X \times X\}$. So $U = X \times X$ and $(X \times X)[x] = X$. This implies that $G = X$ or $G = \emptyset$. So $\mathcal{T}_{\mathcal{I}} = \{\emptyset, X\}$ and the indiscrete uniformity is the only uniformity that generates the indiscrete topology.

b) The discrete uniformity on a set X generates the discrete topology X . But it is not the only uniformity that generates the discrete topology on X . Note that $X \times X$ is the largest element in $\mathcal{D}$. So for any $x \in G \subseteq X$ we have $(X \times X)[x] \subseteq G$. This implies that the indiscrete uniformity also generates the discrete uniformity on X since it contains $X \times X$.

Definition 2.1.9. A topological space X is called a Hausdorff space (T_2 -space) iff for any two distinct points x, y of X , there are two open sets U and V in X with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

Next we give a characterization of Hausdorffness and provide its detailed proof.

Lemma 2.1.10. [18] Let $(X, \mathcal{U})$ be a uniform space. Then $(X, \mathcal{U})$ is Hausdorff iff $\cap\{U \mid U \in \mathcal{U}\} = \Delta$.

Proof:($\Rightarrow$). Suppose $(X, \mathcal{U})$ is Hausdorff. Showing that $(x, y) \notin \Delta$ implies that $(x, y) \notin \cap\{U \mid U \in \mathcal{U}\}$ is the same as showing that $(x, y) \in \cap\{U \mid U \in \mathcal{U}\}$ implies $(x, y) \in \Delta$. Let $(x, y) \notin \Delta$. Then $x \neq y$, so x and y have disjoint neighborhoods since X is Hausdorff by assumption. So there exist $U, V \in \mathcal{U}$ such that $U[x] \cap V[y] = \emptyset$. Then $U \cap V \in \mathcal{U}$ such that $(x, y) \notin U \cap V$. So $(x, y) \in U \cap V$ if $(x, y) \in \Delta$. Since U and V were arbitrary, $\cap\{U \mid U \in \mathcal{U}\} \subseteq \Delta$.

For the other inclusion, if $(x, y) \in \Delta$ then $(x, y) \in \cap\{U \mid U \in \mathcal{U}\}$ since $\Delta \subseteq U$, for all $U \in \mathcal{U}$. So $\Delta \subseteq \cap\{U \mid U \in \mathcal{U}\}$. Hence $\cap\{U \mid U \in \mathcal{U}\} = \Delta$.

($\Leftarrow$). Suppose $\cap\{U \mid U \in \mathcal{U}\} = \Delta$. Let $(x, y) \in X \times X$ such that $x \neq y$. Then $(x, y) \notin \Delta$. So there is $U \in \mathcal{U}$ such that $(x, y) \notin U$. By definition, there is a symmetric $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

Now we show that $V[x]$ and $V[y]$ are disjoint neighborhoods of x and y respectively. Suppose that $V[x] \cap V[y]$ is non-empty.

Then there is $z \in V[x] \cap V[y]$. Then $z \in V[x]$ and $z \in V[y]$. So $(x, z) \in V$ and $(y, z) \in V$. Since V is symmetric, $(z, y) \in V$ and thus $(x, y) \in V \circ V \subseteq U$. So $(x, y) \in U$, which contradicts the fact that $(x, y) \notin U$. Hence $V[x] \cap V[y] = \emptyset$. Therefore the space $(X, \mathcal{U})$ is Hausdorff. $\square$

2.2 Uniformly Continuous Functions

We introduce the concept of uniformly continuous functions. Here we will give three but equivalent definitions of a uniformly continuous function.

Definition 2.2.1. [13] Let $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ be a function between uniform spaces. f is said to be uniformly continuous iff for every $V \in \mathcal{V}$, there is $U \in \mathcal{U}$ such that if $(x, y) \in U$ then $(f(x), f(y)) \in V$ for all $x, y \in X$.

Equivalently: $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ is uniformly continuous iff for every $V \in \mathcal{V}$, $(f \times f)^{-1}(V) \in \mathcal{U}$.

Equivalently: $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ is uniformly continuous iff for every $V \in \mathcal{V}$, there exist $U \in \mathcal{U}$ such that $(f \times f)(U) \subseteq V$.

Of course $f \times f : X \times X \longrightarrow Y \times Y$ is defined by $(f \times f)(x_1, x_2) = (f(x_1), f(x_2))$ for all $(x_1, x_2) \in X \times X$.

Remark 2.2.2. We will use the second definition in the proof of Theorem 2.2.4.

Definition 2.2.3. If f is injective, surjective and f, f^{-1} are uniformly continuous then we say f is a uniform isomorphism.

Theorem 2.2.4. *Let $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ and $g : (Y, \mathcal{V}) \longrightarrow (Z, \mathcal{W})$ be uniformly continuous functions. Then the composition $g \circ f : (X, \mathcal{U}) \longrightarrow (Z, \mathcal{W})$ is also uniformly continuous.*

Proof: Let $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ and $g : (Y, \mathcal{V}) \longrightarrow (Z, \mathcal{W})$ be uniformly continuous functions. We prove by using the second definition of a uniformly continuous function. Let $W \in \mathcal{W}$. Since g is uniformly continuous, $(g \times g)^{-1}(W) \in \mathcal{V}$. Also since f is uniformly continuous then

$(f \times f)^{-1}((g \times g)^{-1}(W)) \in \mathcal{U}$. But $(f \times f)^{-1}((g \times g)^{-1}(W)) = ((f \times f)^{-1} \circ (g \times g)^{-1})(W) = ((g \circ f) \times (g \circ f))^{-1}(W) \in \mathcal{U}$. Hence $g \circ f$ is uniformly continuous. $\square$

Theorem 2.2.5. *Every uniformly continuous function is continuous with respect to the uniform topology.*

Proof: Suppose $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ is a uniformly continuous function. By ([6], definition 7.1), we need to show that for every neighbourhood V of $f(x)$ in Y , there is a neighbourhood U of x in X such that $f(U) \subseteq V$. Let $x \in X$. Then a neighbourhood of $f(x)$ has the form $V[f(x)]$ for some $V \in \mathcal{V}$. Since f is uniformly continuous, there is $U \in \mathcal{U}$ such that $(x, y) \in U$ implies that $(f(x), f(y)) \in V$ for all $x, y \in X$. So $f(x) \in f(U[x])$ and $f(U[x]) \subseteq V[f(x)]$. Hence f is continuous with respect to the uniform topology. $\square$

3. Permutable pairs of Quasi-Uniformities

In this chapter we discuss quasi-uniform spaces. The chapter is divided into two parts. In the first part, we define quasi-uniformities and their relationship with uniformities. In the second part of this chapter, we discuss permutable pairs of quasi-uniformities.

3.1 Quasi-Uniform Spaces

If we remove the symmetry condition (i.e $u2$).) in Definition 1.1.2 of a uniformity, we get the definition of a quasi-uniformity. Hence we obtain a generalization of uniformity.

Definition 3.1.1. [15] *A quasi-uniformity $\mathcal{U}$ on a set X is a collection of subsets of $X \times X$ such that;*

- $q1).$ $U \in \mathcal{U} \implies \Delta \subseteq U$,
- $q2).$ $U \in \mathcal{U} \implies V \circ V \subseteq U$ for some $V \in \mathcal{U}$,
- $q3).$ $U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U}$,
- $q4).$ $U \in \mathcal{U}$ and $U \subseteq V \implies V \in \mathcal{U}$.

We say that $(X, \mathcal{U})$ is a quasi-uniform space and the members $U \in \mathcal{U}$ are called surroundings or entourages of $\mathcal{U}$.

Example 3.1.2. [4] a) Let R be a preorder on a set X , that is, R is a reflexive and transitive binary relation on X . Then $\{R\}$ is a base for a quasi-uniformity on X .

b) The collection $\{S_A \mid A \in \mathcal{T}\}$ is a subbase for a quasi-uniformity on a set X , where $\mathcal{T}$ is a topology on X and $S_A = [(X \setminus A) \times X] \cup [X \times A]$. See more about S_A in Lemma 3.2.4.

