SOME NEW AND GENERALIZED FAMILIES OF PROBABILITY DISTRIBUTIONS WITH APPLICATIONS

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Certification

This undersigned certifies that he has read and hereby recommends for acceptance by the College of Science a dissertation/thesis titled: SOME NEW AND GENERALIZED FAMILIES OF PROBABILITY DISTRIBUTIONS WITH APPLICATIONS, in fulfilment of the requirements for the Degree of Master of Science in Statistics of BIUST.

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Dedication

I dedicate this work to my father and mother for their unwavering support and encouragement.

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Contents

1 Introduction **1**

1.1 Introduction 1

1.2 Some Generators of Distributions 3

1.2.1 Competing risk method 4

1.2.2 T-X family method 4

1.2.3 Beta-G Generator 4

1.2.4 Kumaraswamy-G Generator 5

1.2.5 Marshall-Olkin-G Generator 5

1.2.6 Half-Logistic-G Generator 6

1.3 Some Properties of Distributions 6

1.3.1 Survival, Hazard and Reverse hazard functions 6

1.3.2 Quantile Function 7

1.3.3 Moments 7

1.3.4 Conditional Moments 7

1.3.5 Mean Deviations 7

1.3.6 Bonferroni and Lorenz Curves 8

1.3.7 Entropy 8

1.3.8 Order Statistics 8

1.3.9 Maximum Likelihood Estimation 9

1.3.10 Goodness-of-fit Statistics 9

1.4 Aims and Objectives 10

1.5 Outline of Thesis 11

2 The New Gamma Lindley Log-Logistic Distribution **12**

2.1 Introduction 12

2.2 The Model, Sub-models, Hazard and Quantile Functions 14
2.2.1 Zografos and Balakrishnan Model 14
2.2.2 The Model 14
2.2.3 Series Expansion of the Density Function 15
2.2.4 Sub-models of GLLLoG Distribution 17
2.2.5 Hazard and Quantile Functions 17
2.3 Moments, Conditional Moments and Mean Deviations 19
2.3.1 Moments 19
2.3.2 Conditional Moments 22
2.3.3 Mean Deviation, Lorenz and Bonferroni Curves 25
2.3.4 Mean Deviations 25
2.4 Order Statistics and Rényi Entropy 26
2.4.1 Order Statistics 26
2.4.2 Rényi Entropy 26
2.5 Estimation and Inference 30
2.5.1 Maximum Likelihood Estimation 30
2.5.2 Fisher Information Matrix and Asymptotic Confidence Intervals 31
2.6 Monte Carlo Simulations 31
2.7 Applications 33
2.7.1 Exceedances of Wheaton river flood Data 34
2.7.2 Quarterly earnings per Johnson and Johnson Share 1960-1980 data set 35
2.8 Concluding Remarks 37

3 The New Weibull Generalised-G Family of Distributions 38
3.1 Introduction 38
3.2 The Model, Sub-models, Hazard and Quantile Functions, Shape, Series Expansion of Density Function, and Some Special Cases . . 40
3.2.1 The Model 40
3.2.2 Hazard and Quantile Functions 41
3.2.3 Shapes 41
3.2.4 Series Expansion 42

3.2.5 Some Special Cases 44
3.3 Moments and Incomplete Moments 54
3.3.1 Moments and Generating Function 54
3.3.2 Incomplete Moments 55
3.4 Order Statistics and Rényi Entropy 56
3.4.1 Order Statistics 56
3.4.2 Rényi Entropy 58
3.5 Maximum Likelihood Estimation and Other Estimation Techniques 59
3.5.1 Maximum Likelihood Estimation 60
3.5.2 Ordinary Least Squares 61
3.5.3 Weighted Least Squares 61
3.5.4 Minimum Distance Methods 62
3.5.5 Maximum Product of Spacings Method 63
3.6 Simulation Study 63
3.7 Applications 65
3.7.1 Failure times of 50 components 67
3.7.2 Strengths of the glass fibres data set 69
3.8 Concluding Remarks 71

