

SOME NEW AND GENERALIZED DISTRIBUTIONS WITH APPLICATIONS

BY

KETHAMILE RANNOA

Reg No. 15000146

MSc Statistics

Department of Mathematics and Statistical Sciences
Botswana International University of Science and Technology

E-mail: kethamile.rannona@studentmail.biust.ac.bw

Tel: +267 74184262

A Dissertation/Thesis Submitted to the College of Science in Partial
Fulfillment of the Requirements for the Award of the Degree of Master of
Science in Statistics of BIUST

Supervisors:

Prof. Broderick Oluyede
Department of Mathematics
and Statistical Sciences
College of Science, BIUST
Email: oluyedeo@biust.ac.bw

Prof. Boikanyo Makubate
Department of Mathematics
and Statistical Sciences
College of Science, BIUST
makubateb@biust.ac.bw

Signature:.....

Date:.....20/09/2021

Signature:.....

Date:.....21/09/2021

September, 2021

Declaration and Copyright

I, Kethamile Rannona, declare that this dissertation/thesis is my own original work and that it has not been presented and will not be presented to any other university for a similar or any other degree award.

Signature K. Rannona

This dissertation/thesis is copyright material protected under the Berne Convention, the Copyright Act of 1999 and other international and national enactments, in that behalf, on intellectual property. It must not be reproduced by any means, in full or in part, except for short extracts in fair dealing; for researcher private study, critical scholarly review or discourse with an acknowledgement, without the written permission of the office of the Provost, on behalf of both the author and the BIUST.

Certification

This undersigned certifies that she has read and hereby recommends for acceptance by the College of Science a dissertation/thesis titled: SOME NEW AND GENERALIZED DISTRIBUTIONS WITH APPLICATIONS, in fulfilment of the requirements for the Degree of Master of Science in Statistics of BIUST.

Supervisors:

Prof. Broderick Oluyede
Department of Mathematics
and Statistical Sciences
College of Science, BIUST
Email: oluyedeo@biust.ac.bw

Prof. Boikanyo Makubate
Department of Mathematics
and Statistical Sciences
College of Science, BIUST
makubateb@biust.ac.bw

Signature: 
Date: 20/09/2021

Signature: 
Date: 21/09/2021

Dedication

This dissertation work is dedicated to my wonderful, caring, loving and supportive mother for her unwavering support, prayers and words of encouragement.

Acknowledgements

It is a great honor and privilege for me to share my heartfelt appreciation to my supervisors Prof. Broderick Oluyede and Prof. Boikanyo Makubate. They have been a constant source of guidance and advice for me throughout the duration of my study and they have made it possible for me to finish this thesis.

I express my sincere thanks with gratitude to my parents, siblings and friends for their support, blessings and encouragement.

Finally, I thank the Almighty who has made everything possible.

Contents

1	Introduction	1
1.1	Introduction	1
1.2	Some Generators of Distributions	4
1.2.1	T-X Family Method	4
1.2.2	Compounding Method	4
1.2.3	Exponentiation Method	5
1.2.4	Beta-G Generator	5
1.2.5	Kumaraswamy-G Generator	6
1.2.6	Marshall-Olkin-G Generator	6
1.3	Some Properties of Distributions	6
1.3.1	Survival, Hazard and Reverse Hazard Functions	6
1.3.2	Quantile Function	7
1.3.3	Moments	7
1.3.4	Conditional Moments	7
1.3.5	Mean Deviations	8
1.3.6	Bonferroni and Lorenz Curves	8
1.3.7	Distribution of Order Statistics	8
1.3.8	Entropy	9
1.3.9	Estimation	9
1.3.10	Goodness-of-fit Statistics	9
1.4	Aims and Objectives	10
1.5	Outline of Thesis	10
2	The Topp-Leone-G Power Series Class of Distributions	12
2.1	Introduction	12

2.2	The Model	13
2.2.1	Quantile function	14
2.2.2	Expansion of the Density Function	14
2.2.3	Sub-Classes of the TL-GPS Distribution	16
2.2.4	Some Special Cases of the TL-GPS Class of Distributions . .	16
2.3	Moments, Conditional Moments and Mean Deviations	24
2.3.1	Moments and Generating Function	24
2.3.2	Conditional Moments	25
2.3.3	Mean Deviations, Lorenz and Bonferroni Curves	25
2.3.4	Order Statistics and Rényi Entropy	26
2.4	Estimation	29
2.5	Simulation Study	30
2.6	Applications	31
2.6.1	Growth Hormone Data	34
2.6.2	Repair Lifetimes Data	35
2.7	Concluding Remarks	37

3 The Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G Family of Distributions 38

3.1	Introduction	38
3.2	Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G Family of Distributions	39
3.2.1	Expansion of Density Function	40
3.2.2	Stochastic Orders	44
3.3	Some Special Cases	44
3.3.1	Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-Burr XII (MO-EOEHL-BXII) Distribution	45
3.3.2	Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-Weibull (MO-EOEHL-W) Distribution	46
3.3.3	Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-Uniform (MO-EOEHL-U) Distribution	48
3.4	Statistical Properties	49
3.4.1	Distribution of Order Statistics	49

3.4.2	Entropy	52
3.4.3	Moments and Generating Functions	56
3.4.4	Quantile Function	59
3.5	Maximum Likelihood Estimation	60
3.6	Simulation Study	62
3.7	Applications	62
3.7.1	Kevlar 49/Epoxy Strands Failure at 90% Data	65
3.7.2	Strengths of 1.5 cm Glass Fibres Data	67
3.8	Concluding Remarks	69
4	The Exponentiated Odd Exponential Half-Logistic-G Power Series Class of Distributions	70
4.1	Introduction	70
4.2	Exponentiated Odd Exponential Half-Logistic-G Power Series Distribution	71
4.2.1	Expansion of Density Function	72
4.3	Some Special Cases of the EOEHL-GPS Class of Distributions . . .	74
4.3.1	Exponentiated Odd Exponential Half-Logistic-Log-logistic Power Series (EOEHL-LLoGPS) Class of Distributions . . .	74
4.3.2	Exponentiated Odd Exponential Half-Logistic-Weibull Power Series (EOEHL-WPS) Class of Distributions	79
4.4	Statistical Properties	84
4.4.1	Quantile Function	84
4.4.2	Moments and Generating Function	86
4.4.3	Mean Deviation, Lorenz and Bonferroni Curves	88
4.4.4	Order Statistics and Rényi Entropy	89
4.4.5	Rényi Entropy	91
4.5	Maximum Likelihood Estimation	94
4.6	Simulation Study	96
4.7	Applications	96
4.7.1	Cancer Patients Data	99
4.7.2	Failure Times Data	101
4.8	Concluding Remarks	102

5	The Marshall-Olkin-Type II-Topp-Leone-G Family of Distributions	104
5.1	Introduction	104
5.2	Marshall-Olkin-Type II-Topp-Leone-G Family of Distributions . . .	105
5.2.1	Expansion of Density Function	105
5.3	Some Special Cases	108
5.3.1	Marshall-Olkin-Type II-Topp-Leone-Weibull (MO-TII-TL-W) Distribution	108
5.3.2	Marshall-Olkin-Type II-Topp-Leone-Log-Logistic (MO-TII-TL-LLoG) Distribution	109
5.3.3	Marshall-Olkin-Type II-Topp-Leone-Uniform (MO-TII-TL-U) Distribution	110
5.4	Statistical Properties	111
5.4.1	Distribution of Order Statistics	111
5.4.2	Entropy	113
5.4.3	Moments and Generating Functions	115
5.4.4	Quantile Function	117
5.5	Maximum Likelihood Estimation	118
5.6	Simulation Study	119
5.7	Applications	121
5.7.1	Silicon Nitride Data	122
5.7.2	Kevlar 49/Epoxy Strands Failure at 90% Data	123
5.8	Concluding Remarks	125
6	Conclusions and Suggestions for Further Studies	126
6.1	Conclusions	126
6.2	Suggestions for Further Studies	126

List of Figures

2.1	Plots of TL-WP density and hazard functions	17
2.2	Plots of TL-WB density and hazard functions	19
2.3	Plots of TL-LLP density and hazard functions	21
2.4	Plots of TL-LLB density and hazard functions	23
2.5	Fitted densities and probability plots for the growth hormone data	35
2.6	Fitted densities and probability plots for the repair lifetimes data .	36
3.1	Plots of MO-EOEHL-LLoG density and hazard functions	46
3.2	Plots of MO-EOEHL-W density and hazard functions	47
3.3	Plots of MO-EOEHL-U density and hazard functions	49
3.4	Plots of skewness and kurtosis for the MO-EOEHL-LLoG distribution	58
3.5	Plots of skewness and kurtosis for the MO-EOEHL-LLoG distribtion	58
3.6	Fitted densities and probability plots for the kelvar data	67
3.7	Fitted densities and probability plots for the glass fibre data	68
4.1	Plots of EOEHL-LLoGP density and hazard functions	76
4.2	Plots of EOEHL-LLoGL density and hazard functions	77
4.3	Plots of EOEHL-LLoGG density and hazard functions	79
4.4	Plots of EOEHL-WP density and hazard functions	81
4.5	Plots of EOEHL-WL density and hazard functions	82
4.6	Plots of EOEHL-WG density and hazard functions	84
4.7	Plots of skewness and kurtosis for the EOEHL-LLoGP distribtion .	87
4.8	Plots of skewness and kurtosis for the EOEHL-LLoGP distribtion .	88
4.9	Fitted densities and probability plots for the cancer patients data .	101
4.10	Fitted densities and probability plots for the failure times data . .	102

5.1	Plots of MO-TII-TL-W density and hazard functions	108
5.2	Plots of MO-TII-TL-LLoG density and hazard functions	109
5.3	Plots of MO-TII-TL-U density and hazard functions	110
5.4	Plots of skewness and kurtosis for the MO-TII-TL-W distribtion . .	117
5.5	Plots of skewness and kurtosis for the MO-TII-TL-W distribtion . .	117
5.6	Fitted densities and probability plots for silicon nitride data	123
5.7	Fitted densities and probability plots for kelvar data	125

List of Tables

1	List of Publications, Accepted and Submitted Papers	xiv
1.1	Useful Quantities for Some Power Series Distributions	5
2.1	Sub-Classes of the TL-GPS Distribution	16
2.2	Table of Quantiles for TL-WP Distribution	18
2.3	Table of Quantiles for TL-WB Distribution	20
2.4	Table of Quantiles for the TL-LLP Distribution	22
2.5	Table of Quantiles for the TL-LLB Distribution	24
2.6	Monte Carlo Simulation Results for TL-LLP Distribution	32
2.7	Parameter estimates and goodness-of-fit statistics for various fitted models for growth hormone data set	34
2.8	Parameter estimates and goodness of fit statistics for various fitted models for repair lifetimes data set	36
3.1	Table of Moments for Selected Parameters of the MO-EOEHL-LLoG Distribution	57
3.2	Table of Quantiles for Selected Parameters of the MO-EOEHL-LLoG Distribution	60
3.3	Monte Carlo Simulation Results for MO-EOEHL-LLoG Distribution	63
3.4	Parameter estimates and goodness-of-fit statistics for various models fitted for kelvar data set	66
3.5	Parameter estimates and goodness-of-fit statistics for various models fitted for 1.5 cm glass fibres data set	68
4.1	Sub-Classes of the EOEHL-GPS Distribution	72

4.2	Table of Quantiles for Selected Parameters of the EOEHL-LLoGP Distribution	85
4.3	Table of Moments for Selected Parameters of the EOEHL-LLoGP Distribution	86
4.4	Monte Carlo Simulation Results for EOEHL-LLoGP Distribution .	97
4.5	Parameter estimates and goodness-of-fit statistics for various models fitted for cancer patients data	100
4.6	Parameter estimates and goodness-of-fit statistics for various models fitted for failure times data	101
5.1	Table of Moments for Selected Parameters of the MO-TII-TL-W Distribution	116
5.2	Table of Quantiles for Selected Parameters of the MO-TII-TL-W Distribution	118
5.3	Monte Carlo Simulation Results for MO-TII-TL-W Distribution . .	120
5.4	Parameter estimates and goodness-of-fit statistics for various models fitted for silicon nitride data set	122
5.5	Parameter estimates and goodness-of-fit statistics for various models fitted for kelvar data set	124

Abstract

New generalized distributions namely Topp-Leone-G Power Series class of distributions, Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G family of distributions, Exponentiated Odd Exponential Half Logistic-G Power Series class of distributions and Marshall-Olkin-Type II-Topp-Leone-G family of distributions are presented. Properties of these generalized distributions including series expansion of density functions, hazard functions, quantile functions, moments, conditional moments, mean deviations, Bonferroni and Lorenz curves, Rényi entropy and distribution of order statistics are discussed. The maximum likelihood estimates of the model parameters are given and simulation studies are performed to assess the consistency of the model parameters. Real data sets are used on the presented models to illustrate their usefulness in comparison to some non-nested distributions.

List of Publications, Accepted and Submitted Papers

Table 1: List of Publications, Accepted and Submitted Papers

Distribution	Status	Journal
The Topp-Leone-G Power Series Class of Distributions	Accepted	Pakistan Journal of Statistics and Operations Research
The Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G Family of Distributions	Submitted	Eurasian Bulletin of Mathematics
The Exponentiated Odd Exponential Half Logistic-G Power Series Class of Distributions	Accepted	Journal of Statistics and Management Systems
The Marshall-Olkin-Type II-Topp-Leone-G Family of Distributions	Submitted	Journal of Probability and Statistical Science

Chapter 1

Introduction

1.1 Introduction

Many new classes of distributions obtained by extending the classical distributions are available in the statistical literature. Adding parameters to well known distributions is a common method for gaining new flexible families of distributions. Generalized distributions are used for modeling data in many areas such as reliability, lifetime analysis, finance and insurance. Consequently, several techniques have been proposed to generate new families of distributions and these include the Marshall-Olkin-G by Marshall and Olkin [54], beta-G by Eugene et al. [35], Kumaraswamy-G by Cordeiro and de Castro [27], T-X family by Alzaatreh et al. [8], Weibull-G by Bourguignon et al. [18], Type 1 half-logistic-G by Cordeiro et al [29], the Topp-Leone-G by Al-Shomrani et al. [12] and the Generalized odd log-logistic-G by Cordeiro et al. [30], to name a few. The new generalized distributions often exhibit different shapes of the hazard rate function including; decreasing, increasing, bathtub and upside-down bathtub shapes and are also flexible to model various real data sets.

Topp and Leone [74] developed the Topp-Leone (TL) distribution defined

in the domain $(0,1)$. Numerous authors also studied the TL distribution. Nadarajah and Kotz [59] studied its properties and provided its moments and the characteristic function. Ghitany et al. [36] provided some reliability measures of the TL distribution, Vicaria et al. [75] introduced a two-sided generalized version of the TL distribution and Al-Zahrani [13] derived the goodness-of-fit test for the TL distribution. Many extensions of the Topp-Leone distribution has been proposed. Among them are the Topp-Leone generated (TL-G) family of distributions developed by Al-Shomrani et al. [12] and the Type II Topp-Leone generated (TII-TL-G) family of distributions by Elgarhy et al. [34]. Some notable generalizations of the TII-TL-G distribution include the Type II Topp-Leone inverse Rayleigh distribution by Mohammed and Yahia [55], Type II Topp-Leone generalized Rayleigh by Yahia and Mohammed [76], Type II Topp-Leone inverted Kumaraswamy by ZeinEldin et al. [77], Type II Topp-Leone inverse Exponential by Al-Marzouki [10], Type II Topp-Leone power Lomax by Al-Marzouki et al. [11] and Type II Topp-Leone Dagum by Sakthivel and Dhivakar [67].

Afify et al. [2] proposed the Exponentiated odd Exponential Half Logistic-G family of distributions by generalizing the Half logistic-G distribution. This generalization is more flexible and can exhibit different shapes for the hazard rate and density functions.

Power series distributions are extensively used in finance and actuarial science. The power series class of distributions was proposed and developed by Noack [61]. The truncated at zero power series distributions includes binomial, geometric, logarithmic and poisson distributions (Johnson et al. [42]). In recent years, several authors have used the power series class of distributions to develop new distributions. Some of the power series distributions include the exponential-power series distribution by Chahkandi and Ganjali [19], Weibull-power series distributions by Morais and Barreto-Souza [56], generalized exponential-power series distribution by Mahmoudi and Jafari [50], double bounded kumaraswamy-power series class of distributions by Bidram and Nekoukhou [17], Burr XII power series distributions by Silva and Cordeiro [72], generalized linear failure rate-power series distribution by

Alamatsaz and Shams [70], exponentiated generalized power series family of distributions by Oluyede et al. [65], odd Weibull-Topp-Leone-G power series family of distributions by Oluyede et al. [64] and the Lindley-Weibull power series distribution by Makubate et al. [52]. The power series distribution is compounded with the Topp-Leone-G and Exponentiated Odd Exponential Half-Logistic-G distributions to develop two new power series distributions namely; the Topp-Leone-G Power Series (TL-GPS) class of distributions and the Exponentiated Odd Exponential Half-Logistic-G Power Series (EOEHL-G)PS class of distributions, respectively.

Marshall and Olkin [54] developed a new method of adding an extra shape parameter to any given distribution. The resulting distribution is more flexible compared to the exponential, Weibull and gamma distributions. The Marshall-Olkin technique has been applied to well known distributions and its generalizations are available in the literature. General results of the Marshall-Olkin-G (MO-G) distribution are given by Barreto-Souza et al. [16]. Krishna et al. [44] introduced the Marshall-Olkin Fréchet distribution and its applications, Alizadeh et al. [5] studied the Kumaraswamy Marshall-Olkin-G distribution, Alizadeh et al. [4] developed the Beta Marshall-Olkin-G distribution, Lepetu et al. [49] presented the Marshall-Olkin Log-logistic Extended Weibull distribution, Kumar et al. [45] developed the Marshall-Olkin-Extended Burr Type III distribution, Lazhar [46] developed and studied the properties of the Marshall-Olkin extended generalized Gompertz distribution, Javed et al. [41] developed the Marshall-Olkin Kappa distribution. Oluyede et al. [63] tackled the Marshall-Olkin Log-logistic Erlang-Truncated Exponential distribution, and Chipepa and Oluyede [23] developed and studied the Marshall-Olkin-Gompertz-G distribution. The Marshall-Olkin transformation is utilised to develop two new families of distributions called the Marshall-Olkin-Exponentiated Odd Exponential Half-Logistic-G (MO-EOEHL-G) family of distributions and the Marshall-Olkin-Type II-Topp-Leone-G (MO-TII-TL-G) family of distributions.

1.2 Some Generators of Distributions

In this section, different generators used in this thesis to extend some lifetime distributions are presented.

1.2.1 T-X Family Method

Let $r(t)$ denote the probability density function (pdf) of a random variable $T \in [a, b]$, for $-\infty \leq a < b \leq \infty$ and $W[G(x)]$ be a function of the cumulative density function (cdf) $G(x)$, of a random variable X such that $W[G(x)]$ satisfies the following conditions:

- (i) $W[G(x)] \in [a, b]$,
- (ii) $W[G(x)]$ is differentiable and monotonically non-decreasing, and
- (iii) $W[G(x)] \rightarrow a$ as $x \rightarrow -\infty$ and $W[G(x)] \rightarrow b$ as $x \rightarrow \infty$.

Then, the cdf and pdf of the $T - X$ family of distributions developed by Alzaghal et al. [9] is defined by

$$F(x) = \int_a^{W[G(x)]} r(t)dt, \quad (1.1)$$

and

$$f(x) = r\{W[G(x)]\} \cdot \left\{ \frac{d}{dx} W[G(x)] \right\}, \quad (1.2)$$

respectively, where $W[G(x)]$ satisfies the conditions above.

1.2.2 Compounding Method

A discrete random variable N following a power series distribution assumed to be truncated at zero, has a probability mass function (pmf) given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \dots, \quad (1.3)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite and $a_n \geq 0$ depends only on n . Table 1.1 lists some particular cases of the truncated (at zero) power series distributions including Poisson, Geometric, Logarithmic and Binomial distributions (Johnson et al. [42]). $C'(\theta)$ and $C''(\theta)$ denotes the first and second derivatives of $C(\theta)$ with respect to θ .

Table 1.1: Useful Quantities for Some Power Series Distributions

Distribution	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C(\theta)^{-1}$	a_n	Parameter Space
Poisson	$e^\theta - 1$	e^θ	e^θ	$\log(1 + \theta)$	$(n!)^{-1}$	$(0, \infty)$
Geometric	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(1 + \theta)^{-1}$	1	$(0, 1)$
Logarithmic	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	n^{-1}	$(0, 1)$
Binomial	$(1 + \theta)^m - 1$	$m(1 + \theta)^{m-1}$	$m(m-1)(1 + \theta)^{m-2}$	$(\theta - 1)^{\frac{1}{m}} - 1$	$\binom{m}{n}$	$(0, 1)$

1.2.3 Exponentiation Method

Cordeiro et al. [32] defines the Exponentiated generalized (EG) class of distributions as

$$F(x) = [1 - (1 - G(x))^\alpha]^\beta, \quad (1.4)$$

where $\alpha, \beta > 0$. The pdf of the EG class of distributions has the form

$$f(x) = \alpha\beta[1 - G(x)]^{\alpha-1}[1 - (1 - G(x))^\alpha]^{\beta-1}g(x). \quad (1.5)$$

If we take $\alpha=1$ the the EG distribution reduces to a special case with cdf and pdf, respectively, given by

$$F(x) = [G(x)]^\beta$$

and

$$f(x) = \beta[G(x)]^{\beta-1}g(x).$$

1.2.4 Beta-G Generator

Let $G(x)$ and $g(x)$ be the cdf and pdf of a random variable X . Then the cdf of the beta-G distribution (Eugene et al. [35]) is defined as

$$F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)} t^{a-1}(1 - t)^{b-1} dt, \quad (1.6)$$

and its corresponding pdf is

$$f(x) = \frac{1}{B(a, b)} [G(x)]^{a-1} [1 - G(x)]^{b-1} g(x), \quad (1.7)$$

respectively, for $a > 0, b > 0$, and $B(., .)$ is the beta function.

1.2.5 Kumaraswamy-G Generator

Cordeiro and de Castro [27] defines the Kumaraswamy-G (Kw-G) family of distributions by

$$F(x) = 1 - (1 - G(x)^a)^b, \quad (1.8)$$

with pdf given by

$$f(x) = abg(x)(G(x))^{a-1}(1 - G(x)^a)^{b-1}, \quad (1.9)$$

where $a > 0$ and $b > 0$ are two additional parameters and $G(x)$ is the cdf of any parent distribution.

1.2.6 Marshall-Olkin-G Generator

The Marshall-Olkin (Marshall and Olkin [54]) class of distributions is defined by

$$F(x) = \frac{\delta \bar{G}(x)}{1 - \delta \bar{G}(x)}, \quad (1.10)$$

where $-\infty < x < \infty$, $\delta > 1$ and $\bar{\delta} = 1 - \delta$ is the tilt parameter and $\bar{G}(x) = 1 - G(x)$ is the baseline survival function. Note that δ is called the tilt parameter because the hazard rate functions are such that; $\frac{h_F(x)}{h_G(x)}$ increases when $\delta \geq 0$ and $\frac{h_F(x)}{h_G(x)}$ decreases when $0 < \delta \leq 1$. The corresponding pdf to equation (1.10) is given by

$$f(x) = \frac{\delta g(x)}{(1 - \delta \bar{G}(x))^2}. \quad (1.11)$$

1.3 Some Properties of Distributions

1.3.1 Survival, Hazard and Reverse Hazard Functions

The survival function, also referred to as survivor function or reliability function is defined as the probability of an object of interest surviving beyond a certain time. If X is a continuous random variable with cdf $F(x)$ and pdf $f(x)$, then its survival function is given by

$$S(x) = \bar{F}(x) = \int_t^\infty f(x)dx = 1 - F(x). \quad (1.12)$$

The hazard function is the ratio of the pdf to the survival function, that is

$$h(x) = \frac{f(x)}{S(x)}. \quad (1.13)$$

Hazard plots are most commonly used in reliability applications and they often exhibit different shapes including monotone and non-monotone shapes. The reverse hazard rate function is obtained by dividing the pdf by the cdf, that is

$$r(x) = \frac{f(x)}{F(x)}. \quad (1.14)$$

1.3.2 Quantile Function

Let X be a continuous random variable with cdf $F(x)$. Then, the quantile function $Q(u)$, of X is the generalized inverse of the distribution F and is given by

$$Q(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \quad 0 \leq u \leq 1. \quad (1.15)$$

The quantile functions are mostly used in Monte carlo simulations to generate random numbers and they provide a complete description of the statistical properties of a random variable.

1.3.3 Moments

Moments are very useful in statistics because they can be used to describe the statistical distributions. There are four commonly used moments: mean (the average), variance (indicates how spread the data is around the mean), skewness (describes the shape of the distribution) and kurtosis (indicates the degree of peakedness or flatness of a distribution). The r^{th} moment about the origin of a continuous random variable X is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx, \quad (1.16)$$

and the incomplete moment is defined as

$$I_X(t) = \int_0^t x^r f(x) dx. \quad (1.17)$$

1.3.4 Conditional Moments

The r^{th} conditional moment for a random variable X with cdf $F(x)$ given by

$$E(X^r | X > t) = \frac{1}{\bar{F}(x)} \int_t^{\infty} x^r f(x) dx. \quad (1.18)$$

1.3.5 Mean Deviations

A random variable X with pdf $f(x)$, has the mean deviation about the mean $D(\mu)$ and the mean deviation about the median $D(M)$ defined by

$$D(\mu) = \int_0^{\infty} |x - \mu| f(x) dx, \quad \text{and} \quad D(M) = \int_0^{\infty} |x - M| f(x) dx, \quad (1.19)$$

respectively, where $\mu = E(X)$ and $M = \text{median}(X)$. The deviations can also be expressed as

$$D(\mu) = 2\mu F(\mu) - 2\mu - 2 \int_{\mu}^{\infty} x f(x) dx, \quad (1.20)$$

and

$$D(M) = -\mu + 2 \int_M^{\infty} x f(x) dx. \quad (1.21)$$

1.3.6 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves have not only been applied in economics to study income and poverty, but they are also used in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx$$

and

$$L(p) = \frac{1}{\mu} \int_0^q x f(x) dx,$$

respectively.

1.3.7 Distribution of Order Statistics

Order statistics play a crucial role in probability and statistics. Suppose that $X_1, X_2, \dots, X_n$ are independent and identically distributed (i.i.d) random variables from a distribution with pdf $f(x)$ and $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \dots \leq X_{(n)}$ denotes the corresponding order statistics. Then, the pdf of the i^{th} order statistic, for $i = 1, \dots, n$ is defined as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}. \quad (1.22)$$

1.3.8 Entropy

Entropy is a measure of variation of uncertainty or variation for a random variable. The commonly two known measures of entropy are Shannon entropy [71] and Rényi entropy [66]. Shannon entropy is a special case of Rényi entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f(x)]^v dx \right), v \neq 1, v > 0. \quad (1.23)$$

Rényi entropy tends to Shannon entropy as $v \rightarrow 1$.

1.3.9 Estimation

Estimation methods are used to estimate parameters for statistical models and give out important model summary statistics. One of the most popular method is the maximum likelihood estimation (MLE). Let $X_1, X_2, \dots, X_n$ be a sequence of n random variables with a joint pdf $f(x_1, x_2, x_3, \dots, x_n | \theta)$ considered as a function of θ . Then the likelihood function is written as

$$L(\theta; \underline{x}) = \prod_{i=1}^n f(x_i | \theta). \quad (1.24)$$

The maximum likelihood estimate (mle) of the parameter θ is such that the likelihood function is maximized. The mle of θ is obtained by solving the equation $\left(\frac{\partial \log(L(\theta; \underline{x}))}{\partial \theta} \right) = 0$. The natural logarithm of the likelihood function is used because it is an increasing function and it ensures that the maximum value of the log-likelihood function occurs at the same point as the likelihood function.

1.3.10 Goodness-of-fit Statistics

The goodness-of-fit statistics can be used to determine which model fits a set of data and they include -2log-likelihood ($-2\log(L)$), Akaike Information Criterion ($AIC=2p - 2p \log(L)$), Consistent Akaike Information Criterion ($CAIC=AIC+\frac{2p(p+1)}{n-p-1}$), Bayesian Information Criterion ($BIC=p \log(n) - 2 \log(L)$), where p is the number of estimated parameters in the model, n is the sample size and $L = L(\hat{\theta})$ denotes the value of the likelihood function. The Cramer-von-Mises (W^*), Anderson-Darling (A^*) and Kolmogorov-Smirnov (K-S)

and its p-value. The model with the smallest values of these statistics and the highest p-value is regarded as the best model. Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [20]) are presented for each data set.

1.4 Aims and Objectives

The key aim of this study is to develop new generalized distributions using compounding and Marshall-Olkin-G as generators. The specific objectives are:

1. To propose and develop the following four new models:
 - The Topp-Leone-G Power Series class of distributions
 - The Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G family of Distributions
 - The Exponentiated Odd Exponential Half Logistic-G Power Series class of distributions
 - The Marshall-Olkin-Type II-Topp-Leone-G family of distributions
2. To obtain mathematical and statistical properties of the proposed distributions.
3. To estimate the parameters of the proposed models using the maximum likelihood estimation method.
4. To assess consistency of the maximum likelihood estimates through a simulation study.
5. To illustrate the usefulness of the proposed models using real data sets.

1.5 Outline of Thesis

The outline of this thesis is as follows: In Chapter 2, the generalized distribution, called the Topp-Leone-G Power Series class of distributions achieved through compounding of the Topp-Leone-G distribution and the

power series distribution is presented. This includes its properties, estimates of model parameters, simulation study and real data examples to illustrate the usefulness of the new distribution. Chapter 3 presents the Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G family of distributions obtained using the Marshall-Olkin technique. Its properties, estimation of parameters, simulation study and real data examples to demonstrate the usefulness of the proposed new model are also given in Chapter 3. The Exponentiated Odd Exponential Half Logistic-G Power Series family of distributions obtained by compounding the Exponentiated Odd Exponential Half Logistic and the power series distributions is introduced in Chapter 4. Chapter 4 also contains the properties, estimates of model parameters, simulation study and real data examples to demonstrate the usefulness of the proposed new model. The Marshall-Olkin-Type II-Topp-Leone-G family of distributions developed via the Marshall-Olkin method is discussed in Chapter 5. In addition, Chapter 5 also deals with the properties, estimates of model parameters, simulation study and real data examples to show the usefulness of the new model. The thesis ends with conclusions and scope for future work in Chapter 6.

Chapter 2

The Topp-Leone-G Power Series Class of Distributions

2.1 Introduction

In this chapter, a new class of distributions, referred to as the Topp-Leone-G Power Series (TL-GPS) is presented. The proposed class of distributions is obtained by compounding the Topp-Leone-G distribution and the power series distribution.

Al-Shomrani et al. [12] proposed the Topp-Leone generated (TL-G) family of distributions with cdf, pdf, and survival function given by

$$F_{TL-G}(x; b, \psi) = [1 - \bar{G}(x; \psi)^2]^b, \quad (2.1)$$

$$f_{TL-G}(x; b, \psi) = 2bg(x; \psi)\bar{G}(x; \psi)[1 - \bar{G}(x; \psi)^2]^{b-1}, \quad (2.2)$$

and

$$S_{TL-G}(x; b, \psi) = 1 - F_{TLG}(x; b, \psi) = 1 - [1 - \bar{G}(x; \psi)^2]^b, \quad (2.3)$$

respectively, for $b > 0$, where $G(x; \psi)$ is the baseline cdf depending on a parameter vector ψ , $g(x; \psi) = dG(x; \psi)/dx$, and $\bar{G}(x; \psi) = 1 - G(x; \psi)$ is the

survival function.

To develop the new class of distributions, we are motivated by the flexibility in data fitting obtained from the TL-GPS class of distributions and the applicability of the new class of distributions to data sets that exhibit monotonic or non-monotonic hazard rate shapes. Another motivation for developing the TL-GPS class of distributions is the applicability of the power series distributions in different fields such as finance, economics and actuarial sciences.

2.2 The Model

In this section, the TL-GPS class of distributions is developed and some statistical properties which include the series expansion of the pdf, quantile and hazard functions, sub-classes and some special cases are derived.

Suppose that the random variable X has the Topp-Leone-G distribution with cdf defined by equation (2.1). Let $X = \min(X_1, \dots, X_N)$, then the conditional cdf of $X|N = n$ is given by

$$F_{X|N=n}(x) = 1 - [S_{TL-G}(x; b, \psi)]^n, \quad x > 0. \quad (2.4)$$

The Topp-Leone-G Power Series class of distributions is defined by the marginal cdf of X . The general form of the cdf and pdf of the Topp-Leone-G Power Series class of distributions are given by

$$F_{TL-GPS}(x; \theta, b, \psi) = 1 - \frac{C(\theta S_{TL-G}(x; b, \psi))}{C(\theta)} \quad (2.5)$$

and

$$f_{TL-GPS}(x; \theta, b, \psi) = \frac{\theta f_{TL-G}(x; b, \psi) C'(\theta S_{TL-G}(x; b, \psi))}{C(\theta)}, \quad (2.6)$$

respectively.

On the other hand, if we consider $X_{(n)} = \max(X_1, \dots, X_N)$ and conditioning upon $N = n$, then the conditional distribution of $X_{(n)}$ given $N = n$ is obtained as

$$G_{X_{(n)}|N=n}(x) = [1 - \bar{G}(x; \psi)^2]^{nb},$$

which is also a Topp-Leone-G distribution. The marginal cdf of $X_{(n)}$, say F_{TL-GPS} , is given by

$$F_{TL-GPS}(x; \theta, b, \psi) = \frac{C(\theta F_{TL-G}(x; b, \psi))}{C(\theta)}.$$

The hazard rate function (hrf) is given by

$$h_{TL-GPS}(x; \theta, b, \psi) = \theta f_{TL-G}(x; b, \psi) \frac{C'(\theta S_{TL-G}(x; b, \psi))}{C(\theta S_{TL-G}(x; b, \psi))}. \quad (2.7)$$

Similarly, the reverse hazard rate function (rhrf) becomes

$$\tau_{TL-GPS}(x; \theta, b, \psi) = \frac{\theta f_{TL-G}(x; b, \psi) C'(\theta S_{TL-G}(x; b, \psi))}{C(\theta) - C(\theta S_{TL-G}(x; b, \psi))}. \quad (2.8)$$

2.2.1 Quantile function

In this section, the quantile function of the TL-GPS distribution is presented. The quantile function of the TL-GPS class of distributions is easily obtained by inverting equation (2.5), $F_{TL-GPS}(x; \theta, b, \psi) = u$, $0 \leq u \leq 1$. Note that

$$1 - \frac{C(\theta S_{TL-G}(x; b, \psi))}{C(\theta)} = u,$$

so that

$$C(\theta S_{TL-G}(x; b, \psi)) = C(\theta)(1 - u).$$

This is equivalent to

$$C^{-1}(C(\theta)(1 - u)) = \theta S_{TL-G}(x; b, \psi).$$

Therefore, we obtain the quantile values from the TL-GPS class of distributions by solving the non-linear equation

$$C^{-1}(C(\theta)(1 - u)) - \theta S_{TL-G}(x; b, \psi) = 0, \quad (2.9)$$

using iterative methods in R, SAS or MATLAB software.

2.2.2 Expansion of the Density Function

Expansion of the density function of the TL-GPS class of distributions is presented in this sub-section. equation (2.6) can be written as

$$f_{TL-GPS}(x; \theta, b, \psi) = \sum_{n=1}^{\infty} \frac{n a_n \theta^n}{C(\theta)} 2b g(x; \psi) \bar{G}(x; \psi) [1 - \bar{G}(x; \psi)^2]^{b-1} [1 - (1 - \bar{G}(x; \psi)^2)^b]^{n-1}.$$

Using the generalized binomial expansions

$$[1 - (1 - \bar{G}(x; \psi)^2)^b]^{n-1} = \sum_{i=0}^{\infty} (-1)^i \binom{n-1}{i} [1 - \bar{G}(x; \psi)^2]^{bi},$$

the pdf of the TL-GPS class of distribution can be written as

$$f_{TL-GPS}(x; \theta, b, \psi) = \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} (-1)^i \binom{n-1}{i} \frac{na_n \theta^n}{C(\theta)} 2b \\ \times g(x; \psi) \bar{G}(x; \psi) [1 - \bar{G}(x; \psi)^2]^{b(i+1)-1}.$$

Also, applying the generalized binomial expansion

$$[1 - \bar{G}(x; \psi)^2]^{b(i+1)-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b(i+1)-1}{j} \bar{G}(x; \psi)^{2j}$$

we get

$$f_{TL-GPS}(x; \theta, b, \psi) = \sum_{i,j=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{i+j} \binom{n-1}{i} \binom{b(i+1)-1}{j} \frac{na_n \theta^n}{C(\theta)} 2b \\ \times g(x; \psi) \bar{G}(x; \psi)^{2j+1}.$$

Furthermore, using the binomial expansion

$$\bar{G}(x; \psi)^{2j+1} = [1 - G(x; \psi)]^{2j+1} = \sum_{k=0}^{\infty} (-1)^k \binom{2j+1}{k} G(x; \psi)^k,$$

yields

$$f_{TL-GPS}(x; \theta, b, \psi) = \sum_{i,j,k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{i+j+k} \binom{n-1}{i} \binom{b(i+1)-1}{j} \binom{2j+1}{k} \frac{na_n \theta^n}{C(\theta)} 2b \\ \times g(x; \psi) G(x; \psi)^k \\ = \sum_{i,j,k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{i+j+k} \binom{n-1}{i} \binom{b(i+1)-1}{j} \binom{2j+1}{k} \frac{na_n \theta^n}{C(\theta)} 2b \\ \times \left(\frac{k+1}{k+1} \right) g(x; \psi) G(x; \psi)^k \\ = \sum_{k=0}^{\infty} \eta_k g_k(x; \psi),$$

where

$$g_k(x; \psi) = (k+1)g(x; \psi)G(x; \psi)^k$$

is the Exponentiated-G (Exp-G) distribution with power parameter $k+1$, and

$$\eta_k = \sum_{i,j=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{i+j+k} \binom{n-1}{i} \binom{b(i+1)-1}{j} \binom{2j+1}{k} \frac{na_n \theta^n}{C(\theta)} \frac{2b}{k+1}. \quad (2.10)$$

It follows that the TL-GPS distribution can be expressed as an infinite linear combination of Exp-G densities. Therefore, the mathematical and statistical properties of the TL-GPS distribution follows directly from those of the Exp-G distribution.

2.2.3 Sub-Classes of the TL-GPS Distribution

We derive expressions for cdfs of sub-classes of the TL-GPS class of distributions and these are presented in Table 2.1.

Table 2.1: Sub-Classes of the TL-GPS Distribution

Distribution	a_n	$C(\theta)$	cdf
Topp-Leone G Poisson	$(n!)^{-1}$	$e^\theta - 1$	$1 - \frac{e^{\theta(1-[1-\bar{G}(x;\psi)^2]^b)} - 1}{e^\theta - 1}$
Topp-Leone G Geometric	1	$\theta(1 - \theta)^{-1}$	$1 - \frac{(1-[1-\bar{G}(x;\psi)^2]^b)(1-\theta)}{1-\theta(1-[1-\bar{G}(x;\psi)^2]^b)}$
Topp-Leone G Logarithmic	n^{-1}	$-\log(1 - \theta)$	$1 - \frac{\log(1-\theta(1-[1-\bar{G}(x;\psi)^2]^b))}{\log(1-\theta)}$
Topp-Leone G Binomial	$\binom{m}{n}$	$(1 + \theta)^m - 1$	$1 - \frac{(1+\theta(1-[1-\bar{G}(x;\psi)^2]^b))^m - 1}{(1+\theta)^m - 1}$

2.2.4 Some Special Cases of the TL-GPS Class of Distributions

In this section, we present some special cases of the TL-GPS class of distributions. We consider cases when the baseline distribution are Weibull and log-logistic distributions.

Topp-Leone-Weibull-Poisson Distribution

The cdf and pdf of the Topp-Leone-Weibull Poisson (TL-WP) distribution are given by

$$F_{TL-WP}(x; \theta, b, \alpha, \beta) = 1 - \frac{e^{\theta(1-(1-e^{-2\alpha x^\beta})^b)} - 1}{e^\theta - 1}$$

and

$$f_{TL-WP}(x; \theta, b, \alpha, \beta) = \frac{2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^\beta}(1 - e^{-2\alpha x^\beta})^{b-1}e^{\theta(1-(1-e^{-2\alpha x^\beta})^b)}}{e^\theta - 1},$$

respectively, for $\alpha, \beta, b, \theta > 0$ and $x > 0$. The hrf and rhrf are given by

$$h_{TL-WP}(x; \theta, b, \alpha, \beta) = 2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^\beta}(1 - e^{-2\alpha x^\beta})^{b-1} \frac{e^{\theta(1-(1-e^{-2\alpha x^\beta})^b)}}{e^{\theta(1-(1-e^{-2\alpha x^\beta})^b)} - 1}$$

and

$$\tau_{TL-WP}(x; \theta, b, \alpha, \beta) = 2b\theta\alpha x^{\beta-1}e^{-2\alpha x^\beta}(1 - e^{-2\alpha x^\beta})^{b-1} \frac{e^{\theta(1-(1-e^{-2\alpha x^\beta})^b)}}{e^\theta - e^{\theta(1-(1-e^{-2\alpha x^\beta})^b)}},$$

respectively.

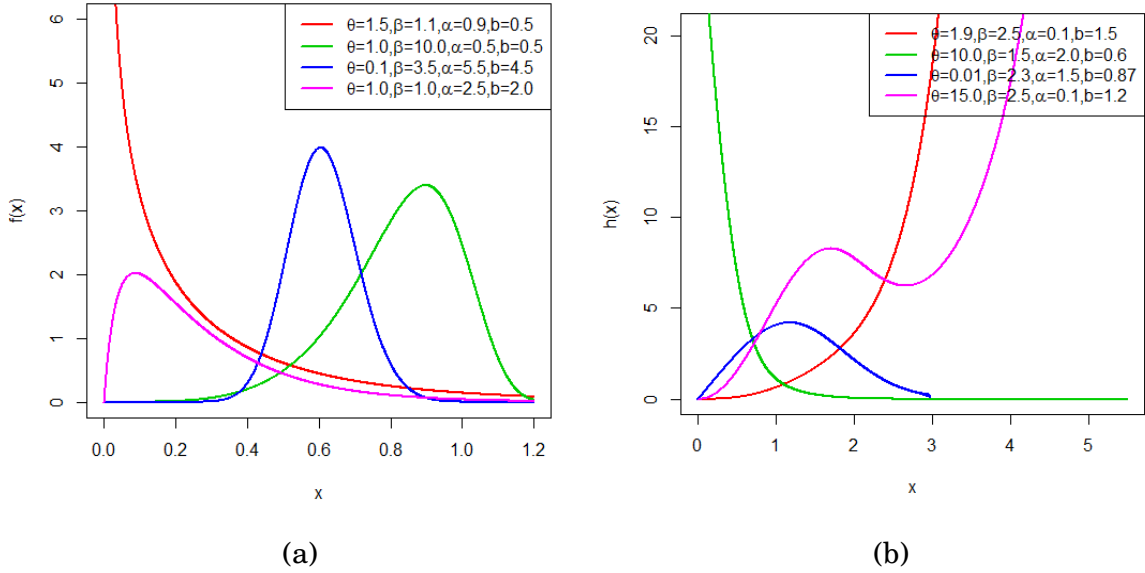


Figure 2.1: Plots of TL-WP density and hazard functions

Figures 2.1(a) and 2.1(b) show the plots of the pdfs and hrfs for the TL-WP distribution for selected parameter values. Plots of the TL-WP pdf exhibit different shapes including almost symmetric, left-skewed, right-skewed, and reverse-J shapes. Plots of the hrf of the TL-WP distribution shows different shapes including increasing, decreasing, upside-down bathtub followed by bathtub and uni-modal shapes.

The quantile function of the TL-WP distribution is obtained by solving the non-linear equation

$$\log[(e^\theta - 1)(1 - u) + 1] - \theta(1 - [1 - e^{-2\alpha x^\beta}]^b) = 0. \quad (2.11)$$

As such, random numbers can be generated from the TL-WP distribution by numerically solving the non-linear equation (2.11). Quantile values of the TL-WP distribution are given in Table 2.2.