Remark 3.1.3. a) Every uniformity is a quasi-uniformity and the converse is not true in general. An obvious example of a quasi-uniformity that is not a uniformity is $\mathcal{U}_{S_A}$ defined in Example 3.2.6. This is true since S_A is not symmetric hence $\mathcal{U}_{S_A}$ can not be symmetric.

b) If we go back to Theorem 2.1.5, we see that in this proof of the underlying topology of a uniform space, the symmetry condition was not needed. Thus, given a quasi-uniformity $\mathcal{U}$ on a set X , the collection

$\mathcal{N}_x = \{U[x] \mid U \in \mathcal{U}\}$ gives a neighborhood base at x for all $x \in X$ thus making X a topological space. This topology is the same as the one defined in Theorem 2.1.5.

Definition 3.1.4. (see [12] definition 2.1.4) A map $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ between quasi-uniform spaces is (quasi-) uniformly continuous iff for each $V \in \mathcal{V}$ there exist $U \in \mathcal{U}$ such that $(f \times f)(U) \subseteq V$.

Remark 3.1.5. a) Let X be a set and $\mathcal{U}$ a uniformity on X . The identity map from $(X, \mathcal{U}) \rightarrow (X, \mathcal{U})$ is always uniformly continuous for every uniformity $\mathcal{U}$.

b) If $(X, \mathcal{U})$ and $(X, \mathcal{V})$ two quasi-uniform spaces then we say that they are isomorphic given that we can find a uniformly continuous bijection, f between the two quasi-uniform spaces whose f^{-1} is also uniformly continuous.

c) For any function f , if f is uniformly continuous function between quasi-uniform spaces then f is also continuous with respect to the underlying topologies.

3.2 Permuting Quasi-Uniformities

We now look at permutable pairs of quasi-uniformities. Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X . Then the set $\mathcal{U} \circ \mathcal{V}$ is a set containing subsets of $X \times X$ generated by $\{U \circ V \mid U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.

Definition 3.2.1. [3] *If $\mathcal{U}$ and $\mathcal{V}$ are two quasi-uniformities on a set X , then they are permutable (or commute, permute) iff $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{U}$.*

Let $\mathcal{Q}(X)$ denote the set of all quasi-uniformities on a set X . Then $\mathcal{Q}(X)$ partially ordered under set-theoretic inclusion is a complete lattice. If $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$ and $\mathcal{U} \vee \mathcal{V} = \mathcal{D}$ then we say $\mathcal{U}$ is a complement of $\mathcal{V}$, where $\vee$ and $\wedge$ denote lattice operations, join and meet respectively of $(\mathcal{Q}(X), \subseteq)$. It is obvious that $\mathcal{I}$ denotes the indiscrete uniformity and $\mathcal{D}$ denote the discrete uniformity. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two uniformities on a set X such that $\mathcal{U}_1$ is contained in $\mathcal{U}_2$. Then the uniformity $\mathcal{U}_1$ is said to be a coarser (smaller or weaker) uniformity than $\mathcal{U}_2$, and $\mathcal{U}_2$ is said to be a finer (larger or stronger) uniformity than $\mathcal{U}_1$.

The following result also appear in [3] and [9] but their proofs are not detailed. For the benefit of the reader here we provide a more detailed proof.

Lemma 3.2.2. [3] *Let $(X, \mathcal{U})$ and $(X, \mathcal{V})$ be quasi-uniform spaces. Then the following statements are equivalent:*

- i). $\mathcal{U} \circ \mathcal{V}$ is a quasi-uniformity.
- ii). $\mathcal{U} \circ \mathcal{V} = \mathcal{U} \wedge \mathcal{V}$
- iii). $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{U}$

Proof: (i) $\Rightarrow$ (ii). ($\subseteq$) Suppose $\mathcal{U} \circ \mathcal{V}$ is a quasi-uniformity. Then by definition we get that $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{V} \cap \mathcal{U}$. So now $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{V} \cap \mathcal{U} \subseteq \mathcal{V} \wedge \mathcal{U}$, that is $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{V} \wedge \mathcal{U}$.

($\supseteq$). Let $D \in \mathcal{U} \wedge \mathcal{V}$. Then there exists $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that $(A \cup B) \circ (A \cup B) \subseteq D$. Thus $D \in \mathcal{U} \circ \mathcal{V}$, hence $\mathcal{U} \wedge \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{U}$.

(ii) $\Rightarrow$ (iii). Let $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \wedge \mathcal{U}$. Let $D \in \mathcal{U} \wedge \mathcal{V}$. Similarly there exists $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that $(A \cup B) \circ (A \cup B) \subseteq D$. Thus $D \in \mathcal{V} \circ \mathcal{U}$ and so $\mathcal{U} \wedge \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{U}$. By assumption we get $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{U}$.

(iii) $\Rightarrow$ (i). Suppose $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{U}$. Take $A \in \mathcal{U}$ and $B \in \mathcal{V}$. Then there exists $A_1 \in \mathcal{U}$ and $B_1 \in \mathcal{V}$ such that $A_1 \circ A_1 \subseteq A$ and $B_1 \circ B_1 \subseteq B$. Also there are $A_2 \in \mathcal{U}$ and $B_2 \in \mathcal{V}$ such that $A_2 \circ B_2 \subseteq A_1 \circ B_1$. So $(A_2 \circ B_2) \circ (A_2 \circ B_2) \subseteq A_2 \circ B_2 \circ A_2 \circ B_2 \subseteq A_2 \circ A_1 \circ A_1 \circ B_2 \subseteq (A_1 \circ B_1) \circ (A_2 \circ B_2) \subseteq A \circ B$. Thus $\mathcal{U} \circ \mathcal{V}$ is a quasi-uniformity. $\square$

In [3], the authors limited the proof of the next result. Here we provide the detailed simple proof.

Proposition 3.2.3. [3] *Two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ are permutable iff both $\mathcal{U} \circ \mathcal{V}$ and $\mathcal{V} \circ \mathcal{U}$ are quasi-uniformities.*

Proof: ($\Rightarrow$). Suppose $\mathcal{U}$ and $\mathcal{V}$ permute. Then $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{U}$. So $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{U}$ and $\mathcal{V} \circ \mathcal{U} \subseteq \mathcal{U} \circ \mathcal{V}$ then by Lemma 3.2.2, $\mathcal{U} \circ \mathcal{V}$ and $\mathcal{V} \circ \mathcal{U}$ are quasi uniformities.

Conversely, suppose both $\mathcal{U} \circ \mathcal{V}$ and $\mathcal{V} \circ \mathcal{U}$ are quasi-uniformities. Then $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{U}$ and $\mathcal{V} \circ \mathcal{U} \subseteq \mathcal{U} \circ \mathcal{V}$ by Lemma 3.2.2. Hence $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{U}$. By definition $\mathcal{U}$ and $\mathcal{V}$ permute. $\square$

For any set X and $A \subseteq X$, set $S_A = [(X \setminus A) \times X] \cup [X \times A]$ and $C_A = \Delta \cup [A \times (X \setminus A)]$.

In [3], de Jager and Künzi stated the following facts without providing the proofs. For completeness, we state and prove them below.

Lemma 3.2.4. *For any subset A of a set X , we have the following:*

- a) S_A is a preorder,
- b) C_A is a partial order.