4 Exponentiated Half Logistic Odd Lindley-G Family of Distributions 72
4.1 Introduction 72
4.2 The Model, Sub-models, Hazard and Quantile Functions, Series Expansion of Density Function, and Some Special Cases 73
4.2.1 The Model 73
4.2.2 Hazard and Quantile Functions 74
4.2.3 Series Expansion of the Density function 75
4.2.4 Some Sub-Families of EHLOL-G Family of Distributions . . 77
4.2.5 Some Special Cases 78
4.3 Moments and Incomplete Moments 84
4.3.1 Moments and Generating Function 84
4.3.2 Incomplete Moments 85
4.4 Order Statistics, Probability Weighted Moments and Rényi Entropy 86

4.4.1 Order Statistics 86
4.4.2 Probability Weighted Moments 87
4.4.3 Rényi Entropy 88
4.5 Maximum Likelihood Estimation 89
4.6 Simulation Study 91
4.7 Applications 92
4.7.1 Relief times of 20 patients 94
4.7.2 Strengths of the glass fibres data set 95
4.8 Concluding Remarks 96

5 Conclusions and Suggestions for Further Studies 98
5.1 Conclusions 98
5.2 Suggestions for Further Studies 98

List of Figures

2.1 Plots of GLLLoG density function 15
2.2 Plots of GLLLoG hazard function 18
2.3 Fitted densities and probability plots for the Wheaton River flooding data set. 35
2.4 Fitted densities and probability plots for Quarterly earnings per Johnson and Johnson Share 1960-1980. 37

3.1 Plots of NWG-U density function 45
3.2 Plots of the NWG-U hazard function 46
3.3 Plots of NWG-LLoG density function 47
3.4 Plots of NWG-LLoG hazard function 47
3.5 Plots of NWG-N density function 48
3.6 Plots of NWG-N hazard function 49
3.7 Plots of NWG-HL density function 50
3.8 Plots of the NWG-HL hazard function 50
3.9 Plots of NWG-Gamma density function 51
3.10 Plots of the NWG-Gamma hazard function 52
3.11 Plots of NWG-W density function 53
3.12 Plots of NWG-W hazard function 53
3.13 Fitted densities and probability plots for the failure times data set. 69
3.14 Fitted densities and probability plots for strength of the glass fibre data set. 71

4.1 Plots of EHLOL-W density function 78
4.2 Plots of EHLOL-W hazard rate function 79
4.3 Plots of EHLOL-U density function 80

4.4 Plots of EHLOL-U hazard rate function 80
4.5 Plots of EHLOL-Kum density function 81
4.6 Plots of EHLOL-Kum hazard rate function 82
4.7 Plots of EHLOL-Fr density function 83
4.8 Plots of EHLOL-Fr hazard rate function 83
4.9 Fitted densities and probability plots for relief times of 20 patients data set. 95
4.10 Fitted densities and probability plots for strengths of the glass fibres data set. 96

List of Tables

1 List of Publications, Accepted and Submitted Papers xiv

2.1 **Table of quantiles for selected parameter values of the GLLLoG Distribution** 18
2.2 **Moments of GLLLoG distribution for some parameter values** 21
2.3 **Monte Carlo Simulation Results** 32
2.4 **Monte Carlo Simulation Results** 33
2.5 **Estimation of the GLLLoG model for Wheaton River Flooding data set** 34
2.6 **Estimation of the GLLLoG model for Quarterly earnings per Johnson and Johnson Share 1960-1980** 36

3.1 **Table of quantiles for selected parameter values of the NWG-W distribution** 54
3.2 **Table of Moments for the NWG-W distribution for selected parameter values** 55
3.3 **Monte Carlo Simulation Results** 64
3.4 **Monte Carlo Simulation Results** 65
3.5 **Estimation of the NWG-W model for failure times of 50 components data set** 68
3.6 **Estimation of the NWG-W model for strengths of the glass fibres data set** 70

4.1 **Table of quantiles for selected parameter values of the EHLOL-W distribution** 84
4.2 **Table of Moments for the EHLOL-W distribution for selected parameter values** 85

4.3 **Monte Carlo Simulation Results** 91
4.4 **Monte Carlo Simulation Results** 92
4.5 **Estimation of the EHLOL-W model for relief times of 20 patients data set** 94
4.6 **Estimation of the EHLOL-W model for strengths of the glass fibres data set** 95