Table 2.2: Table of Quantiles for TL-WP Distribution

	$(\theta, \alpha, \beta, b)$				
u	(0.8,2.1,0.9,1.0)	(1.5,1.2,1.8,2.1)	(3.5,1.2,5.5,0.6)	(0.5,1.0,0.3,0.1)	(2.0,3.0,0.4,1.5)
0.1	0.0112	0.3069	0.7432	0.2066	0.0011
0.2	0.0261	0.3877	0.7814	0.2806	0.0041
0.3	0.0446	0.4529	0.8082	0.3412	0.0100
0.4	0.0971	0.5751	0.8520	0.4552	0.0379
0.6	0.1357	0.6417	0.8732	0.5170	0.0688
0.7	0.1891	0.7193	0.8964	0.5885	0.1264
0.8	0.2704	0.8196	0.9245	0.6804	0.2494
0.9	0.4226	0.9764	0.9666	0.8245	0.6031

Topp-Leone-Weibull-Binomial Distribution

The cdf and pdf of the Topp-Leone-Weibull Binomial (TL-WB) distribution are given by

$$F_{TL-WB}(x; \theta, b, \alpha, \beta, m) = 1 - \frac{(1 + \theta(1 - [1 - e^{-2\alpha x^\beta}]^b))^m - 1}{(1 + \theta)^m - 1}$$

and

$$f_{TL-WB}(x; \theta, b, \alpha, \beta, m) = 2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^\beta}(1 - e^{-2\alpha x^\beta})^{b-1} \times \frac{m(1 + \theta(1 - [1 - e^{-2\alpha x^\beta}]^b))^{m-1}}{(1 + \theta)^m - 1},$$

respectively for $\alpha, \beta, b, \theta > 0$ and $x > 0$. The hrf and rhrf are given by

$$h_{TL-WB}(x; \theta, b, \alpha, \beta, m) = 2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^\beta}(1 - e^{-2\alpha x^\beta})^{b-1} \times \frac{m(1 + \theta(1 - [1 - e^{-2\alpha x^\beta}]^b))^{m-1}}{(1 + \theta(1 - [1 - e^{-2\alpha x^\beta}]^b))^m - 1}$$

and

$$\tau_{TL-WB}(x; \theta, b, \alpha, \beta, m) = 2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^\beta}(1 - e^{-2\alpha x^\beta})^{b-1} \times \frac{m(1 + \theta(1 - [1 - e^{-2\alpha x^\beta}]^b))^{m-1}}{(1 + \theta)^m - (1 + \theta(1 - [1 - e^{-2\alpha x^\beta}]^b))^m},$$

respectively.

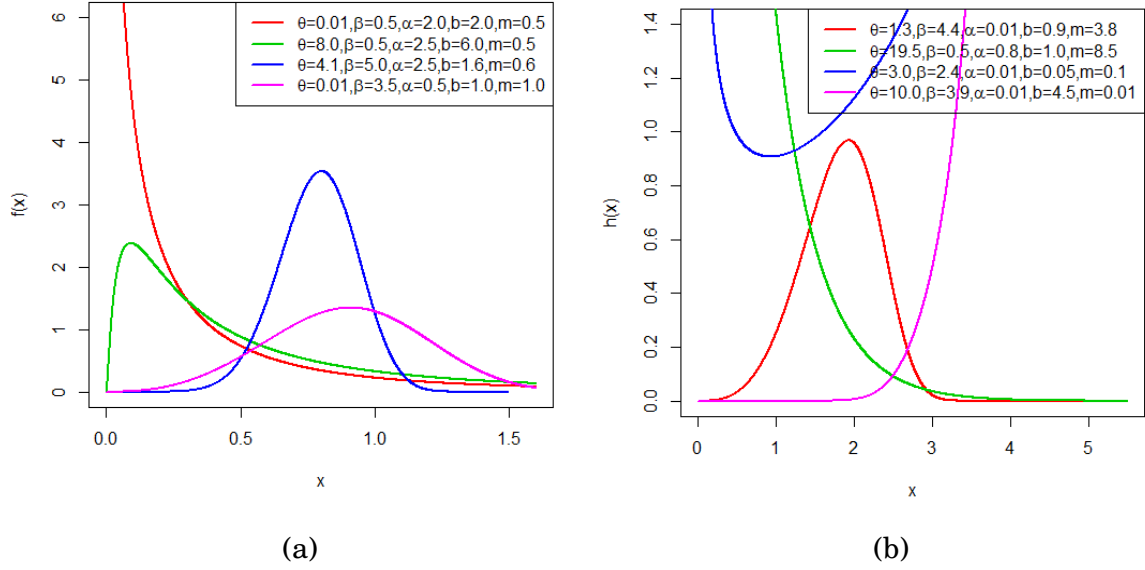


Figure 2.2: Plots of TL-WB density and hazard functions

Figures 2.2(a) and 2.2(b) show the plots of the pdfs and hrfs for the TL-WB distribution for selected parameters values. Plots of the TL-WB pdf exhibit different shapes including symmetric, skewed to the right, skewed to the left and reverse-J shapes. Plots of the hrf of the TL-WB distribution shows different shapes including increasing, decreasing, bathtub and uni-modal shapes.

The quantile function of the TL-WB distribution can be obtained by solving the non-linear equation

$$[((1 + \theta)^m - 1)(1 - u) + 1]^{\frac{1}{m}} - 1 - \theta(1 - [1 - e^{-2\alpha x^\beta}]^b) = 0. \quad (2.12)$$

As such, random numbers can be generated from the TL-WB power series distribution by numerically solving the non-linear equation (2.12). Quantile values of the TL-WB distribution are given in Table 2.3.

Table 2.3: Table of Quantiles for TL-WB Distribution

u	$(\theta, \alpha, \beta, b, m)$				
	(0.8,2.1,0.9,1.0,1.5)	(1.2,1.8,2.1,3.5,1.2)	(5.5,0.6,0.5,1.0,0.3)	(0.1,1.2,1.8,0.4,0.7)	(1.5,0.9,0.5,0.3,1.1)
0.1	0.0167	0.5307	0.0219	0.0069	0.0000
0.2	0.0382	0.5999	0.0819	0.0230	0.0001
0.3	0.0639	0.6504	0.1755	0.0500	0.0001
0.4	0.0947	0.6938	0.3022	0.0925	0.0004
0.5	0.1323	0.7346	0.4653	0.1586	0.0019
0.6	0.1795	0.7756	0.6730	0.2635	0.0082
0.7	0.2420	0.8197	0.9435	0.4393	0.0329
0.8	0.3323	0.8714	1.3200	0.7695	0.1350
0.9	0.4908	0.9435	1.9431	1.5783	0.7004

Topp-Leone-Log-Logistic-Poisson Distribution

The cdf and pdf of the Topp-Leone-Log-Logistic-Poisson (TL-LLP) distribution are given by

$$F_{TL-LLP}(x; \theta, b, c) = 1 - \frac{e^{\theta(1-[1-(1+x^c)^{-2}]^b)} - 1}{e^\theta - 1}$$

and

$$f_{TL-LLP}(x; \theta, b, c) = \frac{2\theta b c x^{c-1} (1+x^c)^{-3} (1 - (1+x^c)^{-2})^{b-1} e^{\theta(1-[1-(1+x^c)^{-2}]^b)}}{e^\theta - 1},$$

respectively for $\theta, b, c > 0$ and $x > 0$. The hrf and rhrf are given by

$$h_{TL-LLP}(x; \theta, b, c) = 2\theta b c x^{c-1} (1+x^c)^{-3} (1 - (1+x^c)^{-2})^{b-1} \frac{e^{\theta(1-[1-(1+x^c)^{-2}]^b)}}{e^{\theta(1-[1-(1+x^c)^{-2}]^b)} - 1}$$

and

$$\tau_{TL-LLP}(x; \theta, b, c) = \frac{2\theta b c x^{c-1} (1+x^c)^{-3} (1 - (1+x^c)^{-2})^{b-1} e^{\theta(1-[1-(1+x^c)^{-2}]^b)}}{e^\theta - e^{\theta(1-[1-(1+x^c)^{-2}]^b)}},$$

respectively.

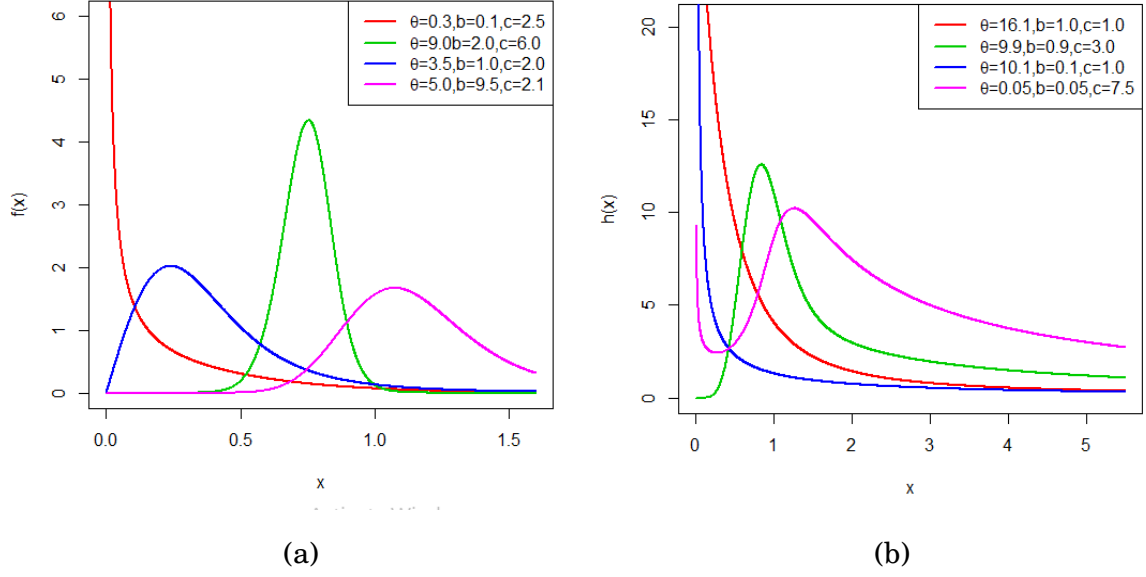


Figure 2.3: Plots of TL-LLP density and hazard functions

Figures 2.3(a) and 2.3(b) show the plots of the pdfs and hrfs for the TL-LLP distribution for selected parameters values. Plots of the TL-LLP pdf exhibit different shapes including reverse-J, almost symmetric, skewed to the left and skewed to the right shapes. Plots of the hrf of the TL-LLP distribution shows different shapes including reverse-J, decreasing, bathtub followed by upside-down bathtub and upside-down bathtub shapes.

The quantile function of the TL-LLP distribution obtained by solving the non-linear equation

$$\ln[(e^\theta - 1)(1 - u) + 1] - \theta(1 - [1 - (1 + x^c)^{-2}]^b) = 0. \quad (2.13)$$

Therefore, random numbers can be generated from the TL-LLP distribution by numerically solving the non-linear equation (2.13). Quantile values of the TL-LLP distribution are given in Table 2.4.

Table 2.4: Table of Quantiles for the TL-LLP Distribution

	(θ, b, c)				
u	(0.8,2.1,0.9)	(1.0,1.5,1.2)	(1.8,2.1,3.5)	(1.2,5.5,0.6)	(0.5,1.0,0.3)
0.1	0.1500	0.1375	0.5753	0.4053	0.0000
0.2	0.2535	0.2229	0.6551	0.6659	0.0004
0.3	0.3646	0.3085	0.7161	0.9585	0.0021
0.4	0.4950	0.4029	0.7715	1.3202	0.0078
0.5	0.6583	0.5143	0.8268	1.8012	0.0259
0.6	0.8776	0.6552	0.8867	2.4944	0.0820
0.7	1.2010	0.8496	0.9572	3.6101	0.2722
0.8	1.7550	1.1574	1.0511	5.7606	1.0841
0.9	3.0603	1.8061	1.2090	11.8717	7.3663

Topp-Leone-Log-Logistic Binomial Distribution

The cdf and the pdf of the Topp-Leone-Log-Logistic Binomial (TL-LLB) distribution are given by

$$F_{TL-LLB}(x; \theta, b, c, m) = 1 - \frac{(1 + \theta(1 - [1 - (1 + x^c)^{-2}]^b))^m - 1}{(1 + \theta)^m - 1}$$

and

$$f_{TL-LLB}(x; \theta, b, c, m) = 2\theta b c x^{c-1} (1 + x^c)^{-3} (1 - (1 + x^c)^{-2})^{b-1} \times \frac{m(1 + \theta(1 - [1 - (1 + x^c)^{-2}]^b))^{m-1}}{(1 + \theta)^m - 1},$$

respectively for $\theta, b, c, m > 0$ and $x > 0$. The hrf and rhrf are given by

$$h_{TL-LLB}(x; \theta, b, c, m) = 2\theta b c x^{c-1} (1 + x^c)^{-3} (1 - (1 + x^c)^{-2})^{b-1} \times \frac{m(1 + \theta(1 - [1 - (1 + x^c)^{-2}]^b))^{m-1}}{(1 + \theta(1 - [1 - (1 + x^c)^{-2}]^b))^m - 1}$$

and

$$\tau_{TL-LLB}(x; \theta, b, c, m) = 2\theta b c x^{c-1} (1 + x^c)^{-3} (1 - (1 + x^c)^{-2})^{b-1} \times \frac{m(1 + \theta(1 - [1 - (1 + x^c)^{-2}]^b))^{m-1}}{(1 + \theta)^m - (1 + \theta(1 - [1 - (1 + x^c)^{-2}]^b))^m},$$

respectively.

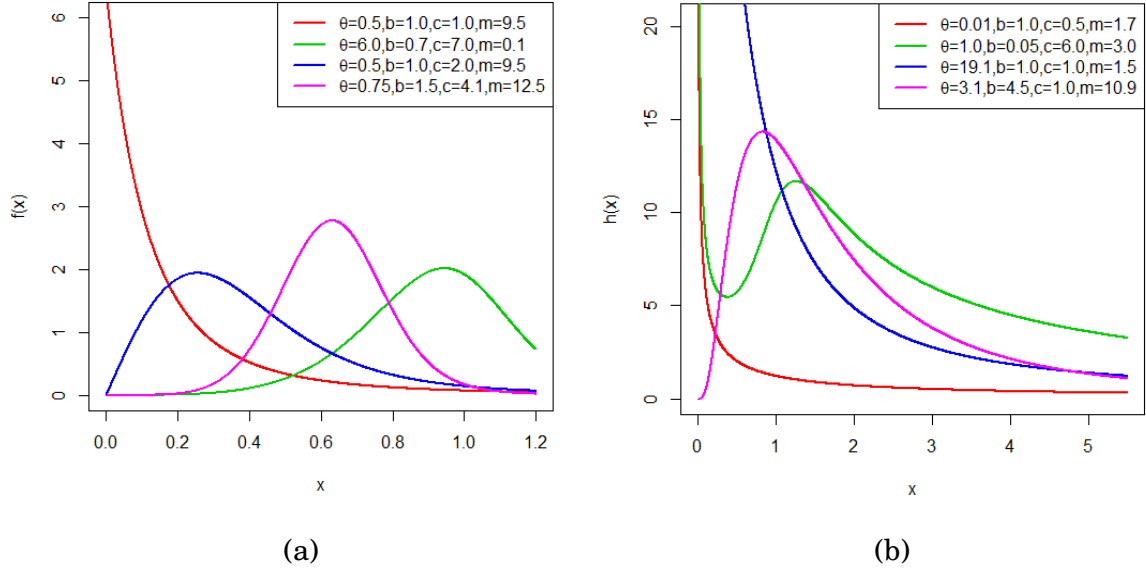


Figure 2.4: Plots of TL-LLB density and hazard functions

Figures 2.4(a) and 2.4(b) show the plots of the pdfs and hrfs for the TL-WP distribution for selected parameters values. Plots of the TL-LLB pdf exhibit different shapes skewed to the right, skewed to the left, reverse-J and almost symmetric shapes. Plots of the hrf of the TL-LLB distribution shows different shapes including reverse-J, decreasing, bathtub followed by an upside-down bathtub and uni-modal shapes.

The quantile function obtained by solving the non-linear equation

$$[\{((1 + \theta)^m - 1)(1 - u) + 1\}^{\frac{1}{m}} - 1 + \theta(1 - [1 - (1 + x^c)^{-2}]^b) = 0. \quad (2.14)$$

Therefore, random numbers can be generated from the TL-LLB distribution by numerically solving the non-linear equation (2.14). Quantile values of the TL-LLB distribution are given in Table 2.5.

Table 2.5: Table of Quantiles for the TL-LLB Distribution

	(θ, b, c, m)				
u	(0.8,2.1,0.9,1.0)	(1.5,1.2,1.8,2.1)	(3.5,1.2,5.5,0.6)	(0.5,1.0,0.3,2.5)	(3.0,4.5,1.8,2.9)
0.1	3.4643	1.6754	19.8517	1.3128	1.9504
0.2	2.1130	1.3088	9.3969	1.0455	1.6754
0.3	1.5031	1.0995	5.7043	0.8869	1.5145
0.4	1.1310	0.9473	3.8018	0.7685	1.3955
0.5	0.8695	0.8223	2.6371	0.6695	1.2965
0.6	0.6689	0.7108	1.8467	0.5799	1.2070
0.7	0.5041	0.6039	1.2700	0.4933	1.1196
0.8	0.3599	0.4926	0.8222	0.4025	1.0262
0.9	0.2213	0.3598	0.4466	0.2940	0.9084

2.3 Moments, Conditional Moments and Mean Deviations

In this subsection, the r^{th} moment, conditional moments, mean deviations, Lorenz and Bonferroni curves of the TL-GPS class of distributions are presented.

2.3.1 Moments and Generating Function

If X follows the TL-GPS distribution and $Y \sim \text{Exp-G}(k)$, then using equation (2.10), the r^{th} moment of the TL-GPS class of distributions is obtained as follows

$$\mu'_r = E(X^r) = \int_0^\infty x^r \cdot f_{TL-GPS}(x; \theta, b, \psi) dx = \sum_{k=0}^\infty \eta_k E(Y^r),$$

where $E[Y^r]$ is the r^{th} moment of the Exp-G distribution with power parameter $k + 1$ and η_k is given by equation (2.10). The moment generating function (mgf) of the TL-GPS class of distributions is given by

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r) = \sum_{k=0}^\infty \eta_k M_Y(t),$$

where $M_Y(t)$ is the mgf of the Exp-G distribution and η_k is given by equation (2.10).

2.3.2 Conditional Moments

The conditional r^{th} moment of the TL-GPS distribution is given by

$$\begin{aligned} E(X^r | X > t) &= \frac{1}{\bar{F}_{TL-GPS}(t; \theta, b, \psi)} \int_t^\infty x^r \cdot f_{TL-GPS}(x; \theta, b, \psi) dx \\ &= \sum_{k=0}^{\infty} \eta_k E(Y^r I_{Y^r > t}), \end{aligned}$$

where

$$E(Y^r I_{Y^r > t}) = \int_t^\infty y^r \cdot g_k(y; \psi) dy.$$

2.3.3 Mean Deviations, Lorenz and Bonferroni Curves

The mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the TL-GPS class of distributions are presented in this subsection.

Mean Deviations

If X has the TL-GPS distribution, then we can derive the mean deviation about the mean $D(\mu)$ and the median deviation about the median $D(M)$ as follows

$$D(\mu) = \int_0^\infty |x - \mu| f_{TL-GPS}(x; \theta, b, \psi) dx = 2\mu f_{TL-GPS}(x; \theta, b, \psi) - 2\mu + 2T(\mu)$$

and

$$D(M) = \int_0^\infty |x - M| f_{TL-GPS}(x; \theta, b, \psi) dx = -\mu + 2T(M),$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(X)$ is the median of $F_{TL-GPS}(x; \theta, b, \psi)$. Note that

$$T(\mu) = \int_\mu^\infty x \cdot f_{TL-GPS}(x; \theta, b, \psi) dx = \sum_{k=0}^{\infty} \eta_k \int_\mu^\infty y \cdot g_k(y; \psi) dy$$

and

$$T(M) = \int_M^\infty x \cdot f_{TL-GPS}(x; \theta, b, \psi) dx = \sum_{k=0}^{\infty} \eta_k \int_M^\infty y \cdot g_k(y; \psi) dy.$$

Bonferroni and Lorenz Curves

In this subsection, we present Bonferroni and Lorenz curves for TL-GPS class of distributions. The Bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu} \int_0^q x \cdot f_{TL-GPS}(x; \theta, b, \psi) dx = \frac{1}{p\mu} \sum_{k=0}^{\infty} \eta_k \int_0^q x \cdot g_k(x; \psi) dx$$

and

$$L(p) = \frac{1}{\mu} \int_0^q x \cdot f_{TL-GPS}(x; \theta, b, \psi) dx = \frac{1}{\mu} \sum_{k=0}^{\infty} \eta_k \int_0^q x \cdot g_k(x; \psi) dx,$$

respectively, where $\int_0^q x \cdot g_k(x; \psi)$ is the first incomplete moment of the Exp-G distribution with power parameter $k + 1$ and η_k is given by equation (2.10).

2.3.4 Order Statistics and Rényi Entropy

In this section, we present the distribution of the order statistic and Rényi entropy of the TL-GPS class of distributions.

Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from the TL-GPS distribution and let $X_{i:n}$ be the corresponding i^{th} order statistics. The pdf of the i^{th} order statistic, $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} [F(x)]^{k+i-1}, \quad (2.15)$$

where $B(.,.)$ is the beta function. Substituting the pdf and cdf of the TL-GPS family of distributions, we write

$$\begin{aligned} f(x)[F(x)]^{k+i-1} &= \sum_{n=1}^{\infty} \frac{na_n\theta^n}{C(\theta)} [1 - (1 - \bar{G}(x; \psi)^2)^b]^{n-1} 2bg(x; \psi) \bar{G}(x; \psi) \\ &\quad \times [1 - \bar{G}(x; \psi)^2]^{b-1} \left[1 - \frac{C(\theta(1 - (1 - \bar{G}(x; \psi)^2)^b))}{C(\theta)} \right]^{k+i-1}. \end{aligned}$$

Using the generalized binomial expansion

$$\left[1 - \frac{C(\theta(1 - (1 - \bar{G}(x; \psi)^2)^b))}{C(\theta)} \right]^{k+i-1} = \sum_{j=0}^{\infty} (-1)^j \binom{k+i-1}{j} \left[\frac{C(\theta(1 - (1 - \bar{G}(x; \psi)^2)^b))}{C(\theta)} \right]^j,$$

and applying the result on power series raised to a positive integer, we get

$$f(x)[F(x)]^{k+i-1} = \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \binom{k+i-1}{j} (-1)^j \frac{na_n \theta^{n+m}}{C^{j+1}(\theta)} b_{m,j} 2bg(x; \psi) \bar{G}(x; \psi) \\ \times [1 - (1 - \bar{G}(x; \psi)^2)^b]^{m+n-1} [1 - \bar{G}(x; \psi)^2]^{b-1},$$

where $b_{m,j} = (ma_0)^{-1} \sum_{l=1}^m (l(j+1) - m) a_l b_{m-l}$ and $b_{0,j} = a_0^j$ (Gradshteyn and Ryzhik [37]). Also, using the following generalized binomial expansion

$$[1 - (1 - \bar{G}(x; \psi)^2)^b]^{m+n-1} = \sum_{p=0}^{\infty} (-1)^p \binom{m+n-1}{p} [1 - \bar{G}(x; \psi)^2]^{bp},$$

we obtain

$$f(x)[F(x)]^{k+i-1} = \sum_{j,p=0}^{\infty} \sum_{n=1}^{\infty} \binom{k+i-1}{j} \binom{m+n-1}{p} (-1)^{j+p} \frac{na_n \theta^{n+m}}{C^{j+1}(\theta)} b_{m,j} \\ \times [1 - \bar{G}(x; \psi)^2]^{b(p+1)-1} 2bg(x; \psi) \bar{G}(x; \psi).$$

Furthermore, applying the generalized binomial expansion

$$[1 - \bar{G}(x; \psi)^2]^{b(p+1)-1} = \sum_{q=0}^{\infty} (-1)^q \binom{b(p+1)-1}{q} \bar{G}(x; \psi)^{2q},$$

yields

$$f(x)[F(x)]^{k+i-1} = \sum_{j,p,q=0}^{\infty} \sum_{n=1}^{\infty} \binom{k+i-1}{j} \binom{m+n-1}{p} \binom{b(p+1)-1}{q} \\ \times (-1)^{j+p+q} \frac{2bna_n \theta^{n+m}}{C^{j+1}(\theta)} b_{m,j} g(x; \psi) \bar{G}(x; \psi)^{2q+1}.$$

Also, applying the binomial expansion

$$\bar{G}(x; \psi)^{2q+1} = \sum_{r=0}^{\infty} (-1)^r \binom{2q+1}{r} G(x; \psi)^r$$

yields

$$f(x)[F(x)]^{k+i-1} = \sum_{j,p,q,r=0}^{\infty} \sum_{n=1}^{\infty} \binom{k+i-1}{j} \binom{m+n-1}{p} \binom{b(p+1)-1}{q} \binom{2q+1}{r} \\ \times (-1)^{j+p+q+r} \frac{2bna_n \theta^{n+m}}{C^{j+1}(\theta)} b_{m,j} \left(\frac{r+1}{r+1} \right) g(x; \psi) G(x; \psi)^r \\ = \sum_{r=0}^{\infty} a_r g_r(x; \psi),$$

(2.16)

where $g_r(x; \psi) = (r + 1)g(x; \psi)G(x; \psi)^r$ is the Exp-G distribution with power parameter $r + 1$, and

$$a_r = \sum_{j,p,q=0}^{\infty} \sum_{n=1}^{\infty} \binom{k+i-1}{j} \binom{m+n-1}{p} \binom{b(p+1)-1}{q} \binom{2q+1}{r} \\ \times (-1)^{j+p+q+r} \frac{2bn a_n \theta^{n+m}}{C^{j+1}(\theta)(r+1)} b_{m,j}.$$

Therefore, substituting equation (2.16) in (2.15) we obtain

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{r=0}^{\infty} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} a_r g_r(x; \psi). \quad (2.17)$$

It follows that the distribution of the i^{th} order statistic from the TL-GPS class of distributions can be obtained directly from the distribution of the i^{th} order statistic from the Exp-G distribution with power parameter $r + 1$.

Rényi Entropy

In this subsection, Rényi entropy of the TL-GPS class of distributions is derived. Entropy measures the uncertainty or variation of a random variable. Rényi entropy [66] is a generalization of Shannon entropy [71]. Rényi entropy of the TL-GPS class of distributions is defined as

$$I_R(\nu) = \frac{1}{1-\nu} \log \left(\int_0^{\infty} (f_{TL-GPS}(x; \theta, b, \psi))^{\nu} dx \right), \quad \nu \neq 1, \quad \nu > 0. \quad (2.18)$$

Note that $f_{TL-GPS}^{\nu}(x; \theta, b, \psi)$ can be written as

$$f_{TL-GPS}^{\nu}(x; \theta, b, \psi) = \left[\theta 2b g(x; \psi) \bar{G}(x; \psi) [1 - \bar{G}(x; \psi)^2]^{(b-1)} \right. \\ \left. \times \frac{C'(\theta(1 - [1 - \bar{G}(x; \psi)^2]^b))}{C(\theta)} \right]^{\nu} \\ = \left[\frac{\theta 2b}{C(\theta)} \right]^{\nu} g(x; \psi)^{\nu} \bar{G}(x; \psi)^{\nu} [1 - \bar{G}(x; \psi)^2]^{\nu(b-1)} \\ \times [C'(\theta(1 - [1 - \bar{G}(x; \psi)^2]^b))]^{\nu}.$$

Considering the following series expansions

$$[C'(\theta(1 - [1 - \bar{G}(x; \psi)^2]^b))]^{\nu} = \sum_{k=0}^{\infty} d_{k,\nu} \theta^k (1 - [1 - \bar{G}(x; \psi)^2]^b)^k,$$

where $d_{k,\nu} = (kb_0)^{-1} \sum_{l=1}^k [\nu(l+1) - k] b_l d_{k-l,\nu}$ and $d_{0,\nu} = b_0^\nu$,

$$(1 - [1 - \bar{G}(x; \psi)^2]^b)^k = \sum_{m=0}^{\infty} (-1)^m \binom{k}{m} [1 - \bar{G}(x; \psi)^2]^{bm},$$

$$[1 - \bar{G}(x; \psi)^2]^{b(m+\nu)-\nu} = \sum_{n=0}^{\infty} (-1)^n \binom{b(m+\nu)-\nu}{n} \bar{G}(x; \psi)^{2n},$$

and

$$\bar{G}(x; \psi)^{2n+\nu} = \sum_{q=0}^{\infty} (-1)^q \binom{2n+\nu}{q} G(x; \psi)^q,$$

we get

$$\begin{aligned} f_{TL-GPS}^\nu(x; \theta, b, \psi) &= \sum_{k,m,n,q=0}^{\infty} \left[\frac{2b}{C(\theta)} \right]^\nu (-1)^{m+n+q} \binom{k}{m} \binom{b(m+\nu)-\nu}{n} \\ &\times \binom{2n+\nu}{q} d_{k,\nu} \theta^{\nu+k} g^\nu(x; \psi) G^q(x; \psi). \end{aligned}$$

Therefore, the Rényi entropy of the TL-GPS class of distributions is given by

$$I_R(\nu) = \frac{1}{1-\nu} \log \left(\sum_{q=0}^{\infty} \eta_q^* e^{(1-\nu)I_{REG}} \right), \quad (2.19)$$

where

$$\eta_q^* = \sum_{k,m,n=0}^{\infty} \left[\frac{2b}{C(\theta)} \right]^\nu (-1)^{m+n+q} \binom{k}{m} \binom{b(m+\nu)-\nu}{n} \binom{2n+\nu}{q} d_{k,\nu} \frac{\theta^{\nu+k}}{[\frac{q}{\nu} + 1]^\nu}, \quad (2.20)$$

and

$$I_{REG} = \frac{1}{1-\nu} \int_0^\infty \left(\left[\frac{q}{\nu} + 1 \right] g(x; \psi) G(x; \psi)^{\frac{q}{\nu}} \right)^\nu dx$$

is the Rényi entropy of the Exp-G distribution with parameter $\frac{q}{\nu}$. As such, we can directly derive the Rényi entropy of the TL-GPS family of distributions from the Rényi entropy of the Exp-G distribution with parameter $\frac{q}{\nu}$.

2.4 Estimation

In this section, we derive the maximum likelihood estimates of the parameter vector $(\theta, b, \psi)^T$ of the TL-GPS class of distributions. Let $X_i \sim TL - GPS(\theta, b, \psi)$ and $\Delta = (\theta, b, \psi)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned} \ell(\Delta) &= n \ln(2b\theta) + (b-1) \sum_{i=1}^n \ln[1 - \bar{G}(x_i; \psi)^2] + \sum_{i=1}^n \ln[\bar{G}(x_i; \psi)] \\ &+ \sum_{i=1}^n \ln[g(x_i; \psi)] - n \ln(C(\theta)) + \sum_{i=1}^n \ln \left(C' \left(\theta (1 - [1 - \bar{G}(x_i; \psi)^2]^b) \right) \right). \end{aligned}$$

The elements of the score vector are given by

$$\begin{aligned}\frac{\partial \ell}{\partial \theta} &= \frac{n}{\theta} - \frac{nC'(\theta)}{C(\theta)} + \sum_{i=1}^n \frac{\left(C'' \left(\theta(1 - [1 - \bar{G}(x_i; \psi)^2]^b) \right) \right) \left(1 - \left(1 - \bar{G}(x_i; \psi)^2 \right)^b \right)}{C' \left(\theta(1 - [1 - \bar{G}(x_i; \psi)^2]^b) \right)}, \\ \frac{\partial \ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \ln[1 - \bar{G}(x_i; \psi)^2] + \sum_{i=1}^n \frac{\theta \left(C'' \left(\theta(1 - [1 - \bar{G}(x_i; \psi)^2]^b) \right) \right)}{C' \left(\theta(1 - [1 - \bar{G}(x_i; \psi)^2]^b) \right)} \\ &\quad \times [1 - \bar{G}(x_i; \psi)^2]^b \ln[1 - \bar{G}(x_i; \psi)^2],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \psi_k} &= (b-1) \sum_{i=1}^n \frac{1}{(1 - \bar{G}(x_i; \psi)^2)} \frac{\partial [1 - \bar{G}(x_i; \psi)^2]}{\partial \psi_k} + \sum_{i=1}^n \frac{\frac{\partial \bar{G}(x_i; \psi)}{\partial \psi_k}}{\bar{G}(x_i; \psi)} + \sum_{i=1}^n \frac{\frac{\partial g(x_i; \psi)}{\partial \psi_k}}{g(x_i; \psi)} \\ &\quad + \sum_{i=1}^n \frac{\left(C'' \left(\theta(1 - (1 - \bar{G}(x_i; \psi)^2)^b) \right) \right)}{C' \left(\theta(1 - (1 - \bar{G}(x_i; \psi)^2)^b) \right)} - 2b\theta [1 - \bar{G}(x_i; \psi)^2]^{b-1} \bar{G}(x_i; \psi) \frac{\partial \bar{G}(x_i; \psi)}{\partial \psi_k}.\end{aligned}$$

The equations obtained by setting the partial derivatives equal to zero are not in closed form. The maximum likelihood estimates of the parameters denoted by $\hat{\Delta}$ are obtained by solving the non-linear equation $\left(\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \psi_k} \right)^T = 0$ using numerical methods such as the Newton-Raphson procedure. The multivariate normal distribution $N(\underline{0}, J^{-1}(\hat{\Delta}))$, where the mean vector $\underline{0} = (0, 0, 0)^T$ and $J^{-1}(\hat{\Delta})$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$ can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

2.5 Simulation Study

In this section, a simulation study was conducted to assess consistency of the maximum likelihood estimators. We considered a special case of the TL-LLP distribution. We simulated for the sample sizes $n = 25, 50, 100, 200, 400, 800$, and 1000, for $N=1000$ for each sample, where n represents the sample sizes for the random variables obtained from the TL-LLP distribution and N represents the number of simulations performed. We estimate the mean, root mean

square error (RMSE), and average bias. The bias and RMSE for the estimated parameter, say, $\hat{\Delta}$, are given by

$$Bias(\hat{\Delta}) = \frac{1}{N} \sum_{i=1}^N (\hat{\Delta}_i - \Delta), \quad \text{and} \quad RMSE(\hat{\Delta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\Delta}_i - \Delta)^2}{N}},$$

respectively. We consider simulations for the following sets of initial parameters values (I: $\theta = 0.5, b = 1.5, c = 1.0$), (II: $\theta = 1.5, b = 1.5, c = 0.5$), (III: $\theta = 0.5, b = 1.0, c = 1.5$), and (IV: $\theta = 1.0, b = 1.5, c = 0.5$). If the model performs better, we expect the mean to approximate the true parameter values, the RMSE, and bias to decay toward zero for an increase in sample size. From the results in Table 2.6, the mean values approximate the true parameter values, RMSE and bias decay towards zero for all the parameter values.

2.6 Applications

In this section, we present examples to illustrate the usefulness and applicability of the TL-GPS class of distributions. This is achieved by applying the special case of Topp-Leone-Log-Logistic Poisson to two real data sets and comparing it to several equal-parameter non-nested models. Model parameters were estimated via the maximum likelihood estimation technique using the R software. The performance of the models were assessed using the following several goodness-of-fit statistics; -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*) and Andersen-Darling (A^*) (as described by Chen and Balakrishnan [21]), Kolmogorov-Smirnov (K-S) statistic and its p-value. Also the sum of squares are computed. The model that has smaller values of these above mentioned goodness-of-fit statistics and larger p-value of the K-S statistics is deemed as the best model.

Tables 2.7 and 2.8 show the model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the two data sets considered. Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [20]) are also presented to show how well our model fits the observed data set compared to the selected non-nested models. For

Table 2.6: Monte Carlo Simulation Results for TL-LLP Distribution

	I: $\theta = 0.5, b = 1.5, c = 1.0$			II: $\theta = 1.5, b = 1.5, c = 0.5$			
sample size	Mean	RMSE	Bias	Mean	RMSE	Bias	
θ	25	1.5971	1.6131	1.0971	2.2845	1.9846	0.7845
	50	1.4729	1.5153	0.9729	2.2459	1.6083	0.7459
	100	1.2283	1.3343	0.7283	2.0671	1.3136	0.5671
	200	0.8945	0.8672	0.3945	1.9149	1.1782	0.4149
	400	0.6985	0.4912	0.1985	1.8147	0.9015	0.3147
	800	0.5767	0.3278	0.0767	1.7387	0.6583	0.2387
	1000	0.5582	0.2816	0.0582	1.7169	0.3868	0.2169
b	25	2.1429	1.0851	0.6429	1.8348	0.9946	0.3348
	50	2.0972	1.0600	0.5972	1.9127	0.9588	0.4127
	100	1.9571	0.9382	0.4571	1.8495	0.8420	0.3495
	200	1.7545	0.5979	0.2545	1.7862	0.7787	0.2862
	400	1.6238	0.3148	0.1238	1.7139	0.5848	0.2139
	800	1.5459	0.1938	0.0459	1.6514	0.4170	0.1514
	1000	1.5336	0.1627	0.0336	1.6269	0.2284	0.1269
c	25	0.9534	0.2776	-0.0466	0.5475	0.2675	0.0475
	50	0.9260	0.2255	-0.0740	0.4895	0.1456	-0.0105
	100	0.9364	0.1784	-0.0636	0.4813	0.1069	-0.0187
	200	0.9588	0.1296	-0.0412	0.4815	0.0950	-0.0185
	400	0.9774	0.0818	-0.0226	0.4815	0.0714	-0.0185
	800	0.9915	0.0570	-0.0085	0.4841	0.0506	-0.0159
	1000	0.9946	0.0505	-0.0054	0.4856	0.0370	-0.0144
	III: $\theta = 0.5, b = 1.0, c = 1.5$			IV: $\theta = 1.0, b = 1.5, c = 0.5$			
θ	25	1.7985	2.7021	1.2985	1.8872	1.7102	0.8872
	50	1.3213	1.3605	0.8213	1.8396	1.5185	0.8396
	100	1.0529	1.0967	0.5529	1.5930	1.2897	0.5930
	200	0.7617	0.6139	0.2617	1.3665	0.9905	0.3665
	400	0.6205	0.3775	0.1205	1.2081	0.6073	0.2081
	800	0.5378	0.2792	0.0378	1.1668	0.4165	0.1668
	1000	0.5307	0.2437	0.0307	1.1387	0.3297	0.1387
b	25	1.5638	1.1103	0.5638	1.9438	0.9746	0.4438
	50	1.3771	0.7673	0.3771	1.9898	1.0018	0.4898
	100	1.2557	0.6358	0.2557	1.8732	0.8563	0.3732
	200	1.1193	0.3231	0.1193	1.7573	0.6714	0.2573
	400	1.0549	0.1742	0.0549	1.6424	0.3878	0.1424
	800	1.0168	0.1206	0.0168	1.6062	0.2516	0.1062
	1000	1.0126	0.1029	0.0126	1.5836	0.1891	0.0836
c	25	1.4752	0.6269	-0.0248	0.5130	0.1866	0.0130
	50	1.4338	0.3859	-0.0662	0.4776	0.1297	-0.0224
	100	1.4436	0.2663	-0.0564	0.4776	0.0965	-0.0224
	200	1.4684	0.1842	-0.0316	0.4806	0.0768	-0.0194
	400	1.4827	0.1224	-0.0173	0.4885	0.0535	-0.0115
	800	1.4971	0.0884	-0.0029	0.4887	0.0372	-0.0113
	1000	1.4989	0.0789	-0.0011	0.4930	0.0296	-0.0070

the probability plot, we plotted $F_{TL-GPS}(x(j); \hat{\theta}, \hat{b}, \hat{\psi})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x(j)$ are the ordered values of the observed data. The measure of closeness of the plots to the diagonal line was determined and is given by the sum of squares

$$SS = \sum_{j=1}^n \left[F_{TL-GPS}(x(j); \hat{\theta}, \hat{b}, \hat{\psi}) \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The non-nested models considered are the Weibull-Poisson by [51], Topp-Leone generalized exponential (TL-GE) by Sangsanit and Bodhisuwan [68], alpha power Weibull (APW) by Nassar et al. [60], Marshall-Olkin Extended Weibull (MOEW) by Cordeiro and Lemonte [28] and Marshall-Olkin log-logistic (MOLL) by Gui [38], Topp-Leone Weibull-Lomax (WLx) by Jamal et al. [40], Transmuted Weibull (TW) by Ahmad et al. [3] distributions. The pdfs of the non-nested models are as follows:

$$f_{MOLL}(x; \alpha, \beta, \gamma) = \alpha^\beta \beta \gamma \frac{x^{\beta-1}}{(x^\beta + \gamma \alpha^\beta)^2},$$

for $\alpha, \beta, \gamma > 0$, and $x > 0$,

$$f_{WP}(x; \theta, \beta, \gamma) = \frac{\theta \gamma \beta^\gamma x^{\gamma-1} \exp(-(\beta x)^\gamma) \exp(\theta(1 - \exp(-(\beta x)^\gamma)))}{\exp(\theta) - 1},$$

for $\theta, \beta, \gamma > 0$, and $x > 0$,

$$f_{TL-GE}(x; \alpha, \beta, \lambda) = 2\alpha\beta\lambda e^{-\lambda x} (1 - (1 - e^{-\lambda x})^\beta (1 - e^{-\lambda x})^{\beta\alpha-1} (2 - (1 - e^{-\lambda x})^\beta))^{\alpha-1},$$

for $\alpha, \beta, \lambda > 0$, and $x > 0$,

$$f_{APW}(x; \alpha, \beta, \theta) = \frac{\log(\alpha)}{(\alpha - 1)} \beta \theta x^{\beta-1} e^{-\theta x^\beta} \alpha^{1-e^{-\theta x^\beta}},$$

for $\alpha, \beta, \theta > 0$ and $x > 0$,

$$f_{MOEW}(x; \alpha, \lambda, \gamma) = \frac{\alpha \gamma \lambda x^{\gamma-1} e^{-\lambda x^\gamma}}{(1 - (1 - \alpha)e^{-\lambda x^\gamma})^2},$$

for $\alpha, \lambda, \gamma > 0$, and $x > 0$,

$$f_{WLx}(x; a, b, \alpha) = 2\alpha ab(1 + bx)^{a\alpha-1} (1 - (1 + bx)^{-a})^{\alpha-1} e^{-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)^\alpha},$$

and

$$f_{TW}(x; \lambda, \beta, \alpha) = \alpha \beta x^{\beta-1} e^{-\lambda x^\beta} (1 - \lambda + 2\lambda e^{-\lambda x^\beta}),$$

for $\lambda, \beta, \alpha > 0$ and $x > 0$.

2.6.1 Growth Hormone Data

This data consists of the estimated time since growth hormone medication until the children reached the targeted age. This data was used by Alizadeh et al.[6] and are as follows: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics are presented in Table 2.7.

Table 2.7: Parameter estimates and goodness-of-fit statistics for various fitted models for growth hormone data set

Model	Estimates			Statistics							
	θ	b	c	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P - value$
TL-LLP	6.5265×10^{-5} (0.1033)	5.2422×10 (17.693)	1.3853 (0.1549)	155.4571	161.4571	162.2313	166.1231	0.0382	0.2660	0.0849	0.9624
WP	θ	β	γ								
	8.8353×10^{-9} (0.0117)	0.159 (0.0149)	1.9932 (0.2438)	165.0	170.9772	171.7514	175.6432	0.1639	1.0262	0.1454	0.4501
APW	α	β	λ								
	6.4700×10^4 (1.4803×10^{-7})	0.8444 (0.1031)	0.7494 (0.1365)	158.4884	164.4884	165.2626	169.1544	0.0710	0.4808	0.0924	0.9262
MOEW	α	γ	λ								
	0.0280 (0.0430)	3.4077 (0.0026)	0.0002 (0.0002)	158.1646	164.1646	164.9388	168.8307	0.0616	0.4330	0.0972	0.8953
WLx	a	b	α								
	0.2548 (0.1825)	2.4039 (5.0117)	4.2177 (2.4857)	162.1874	168.1874	168.9616	172.8535	0.1265	0.8062	0.1329	0.5663
TL-GE	α	β	λ								
	2.3206×10^{-3} (7.5165×10^{-4})	2.7741×10^3 (3.1617×10^{-7})	0.4786 (7.2079×10^{-2})	158.1108	164.1108	164.885	168.7769	0.0724	0.4876	0.1025	0.8561
TW	λ	β	α								
	0.6092 (0.3488)	2.1722 (0.2648)	0.0140 (0.0084)	163.4343	169.4343	170.2085	174.1004	0.1437	0.9030	0.1320	0.5758
MOLL	α	β	γ								
	3.3212 (0.2844)	3.5219 (0.4909)	3.0700 (0.0874)	158.5828	164.5828	165.357	169.2488	0.0560	0.4061	0.0978	0.8911

The estimated variance-covariance matrix for the TL-LLP model on growth hormone data is

$$\begin{bmatrix} 0.0107 & 0.0001 & -0.0013 \\ 0.0001 & 322.661 & 2.4121 \\ -0.0013 & 2.4121 & 0.0240 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\theta \in [(6.5265 \times 10^{-5}) \pm 0.2025]$, $b \in [5.2422 \times 10 \pm 35.2075]$ and $c \in [1.3853 \pm 0.3036]$.