Proof: a) We show that S_A is reflexive and transitive.

i) Let $x \in X$. If $x \in A$ then $(x, x) \in X \times A$. Similarly, if $x \notin A$ then $(x, x) \in (X \setminus A) \times X$. So $(x, x) \in S_A$ for all $x \in X$. That is S_A is reflexive.

ii) Suppose $(x, y), (y, z) \in S_A$ for some $x, y, z \in X$. Then $(x, y), (y, z) \in (X \setminus A) \times X$ or $(x, y), (y, z) \in (X \times A)$. So $x, y \in X \setminus A$ and $y, z \in X$ or $x, y \in X$ and $y, z \in A$. So $(x, z) \in (X \setminus A) \times X$ or $(x, z) \in X \times A$. That is $(x, z) \in S_A$. Hence S_A is transitive. Therefore S_A is a preorder. $\square$

b) We show that C_A is reflexive, antisymmetric and transitive.

i) Let $x \in X$. Then $(x, x) \in \Delta$ for all $x \in X$. So $(x, x) \in C_A$. Hence C_A is reflexive.

ii) Suppose $(x, y), (y, x) \in C_A$. If $(x, y), (y, x) \in \Delta$ then $x = y$. If $(x, y), (y, x) \in A \times (X \setminus A)$ then $x, y \in A$ and $y, x \in X \setminus A$, which is a contradiction. So $(x, y), (y, x) \in C_A$ whenever $(x, y), (y, x) \in \Delta$, hence $x = y$. That is C_A is antisymmetric.

iii) Suppose $(x, y), (y, z) \in C_A$. Then from ii) above we have $(x, y), (y, z) \notin A \times (X \setminus A)$. So $(x, y), (y, z) \in \Delta$. Now $x = y = z$, that is $(x, z) \in \Delta$. Hence $(x, z) \in C_A$. So C_A is transitive. Therefore C_A is a partial order. $\square$

Remark 3.2.5. The condition in Proposition 3.2.3 which says that, both $\mathcal{V} \circ \mathcal{U}$ and $\mathcal{U} \circ \mathcal{V}$ are quasi-uniformities is necessary for $\mathcal{U}$ and $\mathcal{V}$ to permute.

The following example shows us that we can have two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X with $\mathcal{U} \circ \mathcal{V}$ being a quasi-uniformity, while $\mathcal{U}$ and $\mathcal{V}$ do not permute. The same example also highlights the importance of the above Lemma 3.2.4.

Example 3.2.6. We define $\mathcal{U}_{S_A}$ as the quasi-uniformity generated by the base $\{S_A\}$. Consider the set $X = \{a, b, c\}$. Let $Y = \{a\}$ and $Z = \{a, b\}$. Then the complement of $S_Y \circ S_Z$ in $X \times X$ is $(Y \cap Z) \times (X \times (Y \cup Z)) = \{(a, b)\}$, since $Z \setminus Y \neq \emptyset$. Note that $(a, b) \in S_Z$ and $(b, c) \in S_Y$. So $S_Y \circ S_Z = X \times X$, thus $\mathcal{U}_{S_Y} \circ \mathcal{U}_{S_Z} = \{X \times X\}$, that is $\mathcal{U}_{S_Y} \circ \mathcal{U}_{S_Z}$ is a quasi-uniformity. Also $(a, c) \notin S_Y \circ S_Z$, so $\mathcal{U}_{S_Z} \circ \mathcal{U}_{S_Y}$ is strictly finer than $\mathcal{I}$ and it is not a quasi-uniformity on X .

Generalization of example 3.2.6. *Let Y and Z be distinct proper nonempty subsets of a set X . Then $Z \setminus Y \neq \emptyset$ iff $\mathcal{U}_{S_Y} \circ \mathcal{U}_{S_Z} = \mathcal{I}$.*

Proof: ($\Rightarrow$). Let $Y, Z \subseteq X$ such that $Z \setminus Y \neq \emptyset$. Then the complement of $S_Y \cup S_Z$ in $X \times X$ is given by $(Y \cap Z) \times (X \times (Y \cup Z)) = \{(a, b)\}$. Obviously if the complement of $S_Y \cup S_Z$ in $X \times X$ is empty then $S_Y \cup S_Z = X \times X$ and thus $X \times X \subseteq (S_Y \circ \Delta) \cup (\Delta \circ S_Z) \subseteq S_Y \circ S_Z$. Hence $\mathcal{U}_{S_Y} \circ \mathcal{U}_{S_Z} = \mathcal{I}$. Suppose $(Y \cap Z) \times (X \times Y \cup Z) \neq \emptyset$. So there is $y \in (Y \cap Z)$ and $z \in (X \times Y \cup Z)$ and by assumption there is $x \in Z \setminus Y$ such that $(y, x) \in S_Y$ and $(x, z) \in S_Z$. Thus $S_Y \circ S_Z = X \times X$ and so $\mathcal{U}_{S_Y} \circ \mathcal{U}_{S_Z} = \mathcal{I}$.

($\Leftarrow$). We prove the contrapositive of the converse. So suppose that $Z \subseteq Y$. So we can find $z \in Z$ and $x \in X \setminus Y$ and conclude that $(z, x) \notin S_Y \circ S_Z$. Hence $\mathcal{U}_{S_Y} \circ \mathcal{U}_{S_Z} \neq \mathcal{I}$. $\square$

Remark 3.2.7. In ([5], example 3.1), C.H. Yan gave a similar example of two quasi-order relations $\mathcal{R}$ and $\mathcal{T}$ such that $\mathcal{R}$ and $\mathcal{T}$ do not commute, because $\mathcal{R} \circ \mathcal{T}$ is a quasi-order while $\mathcal{T} \circ \mathcal{R}$ is not. So the condition that $\mathcal{R} \circ \mathcal{T}$ and $\mathcal{T} \circ \mathcal{R}$ need to be quasi-uniformities is very important for $\mathcal{R}$ and $\mathcal{T}$ to permute.

The next result appears in [3] without a proof. We state and provide a detailed proof of the proposition here using Lemma 3.2.2.

Proposition 3.2.8. *For any set X , the indiscrete uniformity and discrete uniformity permute with any quasi-uniformity on X .*

Proof: We show that i). $\mathcal{I} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{I}$ and ii). $\mathcal{D} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{D}$ for any uniformity $\mathcal{U}$ on X .

i). Let $O \in \mathcal{I} \circ \mathcal{U}$. Then there exist $V \in \mathcal{I}$ and $U \in \mathcal{U}$ such that $V \circ U \subseteq O$. Since $\mathcal{I} = \{X \times X\}$, $V = X \times X$. So $V \circ U = V = U \circ V \subseteq O$. Hence $O \in \mathcal{U} \circ \mathcal{I}$. So $\mathcal{I} \circ \mathcal{U} \subseteq \mathcal{U} \circ \mathcal{I}$. In a similar way we can show that $\mathcal{U} \circ \mathcal{I} \subseteq \mathcal{I} \circ \mathcal{U}$, thus $\mathcal{I} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{I}$. That is the indiscrete uniformity $\mathcal{I}$ on X permute with any quasi-uniformity on X .

ii). We make use of Lemma 3.2.2. Note that $\mathcal{D} \circ \mathcal{U} = \mathcal{U} = \mathcal{U} \wedge \mathcal{D}$. Hence by Lemma 3.2.2 $\mathcal{D} \circ \mathcal{U} \subseteq \mathcal{U} \circ \mathcal{D}$. Similarly, $\mathcal{U} \circ \mathcal{D} = \mathcal{U} = \mathcal{D} \wedge \mathcal{U}$ and thus $\mathcal{U} \circ \mathcal{D} \subseteq \mathcal{D} \circ \mathcal{U}$. This implies that $\mathcal{D} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{D}$. So $\mathcal{D}$ and $\mathcal{U}$ permute. $\square$

The next theorem appears in [9] with a detailed proof and it involves permuting uniformities. Here we state it so that we can use it in the next section.

Theorem 3.2.9. [9] *Let $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ be uniformities on a set X such that $\mathcal{U}$ and $\mathcal{V}$ permute and $\mathcal{V} \subseteq \mathcal{W}$. Then $\mathcal{U} \vee \mathcal{W}$ and $\mathcal{V}$ permute.*

3.3 Complements in the Lattice of Quasi-Uniformities

In this subsection we discuss complements of quasi-uniformities on X in the lattice of quasi-uniformities, $\mathcal{Q}(X)$. A connection between complementary quasi-uniformities and permutability is shown. It is widely known that $\mathcal{Q}(X)$ partially ordered under set-theoretic inclusion is a complete lattice.