Abstract

Some new generalized distributions called the new Gamma Lindley-Log-logistic (GLLLoG) distribution, new Weibull Generalised-G (NWG-G) family of distributions and the Exponentiated Half-Logistic Odd Lindley-G family of distributions are proposed and presented. These distributions contain several distributions such as Half-Logistic, Odd Lindley, Lindley, Log-Logistic, new-Rayleigh generalized-G, new exponential generalized-G, new Rayleigh-G and the new-Weibull-G family of distributions. A comprehensive investigation of the properties of these generalized distributions including series expansion of pdfs, cdfs, moments, hazard functions, quantile functions, moments, conditional moments, mean deviations, Bonferroni and Lorenz curves, Rényi entropy and distribution of order statistics are presented. Model parameters are estimated using maximum likelihood estimation technique and real datasets are used to illustrate the potentiality of the three new generalized distributions compared with other distributions.

Mathematics Subject Classification : 62E15; Secondary 60E15

Keywords: Generalized Distributions; Generalized-G Distribution; Exponentiated distribution; Odd Lindley; Half Logistic, Gamma Distribution; Lindley distribution; New Weibull Distribution; Log-logistic distribution; Maximum likelihood estimation; Order Statistics; Moments.

List of Publications, Accepted and Submitted Papers

Table 1: List of Publications, Accepted and Submitted Papers

Distribution	Status	Journal
The New Gamma Lindley Log-Logistic (GLLLoG) Distribution	Submitted	EJASA
The New Weibull Generalised-G (NWG-G) Family of Distributions	Submitted	Chilean Journal of Statistics
The Exponentiated Half-Logistic Odd Lindley-G (EHLOL-G) Family of Distributions	Accepted	Journal of Nonlinear Sciences and Applications

Chapter 1

Introduction

1.1 Introduction

Of recent, technical improvements have led to increased demand for using adequate statistical models that can realistically consider the important characteristics of the data and the processes being studied. This has led to the need for generalizations or modifications of the well-known classical distributions so as to accommodate and provide more flexibility in the modelling of different real life applications. In many applied and practical areas such as lifetime analysis, finance and insurance, there is a clear need for extended and generalized forms of distributions. This increased demand for more flexible models has motivated statisticians to work on improving classical models by adding some extra shape, location and scale parameters on these models. For that reason, several generalisations that are suitable for modelling data with different types of the hazard rate functions: decreasing, increasing, bathtub, upside-down bathtub and many others have been studied. Methods for generating new families of distributions have also been studied and are present in statistical literature.

Lindley [43] used a mixture of exponential and length-biased exponential

distributions to illustrate the difference between fiducial and posterior distributions. The resulting mixture is called the Lindley (L) distribution. Unlike the exponential distribution which has a constant hazard rate function, the Lindley distribution possess various shapes of the hazard rate and has proved to be more flexible in modelling and analysis of life time data. The Lindley distribution has also been generalized by Sankaran [61], Zakerzadeh and Dolati [68], Mazucheli and Achcar [47], Nadarajah et al. [51], Oluyede and Yang [55], Gupta and Singh [38], Ghitany et al. ([33],[34]), Warahena-Liyanage and Pararai [66], Bakouch et al. [9], and Gomes-Silva et al. [35], the extended models provided more flexibility in the analysis of life time data.

Weibull [67] developed the Weibull distribution by taking the powers of exponentially distributed random variables. The same technique was employed in devloping the gamma distribution which was also developed by taking the powers of the Laplace transform of the exponential distribution. The resulting models proved to be more flexible than the classical exponential distribution [45]. There are also many generalizations of the Weibull distribution in literature and these include the exponentiated Weibull Family by Mudholkar et al. ([48], [49]), the Weibull-G by Bourguignon et al. [13], the log-generalized Lindley-Weibull distribution with application by Oluyede et al. [52], the generalized inverse Weibull distribution by Gusmao et al. [39], a generalized Weibull distribution for lifetime modeling by Carrasco et al. [14], beta-Weibull and its properties by Famoye et al. [30], the Weibull Lindley distribution by Asgharzadeh et al. [8], the Beta Weibull Poisson distribution by Percontini et al. [58], the Gamma log-logistic Weibull distribution by Foya et al. [31], Beta Weibull log-logistic distribution by Makubate et al. [46], Kumaraswamy Weibull distribution by Cordeiro et al. [24], Exponentiated Weibull Lindley distribution by Cordeiro et al. [22], Weibull Lomax distribution by Tahir et al. [64] and the transmuted Weibull-G by Alizadeh et al. [6].