Based on the results shown in Table 2.7, we observe that the TL-LLP model

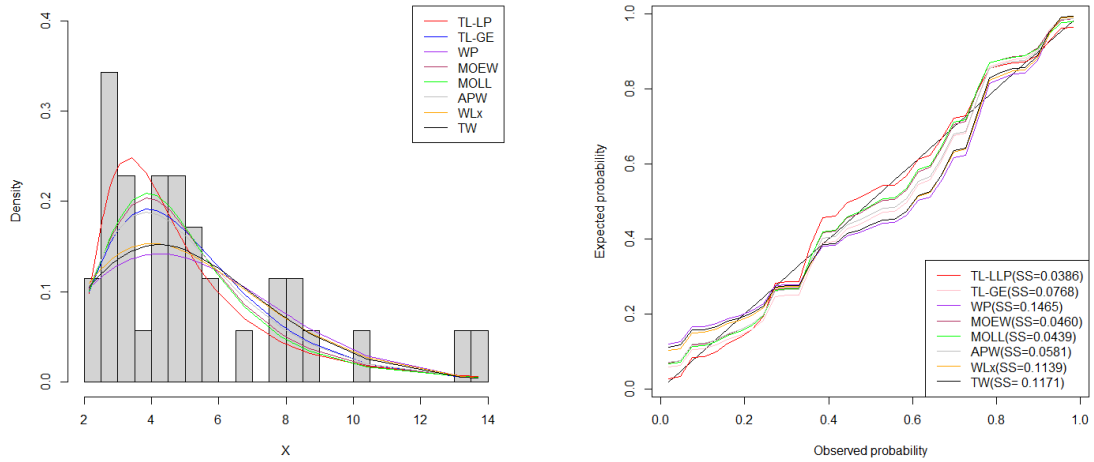


Figure 2.5: Fitted densities and probability plots for the growth hormone data

has the smallest values of all the goodness-of-fit statistics and bigger value for the K-S p-value. We therefore conclude that the TL-LLP distribution performs better than the several models considered in this paper. The fitted densities and probability plots in Figure 2.5 also shows that the TL-LLP model fit the growth hormone data set better than the selected non-nested models.

2.6.2 Repair Lifetimes Data

The second data set represents maintenance on active repair times (in hours) for an airborne communication transceiver reported by Leiva et al. [48] and Chhikara and Folks [22] and are as follows: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics are presented in Table 2.8. The estimated variance-covariance matrix for the TL-LLP model on repair times data set is

$$\begin{bmatrix} 0.0414 & 0.0459 & -0.0030 \\ 0.0459 & 0.4753 & -0.0073 \\ -0.0030 & -0.0073 & 0.0082 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\theta \in [0.0001 \pm 0.3988]$, $b \in [4.4081 \pm 1.3512]$ and $c \in [0.8102 \pm 0.1773]$.

Table 2.8: Parameter estimates and goodness of fit statistics for various fitted models for repair lifetimes data set

Model	Estimates			Statistics							
	θ	b	c	$-2\log L$	AIC	AICC	BIC	W^*	A^*	KS	P - value
TL-LLP	0.0001 (0.2035)	4.4081 (0.6894)	0.8102 (0.0905)	200.1734	206.1734	206.7448	211.6593	0.0547	0.3276	0.0882	0.8667
WP	θ 3.9697×10^{-8} (0.0219)	β 0.2953 (0.0516)	γ 0.8978 (0.0957)	208.9395	214.9395	215.5110	220.4255	0.1297	0.9003	0.1209	0.5125
WLx	a 0.2222 (0.1093)	b 7.0310 (11.2084)	α 2.1228 (0.9099)	202.2696	208.2696	208.841	213.7555	0.0697	0.4745	0.1059	0.6802
APW	α 0.0295 (0.0566)	β 1.1011 (0.1201)	λ 0.0924 (0.0521)	204.9274	210.9274	211.4988	216.4133	0.0986	0.6520	0.1111	0.6209
MOEW	α 0.0332 (0.0540)	γ 1.4861 (0.2084)	λ 0.0126 (0.0208)	201.7121	207.7121	208.2836	213.1981	0.0700	0.4330	0.0922	0.8289
TL-GE	2.1603×10^{-4} (4.2209×10^{-5})	4.4317×10^3 (2.8118×10^{-8})	0.2689 (5.2662×10^{-2})	209.9544	215.9544	216.5258	221.4403	0.1455	1.0116	0.1520	0.2381
TW	λ 0.6735 (0.3038)	β 0.9792 (0.1047)	α 0.1986 (0.0602)	206.9540	212.954	213.5254	218.4399	0.1150	0.7817	0.1161	0.5650
MOLL	α 0.9457 (114.6974)	β 1.5439 (0.1858)	γ 2.8641 (536.30783)	202.3421	208.3421	208.9135	213.828	0.0750	0.4519	0.0939	0.8122

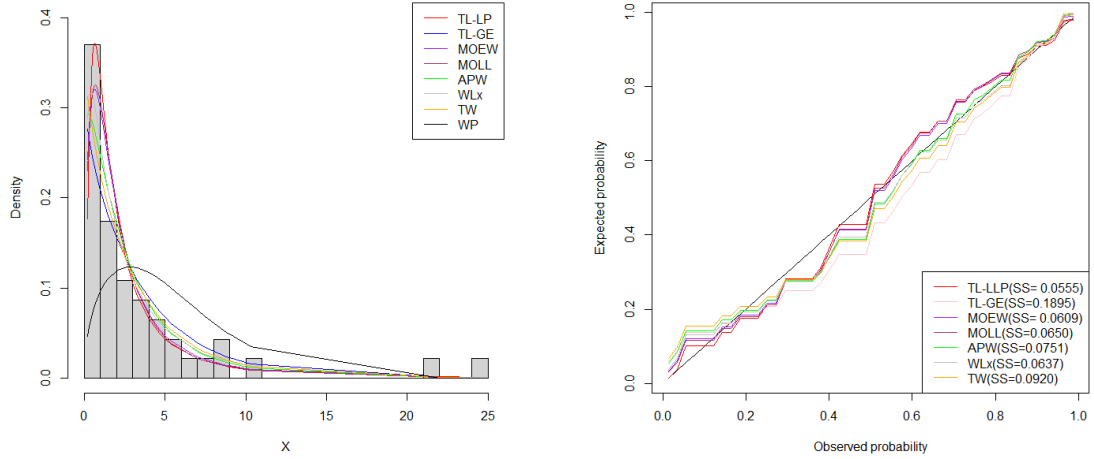


Figure 2.6: Fitted densities and probability plots for the repair lifetimes data

From the results shown in Table 2.8, the TL-LLP model has the smallest values of all the goodness-of-fit statistics and bigger value for the K-S p-value. As such, we conclude that the TL-LLP distribution performs better than the several models considered in this paper. The fitted densities and probability plots in Figure 2.6 also affirms that the TL-LLP model fits the repair times data set

better than the selected models.

2.7 Concluding Remarks

We developed a new class of distributions, called the Topp-Leone-G Power Series (TL-GPS) class of distributions. We presented some sub-classes and some special cases of the new proposed distribution. Structural properties were also derived including moments, mean deviations, distribution of order statistics, Rényi entropy, and maximum likelihood estimates. We also presented two real data examples to show the usefulness of the new class of distributions. The proposed model performs better than the several models on the selected data sets.

Chapter 3

The

Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G Family of Distributions

3.1 Introduction

Numerous generalized families of distributions have been developed and are used for modelling data in areas such as engineering, economics, biological sciences, finance and insurance. A new generalized distribution referred to the Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G (MO-EOEHL-G) family of distributions is discussed in this chapter. The Marshall-Olkin transformation and the Exponentiated Odd Exponential Half Logistic-G distribution are used to obtain the new family of distributions. The Marshall and Olkin (MO-G) class of distributions ([54]) has cdf and pdf given by

$$F_{MO-G}(x; \delta, \xi) = 1 - \frac{\delta \bar{G}(x; \xi)}{1 - \bar{\delta} \bar{G}(x; \xi)}, \quad (3.1)$$

and

$$f_{MO-G}(x; \delta, \xi) = \frac{\delta g(x; \xi)}{[1 - \delta \bar{G}(x; \xi)]^2}, \quad (3.2)$$

respectively, where δ is the tilt parameter, $\bar{G}(x; \xi) = 1 - G(x; \xi)$ and $g(x; \xi)$ are the survival function and pdf of the baseline distribution.

Afify et al. [2] generalized the Half logistic-G (HL-G) family of distributions to obtain a new family of distributions called the Exponentiated Odd Exponential Half Logistic-G (EOEHL-G) family of distributions with cdf and pdf given as

$$F_{EOEHL-G}(x; \alpha, \lambda, \xi) = \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha, \quad (3.3)$$

and

$$f_{EOEHL-G}(x; \alpha, \lambda, \xi) = \frac{2\alpha \lambda g(x; \xi) \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha-1}}{\bar{G}^2(x; \xi) \left[1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha+1}}, \quad (3.4)$$

respectively, where $\alpha, \lambda > 0$ and $G(x; \xi)$ is the baseline distribution.

Motivations for developing the new family of distributions are the desirable properties exhibited by the MO-G and EOEHL-G distributions. Another motivation for developing the MO-EOEHL-G family of distributions is the flexibility in data fitting obtained from the new model and its applicability to data sets with monotonic or non-monotonic hazard rate functions. Furthermore, the proposed distribution can fit data that is heavy tailed.

3.2 Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G Family of Distributions

In this section, we present the Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G Family of Distributions, its density expansion and stochastic orders. Using the generalizations given in equations (3.1) and (3.3) we develop the MO-EOEHL-G family of distributions. As such, the cdf, pdf and hrf of the

MO-EOEHL-G family of distributions are given by

$$F_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) = 1 - \frac{\delta \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right]^\alpha \right)}{1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right]^\alpha \right)},$$

$$f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) = \frac{2\alpha\delta\lambda g(x; \xi) \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \left[1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]^{\alpha-1}}{\bar{G}^2(x; \xi) \left[1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]^{\alpha+1}} \times \left(1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right]^\alpha \right) \right)^{-2} \quad (3.5)$$

and

$$h_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) = \frac{2\alpha\delta^2\lambda g(x; \xi) \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \left[1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]^{\alpha-1}}{\bar{G}^2(x; \xi) \left[1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]^{\alpha+1}} \times \frac{1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right]^\alpha}{\left(1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right]^\alpha \right) \right)},$$

respectively, for $\alpha, \lambda, \delta > 0$, $\bar{\delta} = 1 - \delta$ and ξ is a vector of parameters from the baseline distribution function $G(\cdot)$.

3.2.1 Expansion of Density Function

We derive the series expansion of the MO-EOEHL-G family of distributions using the general results for the Marshall and Olkin's family of distributions by Barreto-Souza et al. [16]. Consider the pdf of the MO-EOEHL-G distribution

$$f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) = \frac{\delta f_{EOEHL-G}(x; \alpha, \lambda, \xi)}{(1 - \bar{\delta} \bar{F}_{EOEHL-G}(x; \alpha, \lambda, \xi))^2}, \quad (3.6)$$

which can be written as

$$f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) = \frac{f_{EOEHL-G}(x; \alpha, \lambda, \xi)}{\delta \left(1 - \frac{\delta-1}{\delta} F(x; \alpha, \lambda, \xi) \right)^2}, \quad (3.7)$$

where $f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi)$ and $F_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi)$ are the pdf and cdf of the EOEHL-G family of distributions, respectively. We also apply the series expansion

$$(1 - z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^j, \quad (3.8)$$

for $|z| < 1$, $k > 0$. If $\delta \in (0, 1)$ we obtain

$$f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) = f_{EOEHL-G}(x; \alpha, \lambda, \xi) \sum_{j=0}^{\infty} \sum_{k=0}^j w_{j,k} F_{EOEHL-G}^j(x; \alpha, \lambda, \xi)^{j-k}$$

where $w_{j,k} = w_{j,k}(\delta) = \delta(j+1)(1-\delta)^j(-1)^{j-k} \binom{j}{k}$. For $\delta > 1$

$$f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) = f_{EOEHL-G}(x; \alpha, \lambda, \xi) \sum_{j=0}^{\infty} v_j F_{EOEHL-G}^j(x; \alpha, \lambda, \xi)$$

where $v_j = v_j(\delta) = \frac{(j+1)(1-\frac{1}{\delta})}{\delta}$.

For $\delta \in (0, 1)$,

$$\begin{aligned} f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) &= \frac{2\alpha\lambda g(x; \xi) \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right)}{\bar{G}^2(x; \xi) \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right)\right]^{\alpha+1}} \\ &\times \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right)\right]^{\alpha-1} \\ &\times \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} w_{j,k} \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right)} \right]^{\alpha(j-k)}. \end{aligned}$$

Applying the generalized binomial series expansions,

$$\begin{aligned} \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right)\right]^{-(\alpha(j-k+1)-1)} &= \sum_{m=0}^{\infty} \binom{-(\alpha(j-k+1)-1)}{m} \\ &\times \exp\left(-\lambda m \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right), \end{aligned}$$

$$\begin{aligned} \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right)\right]^{\alpha(j-k+1)+1} &= \sum_{n=0}^{\infty} (-1)^n \binom{\alpha(j-k+1)+1}{n} \\ &\times \exp\left(-\lambda n \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right), \end{aligned}$$

$$\exp \left(-\lambda(m+n+1) \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) = \sum_{p=0}^{\infty} (-1)^p \frac{\lambda^p (m+n+1)^p}{p!} \left[\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right]^p$$

and

$$[\bar{G}(x; \xi)]^{-(p+2)} = [1 - G(x; \xi)]^{-(p+2)} = \sum_{q=0}^{\infty} \binom{-(p+2)}{q} G^q(x; \xi),$$

we get

$$\begin{aligned} f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) &= \sum_{j,m,n,p,q=0}^{\infty} \sum_k^j \frac{2\alpha(-1)^{n+p}\lambda^{p+1}[m+n+1]^p w_{j,k}}{p!(p+q+1)} \\ &\times \binom{-(\alpha(j-k+1)-1)}{m} \binom{\alpha(j-k+1)+1}{n} \\ &\times \binom{-(p+2)}{q} (p+q+1) g(x; \xi) [G(x; \xi)]^{p+q} \\ &= \sum_{p,q=0}^{\infty} w_{p,q}^* g_{p+q}(x; \xi). \end{aligned}$$

It then follows that for $\delta \in (0, 1)$, the MO-EOEHL-G family of distributions can be expressed as a linear combination of the Exponentiated-G (Exp-G) distribution with power parameter $(p+q+1)$ and linear component

$$\begin{aligned} w_{p,q}^* &= \sum_{j,m,n}^{\infty} \sum_k^j \frac{2\alpha(-1)^{n+p}[m+n+1]^p w_{j,k}}{p!(p+q+1)} \\ &\times \binom{-(\alpha(j-k+1)-1)}{m} \binom{\alpha(j-k+1)+1}{n} \binom{-(p+2)}{q}. \end{aligned} \quad (3.9)$$

Furthermore, for $\delta > 1$, we get

$$\begin{aligned} f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) &= \frac{2\alpha\lambda g(x; \xi) \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{\bar{G}^2(x; \xi) \left[1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]^{\alpha+1}} \\ &\times \left[1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]^{\alpha-1} \\ &\times \sum_{j=0}^{\infty} v_j \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right]^{\alpha j}. \end{aligned}$$

Applying the generalized binomial series expansions,

$$\begin{aligned}
\left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right)\right]^{-(\alpha(j+1)-1)} &= \sum_{m=0}^{\infty} \binom{-(\alpha(j+1)-1)}{m} \\
&\times \exp\left(-\lambda m \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right), \\
\left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right)\right]^{(\alpha(j+1)+1)} &= \sum_{n=0}^{\infty} (-1)^n \binom{\alpha(j+1)+1}{n} \\
&\times \exp\left(-\lambda n \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right), \\
\exp\left(-\lambda(m+n+1) \frac{G(x; \xi)}{\bar{G}(x, \xi)}\right) &= \sum_{p=0}^{\infty} (-1)^p \frac{\lambda^p (m+n+1)^p}{p!} \left[\frac{G(x; \xi)}{\bar{G}(x, \xi)}\right]^p
\end{aligned}$$

and

$$[\bar{G}(x; \xi)]^{-(p+2)} = [1 - G(x; \xi)]^{-(p+2)} = \sum_{q=0}^{\infty} \binom{-(p+2)}{q} G^q(x; \xi),$$

we obtain

$$\begin{aligned}
f_{MO-EOEHL-G}(x; \alpha, \lambda, \delta, \xi) &= \sum_{j, m, n, p, q=0}^{\infty} \frac{2\alpha(-1)^{n+p}\lambda^{p+1}[m+n+1]^p v_j}{p!(p+q+1)} \\
&\times \binom{-(\alpha(j+1)-1)}{m} \binom{\alpha(j+1)+1}{n} \\
&\times \binom{-(p+2)}{q} (p+q+1) g(x; \xi) [G(x; \xi)]^{p+q} \\
&= \sum_{p, q=0}^{\infty} v_{p, q}^* g_{p+q}(x; \xi).
\end{aligned}$$

Also, for $\delta > 1$ MO-EOEHL-G family of distributions can be expressed as a linear combination of the Exp-G distribution with power parameter $(p+q+1)$ and linear component

$$\begin{aligned}
v_{p, q}^* &= \sum_{j, m, n}^{\infty} \frac{2\alpha(-1)^{p+q}[m+p+1]^p v_j}{p!(p+q+1)} \\
&\times \binom{-(\alpha(j-k+1)-1)}{m} \binom{\alpha(j-k+1)+1}{n} \binom{-(p+2)}{q}. \quad (3.10)
\end{aligned}$$

3.2.2 Stochastic Orders

We present in this section, the stochastic orders for the MO-EOEHL-G family of distributions. If we have two random variables X and Y , then X is stochastically smaller than Y if $\bar{F}_X(r) \leq \bar{F}_Y(r)$ for all r , where $\bar{F}_X(r)$ and $\bar{F}_Y(r)$ are the survival functions of X and Y respectively. This is denoted by $X <_s Y$. The hazard rate order and likelihood ratio order given by $X <_{hr} Y$ if $h_X(r) \geq h_Y(r)$ for all r , and $X <_{lr} Y$ if $\frac{f_X(r)}{f_Y(r)}$ is decreasing in r , respectively, can also be used. From Shaked and Shanthikumar [69], $X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_s Y$.

Theorem 3.1. *Let X_1 and X_2 be two random variables such that $X_1 \sim \text{MO-EOEHL-G}(\alpha, \lambda, \delta_1, \xi)$ and $X_2 \sim \text{MO-EOEHL-G}(\alpha, \lambda, \delta_2, \xi)$. If $\delta_2 \leq \delta_1$, then $\frac{f(x; \alpha, \lambda, \delta_1, \xi)}{f(x; \alpha, \lambda, \delta_2, \xi)}$ is decreasing.*

Proof. Using equation (3.5) we have

$$\frac{f(x; \alpha, \lambda, \delta_1, \xi)}{f(x; \alpha, \lambda, \delta_2, \xi)} = \frac{\delta_1}{\delta_2} \left[\frac{1 - \bar{\delta}_1 \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right)} \right)^\alpha \right)}{1 - \bar{\delta}_2 \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right)} \right)^\alpha \right)} \right]^2,$$

such that

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{f(x; \alpha, \lambda, \delta_1, \xi)}{f(x; \alpha, \lambda, \delta_2, \xi)} \right) &= \frac{\delta_1}{\delta_2} (\delta_2 - \delta_1) \frac{1 - \bar{\delta}_2 \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right)} \right)^\alpha \right)}{\left[1 - \bar{\delta}_1 \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right)} \right)^\alpha \right]^3} \\ &\quad \times \frac{2\alpha \lambda g(x; \xi) \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right) \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right) \right]^{\alpha-1}}{\bar{G}^2(x; \xi) \left[1 + \exp\left(-\lambda \frac{G(x; \xi)}{G(x; \xi)}\right) \right]^{\alpha+1}}, \end{aligned}$$

which is negative if $\delta_2 \leq \delta_1$. Therefore $\frac{f(x; \alpha, \lambda, \delta_1, \xi)}{f(x; \alpha, \lambda, \delta_2, \xi)}$ is decreasing, implying that $X_1 <_{lr} X_2$, $X_1 <_{hr} X_2$ and $X_1 <_s X_2$. As such, the random variables X_1 and X_2 are stochastically ordered. $\square$

3.3 Some Special Cases

In this section, we present some special cases of the MO-EOEHL-G family of distributions. We considered cases when the baseline distributions are Burr

XII, Weibull and Uniform distributions.

3.3.1 Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-Burr XII (MO-EOEHL-BXII) Distribution

Consider the Burr XII distribution as the baseline distribution with pdf and cdf given by $g(x; c, k) = ckx^{c-1}(1+x^c)^{-k-1}$ and $G(x; c) = 1 - (1+x^c)^{-k}$, respectively, for $c, k > 0$. Therefore, the MO-EOEHL-BXII distribution have the cdf, pdf and hrf given by

$$F_{MO-EOEHL-BXII}(x; \alpha, \lambda, \delta, c, k) = 1 - \frac{\delta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)} \right]^\alpha \right)}{1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)} \right]^\alpha \right)},$$

$$f_{MO-EOEHL-BXII}(x; \alpha, \lambda, \delta, c, k) = \frac{2\alpha\lambda\delta ckx^{c-1}(1+x^c)^{-k-1} \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)}{(1+x^c)^{-2k} \left[1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right) \right]^{\alpha+1}} \\ \times \frac{\left[1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right) \right]^{\alpha-1}}{\left[1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)} \right]^\alpha \right)^2},$$

and

$$h_{MO-EOEHL-BXII}(x; \alpha, \lambda, \delta, c, k) = \frac{2\alpha\lambda\delta^2 ckx^{c-1}(1+x^c)^{k-1} \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)}{(1+x^c)^{-2k} \left[1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right) \right]^{\alpha+1}} \\ \times \frac{\left[1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right) \right]^{\alpha-1}}{1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)} \right]^\alpha \right)} \\ \times \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-k}}{(1+x^c)^{-k}} \right] \right)} \right]^\alpha \right),$$

respectively for $\alpha, \lambda, \delta, c, k > 0$. By letting $k=1$ we obtain the Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-Log-logistic (MO-EOEHL-LLoG) distribution from the MO-EOEHL-BXII distribution.

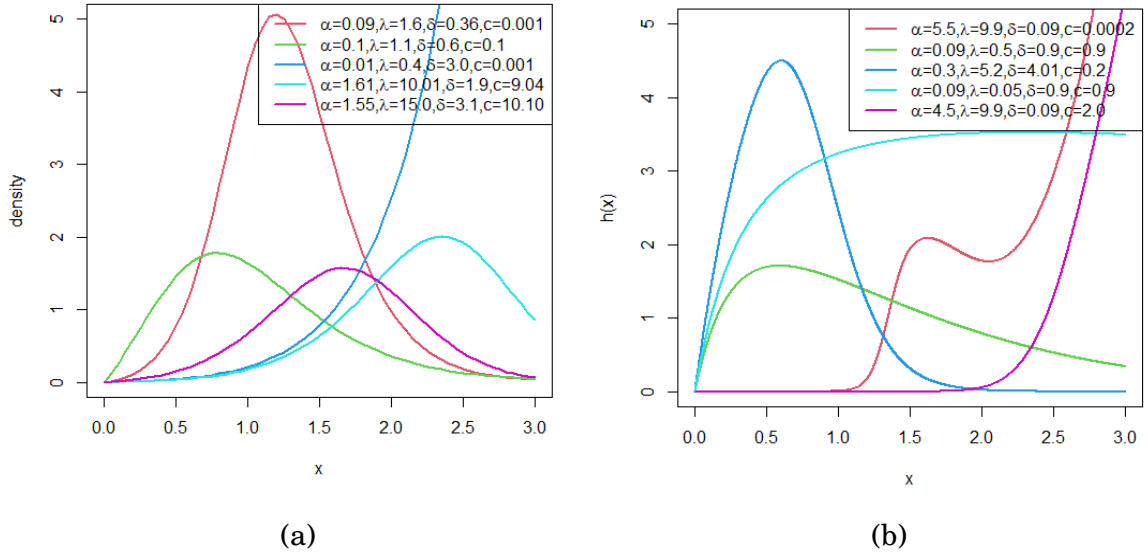


Figure 3.1: Plots of MO-EOEHL-LLoG density and hazard functions

Figures 3.1(a) and 3.1(b) show the plots of the pdfs and hrfs for the MO-EOEHL-LLoG distribution for selected parameter values. The pdf exhibit different shapes including almost symmetric, left-skewed, right-skewed, and J shapes. The hrf shows increasing, decreasing, upside-down bathtub followed by bathtub and uni-modal shapes.

3.3.2 Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-Weibull (MO-EOEHL-W) Distribution

Suppose the Weibull distribution is the baseline distribution with pdf and cdf given by $g(x; \theta) = \theta x^{\theta-1} e^{-x^\theta}$ and $G(x; \theta) = 1 - e^{-x^\theta}$, respectively, for $\theta > 0$. Then the cdf, pdf and hrf of the MO-EOEHL-W distribution are given by

$$F_{MO-EOEHL-W}(x; \alpha, \lambda, \delta, \theta) = 1 - \frac{\delta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - e^{-x^\theta}}{e^{-x^\theta}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - e^{-x^\theta}}{e^{-x^\theta}} \right] \right)} \right]^\alpha \right)}{1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - e^{-x^\theta}}{e^{-x^\theta}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - e^{-x^\theta}}{e^{-x^\theta}} \right] \right)} \right]^\alpha \right)},$$

$$f_{MO-EOEHL-W}(x; \alpha, \lambda, \delta, \theta) = \frac{2\alpha\lambda\delta\theta x^\theta e^{-x^\theta} \exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)}{e^{-2x^\theta} \left[1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)\right]^{\alpha+1}} \times \frac{\left[1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)\right]^{\alpha-1}}{\left[1 - \delta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)}\right)^\alpha\right]^2}$$

and

$$h_{MO-EOEHL-W}(x; \alpha, \lambda, \delta, \theta) = \frac{2\alpha\lambda\delta^2\theta x^{\theta-1} e^{-x^\theta} \exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)}{e^{-2x^\theta} \left[1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)\right]^{\alpha+1}} \times \frac{\left[1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)\right]^{\alpha-1}}{1 - \delta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)}\right)^\alpha\right)} \times \left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\theta}}{e^{-x^\theta}}\right]\right)}\right]^\alpha\right),$$

respectively, for $\alpha, \lambda, \delta, \theta > 0$.

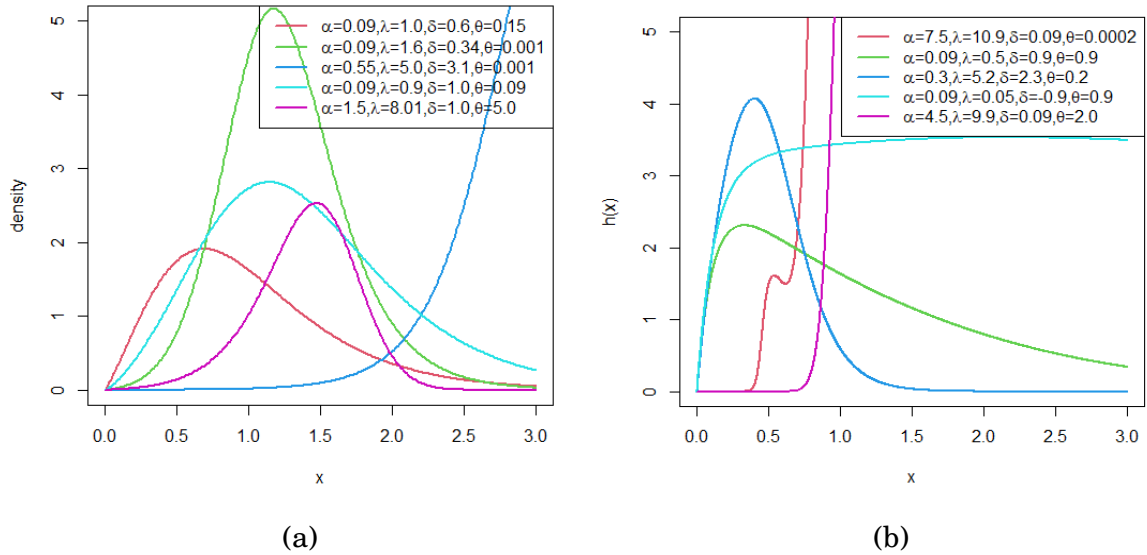


Figure 3.2: Plots of MO-EOEHL-W density and hazard functions

Figures 3.2(a) and 3.2(b) show the plots of the pdfs and hrfs for the MO-EOEHL-W distribution for selected parameter values. The pdf exhibit

different shapes including almost symmetric, left-skewed, right-skewed, and J shapes. The hrf shows increasing, upside-down bathtub followed by bathtub and uni-modal shapes.

3.3.3 Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-Uniform (MO-EOEHL-U) Distribution

Consider the Uniform distribution as the baseline distribution with pdf and cdf given by $g(x; \theta) = 1/\theta$ and $G(x; \theta) = x/\theta$, respectively, for $0 < x < \theta$. Therefore, the MO-EOEHL-U distribution have the cdf, pdf and hrf given by

$$F_{MO-EOEHL-U}(x; \alpha, \lambda, \delta, \theta) = 1 - \frac{\delta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)} \right]^\alpha \right)}{1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)} \right]^\alpha \right)},$$

$$f_{MO-EOEHL-U}(x; \alpha, \lambda, \delta, \theta) = \frac{2\alpha\lambda\delta(1/\theta) \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)}{\left(1 - \frac{x}{\theta} \right)^2 \left[1 + \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right) \right]^{\alpha+1}} \\ \times \frac{\left[1 - \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right) \right]^{\alpha-1}}{\left[1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)} \right]^\alpha \right) \right]^2}$$

and

$$h_{MO-EOEHL-U}(x; \alpha, \lambda, \delta, \theta) = \frac{2\alpha\lambda\delta^2(1/\theta) \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)}{\left(1 - \frac{x}{\theta} \right)^2 \left[1 + \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right) \right]^{\alpha+1}} \\ \times \frac{\left[1 - \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right) \right]^{\alpha-1}}{1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)} \right]^\alpha \right)} \\ \times \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{x}{\theta-x} \right] \right)} \right]^\alpha \right),$$

respectively, for α, λ, δ and $\theta > 0$.

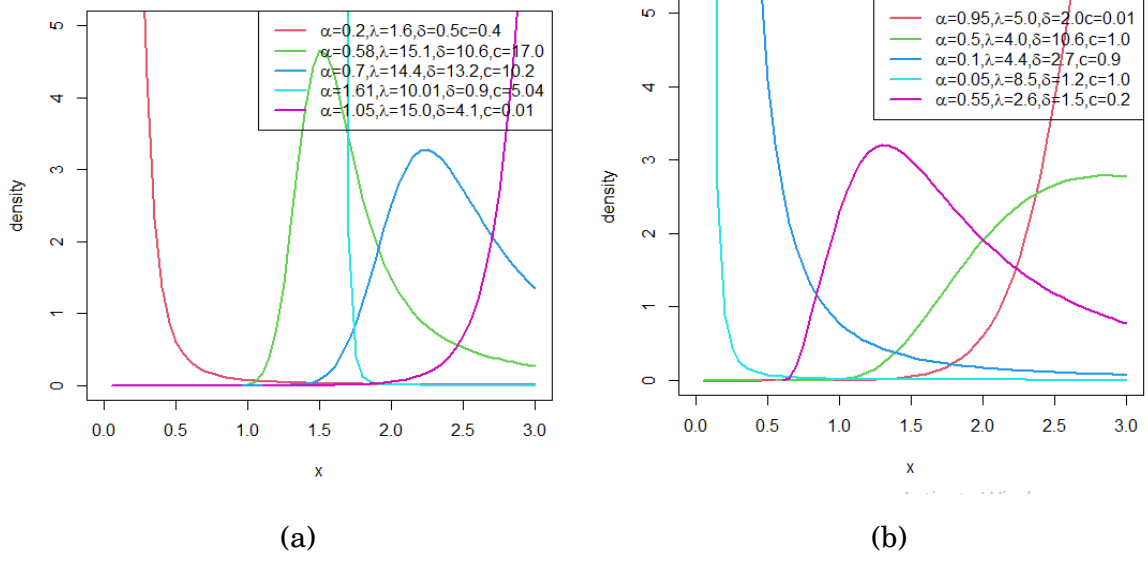


Figure 3.3: Plots of MO-EOEHL-U density and hazard functions

Figures 3.3(a) and 3.3(b) show the plots of the pdfs and hrfs for the MO-EOEHL-U distribution for selected parameter values. The pdf exhibit different shapes including right and left-skewed, reverse J and J shapes. The hrf exhibit increasing, decreasing, and upside-down bathtub shapes.

3.4 Statistical Properties

In this section, we present the statistical properties of the MO-EOEHL-G family of distributions including the order statistics, Rényi entropy, moments and the quantile function.

3.4.1 Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (iid) random variables from the MO-EOEHL-G family of distributions. The pdf of the i^{th} order statistic $X_{i:n}$, is given by

$$f_{i:n}(x; \alpha, \lambda, \xi) = \delta n! f_{EOEHL-G}(x; \alpha, \lambda, \xi) \sum_{k=0}^{n-i} \frac{(-1)^k}{(i-1)!(n-1)!} \times \frac{F_{EOEHL-G}^{k+i-1}(x; \alpha, \lambda, \xi)}{[1 - \delta F_{EOEHL-G}(x; \alpha, \lambda, \xi)]^{k+i-1}}.$$

If $\delta \in (0, 1)$, we have

$$f_{i:n}(x; \alpha, \lambda, \xi) = f_{EOEHL-G}(x; \alpha, \lambda, \xi) \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^j B_{j,k,l} F_{EOEHL-G}^{j+k-l+i-1}(x; \alpha, \lambda, \xi), \quad (3.11)$$

where $B_{j,k,l} = B_{j,k,l}(\delta) = \delta n! (-1)^{j+k-1} (1-\delta)^j \binom{j}{k} \binom{k+i+j}{j}$.

Substituting the pdf and cdf of the EOEHL-G distribution into equation (3.11) yields

$$\begin{aligned} f_{i:n}(x; \alpha, \lambda, \xi) &= \frac{2\alpha \lambda g(x; \xi) \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha-1}}{\bar{G}^2(x; \xi) \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha+1}} \\ &\quad \times \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^j B_{j,k,l} \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^{\alpha(j+k-l+i-1)}. \end{aligned}$$

Applying the series of expansions,

$$\begin{aligned} \left[1 + \exp\left(\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{-(\alpha(j+k-l+i)+1)} &= \sum_{m=0}^{\infty} \binom{-(\alpha(j+k-l+i)+1)}{m} \\ &\quad \times \exp\left(-\lambda m \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right), \end{aligned}$$

$$\begin{aligned} \left[1 - \exp\left(\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{-(\alpha(j+k-l+i)-1)} &= \sum_{n=0}^{\infty} (-1)^n \binom{-(\alpha(j+k-l+i)-1)}{n} \\ &\quad \times \exp\left(-\lambda n \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right), \end{aligned}$$

$$\exp\left(-\lambda(m+n+1) \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) = \sum_{p=0}^{\infty} (-1)^p \frac{\lambda^p (m+n+1)^p}{p!} \left[\frac{G(x; \xi)}{\bar{G}(x; \xi)}\right]^p$$

and

$$[\bar{G}(x; \xi)]^{-(p+2v)} = [1 - G(x; \xi)]^{-(p+2v)} = \sum_{q=0}^{\infty} \binom{-(p+2v)}{q} G^q(x; \xi),$$

gives

$$\begin{aligned}
f_{i:n}(x; \alpha, \lambda, \xi) &= \sum_{j,m,n,p,q=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^j \frac{2\alpha(-1)^{n+p}\lambda^{p+1}[m+n+1]^p B_{j,k,l}}{p!(p+q+1)} \\
&\times \binom{-(\alpha(j+k-l+i)+1)}{m} \binom{-(\alpha(j+k-l+i)-1)}{n} \\
&\times \binom{-(p+2v)}{q} (p+q+1)g(x; \xi)[G(x; \xi)]^{p+q} \\
&= \sum_{p,q=0}^{\infty} B_{p,q}^* g_{p+q}(x; \xi),
\end{aligned}$$

where $g_{p+q} = (p+q+1)g(x; \xi)[G(x; \xi)]^{p+q}$ is an **Exp-G** distribution with power parameter $(p+q+1)$ and

$$\begin{aligned}
B_{p,q}^* &= \sum_{j,m,n=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^j \frac{2\alpha(-1)^{n+p}\lambda^{p+1}[m+n+1]^p B_{j,k,l}}{p!(p+q+1)} \\
&\times \binom{-(\alpha(j+k-l+i)+1)}{m} \binom{-(\alpha(j+k-l+i)-1)}{n} \\
&\times \binom{-(p+2v)}{q}.
\end{aligned}$$

If $\delta > 1$, we write $1 - \bar{\delta}F_{EOEHL-G}(x; \alpha, \lambda, \xi) = \delta[1 - \frac{(\delta-1)F_{EOEHL-G}(x; \alpha, \lambda, \xi)}{\delta}]$, such that

$$f_{i:n}(x; \alpha, \lambda, \xi) = f_{EOEHL-G}(x; \alpha, \lambda, \xi) \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} U_{j,k} F_{EOEHL-G}^{j+k+i-1}(x; \alpha, \lambda, \xi),$$

where $U_{j,k} = U_{j,k}(\delta) = \frac{(-1)^k(\delta-1)^j n!}{\delta^{k+j+i}(i-1)!(n-i)!} \binom{k+i+j}{j}$.

Therefore,

$$\begin{aligned}
f_{i:n}(x; \alpha, \lambda, \xi) &= \frac{2\alpha\lambda g(x; \xi) \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) [1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)]^{\alpha-1}}{\bar{G}^2(x; \xi) [1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)]^{\alpha+1}} \\
&\times \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} U_{j,k} \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^{\alpha(j+k+i-1)}.
\end{aligned}$$

Applying the series of expansions,

$$\begin{aligned}
\left[1 + \exp\left(\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{-(\alpha(j+k+i)+1)} &= \sum_{m=0}^{\infty} \binom{-(\alpha(j+k+i)+1)}{m} \\
&\times \exp\left(-\lambda m \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right),
\end{aligned}$$

$$\begin{aligned}
\left[1 - \exp\left(\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{-(\alpha(j+k+i)-1)} &= \sum_{n=0}^{\infty} (-1)^n \binom{-(\alpha(j+k+i)-1)}{n} \\
&\times \exp\left(-\lambda m \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right), \\
\exp\left(-\lambda(m+n+1) \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) &= \sum_{p=0}^{\infty} (-1)^p \frac{\lambda^p (m+n+1)^p}{p!} \left[\frac{G(x; \xi)}{\bar{G}(x; \xi)}\right]^p
\end{aligned}$$

and

$$[\bar{G}(x; \xi)]^{-(p+2v)} = [1 - G(x; \xi)]^{-(p+2v)} = \sum_{q=0}^{\infty} \binom{-(p+2v)}{q} G^q(x; \xi),$$

we can write

$$\begin{aligned}
f_{i:n}(x; \alpha, \lambda, \xi) &= \sum_{j,m,n,p,q=0}^{\infty} \sum_{k=0}^{n-i} \frac{2\alpha(-1)^{n+p}\lambda^{p+1}[m+n+1]^p U_{j,k}}{p!(p+q+1)} \\
&\times \binom{-(\alpha(j+k+i)+1)}{m} \binom{-(\alpha(j+k+i)-1)}{n} \\
&\times \binom{-(p+2v)}{q} (p+q+1) g(x; \xi) [G(x; \xi)]^{p+q} \\
&= \sum_{p,q=0}^{\infty} U_{p,q}^* g_{p+q}(x; \xi),
\end{aligned}$$

where $g_{p+q} = (p+q+1)g(x; \xi)[G(x; \xi)]^{p+q}$ is an **Exp-G** distribution with power parameter $(p+q+1)$ and

$$\begin{aligned}
U_{p,q}^* &= \sum_{j,m,n=0}^{\infty} \sum_{k=0}^{n-i} \frac{2\alpha(-1)^{n+p}\lambda^{p+1}[m+n+1]^p U_{j,k}}{p!(p+q+1)} \\
&\times \binom{-(\alpha(j+k+i)+1)}{m} \binom{-(\alpha(j+k+i)-1)}{n} \\
&\times \binom{-(p+2v)}{q}.
\end{aligned}$$

3.4.2 Entropy

Entropy measures the uncertainty of a random variable X with pdf $f(x)$. There are two popular measures of entropy, namely Shannon entropy [71] and Rényi entropy [66]. Rényi entropy is defined as

$$I_R(v) = (1-v)^{-1} \log \left[\int_0^{\infty} f^v(x; \alpha, \lambda, \delta, \xi) dx \right],$$

where $v > 0$, and $v \neq 1$. Using expansion (3.8), for $\delta \in (0, 1)$

$$f_{MO-EOEHL-G}^v(x; \alpha, \lambda, \delta, \xi) = \frac{\delta^v f^v(x; \alpha, \lambda, \xi)}{\Gamma(2v)} \sum_{j=0}^{\infty} (1-\delta)^j \Gamma(2v+j) \\ \times \frac{[1 - F_{EOEHL-G}(x; \alpha, \lambda, \xi)]^j}{j!},$$

and for $\delta > 1$

$$f_{MO-EOEHL-G}^v(x; \alpha, \lambda, \delta, \xi) = \frac{\delta^v f^v(x; \alpha, \lambda, \xi)}{\delta^v \Gamma(2v)} \sum_{j=0}^{\infty} (\delta-1)^j \Gamma(2v+j) \\ \times \frac{[F_{EOEHL-G}(x; \alpha, \lambda, \xi)]^j}{j!}.$$

Thus, for $\delta \in (0, 1)$ we have,

$$I_R(v) = (1-v)^{-1} \log \left[\sum_{j=0}^{\infty} e_j \int_0^{\infty} f_{EOEHL-G}^v(x; \alpha, \lambda, \xi) [1 - F_{EOEHL-G}(x; \alpha, \lambda, \xi)]^j dx \right],$$

where $e_j = e_j(\delta) = \frac{\delta^v (1-\delta)^j \Gamma(2v+j)}{\Gamma(2v)j!}$.

Now,

$$I_R(v) = (1-v)^{-1} \log \left[\sum_{j=0}^{\infty} e_j \int_0^{\infty} \frac{(2\alpha\lambda)^v g^v(x; \xi) \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)})}{\bar{G}^{2v}(x; \xi) \left[1 + \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)}) \right]^{v(\alpha+1)}} \right. \\ \left. \times \left[1 - \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)}) \right]^{v(\alpha-1)} \left(1 - \left[\frac{1 - \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)})}{1 + \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)})} \right]^{\alpha} \right)^j dx \right].$$

Using the generalized binomial expansions

$$\left(1 - \left[\frac{1 - \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)})}{1 + \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)})} \right]^{\alpha} \right)^j = \sum_{k=0}^{\infty} (-1)^k \binom{j}{k} \left[\frac{1 - \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)})}{1 + \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)})} \right]^{\alpha k},$$

$$\left[1 + \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)}) \right]^{-(\alpha(k+v)+v)} = \sum_{m=0}^{\infty} \binom{-(\alpha(k+v)+v)}{m} \exp\left(-\lambda m \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right),$$

$$\left[1 - \exp(-\lambda v \frac{G(x;\xi)}{\bar{G}(x;\xi)}) \right]^{\alpha(k+v)-v} = \sum_{n=0}^{\infty} (-1)^n \binom{\alpha(k+v)+v}{n} \exp\left(-\lambda n \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right),$$

$$\exp \left(-\lambda(m+n+v) \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) = \sum_{p=0}^{\infty} (-1)^p \frac{\lambda^p (m+n+v)^p}{p!} \left[\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right]^p$$

and

$$[\bar{G}(x; \xi)]^{-(p+2v)} = [1 - G(x; \xi)]^{-(p+2v)} = \sum_{q=0}^{\infty} \binom{-(p+2v)}{q} G^q(x; \xi),$$

yields

$$\begin{aligned} I_R(v) &= (1-v)^{-1} \log \left[\sum_{j,k,m,n,p,q=0}^{\infty} e_j \frac{(2\alpha)^v (-1)^{k+n+p} \lambda^{p+v} [m+n+v]^p}{\left(\frac{p+q}{v} + 1\right)^v p!} \right. \\ &\quad \times \binom{j}{k} \binom{-(\alpha(k+v)+v)}{m} \binom{\alpha(k+v)+v}{n} \binom{-(p+2v)}{q} \\ &\quad \times \left. \int_0^{\infty} \left(\left(\frac{p+q}{v} + 1\right) g(x; \xi) [G(x; \xi)]^{\frac{p+q}{v}} \right)^v dx \right] \\ &= \times (1-v)^{-1} \log \left[\sum_{p,q=0}^{\infty} e_{p,q}^* e^{(1-v)I_{REG}} \right], \end{aligned}$$

where

$$\begin{aligned} e_{p,q}^* &= \sum_{j,k,m,n=0}^{\infty} e_j \frac{(2\alpha)^v (-1)^{k+n+p} \lambda^{p+v} [m+n+v]^p}{\left(\frac{p+q}{v} + 1\right)^v p!} \\ &\quad \times \binom{j}{k} \binom{-(\alpha(k+v)+v)}{m} \binom{\alpha(k+v)+v}{n} \binom{-(p+2v)}{q} \end{aligned}$$

and

$$I_{REG} = (1-v)^{-1} \log \left[\int_0^{\infty} \left(\left(\frac{p+q}{v} + 1\right) g(x; \xi) [G(x; \xi)]^{\frac{p+q}{v}} \right)^v dx \right]$$

is the Rényi entropy of the Exp-G distribution with power parameter $\frac{p+q}{v} + 1$.

Furthermore, for $\delta > 1$

$$I_R(v) = (1-v)^{-1} \log \left[\sum_{j=0}^{\infty} h_j \int_0^{\infty} f_{EOEHL-G}^v(x; \alpha, \lambda, \xi) F_{EOEHL-G}^j(x; \alpha, \lambda, \xi) dx \right],$$

where $h_j = h_j(\delta) = \frac{(\delta-1)^j \Gamma(2v+j)}{\delta^{v+j} \Gamma(2v) j!}$.