Definition 3.3.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X . Then $\mathcal{U}$ is a complement of $\mathcal{V}$ if and only if $\mathcal{U} \vee \mathcal{V} = \mathcal{D}$ and $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$.*

For the corollary below we need the following proposition.

Proposition 3.3.2. [1] *All quasi-uniformity $\mathcal{U}$ of a complete lattice $(\mathcal{Q}(X), \subseteq)$ that are generated by a preorder R , have complements in $(\mathcal{Q}(X), \subseteq)$ that are generated by preorders.*

Proof: The detailed proof of this proposition appears in [1].

Definition 3.3.3. [17] *A set X is said to be finite if X is equivalent to A_n for some positive integer n , where $A_n = \{1, 2, 3, \dots, n\}$.*

Corollary 3.3.4. [1] *If X is a finite set then, every member of $(\mathcal{Q}(X), \subseteq)$ has a complement.*

Proof: Suppose X is a finite set and $\mathcal{U}$ is a quasi-uniformity on X . Then $\mathcal{U}$ contains a smallest surrounding say R which is a preorder and $\{R\}$ is a base for $\mathcal{U}$ by Example 3.1.2. So by the above proposition $\mathcal{U}$ has a complement. $\square$

Remark 3.3.5. Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities on a set X . If $\mathcal{U}$ and $\mathcal{V}$ permute and the complement of $\mathcal{U}$ in $(\mathcal{Q}(X), \subseteq)$ is $\mathcal{V}$ then we say $\mathcal{U}$ and $\mathcal{V}$ are permutable complements.

Note that Theorem 3.2.9 was stated in ([9], Theorem 3.5) using uniformities, here we state it using quasi-uniformities and putting an extra condition on $\mathcal{U}$ and $\mathcal{V}$ so that we get the proof without doing so much work.

Theorem 3.3.6. *Let $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ be members of $(\mathcal{Q}(X), \subseteq)$ such that $\mathcal{U}$ and $\mathcal{V}$ are complements. Then $\mathcal{U} \vee \mathcal{V}$ and $\mathcal{W}$ permute.*

Proof: Suppose $\mathcal{U}$ and $\mathcal{V}$ are complements in $(\mathcal{Q}(X), \subseteq)$ and $\mathcal{W}$ is another quasi-uniformity on X . Then we show that $(\mathcal{U} \vee \mathcal{V}) \circ \mathcal{W} = \mathcal{W} \circ (\mathcal{U} \vee \mathcal{V})$. Since $\mathcal{U}$ and $\mathcal{V}$ are complements, their join is $\mathcal{D}$. That is, $\mathcal{U} \vee \mathcal{V} = \mathcal{D}$. By Proposition 3.2.8, $\mathcal{D}$ and $\mathcal{W}$ permute. Hence $(\mathcal{U} \vee \mathcal{V}) \circ \mathcal{W} = \mathcal{W} \circ (\mathcal{U} \vee \mathcal{V})$. $\square$

Remark 3.3.7. From the above theorem we can easily see again that if $\mathcal{U}$ and $\mathcal{V}$ are complements in $(\mathcal{Q}(X), \subseteq)$ and $\mathcal{W}$ is another quasi-uniformity on X , then $\mathcal{U} \wedge \mathcal{V}$ and $\mathcal{W}$ permute since $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$ and by Proposition 3.2.8, $\mathcal{I}$ and $\mathcal{W}$ permute.

We easily get the following corollary.

Corollary 3.3.8. *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ and $\mathcal{Z}$ be members of $(\mathcal{Q}(X), \subseteq)$. If $\mathcal{U}$ is a complement of $\mathcal{V}$ and $\mathcal{W}$ is a complement of $\mathcal{Z}$. Then we have that:*

- i) $\mathcal{U} \vee \mathcal{V}$ and $\mathcal{W} \wedge \mathcal{Z}$ permute,
- ii) $\mathcal{U} \wedge \mathcal{V}$ and $\mathcal{W} \vee \mathcal{Z}$ also permute.

Proof: The proof easily follows from Theorem 3.3.6. $\square$

4. Some Special Quasi-Uniformities

In this chapter, we study some special quasi-uniformities. We begin the chapter by looking at quasi-uniformities that permute with their conjugates. The notion of adjacent quasi-uniformities is discussed in the second part of this chapter.

In the final part of this chapter we discuss independent quasi-uniformities. We then conclude the chapter by making some observations and remarks.

4.1 Quasi-Uniformities and their Conjugates

Next we look at quasi-uniformities that permute with their conjugates. Remember that the conjugate of a quasi-uniformity $\mathcal{U}$, is defined and denoted by $\mathcal{U}^{-1} = \{U^{-1} \mid U \in \mathcal{U}\}$. We say $\mathcal{U} \circ \mathcal{V}$ is symmetric given that $\mathcal{U} \circ \mathcal{V} = (\mathcal{U} \circ \mathcal{V})^{-1}$.

Definition 4.1.1. [4] *Let X be a set. If $(X, \mathcal{U})$ is a quasi-uniform then $(X, \mathcal{U}^{-1})$ is a quasi-uniform space, where $\mathcal{U}^{-1} = \{U^{-1} \mid U \in \mathcal{U}\}$.*

Remark 4.1.2. The quasi-uniformity $\mathcal{U}^{-1}$ is called the conjugate of $\mathcal{U}$.

Lemma 4.1.3. *Let $(X, \mathcal{U})$ be a quasi-uniform space. If $\mathcal{U}^{-1} = \mathcal{U}$ then $\mathcal{U}$ is a uniformity.*

Proof: For the forward implication, we suppose $\mathcal{U}$ is a quasi-uniformity and $\mathcal{U}^{-1} = \mathcal{U}$. Since $\mathcal{U}$ is a quasi-uniformity, $\mathcal{U}$ satisfies conditions of Definition 3.1.1. So we only need to show that for any $U \in \mathcal{U}$ we have $U^{-1} \in \mathcal{U}$. But $\mathcal{U}^{-1} = \{U^{-1} \mid U \in \mathcal{U}\}$ and $\mathcal{U}^{-1} = \mathcal{U}$. Hence for any $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$. So $\mathcal{U}$ is a uniformity on X . $\square$

Corollary 4.1.4. [4] *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $\mathcal{U} \circ \mathcal{U}^{-1}$ is a quasi-uniformity if and only if $\mathcal{U} \circ \mathcal{U}^{-1}$ is a uniformity.*

Proof: ($\Rightarrow$). Suppose $\mathcal{U} \circ \mathcal{U}^{-1}$ is a quasi-uniformity. We show that if $O \in \mathcal{U} \circ \mathcal{U}^{-1}$ then $O^{-1} \in \mathcal{U} \circ \mathcal{U}^{-1}$ by Definition 1.1.2. Let $O \in \mathcal{U} \circ \mathcal{U}^{-1}$. Then there is $U \in \mathcal{U}$ and $U^{-1} \in \mathcal{U}^{-1}$ such that $U \circ U^{-1} \subseteq O$. For contrary, suppose there is $(x, y) \in U \circ U^{-1}$ and $(y, x) \notin U \circ U^{-1}$. Then there is $z \in X$ such that $(x, z) \in U^{-1}$ and $(z, y) \in U$ and $(y, z) \notin U^{-1}$ and $(z, x) \notin U$.

So we have that $(x, z) \in U^{-1}$ and $(z, x) \notin U$ and $(z, y) \in U$ and $(y, z) \notin U^{-1}$. Hence a contradiction, that means we cannot find $(x, y) \in O$ and $((y, x) \in O$. Thus $O^{-1} = O \in \mathcal{U} \circ \mathcal{U}^{-1}$. So $\mathcal{U} \circ \mathcal{U}^{-1}$ is a uniformity.