The Half-Logistic distribution was developed by Balakrishnan [10] by transforming the Logistic distribution. The density function of the Half-Logistic distribution has uni-modal or reversed-J shape, this act as a limitation in life time data analysis since most real data exhibits hazard rate functions that are

non-monotonic. Due to these limitations, there is a great need to modify and generalize it so as to add flexibility in data analysis. Some generalizations of the Half-Logistic model includes the Odd exponentiated Half-Logistic-G family by Afify et al. [1], the Exponentiated Half Logistic family by Cordeiro et al. [20], the type 1 Half-Logistic family by Cordeiro et al. [19] and the Kumaraswamy type 1 Half-Logistic Family by El-sayed and Mahmoud [28].

Many methods for generating new family of distributions have also been introduced and studied including the transformer (T-X) by Alzaghal et al. [7], Weibull-G by Bourguignon et al. [13], beta-G by Eugene et al. [29], Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [25], McDonald-G (Mc-G) by Alexander et al. [3], Lomax generator by Cordeiro et al. [26], Kumaraswamy odd log-logistic family by Alizadeh et al. [4], and Kumaraswamy Marshall-Olkin family by Alizadeh et al. [5], the exponentiated generalized class of distributions by Cordeiro et al. [23] allows for greater flexibility in the analysis of heavy tail data and is widely applied in many areas of engineering and biology. Barreto-Souza et al. [11] stated that adding parameters to an established baseline distribution is a proven technique for obtaining more flexible new families of distributions.

The versatility and flexibility of the Lindley, Log-Logistic and Weibull distributions in modeling lifetime data and also the need for modifications of the Half-Logistic model has motivated us to propose and study some new distributions called the new Gamma Lindley Log-Logistic (GLLLoG) distribution and the new Weibull generalized-G (NWG-G) family of distributions and the Exponentiated Half-Logistic Odd Lindley-G (EHLOL-G) family of distributions. The proposed new distributions generalizes the Weibull, Log-Logistic, Half-logistic, Odd-Lindley and the Lindley distributions.

1.2 Some Generators of Distributions

In this section, we discuss in brief the generators used in this thesis to extend some lifetime distributions.

1.2.1 Competing risk method

In this method, the survival functions $S_{X_1}(x)$ and $S_{X_2}(x)$ of two continuous independent random variables say X_1 and X_2 respectively are combined to obtain a new distribution as follows, let $X = min(X_1, X_2)$, then the cdf of X is given by

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= 1 - [1 - F_1(x)][1 - F_2(x)] \\ &= 1 - (S_{X_1}(x))(S_{X_2}(x)). \end{aligned} \tag{1.1}$$

For more examples on the use of competing risks method, see Codeiro et al. [21], Oluyede et al. [54] and Jiang [41].

1.2.2 T-X family method

The $T - X$ family of distributions was developed by Alzaghal et al. [7] and is defined as follows,

$$F(x) = \int_a^{W(G(x))} r(t)dt, \tag{1.2}$$

and

$$f(x) = \Big\{\frac{d}{dx}W[G(x)]\Big\}r\{W[G(x)]\}, \tag{1.3}$$

where $r(t)$ is the pdf of a random variable $T \in [a, b]$, for $-\infty \leq a < b \leq \infty$ and $WG(x))$ is a function of the cdf $G(x)$ of any random variable X so that $W(G(x))$ satisfies the following conditions:

(1) $W(G(x)) \in [a, b]$,

(2) $W(G(x))$ is differentiable and monotonically non-decreasing, and

(3) $W(G(x)) \to a$ as $x \to -\infty$ and $W(G(x)) \to b$ as $x \to \infty$.

1.2.3 Beta-G Generator

Suppose $g(x)$ and $G(x)$ are the probability density function (pdf) and the cumulative distribution function (cdf) of a continuous random variable X respectively; then the cdf and pdf of the beta-G distribution are given by

$$F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)} t^{a-1}(1 - t)^{b-1}dt, \tag{1.4}$$

and

$$f(x) = \frac{1}{B(a,b)}[G(x)]^{a-1}[1-G(x)]^{b-1}g(x), \tag{1.5}$$

respectively, for $a > 0, b > 0$, where $I_y(a,b) = B_y(a,b)/B(a,b)$ is the incomplete beta ratio, $B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(.)$ is the gamma function.