Now,

$$I_R(v) = (1-v)^{-1} \log \left[\sum_{j=0}^{\infty} h_j \int_0^{\infty} \frac{(2\alpha\lambda)^v g^v(x; \xi) \exp\left(-\lambda v \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{\bar{G}^{2v}(x; \xi) \left[1 + \exp\left(-\lambda v \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{v(\alpha+1)}} \right. \\ \left. \times \left[1 - \exp\left(-\lambda v \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{v(\alpha-1)} \left[\frac{1 - \exp\left(-\lambda v \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda v \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}\right]^{\alpha j} dx \right].$$

Considering the generalized binomial expansions

$$\left[1 + \exp\left(-\lambda v \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{-(\alpha(j+v)+v)} = \sum_{m=0}^{\infty} \binom{-(\alpha(j+v)+v)}{m} \exp\left(-\lambda m \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right),$$

$$\left[1 - \exp\left(-\lambda v \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha(j+v)-v} = \sum_{n=0}^{\infty} \binom{\alpha(j+v)+v}{n} \exp\left(-\lambda n \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right),$$

$$\exp\left(-\lambda(m+n+v) \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) = \sum_{p=0}^{\infty} (-1)^p \frac{\lambda^p (m+n+v)^p}{p!} \left[\frac{G(x; \xi)}{\bar{G}(x; \xi)}\right]^p$$

and

$$[\bar{G}(x; \xi)]^{-(p+2v)} = [1 - G(x; \xi)]^{-(p+2v)} = \sum_{q=0}^{\infty} \binom{-(p+2v)}{q} G^q(x; \xi),$$

we obtain

$$I_R(v) = (1-v)^{-1} \log \left[\sum_{j,m,n,p,q,0}^{\infty} h_j \frac{(2\alpha)^v (-1)^{n+p} \lambda^{p+v} [m+n+v]^p}{\left(\frac{p+q}{v} + 1\right)^v p!} \right. \\ \times \binom{-(\alpha(j+v)+v)}{m} \binom{\alpha(j+v)+v}{n} \binom{-(p+2v)}{q} \\ \left. \times \int_0^{\infty} \left(\left(\frac{p+q}{v} + 1\right) g(x; \xi) [G(x; \xi)]^{\frac{p+q}{v}}\right)^v dx \right] \\ = \times (1-v)^{-1} \log \left[\sum_{p,q=0}^{\infty} h_{p,q}^* e^{(1-v)I_{REG}} \right],$$

where

$$h_{p,q}^* = \sum_{j,m,n=0}^{\infty} \frac{(2\alpha)^v (-1)^{n+p} \lambda^{p+v} [m+n+v]^p}{\left(\frac{p+q}{v} + 1\right)^v p!} \\ \times \binom{-(\alpha(j+v)+v)}{m} \binom{\alpha(j+v)+v}{n} \binom{-(p+2v)}{q}$$

and

$$I_{REG} = (1 - v)^{-1} \log \left[\int_0^\infty \left(\left(\frac{p+q}{v} + 1 \right) g(x; \xi) [G(x; \xi)]^{\frac{p+q}{v}} \right)^v dx \right]$$

is the Rényi entropy of the Exp-G distribution with power parameter $\frac{p+q}{v} + 1$.

3.4.3 Moments and Generating Functions

Let $X \sim \text{MO-EOEHL-G}(\alpha, \lambda, \delta, \xi)$, then the r^{th} moment can be obtained using equations (3.9) and (3.10). For $\delta \in (0, 1)$,

$$E[X^r] = \sum_{p,q=0}^{\infty} w_{p,q}^* E(W_{p+q}^r),$$

where $w_{p,q}^*$ is given by equation (3.9) and $E(W_{p+q}^r)$ is the r^{th} moment of W_{p+q} which follows an Exp-G distribution with power parameter $(p + q + 1)$.

For $\delta > 1$,

$$E[X^r] = \sum_{p,q=0}^{\infty} v_{p,q}^* E(W_{p+q}^r),$$

where $w_{p,q}^*$ is given by equation (3.10) and $E(W_{p+q}^r)$ is the r^{th} moment of W_{p+q} which follows an Exp-G distribution with power parameter $(p + q + 1)$.

The s^{th} incomplete moments are as follows: For $\delta \in (0, 1)$

$$I_x(t) = \int_0^t x^s f_{\text{MO-EOEHL-G}}(x; \alpha, \lambda, \delta, \xi) dx = \sum_{p,q}^{\infty} w_{p,q}^* I_{p+q}(t),$$

where $I_{p+q}(t) = \int_0^t x^s g_{p+q}(x; \xi) dx$ and $w_{p,q}^*$ is defined by equation (3.9).

For $\delta > 1$

$$I_x(t) = \int_0^t x^s f_{\text{MO-EOEHL-G}}(x; \alpha, \lambda, \delta, \xi) dx = \sum_{p,q}^{\infty} v_{p,q}^* I_{p+q}(t),$$

where $I_{p+q}(t) = \int_0^t x^s g_{p+q}(x; \xi) dx$ and $v_{p,q}^*$ is defined by equation (3.10).

The moment generating function (mgf) of X is given by: For $\delta \in (0, 1)$

$$M_X(t) = \sum_{p,q=0}^{\infty} w_{p,q}^* E[e^{tW_{p+q}}],$$

where $E[e^{tW_{p+q}}]$ is the mgf of the Exp-G distribution with power parameter $(p + q + 1)$ and $w_{p,q}^*$ is given by equation (3.9).

For $\delta > 1$

$$M_X(t) = \sum_{p,q=0}^{\infty} v_{p,q}^* E[e^{tW_{p+q}}],$$

where $E[e^{tW_{p+q}}]$ is the mgf of the Exp-G distribution with power parameter $(p + q + 1)$ and $w_{p,q}^*$ is given by equation (3.10).

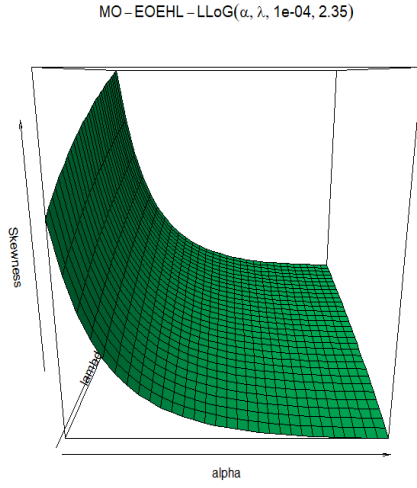
A table of moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) for selected parameter values of the MO-EOEHL-LLoG distribution are given in Table 3.1.

Table 3.1: Table of Moments for Selected Parameters of the MO-EOEHL-LLoG Distribution

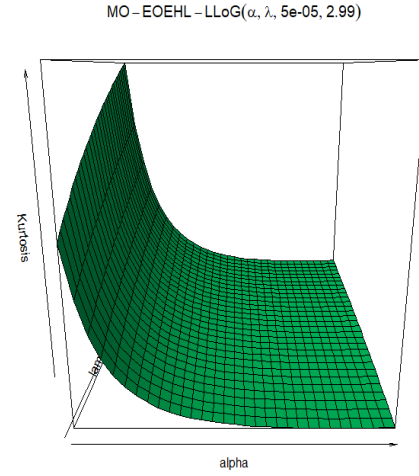
	$(\alpha, \lambda, \delta, c)$				
	(0.2,1.1,0.5,2)	(0.9,0.3,1,0.5)	(1.5,2.5,0.2,0.1)	(1,0.9,0.2,0.5)	(1.3,0.1,0.5,1.7)
E(X)	0.0993	0.0947	0.0576	0.1222	0.0786
E(X ²)	0.0749	0.0495	0.0281	0.0590	0.0383
E(X ³)	0.0600	0.0334	0.0185	0.0383	0.0249
E(X ⁴)	0.0500	0.0252	0.0138	0.0283	0.0184
E(X ⁵)	0.0429	0.0202	0.0110	0.0224	0.0145
SD	0.2550	0.2014	0.1575	0.2099	0.1792
CV	2.5673	2.1264	2.7328	1.7176	2.2800
CS	2.3927	2.5759	3.5992	2.2010	2.9304
CK	7.1774	9.0873	16.3703	7.3018	11.4589

3D plots of skewness and kurtosis for the MO-EOEHL-LLoG distribution are shown in Figures 3.4 and 3.5. We observe that

- When we fix the parameters for λ, δ and c , the skewness and kurtosis of MO-EOEHL-LLoG decreases as α increases.
- When we fix the parameters for α, δ and c , the skewness and kurtosis of MO-EOEHL-LLoG increases as λ increases.
- When we fix the parameters for α, λ and c , the skewness and kurtosis of MO-EOEHL-LLoG increases as δ increases.
- When we fix the parameters for α, λ and δ , the skewness and kurtosis of MO-EOEHL-LLoG increases as c increases.



(a)



(b)

Figure 3.4: Plots of skewness and kurtosis for the MO-EOEHL-LLoG distribution

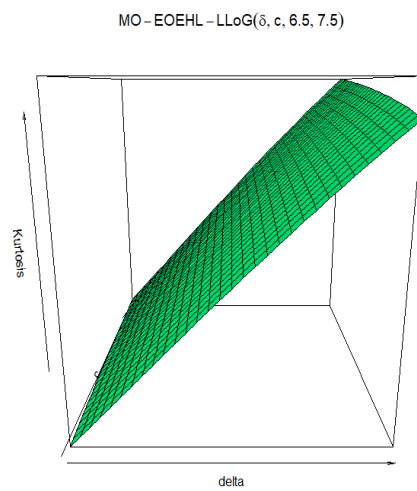
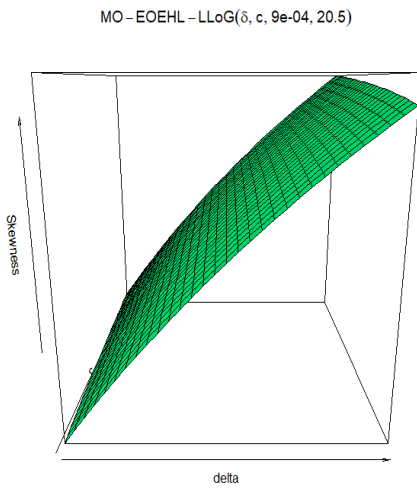


Figure 3.5: Plots of skewness and kurtosis for the MO-EOEHL-LLoG distribution

3.4.4 Quantile Function

The quantile function of the MO-EOEHL-G family of distributions is obtained by solving the non-linear equation:

$$1 - \frac{\delta \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)} \right]^\alpha \right)}{1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)} \right]^\alpha \right)} = u,$$

for $0 \leq u \leq 1$. Simplifying the equation we get

$$(1 - u)(1 - \bar{\delta}) + \bar{\delta}(1 - u) \left[\frac{1 - \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)} \right]^\alpha = \delta \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)} \right]^\alpha \right),$$

such that

$$\frac{\delta - (1 - u)(1 - \bar{\delta}) \left[\frac{1 - \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)} \right]^\alpha}{\bar{\delta}(1 - u) + \delta} = \left[\frac{1 - \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right)} \right]^\alpha.$$

Simplifying the equation further we obtain

$$\frac{1 - \left[\frac{\delta - (1 - u)(1 - \bar{\delta})}{\bar{\delta}(1 - u) + \delta} \right]^{\frac{1}{\alpha}}}{1 + \left[\frac{\delta - (1 - u)(1 - \bar{\delta})}{\bar{\delta}(1 - u) + \delta} \right]^{\frac{1}{\alpha}}} = \exp \left(-\lambda \frac{G(x;\xi)}{G(x;\xi)} \right),$$

hence

$$G(x; \xi) = \left[\frac{\ln \left[1 - \left[\frac{\delta - (1 - u)(1 - \bar{\delta})}{\bar{\delta}(1 - u) + \delta} \right]^{\frac{1}{\alpha}} \right]}{\ln \left[1 + \left[\frac{\delta - (1 - u)(1 - \bar{\delta})}{\bar{\delta}(1 - u) + \delta} \right]^{\frac{1}{\alpha}} \right] - \lambda} \right].$$

Therefore, the quantiles of the MO-EOEHL-G family of distributions may be obtained by solving the non-linear equation

$$x(u) = G^{-1} \left[\frac{\ln \left[1 - \left[\frac{\delta - (1 - u)(1 - \bar{\delta})}{\bar{\delta}(1 - u) + \delta} \right]^{\frac{1}{\alpha}} \right]}{\ln \left[1 + \left[\frac{\delta - (1 - u)(1 - \bar{\delta})}{\bar{\delta}(1 - u) + \delta} \right]^{\frac{1}{\alpha}} \right] - \lambda} \right].$$

The equation can be solved using software like R, SAS or MATLAB. Quantiles for selected parameter values for the MO-EOEHL-LLoG distribution are shown in Table 3.2.

Table 3.2: Table of Quantiles for Selected Parameters of the MO-EOEHL-LLoG Distribution

	$(\lambda, \alpha, \delta, c)$				
u	(0.2,0.5,0.1,1)	(0.4,1,1.5,0.9)	(0.7,1.5,0.1,2)	(2,1.5,0.9,1.1)	(0.8,1.8,0.1,1.7)
0.1	0.0012	0.6933	0.3760	0.2368	0.3930
0.2	0.0060	1.4522	0.4908	0.3702	0.5110
0.3	0.0169	2.2403	0.5847	0.4891	0.6075
0.4	0.0391	3.0728	0.6736	0.6060	0.6991
0.5	0.0826	3.9790	0.7653	0.7279	0.7941
0.6	0.1702	5.0067	0.8670	0.8626	0.9001
0.7	0.3581	6.2435	0.9890	1.0214	1.0284
0.8	0.8181	7.8829	1.1521	1.2279	1.2027
0.9	2.2826	10.5431	1.4195	1.5548	1.4960

3.5 Maximum Likelihood Estimation

The maximum likelihood estimation method is used in this section to estimate the model parameters of the MO-EOEHL-G family of distributions. Let $X_i \sim MO - EOEHL - G(\alpha, \lambda, \delta, \xi)$ and $\Delta = (\alpha, \lambda, \delta, \xi)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\begin{aligned}
\ell(\Delta) = & n \log(2\alpha\lambda\delta) + \sum_{i=1}^n \log[g(x; \xi)] - \lambda \sum_{i=1}^n \frac{G(x; \xi)}{\bar{G}(x; \xi)} \\
& + (\alpha - 1) \sum_{i=1}^n \log \left[1 - \exp \left(- \lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right] - 2 \sum_{i=1}^n \log[\bar{G}(x; \xi)] \\
& - (\alpha + 1) \sum_{i=1}^n \log \left[1 + \exp \left(- \lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right] \\
& - 2 \sum_{i=1}^n \log \left[1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(- \lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(- \lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right)^\alpha \right) \right].
\end{aligned}$$

The elements of the score vector $U(\Delta)$ are given as

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \log \left[1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right] - \sum_{i=1}^n \log \left[1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right] \\ &\quad - 2 \sum_{i=1}^n \frac{\bar{\delta} \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right]^\alpha \log \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right]}{\left[1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right)^\alpha \right) \right]},\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n \frac{G(x; \xi)}{\bar{G}(x; \xi)} + (\alpha - 1) \sum_{i=1}^n \frac{\exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \frac{G(x; \xi)}{\bar{G}(x; \xi)}}{\left[1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]} \\ &\quad + (\alpha + 1) \sum_{i=1}^n \frac{\exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \frac{G(x; \xi)}{\bar{G}(x; \xi)}}{\left[1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]} \\ &\quad - 2 \sum_{i=1}^n \frac{2\alpha \delta \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \frac{G(x; \xi)}{\bar{G}(x; \xi)} \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right]^{\alpha-1}}{\left[1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right)^\alpha \right) \right] \left(1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right)^2}, \\ \frac{\partial \ell}{\partial \delta} &= \frac{n}{\delta} - 2 \sum_{i=1}^n \frac{\left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right)^\alpha \right)}{\left[1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right)^\alpha \right) \right]}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \xi_k} &= \sum_{i=1}^n \frac{\frac{\partial [g(x; \xi)]}{\partial \xi_k}}{g(x; \xi)} - \lambda \sum_{i=1}^n \frac{\bar{G}(x; \xi) \frac{\partial [G(x; \xi)]}{\partial \xi_k} - G(x; \xi) \frac{\partial [\bar{G}(x; \xi)]}{\partial \xi_k}}{\bar{G}^2(x; \xi)} \\ &\quad + (\alpha - 1) \sum_{i=1}^n \frac{\frac{\partial [1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)]}{\partial \xi_k}}{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} - 2 \sum_{i=1}^n \frac{\frac{\partial [\bar{G}(x; \xi)]}{\partial \xi_k}}{\bar{G}(x; \xi)} \\ &\quad - (\alpha + 1) \sum_{i=1}^n \frac{\frac{\partial [1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)]}{\partial \xi_k}}{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} - 2 \sum_{i=1}^n \frac{\frac{\partial \left[1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right)^\alpha \right]}{\partial \xi_k}}{\left[1 - \bar{\delta} \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)} \right)^\alpha \right) \right]},\end{aligned}$$

respectively. The maximum likelihood estimates (MLEs) of the parameters denoted by $\hat{\Delta}$ are obtained by solving the non-linear equation

$\left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \xi_k}\right)^T = 0$. These partial derivatives are not in closed form, therefore numerical methods such as Newton-Raphson procedure can be used to obtain the MLEs of the model parameters.

3.6 Simulation Study

In this section, we examine the consistency of the MO-EOEHL-LLoG distribution by conducting a simulation study for various sample sizes, $n= 50, 100, 200, 400$, and 800 for $N=1000$ for each sample. The results of the simulation study are shown in Table 3.3. The average bias and RMSE are given by

$$Bias(\hat{\psi}) = \frac{1}{N} \sum_{i=1}^N (\hat{\psi}_i - \psi), \quad \text{and} \quad RMSE(\hat{\psi}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\psi}_i - \psi)^2}{N}},$$

respectively. From the results, we can verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

3.7 Applications

In this section, two real data examples to demonstrate the usefulness and flexibility of the MO-EOEHL-G family of distributions are considered. The new special case of the proposed model is compared to other competing models. The performance of the models were assessed using the following goodness-of-fit statistics; -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*) and Andersen-Darling(A^*) (as described by Chen and Balakrishnan [21]), Kolmogorov-Smirnov (K-S) statistic and its p-value. These statistics are used to verify which distribution fits better to the data. The model that has smaller values of these above mentioned goodness-of-fit statistics and larger p-value of the K-S statistics is deemed as the best model.

Model parameters were estimated via the maximum likelihood estimation

Table 3.3: Monte Carlo Simulation Results for MO-EOEHL-LLoG Distribution

	$\lambda = 1.1, \alpha = 1.1, \delta = 1.1, c = 0.8$			$\lambda = 1.2, \alpha = 1.1, \delta = 1.2, c = 1.0$			
n	Mean	RMSE	Bias	Mean	RMSE	Bias	
λ	50	1.7463	1.9153	0.6463	1.8164	1.8905	0.6164
	100	1.4806	1.4412	0.3806	1.6209	1.5276	0.4209
	200	1.2109	1.0406	0.1109	1.2957	1.0852	0.0957
	400	1.1343	0.7403	0.0343	1.2182	0.7624	0.0182
	800	1.1051	0.5242	0.0051	1.2159	0.5489	0.0159
α	50	4.1420	32.1811	3.0420	3.1186	19.1776	2.0186
	100	1.3140	1.6605	0.2140	1.3619	1.7669	0.2619
	200	1.0720	0.5852	-0.0280	1.0612	0.5632	-0.0388
	400	1.0737	0.3107	-0.0263	1.0689	0.3015	-0.0311
	800	1.0772	0.2032	-0.0228	1.0790	0.1968	-0.0210
δ	50	9.6811	32.0698	8.5811	9.8044	30.9684	8.6044
	100	5.6854	18.5304	4.5854	7.1306	23.4855	5.9306
	200	3.1550	11.0776	2.0550	3.7238	14.9862	2.5238
	400	1.6782	2.3176	0.5782	1.8443	2.7742	0.6443
	800	1.3281	1.0799	0.2281	1.4923	1.3012	0.2923
c	50	1.0608	0.8189	0.2608	1.3358	1.0436	0.3358
	100	1.0335	0.7283	0.2335	1.2687	0.9000	0.2687
	200	0.9919	0.5451	0.1919	1.2551	0.7179	0.2551
	400	0.9073	0.3599	0.1073	1.1365	0.4511	0.1365
	800	0.8540	0.2127	0.0540	1.0591	0.2582	0.0591
	$\lambda = 0.5, \alpha = 1.0, \delta = 0.5, c = 1.1$			$\lambda = 1.0, \alpha = 1.1, \delta = 1.1, c = 0.8$			
λ	50	1.1649	1.6816	0.6649	1.6299	1.8687	0.6299
	100	0.9766	1.3533	0.4766	1.4532	1.4653	0.4532
	200	0.7471	0.9519	0.2471	1.1341	0.9968	0.1341
	400	0.5941	0.5737	0.0941	1.0394	0.6995	0.0394
	800	0.5520	0.3883	0.0520	1.0157	0.4969	0.0157
α	50	1.7384	6.4616	0.7384	4.2643	35.5467	3.1643
	100	1.1633	1.4771	0.1633	1.3863	2.0089	0.2863
	200	1.0242	0.6582	0.0242	1.0823	0.5850	-0.0177
	400	0.9835	0.3314	-0.0165	1.0755	0.3079	-0.0245
	800	0.9875	0.2200	-0.0125	1.0795	0.2022	-0.0205
δ	50	8.2032	45.2376	7.7032	9.7748	32.7755	8.6748
	100	4.2808	17.1108	3.7808	6.4211	21.3858	5.3211
	200	2.0451	11.6497	1.5451	3.0641	10.4275	1.9641
	400	0.8073	1.4066	0.3073	1.6525	2.2047	0.5525
	800	0.6241	0.5535	0.1241	1.3372	1.0834	0.2372
c	50	1.4359	0.9822	0.3359	1.0723	0.8169	0.2723
	100	1.3802	0.8595	0.2802	0.9882	0.6589	0.1882
	200	1.3111	0.6577	0.2111	0.9647	0.4903	0.1647
	400	1.2464	0.4837	0.1464	0.9036	0.3452	0.1036
	800	1.1735	0.3210	0.0735	0.8498	0.2077	0.0498

technique using the R software. Tables 3.4 and 3.5 show the model parameter estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the two data sets considered. Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [20]) are also presented to show how well our model fits the observed data sets. For the probability plot, we plotted $F_{MO-EOEHL-G}(x_{(j)}; \hat{\lambda}, \hat{\alpha}, \hat{\delta}, \hat{\xi})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness of the plots to the diagonal line was determined and is given by the sum of squares

$$SS = \sum_{j=1}^n \left[F_{MO-EOEHL-G}(x_{(j)}; \hat{\lambda}, \hat{\alpha}, \hat{\delta}, \hat{\xi}) \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The MO-EOEHL-LLoG distribution was compared to the following distributions: Kumaraswamy odd Lindley-Log logistic (KOL-LLoG) by Chipepa et al. [24], Topp-Leone- Marshall-Olkin-Weibull (TLMO-W) by Chipepa et al. [25], Beta-Weibull (BW) by Cordeiro et al. [31], the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. [43], Topp-Leone-Weibull-Lomax (TL-WLx) by Jamal et al. [40], odd log-logistic exponentiated Weibull (OLLEW) by Afify et al. [1] and Marshall-Olkin-Gompertz-Weibull (MO-Gom-W) by Chipepa and Oluyede [23] distributions.

The pdfs of the non-nested models are as follows:

$$f_{TLMO-W}(b, \delta, \lambda, \gamma) = \frac{2b\delta^2\lambda\gamma x^{\gamma-1}e^{-2\lambda x^\gamma}}{(1 - \delta e^{\lambda x^\gamma})^3} \left[1 - \frac{\delta^2 e^{-2\lambda x^\gamma}}{(1 - \delta e^{\lambda x^\gamma})^2} \right]^{b-1},$$

for $b, \delta, \lambda, \gamma > 0$,

$$\begin{aligned} f_{KOL-LLoG}(x; a, b, \lambda, c) &= ab \left[\frac{\lambda^2}{(1 + \lambda)} \frac{cx^{c-1}}{(1 + x^c)^{-1}} \exp \left(-\lambda \frac{(1 - (1 + x^{vc})^{-1})}{(1 + x^c)^{-1}} \right) \right] \\ &\times \left[1 - \frac{\lambda + (1 + x^c)^{-1}}{(1 + \lambda)(1 + x^c)^{-1}} \exp \left(-\lambda \frac{(1 - (1 + x^{vc})^{-1})}{(1 + x^c)^{-1}} \right) \right]^{a-1} \\ &\times \left(1 - \left[1 - \frac{\lambda + (1 + x^c)^{-1}}{(1 + \lambda)(1 + x^c)^{-1}} \exp \left(-\lambda \frac{(1 - (1 + x^{vc})^{-1})}{(1 + x^c)^{-1}} \right) \right]^a \right)^{b-1}, \end{aligned}$$

for $a, b, \lambda, c > 0$,

$$f_{BW}(x; a, b, \alpha, \beta) = \frac{\beta\alpha^\beta}{B(a, b)} x^{\beta-1} e^{-b(\alpha x)^\beta} (1 - e^{-(\alpha x)^\beta})^{a-1},$$

for $a, b, \alpha, \beta > 0$,

$$f_{ELLOW}(x; \alpha, \beta, \gamma, \theta, \lambda) = \frac{\alpha\theta^2\gamma\lambda^\gamma x^{\gamma-1} e^{-(\lambda x)^\gamma} (e^{-(\lambda x)^\gamma})^{\alpha\theta-1} (1 - e^{-(\lambda x)^\gamma})^{\alpha-1}}{(\theta + \beta)((1 - e^{-(\lambda x)^\gamma})^\alpha + e^{-\alpha(\lambda x)^\gamma})^{\theta-1}} \\ \times \left(1 - \beta \log \left[\frac{e^{-(\lambda x)^\gamma}}{(1 - e^{-(\lambda x)^\gamma})^\alpha + e^{-\alpha(\lambda x)^\gamma}} \right] \right),$$

for $\alpha, \beta, \gamma, \theta, \lambda > 0$,

$$f_{TL-WLx}(x; a, b, \alpha, \theta) = 2\theta\alpha ab(1 + bx)^{a\alpha-1} (1 - (1 + bx)^{-a})^{\alpha-1} \\ \times \exp \left(-2 \left(\frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right) \right) \\ \times \left[1 - \exp \left(-2 \left(\frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right) \right) \right]^{\theta-1},$$

for $a, b, \alpha, \theta > 0$,

$$f_{MO-Gom-W}(x; \theta, \gamma, \delta, \lambda) = \frac{\delta\theta\lambda x^{\lambda-1} e^{-x^\lambda} [e^{-x^\lambda}]^{-\gamma-1} e^{\frac{\theta}{\gamma}(1-[e^{-x^\lambda}]^{-\gamma})}}{(1 - \delta(e^{\frac{\theta}{\gamma}(1-[e^{-x^\lambda}]^{-\gamma})}))^2}$$

for $\theta, \gamma, \delta, \lambda > 0$ and

$$f_{OLLEW}(x; \alpha, \beta, \gamma, \theta) = \frac{\theta\beta\gamma x^{\beta-1} e^{-(x/\alpha)^\beta} [1 - e^{-(x/\alpha)^\beta}]^{\gamma\theta-1} (1 - [1 - e^{-(x/\alpha)^\beta}]^\gamma)^{\theta-1}}{\alpha\beta([1 - e^{-(x/\alpha)^\beta}]^{\theta\gamma} + (1 - [1 - e^{-(x/\alpha)^\beta}]^\gamma)^\theta)^2},$$

for $\alpha, \beta, \gamma, \theta > 0$. For the ELLLOW distribution, we considered the case when $\alpha = 1$.

3.7.1 Kevlar 49/Epoxy Strands Failure at 90% Data

The first data set used consist of 101 observations representing the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all had failed. The failure times are in hours and are shown below (see Andrews and Herzberg [14] or Barlow et al. [15], for details): 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54,

1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics are presented in Table 3.4.

Table 3.4: Parameter estimates and goodness-of-fit statistics for various models fitted for kelvar data set

Model	Estimates				Statistics							
	λ	α	δ	c	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P - value$
MO-EOEHL-LLoG	1.8617 (0.9833)	0.9200 (0.3337)	1.8617 (1.8560)	0.7330 (0.2511)	203.6	211.6	212.1	222.1	0.1092	0.6956	0.0674	0.7494
KOL-LLoG	a	b	λ	c	205.1	213.1	213.5	223.5	0.1481	0.8782	0.0787	0.5593
	0.9438 (0.4859)	3.1447 (12.7272)	0.6028 (1.4993)	0.8525 (0.40008)								
TLMOW	b	δ	λ	γ	204.0	212.0	212.4	222.5	0.1169	0.7321	0.0678	0.7421
	1.1928 (0.3539)	5.8027 (7.4228)	1.6659 (1.0843)	0.5780 (0.2312)								
OLLEW	α	β	γ	θ	205.5	213.5	214.0	224.0	0.1616	0.9432	0.0798	0.5407
	1.6251 (2.6163)	1.1121 (0.4424)	0.6081 (0.9012)	1.1687 (0.9463)								
BW	a	b	α	β	204.8	212.8	213.2	223.3	0.4673	3.3332	0.0723	0.6662
	0.7077 (0.2179)	0.1531 (0.1538)	5.2436 (5.7741)	1.0548 (0.1488)								
ELOLLW	β	λ	θ	γ	205.2	213.2	213.6	223.655	0.1668	0.9601	0.0816	0.5124
	4.0660 (2.6163)	0.3012 (0.4424)	4.2525 (0.9012)	0.8374 (0.94630)								
TLWLx	a	b	α	θ	205.4	213.4	213.8	223.9	0.1625	0.9447	0.0835	0.4825
	0.6616 (0.6595)	0.7096 (0.9640)	1.3633 (0.9470)	0.6079 (0.4456)								
MO-Gom-W	θ	γ	δ	λ	203.8	211.8	212.3	222.3	0.1119	0.7095	0.0676	0.7458
	1.9358 (0.8469)	0.0525 (0.3110)	3.9118 (4.2949)	0.6728 (0.2537)								

The estimated variance-covariance for the MO-EOEHL-LLoG model on kelvar data is given by

$$\begin{bmatrix} 0.96694 & 0.1762 & 1.6514 & -0.2356 \\ 0.1762 & 0.1113 & 0.0980 & -0.0562 \\ 1.6514 & 0.0980 & 3.4447 & -0.3703 \\ -0.2356 & -0.0562 & -0.3703 & 0.0630 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\lambda \in [1.8617 \pm 1.9273]$, $\alpha \in [0.9200 \pm 0.6540]$, $\delta \in [1.8617 \pm 3.6377]$ and $c \in [0.7330 \pm 0.4922]$.

Results from Table 3.4 shows that the MO-EOEHL-G model performs better than the non-nested models because it has the lowest values for the goodness-of-fit statistics and the highest K-S p-value. Figure 3.6 shows the

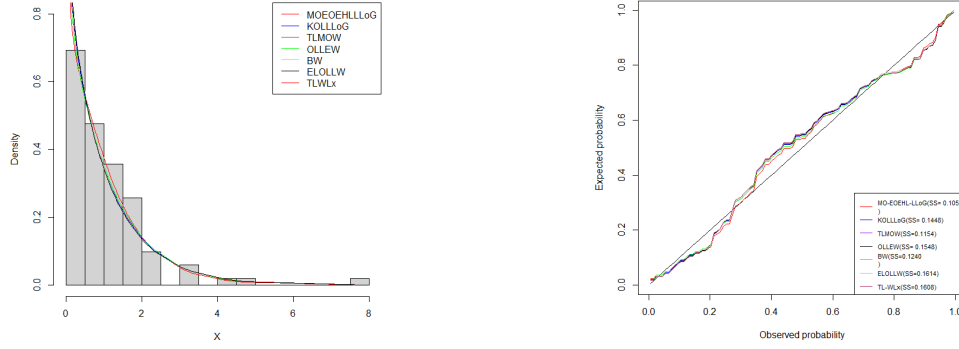


Figure 3.6: Fitted densities and probability plots for the kelvar data

fitted densities and probability plots for the MO-EOEHL-G model and other competing models. The value of sum of squares from the probability plots in Figure 3.6 is smaller for the MO-EOEHL-LLoG distribution.

3.7.2 Strengths of 1.5 cm Glass Fibres Data

We then considered the data set representing the strengths of 1.5 cm glass fibres. The data set was also analyzed by Bourguignon et al. [18] and Smith and Naylor [73]. The data are 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics are presented in Table 3.5.

The estimated variance-covariance for the MO-EOEHL-LLoG model on 1.5 cm glass fibre data is given by

$$\begin{bmatrix} 0.2465 & 0.0224 & 5.2060 & -0.5776 \\ 0.0224 & 0.1816 & -2.7742 & -0.0816 \\ 5.2060 & -2.7742 & 177.3965 & -11.3294 \\ -0.5776 & -0.0816 & -11.3294 & 1.3852 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\lambda \in [0.5213 \pm 0.9731]$, $\alpha \in [0.6517 \pm 0.8352]$, $\delta \in [9.6450 \pm 26.1053]$ and $c \in [3.5927$

Table 3.5: Parameter estimates and goodness-of-fit statistics for various models fitted for 1.5 cm glass fibres data set

Model	Estimates				Statistics							
	λ	α	δ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P - value$
MO-EOEHL-LLoG	0.5213 (0.4965)	0.6517 (0.4261)	9.6450 (13.3190)	3.5927 (1.1770)	23.6	31.6	32.2	40.1	0.0796	0.4651	0.0957	0.6113
KOL-LLoG	a	b	λ	c	27.4	35.4	36.1	44.0	0.1507	0.8453	0.1290	0.2449
	0.5538 (0.3184)	26.2535 (0.0328)	0.0038 (0.0116)	5.9048 (2.7700)								
TLMOW	b	δ	λ	γ	24.6	32.6	33.3	41.1	0.0918	0.5322	0.1063	0.4746
	0.9632 (0.3630)	80.0679 (246.1330)	1.2887 (1.6535)	2.2492 (1.2012)								
OLLEW	α	β	γ	θ	28.0	36.0	36.7	44.6	0.1864	1.0314	0.1320	0.2223
	1.9920 (0.2975)	8.7485 (3.9398)	0.3021 (0.2668)	1.6872 (0.7436)								
BW	a	b	α	β	26.1	34.1	34.8	42.7	2.0396	11.1272	0.1133	0.3942
	0.4493 (0.1819)	0.0496 (0.0470)	0.9199 (0.1499)	7.0127 (0.8894)								
ELOLLW	β	λ	θ	γ	28.4	36.4	37.1	45.0	0.1953	1.0753	0.1367	0.1895
	58.4191 (11.8176)	0.3423 (0.0458)	31.2427 (22.0974)	4.9441 (0.6591)								
TLWLx	a	b	α	θ	29.1	37.1	37.8	45.7	0.1928	1.0740	0.1428	0.1534
	4.2885 (3.5333)	0.0937 (0.0833)	5.3964 (1.3872)	0.7779 (0.3051)								
MO-Gom-W	θ	γ	δ	λ	24.0	32.0	32.8	40.6	0.1008	0.5684	0.0999	0.5552
	0.7736 (0.8422)	0.0243 (0.2000)	18.8319 (30.1444)	2.9458 (2.0111)								

$\pm 2.3069]$.

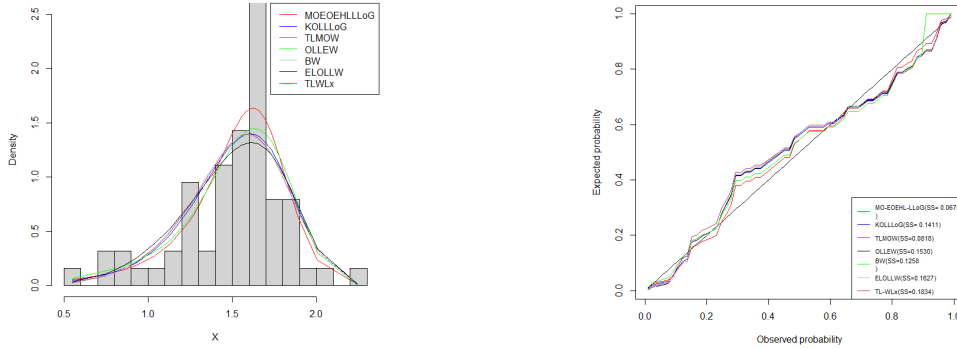


Figure 3.7: Fitted densities and probability plots for the glass fibre data

From Table 3.5, we observe that the MO-EOEHL-LLoG model has the smallest values of all the goodness-of-fit statistics and bigger value for the K-S p-value and therefore it performs better the other models. Figure 3.7 also shows the fitted densities and probability plots for the MO-EOEHL-LLoG model and other competing models and that it performs better than the various non-nested

models on glass fibres data.

3.8 Concluding Remarks

A new family of generalized distributions called the Marshall-Olkin-Exponentiated Half Logistic-G family of distributions was developed. We presented some special cases of the proposed distribution. The proposed distribution can be applied to heavy tailed data, and has monotonic and non-monotonic hazard rate shapes. The family of distributions can be expressed as a linear combination of the Exp-G distribution and its statistical properties can be derived directly from those of the Exp-G distribution. The proposed model performs better than the several competing non-nested models on the data sets presented.

Chapter 4

The Exponentiated Odd Exponential Half-Logistic-G Power Series Class of Distributions

4.1 Introduction

In this chapter, a new power series class of distributions, referred to as the Exponentiated Odd Exponential Half Logistic-G Power Series (EOEHL-GPS) class of distributions is presented. This new model is obtained by compounding the power series distribution with the Exponentiated Odd Exponential Half Logistic-G family of distributions. We are motivated by the desirable properties and flexibility obtained from compounding continuous distributions with power series distributions. Such properties includes hazard function that shows increasing, unimodal and bathtub shapes. One other crucial reason for developing the new power series class of distributions is the modeling of lifetime data using a model that takes into consideration not only scale and shape but also skewness, kurtosis and tail variation. We are also motivated by the wide

applications of power series distributions in different areas such as finance, actuarial sciences and economics.

The Exponentiated odd Exponential Half Logistic-G (EOEHL-G) family of distributions proposed by Afify et al. [2] has cdf and pdf given by

$$F_{EOEHL-G}(x; \alpha, \lambda, \xi) = \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha, \quad (4.1)$$

and

$$f_{EOEHL-G}(x; \alpha, \lambda, \xi) = \frac{2\alpha\lambda g(x; \xi) \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha-1}}{\bar{G}^2(x; \xi) \left[1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha+1}}, \quad (4.2)$$

where $\alpha, \lambda > 0$ and $G(x; \xi)$ is the baseline distribution.

4.2 Exponentiated Odd Exponential Half-Logistic-G Power Series Distribution

In this section, the EOEHL-GPS class of distributions is presented together with its hazard rate function, sub-classes and density expansion.

Given N , let $X_1, \dots, X_N$ be independent and identically distributed random variables following the EOEHL-G distribution. Let $X_{(1)} = \min(X_1, \dots, X_N)$, then the conditional cdf of $X_{(1)}|N = n$ is given by

$$F_{X_{(1)}|N=n}(x) = 1 - [S_{EOEHL-G}(x; \alpha, \lambda, \theta, \xi)]^n, \quad (4.3)$$

where $S_{EOEHL-G}(x; \alpha, \lambda, \theta, \xi)$ is the survival function of the EOEHL-G distribution, $\alpha, \lambda, \theta > 0$ and ξ is the parameter vector. The Exponentiated Odd Exponential Half-Logistic-G Power Series (EOEHL-GPS) class of distributions denoted by EOEHL-GPS(α, λ, θ) is defined by the marginal cdf of $X_{(1)}$, that is

$$F_{X_{(1)}}(x) = 1 - \frac{C\left(\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}\right]^\alpha\right)\right)}{C(\theta)}, \quad (4.4)$$

for $\alpha, \lambda, \theta > 0$ and vector parameter ξ . The pdf is given by

$$f_{X_{(1)}}(x) = \frac{2\theta\alpha\lambda g(x; \xi) \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha-1}}{\bar{G}^2(x; \xi) \left[1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha+1}} \times \frac{C' \left(\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha \right) \right)}{C(\theta)}, \quad (4.5)$$

and the hrf is given by

$$h_{X_{(1)}}(x) = \frac{2\theta\alpha\lambda g(x; \xi) \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) \left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha-1}}{\bar{G}^2(x; \xi) \left[1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)\right]^{\alpha+1}} \times \frac{C' \left(\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha \right) \right)}{C \left(\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha \right) \right)}.$$

Table 4.1 shows some sub-classes of the EOEHL-GPS distribution.

Table 4.1: Sub-Classes of the EOEHL-GPS Distribution

Distribution	a_n	$C(\theta)$	cdf
EOEHL-G Poisson	$(n!)^{-1}$	$e^\theta - 1$	$1 - \frac{\exp\left(\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}\right]^\alpha\right)\right) - 1}{e^\theta - 1}$
EOEHL-G Geometric	1	$\theta(1 - \theta)^{-1}$	$1 - \frac{(1 - \theta) \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}\right]^\alpha\right)}{1 - \theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}\right]^\alpha\right)}$
EOEHL-G Logarithmic	n^{-1}	$-\log(1 - \theta)$	$1 - \frac{\log\left(1 - \theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}\right]^\alpha\right)\right)}{\log(1 - \theta)}$
EOEHL-G Binomial	$\binom{m}{n}$	$(1 + \theta)^m - 1$	$1 - \frac{\left(1 + \theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}\right]^\alpha\right)\right)^m - 1}{(1 + \theta)^m - 1}$

4.2.1 Expansion of Density Function

In this subsection, we present the series expansion of the EOEHL-GPS class of distributions. equation (4.5) can be written as

$$f_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) = \sum_{n=1}^{\infty} \frac{na_n \theta^n}{C(\theta)} \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^{\alpha} \right)^{n-1} \\ \times \frac{2\alpha \lambda g(x; \xi) \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) [1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)]^{\alpha-1}}{\bar{G}^2(x; \xi) [1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)]^{\alpha+1}}.$$

Using the generalized binomial expansions:

$$\left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^{\alpha} \right)^{n-1} = \sum_{i=0}^{\infty} (-1)^i \binom{n-1}{i} \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^{\alpha i},$$

$$\left[1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) \right]^{-(\alpha(i+1)+1)} = \sum_{j=0}^{\infty} \binom{-(\alpha(i+1)+1)}{j} \exp\left(-\lambda j \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right),$$

$$\left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) \right]^{(\alpha(i+1)-1)} = \sum_{k=0}^{\infty} \binom{(\alpha(i+1)-1)}{k} \exp\left(-\lambda k \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right),$$

$$\exp\left(-\lambda(j+k+1) \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) = \sum_{l=0}^{\infty} \frac{(-1)^l \lambda^l (j+k+1)^l}{l!} \left[\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right]^l$$

and

$$[\bar{G}(x; \xi)]^{-(l+2)} = [1 - G(x; \xi)]^{-(l+2)} = \sum_{m=0}^{\infty} \binom{-(l+2)}{m} G^m(x; \xi),$$

we write

$$f_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) = \sum_{i,j,k,l,m=0}^{\infty} \sum_{n=1}^{\infty} \frac{na_n \theta^n}{C(\theta)} \frac{(-1)^{k+l} 2\alpha \lambda^{l+1} (j+k+1)^l}{l!(l+m+1)} \\ \times \binom{n-1}{i} \binom{-(\alpha(i+1)+1)}{j} \binom{(\alpha(i+1)-1)}{k} \\ \times \binom{-(l+2)}{m} (l+m+1) g(x; \xi) G^{l+m}(x; \xi) \\ = \sum_{l,m=0}^{\infty} \eta_{l,m} g_{l+m}(x; \xi),$$

where $g_{l+m}(x; \xi) = (l+m+1)g(x; \xi)G^{l+m}(x; \xi)$ is the Exp-G distribution with power parameter $(l+m)$ and

$$\begin{aligned} \eta_{l,m} &= \sum_{i,j,k=0}^{\infty} \sum_{n=1}^{\infty} \frac{na_n \theta^n}{C(\theta)} \frac{(-1)^{k+l} 2\alpha \lambda^{l+1} (j+k+1)^l}{l!(l+m+1)} \\ &\times \binom{n-1}{i} \binom{-(\alpha(i+1)+1)}{j} \binom{(\alpha(i+1)-1)}{k} \binom{-(l+2)}{m}. \end{aligned} \quad (4.6)$$

It then follows that the EOEHL-GPS class of distributions can be expressed as a linear combination of the Exp-G distribution.

4.3 Some Special Cases of the EOEHL-GPS Class of Distributions

In this section, we look at some special cases of the EOEHL-GPS class of distributions. We consider cases when the baseline distributions are Log-logistic and Weibull distributions.