($\Leftarrow$). Suppose $\mathcal{U} \circ \mathcal{U}^{-1}$ is a uniformity. Then obviously $\mathcal{U} \circ \mathcal{U}^{-1}$ is a quasi-uniformity by Definition 1.1.2. $\square$

Lemma 4.1.5. [3] *If $\mathcal{U}$ and $\mathcal{V}$ are two uniformities on a set X then $\mathcal{U}$ and $\mathcal{V}$ permute iff $\mathcal{U} \circ \mathcal{V}$ is symmetric.*

Proof:($\Rightarrow$). Let $\mathcal{U}$ and $\mathcal{V}$ be uniformities on a set X . Suppose $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{U}$. Since $\mathcal{U} = \mathcal{U}^{-1}$ and $\mathcal{V} = \mathcal{V}^{-1}$ then we have $\mathcal{U} \circ \mathcal{V} = \mathcal{U}^{-1} \circ \mathcal{V}^{-1} = (\mathcal{V} \circ \mathcal{U})^{-1} = (\mathcal{U} \circ \mathcal{V})^{-1}$. That is, $\mathcal{U} \circ \mathcal{V}$ is symmetric.

($\Leftarrow$). Suppose $\mathcal{U} \circ \mathcal{V}$ is symmetric. Then $\mathcal{U} \circ \mathcal{V} = (\mathcal{U} \circ \mathcal{V})^{-1}$. Now $\mathcal{U} \circ \mathcal{V} = (\mathcal{U} \circ \mathcal{V})^{-1} = \mathcal{V}^{-1} \circ \mathcal{U}^{-1} = \mathcal{V} \circ \mathcal{U}$. So $\mathcal{U}$ and $\mathcal{V}$ permute. $\square$

Example 4.1.6. Let $X = \{a, b\}$. Consider the two sets $\mathcal{U} = \{X \times X\}$ and $\mathcal{V} = \{\Delta, \Delta \cup \{(a, b)\}, \Delta \cup \{(b, a)\}, \Delta \cup \{(a, b), (b, a)\}\}$. Note that $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on X . In fact $\mathcal{U}$ is the indiscrete and $\mathcal{V}$ is the discrete uniformity on X . By Proposition 3.2.8, $\mathcal{U}$ and $\mathcal{V}$ permute. Now $\mathcal{U} \circ \mathcal{V} = \{X \times X\} = (\{X \times X\})^{-1} = (\mathcal{U} \circ \mathcal{V})^{-1}$. So $\mathcal{U} \circ \mathcal{V}$ is symmetric.

Remark 4.1.7. For any two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$, if $\mathcal{U} \circ \mathcal{V}$ is a quasi-uniformity then the composition of their conjugates, $\mathcal{U}^{-1} \circ \mathcal{V}^{-1}$ is a quasi-uniformity.

Corollary 4.1.8. *Let $\mathcal{U}$ and $\mathcal{V}$ be permutable quasi-uniformities on a set X . Then their conjugates $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$ also permute.*

Proof: Suppose $\mathcal{U}$ and $\mathcal{V}$ permute. Then $\mathcal{U} \circ \mathcal{V}$ and $\mathcal{V} \circ \mathcal{U}$ are quasi-uniformities. By Definition 4.1.1, $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$ are quasi-uniformities on X . So $\mathcal{U}^{-1} \circ \mathcal{V}^{-1}$ is a quasi-uniformity by the above remark. Therefore $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$ also permute. $\square$

Remark 4.1.9. If $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities such that $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{U}$ then $\mathcal{U}^{-1} \circ \mathcal{V}^{-1} = \mathcal{V}^{-1} \circ \mathcal{U}^{-1}$.

Proposition 4.1.10. [3] *Suppose $\mathcal{U}$ is a quasi-uniformity on a set X . Let $\emptyset \neq A \subseteq X$ such that $C_A \in \mathcal{U}$ where $C_A = \Delta \cup (A \times (X \setminus A))$. Then $\mathcal{U} \circ \mathcal{U}^{-1} = \mathcal{U} \cap \mathcal{U}^{-1}$ iff there is $U \in \mathcal{U}$ such that $|U[x]| \leq 2$ for some $x \in X$.*

Proof:($\Leftarrow$). Let $U \in \mathcal{U}$, $|U[x]| \leq 2$ for some $x \in X$. Let $V \in \mathcal{U}$ be such that $V \subseteq U$. For a contradiction suppose $V \circ V^{-1} \neq V \cup V^{-1}$. Then there exists $(x, y) \in V \circ V^{-1} \setminus V \cup V^{-1}$. So there is $z \in X$ such that $(x, z) \in V$ and $(y, z) \in V^{-1}$. On the other hand $x \neq z$ and $y \neq z$ since $(x, y) \notin V \cup V^{-1}$. Since $|U[x]| \leq 2$ then $|V[x]| \leq 2$ so $x = y$. That means $(x, y) \in V \cup V^{-1}$ which contradicts our assumption that $V \circ V^{-1} \neq V \cup V^{-1}$. Therefore $V \circ V^{-1} = V \cup V^{-1}$. So for every $V \in \mathcal{U}$, $V \circ V^{-1} = V \cup V^{-1}$ and thus $\mathcal{U} \circ \mathcal{U}^{-1} = \mathcal{U} \cap \mathcal{U}^{-1}$.

($\Rightarrow$). Suppose $\mathcal{U} \circ \mathcal{U}^{-1} = \mathcal{U} \cap \mathcal{U}^{-1}$. By the above remark, we have that $\mathcal{U} \circ \mathcal{U}^{-1} = \mathcal{U}^{-1} \circ \mathcal{U}$. By assumption there is $U \in \mathcal{U}$ such that $U \subseteq C_A$ and there is $V \in \mathcal{U}$ such that $V \subseteq U$ and $V \circ V^{-1} \subseteq U^{-1} \circ U$. Let $x \in X$ and $y, z \in V[x]$ such that $x \neq y$ and $x \neq z$. Then $x \in A$ and $y, z \notin A$ since $V \subseteq U \subseteq C_A$. So $V[x] \subseteq C_A[x]$. Thus $x \neq y$ and $x \neq z$. Also $(y, z) \in V \circ V^{-1} \subseteq U^{-1} \circ U$. So there exists $v \in X$ with $y, v \in U$ and $(z, v) \in U^{-1}$, that is $(v, z) \in U$. By assumption $y, z \notin X \setminus A$ and $U \subseteq C_A$ so $y = v$ and $z = v$. Hence $y = z$. Therefore $|U[x]| \leq 2$ whenever $x \in X$ which concludes our proof. $\square$

Lemma 4.1.11. *Let $(X, \mathcal{U})$ be a quasi-uniformity such that for some non-empty $A \subseteq X$, $C_A \in \mathcal{U}$. Then $\mathcal{U}$ and $\mathcal{U}^{-1}$ permute iff $\exists U \in \mathcal{U}$ such that $|U[x]| \leq 2$ and $|U^{-1}[x]| \leq 2$ where $x \in X$.*

Proof:($\Rightarrow$). Suppose $\mathcal{U}$ and $\mathcal{U}^{-1}$ are permutable. Then $\mathcal{U}^{-1} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{U}^{-1}$ by definition. By Proposition 4.1.10, there exist $V_1 \in \mathcal{U}$ and $V_2 \in \mathcal{U}$ such that $|V_1[x]| \leq 2$ and $|V_2[x]| \leq 2$. Now let $U = V_1 \cap V_2$. Then $|U[x]| \leq 2$. So $\exists U \in \mathcal{U}$ such that $|U^{-1}[x]| \leq 2$.

($\Leftarrow$). Suppose $\exists U \in \mathcal{U}$ such that $|U^{-1}[x]| \leq 2$ and $|U[x]| \leq 2$ for some $x \in X$. By Proposition 4.1.10 we have that $\mathcal{U}^{-1} \circ \mathcal{U} = \mathcal{U} \cap \mathcal{U}^{-1} = \mathcal{U} \circ \mathcal{U}^{-1}$. Hence $\mathcal{U}$ and $\mathcal{U}^{-1}$ permute.