4.3.1 Exponentiated Half-Logistic-Log-logistic (EOEHL-LLoGPS) Class of Distributions

The cdf and pdf of the log-logistic distribution are given by $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$, respectively, for $c > 0$ and $x > 0$. As such, the cdf, pdf and hrf of the EOEHL-LLoGPS distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{C \left(\theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)} \right] \right)^{\alpha} \right)}{C(\theta)},$$

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda cx^{c-1}(1+x^c)^{-2} \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{(1+x^c)^{-2} \left[1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right) \right]^{\alpha+1}} \\ &\times \left[1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right) \right]^{\alpha-1} \\ &\times \frac{C' \left(\theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)} \right] \right)^{\alpha} \right)}{C(\theta)}, \end{aligned}$$

and

$$\begin{aligned}
h_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda cx^{c-1}(1+x^c)^{-2} \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{\bar{G}^2(x; \xi) \left[1 + \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)\right]^{\alpha+1}} \\
&\times \left[1 - \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)\right]^{\alpha-1} \\
&\times \frac{C'\left(\theta\left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}\right]^\alpha\right)\right)}{C\left(\theta\left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}\right]^\alpha\right)\right)},
\end{aligned}$$

respectively, for $\alpha, \lambda, \theta, c > 0$.

Exponentiated Odd Exponential Half-Logistic-Log-logistic Poisson (EOEHL-LLoGP) Distribution

The cdf, pdf and hrf of the EOEHL-LLoGP distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{\exp\left(\theta\left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}\right]^\alpha\right)\right) - 1}{\exp(\theta) - 1},$$

$$\begin{aligned}
f_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda cx^{c-1}(1+x^c)^{-2} \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{(1+x^c)^{-2} \left[1 + \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)\right]^{\alpha+1}} \\
&\times \left[1 - \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)\right]^{\alpha-1} \\
&\times \frac{\exp\left(\theta\left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}\right]^\alpha\right)\right)}{\exp(\theta) - 1},
\end{aligned}$$

and

$$\begin{aligned}
h_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda cx^{c-1}(1+x^c)^{-2} \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{(1+x^c)^{-2} \left[1 + \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)\right]^{\alpha+1}} \\
&\times \left[1 - \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)\right]^{\alpha-1} \\
&\times \frac{\exp\left(\theta\left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}\right]^\alpha\right)\right)}{\exp\left(\theta\left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}\right]^\alpha\right)\right) - 1},
\end{aligned}$$

respectively, for $\alpha, \lambda, \theta, c > 0$.

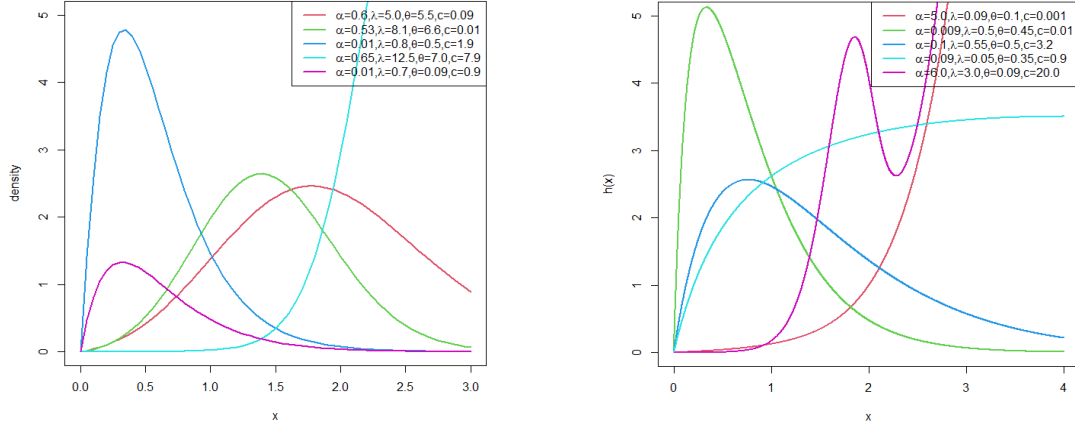


Figure 4.1: Plots of EOEHL-LLoGP density and hazard functions

Figure 4.1 show the plots of the pdfs and hrfs for the EOEHL-LLoGP distribution. The pdf can take various shapes including almost symmetric, left-skewed, right-skewed, and J shapes. The hrf shows increasing, upside-down bathtub followed by bathtub and uni-modal shapes.

Exponentiated Odd Exponential Half-Logistic-Log-logistic Logarithmic (EOEHL-LLoGL) Distribution

The cdf, pdf and hrf of the EOEHL-LLoGL distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{\log \left(1 - \theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)} \right]^\alpha \right) \right)}{\log(1 - \theta)},$$

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda c x^{c-1} (1+x^c)^{-2} \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{(1+x^c)^{-2} \left[1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right) \right]^{\alpha+1}} \\ &\times \left[1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right) \right]^{\alpha-1} \\ &\times \frac{\left(1 - \theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)} \right]^\alpha \right) \right)^{-1}}{-\log(1 - \theta)}, \end{aligned}$$

and

$$\begin{aligned}
h_{X_{(1)}}(x) = & \frac{2\theta\alpha\lambda cx^{c-1}(1+x^c)^{-2} \exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{(1+x^c)^{-2}\left[1+\exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)\right]^{\alpha+1}} \\
& \times \left[1-\exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)\right]^{\alpha-1} \\
& \times \frac{\left(1-\theta\left(1-\left[\frac{1-\exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}\right]^{\alpha}\right)\right)^{-1}}{-\log(1-\theta)} \\
& \times \left(\frac{\log\left(1-\theta\left(1-\left[\frac{1-\exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]\right)}\right]^{\alpha}\right)\right)}{\log(1-\theta)}\right)^{-1},
\end{aligned}$$

respectively, for $\alpha, \lambda, \theta, c > 0$.

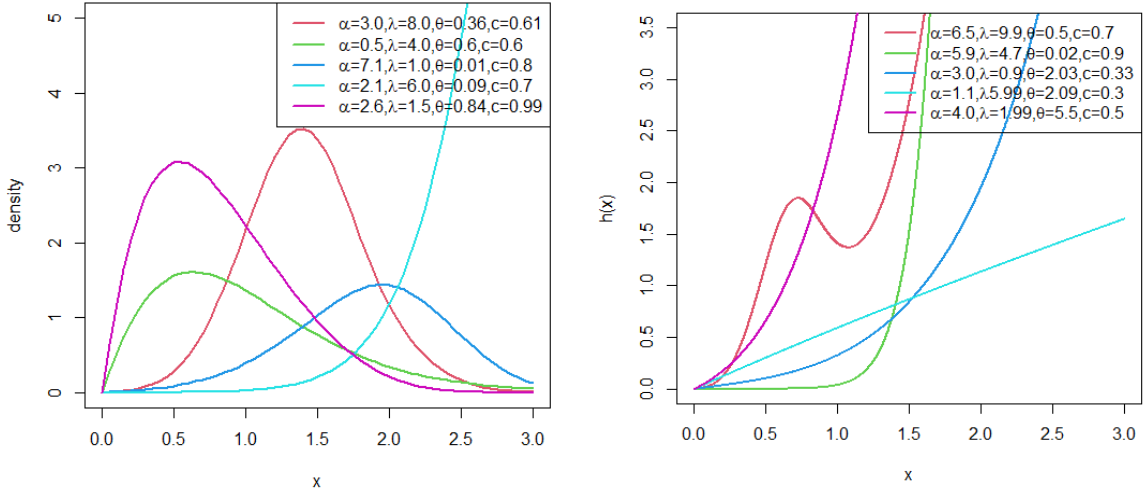


Figure 4.2: Plots of EOEHL-LLoGL density and hazard functions

Plots of the pdfs and hrfs for the EOEHL-LLoGL distribution are shown in Figure 4.2. The pdf exhibit different shapes including almost symmetric, left-skewed, right-skewed, and J shapes. The hrf shows increasing, upside-down bathtub followed by bathtub and J shapes.

Exponentiated Odd Exponential Half-Logistic-Log-logistic Geometric (EOEHL-LLoGG) Distribution

The cdf, pdf and hrf of the EOEHL-LLoGG distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{(1 - \theta) \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)} \right]^\alpha \right)}{1 - \theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)} \right]^\alpha \right)},$$

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda cx^{c-1}(1+x^c)^{-2} \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{(1+x^c)^{-2} \left[1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right) \right]^{\alpha+1}} \\ &\times \left[1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right) \right]^{\alpha-1} \\ &\times \frac{(1 - \theta)}{\left(1 - \theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)} \right]^\alpha \right) \right)^2}, \end{aligned}$$

and

$$\begin{aligned} h_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda cx^{c-1}(1+x^c)^{-2} \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{(1+x^c)^{-2} \left[1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right) \right]^{\alpha+1}} \\ &\times \left[1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right) \right]^{\alpha-1} \\ &\times \frac{1}{1 - \theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)} \right]^\alpha \right)} \\ &\times \frac{1}{\left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right] \right)} \right]^\alpha \right)}, \end{aligned}$$

respectively, for $\alpha, \lambda, \theta, c > 0$.

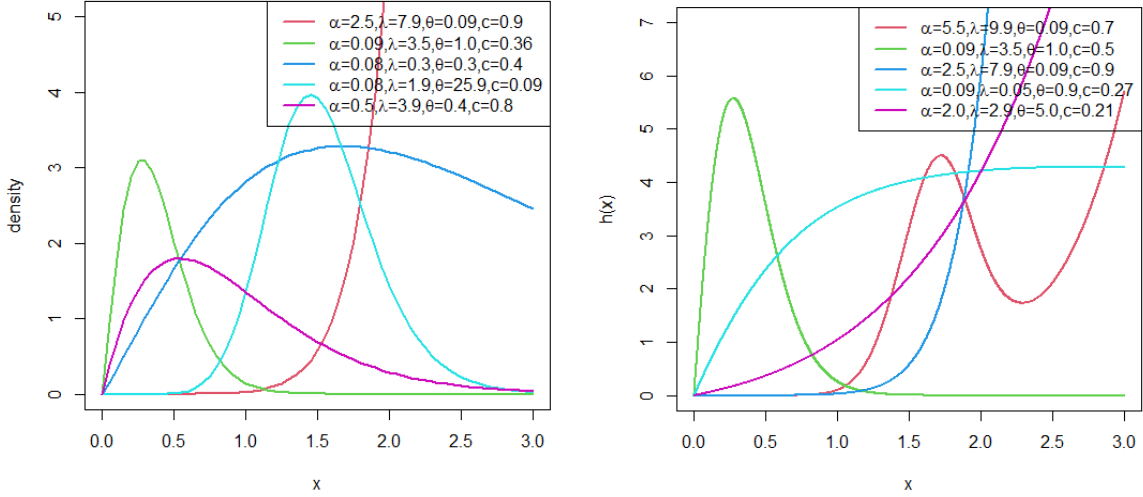


Figure 4.3: Plots of EOEHL-LLoGG density and hazard functions

Figure 4.3 shows the plots of the pdfs and hrfs of the EOEHL-LLoGG distribution for selected parameters values. The pdf can take various shapes including J, left and right skewed and almost symmetric. The EOEHL-LLoGG distribution exhibits increasing, unimodal, upside-down bathtub followed by bathtub and J hazard rate shapes.

4.3.2 Exponentiated Odd Exponential Half-Logistic-Weibull Power Series (EOEHL-WPS) Class of Distributions

The cdf and pdf of the weibull distribution are given by $G(x; c) = 1 - e^{-x^\gamma}$ and $g(x; c) = \gamma x^{\gamma-1} e^{-x^\gamma}$, respectively, for $\gamma > 0$ and $x > 0$. Therefore, the cdf, pdf and hrf of the EOEHL-WPS distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{C\left(\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1 + \exp\left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)}{C(\theta)},$$

$$\begin{aligned}
f_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda\gamma x^{\gamma-1} e^{-x^\gamma} \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{e^{-2x^\gamma} \left[1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha+1}} \\
&\times \left[1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha-1} \\
&\times \frac{C'\left(\theta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)}{C(\theta)},
\end{aligned}$$

and

$$\begin{aligned}
h_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda\gamma x^{\gamma-1} e^{-x^\gamma} \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{e^{-2x^\gamma} \left[1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha+1}} \\
&\times \left[1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha-1} \\
&\times \frac{C'\left(\theta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)}{C\left(\theta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)},
\end{aligned}$$

respectively, for $\alpha, \lambda, \theta, \gamma > 0$.

Exponentiated Odd Exponential Half-Logistic-Weibull Poisson (EOEHL-WP) Distribution

The cdf, pdf and hrf of the EOEHL-WP distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{\exp\left(\theta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right) - 1}{\exp(\theta) - 1},$$

$$\begin{aligned}
f_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda\gamma x^{\gamma-1} e^{-x^\gamma} \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{e^{-2x^\gamma} \left[1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha+1}} \\
&\times \left[1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha-1} \\
&\times \frac{\exp\left(\theta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)}{\exp(\theta) - 1},
\end{aligned}$$

and

$$h_{X_{(1)}}(x) = \frac{2\theta\alpha\lambda\gamma x^{\gamma-1} e^{-x^\gamma} \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{e^{-2x^\gamma} \left[1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha+1}} \times \left[1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha-1} \times \frac{\exp\left(\theta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)}{\exp\left(\theta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right) - 1},$$

respectively, for $\alpha, \lambda, \theta, \gamma > 0$.

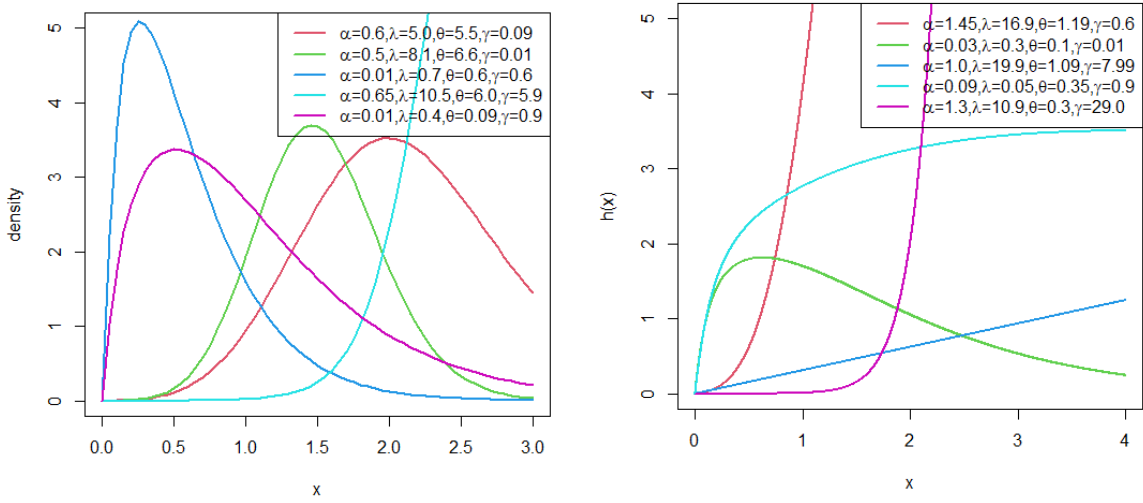


Figure 4.4: Plots of EOEHL-WP density and hazard functions

Plots of the pdfs and hrfs for the EOEHL-WP distribution are shown in Figure 4.4. The pdf exhibit different shapes including almost symmetric, left-skewed, right-skewed, and J shapes. The hrf shows increasing, upside-down bathtub followed by bathtub, unimodal and J shapes.

Exponentiated Odd Exponential Half-Logistic-Weibull Logarithmic (EOEHL-WL) Distribution

The cdf, pdf and hrf of the EOEHL-WL distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{\log\left(1 - \theta\left(1 - \left[\frac{1-\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1+\exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)}{\log(1 - \theta)},$$

$$f_{X_{(1)}}(x) = \frac{2\theta\alpha\lambda\gamma x^{\gamma-1} e^{-x^\gamma} \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{e^{-2x^\gamma} \left[1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha+1}} \\ \times \left[1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha-1} \\ \times \left(1 - \theta\left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)^{-1} \\ \times \frac{1}{-\log(1-\theta)},$$

and

$$h_{X_{(1)}}(x) = \frac{2\theta\alpha\lambda\gamma x^{\gamma-1} e^{-x^\gamma} \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{e^{-2x^\gamma} \left[1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha+1}} \\ \times \left[1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)\right]^{\alpha-1} \\ \times \left(1 - \theta\left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)^{-1} \\ \times \frac{1}{-\log(1-\theta)} \\ \times \left(\frac{\log\left(1 - \theta\left(1 - \left[\frac{1 - \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}{1 + \exp\left(-\lambda\left[\frac{1-e^{-x^\gamma}}{e^{-x^\gamma}}\right]\right)}\right]^\alpha\right)\right)}{\log(1-\theta)}\right)^{-1},$$

respectively, for $\alpha, \lambda, \theta, \gamma > 0$.

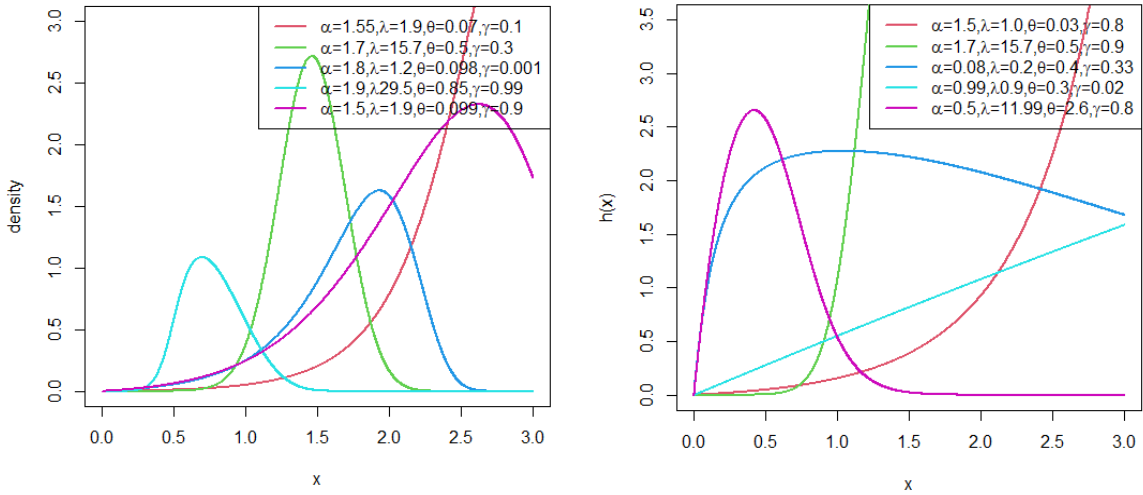


Figure 4.5: Plots of EOEHL-WL density and hazard functions

Figure 4.5 shows the plots of the pdfs and hrfs of the EOEHL-WL distribution. It shows that the density has different shapes such as left or right-skewed, J and almost symmetric. The EOEHL-WL model can fit data sets that have increasing, unimodal, upside-down bathtub and J hazard rate functions.

Exponentiated Odd Exponential Half-Logistic-Weibull Geometric (EOEHL-WG) Distribution

The cdf, pdf and hrf of the EOEHL-WG distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{(1 - \theta) \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)} \right]^\alpha \right) (1 - \theta)}{1 - \theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)} \right]^\alpha \right)},$$

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda\gamma x^{\gamma-1} e^{-x^\gamma} \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)}{e^{-2x^\gamma} \left[1 + \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right) \right]^{\alpha+1}} \\ &\times \left[1 - \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right) \right]^{\alpha-1} \\ &\times \frac{(1 - \theta)}{\left(1 - \theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)} \right]^\alpha \right) \right)^2}, \end{aligned}$$

and

$$\begin{aligned} h_{X_{(1)}}(x) &= \frac{2\theta\alpha\lambda\gamma x^{\gamma-1} e^{-x^\gamma} \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)}{e^{-2x^\gamma} \left[1 + \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right) \right]^{\alpha+1}} \\ &\times \left[1 - \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right) \right]^{\alpha-1} \\ &\times \frac{1}{1 - \theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)} \right]^\alpha \right)} \\ &\times \frac{1}{\left(1 - \left[\frac{1 - \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)}{1 + \exp \left(-\lambda \left[\frac{1 - e^{-x^\gamma}}{e^{-x^\gamma}} \right] \right)} \right]^\alpha \right)}, \end{aligned}$$

respectively, for $\alpha, \lambda, \theta, \gamma > 0$.

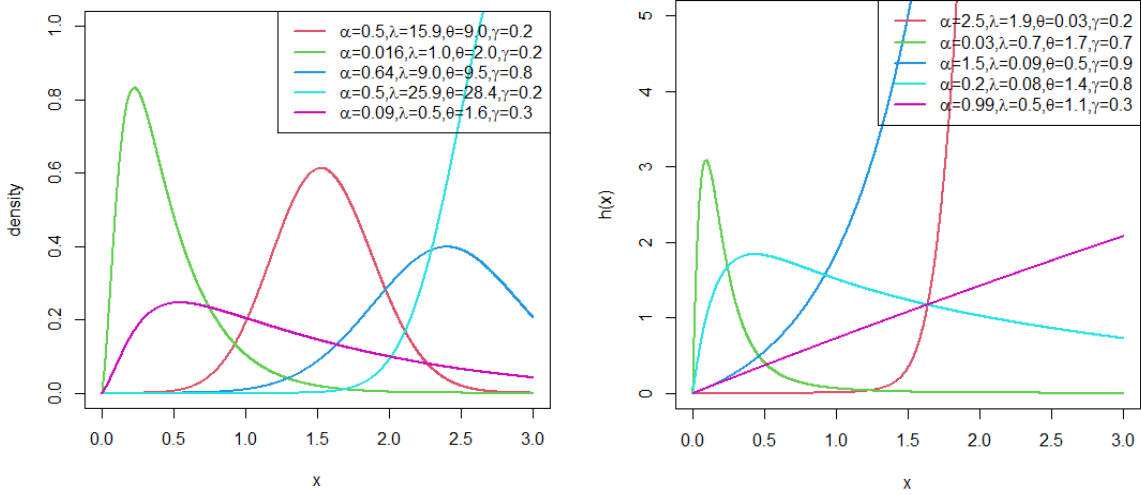


Figure 4.6: Plots of EOEHL-WG density and hazard functions

Plots of the pdfs and hrfs for the EOEHL-WG distribution are shown in Figure 4.6. The pdf exhibit different shapes including almost symmetric, left-skewed, right-skewed, and J shapes. The hrf shows increasing, upside-down bathtub, unimodal and J shapes.

4.4 Statistical Properties

In this section, we present the quantile function, moments, mean deviations, Lorenz and Bonferroni curves, order statistics and Rényi entropy for the EOEHL-GPS class of distributions.

4.4.1 Quantile Function

The quantile function of the EOEHL-GPS class of distributions is obtained by inverting the cdf defined by equation (4.4), that is $F_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) = u$, $0 \leq u \leq 1$. Note that

$$1 - \frac{C\left(\theta\left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)}\right]^\alpha\right)\right)}{C(\theta)} = u,$$

so that

$$C\left(\theta\left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)}\right]^\alpha\right)\right) = C(\theta)(1 - u).$$

This is equivalent to

$$\left[1 - \frac{C^{-1}C(\theta)(1-u)}{\theta}\right]^{\frac{1}{\alpha}} = \left[\frac{1 - \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)}\right].$$

Simplifying the equation, we get

$$\frac{1 - \left[1 - \frac{C^{-1}C(\theta)(1-u)}{\theta}\right]^{\frac{1}{\alpha}}}{1 + \left[1 - \frac{C^{-1}C(\theta)(1-u)}{\theta}\right]^{\frac{1}{\alpha}}} = \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right).$$

Hence

$$G(x;\xi) = \left[\frac{\ln \left[1 - \left[1 - \frac{C^{-1}C(\theta)(1-u)}{\theta} \right]^{\frac{1}{\alpha}} \right]}{\ln \left[1 + \left[1 - \frac{C^{-1}C(\theta)(1-u)}{\theta} \right]^{\frac{1}{\alpha}} \right] - \lambda} \right].$$

Therefore, we obtain the quantile values from the EOEHL-GPS class of distributions by solving the non-linear equation

$$X(u) = G^{-1} \left[\frac{\ln \left[1 - \left[1 - \frac{C^{-1}C(\theta)(1-u)}{\theta} \right]^{\frac{1}{\alpha}} \right]}{\ln \left[1 + \left[1 - \frac{C^{-1}C(\theta)(1-u)}{\theta} \right]^{\frac{1}{\alpha}} \right] - \lambda} \right]$$

using softwares such as, SAS or MATLAB. Quantiles for selected parameter values for the EOEHL-LLoGP distribution are shown in Table 4.2.

Table 4.2: Table of Quantiles for Selected Parameters of the EOEHL-LLoGP Distribution

	$(\alpha, \lambda, \theta, c)$				
u	(0.4,1.7,0.5,1.2)	(1.1,0.2,0.4,1.8)	(1.3,0.4,1.2,1.5)	(0.9,1.4,0.6,0.9)	(2,0.1,1.4,0.7)
0.1	0.0060	1.0296	0.6951	0.0627	9.4634
0.2	0.0265	1.4820	1.0179	0.1528	16.5030
0.3	0.0644	1.8519	1.2920	0.2626	23.6071
0.4	0.1230	2.1904	1.5522	0.3934	31.3407
0.5	0.2061	2.5216	1.8166	0.5503	40.2408
0.6	0.3195	2.8650	2.1019	0.7433	51.0902
0.7	0.4734	3.2448	2.4311	0.9921	65.3317
0.8	0.6908	3.7050	2.8499	1.3428	86.3064
0.9	1.0457	4.3722	3.4943	1.9448	125.3274

4.4.2 Moments and Generating Function

If we let $X \sim \text{EOEHL-GPS}(\alpha, \lambda, \theta, \xi)$ and $Y \sim \text{Exp-G}(l + m)$, then the r^{th} moment of the EOEHL-GPS class of distributions can be obtained from equation (4.6) as

$$E[X^r] = \int_{-\infty}^{\infty} x^r \cdot f_{\text{EOEHL-GPS}}(x; \alpha, \lambda, \theta, \xi) dx = \sum_{l,m=0}^{\infty} \eta_{l,m} E[Y^r],$$

where $E[Y^r]$ denotes the r^{th} moment of the Exp-G distribution with power parameter $(l + m)$ and $\eta_{l,m}$ is as defined in equation (4.6). The incomplete moments can be obtained as

$$I_X(t) = \int_0^t x^r \cdot f_{\text{EOEHL-GPS}}(x; \alpha, \lambda, \theta, \xi) dx = \sum_{l,m=0}^{\infty} \eta_{l,m} I_{l+m}(t),$$

where $I_{l+m}(t) = \int_0^t x^r \cdot g_{l+m}(x; \xi) dx$. Furthermore, the moment generating function (mgf) of X is given by

$$M_X(t) = \sum_{l,m=0}^{\infty} \eta_{l,m} M_Y(t),$$

where $M_Y(t)$ is the mgf of the Exp-G distribution with power parameter $(l + m)$ and $\eta_{l,m}$ is given by equation (4.6).

A table of moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) for selected parameter values of the EOEHL-LLoGP distribution are given in Table 4.3.

Table 4.3: Table of Moments for Selected Parameters of the EOEHL-LLoGP Distribution

	$(\alpha, \lambda, \theta, c)$				
	(1.3,0.6,1.6,0.4)	(0.3,0.1,1,1.7)	(0.5,0.3,1.2,1.5)	(0.4,0.9,1.2,1.5)	(0.3,1.1,0.9,0.7)
E(X)	0.1059	0.1609	0.2058	0.2423	0.1066
E(X ²)	0.0615	0.0934	0.1253	0.1350	0.0538
E(X ³)	0.0432	0.0656	0.0897	0.0923	0.0356
E(X ⁴)	0.0332	0.0505	0.0697	0.0698	0.0266
E(X ⁵)	0.0270	0.0410	0.0569	0.0560	0.0212
SD	0.2242	0.2598	0.2881	0.2763	0.2060
CV	2.1168	1.6150	1.3999	1.1405	1.9321
CS	2.3079	1.6450	1.2437	1.0726	2.3872
CK	7.4001	4.5617	3.2441	3.0069	8.1411

3D plots of skewness and kurtosis for the EOEHL-LLoGP distribution are given in Figures 3.4 and 3.5. We observe that

- When we fix the parameters for λ, θ and c , the skewness and kurtosis of EOEHL-LLoGP increases as α increases.
- When we fix the parameters for α, θ and c , the skewness and kurtosis of EOEHL-LLoGP increases as λ increases.
- When we fix the parameters for α, λ and c , the skewness and kurtosis of EOEHL-LLoGP decreases as θ increases.
- When we fix the parameters for α, λ and θ , the skewness and kurtosis of EOEHL-LLoGP increases as c increases.

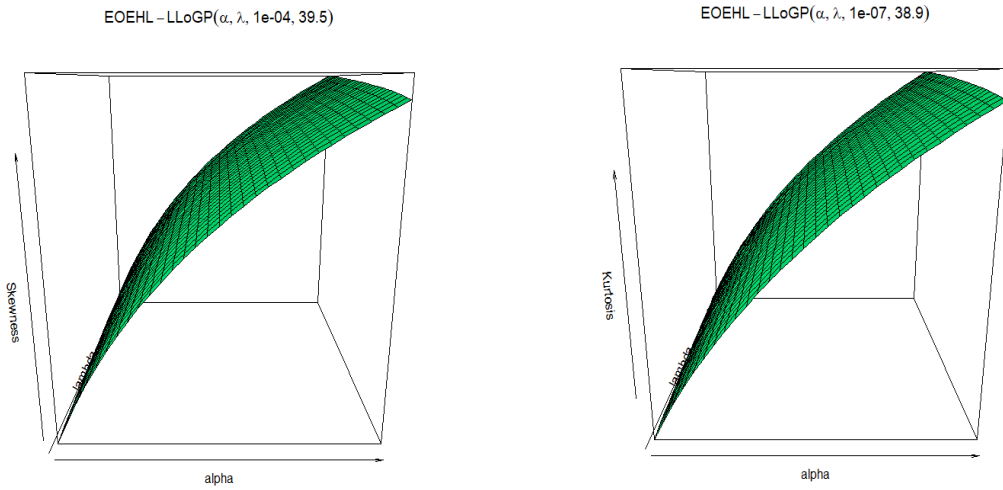


Figure 4.7: Plots of skewness and kurtosis for the EOEHL-LLoGP distribution

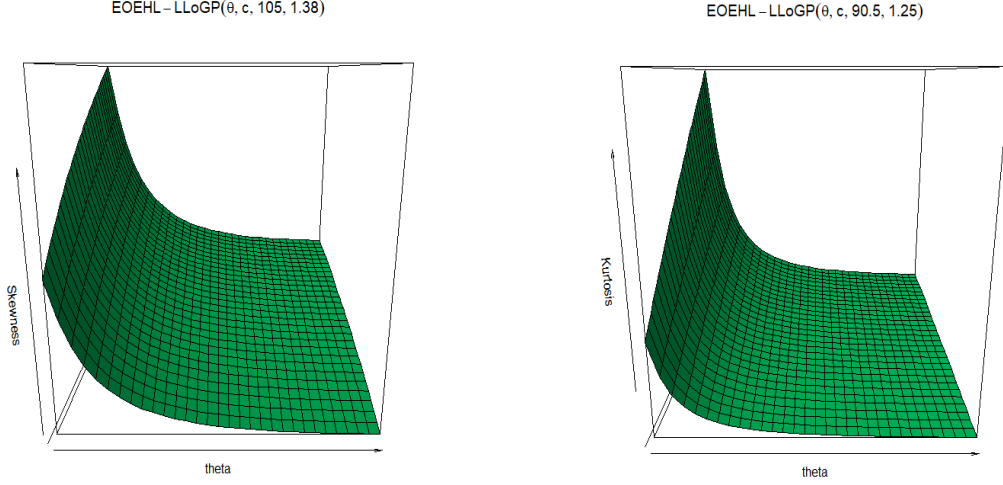


Figure 4.8: Plots of skewness and kurtosis for the EOEHL-LLoGP distribution

4.4.3 Mean Deviation, Lorenz and Bonferroni Curves

The mean deviation about the mean $D(\mu)$ and the mean deviation about the median $D(M)$, are defined as

$$D(\mu) = \int_0^\infty |x - \mu| \cdot f_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) dx,$$

and

$$D(M) = \int_0^\infty |x - M| \cdot f_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) dx,$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(x)$. Note that

$$\begin{aligned} D(\mu) &= 2\mu F_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) - 2\mu + 2 \int_\mu^\infty x \cdot f_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) dx \\ &= 2\mu F_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) - 2\mu + 2 \sum_{l,m=0}^\infty \eta_{l,m} I_{l+m}^*(t), \end{aligned}$$

and

$$D(M) = \int_M^\infty x \cdot f_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) dx - \mu = 2 \sum_{l,m=0}^\infty \eta_{l,m} I_{l+m}^*(t) - \mu.$$

Lorenz and Bonferroni curves are given by

$$L(p) = \frac{1}{\mu} \int_0^t x \cdot f_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) dx = \frac{1}{\mu} \sum_{l,m=0}^{\infty} \eta_{l,m} I_{l+m}^*(t),$$

and

$$B(p) = \frac{1}{p\mu} \int_0^t x \cdot f_{EOEHL-GPS}(x; \alpha, \lambda, \theta, \xi) dx = \frac{1}{p\mu} \sum_{l,m=0}^{\infty} \eta_{l,m} I_{l+m}^*(t),$$

respectively, where $I_{l+m}^*(t) = \int_0^t x \cdot g_{l+m}(x; \xi) dx$ is the first incomplete moment of the Exp-G distribution and $\eta_{l,m}$ is given in equation (4.6).

4.4.4 Order Statistics and Rényi Entropy

In this section, we present the distribution of the i^{th} order statistic and Rényi entropy for the EOEHL-GPS class of distributions.

Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (iid) random variables from the EOEHL-GPS class of distributions and $X_{1:n} < X_{2:n} < \dots < X_{n:n}$ denote the corresponding order statistics. The pdf of the i^{th} order statistic, $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f_{X_{(1)}}(x) \sum_{j=0}^{\infty} \binom{n-i}{j} [F_{X_{(1)}}(x)]^{j+i-1}.$$

Substituting the pdf and cdf of the EOEHL-GPS class of distributions, we have

$$\begin{aligned} f_{X_{(1)}}(x) [F_{X_{(1)}}(x)]^{j+i-1} &= \sum_{n=1}^{\infty} \frac{n a_n \theta^n}{C(\theta)} \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)} \right]^{\alpha} \right)^{n-1} \\ &\times \frac{2\alpha \lambda g(x; \xi) \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right) \left[1 - \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right) \right]^{\alpha-1}}{\bar{G}^2(x; \xi) \left[1 + \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right) \right]^{\alpha+1}} \\ &\times \left[1 - \frac{C\left(\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)} \right)^{\alpha} \right)}{C(\theta)} \right]^{j+i-1}. \end{aligned}$$

If we let $Y = \left[\frac{1 - \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x;\xi)}{G(x;\xi)}\right)} \right]^{\alpha}$ and using the generalized binomial expansion

$$\begin{aligned} \left[1 - \frac{C(\theta(1-Y))}{C(\theta)}\right]^{j+i-1} &= \sum_{k=0}^{\infty} (-1)^k \binom{j+i-1}{k} \\ &\times \left[\frac{C(\theta(1-Y))}{C(\theta)}\right]^k \end{aligned}$$

and the power series raised to a positive integer result by Gradshetyn and Ryzhik [37] we get

$$\begin{aligned} \left[\frac{C(\theta(1-Y))}{C(\theta)}\right]^k &= \frac{[\sum_{w=0}^{\infty} d_{w,k}(\theta(1-Y))^w]^k}{C^k(\theta)} \\ &= \sum_{w=0}^{\infty} \frac{d_{w,k}\theta^w}{C^k(\theta)}(1-Y)^w, \end{aligned}$$

where $d_{w,k} = (wb_0)^{-1} \sum_{l=1}^w [k(l+1) - w] b_l d_{w-l,k}$ and $d_{0,k} = b_0^k$.

Hence

$$\begin{aligned} f_{X_{(1)}}(x)[F_{X_{(1)}}(x)]^{j+i-1} &= \sum_{k,w=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k \binom{j+i-1}{k} \frac{na_n\theta^{n+w}}{C^{k+1}(\theta)} d_{w,k} (1-Y)^{w+n-1} \\ &\times \frac{2\alpha\lambda g(x;\xi) \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right) [1 - \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)]^{\alpha-1}}{\bar{G}^2(x;\xi) [1 + \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)]^{\alpha+1}}, \end{aligned}$$

Substituting back the value of Y and using the generalized series expansions

$$\begin{aligned} \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)}\right]^{\alpha}\right)^{w+n-1} &= \sum_{m=0}^{\infty} (-1)^m \binom{w+n-1}{m} \\ &\times \left[\frac{1 - \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)}\right]^{\alpha m}, \end{aligned}$$

$$\left[1 + \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)\right]^{-(\alpha(m+1)+1)} = \sum_{p=0}^{\infty} \binom{-(\alpha(m+1)+1)}{p} \exp\left(-\lambda p \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right),$$

$$\left[1 - \exp\left(-\lambda \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)\right]^{\alpha(m+1)-1} = \sum_{q=0}^{\infty} \binom{-(\alpha(m+1)+1)}{q} \exp\left(-\lambda q \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right),$$

$$\exp \left(-\lambda(p+q+1) \frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) = \sum_{r=0}^{\infty} \frac{(-1)^r \lambda^r (p+q+1)^r}{r!} \left[\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right]^r$$

and

$$[\bar{G}(x; \xi)]^{-(r+2)} = [1 - G(x; \xi)]^{-(r+2)} = \sum_{s=0}^{\infty} \binom{-(r+2)}{s} G^s(x; \xi),$$

gives

$$\begin{aligned} f_{i:n} &= \frac{n!}{(i-1)!(n-i)!} \sum_{j,k,m,p,q,r,s,w=0}^{\infty} \sum_{n=1}^{\infty} \binom{n-i}{j} \binom{j+i-1}{k} \\ &\times \binom{-(\alpha(m+1)+1)}{p} \binom{-(\alpha(m+1)+1)}{q} \binom{-(r+2)}{s} \\ &\times (-1)^{j+k+q+r} \frac{2\alpha\lambda^{r+1}(p+q+1)^r n a_n \theta^{n+w}}{C^{k+1}(\theta)(r+s+1)} d_{w,k} \\ &\times (r+s+1) g(x; \xi) G^{r+s}(x; \xi) \\ &= \sum_{r,s=0}^{\infty} U_{r,s} g_{r+s}(x; \xi), \end{aligned}$$

where $g_{r+s}(x; \xi) = (r+s+1)g(x; \xi)G^{r+s}(x; \xi)$ is the Exp-G distribution with power parameter $(r+s)$ and

$$\begin{aligned} U_{r,s} &= \frac{n!}{(i-1)!(n-i)!} \sum_{j,k,m,p,q=0}^{\infty} \sum_{j=0}^{\infty} \binom{n-i}{j} \binom{j+i-1}{k} \\ &\times \binom{-(\alpha(m+1)+1)}{p} \binom{-(\alpha(m+1)+1)}{q} \binom{-(r+2)}{s} \\ &\times (-1)^{j+k+q+r} \frac{2\alpha\lambda^{r+1}(p+q+1)^r n a_n \theta^{n+w}}{C^{k+1}(\theta)(r+s+1)} d_{w,k}. \end{aligned}$$

It then follows that the distribution of the i^{th} order statistic from the EOEHL-GPS class of distributions can be obtained from the distribution of the i^{th} order statistic of the Exp-G distribution with power parameter $(r+s)$.

4.4.5 Rényi Entropy

Entropy is a measure of variation of uncertainty for a random variable X . There are two well known measures of entropy, namely Shannon entropy [71] and Rényi entropy [66]. Rényi entropy is defined as

$$I_R(v) = \frac{1}{1-v} \log \int_0^\infty [f_{X_{(1)}}(x)]^v dx,$$

where $v > 0$ and $v \neq 1$. $[f_{X_{(1)}}(x)]^v$ can be written as

$$[f_{X_{(1)}}(x)]^v = \frac{(2\theta\alpha\lambda)^v g^v(x; \xi) \exp\left(-\lambda v \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) [1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)]^{v(\alpha-1)}}{\bar{G}^{2v}(x; \xi) [1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)]^{v(\alpha+1)}} \\ \times \left[\frac{C' \left(\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha \right) \right)}{C(\theta)} \right]^v.$$

Applying the power series raised to a positive integer (see Gradshyten and Ryzhik [37] for details), we get

$$\left[\frac{C' \left(\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha \right) \right)}{C(\theta)} \right]^v = \sum_{w=0}^{\infty} \frac{d_{w,v} \theta^w}{C^v(\theta)} \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha \right)^w$$

where $d_{w,v} = (wb_0)^{-1} \sum_{l=1}^w [v(l+1) - w] b_l d_{w-l,v}$ and $d_{0,v} = b_0^v$.

Using the generalized binomial expansions

$$\left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha \right)^w = \sum_{j=0}^{\infty} (-1)^j \binom{w}{j} \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^{\alpha j},$$

$$\left[1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) \right]^{-(\alpha(j+v)+v)} = \sum_{k=0}^{\infty} \binom{-(\alpha(j+v)+v)}{k} \exp\left(-\lambda k \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right),$$

$$\left[1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) \right]^{\alpha(j+v)-v} = \sum_{m=0}^{\infty} \binom{\alpha(j+v)-v}{m} \exp\left(-\lambda m \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right),$$

$$\exp\left(-\lambda(k+m+v) \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right) = \sum_{p=0}^{\infty} \frac{(-1)^p \lambda^p (k+m+v)^p}{p!} \left[\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right]^p$$

and

$$[\bar{G}(x; \xi)]^{-(p+2v)} = [1 - G(x; \xi)]^{-(p+2v)} = \sum_{q=0}^{\infty} \binom{-(p+2v)}{q} G^q(x; \xi),$$

we obtain

$$\begin{aligned} [f_{X_{(1)}}(x)]^v &= \sum_{j,k,m,p,q,w=0}^{\infty} \frac{(2\alpha)^v \theta^{v+w} (-1)^{j+m+p} \lambda^{p+v} (k+m+v)^p d_{w,v}}{C^v(\theta) p!} \\ &\times \binom{w}{j} \binom{-(\alpha(j+v)+v)}{k} \binom{\alpha(j+v)-v}{m} \binom{-(p+2v)}{q} \\ &\times g^v(x; \xi) G^{p+q}(x; \xi). \end{aligned}$$

Consequently, Rényi entropy for the EOEHL-GPS class of distributions can be written as

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left(\sum_{j,k,m,p,q,w=0}^{\infty} \frac{(2\alpha)^v \theta^{v+w} (-1)^{j+m+p} \lambda^{p+v} (k+m+v)^p d_{w,v}}{C^v(\theta) p!} \right. \\ &\times \binom{w}{j} \binom{-(\alpha(j+v)+v)}{k} \binom{\alpha(j+v)-v}{m} \binom{-(p+2v)}{q} \\ &\times \left. \int_0^{\infty} g^v(x; \xi) G^{p+q}(x; \xi) dx \right) \\ &= \frac{1}{1-v} \log \left(\sum_{j,k,m,p,q,w=0}^{\infty} \frac{(-1)^{j+m+p} \lambda^{p+v} (k+m+v)^p d_{w,v}}{C^v(\theta) p! \left[1 + \frac{p+q}{v}\right]^v} \right. \\ &\times \binom{w}{j} \binom{-(\alpha(j+v)+v)}{k} \binom{\alpha(j+v)-v}{m} \binom{-(p+2v)}{q} \\ &\times \left. \int_0^{\infty} \left(\left[1 + \frac{p+q}{v}\right] g(x; \xi) G^{\frac{p+q}{v}}(x; \xi) \right)^v dx \right) \\ &= \log \left(\sum_{p,q=0}^{\infty} z_{p,q}^* e^{1-v(I_{REG})} \right), \end{aligned}$$

where

$$I_{REG} = \frac{1}{1-v} \int_0^{\infty} \left(\left[1 + \frac{p+q}{v}\right] g(x; \xi) G^{\frac{p+q}{v}}(x; \xi) \right)^v dx$$

is the Rényi entropy of Exp-G distribution with power parameter $\frac{p+q}{v}$ and

$$\begin{aligned}
z_{p,q}^* &= \log \left(\sum_{j,k,m,p,q,w=0}^{\infty} \frac{(-1)^{j+m+p} \lambda^{p+v} (k+m+v)^p d_{w,v}}{C^v(\theta) p! \left[1 + \frac{p+q}{v}\right]^v} \right. \\
&\quad \times \binom{w}{j} \binom{-(\alpha(j+v)+v)}{k} \binom{\alpha(j+v)-v}{m} \binom{-(p+2v)}{q} \Big).
\end{aligned}$$

4.5 Maximum Likelihood Estimation

In this section, the maximum likelihood estimation technique is employed to estimate the model parameters. Let $X_i \sim \text{EOEHL-GPS}(\alpha, \lambda, \theta, \xi)$ and $\Delta = (\alpha, \lambda, \theta, \xi)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is

$$\begin{aligned}
\ell(\Delta) &= n \log(2\alpha\lambda\theta) + \sum_{i=1}^n \log[g(x_i; \xi)] - \lambda \sum_{i=1}^n \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \\
&\quad + (\alpha - 1) \sum_{i=1}^n \log \left[1 - \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right] - 2 \sum_{i=1}^n \log[\bar{G}(x_i; \xi)] \\
&\quad - (\alpha + 1) \sum_{i=1}^n \log \left[1 + \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right] \\
&\quad + \sum_{i=1}^n \log \left[C' \left(\theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)} \right]^\alpha \right) \right) \right] - n \log[C(\theta)].
\end{aligned}$$

The elements of the score vector $U = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \xi_k} \right)$ are given by

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \log \left[1 - \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right] - \sum_{i=1}^n \log \left[1 + \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right] \\
&\quad + \sum_{i=1}^n \frac{\theta C'' \left[\theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)} \right)^\alpha \right] \right]}{C' \left[\theta \left(1 - \left[\frac{1 - \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)} \right)^\alpha \right] \right]} \left[\frac{1 - \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)} \right]^\alpha \\
&\quad \times \log \left[\frac{1 - \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)}{1 + \exp \left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)} \right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} + (\alpha - 1) \sum_{i=1}^n \frac{\exp\left(\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right) \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}}{\left[1 - \exp\left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right)\right]} \\
&- (\alpha + 1) \sum_{i=1}^n \frac{\exp\left(\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right) \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}}{\left[1 + \exp\left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right)\right]} + \sum_{i=1}^n \frac{\alpha \theta C'' \left[\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right)^\alpha \right]}{C' \left[\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right)^\alpha \right]} \\
&\times \frac{\exp\left(\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right) \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^{\alpha-1}}{\left[1 - \exp\left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right)\right]^2}, \\
\\
\frac{\partial \ell}{\partial \theta} &= \frac{n}{\theta} - \frac{n C'(\theta)}{C(\theta)} + \sum_{i=1}^n \frac{C'' \left[\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right)^\alpha \right]}{C' \left[\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right)^\alpha \right]} \\
&\times \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right]^\alpha \right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell}{\partial \xi_k} &= \frac{\frac{\partial g(x_i; \xi)}{\partial \xi_k}}{g(x_i; \xi)} - \lambda \sum_{i=1}^n \frac{\bar{G}(x_i; \xi) \frac{\partial G(x_i; \xi)}{\partial \xi_k} - G(x_i; \xi) \frac{\partial \bar{G}(x_i; \xi)}{\partial \xi_k}}{\bar{G}^2(x_i; \xi)} \\
&+ (\alpha - 1) \sum_{i=1}^n \frac{\frac{\partial}{\partial \xi_k} \left(1 - \exp\left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right)\right)}{\left[1 - \exp\left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right)\right]} - 2 \sum_{i=1}^n \frac{\frac{\partial \bar{G}(x_i; \xi)}{\partial \xi_k}}{\bar{G}(x_i; \xi)} \\
&- (\alpha + 1) \sum_{i=1}^n \frac{\frac{\partial}{\partial \xi_k} \left(1 + \exp\left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right)\right)}{\left[1 + \exp\left(-\lambda \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}\right)\right]} \\
&+ \sum_{i=1}^n \frac{1}{C' \left[\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right)^\alpha \right]} \frac{\partial \left(C' \left[\theta \left(1 - \left[\frac{1 - \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)}{1 + \exp\left(-\lambda \frac{G(x; \xi)}{\bar{G}(x; \xi)}\right)} \right)^\alpha \right] \right)}{\partial \xi_k}.
\end{aligned}$$

respectively. The equations obtained by setting the score vector to zero are not in closed form and the parameter estimates can only be obtained through iterative methods using R, MATLAB and SAS software.

4.6 Simulation Study

In this section, we conducted a simulation study to assess the performance of the maximum likelihood estimates of the EOEHL-LLoGP distribution. We simulated for the sample sizes $n= 25, 50, 100, 200, 400$, and 800 for $N=1000$ for each sample. Simulation results are shown in Table 4.4. We estimate the mean, root mean square error (RMSE), and average bias. The average bias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by

$$Bias(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta), \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. From the simulation results, it is seen that as the sample size increases, the mean values approximate the true parameter values and also the RMSE and bias decay towards zero. Therefore, we can conclude that our model produces consistent EOEHL-LLoGP model parameter estimates.

4.7 Applications

In this section, two real data examples are presented to demonstrate the versatility of the EOEHL-GPS class of distribution in data fitting. The new proposed model is compared to other competing models. Model performance is assessed using the goodness-of-fit statistics; -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer-von Mises (W^*) and Andersen-Darling (A^*) as described by Chen and Balakrishnan [21], Kolmogorov-Smirnov (K-S) statistic and its p-value. The model with the smallest values of these statistics and a bigger p-value for the K-S statistic is the better model.

We used the R software to estimate the model parameters. The model parameter estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the two data sets considered are shown in Tables 4.5 and 4.6. Plots of the fitted densities, the histogram of the data and probability plots (as described by Chambers et al. [20]) are provided to illustrate how well our model fits the selected data sets.

Table 4.4: Monte Carlo Simulation Results for EOEHL-LLoGP Distribution

n	$\alpha = 0.5, \lambda = 0.5, \theta = 1.5, c = 1.7$			$\alpha = 0.5, \lambda = 0.7, \theta = 0.5, c = 0.8$		
	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	4.7011	28.9538	0.9385	1.0899	0.4385
	50	0.5620	0.4238	0.5421	0.3798	0.0421
	100	0.4732	0.2803	0.4552	0.2116	-0.0448
	200	0.4440	0.1793	0.4657	0.1565	-0.0343
	400	0.4653	0.1395	0.4503	0.1187	-0.0497
	800	0.4779	0.1082	0.4998	0.0402	-0.0002
λ	25	0.9938	1.1827	0.8811	0.7908	0.1811
	50	0.5357	0.5316	0.5105	0.5342	-0.1895
	100	0.4567	0.4400	0.4334	0.4492	-0.2666
	200	0.4349	0.3629	0.4599	0.3863	-0.2401
	400	0.4780	0.3452	0.4277	0.3855	-0.2723
	800	0.4913	0.2847	0.5199	0.1921	-0.1801
θ	25	1.8789	1.7174	1.3505	1.2702	0.8505
	50	1.8088	1.2205	1.5813	1.5508	1.0813
	100	1.7903	1.2047	1.4566	1.4098	0.9566
	200	1.6668	1.0190	1.3449	1.1980	0.8449
	400	1.6111	0.9058	1.3997	1.2076	0.8997
	800	1.5581	0.7068	1.1217	0.6347	0.6217
c	25	2.1263	1.6898	1.0440	0.8228	0.2440
	50	2.3231	1.4345	1.1860	0.7095	0.3860
	100	2.3155	1.2319	1.1891	0.6298	0.3891
	200	2.1910	0.9687	1.1016	0.5322	0.3016
	400	2.0085	0.7545	1.0771	0.4273	0.2771
	800	1.8798	0.5388	0.8864	0.0993	0.0864
n	$\alpha = 0.6, \lambda = 0.6, \theta = 0.5, c = 1.2$			$\alpha = 0.5, \lambda = 0.5, \theta = 0.5, c = 1.7$		
	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	4.6517	38.5217	1.3138	5.8991	0.8138
	50	0.6536	0.5040	0.5358	0.4226	0.0358
	100	0.5191	0.3112	0.4420	0.2052	-0.0580
	200	0.4976	0.1909	0.4354	0.1471	-0.0646
	400	0.5189	0.1508	0.4479	0.1138	-0.0521
	800	0.5496	0.1117	0.4664	0.0857	-0.0336
λ	25	0.8341	0.9913	0.6649	0.7983	0.1649
	50	0.4339	0.5163	0.3787	0.4532	-0.1213
	100	0.3589	0.4351	0.3189	0.3407	-0.1811
	200	0.3506	0.3582	0.3246	0.2885	-0.1754
	400	0.3894	0.3094	0.3516	0.2463	-0.1484
	800	0.4612	0.2489	0.4036	0.1988	-0.0964
θ	25	1.5053	1.4190	1.4756	1.4117	0.9756
	50	1.7425	1.7265	1.4931	1.4292	0.9931
	100	1.4299	1.3939	1.2721	1.2400	0.7721
	200	1.2759	1.2327	1.0865	0.9639	0.5865
	400	1.1310	0.9778	0.9642	0.7674	0.4642
	800	0.9104	0.7231	0.7900	0.5623	0.2900
c	25	1.5754	1.1564	2.3751	1.7734	0.6751
	50	1.8000	1.0713	2.4410	1.3282	0.7410
	100	1.8580	1.0291	2.4367	1.2053	0.7367
	200	1.7011	0.7858	2.2732	0.9525	0.5732
	400	1.5653	0.6074	2.0954	0.6750	0.3954
	800	1.4060	0.3832	1.9324	0.4542	0.2324

The EOEHL-LLoGP distribution was compared to the following non-nested models: Kumaraswamy odd Lindley-Log logistic (KOL-LLoG) by Chihepa et al. [24], beta odd Lindley-uniform (BOL-U) by Chihepa et al. [26], Kumaraswamy Weibull (KwW) by Cordeiro et al. [33], beta-Weibull (BW) by Cordeiro et al. [31], the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. [43], Topp-Leone-Weibull-Lomax (TL-WLx) by Jamal et al. [40], Exponentiated generalized exponential logarithmic (EGEL) by Oluyede et al. [65] and the odd Weibull-Topp-Leone-log logistic logarithmic (OW-TL-LLoL) by Oluyede et al. [64] distributions. The pdfs of the non-nested models are as follows:

$$f_{KOL-LLoG}(x; a, b, \lambda, c) = ab \left[\frac{\lambda^2}{(1+\lambda)} \frac{cx^{c-1}}{(1+x^c)^{-1}} \exp \left(-\lambda \frac{(1-(1+x^{vc})^{-1})}{(1+x^c)^{-1}} \right) \right] \\ \times \left[1 - \frac{\lambda + (1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}} \exp \left(-\lambda \frac{(1-(1+x^{vc})^{-1})}{(1+x^c)^{-1}} \right) \right]^{a-1} \\ \times \left(1 - \left[1 - \frac{\lambda + (1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}} \exp \left(-\lambda \frac{(1-(1+x^{vc})^{-1})}{(1+x^c)^{-1}} \right) \right]^a \right)^{b-1},$$

for $a, b, \lambda, c > 0$,

$$f_{BOL-U}(x; a, b, \lambda, \theta) = \frac{1}{B(a, b)} \left[1 - \frac{\lambda + (1-x/\theta)}{(1+\lambda)(1-x/\theta)} \exp \left\{ -\lambda \frac{x}{(\theta-x)} \right\} \right]^{a-1} \\ \times \left[\frac{\lambda + (1-x/\theta)}{(1+\lambda)(1-x/\theta)} \exp \left\{ -\lambda \frac{x}{(\theta-x)} \right\} \right]^{b-1} \\ \times \frac{\lambda^2}{(1+\lambda)} \frac{\theta^2}{(\theta-x)^3} \exp \left\{ -\lambda \frac{x}{(\theta-x)} \right\},$$

for $a, b, \lambda, \theta > 0$,

$$f_{KwW}(x; a, b, \alpha, \beta) = ab\alpha^\beta x^{\beta-1} e^{-(\alpha x)^\beta} (1 - e^{-(\alpha x)^\beta})^{a-1} (1 - (1 - e^{-(\alpha x)^\beta})^a)^{b-1},$$

for $a, b, \alpha, \beta > 0$,

$$f_{BW}(x; a, b, \alpha, \beta) = \frac{\beta\alpha^\beta}{B(a, b)} x^{\beta-1} e^{-b(\alpha x)^\beta} (1 - e^{-(\alpha x)^\beta})^{a-1},$$

for $a, b, \alpha, \beta > 0$,

$$f_{ELOLLW}(x; \alpha, \beta, \gamma, \theta, \lambda) = \frac{\alpha\theta^2\gamma\lambda^\gamma x^{\gamma-1} e^{-(\lambda x)^\gamma} (e^{-(\lambda x)^\gamma})^{\alpha\theta-1} (1 - e^{-(\lambda x)^\gamma})^{\alpha-1}}{(\theta + \beta)((1 - e^{-(\lambda x)^\gamma})^\alpha + e^{-\alpha(\lambda x)^\gamma})^{\theta-1}} \\ \times \left(1 - \beta \log \left[\frac{e^{-(\lambda x)^\gamma}}{(1 - e^{-(\lambda x)^\gamma})^\alpha + e^{-\alpha(\lambda x)^\gamma}} \right] \right),$$

for $\alpha, \beta, \gamma, \theta, \lambda > 0$, and

$$\begin{aligned} f_{TL-WLx}(x; a, b, \alpha, \theta) &= 2\theta\alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1} \\ &\times \exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right) \\ &\times \left[1 - \exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right)\right]^{\theta-1}, \end{aligned}$$

$$f_{EGEL}(x; p, \alpha, \beta, \lambda) = \frac{p\alpha\beta\lambda e^{-\alpha\lambda x}[1 - e^{-\alpha\lambda x}]^{\beta-1}}{\ln(1-p)[p(1 - e^{-\alpha\lambda x})^{\beta} - 1]},$$

for $p, \alpha, \beta, \lambda > 0$ and

$$\begin{aligned} f_{OW-TL-LLoGL}(x; \alpha, \lambda, \gamma, \theta) &= \frac{2\theta\gamma\alpha\lambda x^{\lambda-1}(1+x^{\lambda})^{-3}[1-(1+x^{\lambda})^{-2}]^{\gamma\alpha-1}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]^{\alpha+1}} \\ &\times \exp\left(-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]}\right]^{\alpha}\right) \\ &\times \frac{\left(1 - \theta\left(\exp\left(-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]}\right]^{\alpha}\right)\right)\right)^{-1}}{-\log(1-\theta)} \end{aligned}$$

for $\alpha, \lambda, \gamma, \theta > 0$ for $\alpha, \lambda, \gamma, \theta > 0$. For the ELLLOW distribution, we considered the case when $\alpha = 1$.

4.7.1 Cancer Patients Data

The first data set used presents the remission times (in months) of a random sample of 128 bladder cancer patients. The data was previously used by Lee and Wang [47]. The observations are as follows: 0.08, 4.98, 25.74, 3.7, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.5, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.4, 11.98, 6.54, 6.93, 6.94, 0.4, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26, 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.5, 2.02, 0.2, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26, 4.23, 17.12, 2.87, 18.1,

8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.9, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics are presented in Table 4.5.

Table 4.5: Parameter estimates and goodness-of-fit statistics for various models fitted for cancer patients data

Model	Estimates				Statistics							
	α	λ	θ	c	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P - value$
EOEHL-LLoGP	2.1526 (0.8450)	0.3148 (0.2372)	3.0730 (2.5424)	0.6589 (0.1758)	819.3	827.3	827.6	838.7	0.0201	0.1334	0.0357	0.9968
KOL-LLoG	a	b	λ	c	820.8	828.8	829.2	840.2	0.0389	0.2541	0.0430	0.9719
	3.1393 (2.9508)	1.4470 (3.0564)	0.8238 (0.3651)	0.5046 (0.3664)								
BOL-U	a	b	λ	θ	826.4	834.4	834.7	845.8	0.1151	0.6907996	0.0735	0.4940
	1.1988 (0.1388)	1.3360 (0.1576)	6.9105×10^5 (3.0371×10^{-7})	7.6215×10^6 (2.7537×10^{-8})								
KwW	a	b	α	β	821.1	829.1	829.5	840.5	0.0415	0.2732	0.0447	0.9603
	4.1177 (5.8351)	2.9413 (8.1437)	0.2159 (0.2491)	0.4589 (0.5152)								
BW	a	b	α	β	821.4	829.4	829.7	840.8	0.0436	0.2883	0.0450	0.9582
	2.7348 (1.6010)	0.9083 (1.5186)	0.32157 (0.4382)	0.66613 (0.2455)								
ELOLLW	a	b	α	θ	822.8	830.8	831.1	842.2	0.0665	0.4115	0.053392	0.8589
	423.99 (8.9215×10^{-9})	17.1480 (3.0710×10^{-5})	0.0549 (0.0139)	0.7252 (0.0465)								
TL-WLx	a	b	α	θ	820.8	828.8	829.1	840.2	0.0372	0.2469	0.0429	0.9728
	0.5080 (0.4034)	0.2070 (0.2299)	1.0718 (0.6875)	1.5290 (1.1200)								
EGEL	p	α	β	λ	826.5	834.5	834.8	845.9	0.1093	0.6572	0.0760	0.4501
	2.9670×10^{-8} (0.0176)	7.8848×10^{-3} (8.7441×10^{-4})	1.3063 (0.1629)	16.0600 (4.8688×10^{-7})								
OW-TL-LLoGL	α	λ	γ	θ	824.8	832.8	833.1	844.2	0.0858	0.5411	0.0603	0.7398
	1.3860 (0.4924)	0.3659 (0.1017)	6.9888 (2.2124)	0.0001 (0.1091)								

The estimated variance-covariance for the EOEHL-LLoGP model on cancer patients data is given by

$$\begin{bmatrix} 0.7141 & 0.1797 & -0.0803 & -0.1411 \\ 0.1797 & 0.0563 & -0.2712 & -0.0340 \\ -0.0803 & -0.2712 & 6.4637 & -0.0605 \\ -0.1411 & -0.0330 & -0.0605 & 0.0309 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [2.1526 \pm 1.6563]$, $\lambda \in [0.3148 \pm 0.4649]$, $\theta \in [0.6589 \pm 4.9830]$ and $c \in [3.5927 \pm 0.3446]$.

Results from Table 4.5 indicate that the EOEHL-LLoGP model performs better than the non-nested models because it has the smallest values for the goodness-of-fit statistics and bigger value for the K-S p-value. The fitted densities and probability plots shown in Figure 4.9 also show that the EOEHL-LLoGP model fit the data set better than the selected non-nested models.

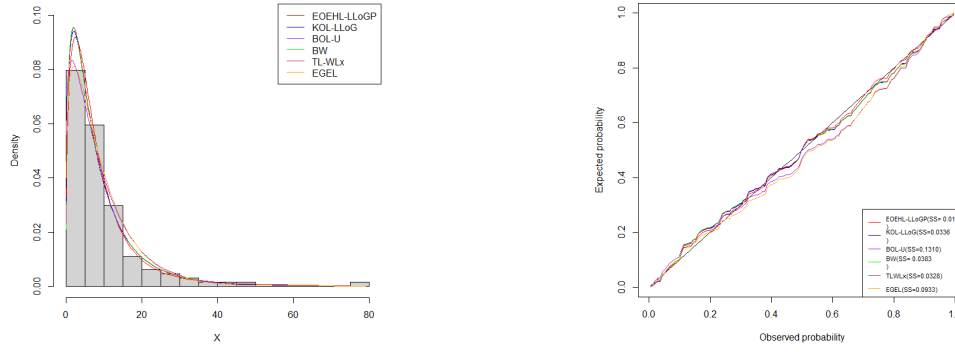


Figure 4.9: Fitted densities and probability plots for the cancer patients data

4.7.2 Failure Times Data

We then considered as the second example, the data set reported by Murthy et al. [57] representing failure times (per 1000h) of 50 components. The observations are as follows: 0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183, 0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904, 0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228, 3.931, 9.337, 0.086, 0.379, 1.600, 4.073, 10.940, 0.102, 0.381, 2.006, 4.393, 11.020, 0.103, 0.538, 2.054, 4.534, 13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116, 0.574, 3.058, 6.274, 15.08. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics are presented in Table 4.6.

Table 4.6: Parameter estimates and goodness-of-fit statistics for various models fitted for failure times data

Model	Estimates				Statistics							
	α	λ	θ	c	$-2 \log L$	AIC	AICC	BIC	W^*	A^*	KS	P-value
EOEHL-LLoGP	0.1523 (1.7126×10^{-2})	6.9658×10^{-5} (3.4050×10^{-5})	2.1319 (3.0772×10^{-4})	3.7337 (7.0761×10^{-4})	201.0	209.0	209.9	216.7	0.1196	0.7907	0.1267	0.3672
KOL-LLoG	a 4.9595 (9.4906)	b 0.1242 (0.1406)	λ 7.2484 (7.4713)	c 0.4817 (0.1214)	202.1	210.1	211.0	217.7	0.1459	0.9019	0.1293	0.3434
BOL-U	a 0.4802 (0.1010)	b 1.6606 (6.3842)	λ 3.9497 (14.9488)	θ 41.5651 (33.9511)	203.9	211.9	212.8	219.6	0.1390	0.9235	0.1505	0.1874
KwW	a 0.2111 (0.3767)	b 8.4430 (0.6916)	α 0.0112 (0.0042)	β 2.9490 (5.2812)	204.0	212.0	212.9	219.6	0.1445	0.9133	0.1289	0.3474
BW	a 0.7058 (1.3924)	b 0.7040 (1.4642)	α 0.4093 (1.5653)	β 0.8208 (1.0624)	204.7	212.7	213.6	220.3	0.1485	0.9418	0.1367	0.2812
ELoLLW	a 1.1051 (3.0861)	b 0.0423 (0.0296)	λ 5.0896 (0.6698)	θ 0.6522 (0.0862)	204.6	212.6	213.6	220.3	0.1512	0.9494	0.1284	0.3513
TL-WLx	a 0.3186 (0.0994)	b 0.4985 (0.3619)	α 5.4403 (4.4655)	β 0.0943 (0.0799)	202.4	210.4	211.3	218.0	0.1233	0.8330	0.1446	0.2243
EGEL	p 6.9963×10^{-8} (0.0456)	α 0.3229 (66.8740)	β 0.5368 (0.0902)	λ 0.6011 (124.4800)	204.7	212.7	213.6	220.4	0.1471	0.9418	0.1429	0.2353
OW-TL-LLoGL	α 0.2999 (0.0898)	λ 0.9479 (0.2491)	γ 3.1083 (0.5405)	θ 1.4661×10^{-8} (0.0191)	206.2	214.2	215.1	221.8	0.1246	0.7833	0.2114	0.0194

The estimated variance-covariance for the EOEHL-LLoGP model on failure

times data is given by

$$\begin{bmatrix} 2.9332 \times 10^{-4} & 1.8308 \times 10^{-7} & -5.2702 \times 10^{-6} & 1.2119 \times 10^{-5} \\ 1.8308 \times 10^{-7} & 1.1594 \times 10^{-9} & -3.2894 \times 10^{-9} & 7.5643 \times 10^{-9} \\ 1.8308 \times 10^{-7} & 1.1594 \times 10^{-9} & -3.2894 \times 10^{-9} & 7.5643 \times 10^{-9} \\ 1.2119 \times 10^{-5} & 7.5643 \times 10^{-9} & -2.1775 \times 10^{-7} & 5.0070 \times 10^{-7} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [0.1523 \pm 3.356788 \times 10^{-2}]$, $\lambda \in [6.9658 \times 10^{-5} \pm 6.673785 \times 10^{-5}]$, $\theta \in [2.1319 \pm 6.031353 \times 10^{-4}]$ and $c \in [3.7337 \pm 1.386911 \times 10^{-3}]$.

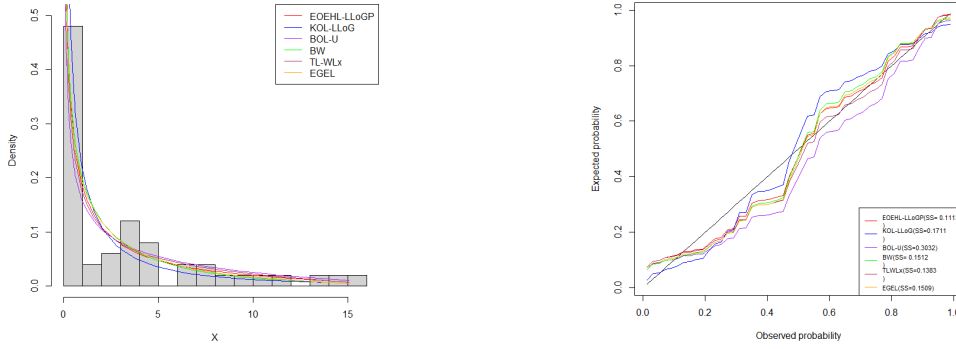


Figure 4.10: Fitted densities and probability plots for the failure times data

Futhermore, results shown in Table 4.6 affirms that the EOEHL-LLoGP model performs better than the non-nested models considered in this paper. Also, the fitted densities and probability plots shown in Figure 4.10 show that the EOEHL-LLoGP model fit the data set better than the selected non-nested models.

4.8 Concluding Remarks

A new class of distributions, namely, the Exponentiated Odd Exponential Half-Logistic-G Power Series (EOEHL-GPS) class of distributions was developed. We present some special classes of the new proposed distribution. Structural properties including moments, distribution of order statistics, Rényi entropy, and maximum likelihood estimates were obtained. A simulation study was carried out to evaluate the consistency of the maximum likelihood estimates. We also present two real data examples to demonstrate the

usefulness of the new class of distributions. The proposed model performs better than the several non-nested models on the data sets used.

Chapter 5

The Marshall-Olkin-Type II-Topp-Leone-G Family of Distributions

5.1 Introduction

The Marshall-Olkin ([54]) technique is applied in this chapter to obtain a new generalized distribution called the Marshall-Olkin-Type II-Topp-Leone-G (MO-TII-TL-G) family of distributions. We are motivated by the interesting properties of the MO-G and TII-TL-G distributions to develop a new family of distributions which is a combination of these two distributions. We develop a new family of distributions which is flexible because it can be applied to data sets of varying skewness and kurtosis. Also, it can model different types of hazard rate functions including monotonic as well as non-monotonic shapes.

The Marshall-Olkin-G (MO-G) distribution has cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{MO-G}(x; \delta, \xi) = 1 - \frac{\delta \bar{G}(x; \xi)}{1 - \delta \bar{G}(x; \xi)} \quad (5.1)$$

and

$$f_{MO-G}(x; \delta, \xi) = \frac{\delta g(x; \xi)}{[1 - \delta \bar{G}(x; \xi)]^2}, \quad (5.2)$$

respectively, where $\delta > 0$ is the tilt parameter and $\bar{G}(x; \xi) = 1 - G(x; \xi)$ is the survival function of the baseline distribution.

The Type II Topp-Leone generated (TII-TL-G) family of distributions proposed by Elgarhy et al. [34] has cdf and pdf, respectively, specified by

$$F_{TII-TL-G}(x; b, \xi) = 1 - [1 - G^2(x; \xi)]^b \quad (5.3)$$

and

$$f_{TII-TL-G}(x; b, \xi) = 2bg(x; \xi)G(x; \xi)[1 - G^2(x; \xi)]^{b-1},$$

where $b > 0$ and $G(x; \xi)$ is the cdf of the baseline distribution.

5.2 Marshall-Olkin-Type II-Topp-Leone-G Family of Distributions

In this section, we develop some properties of the MO-TII-TL-G distribution including the hazard function and expansion of the density. We derive the new MO-TII-TL-G family of distributions by using the generalizations in equations (5.1) and (5.3). The cdf, pdf and hrf of the MO-TII-TL-G family of distributions are given by

$$F_{MO-TII-TL-G}(x; \delta, b, \xi) = 1 - \frac{\delta[1 - G^2(x; \xi)]^b}{1 - \bar{\delta}[1 - G^2(x; \xi)]^b},$$

$$f_{MO-TII-TL-G}(x; \delta, b, \xi) = \frac{2\delta bg(x; \xi)G(x; \xi)[1 - G^2(x; \xi)]^{b-1}}{(1 - \bar{\delta}[1 - G^2(x; \xi)]^b)^2} \quad (5.4)$$

and

$$h_{MO-TII-TL-G}(x; \delta, b, \xi) = \frac{2\delta bg(x; \xi)G(x; \xi)[1 - G^2(x; \xi)]^{b-1}}{(1 - \bar{\delta}[1 - G^2(x; \xi)]^b)(\delta[1 - G^2(x; \xi)]^b)},$$

respectively, for $b, \delta > 0$, $\bar{\delta} = 1 - \delta$ and ξ is a vector of parameters from the baseline distribution function $G(\cdot)$.

5.2.1 Expansion of Density Function

The series expansion of the MO-TII-TL-G family of distributions is derived by making use of the general results of the Marshall and Olkin's family of

distributions by Barreto-Souza et al. [16]. The pdf of the MO-TII-TL-G given by

$$f_{MO-TII-TL-G}(x; \delta, b, \xi) = \frac{\delta f_{TII-TL-G}(x; b, \xi)}{(1 - \delta \bar{F}_{TII-TL-G}(x; b, \xi))^2},$$

can be written as

$$f_{MO-TII-TL-G}(x; \delta, b, \xi) = \frac{f_{TII-TL-G}(x; b, \xi)}{\delta(1 - \frac{\delta-1}{\delta} F_{TII-TL-G}(x; b, \xi))^2},$$

where $f_{TII-TL-G}(x; b, \xi)$ and $F_{TII-TL-G}(x; b, \xi)$ are the pdf and cdf of the TII-TL-G family of distributions, respectively. We also make use of the series expansion

$$(1 - z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^j, \quad (5.5)$$

which is valid for $|z| < 1$, $k > 0$. If $\delta \in (0, 1)$ we obtain

$$f_{MO-TII-TL-G}(x; \delta, b, \xi) = f_{TII-TL-G}(x; b, \xi) \sum_{j=0}^{\infty} \sum_{k=0}^j w_{j,k} F_{TII-TL-G}(x; b, \xi)^{j-k},$$

where $w_{j,k} = w_{j,k}(\delta) = \delta(j+1)(1-\delta)^j(-1)^{j-k} \binom{j}{k}$. For $\delta > 1$, we have

$$f_{MO-TII-TL-G}(x; \delta, b, \xi) = f_{TII-TL-G}(x; b, \xi) \sum_{j=0}^{\infty} v_j F_{TII-TL-G}^j(x; b, \xi),$$

where $v_j = v_j(\delta) = \frac{(j+1)(1-\frac{1}{\delta})}{\delta}$.

For $\delta \in (0, 1)$, equation (5.4) becomes

$$\begin{aligned} f_{MO-TII-TL-G}(x; \delta, b, \xi) &= 2bg(x; \xi)G(x; \xi)[1 - G^2(x; \xi)]^{b-1} \\ &\times \sum_{j=0}^{\infty} \sum_{k=0}^j w_{j,k} (1 - [1 - G^2(x; \xi)]^b)^{j-k}. \end{aligned}$$

By applying the generalized binomial series expansions

$$(1 - [1 - G^2(x; \xi)]^b)^{j-k} = \sum_{l=0}^{\infty} (-1)^l \binom{j-k}{l} [1 - G^2(x; \xi)]^{bl}$$

and

$$[1 - G^2(x; \xi)]^{b(l+1)-1} = \sum_{m=0}^{\infty} (-1)^m \binom{b(l+1)-1}{m} G^{2m}(x; \xi),$$

we can write

$$\begin{aligned}
f_{MO-TII-TL-G}(x; \delta, b, \xi) &= \sum_{j,l,m=0}^{\infty} \sum_{k=0}^j \frac{2b(-1)^{l+m} w_{j,k}}{2m+2} \binom{j-k}{l} \binom{b(l+1)-1}{m} \\
&\times (2m+2)g(x; \xi) G^{2m+1}(x; \xi) \\
&= \sum_{m=0}^{\infty} w_m^* g_m(x; \xi).
\end{aligned} \tag{5.6}$$

It then follows that for $\delta \in (0, 1)$, the MO-TII-TL-G family of distributions can be expressed as a linear combination of the Exp-G distribution with power parameter $(2m+1)$ and linear component

$$w_m^* = \sum_{j,l} \sum_{k=0}^j \frac{2b(-1)^{l+m} w_{j,k}}{2m+2} \binom{j-k}{l} \binom{b(l+1)-1}{m}. \tag{5.7}$$

Furthermore, for $\delta > 1$, equation (5.4) can be written as

$$\begin{aligned}
f_{MO-TII-TL-G}(x; \delta, b, \xi) &= 2bg(x; \xi)G(x; \xi)[1 - G^2(x; \xi)]^{b-1} \\
&\times \sum_{k=0}^j v_j (1 - [1 - G^2(x; \xi)]^{b-1})^j.
\end{aligned}$$

Applying the series expansions

$$(1 - [1 - G^2(x; \xi)]^{b-1})^j = \sum_{l=0}^{\infty} (-1)^l \binom{j}{l} [1 - G^2(x; \xi)]^{bl}$$

and

$$[1 - G^2(x; \xi)]^{b(l+1)-1} = \sum_{m=0}^{\infty} (-1)^m \binom{b(l+1)-1}{m} G^{2m}(x; \xi),$$

we get

$$\begin{aligned}
f_{MO-TII-TL-G}(x; \delta, b, \xi) &= \sum_{j,l,m=0}^{\infty} \frac{2b(-1)^{l+m} v_j}{2m+2} \binom{j}{l} \binom{b(l+1)-1}{m} \\
&\times (2m+2)g(x; \xi) G^{2m+1}(x; \xi) \\
&= \sum_{m=0}^{\infty} v_m^* g_m(x; \xi).
\end{aligned} \tag{5.8}$$

Therefore, for $\delta > 1$ the MO-TII-TL-G family of distributions can be expressed as n infinite linear combination of the Exp-G distribution with power parameter $(2m+1)$ and linear component

$$v_m^* = \sum_{j,l} \frac{2b(-1)^{l+m} v_j}{2m+2} \binom{j}{l} \binom{b(l+1)-1}{m}. \tag{5.9}$$

5.3 Some Special Cases

In this Section, we present some special cases of the MO-TII-TL-G family of distributions. We considered cases when the baseline distributions are weibull, uniform and log-logistic distributions.

5.3.1 Marshall-Olkin-Type II-Topp-Leone-Weibull (MO-TII-TL-W) Distribution

Consider the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$ and $G(x; \lambda) = 1 - e^{-x^\lambda}$, respectively, for $\lambda > 0$. The cdf, pdf and hrf of the MO-TII-TL-W distribution are given by

$$F_{MO-TII-TL-W}(x; \delta, b, \lambda) = 1 - \frac{\delta[1 - (1 - e^{-x^\lambda})^2]^b}{1 - \delta[1 - (1 - e^{-x^\lambda})^2]^b},$$

$$f_{MO-TII-TL-W}(x; \delta, b, \lambda) = \frac{2\delta b \lambda x^{\lambda-1} e^{-x^\lambda} (1 - e^{-x^\lambda}) [1 - (1 - e^{-x^\lambda})^2]^{b-1}}{(1 - \delta[1 - (1 - e^{-x^\lambda})^2]^b)^2}$$

and

$$h_{MO-TII-TL-W}(x; \delta, b, \lambda) = \frac{2\delta b \lambda x^{\lambda-1} e^{-x^\lambda} (1 - e^{-x^\lambda}) [1 - (1 - e^{-x^\lambda})^2]^{b-1}}{(1 - \delta[1 - (1 - e^{-x^\lambda})^2]^b)(\delta[1 - (1 - e^{-x^\lambda})^2]^b)}$$

respectively, for $\delta, b, \lambda > 0$.

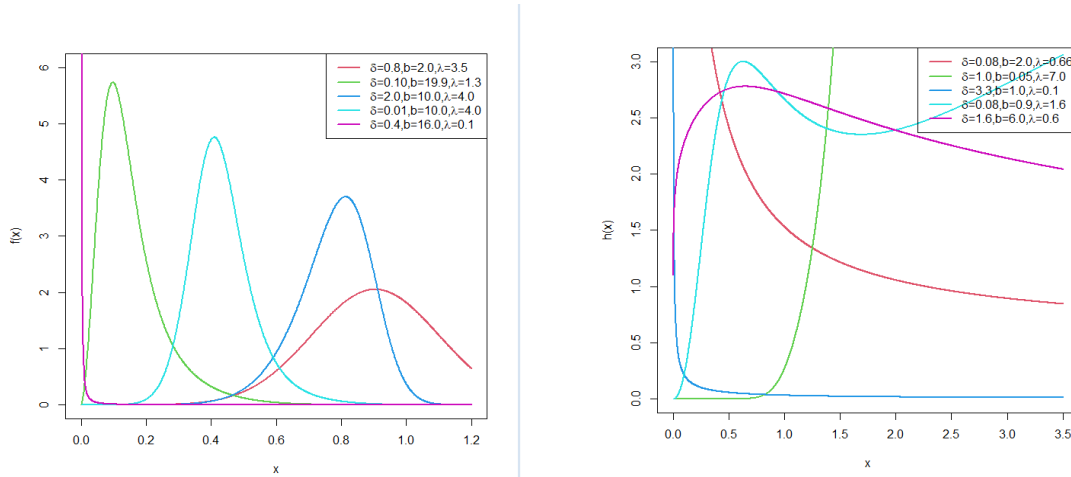


Figure 5.1: Plots of MO-TII-TL-W density and hazard functions

Figures 5.1 show the plots of the pdfs and hrfs for the MO-TII-TL-W. The pdf exhibit different shapes including right or left-skewed, reverse J and unimodal.

The hrf exhibit reverse-J, increasing, decreasing, upside-down bathtub and upside-down bathtub followed by bathtub shapes.

5.3.2 Marshall-Olkin-Type II-Topp-Leone-Log-Logistic (MO-TII-TL-LLoG) Distribution

Suppose that the Log-logistic distribution is the baseline distribution with pdf and cdf given $g(x; c, k) = cx^{c-1}(1+x^c)^{-2}$ and $G(x; c) = 1 - (1+x^c)^{-1}$, respectively, for $c, k > 0$. Therefore, the MO-TII-TL-LLoG distribution have cdf, pdf and hrf given by

$$F_{MO-TII-TL-LLoG}(x; \delta, b, c) = 1 - \frac{\delta[1 - (1 - (1+x^c)^{-1})^2]^b}{1 - \delta[1 - (1 - (1+x^c)^{-1})^2]^b},$$

$$f_{MO-TII-TL-LLoG}(x; \delta, b, c) = \frac{2\delta b c x^{c-1}(1+x^c)^{-2}(1 - (1+x^c)^{-1})}{(1 - \delta[1 - (1 - (1+x^c)^{-1})^2]^b)^2} \times [1 - (1 - (1+x^c)^{-1})^2]^{b-1}$$

and

$$h_{MO-TII-TL-LLoG}(x; \delta, b, c) = \frac{2\delta b c x^{c-1}(1+x^c)^{-2}(1 - (1+x^c)^{-1})}{(1 - \delta[1 - (1 - (1+x^c)^{-1})^2]^b)} \times \frac{[1 - (1 - (1+x^c)^{-1})^2]^{b-1}}{(\delta[1 - (1 - (1+x^c)^{-1})^2]^b)}$$

respectively, for $\delta, b, c > 0$.

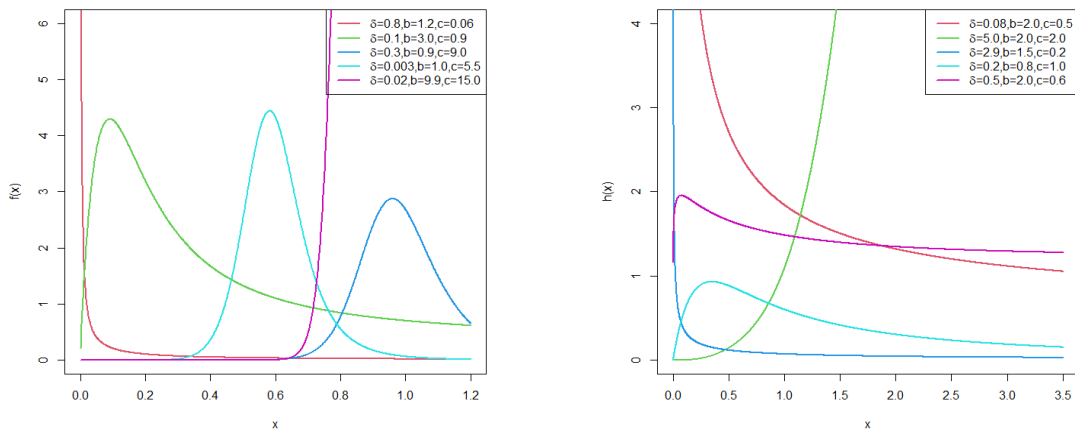


Figure 5.2: Plots of MO-TII-TL-LLoG density and hazard functions

Plots for the MO-TII-TL-LLoG pdf shows that the distribution can take various shapes that include: reverse-J, J, almost symmetric and left or right-skewed. The hazard rate function exhibits both monotonic and non-monotonic hazards rate shapes.

5.3.3 Marshall-Olkin-Type II-Topp-Leone-Uniform (MO-TII-TL-U) Distribution

Consider the Uniform distribution as the baseline distribution with pdf and cdf given by $g(x; \theta) = 1/\theta$ and $G(x; \theta) = x/\theta$, respectively, for $0 < x < \theta$. Therefore, the MO-TII-TL-U distribution have cdf, pdf and hrf given by

$$F_{MO-TII-TL-U}(x; \delta, b, \theta) = 1 - \frac{\delta[1 - (x/\theta)^2]^b}{1 - \delta[1 - (x/\theta)^2]^b},$$

$$f_{MO-TII-TL-U}(x; \delta, b, \theta) = \frac{2\delta b(1/\theta)(x/\theta)[1 - (x/\theta)^2]^{b-1}}{(1 - \delta[1 - (x/\theta)^2]^b)^2}$$

and

$$h_{MO-TII-TL-U}(x; \delta, b, \theta) = \frac{2\delta b(1/\theta)(x/\theta)[1 - (x/\theta)^2]^{b-1}}{(1 - \delta[1 - (x/\theta)^2]^b)(\delta[1 - (x/\theta)^2]^b)}$$

respectively, for $\delta, b, \lambda > 0$.

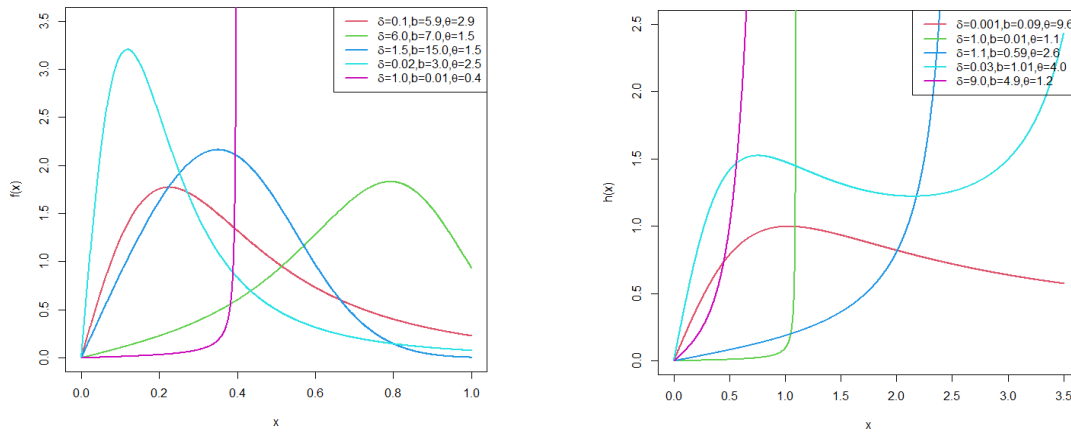


Figure 5.3: Plots of MO-TII-TL-U density and hazard functions

Figures 5.3 shows that the pdf of the MO-TII-TL-U can take various shapes that includes right or left-skewed, J and unimodal. The hrf shows J, increasing, upside-down bathtub and upside-down bathtub followed by bathtub shapes.

5.4 Statistical Properties

The order statistics, Rényi entropy, moments and the quantile function of the MO-TII-TL-G family of distributions are presented in this section.

5.4.1 Distribution of Order Statistics

Suppose that $X_1, X_2, \dots, X_n$ are independent and identically distributed (iid) random variables from the MO-TII-TL-G family of distributions. The pdf of the i^{th} order statistic $X_{i:n}$, is given by

$$f_{i:n}(x; \delta, b, \xi) = \delta n! f_{TII-TL-G}(x; b, \xi) \sum_{k=0}^{n-i} \frac{(-1)^k}{(i-1)!(n-1)!} \\ \times \frac{F_{TII-TL-G}^{k+i-1}(x; b, \xi)}{[1 - \delta F_{TII-TL-G}(x; b, \xi)]^{k+i-1}}.$$

If $\delta \in (0, 1)$, we have

$$f_{i:n}(x; \delta, b, \xi) = f_{TII-TL-G}(x; b, \xi) \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^j B_{j,k,l} F_{TII-TL-G}^{j+k-l+i-1}(x; b, \xi), \quad (5.10)$$

where $B_{j,k,l} = B_{j,k,l}(\delta) = \delta n! (-1)^{j+k-1} (1-\delta)^j \binom{j}{k} \binom{k+i+j}{j}$.

Substituting the pdf and cdf of the TII-TL-G distribution into equation (5.10) we get

$$f_{i:n}(x; \delta, b, \xi) = 2bg(x; \xi)G(x; \xi)[1 - G^2(x; \xi)]^{b-1} \\ \times \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^j B_{j,k,l} (1 - [1 - G^2(x; \xi)]^b)^{j+k-l+i-1}.$$

Using the series of expansions,

$$(1 - [1 - G^2(x; \xi)]^b)^{j+k-l+i-1} = \sum_{m=0}^{\infty} (-1)^m \binom{j+k-l+i-1}{m} [1 - G^2(x; \xi)]^{bm}$$

and

$$[1 - G^2(x; \xi)]^{b(m+1)-1} = \sum_{p=0}^{\infty} (-1)^p \binom{b(m+1)-1}{p} G^{2p}(x; \xi),$$

we get

$$f_{i:n}(x; \delta, b, \xi) = \sum_{j,m,p=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^j \frac{2b(-1)^{m+p} B_{j,k,l}}{2p+2} \binom{j+k-l+i-1}{m} \\ \times \binom{b(m+1)-1}{p} (2p+2)g(x; \xi)G^{2p+1}(x; \xi) \\ = \sum_{p=0}^{\infty} B_p^* g_p(x; \xi),$$

where

$$B_p^* = \sum_{j,m=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^j \frac{2b(-1)^{m+p} B_{j,k,l}}{2p+2} \binom{j+k-l+i-1}{m} \binom{b(m+1)-1}{p}$$

and $g_p(x; \xi) = (2p+2)g(x; \xi)G^{2p+1}(x; \xi)$ is an Exp-G distribution with power parameter $(2p+1)$. Therefore, the distribution of the order statistics of the MO-TII-TL-G family of distributions can be obtained from those of the Exp-G distribution with parameter $(2p+1)$.