□

4.2 Independent Quasi-Uniformities

In this chapter we discuss independent quasi-uniformities. In particular we present a known result from ([3], corollary 4) in terms of independent quasi-uniformities.

We will need the following definitions in the corollary below.

Definition 4.2.1. *A quasi-uniformity $\mathcal{U}$ on the set X is said to be an atom if for any quasi-uniformity $\mathcal{V}$ on X such that $\mathcal{I} \subseteq \mathcal{V} \subseteq \mathcal{U}$ then $\mathcal{V} = \mathcal{U}$.*

Example 4.2.2. Let $X = \{a, b, c\}$ and take $A = \{a\}$. Then $A \subseteq X$ and $X \setminus A = \{b, c\}$. So $S_A = \{(b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (a, a), (a, b), (a, c)\}$. Define $\mathcal{F} = \{F \subseteq X \times X \mid S_A \subseteq F\}$. Then $\mathcal{F}$ is an atom of $(\mathcal{Q}(X), \subseteq)$.

Lemma 4.2.3. *$\mathcal{F}$ is a transitive atom of $(\mathcal{Q}(X), \subseteq)$.*

Proof: Note that a quasi-uniformity is transitive if it is generated by a transitive base. By Definition 1.1.5 of a base, S_A is a base for $\mathcal{F}$. Now we need to show that S_A is transitive. By Lemma 3.2.4, S_A is a preorder hence transitive. That is $\mathcal{F}$ is transitive. □

Definition 4.2.4. Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities on a set X . Then $\mathcal{U}$ and $\mathcal{V}$ are said to be independent provided that $\mathcal{U}$ and $\mathcal{V}$ permute and $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$.

Proposition 4.2.5. Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities on X . Then $\mathcal{U} \wedge \mathcal{V} \subseteq \mathcal{U} \circ \mathcal{V}$.

Proof: Let $O \in \mathcal{U} \wedge \mathcal{V}$. Then there exist $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $(U \cup V) \circ (U \cup V) \subseteq O$, hence $O \in \mathcal{U} \circ \mathcal{V}$. Thus $\mathcal{U} \wedge \mathcal{V} \subseteq \mathcal{U} \circ \mathcal{V}$. $\square$

Next we present a known result (see [3], corollary 4) in terms of independent quasi-uniformities and provide a more detailed proof.

Corollary 4.2.6. Let $\mathcal{U}$ and $\mathcal{V}$ be independent quasi-uniformities on a set X and let $\mathcal{A}$ and $\mathcal{B}$ be quasi-uniformities on a set X such that $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$. Then $\mathcal{A}$ and $\mathcal{B}$ permute.

Proof: Suppose that $\mathcal{U}$ and $\mathcal{V}$ are independent quasi-uniformities on a set X and further let $\mathcal{A}$ and $\mathcal{B}$ be quasi-uniformities on the set X such that $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$. By definition of independence, $\mathcal{U}$ and $\mathcal{V}$ permute and $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$. By Lemma 3.2.2, $\mathcal{U} \circ \mathcal{V}$ is a quasi-uniformity and thus $\mathcal{U} \circ \mathcal{V} = \mathcal{U} \wedge \mathcal{V}$. So by Proposition 4.2.5, $\mathcal{A} \wedge \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$ and $\mathcal{A} \circ \mathcal{B} \subseteq \mathcal{U} \circ \mathcal{V} = \mathcal{U} \wedge \mathcal{V} \subseteq \mathcal{A} \wedge \mathcal{B}$. Thus $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \circ \mathcal{B}$ and hence $\mathcal{A} \circ \mathcal{B}$ is a quasi-uniformity by Lemma 3.2.2. Similarly, we can show that $\mathcal{B} \wedge \mathcal{A} = \mathcal{B} \circ \mathcal{A}$, thus $\mathcal{B} \circ \mathcal{A}$ is a quasi-uniformity, hence by definition of permutability, $\mathcal{A}$ and $\mathcal{B}$ permute. $\square$

The above Corollary 4.2.6 can be stated in the following ways:

Remark 4.2.7. 1) If $\mathcal{A}$ and $\mathcal{B}$ are atoms of $(\mathcal{Q}(X), \subseteq)$ on a set X and $\mathcal{U}$ and $\mathcal{V}$ are permuting quasi-uniformities in $\mathcal{Q}(X)$ such that $\mathcal{U}$ is finer than $\mathcal{A}$ and $\mathcal{V}$ is finer than $\mathcal{B}$ then $\mathcal{A}$ and $\mathcal{B}$ permute.

2) Two atoms $\mathcal{A}$ and $\mathcal{B}$ of $(\mathcal{Q}(X), \subseteq)$ permute if there exist two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ of $(\mathcal{Q}(X), \subseteq)$ such that $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$.

We get the following lemma from Remark 3.3.5 and it gives us a nice relationship between complementary, permuting and independent quasi-uniformities.

Lemma 4.2.8. *Permutable complements are independent.*

Proof: Suppose $\mathcal{U}$ and $\mathcal{V}$ are permutable complements. Then we have to show that $\mathcal{U}$ and $\mathcal{V}$ permute and $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$. But everything follows from Remark 3.3.5, hence $\mathcal{U}$ and $\mathcal{V}$ are independent quasi-uniformities. $\square$

4.3 Adjacent Quasi-uniformities

Adjacent quasi-uniformities were also studied in [9] by Hans Weber. Here we show that conjugates of adjacent quasi-uniformities are adjacent and conclude by linking adjacent quasi-uniformities with atoms.

Definition 4.3.1. [1] *Any two comparable distinct quasi-uniformities on a set X such that there is no quasi-uniformity on X which is strictly in between them are said to be adjacent or neighbors. That is, if $\mathcal{U} \subset \mathcal{V}$ or $\mathcal{V} \subset \mathcal{U}$ then $\mathcal{U}$ and $\mathcal{V}$ are adjacent quasi-uniformities on X if and only if we cannot find another quasi-uniformity $\mathcal{W}$ on X such that $\mathcal{U} \subset \mathcal{W} \subset \mathcal{V}$ or $\mathcal{V} \subset \mathcal{W} \subset \mathcal{U}$.*

The lemma below was stated in [1] without a proof. For the benefit of the reader, we state and prove the result here.

Lemma 4.3.2. *Let $\mathcal{U}$ and $\mathcal{V}$ be adjacent quasi-uniformities on a set X . Then $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$ are also adjacent quasi-uniformities on X .*

Proof: Suppose $\mathcal{U}$ and $\mathcal{V}$ are adjacent quasi-uniformities on a set X such that $\mathcal{U} \subset \mathcal{V}$. Then by Definition 1.1.3, $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$ are quasi-uniformities on X . We show that $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$ are also adjacent quasi-uniformities on X . For a contrary, assume $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$ are not adjacent quasi-uniformities on X . Then there is a quasi-uniformity $\mathcal{W}^{-1}$ on X such that $\mathcal{W}^{-1}$ strictly between $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$.

Let $(y, x) \in \mathcal{U}^{-1}$. Then $(x, y) \in \mathcal{U}$ and $(y, x) \in \mathcal{W}^{-1}$. So $(x, y) \in \mathcal{W}$. Also $(y, x) \in \mathcal{V}^{-1}$ and so $(x, y) \in \mathcal{V}$. That is $\mathcal{U} \subset \mathcal{W} \subset \mathcal{V}$. Now we have that $\mathcal{U}$ and $\mathcal{V}$ are not adjacent quasi-uniformities on X , hence a contradiction. If $\mathcal{V} \subset \mathcal{U}$ then a similar argument works. Therefore $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$ are adjacent quasi-uniformities on X . $\square$

Remark 4.3.3. a) By the above lemma, $\mathcal{U}$ is an atom if and only if $\mathcal{I}$ and $\mathcal{U}$ are adjacent quasi-uniformities.

b) Also, if $\mathcal{U}$ is an atom then $\mathcal{U}^{-1}$ is also an atom. The proof follows from a).