For $\delta > 1$, we write $1 - \bar{\delta}F_{TII-TL-G}(x; b, \xi) = \delta[1 - \frac{(\delta-1)F_{TII-TL-G}(x; b, \xi)}{\delta}]$, so that

$$f_{i:n}(x; \delta, b, \xi) = f_{TII-TL-G}(x; b, \xi) \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} U_{j,k} F_{TII-TL-G}^{j+k+i-1}(x; b, \xi),$$

where $U_{j,k} = U_{j,k}(\delta) = \frac{(-1)^k (\delta-1)^j n!}{\delta^{k+j+i} (i-1)! (n-i)!} \binom{k+i+j}{j}$.

$$\begin{aligned} f_{i:n}(x; \delta, b, \xi) &= 2bg(x; \xi)G(x; \xi)[1 - G^2(x; \xi)]^{b-1} \\ &\times \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} U_{j,k} (1 - [1 - G^2(x; \xi)]^b)^{j+k+i-1}. \end{aligned}$$

Applying the series of expansions,

$$(1 - [1 - G^2(x; \xi)]^b)^{j+k+i-1} = \sum_{m=0}^{\infty} (-1)^m \binom{j+k+i-1}{m} [1 - G^2(x; \xi)]^{bm}$$

and

$$[1 - G^2(x; \xi)]^{b(m+1)-1} = \sum_{p=0}^{\infty} (-1)^p \binom{b(m+1)-1}{p} G^{2p}(x; \xi),$$

we obtain

$$\begin{aligned} f_{i:n}(x; \delta, b, \xi) &= \sum_{j,m,p=0}^{\infty} \sum_{k=0}^{n-i} \frac{2b(-1)^{m+p} U_{j,k}}{2p+2} \binom{j+k+i-1}{m} \\ &\times \binom{b(m+1)-1}{p} (2p+2)g(x; \xi)G^{2p+1}(x; \xi) \\ &= \sum_{p=0}^{\infty} U_p^* g_p(x; \xi), \end{aligned}$$

where

$$U_p^* = \sum_{j,m=0}^{\infty} \sum_{k=0}^{n-i} \frac{2b(-1)^{m+p} U_{j,k}}{2p+2} \binom{j+k+i-1}{m} \binom{b(m+1)-1}{p}$$

and $g_p(x; \xi) = (2p + 2)g(x; \xi)G^{2p+1}(x; \xi)$ is an Exp-G distribution with power parameter $(2p + 1)$.

Also, for $\delta > 1$, the distribution of the order statistics of MO-TII-TL-G family of distributions can be obtained from those of the Exp-G since its i^{th} order statistic is an infinite linear combination of Exp-G densities with parameter $(2p + 1)$.

5.4.2 Entropy

Entropy measures variation of uncertainty of a random variable. Rényi entropy [66] is a generalization of Shannon entropy [71]. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty f_{MO-TII-TL-G}^v(x; \delta, b, \xi) dx \right),$$

where $v > 0$, and $v \neq 1$. Using expansion (5.5), for $\delta \in (0, 1)$

$$\begin{aligned} f^v(x; \delta, b, \xi) &= \frac{\delta^v f_{TII-TL-G}^v(x; b, \xi)}{\Gamma(2v)} \sum_{j=0}^{\infty} (1-\delta)^j \Gamma(2v+j) \\ &\times \frac{[1 - F_{TII-TL-G}(x; b, \xi)]^j}{j!} \end{aligned}$$

and for $\delta > 1$

$$\begin{aligned} f^v(x; \delta, b, \xi) &= \frac{\delta^v f_{TII-TL-G}^v(x; b, \xi)}{\delta^v \Gamma(2v)} \sum_{j=0}^{\infty} (\delta - 1)^j \Gamma(2v+j) \\ &\times \frac{[F_{TII-TL-G}(x; b, \xi)]^j}{j!}. \end{aligned}$$

Therefore, the Rényi entropy for $\delta \in (0, 1)$ is

$$I_R(v) = \frac{1}{1-v} \log \left(\sum_{j=0}^{\infty} e_j \int_0^\infty f_{TII-TL-G}^v(x; b, \xi) [1 - F_{TII-TL-G}(x; b, \xi)]^j dx \right),$$

where $e_j = e_j(\delta) = \frac{\delta^v (1-\delta)^j \Gamma(2v+j)}{\Gamma(2v)j!}$.

Thus,

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left(\sum_{j=0}^{\infty} e_j \int_0^\infty (2b)^v g^v(x; \xi) G^v(x; \xi) [1 - G^2(x; \xi)]^{v(b-1)} \right. \\ &\times \left. [1 - G^2(x; \xi)]^{bj} dx \right). \end{aligned}$$

Using the generalized binomial expansion

$$[1 - G^2(x; \xi)]^{b(j+v)-v} = \sum_{k=0}^{\infty} (-1)^k \binom{b(j+v)-v}{k} G^{2k}(x; \xi),$$

we can write

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left(\sum_{j,k=0}^{\infty} e_j \frac{(2b)^v (-1)^k}{\left(\frac{2k+v}{v} + 1\right)^v} \binom{b(j+v)-v}{k} \right. \\ &\quad \times \left. \int_0^{\infty} \left[\left(\frac{2k+v}{v} + 1 \right) g(x; \xi) G^{\frac{2k+v}{v}}(x; \xi) \right]^v dx \right) \\ &= \frac{1}{1-v} \log \left(\sum_{k=0}^{\infty} e_k^* e^{(1-v)I_{REG}} \right), \end{aligned}$$

where,

$$e_k^* = \sum_{j=0}^{\infty} e_j \frac{(2b)^v (-1)^k}{\left(\frac{2k+v}{v} + 1\right)^v} \binom{b(j+v)-v}{k}$$

and

$$I_{REG} = \frac{1}{1-v} \log \left(\int_0^{\infty} \left[\left(\frac{2k+v}{v} + 1 \right) g(x; \xi) G^{\frac{2k+v}{v}}(x; \xi) \right]^v dx \right),$$

is the Rényi entropy of the Exp-G distribution with power parameter $\frac{2k+v}{v}$.

For $\delta > 1$

$$I_R(v) = \frac{1}{1-v} \log \left(\sum_{j=0}^{\infty} h_j \int_0^{\infty} f_{TH-TL-G}^v(x; b, \xi) F_{TH-TL-G}^j(x; b, \xi) dx \right),$$

where $h_j = h_j(\delta) = \frac{(\delta-1)^j \Gamma(2v+j)}{\delta^{v+j} \Gamma(2v) j!}$.

Now

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left(\sum_{j=0}^{\infty} h_j \int_0^{\infty} (2b)^v g^v(x; \xi) G^v(x; \xi) [1 - G^2(x; \xi)]^{v(b-1)} \right. \\ &\quad \times \left. (1 - [1 - G^2(x; \xi)]^b)^j dx \right). \end{aligned}$$

Applying the generalized binomial expansions

$$(1 - [1 - G^2(x; \xi)]^b)^j = \sum_{k=0}^{\infty} (-1)^k \binom{j}{k} [1 - G^2(x; \xi)]^{bk}$$

and

$$[1 - G^2(x; \xi)]^{b(k+v)-v} = \sum_{l=0}^{\infty} (-1)^l \binom{b(k+v)-v}{l} G^{2l}(x; \xi),$$

we get

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v} \log \left(\sum_{j,k,l=0}^{\infty} h_j \frac{(2b)^v (-1)^{k+l}}{\left(\frac{2k+v}{v} + 1\right)^v} \binom{j}{k} \binom{b(k+v)-v}{l} \right. \\
&\quad \times \left. \int_0^{\infty} \left[\left(\frac{2l+v}{v} + 1 \right) g(x; \xi) G^{\frac{2l+v}{v}}(x; \xi) \right]^v dx \right) \\
&= \frac{1}{1-v} \log \left(\sum_{l=0}^{\infty} h_l^* e^{(1-v)I_{REG}} \right),
\end{aligned}$$

where,

$$h_l^* = \sum_{j,k=0}^{\infty} h_j \frac{(2b)^v (-1)^{k+l}}{\left(\frac{2k+v}{v} + 1\right)^v} \binom{j}{k} \binom{b(k+v)-v}{l}$$

and

$$I_{REG} = \frac{1}{1-v} \log \left(\int_0^{\infty} \left[\left(\frac{2l+v}{v} + 1 \right) g(x; \xi) G^{\frac{2l+v}{v}}(x; \xi) \right]^v dx \right),$$

is the Rényi entropy of the Exp-G distribution with power parameter $\frac{2l+v}{v}$.

5.4.3 Moments and Generating Functions

Let $X \sim \text{MO-TII-TL-G}(\delta, b, \xi)$ and $Y \sim \text{EXP-G}(2m+1)$, then the r^{th} moment of X can be obtained as follows; For $\delta \in (0, 1)$,

$$E[X^r] = \sum_{m=0}^{\infty} w_m^* E[Y^r],$$

where w_m^* is given by equation (5.7) and $E[Y^r]$ is the r^{th} moment of Y which follows an Exp-G distribution with power parameter $(2m+1)$.

For $\delta > 1$,

$$E[X^r] = \sum_{m=0}^{\infty} v_m^* E[Y^r],$$

where v_m^* is given by equation (5.9) and $E[Y^r]$ is the r^{th} moment of Y which follows an Exp-G distribution with power parameter $(2m+1)$.

The moment generating function (mgf) of X is given by: For $\delta \in (0, 1)$

$$M_X(t) = \sum_{m=0}^{\infty} w_m^* E[e^{tY}],$$

where $E[e^{tY}]$ is the mgf of the Exp-G distribution with power parameter $(2m+1)$ and w_m^* is given by equation (5.7).

For $\delta > 1$

$$M_X(t) = \sum_{m=0}^{\infty} v_m^* E[e^{tY}],$$

where $E[e^{tY}]$ is the mgf of the Exp-G distribution with power parameter $(2m+1)$ and v_m^* is given by equation (5.9).

A table of moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) for selected parameter values of the MO-TII-TL-W distribution are given in Table 5.1.

Table 5.1: Table of Moments for Selected Parameters of the MO-TII-TL-W Distribution

	(δ, b, λ)				
	(1.5,0.9,0.3)	(2,0.9,0.2)	(1.5,0.7,0.5)	(1.1,2,0.2)	(1.2,2.9,0.4)
E(X)	0.0849	0.0526	0.0955	0.1085	0.2142
E(X ²)	0.0491	0.0294	0.0599	0.0565	0.1158
E(X ³)	0.0345	0.0204	0.0435	0.0380	0.0779
E(X ⁴)	0.0265	0.0156	0.0341	0.0286	0.0584
E(X ⁵)	0.0215	0.0126	0.0280	0.0229	0.0466
SD	0.2047	0.1632	0.2253	0.2116	0.2645
CV	2.4110	3.1042	2.3583	1.9504	1.2352
CS	2.7000	3.6886	2.4518	2.3399	1.2518
CK	9.5510	16.5919	7.9505	7.8199	3.5090

3D plots of skewness and kurtosis for the MO-TII-TL-W distribution are given in Figures 5.4 and 5.5. We observe that

- When we fix the parameters for b and λ the skewness and kurtosis of MO-TII-TL-W increases as δ increases.
- When we fix the parameters for δ and b , the skewness and kurtosis of MO-TII-TL-W increases as λ increases.
- When we fix the parameters for δ and λ the skewness and kurtosis of MO-TII-TL-W decreases as b increases.

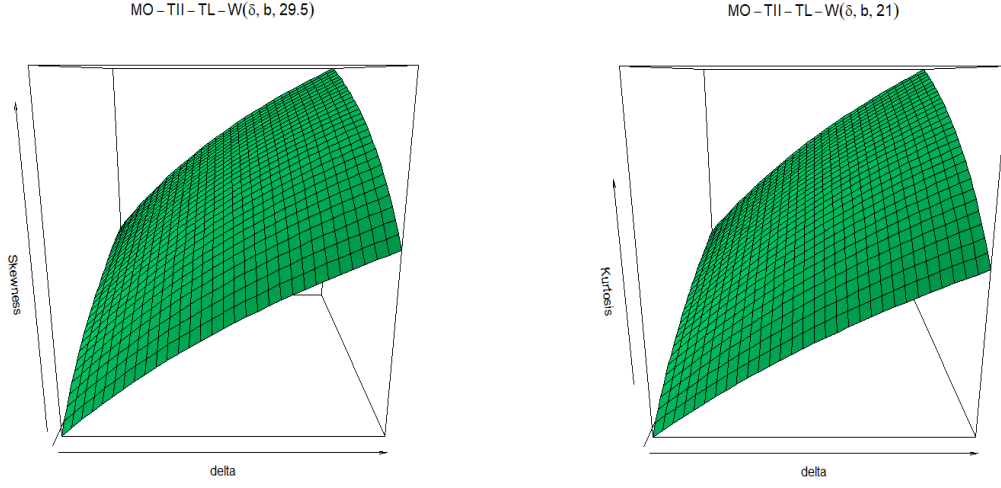


Figure 5.4: Plots of skewness and kurtosis for the MO-TII-TL-W distribtion

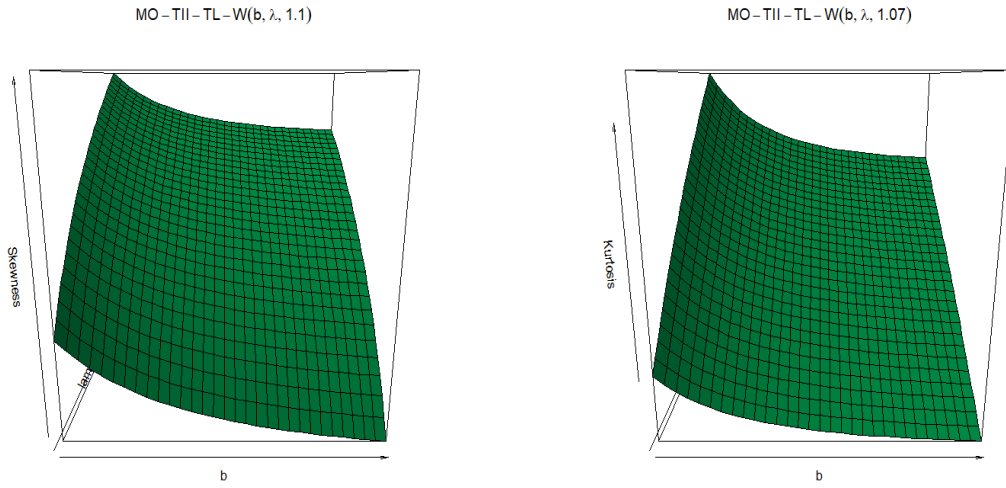


Figure 5.5: Plots of skewness and kurtosis for the MO-TII-TL-W distribtion

5.4.4 Quantile Function

The quantile function of the MO-TII-TL-G family of distributions is obtained by solving the non-linear equation:

$$1 - \frac{\delta[1 - G^2(x; \xi)]^b}{1 - \delta[1 - G^2(x; \xi)]^b} = u,$$

for $0 \leq u \leq 1$. Simplifying the equation we obtain

$$(1 - u) - (1 - u)\bar{\delta}[1 - G^2(x; \xi)]^b = \delta[1 - G^2(x; \xi)]^b,$$

such that

$$\frac{1 - u}{\bar{\delta}(1 - u)} = [1 - G^2(x; \xi)]^b.$$

Further simplifying the equation yields

$$\left(1 - \left[\frac{1 - u}{\bar{\delta}(1 - u)}\right]^{\frac{1}{b}}\right)^{\frac{1}{2}} = G^2(x; \xi).$$

Therefore, the quantiles of the MO-TII-TL-G family of distributions may be obtained by solving the non-linear equation

$$x(u) = G^{-1}\left(1 - \left[\frac{1 - u}{\bar{\delta}(1 - u)}\right]^{\frac{1}{b}}\right)^{\frac{1}{2}}.$$

The equation can be solved using software like R, SAS or MATLAB. Quantiles for selected parameter values for the MO-TII-TL-W distribution are shown in Table 5.2.

Table 5.2: Table of Quantiles for Selected Parameters of the MO-TII-TL-W Distribution

	(δ, b, λ)				
u	(1.1,1.5,0.9)	(1.2,0.1,1.1)	(0.5,0.3,0.8)	(0.4,1.7,0.8)	(0.5,0.2,1.2)
0.1	0.2795	1.7615	0.4427	0.1116	0.7135
0.2	0.4515	2.9591	0.8084	0.1910	1.0827
0.3	0.6166	4.1925	1.2195	0.2730	1.4424
0.4	0.7886	5.5368	1.7135	0.3653	1.8308
0.5	0.9777	7.0553	2.3378	0.4750	2.2780
0.6	1.1964	8.8376	3.1709	0.6132	2.8240
0.7	1.4658	11.0446	4.3652	0.8005	3.5371
0.8	1.8316	14.0305	6.2799	1.0836	4.5652
0.9	2.4383	18.9112	10.1294	1.6166	6.3653

5.5 Maximum Likelihood Estimation

Let $X_i \sim MO - TII - TL - G(\delta, b, \xi)$ and $\Delta = (\delta, b, \xi)^T$ be the parameter vector.

The log-likelihood $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\begin{aligned} \ell(\Delta) &= n \log(2b\delta) + \sum_{i=1}^n \log[g(x_i; \xi)] + \sum_{i=1}^n \log[G(x_i; \xi)] \\ &+ (b - 1) \sum_{i=1}^n \log[1 - G^2(x_i; \xi)] - 2 \sum_{i=1}^n \log[1 - \bar{\delta}[1 - G^2(x_i; \xi)]^b]. \end{aligned}$$

The score vector $U = \left(\frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \xi_k} \right)$ has elements given by

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} - 2 \sum_{i=1}^n \frac{[1 - G^2(x_i; \xi)]^b}{[1 - \delta[1 - G^2(x_i; \xi)]^b]},$$

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^n \log[1 - G^2(x_i; \xi)] - 2 \sum_{i=1}^n \frac{[1 - G^2(x_i; \xi)]^b \log[1 - G^2(x_i; \xi)]}{[1 - \delta[1 - G^2(x_i; \xi)]^b]}$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \xi_k} &= \sum_{i=1}^n \frac{\frac{\partial}{\partial \xi_k} g(x_i; \xi)}{g(x_i; \xi)} + \sum_{i=1}^n \frac{\frac{\partial}{\partial \xi_k} G(x_i; \xi)}{G(x_i; \xi)} + (b-1) \sum_{i=1}^n \frac{\frac{\partial}{\partial \xi_k} [1 - G^2(x_i; \xi)]}{[1 - G^2(x_i; \xi)]} \\ &- 2 \sum_{i=1}^n \frac{\frac{\partial}{\partial \xi_k} [1 - \delta[1 - G^2(x_i; \xi)]^b]}{[1 - \delta[1 - G^2(x_i; \xi)]^b]}. \end{aligned}$$

The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $\left(\frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \xi_k} \right)^T = 0$ using a numerical method such as the Newton-Raphson procedure. The multivariate normal distribution $N_{q+2}(\underline{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\underline{0} = (0, 0, 0)^T$ and $J(\hat{\Delta})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$, can be used to construct confidence intervals and confidence regions for the individual model parameters.

5.6 Simulation Study

In this section, a simulation study was carried out to assess the performance of the maximum likelihood estimates. Different simulations were conducted for the sample sizes $n = 100, 200, 400, 800, 1000$ and 1200 , for $N=1000$ for each sample. We estimate the mean, root mean square error (RMSE), and average bias. The average bias and RMSE for the estimated parameter, are given by

$$Bias(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta) \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. The results of the simulation study are shown in Table 5.3 and from the results we see that as the sample size increases, the mean approximates the true parameter values, the RMSE and bias decay towards zero for all parameters. Consequently, we conclude that the MO-TII-TL-W model gives out consistent model parameter estimates.

Table 5.3: Monte Carlo Simulation Results for MO-TII-TL-W Distribution

		$\delta = 0.5, b = 0.5, \lambda = 0.5$			$\delta = 0.2, b = 0.6, \lambda = 0.6$		
	n	Mean	RMSE	Bias	Mean	RMSE	Bias
δ	100	0.7653	0.6525	0.2653	0.3205	0.2903	0.1205
	200	0.6435	0.3659	0.1435	0.2677	0.1616	0.0677
	400	0.5632	0.2150	0.0632	0.2336	0.0995	0.0336
	800	0.5272	0.1346	0.0272	0.2163	0.0572	0.0163
	1000	0.5264	0.1268	0.0264	0.2125	0.0532	0.0125
	1200	0.5170	0.1094	0.0170	0.2097	0.0494	0.0097
b	100	0.6564	0.3851	0.1564	0.7973	0.4672	0.1973
	200	0.5834	0.2546	0.0834	0.7196	0.3185	0.1196
	400	0.5380	0.1610	0.0380	0.6589	0.2088	0.0589
	800	0.5186	0.1089	0.0186	0.6342	0.1292	0.0342
	1000	0.5182	0.1026	0.0182	0.6251	0.1205	0.0251
	1200	0.5099	0.0893	0.0099	0.6180	0.1115	0.0180
λ	100	0.4664	0.0704	-0.0336	0.5773	0.0642	-0.0227
	200	0.4782	0.0532	-0.0218	0.5796	0.0486	-0.0204
	400	0.4859	0.0361	-0.0141	0.5844	0.0344	-0.0156
	800	0.4910	0.0268	-0.0090	0.5888	0.0233	-0.0112
	1000	0.4909	0.0246	-0.0092	0.5899	0.0221	-0.0101
	1200	0.4937	0.0220	-0.0063	0.5915	0.0205	-0.0085
		$\delta = 0.2, b = 1.1, \lambda = 1.1$			$\delta = 0.9, b = 0.2, \lambda = 0.2$		
δ	100	0.2988	0.3088	0.0988	1.6470	1.3322	0.7470
	200	0.2477	0.1600	0.0477	1.1732	0.4518	0.2732
	400	0.2223	0.0953	0.0223	1.0286	0.2356	0.1286
	800	0.2117	0.0603	0.0117	0.9581	0.1338	0.0581
	1000	0.2093	0.0537	0.0093	0.9372	0.0447	0.0372
	1200	0.2044	0.0480	0.0044	0.9185	0.0992	0.0185
b	100	1.3267	0.7280	0.2267	0.3230	0.2169	0.1230
	200	1.2255	0.5035	0.1255	0.2430	0.0853	0.0430
	400	1.1573	0.3345	0.0573	0.2196	0.0484	0.0196
	800	1.1354	0.2251	0.0354	0.2076	0.0233	0.0076
	1000	1.1290	0.2025	0.0290	0.2020	0.0056	0.0020
	1200	1.1095	0.1855	0.0095	0.2104	0.0222	0.0104
λ	100	1.0952	0.1276	-0.0048	0.1811	0.0293	-0.0189
	200	1.0996	0.0933	-0.0004	0.1912	0.0161	-0.0088
	400	1.0991	0.0660	-0.0009	0.1951	0.0106	-0.0049
	800	1.0978	0.0456	-0.0022	0.1976	0.0048	-0.0024
	1000	1.0984	0.0410	-0.0016	0.1990	0.0030	-0.0010
	1200	1.0999	0.0378	-0.0001	0.1966	0.0056	-0.0034

5.7 Applications

In this section, we present two real data examples to illustrate the applicability of the MO-TII-TL-W distribution. Model parameters were estimated using the maximum likelihood estimation technique, with the aid of the R software for fitting the data and model diagnostics. The performance of the models were assessed using the following goodness-of-fit statistics; -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*) and Andersen-Darling(A^*) (as described by Chen and Balakrishnan [21]), Kolmogorov-Smirnov (K-S) statistic and its p-value. The smaller the goodness-of-fit statistics, the better the model.

We present the model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics in Tables 5.4 and 5.5. In addition, the plots of the fitted densities and probability plots (Chambers et al. [20]) are provided to demonstrate how well our model fits the two data sets.

The non-nested models compared to the MO-TI-TL-W distribution are the Weibull exponential (WE) distribution by Oguntunde et al. [62], Odd Lindley Fréchet (OLiFr) distribution by Mansour et al. [53], Type II generalized Topp-Leone-Rayleigh (TIGTL-R) distribution by Hassan et al. [39], Topp-Leone generalized exponential (TL-GE) distribution by Sangsanit and Bodhisuwan [68], Marshall-Olkin extended Frchet (MOEFr) and Marshall-Olkin extended generalized exponential (MOEGE) distribution given by Barreto-Souza et al. [16]. The pdfs of the non-nested models are given by:

$$f_{WE}(x; \alpha, \beta, \lambda) = \alpha\beta(\lambda e^{-\lambda x}) \left[\frac{(1 - e^{-\lambda x})^{\beta-1}}{(e^{-\lambda x})^{\beta+1}} \right] e^{-\alpha \left[\frac{(1 - e^{-\lambda x})}{(e^{-\lambda x})} \right]^{\beta}},$$

for $\alpha, \beta, \lambda > 0$,

$$f_{OLiFr}(x; \theta, \alpha, \beta) = \frac{\theta\alpha^{\beta}\theta^2 x^{-\beta-1} e^{-(\frac{\alpha}{x})^{\beta}}}{(1 + \theta)(1 - e^{-(\frac{\alpha}{x})^{\beta}})^3} \exp \left(\frac{-\theta e^{-(\frac{\alpha}{x})^{\beta}}}{1 - e^{-(\frac{\alpha}{x})^{\beta}}} \right),$$

for $\theta, \alpha, \beta > 0$,

$$f_{TIGTL-R}(x; \alpha, \beta, \delta) = 4\alpha\beta\delta x e^{-\delta^2} [1 - e^{-\delta^2}]^{2\beta-1} (1 - (1 - e^{-\delta^2})^{2\beta})^{\alpha-1},$$

for $\alpha, \beta, \delta > 0$,

$$f_{TL-GE}(x; \alpha, \beta, \lambda) = 2\alpha\beta\lambda e^{-\lambda x} (1 - (1 - e^{-\lambda x})^\beta (1 - e^{-\lambda x})^{\beta\alpha-1} (2 - (1 - e^{-\lambda x})^\beta))^{\alpha-1},$$

for $\alpha, \beta, \lambda > 0$,

$$f_{MOEFr}(x; \alpha, \delta, \lambda) = \frac{\alpha\lambda\delta^\lambda x^{-(\lambda+1)} e^{-(\delta/x)^\lambda}}{[1 - \bar{\alpha}(1 - e^{-(\delta/x)^\lambda})]^2},$$

for $\alpha, \delta, \lambda > 0$ and

$$f_{MOEGE}(x; \alpha, \gamma, \lambda) = \frac{\alpha\gamma e^{-\lambda x} (1 - e^{-\lambda x})^{\gamma-1}}{(1 - \bar{\alpha}[1 - e^{-\lambda x}])^2}$$

for $\alpha, \gamma, \lambda > 0$.

5.7.1 Silicon Nitride Data

The first data set is on fracture toughness of silicon nitride measured in MPa $m^{1/2}$. The data set was also analyzed by Nadarajah and Kotz [58] and also by Ali et al. [7]. The data are 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

Table 5.4: Parameter estimates and goodness-of-fit statistics for various models fitted for silicon nitride data set

Model	Estimates			Statistics							
	δ	b	λ	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P - value$
MO-TII-TL-W	71.2032 (0.0014)	0.3663 (0.1045)	1.6974 (0.1695)	336.9	342.9	343.1	351.2	0.0464	0.3294	0.0473	0.9527
WE	α	β	λ								
	25.6129 (0.0014)	4.1621 (0.2700)	0.0798 (0.0032)	337.2	343.2	343.4	351.5	0.0820	0.4981	0.0689	0.6252
OLIFr	θ	α	β								
	0.1591 (0.1063)	2.1951 (0.4366)	3.3991 (0.2320)	339.2	345.2	345.4	353.5	0.1426	0.8740	0.0818	0.4037
TIIGTL-R	α	β	δ								
	3585.2000 (2.1799×10^{-7})	1.2646 (0.0966)	1.8076×10^{-3} (4.4088×10^{-4})	337.5	343.5	343.7	351.8	0.0942	0.5795	0.0725	0.5599
TL-GE	α	β	λ								
	0.3892 (0.2426)	56.9230 (42.6845)	0.7549 (0.0779)	359.6	365.6	365.8	373.9	0.4620	2.7741	0.1414	0.01713
MOEFr	α	δ	λ								
	2407.7000 (7.7867×10^{-6})	1.4344 (0.1296)	7.0579 (0.5495)	356.6	362.6	362.9	371.0	38.2495	235.4782	0.9989	$< 2.2 \times 10^{-16}$
MOEGE	α	γ	λ								
	0.0107 (7.0715×10^{-3})	20.7600 (3.5685×10^{-5})	1.7321 (0.1434)	340.3	346.3	346.5	354.6	0.4722	2.9106	0.9893	$< 2.2 \times 10^{-16}$

The estimated variance-covariance for the MO-TII-TL-W model on silicon nitride data is given by

$$\begin{bmatrix} 2.0274 \times 10^{-6} & -1.4830 \times 10^{-4} & 2.4119 \times 10^{-4} \\ -1.4830 \times 10^{-4} & 0.0109 & -0.0176 \\ 2.4119 \times 10^{-4} & -0.0176 & 0.0287 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [71.2032 \pm 0.0028]$, $b \in [0.3663 \pm 0.2048]$ and $\lambda \in [1.6974 \pm 0.3323]$.

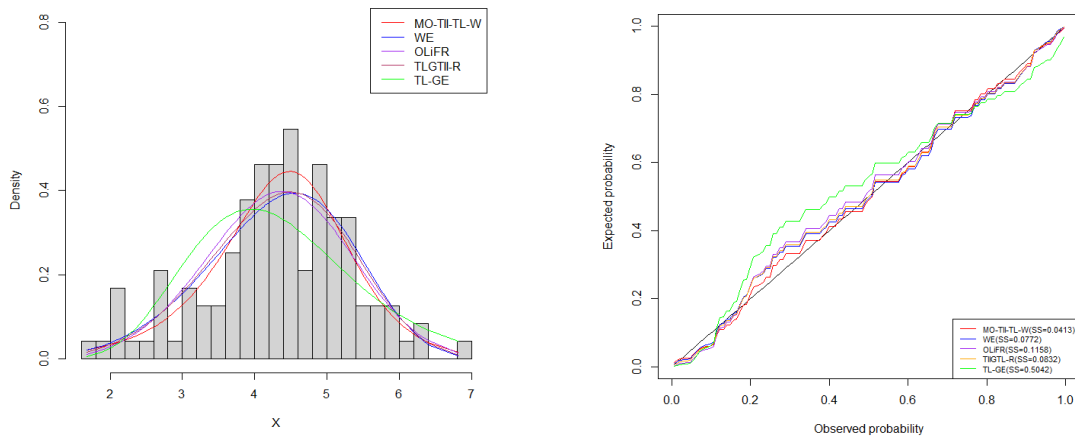


Figure 5.6: Fitted densities and probability plots for silicon nitride data

From the results of the goodness-of-fit statistics and the p-value shown in Table 5.4, we conclude that the MO-TII-TL-W distribution performs better than the non-nested models considered in this paper. Figure 5.6 which shows the fitted densities and probability plots also shows that the MO-TII-TL-W model fit the silicon nitride data set better than the non-nested models.

5.7.2 Kevlar 49/Epoxy Strands Failure at 90% Data

The second data set is on 101 observations representing the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all had failed. The failure times are in hours and are shown below (see Andrews and Herzberg [14] or Barlow et al. [15], for details): 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29,

0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

Table 5.5: Parameter estimates and goodness-of-fit statistics for various models fitted for kelvar data set

Model	Estimates			Statistics							
	δ	b	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
MO-TII-TL-W	5.0534 (4.2022)	4.4047 (1.4263)	0.4231 (0.0890)	204.1	210.1	210.4	217.9	0.1589	0.9161	0.0799	0.5400
WE	α	β	λ	206.0	212.0	212.2	219.8	0.1973	1.1051	0.0903	0.382
	147.4900 (1.6905×10^{-6})	0.9232 (0.0711)	4.5027×10^{-3} (1.7793×10^{-3})								
OLiFr	θ	α	β	207.2	213.2	213.5	221.1	0.2858	1.5263	0.1065	0.2021
	0.2650 (0.1627)	0.0377 (0.0353)	0.6349 (0.0480)								
THGTL-R	α	β	δ	206.0	212.0	212.2	219.8	0.1922	1.0825	0.0898	0.3895
	41.4010 (1.5381×10^{-6})	0.2278 (0.0178)	2.7764×10^{-4} (1.6758×10^{-4})								
TL-GE	α	β	λ	205.4	211.4	211.7	219.3	0.1518	0.8982	0.0801	0.5366
	0.4742 (0.5673)	1.7241 (1.9169)	0.5110 (0.1540)								
MOEFr	α	δ	λ	225.2	231.2	231.5	239.1	33.5614	199.4422	0.9960	$< 2.2 \times 10^{-16}$
	312.8500 (4.4813×10^{-6})	6.5773×10^{-3} (2.6809×10^{-3})	1.2631 (0.1033)								
MOEGE	α	γ	λ	204.8	210.8	211.0	218.6	0.1568	1.4421	0.4186	8.882×10^{-16}
	0.5942 (0.3306)	0.7307 (0.1849)	1.0456 (0.2071)								

The estimated variance-covariance for the MO-TII-TL-W model on kelvar data is given by

$$\begin{bmatrix} 17.6587 & 5.8581 & -0.3385 \\ 5.8581 & 2.0343 & -0.1162 \\ -0.3385 & -0.1162 & 0.0079 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [5.0534 \pm 8.2364]$, $b \in [4.4047 \pm 2.7956]$ and $\lambda \in [0.4231 \pm 0.1744]$.

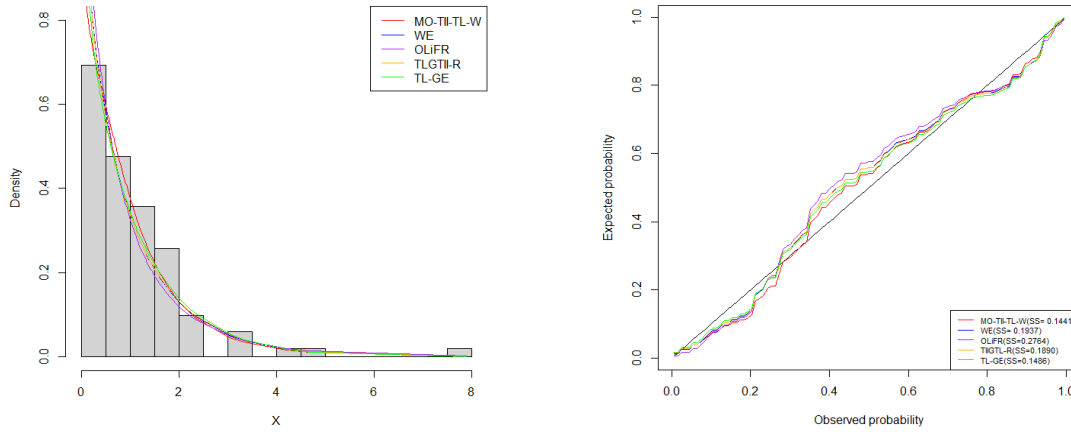


Figure 5.7: Fitted densities and probability plots for kelvar data

Furthermore, the results of the goodness-of-fit statistics and the p-value shown in Table 5.5 also confirms that the MO-TII-TL-W distribution performs better than the non-nested models considered in this paper. Figure 5.7 which shows the fitted densities and probability plots also indicates that the MO-TII-TL-W model fit the kelvar data set better than the non-nested models.

5.8 Concluding Remarks

A new generalized distribution referred to as the Marshall-Olkin-Type II-Topp-Leone-G family of distributions is developed and presented. The MO-TII-TL-G family of distributions has hazard function with flexible behavior and the new proposed distribution can be expressed as an infinite linear combination of the Exp-G distribution. Closed form expressions for the moments, distribution of order statistics and entropy were obtained. Maximum likelihood estimation method was used to estimate the model parameters. The performance of the special case of the MO-TII-TL-G family of distributions was examined by conducting various simulations for different sample sizes and lastly, the special case of the MO-TII-TL-G family of distributions was fitted to two real data sets to illustrate the applicability and usefulness of the proposed family of distributions.

Chapter 6

Conclusions and Suggestions for Further Studies

6.1 Conclusions

Four new generalized distributions namely, the Topp-Leone-G class of distributions, Marshall-Olkin-Exponentiated Odd Exponential Half Logistic-G family of distributions, the Exponentiated Odd Exponential Half-Logistic-G Power Series class of distributions and the Marshall-Olkin-Type II-Topp-Leone-G family of distributions were studied and developed. Distributional properties of these models were derived. Maximum likelihood estimation method was used to estimate the model parameters and the maximum likelihood estimators were examined by conducting a Monte Carlo simulation study for various sample sizes. The usefulness of the proposed distributions was demonstrated by real data sets and we conclude that the proposed models are useful because they performed better than the non-nested models considered.

6.2 Suggestions for Further Studies

Future research could include performing the likelihood ratio tests on the proposed models. Other methods of estimation such as Bayesian estimation,

ordinary least squares, weighted least squares, maximum product of spacings, Cramér-von Mises method, Anderson-Darling method and percentile method may also be considered. Furthermore, possible research may cover the inclusion of covariates or concomitant variables in the models.

Bibliography

- [1] Afify, A. Z., Alizadeh, M., Zayed, M., Ramires, T. G., and Louzada, F. (2018). The Odd Log-Logistic Exponentiated Weibull Distribution: Regression Modeling, Properties, and Applications. *Iranian Journal of Science and Technology, Transactions A: Science*, 42(4), 2273-2288.
- [2] Afify, A. Z., Altun, E., Alizadeh, M., Ozel, G., and Hamedani G.G. (2017). The Odd Exponentiated Half-Logistic-G Family: Properties, Characterization and Applications. *Chilean Journal of Statistics*, 8(2), 65-91.
- [3] Ahmad, K., Ahmad, S.P., and Ahmed, A. (2015). Structural Properties of Transmuted Weibull Distribution. *Journal of Modern Applied Statistical Methods*, 14(2), 141-158.
- [4] Alizadeh, A., Cordeiro, G. M., Brito, E., and Demétrio, C. G. (2015). The Beta Marshall-Olkin Family of Distributions. *Journal of Statistical Distributions and Applications*, 2(1), 1-18.
- [5] Alizadeh, A., Emadi, M., Doostparast, M., Cordeiro, G. M., Ortega, E., and Pescim, R. (2015). A New Family of Distributions: The Kumaraswamy Odd Log-Logistic, Properties and Applications. *Hacettepe Journal of Mathematics and Statistics*, 44(6), 1491-1512.
- [6] Alizadeh, M., Bagheri, S., Bahrami, S. E., Ghobadi, S., and Nadarajah, S. (2017). Exponentiated Power Lindley Power Series Class of Distribution: Theory and Applications. *Communications in Statistics-Simulation and Computation*, 47(9), 2499-2531.

- [7] Alizadeh, M., Tahir, M. H., Cordeiro, G. M., Mansoor, M., Zubair, M., and Hamedani, G. G. (2015). The Kumaraswamy Marshall-Olkin Family of Distributions. *Journal of Egyptian Mathematical Society*, 23(3), 546-557.
- [8] Alzaatreh, A., Famoye, F., and Lee, C. (2013). A New Method for Generating Families of Continuous Distributions. *Metron*, 71(1), 63-79.
- [9] Alzaghal, A., Famoye, F. and Lee, C. (2013). Exponentiated T-X Family of Distributions with Some Applications. *International Journal of Probability and Statistics*, 2(3), 31-49.
- [10] Al-Marzouki, S. (2020). Statistical Properties of Type II Topp Leone Inverse Exponential Distribution. *Journal of Nonlinear Sciences and Applications*, 14(1), 1-7.
- [11] Al-Marzouki S., Jamal, F., Chesneau C., and Elgarhy M. (2020). Type II Topp Leone Power Lomax Distribution with Applications. *Mathematics*, 8(1), 4.
- [12] Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S., and Shahbaz, M. Q. (2016). Topp Leone Family of Distributions: Some Properties and Application. *Pakistan Journal of Statistics and Operation Research*, 12(3), 443-451.
- [13] Al-Zahrani, B. (2012). Goodness-of-fit for the Topp-Leone Distribution with Unknown Parameters. *Applied Mathematical Sciences*, 6(128), 6355-6363.
- [14] Andrews, D. F., and Herzberg, A. M. (2012). Data: A Collection Problems from Many Fields for the Student and Research Worker. *Springer Science and Business Media*.
- [15] Barlow, R. E., Toland, R. H., and Freeman, T. (1984). A Bayesian Analysis of Stress-Rupture Life of Kevlar/Epoxy Spherical Pressure Vessels. *Proceedings of the Canadian Conference in Applied Statistics*.
- [16] Barreto-Souza, W., Lemonte, A. J. and Cordeiro, G.M., (2013). General Results for the Marshall and Olkins Family of Distributions. *Annals of the Brazilian Academy of Sciences*, 85(1), 3-21.

- [17] Bidram, H., and Nekoukhou, V. (2013). Double Bounded Kumaraswamy-Power Series Class of Distributions. *Statistics and Operations Research Transactions*, 37(2), 211-230.
- [18] Bourguignon, M., Silva, R. B., and Cordeiro, G. M. (2014). The Weibull-G Family of Probability Distributions. *Journal of Data Science*, 12, 53-68.
- [19] Chahkandi, M., and Ganjali, M. (2009). On Some Lifetime Distributions with Decreasing Failure Rate. *Computational Statistics & Data Analysis*, 53(12), 4433-4440.
- [20] Chambers, J., Cleveland, W., Kleiner, B., and Tukey, P. (1983). Graphical Methods for Data Analysis, *Chapman and Hall*.
- [21] Chen, G., and Balakrishnan, N. (1995). A General Purpose Approximate Goodness-of-Fit Test. *Journal of Quality Technology*, 27(2), 154-161.
- [22] Chhikara, R. S and Folks, J. L. (1977). The Inverse Gaussian Distribution as a Lifetime Model. *Technometrics*, 19(4), 461-468.
- [23] Chipepa, F., and Oluyede, B. (2021). The Marshall-Olkin-Gompertz-G Family of Distributions: Properties and Applications. *Journal of Nonlinear Sciences and Applications*, 14(4), 250-267.
- [24] Chipepa, F., Oluyede, B., and Makubate, B. (2019). A New Generalized Family of Odd Lindley-G Distributions with Applications. *International Journal of Statistics and Probability*, 8(6), 1-22.
- [25] Chipepa, F., Oluyede, B., and Makubate, B. (2020). The Topp-Leone-Marshall-Olkin-G Family of Distributions with Applications. *International Journal of Statistics and Probability*, 9(4), 1-15.
- [26] Chipepa, F., Oluyede, B., Makubate, B., and Fagbamigbe A. F. (2019). The Beta Odd Lindley-G Family of Distributions with Applications. *Journal of Probability and Statistical Science*, 17(1), 51-83.

- [27] Cordeiro, G. M., and de Castro, M. (2011). A New Family of Generalized Distributions. *Journal of Statistical Computation and Simulation* 81(7), 883-898.
- [28] Cordeiro, G. M., and Lemonte, A. J. (2013). On the Marshall-Olkin Extended Weibull Distribution. *Statistical papers*, 54(2), 333-353.
- [29] Cordeiro, G. M., Alizadeh, M., and Diniz Marinho, P. R. (2016). The Type I Half-Logistic Family of Distributions. *Journal of Statistical Computation and Simulation*, 86(4), 707-728.
- [30] Cordeiro, G.M., Alizadeh, M., Ozel, G., Hosseini, B., Ortega, E.M.M., and Altun, E. (2017). The Generalized Odd Log-Logistic Family of Distributions: Properties, Regression Models and Applications. *Journal of Statistical Computation and Simulation* 87(5), 908-932.
- [31] Cordeiro, G. M., Gomes, A., da Silva, C., and Ortega, E. M. M. (2013). The Beta Exponentiated Weibull Distribution. *Journal of Statistical Computation and Simulations*, 38(1), 114-138.
- [32] Cordeiro, G. M., Ortega, E. M. M. and da Cunha, D. C. C. (2013). The Exponentiated Generalized Class of Distributions. *Journal of Data Science*, 11(1), 1-27.
- [33] Cordeiro, G. M. Ortega, E. M. M., and Nadarajaah, S. (2010). The Kumaraswamy Weibull Distribution with Application to Failure Data. *Journal of the Franklin Institute*, 347(8), 1399 -1429.
- [34] Elgarhy, M., Arslan Nasir, M., Jamal, F., and Ozel, G. (2018). The Type II Topp-Leone Generated Family of Distributions: Properties and Applications. *Journal of Statistics and Management Systems*, 21(8), 1529-1551.
- [35] Eugene, N., Lee, C., and Famoye, F. (2002). Beta-Normal Distribution and its Applications. *Communications in Statistics - Theory and Methods*, 31(4), 497-512

- [36] Ghitany, M. E., Kotz, S., and Xie, M. (2005). On Some Reliability Measures and their Stochastic Orderings for the Topp-Leone Distribution. *Journal of Applied Statistics*, 32(7), 715-722.
- [37] Gradshetyn, I. S., and Ryzhik, I. M. (2000). Tables of Integrals, Series and Products, sixth Edition, Academic Press, San Diego.
- [38] Gui, W. (2013). Marshall-Olkin Extended Log-Logistic Distribution and its Application in Minification Processes. *Applied Mathematical Sciences*, 7(80), 3947-3961.
- [39] Hassan, A. S., Elgarhy, M., and Zubair, A. (2019). Type II Generalized Topp-Leone Family of Distributions: Properties and Applications. *Journal of Data Science*, 17(4), 638-659.
- [40] Jamal, F., Reyad, H. M., Nasir, M., Chesneau, A. C., Shah, M. A. A., and Ahmed, S. O. (2019). Topp-Leone Weibull-Lomax Distribution: Properties, Regression Model and Applications. hal-02270561.
- [41] Javed, M., Nawaz, T., and Irfan, M. (2018). The Marshall-Olkin Kappa Distribution. *Journal of King Saud University-Science*, 31(4), 684-691.
- [42] Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). Continuous Distributions, Volume 1, John Wiley and Sons, New York, NY.
- [43] Korkmaz, M. C., Yousof, H. M., and Hamedani, G. G. (2018). The Exponential Lindley Odd Log-Logistic-G Family: Properties, Characterizations and Applications. *Journal of Statistical Theory and Applications*, 17(3), 554-571.
- [44] Krishna, E., Jose, K. K., and Ristic, M. M. (2013). Applications of Marshall-Olkin Fréchet Distribution. *Communications in Statistics-Simulation and Computation*, 42(1), 76-89.
- [45] Kumar, D. (2016). Ratio and Inverse Moments of Marshall-Olkin Extended Burr Type III Distribution Based on Lower Generalized Order Statistics. *Journal of Data Science*, 14(1), 53-66.

- [46] Lazhar, B. (2017). Marshall-Olkin Extended Generalized Gompertz Distribution. *Journal of Data Science*, 15(2), 239-266.
- [47] Lee, E. T., and Wang, J. W. (2003). Statistical Methods for Survival Data Analysis, (3rd edition), Wiley, Hoboken.
- [48] Leiva, V., Barros M., and Paula, G. A. (2009). Generalized Birnbaum-Saunders Models using R. *Brazilian Statistical Association*, No 53, 1511-1528.
- [49] Lepetu, L., Oluyede, B. O., Makubate, B., Foya, S., and Mdlongwa, P. (2017). Marshall-Olkin Log-logistic Extended Weibull: Theory, Properties and Applications. *Journal of Data Science*, 15, 691-722.
- [50] Mahmoudi, E., and Jafari, A. A. (2012). Generalized Exponential Power Series Distributions. *Computational Statistics & Data Analysis*, 56(12), 4047-4066.
- [51] Mahmoudi, E., and Seahdar, A. (2013). Exponentiated Weibull-Poisson Distribution: Model, Properties and Applications. *Mathematics and Computers in Simulation*, 92, 76-97.
- [52] Makubate, B., Moakofi, T., and Oluyede, B. (2021). A New Generalized Lindley-Weibull Class of Distributions: Theory, Properties and Applications. *Mathematica Slovaca*, 71(1), 211-234.
- [53] Mansour, M., M., Elrazik, E. M. A., Altun, E., Afify, A., Z., and Iqbal, Z. (2018). A New Three-Parameter Fréchet Distribution: Properties and Applications. *Pakistan Journal of Statistics* , 34(6), 441-458.
- [54] Marshall, A. N., and Olkin I. (1997). A New Method for Adding a Parameter to a Family of Distributions with Applications to the Exponential and Weibull Families. *Biometrika*, 84, 641-652.
- [55] Mohammed, H. F., and Yahia, N. (2019). On Type II Topp-Leone Inverse Rayleigh Distribution. *Applied Mathematical Sciences*, 13(13), 607615.

- [56] Morais, A. L., and Barreto-Souza, W. (2011). A Compound Class of Weibull and Power Series Distributions. *Computational Statistics & Data Analysis*, 55(3), 1410-1425.
- [57] Murthy, D. N. P., Xie, M., and Jiang, R. (2004). Weibull Models. John Wiley.
- [58] Nadarajah, S., and Kotz, S. (2007). On the Alternative to Weibull Function. *Engineering Fracture Mechanics*, 74(3), 451-456.
- [59] Nadarajah, S., and Kotz, S. (2003). Moments of Some J-Shaped Distributions. *Journal of Applied Statistics*, 30(3), 311-317.
- [60] Nassar, M., Alzaatreh, A., Mead, M., and Abo-Kasem, O. E. (2016). Alpha Power Weibull Distribution: Properties and Applications. *Communication in Statistics-Theory and Methods*, 46(20), 10236-10252.
- [61] Noack, A. (1950). A Class of Random Variables with Discrete Distributions. *The Annals of Mathematical Statistics*, 21(1), 127-132.
- [62] Oguntunde, P. E., Balogun, O.S., Okagbue, H.I., and Bishop, S. A. (2015). The Weibull Exponential Distribution: Its Properties and Applications. *Journal of Applied Sciences*, 15(11), 1305-1311.
- [63] Oluyede, B., Jimoh. H., Wanduku, D., and Makubate. B. (2020). A New Generalized Log-Logistic Erlang Truncated Exponential Distribution with Applications. *Electronic Journal of Applied Statistical Analysis*, 13 (2), 293-349.
- [64] Oluyede, B., Chipepa. F., and Wanduku, D. (2021). The Odd Weibull-Topp-Leone-G Power Series Family of Distributions: Model, Properties, and Applications. *Journal of Nonlinear Science and Application*, 14, 60-65.
- [65] Oluyede, B., Mashabe, B., Fagbamigbe, A., Makubate, B., and Wanduku, D. (2020). The Exponentiated Generalized Power Series: Family of Distributions: Theory, Properties and Applications. *Heliyon*, 6(8).

- [66] Rényi, A. (1960). On Measures of Entropy and Information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, 1, 54 -561.
- [67] Sakthivel, K. M., and Dhivakar, K. (2021). Type II Topp-Leone Dagum Distribution for Modeling Failure Times Data. *Malaya Journal of Matematik*, 9(1), 343-353.
- [68] Sangsanit, Y., and Bodhisuwan, W. (2016). The Topp-Leone Generator of Distributions: Properties and Inferences. *Songklanakarin Journal of Science and Technology*, 38(5), 537-548.
- [69] Shaked, M., and Shanthikumar, J. G. (2007). Stochastic Orders. Springer Science & Business Media.
- [70] Shams, H. S., and Alamatsaz, M. H. (2016). Generalized Linear Failure Rate Power Series Distribution. *Communications in Statistics - Theory and Methods*, 45(8), 2204-2227.
- [71] Shannon, C. E. (1951). Prediction and Entropy of Printed English. *The Bell System Technical Journal*, 30(1), 50-64.
- [72] Silva, R. B., and Cordeiro, G. M. (2015). The Burr XII Power Series Distributions: A New Compounding Family. *Brazilian Journal of Probability and Statistics*, 29(3), 565-589.
- [73] Smith, R. L., and Naylor, J. C. (1987). A Comparison of Maximum Likelihood and Bayesian Estimators for the Three-Parameter Weibull Distribution. *Applied Statistics*, 36(3), 358-369.
- [74] Topp, C. W., and Leone, F. C. (1955). A Family of J-shaped Frequency Functions. *Journal of the American Statistical Association*, 50(269), 209-219.
- [75] Vicaria, D., Dorp, J. R. V., and Kotz, S. (2008). Two-Sided Generalized Topp and Leone Distributions. *Journal of Applied Statistics*, 35(10), 1115-1129.

- [76] Yahia, N., and Mohammed, H. F. (2019). The Type II Topp-Leone Generalized Inverse Rayleigh Distribution. *International Journal of Contemporary Mathematical Sciences*, 14(3), 113122.
- [77] ZeinEldin, R. A., Jamal, F., Chesneau. C., and Elgarhy, M. (2019). Type II ToppLeone Inverted Kumaraswamy Distribution with Statistical Inference and Applications. *Symmetry*, 11(12), 1459.

Appendix

#####CHAPTER 2 APPENDIX

TL-LLP R Algorithms

###Pdf and hazard plot of TL-LLP Distribution

define TLLLP pdf

```
TLLLP_pdf=function(theta,b,c,x){  
  y= 2*b*theta*c*x^(c-1)*(1+x^c)^(-3)*(1-(1+x^c)^(-2))^(b-1)  
  *exp(theta*(1-(1-(1+x^c)^(-2))^(b))))/(exp(theta)-1)  
  return(y)  
}
```

Plot of the pdfs

```
##png(filename="TLLLP_Plot_pdf.png")  
x=seq(0,1.6,by=0.001)  
y1=TLLLP_pdf(0.3, 0.1, 2.5, x)  
plot(x,y1,ylim=c(0,6.0),col=2,'l',lwd=2,xlab="x",ylab="f(x)")  
y2=TLLLP_pdf(9.0, 2.0, 6.0, x)  
lines(x,y2,col=3,lwd=2)  
y3=TLLLP_pdf(3.5, 1.0, 2.0, x)  
lines(x,y3,col=4,lwd=2)  
y4=TLLLP_pdf(5.0, 9.5, 2.1, x)  
lines(x,y4,col=6,lwd=2)  
legend("topright",c(  
  expression(paste(theta,'=0.3',b,'=0.1',c,'=2.5')),  
  expression(paste(theta,'=9.0',b,'=2.0',c,'=6.0')),  
  expression(paste(theta,'=3.5',b,'=1.0',c,'=2.0')),  
  expression(paste(theta,'=5.0',b,'=9.5',c,'=2.1'))),  
  col=c(2,3,4,6),lwd=c(2,2,2,2))
```

define TLLLP hazard function

```
TLLLP_hazard=function(theta,b,c,x){  
  y= 2*b*theta*c*x^(c-1)*(1+x^c)^(-3)*(1-(1+x^c)^(-2))^(b-1)  
  *exp(theta*(1-(1-(1+x^c)^(-2))^(b))))  
  /(exp(theta*(1-(1-(1+x^c)^(-2))^(b))))-1)  
  return(y)  
}
```

Plot of the pdfs

```
##png(filename="TLLLP_Plot_hazard.png")  
x=seq(0,5.5,by=0.001)  
y1=TLLLP_hazard(16.01, 1.0, 1.0, x)  
plot(x,y1,ylim=c(0,20.4),col=2,'l',lwd=2,xlab="x",ylab="h(x)")  
y2=TLLLP_hazard(9.9, 1.9, 3.0, x)  
lines(x,y2,col=3,lwd=2)  
y3=TLLLP_hazard(10.1, 0.1, 1.0, x)  
lines(x,y3,col=4,lwd=2)  
y4=TLLLP_hazard(0.05, 0.05, 7.5, x)  
lines(x,y4,col=6,lwd=2)
```

```

legend("topright",c(
  expression(paste(theta,'=16.1',',', b,'=1.0',',', c,'=1.0')),
  expression(paste(theta,'=9.9',',', b,'=0.9',',', c,'=3.0')),
  expression(paste(theta,'=10.1',',', b,'=0.1',',', c,'=1.0')),
  expression(paste(theta,'=0.05',',', b,'=0.05',',', c,'=7.5'))),
  col=c(2,3,4,6),lwd=c(2,2,2,2))

###TL-LLP Quantile Function

library(numDeriv)
library(Matrix)
library(zipfR)
#Define the quantile of TLLLP distribution
TLLLP_quantile=function(theta,b,c,u){
  f=function(x){
    log((exp(theta)-1)*(1-u)+1)-theta*(1-(1-(1+x^c)^(-2))^(b)) }
  x=uniroot(f,c(0,100))$root
  return(x)
}
### table of quantile
parameter_matrix_quantile=matrix(c(0.8,2.1,0.9,1.0,1.5,1.2,
1.8,2.1,3.5,1.2,5.5,0.6,0.5,1.0,0.3),ncol=3,byrow=T)
parameter_matrix_quantile
list_quantile=seq(0.1,0.9,0.1)
list_quantile
Table_quantile=matrix(0,nrow=length(list_quantile)
,ncol=dim(parameter_matrix_quantile)[1])
row.names(Table_quantile)=list_quantile

for(iter in 1:dim(parameter_matrix_quantile)[1]){
  par=parameter_matrix_quantile[iter,]
  for(i in 1:length(list_quantile)){
    Table_quantile[i,iter]=TLLLP_quantile(par[1],par[2],par[3],
list_quantile[i])
  }
}
Table_quantile

###Application for TL-LLP Distribution

rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)

x<-c( 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05,
3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60,
4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00,
8.16, 8.21, 8.72, 10.40, 13.20, 13.70)

```

```

TLLLP_LL=function(theta,b,c){-sum(log(
2*b*theta*c*x^(c-1)*(1+x^c)^(-3)*(1-(1+x^c)^(-2))^(b-1)
*exp(theta*(1-(1-(1+x^c)^(-2))^(b)))/(exp(theta)-1)))}
mle.result<-mle2(TLLLP_LL,hessian = NULL,start
=list(theta=0.3785e-15 , b=0.0556, c=0.0669),
optimizer="nlminb",lower=0)
summary(mle.result)

###Goodness-of-fit

rm(list=ls())
library(stats4)
library(AdequacyModel)
library(bbmle)
library(stats)

data=c(2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05,
3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60,
4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00,
8.16, 8.21, 8.72, 10.40, 13.20, 13.70)

TLLLP_pdf<-function(par,x){

  theta=par[1]
  b=par[2]
  c=par[3]

  2*theta*b*c*x^(c-1)*(1+x^c)^{-3}*(1-(1+x^c)^(-2))^(b-1)
  *exp(theta*(1-(1-(1+x^c)^(-2))^(b)))/(exp(theta)-1)}

TLLLP_cdf<-function(par,x){

  theta=par[1]
  b=par[2]
  c=par[3]

  1 - (exp(theta*(1-(1-(1+x^c)^(-2))^(b)))-1)/(exp(theta)-1)
  }

goodness.fit(pdf=TLLLP_pdf, cdf=TLLLP_cdf,mle=c(6.5265e-05,
5.2422e+01, 1.3853e+00 ),data = data,method = "BFGS",
domain = c(0,1), lim_inf = c(0,0,0,0),
lim_sup = c(1000000000000,10000000000,1000000000000))

#####CHAPTER 3 APPENDIX
###MO-EOEHL-LLoG R Algorithms

###Pdf and hazard plot of MO-EOEHL-LLoG Distribution

##define MOEOEHLLLoG cdf
MOEOEHLLLoG.cdf <- function(x,alpha,lambda,delta,c){
1-((delta*(1-((1-exp(-lambda*((1-(1+x^c)^(-1)))/((1+x^c)^(-1))))))

```

```

/((1+exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1)))))^alpha))
/((1-(1-delta)*(1-((1-exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1))
))))))
/(1+exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1)))))^alpha))))}

##define MOEOEHLLoG pdf
MOEOEHLLoG.pdf <- function(x,alpha,lambda,delta,c){
((2*alpha*delta*lambda*c*x^(c-1)*(1+x^c)^(-2)
*exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1)))))
*(1-exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1)))))^(alpha-1))
/((1+x^c)^(-2)*(1+exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1))
/((1+x^c)^(-1)))))^(alpha+1)))*(1-(1-delta)
*(1-((1-exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1)))))
/(1+exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1)))))^alpha))^(c-2))

#####plots of pdf
x=seq(0,3.0,by=0.05)
z1=MOEOEHLLoG.pdf(0.09, 1.6, 0.36, 0.001, x)
plot(x,z1,ylim=range(c(0,5)),col=2,'l',lwd=2,xlab="x",
ylab="density")
z2=MOEOEHLLoG.pdf(0.1, 1.1, .6, 0.1, x)
lines(x,z2,col=3,lwd=2)
z3=MOEOEHLLoG.pdf(.01, .4, 3.0, 0.001, x)
lines(x,z3,col=4,lwd=2)
z4=MOEOEHLLoG.pdf(1.61, 10.01, 1.9, 9.04, x)
lines(x,z4,col=5,lwd=2)
z5=MOEOEHLLoG.pdf(1.55, 15.0, 3.1, 10.10, x)
lines(x,z5,col=6,lwd=2)
legend("topright",c(
expression(paste(alpha,'=0.09,', lambda,'=1.6,', delta,'=0.36,',
c,'=0.001')),
expression(paste(alpha,'=0.1,', lambda,'=1.1,', delta,'=0.6,',
c,'=0.1')),
expression(paste(alpha,'=0.01,', lambda,'=0.4,', delta,'=3.0,',
c,'=0.001')),
expression(paste(alpha,'=1.61,', lambda,'=10.01,', delta,'=1.9,',
c,'=9.04')),
expression(paste(alpha,'=1.55,', lambda,'=15.0,', delta,'=3.1,',
c,'=10.10'))),
col=c(2,3,4,5,6),lwd=c(2,2,2,2,2))

###define MOEOEHLLoG hazard
MOEOEHLLoG.hazard <-function(x,alpha,lambda,delta,c){
MOEOEHLLoG.pdf(x,alpha,lambda,delta,c)
/(1 - MOEOEHLLoG.cdf(x,alpha,lambda,delta,c))}

#####plots of hazard
x=seq(0,3.0,by=0.001)
z1=MOEOEHLLoG.hazard(5.5, 9.9, 0.09, .0002, x)
plot(x,z1,ylim=range(c(0,5)),col=2,'l',lwd=2,xlab="x",
ylab="h(x)")

```

```

z2=MOEOEHLLoG_hazard(0.09, .5, .9, 0.9, x)
lines(x,z2,col=3,lwd=2)
z3=MOEOEHLLoG_hazard(0.3, 5.2, 4.01, .2, x)
lines(x,z3,col=4,lwd=2)
z4=MOEOEHLLoG_hazard(0.09, .05, .9, 0.9, x)
lines(x,z4,col=5,lwd=2)
z5=MOEOEHLLoG_hazard(4.5, 9.9, 0.09, 2.0, x)
lines(x,z5,col=6,lwd=2)
legend("topright",c(
  expression(paste(alpha,'=5.5,', lambda,'=9.9,', delta,'=0.09,',
    c,'=0.0002')),
  expression(paste(alpha,'=0.09,', lambda,'=0.5,', delta,'=0.9,',
    c,'=0.9')),
  expression(paste(alpha,'=0.3,', lambda,'=5.2,', delta,'=4.01,',
    c,'=0.2')),
  expression(paste(alpha,'=0.09,', lambda,'=0.05,', delta,'=0.9,',
    c,'=0.9')),
  expression(paste(alpha,'=4.5,', lambda,'=9.9,', delta,'=0.09,',
    c,'=2.0'))),
col=c(2,3,4,5,6),lwd=c(2,2,2,2,2))

```

####Skewness and kurtosis for MO-EOEHL-LLoG

#####skewness

###CDF function

```

F=function(x,par){
  alpha=par[1];lambda =par[2];delta = par[3]; c=par[4]
  C = 1-(((delta*(1-((1-exp(-lambda*((1-(1+x^c))^(-1)))/((1+x^c))^(-1)
  ))))
  /(1+exp(-lambda*((1-(1+x^c))^(-1)))/((1+x^c))^(-1))))^alpha))/
  (((1-(1-delta)*(1-((1-exp(-lambda*((1-(1+x^c))^(-1)))/((1+x^c))^(-1)
  ))))
  /(1+exp(-lambda*((1-(1+x^c))^(-1)))/((1+x^c))^(-1))))^alpha))))
return(C)}

```

#PDF function:

```

f = function(x,par){
  alpha=par[1];lambda =par[2];delta = par[3]; c=par[4]
  D = ((2*alpha*delta*lambda*c*x^(c-1)*(1+x^c)^(-2)
  *exp(-lambda*((1-(1+x^c))^(-1)))/((1+x^c))^(-1)))
  *(1-exp(-lambda*((1-(1+x^c))^(-1)))/((1+x^c))^(-1))))^(alpha-1))
  /(((1+x^c)^(-2)*(1+exp(-lambda*((1-(1+x^c))^(-1)))/((1+x^c))^(-1)
  /((1+x^c))^(-1))))^(alpha+1)))*(1-(1-delta)*
  (1-((1-exp(-lambda*((1-(1+x^c))^(-1)))/((1+x^c))^(-1))))
  /(1+exp(-lambda*((1-(1+x^c))^(-1)))/((1+x^c))^(-1))))^alpha))^(alpha-2)
return(D)}

```

#Hazard function:

```

h = function(x,par) f(x,par)/(1-F(x,par))
library(moments)
alpha = seq(1,4,.1)

```

```

lambda = seq(1,4,.1)
G.skew <- function(alpha, lambda) {
  x <- seq(0, 10, le = 2*10^4)
  U=f(x,c(alpha,lambda, 0.0001,2.35))
  Skewness = skewness(U) # or kurt =kurtosis(U)
  return(Skewness)}
#Skew
G.skew <-Vectorize(G.skew, c('alpha', 'lambda'))
Skewness<-outer(alpha,lambda,G.skew)
persp(alpha,lambda,Skewness,main=expression(paste(
MO-EOEHL-LLoG(alpha,lambda,0.0001,2.35))),
col = "springgreen", shade = 0.5)

#####kurtosis

###CDF function
F=function(x,par){
  alpha=par[1];lambda =par[2];delta = par[3]; c=par[4]
  C = 1-((delta*(1-((1-exp(-lambda*((1-(1+x^c)^(-1)))/((1+x^c)^(-1)
))))))
  /(1+exp(-lambda*((1-(1+x^c)^(-1)))/((1+x^c)^(-1))))^alpha))/
  ((1-(1-delta)*(1-((1-exp(-lambda*((1-(1+x^c)^(-1)))/((1+x^c)^(-1)
))))))
  /(1+exp(-lambda*((1-(1+x^c)^(-1)))/((1+x^c)^(-1))))^alpha)))
return(C)}
#PDF function:
f = function(x,par){
  alpha=par[1];lambda =par[2];delta = par[3]; c=par[4]
  D = ((2*alpha*delta*lambda*c*x^(c-1)*(1+x^c)^(-2)
*exp(-lambda*((1-(1+x^c)^(-1)))/((1+x^c)^(-1))))
*(1-exp(-lambda*((1-(1+x^c)^(-1)))/((1+x^c)^(-1))))^(alpha-1))
/((1+x^c)^(-2)*(1+exp(-lambda*((1-(1+x^c)^(-1))
/((1+x^c)^(-1))))^(alpha+1)))*(1-(1-delta)*
(1-((1-exp(-lambda*((1-(1+x^c)^(-1)))/((1+x^c)^(-1))))
/(1+exp(-lambda*((1-(1+x^c)^(-1)))/((1+x^c)^(-1))))^alpha))^(alpha-2)
return(D)}
#Hazard function:
h = function(x,par) f(x,par)/(1-F(x,par))
#The program is developed in R to obtain the value of skewness:
library(moments)
alpha = seq(1,4,.1)
lambda = seq(1,4,.1)
G.kurt <- function(alpha, lambda) {
  x <- seq(0, 10, le = 2*10^4)
  U=f(x,c(alpha,lambda, 0.00005,2.99))
  Kurtosis = kurtosis(U) # or kurt =kurtosis(U)
  return( Kurtosis )}
#Kurt
G.kurt <-Vectorize(G.kurt, c('alpha', 'lambda'))
Kurtosis <-outer(alpha,lambda,G.kurt)
persp(alpha,lambda,Kurtosis,main=expression(paste(

```

```

MO-EOEHL-LLoG(alpha,lambda,0.00005,2.99)),
col = "springgreen", shade = 0.5)

#####CHAPTER 4 APPENDIX
##EOEHL-LLoGP R Algorithms

###Pdf and hazard plot of EOEHL-LLoGP Distribution

##define EOEHL-LLoGP cdf
EOEHL_LLoGP_cdf <- function(x,alpha,lambda,theta,c){
1-((exp(theta*(1-((1-exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1)
))))
/(1+exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1))))))^alpha))-1)
/(exp(theta)-1))}

##define EOEHL-LLoGP pdf
EOEHL_LLoGP_pdf <- function(x,alpha,lambda,theta,c){
(2*theta*alpha*lambda*c*x^(c-1)*(1+x^c)^(-2)
*exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1))))
*(1-exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1))))^(alpha-1))
/((1+x^c)^(-2)*(1+exp(-lambda*((1-(1+x^c)^(-1)
/((1+x^c)^(-1))))^(alpha+1))
*(exp(theta*(1-((1-exp(-lambda*((1-(1+x^c)^(-1)
/((1+x^c)^(-1))))/(1+exp(-lambda*((1-(1+x^c)^(-1)
/((1+x^c)^(-1))))^(alpha)))/(exp(theta)-1))}

x=seq(0,3.0,by=0.05)
z1=EOEHL_LLoGP_pdf(0.6, 5.0, 5.50, 0.09, x)
plot(x,z1,ylim=range(c(0,5)),col=2,'l',lwd=2,xlab="x",
ylab="density")
z2=EOEHL_LLoGP_pdf(0.53, 8.1, 6.6, 0.01, x)
lines(x,z2,col=3,lwd=2)
z3=EOEHL_LLoGP_pdf(0.01, 0.8, 0.5, 1.9, x)
lines(x,z3,col=4,lwd=2)
z4=EOEHL_LLoGP_pdf(0.65, 12.5, 7.0, 7.9, x)
lines(x,z4,col=5,lwd=2)
z5=EOEHL_LLoGP_pdf(0.01, 0.7, 0.09, 0.9, x)
lines(x,z5,col=6,lwd=2)
legend("topright",c(
expression(paste(alpha,'=0.6,', lambda,'=5.0,', theta,'=5.5,',
c,'=0.09')),
expression(paste(alpha,'=0.53,', lambda,'=8.1,', theta,'=6.6,',
c,'=0.01')),
expression(paste(alpha,'=0.01,', lambda,'=0.8,', theta,'=0.5,',
c,'=1.9')),
expression(paste(alpha,'=0.65,', lambda,'=12.5,', theta,'=7.0,',
c,'=7.9')),
expression(paste(alpha,'=0.01,', lambda,'=0.7,', theta,'=0.09,',
c,'=0.9'))),
col=c(2,3,4,5,6),lwd=c(2,2,2,2,2))

```

```

## define EOEHL-LLoGP hazard
EOEHL_LLoGP_hazard <-function(x,alpha,lambda,theta,c){
  EOEHL_LLoGP_pdf(x,alpha,lambda,theta,c)
  /(1 - EOEHL_LLoGP_cdf(x,alpha,lambda,theta,c))}

x=seq(0,4.0,by=0.001)
z1=EOEHL_LLoGP_hazard(5.0, 0.09, 0.1, 0.001, x)
plot(x,z1,ylim=range(c(0,5)),col=2,'l',lwd=2,xlab="x",ylab="h(x)")
z2=EOEHL_LLoGP_hazard(0.009, 0.5, 0.45, 0.01, x)
lines(x,z2,col=3,lwd=2)
z3=EOEHL_LLoGP_hazard(0.1, 0.55, 0.5, 3.2, x)
lines(x,z3,col=4,lwd=2)
z4=EOEHL_LLoGP_hazard(0.09, 0.05, 0.35, 0.9, x)
lines(x,z4,col=5,lwd=2)
z5=EOEHL_LLoGP_hazard(6.0, 3.0, 0.09, 20, x)
lines(x,z5,col=6,lwd=2)
legend("topright",c(
  expression(paste(alpha,'=5.0',lambda,'=0.09',theta,'=0.1',
    c,'=0.001')),
  expression(paste(alpha,'=0.009',lambda,'=0.5',theta,'=0.45',
    c,'=0.01')),
  expression(paste(alpha,'=0.1',lambda,'=0.55',theta,'=0.5',
    c,'=3.2')),
  expression(paste(alpha,'=0.09',lambda,'=0.05',theta,'=0.35',
    c,'=0.9')),
  expression(paste(alpha,'=6.0',lambda,'=3.0',theta,'=0.09',
    c,'=20.0'))),
  col=c(2,3,4,5,6),lwd=c(2,2,2,2,2))

#####Application

rm(list=ls())
library(stats4)
library(bbmle)
library(numDeriv)
x<-c(0.08, 4.98, 25.74, 3.7, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4,
2.02, 21.73, 2.09, 6.97, 0.5, 5.17, 14.77, 4.18, 10.75, 2.83,
11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34,
16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74,
2.64, 7.59, 43.01, 5.49, 1.4, 11.98, 6.54, 6.93, 6.94, 0.4,
5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65,
8.66, 2.26, 7.26, 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76,
12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14,
5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05,
5.41, 79.05, 7.93, 4.5, 2.02, 0.2, 7.09, 25.82, 3.82, 14.83,
2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32,
34.26, 4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54,
7.32, 0.9, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07)

EOEHL_LLoP_LL<-function(alpha,lambda,theta,c)

```

```
{-sum(log(2*theta*alpha*lambda*c*x^(c-1)*(1+x^c)^(-2)
*exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1))))
*(1-exp(-lambda*((1-(1+x^c)^(-1))/((1+x^c)^(-1))))^(alpha-1))
/((1+x^c)^(-2)*(1+exp(-lambda*((1-(1+x^c)^(-1))
/((1+x^c)^(-1))))^(alpha+1))
*(exp(theta*(1-((1-exp(-lambda*((1-(1+x^c)^(-1))
/((1+x^c)^(-1))))/(1+exp(-lambda*((1-(1+x^c)^(-1))
/((1+x^c)^(-1))))^(alpha)))/(exp(theta)-1))}
mle.result<-mle2(EOEHLLoPLL,hessian= NULL,start=list
(alpha=0.0229,lambda=0.456,theta=0.2901,c=0.2514),
optimizer="nlminb",lower=0)
summary(mle.result)
```

```
#####CHAPTER 5 APPENDIX
#####MO-TII-TL-W R Algorithms
```

```
###Pdf and hazard plots of MO-TII-TL-W Distribution
```

```
##define MOTIITLW cdf
MOTIITLW_cdf <- function(delta,b,lambda,x){
  1 - (delta*(1-(1-exp(-x^lambda))^(2))^(b))/(1-(1-delta)
*(1-(1-exp(-x^lambda))^(2))^(b))}

##define MO-TII-TL-W pdf
MOTIITLW_pdf <- function(delta,b,lambda,x){
  (2*b*delta*lambda*(x^(lambda-1))*exp(-x^lambda)
*(1-exp(-x^lambda))*((1-(1-exp(-x^lambda))^(2))^(b-1))
/(1-(1-delta)*(1-(1-exp(-x^lambda))^(2))^(b))^(2) }

##pdf plots
x=seq(0,1.2,by=0.001)
y1=MOTIITLW_pdf(0.8, 2.0, 3.5, x)
plot(x,y1,ylim=c(0,6.0),col=2,'l',lwd=2,xlab="x",ylab="f(x)")
y2=MOTIITLW_pdf(0.10, 19.9, 1.3, x)
lines(x,y2,col=3,lwd=2)
y3=MOTIITLW_pdf( 2.0, 10.0, 4.0,x)
lines(x,y3,col=4,lwd=2)
y4=MOTIITLW_pdf(0.01, 10.0, 4.0, x)
lines(x,y4,col=5,lwd=2)
y5=MOTIITLW_pdf( 0.4, 16.0, 0.1,x)
lines(x,y5,col=6,lwd=2)
legend("topright",c(
  expression(paste(delta,'=0.8,', b,'=2.0,', lambda,'=3.5')),
  expression(paste(delta,'=0.10,', b,'=19.9,', lambda,'=1.3')),
  expression(paste(delta,'=2.0,', b,'=10.0,', lambda,'=4.0')),
  expression(paste(delta,'=0.01,', b,'=10.0,', lambda,'=4.0')),
  expression(paste(delta,'=0.4,', b,'=16.0,', lambda,'=0.1'))),
col=c(2,3,4,5,6),lwd=c(2,2,2,2,2))

## define MOTIITLW hazard
MOTIITLW_hazard <-function(delta,b,lambda,x){
  MOTIITLW_pdf(delta,b,lambda,x)
```

```

/(1 -MOTIITLW_cdf(delta ,b,lambda,x))}

##hazard plots
x=seq(0,3.5,by=0.001)
y1=MOTIITLW_hazard(0.08, 2.0, 0.66, x)
plot(x,y1,ylim=c(0,3.0),col=2,'l',lwd=2,xlab="x",ylab="h(x)")
y2=MOTIITLW_hazard(1.0, 0.05, 7.0, x)
lines(x,y2,col=3,lwd=2)
y3=MOTIITLW_hazard(3.3, 1.0, 0.1, x)
lines(x,y3,col=4,lwd=2)
y4=MOTIITLW_hazard(0.08, 0.9, 1.6, x)
lines(x,y4,col=5,lwd=2)
y5=MOTIITLW_hazard(1.6, 6.0, 0.6, x)
lines(x,y5,col=6,lwd=2)
legend("topright",c(
  expression(paste(delta,'=0.08',', b','=2.0',', lambda','=0.66')),
  expression(paste(delta,'=1.0',', b','=0.05',', lambda','=7.0')),
  expression(paste(delta,'=3.3',', b','=1.0',', lambda','=0.1')),
  expression(paste(delta,'=0.08',', b','=0.9',', lambda','=1.6')),
  expression(paste(delta,'=1.6',', b','=6.0',', lambda','=0.6'))),
  col=c(2,3,4,5,6),lwd=c(2,2,2,2,2))

####Moments and Quantile

rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library(rootSolve)

##define MO-TII-TL-W pdf
MOTIITLW_pdf <- function(delta ,b,lambda,x){
  (2*b*delta*lambda*(x^(lambda-1))*exp(-x^lambda)
  *(1-exp(-x^lambda))*((1-(1-exp(-x^lambda))^(2))^(b-1)))
  /(1-(1-delta)*(1-(1-exp(-x^lambda))^(2))^(b))^(2) }

##define MOTIITLW cdf
MOTIITLW_cdf <- function(delta ,b,lambda,x){
  1 - (delta*(1-(1-exp(-x^lambda))^(2))^(b))
  /(1-(1-delta)*(1-(1-exp(-x^lambda))^(2))^(b)) }

## define MOTIITLW hazard
MOTIITLW_hazard <-function(delta ,b,lambda,x){
  MOTIITLW_pdf(delta ,b,lambda,x)
  /(1 -MOTIITLW_cdf(delta ,b,lambda,x)) }

MOTIITLW_neglogl=function(delta ,b,lambda,x){
  -sum(log(MOTIITLW_pdf(delta ,b,lambda,x))) }

MOTIITLW_hazard2=function(delta ,b,lambda,x){

```

```

    f=MOTIITLW.pdf( delta ,b,lambda ,x)
    F=MOTIITLW.cdf( delta ,b,lambda ,x)
    y=f/(1-F)
    return(y)
  }
  MOTIITLW.quantile=function( parameter ,u){
delta=parameter[1]
b=parameter[2]
lambda=parameter[3]
f=function(x){
  MOTIITLW.cdf( delta ,b,lambda ,x)-u
}
x=min( uniroot. all ( f ,lower=0,upper=1000000000000,tol=0.0001))
return(x)
}
MOTIITLW.QuantileTable=function( parameter_matrix){
  u=seq(0.1,0.9,0.1)
  size=dim( parameter_matrix)[1]
  Table_Quantile=matrix(NA,nrow=length(u),ncol=size)
  row.names( Table_Quantile)=u
  colnames( Table_Quantile)=apply( parameter_matrix,1,function(x)
  {paste0( '(' ,paste0(x,collapse=',') ,")" )})
  Table_Quantile
  for( iter in 1:size){
    parameter=parameter_matrix[ iter ,]
    for(i in 1:length(u)){
      Table_Quantile[i,iter]=MOTIITLW.quantile(parameter,u[i])
    }
  }
  return( Table_Quantile )
}
## table of quantile
parameter_matrix=as.matrix(rbind(
  par1=c(1.1,1.5,0.9),
  par2=c(1.2,0.1,1.1),
  par3=c(.5,0.3,0.8),
  par4=c(0.4,1.7,0.8),
  par5=c(0.5,0.2,1.2)
))
MOTIITLW.QuantileTable( parameter_matrix )
sink("C:/Users/RK15000146/Desktop/MO-TII-TL-G/
MOTIITLW_quantile_table.txt")
print( parameter_matrix )
print( MOTIITLW.QuantileTable( parameter_matrix ) )
sink()

###moments

MOTIITLW.moments=function( parameter ,k){
  delta=parameter[1]
  b=parameter[2]

```

```

lambda=parameter[3]
f=function(x){
  (x^k)*(MOTIITLW_pdf(delta,b,lambda,x))
}
y=integrate(f,lower=0,upper=1,subdivisions = 10000000)
return(y$value)
}
MOTIITLW_MomentsTable=function(parameter_matrix,K){
  list=c(paste0('EX',1:K),'SD','CV','CS','CK')
  Table_Moments=matrix(NA,nrow=length(list),
    ncol=dim(parameter_matrix)[1])
  row.names(Table_Moments)=list
  colnames(Table_Moments)=apply(parameter_matrix,1,function(x)
    {paste0('(' ,paste0(x,collapse=',') ,")")})
  Table_Moments
  for(iter in 1:dim(parameter_matrix)[1]){
    parameter=parameter_matrix[iter,]
    for(i in 1:K){
      Table_Moments[i,iter]=MOTIITLW_moments(parameter,i)
    }

    Table_Moments['SD',iter]=sqrt(Table_Moments['EX2',iter]
      -Table_Moments['EX1',iter]^2)
    Table_Moments['CV',iter]=sqrt(Table_Moments['EX2',iter]
      /Table_Moments['EX1',iter]^2-1)
    Table_Moments['CS',iter]=(Table_Moments['EX3',iter]
      -3*Table_Moments['EX1',iter]*Table_Moments['EX2',iter]
      +2*Table_Moments['EX1',iter]^3)/((Table_Moments['EX2',iter]
      -Table_Moments['EX1',iter]^2)^(3/2))
    Table_Moments['CK',iter]=(Table_Moments['EX4',iter]
      -4*Table_Moments['EX1',iter]*Table_Moments['EX3',iter]
      +6*Table_Moments['EX1',iter]^2*Table_Moments['EX2',iter]
      -3*Table_Moments['EX1',iter]^4)/((Table_Moments['EX2',iter]
      -Table_Moments['EX1',iter]^2)^2)
  }
  return(Table_Moments)
}

## Moments Table
parameter_matrix=as.matrix(rbind(
  par1=c(1.5,0.9,0.3),
  par2=c(2.0,0.9,0.2),
  par3=c(1.5,0.7,0.5),
  par4=c(1.1,2.0,0.2),
  par5=c(1.2,2.9,0.4)
))
K=5
MOTIITLW_MomentsTable(parameter_matrix,K)

sink("C:/Users/RK15000146/Desktop/MO-TII-TL-G/
MOTIITLW_moment_table.txt")

```

```

print(parameter_matrix)
print(MOTIITLW_MomentsTable(parameter_matrix,K))
sink()

###Skewness and Kurtosis Plots

#####skewness
##CDF Function
F=function(x,par){
  delta=par[1];b =par[2];lambda = par[3]
  C = 1 -(delta*(1-(1-exp(-x^lambda))^(2))^(b))/(1-(1-delta)
    *(1-(1-exp(-x^lambda))^(2))^(b))
  return(C)}
#PDF function:
f = function(x,par){
  delta=par[1];b =par[2];lambda = par[3]
  D = (2*b*delta*lambda*(x^(lambda-1))*exp(-x^lambda)
    *(1-exp(-x^lambda))*((1-(1-exp(-x^lambda))^(2))^(b-1)))
    /(1-(1-delta)*(1-(1-exp(-x^lambda))^(2))^(b))^(2)
  return(D)}
#Hazard function:
h = function(x,par) f(x,par)/(1-F(x,par))
library(moments)
delta = seq(1,4,.1)
b = seq(1,4,.1)
G_skew <- function(delta , b) {
  x <- seq(0, 10, le = 2*10^4)
  U=f(x,c(delta ,b, 29.5))
  Skewness = skewness(U) # or kurt =kurtosis(U)
  return(Skewness)}
#Skew
G_skew <-Vectorize(G_skew, c('delta', 'b'))
Skewness<-outer(delta ,b,G_skew)
persp(delta ,b,Skewness,main=expression(paste(
MO-TII-TL-W(delta ,b,29.5))),col = "springgreen", shade = 0.5)

#####kurtosis
##CDF Function
F=function(x,par){
  delta=par[1];b =par[2];lambda = par[3]
  C = 1 -(delta*(1-(1-exp(-x^lambda))^(2))^(b))/(1-(1-delta)
    *(1-(1-exp(-x^lambda))^(2))^(b))
  return(C)}
#PDF function:
f = function(x,par){
  delta=par[1];b =par[2];lambda = par[3]
  D = (2*b*delta*lambda*(x^(lambda-1))*exp(-x^lambda)
    *(1-exp(-x^lambda))*((1-(1-exp(-x^lambda))^(2))^(b-1)))
    /(1-(1-delta)*(1-(1-exp(-x^lambda))^(2))^(b))^(2)
  return(D)}
#Hazard function:

```

```

h = function(x,par) f(x,par)/(1-F(x,par))
library(moments)
delta = seq(1,4,.1)
b = seq(1,4,.1)
G_kurt <- function(delta, b) {
  x <- seq(0, 10, le = 2*10^4)
  U=f(x,c(delta,b, 21.0))
  Kurtosis = kurtosis(U) # or kurt =kurtosis(U)
  return( Kurtosis )}
#Kurt
G_kurt <-Vectorize(G_kurt, c('delta', 'b'))
Kurtosis <-outer(delta,b,G_kurt)
persp(delta,b,Kurtosis ,main=expression(paste(
MO-TII-TL-W(delta,b,21.0))), col = "springgreen", shade = 0.5)

```

#####Application

```

rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
x<-c(0.01,0.01,0.02,0.02,0.02,0.03,0.03,0.04,0.05,0.06,0.07,
0.07,0.08,0.09,0.09,0.10,0.10,0.11,0.11,0.12,0.13,0.18,0.19,
0.20,0.23,0.24,0.24,0.29,0.34,0.35,0.36,0.38,0.40,0.42,0.43,
0.52,0.54,0.56,0.60,0.60,0.63,0.65,0.67,0.68,0.72,0.72,0.72,
0.73,0.79,0.79,0.80,0.80,0.83,0.85,0.90,0.92,0.95,0.99,1.00,
1.01,1.02,1.03,1.05,1.10,1.10,1.11,1.15,1.18,1.20,1.29,1.31,
1.33,1.34,1.40,1.43,1.45,1.50,1.51,1.52,1.53,1.54,1.54,1.55,
1.58,1.60,1.63,1.64,1.80,1.80,1.81,2.02,2.05,2.14,2.17,2.33,
3.03,3.03,3.34,4.20,4.69,7.89)

MOTITLW_LL<-function(delta,b,lambda){-sum(log(
(2*b*delta*lambda*(x^(lambda-1))*exp(-x^lambda)
*(1-exp(-x^lambda))*((1-(1-exp(-x^lambda))^(2))^(b-1)))
/(1-(1-delta)*(1-(1-exp(-x^lambda))^(2))^(b))^(2) )}
mle.result<-mle2(MOTITLW_LL,hessian = NULL,start=list(
delta=0.00001, b=0.001, lambda=0.00001),optimizer="nlminb",
lower=0)
summary(mle.result)

###Goodness of fit

library(AdequacyModel)

data=c(0.01,0.01,0.02,0.02,0.02,0.03,0.03,0.04,0.05,0.06,0.07,
0.07,0.08,0.09,0.09,0.10,0.10,0.11,0.11,0.12,0.13,0.18,0.19,
0.20,0.23,0.24,0.24,0.29,0.34,0.35,0.36,0.38,0.40,0.42,0.43,
0.52,0.54,0.56,0.60,0.60,0.63,0.65,0.67,0.68,0.72,0.72,0.72,

```

```
0.73,0.79,0.79,0.80,0.80,0.83,0.85,0.90,0.92,0.95,0.99,1.00,
1.01,1.02,1.03,1.05,1.10,1.10,1.11,1.15,1.18,1.20,1.29,1.31,
1.33,1.34,1.40,1.43,1.45,1.50,1.51,1.52,1.53,1.54,1.54,1.55,
1.58,1.60,1.63,1.64,1.80,1.80,1.81,2.02,2.05,2.14,2.17,2.33,
3.03,3.03,3.34,4.20,4.69,7.89)
```

```
MOTIITLW_pdf<-function(par,x){
  delta=par[1]
  b=par[2]
  lambda=par[3]
```

```
(2*b*delta*lambda*(x^(lambda-1))*exp(-x^lambda)
*(1-exp(-x^lambda))*((1-(1-exp(-x^lambda))^(2))^(b-1)))
/(1-(1-delta)*(1-(1-exp(-x^lambda))^(2))^(b))^(2) }
```

```
MOTIITLW_cdf<-function(par,x){
  delta=par[1]
  b=par[2]
  lambda=par[3]
```

```
1 - (delta*(1-(1-exp(-x^lambda))^(2))^(b))/
(1-(1-delta)*(1-(1-exp(-x^lambda))^(2))^(b)) }
```

```
goodness.fit(pdf=MOTIITLW_pdf, cdf=MOTIITLW_cdf,mle=c(5.053405,
4.404747,0.423137 ),data = data,method = "BFGS",
domain = c(0,1), lim_inf = c(0,0,0,0),
lim_sup = c(1000000000000,10000000000,1000000000000))
```