5. Quasi-Uniformities on an Ordered Space

In this chapter we construct some quasi-uniformities on an ordered topological space $(X, \mathcal{T}, \leq)$. We conclude the chapter by providing the direction we intend to take in future work where we will try to establish the link of these quasi-uniformities to permutability.

5.1 An Ordered Uniform Space

An ordered topological space is a triple $(X, \mathcal{T}, \leq)$ where X is a set equipped with a topology $\mathcal{T}$ and a order $\leq$.

A binary relation $\leq$ on a set S is called a partial order if $\leq$ is reflexive, antisymmetric and transitive. The pair $(S, \leq)$ is then called a partially ordered set or a poset.

Let $(X, \mathcal{T})$ be a topological space equipped with a partial order $\leq$ and let $\mathcal{U}$ be a uniformity on X . Then for any $x \in X$, define $\uparrow(x)$ and $\downarrow(x)$ as follows:

$$\uparrow(x) = \{y \in X \mid x \leq y\}$$

$$\downarrow(x) = \{y \in X \mid y \leq x\}.$$

By definition 1.1.1, we can see that

$$U[\uparrow(x)] = \{y \in X \mid \exists a \in \uparrow(x) \text{ with } (a, y) \in U\} \text{ and}$$

$$U[\downarrow(x)] = \{y \in X \mid \exists a \in \downarrow(x) \text{ with } (a, y) \in U\}.$$

For any $U \in \mathcal{U}$, set

$$U^\uparrow = \{(x, y) \in X \times X \mid \uparrow(y) \subseteq U(\uparrow(x))\} \text{ and}$$

$$U^\downarrow = \{(x, y) \in X \times X \mid \downarrow(y) \subseteq U(\downarrow(x))\}.$$

Remark 5.1.1. By construction we have that $\uparrow(x), \downarrow(x), U(\uparrow(x)), U(\downarrow(x)) \subseteq X$ and $U^\uparrow, U^\downarrow \subseteq X \times X$.

Lemma 5.1.2. (compare with [11] lemma 4.3.1) The set $\mathcal{B}^\uparrow = \{U^\uparrow \mid U \in \mathcal{U}\}$ forms a basis for a quasi-uniformity on X .

Proof: We show that $\mathcal{B}^\uparrow$ forms a basis for a quasi-uniformity on X . Let $x \in X$. By definition $U(\uparrow(x)) = \{y \in X \mid \exists a \in \uparrow(x) \text{ with } (a, y) \in U\}$, so if $a \in \uparrow(x)$ then obviously $a \in U(\uparrow(x))$. So $\uparrow(x) \subseteq U(\uparrow(x))$ which implies that $(x, x) \in U^\uparrow$ by construction of $U^\uparrow$. Hence $\Delta \subseteq U^\uparrow$ for every $U^\uparrow \in \mathcal{B}^\uparrow$.

To complete the proof, by definition of a base, we need to show that for every $U^\uparrow \in \mathcal{B}^\uparrow$ there is a $V^\uparrow \in \mathcal{B}^\uparrow$ such that $V^\uparrow \circ V^\uparrow \subseteq U^\uparrow$. Let $U^\uparrow \in \mathcal{B}^\uparrow$. Since $U \in \mathcal{U}$ and $\mathcal{U}$ is quasi-uniformity, there is $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. We then have $V^\uparrow \in \mathcal{B}^\uparrow$ by definition of $\mathcal{B}^\uparrow$. For any $(x, y) \in V^\uparrow \circ V^\uparrow$, we can always find $z \in X$ such that $(x, z) \in V^\uparrow$ and $(z, y) \in V^\uparrow$ which implies $\uparrow(z) \subseteq V(\uparrow(x))$ and $\uparrow(y) \subseteq V(\uparrow(z))$. Next we show that $(x, y) \in U^\uparrow$ by showing that $V(\uparrow(z)) \subseteq U(\uparrow(x))$. Begin with $u \in V(\uparrow(z))$. Then $(u, a) \in V$ for some $a \in \uparrow(z)$. So $a \in V(\uparrow(x))$ since $\uparrow(z) \subseteq V(\uparrow(x))$, since $(a, b) \in V$ for some $b \in \uparrow(x)$. Now $(u, a), (a, b) \in V$, so $(u, b) \in V \circ V$.

Since $V \circ V \subseteq U$, we have $(u, b) \in U$ and thus $u \in U(\uparrow(x))$. Hence $V(\uparrow(z)) \subseteq U(\uparrow(x))$. But now $\uparrow(y) \subseteq V(\uparrow(z)) \subseteq U(\uparrow(x))$, so $\uparrow(y) \subseteq U(\uparrow(x))$ which implies that $(x, y) \in U^\uparrow$. Hence $V^\uparrow \circ V^\uparrow \subseteq U^\uparrow$. So $\mathcal{B}^\uparrow$ is a base for a quasi-uniformity on X . $\square$

As may be expected, we also obtain:

Lemma 5.1.3. The set $\mathcal{B}^\downarrow = \{U^\downarrow \mid U \in \mathcal{U}\}$ forms a basis for a quasi-uniformity on X .

Proof: The proof is similar to that of Lemma 5.1.2. $\square$

Remark 5.1.4. We will denote these quasi-uniformities generated by $\mathcal{B}^\uparrow$ and $\mathcal{B}^\downarrow$ in Lemma 5.1.2 and Lemma 5.1.3 with $\mathcal{U}^\uparrow$ and $\mathcal{U}^\downarrow$ respectively.

Lemma 5.1.5. *Let $(X, \mathcal{U})$ and $(X, \mathcal{V})$ be two uniform spaces equipped with order $\leq$ and further let $V^\Downarrow \in \mathcal{V}^\Downarrow, U^\Uparrow \in \mathcal{U}^\Uparrow$. Then we have*

$$a). (V^\Downarrow)^{-1} \circ (V^\Downarrow)^{-1} = (V^\Downarrow \circ V^\Downarrow)^{-1},$$

$$b). (U^\Uparrow)^{-1} \circ (U^\Uparrow)^{-1} = (U^\Uparrow \circ U^\Uparrow)^{-1}.$$

Proof: a) $(\subseteq)$: Let $(x, y) \in (V^\Downarrow)^{-1} \circ (V^\Downarrow)^{-1}$. Then there exist $z \in X$ such that $(x, z) \in (V^\Downarrow)^{-1}$ and $(z, y) \in (V^\Downarrow)^{-1}$. Now we have that $(y, z) \in V^\Downarrow$ and $(z, x) \in V^\Downarrow$. So $(y, x) \in V^\Downarrow \circ V^\Downarrow$. Hence $(x, y) \in (V^\Downarrow \circ V^\Downarrow)^{-1}$. That is $(V^\Downarrow)^{-1} \circ (V^\Downarrow)^{-1} \subseteq (V^\Downarrow \circ V^\Downarrow)^{-1}$.

$(\supseteq)$: Let $(x, y) \in (V^\Downarrow \circ V^\Downarrow)^{-1}$. Then $(y, x) \in V^\Downarrow \circ V^\Downarrow$. Then there exist $z \in X$ such that $(y, z) \in V^\Downarrow$ and $(z, x) \in V^\Downarrow$. Now $(x, z) \in (V^\Downarrow)^{-1}$ and $(z, y) \in (V^\Downarrow)^{-1}$. Hence $(x, y) \in (V^\Downarrow)^{-1} \circ (V^\Downarrow)^{-1}$. This implies that $(V^\Downarrow \circ V^\Downarrow)^{-1} \subseteq (V^\Downarrow)^{-1} \circ (V^\Downarrow)^{-1}$. So $(V^\Downarrow)^{-1} \circ (V^\Downarrow)^{-1} = (V^\Downarrow \circ V^\Downarrow)^{-1}$. $\square$

b) The proof is similar to the one for part a). $\square$

Remark 5.1.6. We will use the above Lemma 5.1.5 in the next Proposition 5.1.7.

Proposition 5.1.7. *(compare with [11] proposition 4.3.3) Let $(X, \mathcal{U})$ be a uniform space and define $\mathcal{B}$ by $\mathcal{B} = \{U^\Uparrow \cap (U^\Downarrow)^{-1} \mid U \in \mathcal{U}\}$. Then $\mathcal{B}$ is a base for a quasi-uniformity on X .*

Proof: By Lemma 5.1.2 and Lemma 5.1.3, it is easy to see that $\Delta \subseteq U^\Uparrow \cap (U^\Downarrow)^{-1}$ for every $U^\Uparrow \cap (U^\Downarrow)^{-1} \in \mathcal{B}$.

Since $\mathcal{U}^\Uparrow$ and $\mathcal{U}^\Downarrow$ are both quasi-uniformities, for any $U^\Uparrow \in \mathcal{U}^\Uparrow$ there is $V^\Uparrow \in \mathcal{U}^\Uparrow$ such that $V^\Uparrow \circ V^\Uparrow \subseteq U^\Uparrow$. Also for every $U^\Downarrow \in \mathcal{U}^\Downarrow$ there is $V^\Downarrow \in \mathcal{U}^\Downarrow$ such that $V^\Downarrow \circ V^\Downarrow \subseteq U^\Downarrow$.

We need to show that $(V^\Uparrow \cap (V^\Downarrow)^{-1}) \circ (V^\Uparrow \cap (V^\Downarrow)^{-1}) \subseteq (U^\Uparrow \cap (U^\Downarrow)^{-1})$. Take $(x, y) \in (V^\Uparrow \cap (V^\Downarrow)^{-1}) \circ (V^\Uparrow \cap (V^\Downarrow)^{-1})$. Then there is $z \in X$ such that $(x, z) \in (V^\Uparrow \cap (V^\Downarrow)^{-1})$ and $(z, y) \in (V^\Uparrow \cap (V^\Downarrow)^{-1})$ hence $(z, y) \in V^\Uparrow$ and $(z, y) \in (V^\Downarrow)^{-1}$ and also $(x, z) \in V^\Uparrow$ and $(x, z) \in (V^\Downarrow)^{-1}$. Now we have $(x, z) \in V^\Uparrow$ and $(z, y) \in V^\Uparrow$ which implies that $(x, y) \in V^\Uparrow \circ V^\Uparrow$. So $(x, y) \in U^\Uparrow$ since $V^\Uparrow \circ V^\Uparrow \subseteq U^\Uparrow$. We can also show that $(x, y) \in (V^\Downarrow)^{-1} \circ (V^\Downarrow)^{-1}$ by a similar argument. Hence $(x, y) \in (V^\Downarrow \circ V^\Downarrow)^{-1}$ since $(x, y) \in (V^\Downarrow)^{-1} \circ (V^\Downarrow)^{-1} = (V^\Downarrow \circ V^\Downarrow)^{-1}$ by Lemma 5.1.5. Now $(y, x) \in V^\Downarrow \circ V^\Downarrow \subseteq U^\Downarrow$ hence $(y, x) \in U^\Downarrow$. So $(x, y) \in (U^\Downarrow)^{-1}$, that is $(x, y) \in U^\Uparrow \cap (U^\Downarrow)^{-1}$. Therefore $(V^\Uparrow \cap (V^\Downarrow)^{-1}) \circ (V^\Uparrow \cap (V^\Downarrow)^{-1}) \subseteq (U^\Uparrow \cap (U^\Downarrow)^{-1})$. Hence $\mathcal{B}$ is a base for a quasi-uniformity on X . We denote this quasi-uniformity by $\mathcal{U}^{\Uparrow\Downarrow}$. $\square$

Even if the proofs of the following lemmas are similar, for the benefit of the reader we present all the proofs here.

Lemma 5.1.8. (compare with [11]) Let $(X, \mathcal{U})$ be a uniform space equipped with order $\leq$ and further let $U^\uparrow \in \mathcal{U}^\uparrow$ and $U^\downarrow \in \mathcal{U}^\downarrow$. Then for any $x \in X$ we have

$$(U^\downarrow)^{-1}(x) = \bigcap_{a \in \downarrow(x)} \uparrow(U(a)).$$

Proof: Take $y \in (U^\downarrow)^{-1}(x)$. Then $(x, y) \in (U^\downarrow)^{-1}$ and hence $(y, x) \in U^\downarrow$. So $\downarrow(x) \subseteq U(\downarrow(y))$. That is, for every $a \in \downarrow(x)$, $a \in U(\uparrow(y))$. Then for every $a \in \downarrow(x)$, there is $b \in \downarrow(y)$ such that $a \in U(b)$. So $(b, a) \in U$. Now we have for every $a \in \downarrow(x)$, there is $b \in U(a)$ such that $b \leq y$. Hence for every $a \in \downarrow(x)$ there is $y \in \uparrow(U(a))$. That is $y \in \bigcap_{a \in \downarrow(x)} \uparrow(U(a))$. So $(U^\downarrow)^{-1}(x) \subseteq \bigcap_{a \in \downarrow(x)} \uparrow(U(a))$.

For the other inclusion, we begin with $y \in \bigcap_{a \in \downarrow(x)} \uparrow(U(a))$ and we can easily show that $y \in (U^\downarrow)^{-1}(x)$ in a similar argument. So $\bigcap_{a \in \downarrow(x)} \uparrow(U(a)) \subseteq (U^\downarrow)^{-1}(x)$. Hence $(U^\downarrow)^{-1}(x) = \bigcap_{a \in \downarrow(x)} \uparrow(U(a))$. $\square$

Lemma 5.1.9. (compare with [11]) Let $(X, \mathcal{U})$ be a uniform space equipped with order $\leq$ and further let $U^\uparrow \in \mathcal{U}^\uparrow$ and $U^\downarrow \in \mathcal{U}^\downarrow$. Then for any $x \in X$ we have

$$(U^\uparrow)^{-1}(x) = \bigcap_{a \in \uparrow(x)} \downarrow(U(a)).$$

Proof: Let $y \in (U^\uparrow)^{-1}(x)$. Then $(x, y) \in (U^\uparrow)^{-1}$ and so $(y, x) \in U^\uparrow$. By definition of $U^\uparrow$, $\uparrow(x) \subseteq U(\uparrow(y))$. That is, for every $a \in \uparrow(x)$, $a \in U(\uparrow(y))$. This implies that for every $a \in \uparrow(x)$, there is $b \in \uparrow(y)$, hence $(a, b) \in U$. Now we have that for every $a \in \uparrow(x)$, $y \in \downarrow(U(a))$. That is $y \in \bigcap_{a \in \uparrow(x)} \downarrow(U(a))$. Hence $(U^\uparrow)^{-1}(x) \subseteq \bigcap_{a \in \uparrow(x)} \downarrow(U(a))$.

A similar argumentation gives us that $\bigcap_{a \in \uparrow(x)} \downarrow(U(a)) \subseteq (U^\uparrow)^{-1}(x)$ and therefore $(U^\uparrow)^{-1}(x) = \bigcap_{a \in \uparrow(x)} \downarrow(U(a))$. $\square$

5.2 Link to permutability

The quasi-uniformities $\mathcal{U}^\uparrow$ and $\mathcal{U}^\downarrow$ are of independent interest. They have been used in studying quasi-uniformizability of biotopological spaces. In the future, we want to look at them from a permutability point of view. For instance, one may consider questions like:

Question 1: Does $\mathcal{U}^\uparrow$ permute with $\mathcal{U}^\downarrow$?

Question 2: Does $\mathcal{U}^{\uparrow\downarrow}$ permute with $\mathcal{U}^\downarrow$?

Question 3: Does $\mathcal{U}^{\uparrow\downarrow}$ permute with $\mathcal{U}^\uparrow$?

Question 4: Are these quasi-uniformities above independent quasi-uniformities? Are they complements of each of each other? Are they adjacent quasi-uniformities?

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