1.2.4 Kumaraswamy-G Generator

Cordeiro and de Castro [25], put forward the Kw-G distribution which states that from an arbitrary parent cumulative distribution function (cdf) $G(x)$, a new distribution with cdf say $F(x)$ is given by

$$F(x) = 1 - (1 - G(x)^a)^b, \tag{1.6}$$

and its corresponding probability density function (pdf) $f(x)$ is given by

$$f(x) = abg(x)(G(x))^{a-1}(1 - G(x)^a)^{b-1} \tag{1.7}$$

for $x > 0, a > 0$ and $b > 0$.

1.2.5 Marshall-Olkin-G Generator

Marshall and Olkin [44] developed the Marshall-Olkin (MO) extended family of distributions. For any baseline distribution with pdf $f(x)$ and cdf $F(x)$, Marshall and Olkin [44] defined a new survival function $F(x)$ by introducing an additional parameter $\alpha > 0$. The survival function is defined by

$$\overline{F}_{MO}(x) = \frac{\alpha\overline{F}(x)}{1 - \overline{\alpha}\overline{F}(x)}, \tag{1.8}$$

where $-\infty < x < \infty, \alpha > 0, \overline{\alpha} = 1 - \alpha, \alpha$ is known as the tilt parameter since the hazard rate function of the new family is shifted either above or below the hazard rate function of the baseline distribution for $0 < \alpha < 1$ and $\alpha > 1$, respectively. The cdf and the pdf of the MO-extended distribution are given by

$$F_{MO}(x) = \frac{F(x)}{1 - \overline{\alpha}\overline{F}(x)}, \tag{1.9}$$

and

$$f_{MO}(x) = \frac{\alpha f(x)}{[1-\overline{\alpha}\overline{F}(x)]^2}, \tag{1.10}$$

respectively, where $f(x)$ and $F(x)$ are any respective baseline pdf and cdf.

1.2.6 Half-Logistic-G Generator

Cordeiro et al. [19], developed the type 1 half-logistic family of distributions with the cdf given by

$$\begin{aligned} G(x;\lambda,\theta) &= \int_0^{-log(1-F(x;\theta))} \frac{2\lambda \exp\{-\lambda x\}}{(1+\exp\{-\lambda x\})} dx \\ &= \frac{1-[1-F(x;\theta)]^{\lambda}}{1+[1-F(x;\theta)]^{\lambda}}, \end{aligned} \tag{1.11}$$

where $F(x;\theta)$ is the cdf of the baseline distribution and $\lambda > 0$, is the shape parameter.

1.3 Some Properties of Distributions

1.3.1 Survival, Hazard and Reverse hazard functions

For a continuous random variable X with a cdf $F(x)$ and pdf $f(x) = d(F(x)/dx$, then the survival function which gives the probability of a device, patient or any other objects of interest surviving more than any specified time x is given by

$$S(x) = \overline{F}(x) = 1 - F(x) = \int_t^{\infty} f(x)dx. \tag{1.12}$$

$F(x)$ gives the probability of surviving up to time x.
The hazard and the reverse hazard functions are given by

$$h(x) = \frac{f(x)}{S(x)}, \tag{1.13}$$

and

$$r(x) = \frac{f(x)}{F(x)}, \tag{1.14}$$

respectively.

1.3.2 Quantile Function

Let X be a random variable with the cdf $F(x)$, where $x \in R$. The quantile function $Q(u)$ of X is the solution of

$$F(Q(u)) = u, u \in (0,1), \tag{1.15}$$

where $Q(u)$ is the inverse of the cdf.

1.3.3 Moments

The r^{th} moment about the origin is given by $E(X^r)$, and the r^{th} moment about the mean is given by $E(X-\mu)^r$. The r^{th} moment about the origin for a continuous distribution with pdf $f(x)$ is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx. \tag{1.16}$$

It is of great importance to derive the moments when a new distribution is being studied since they play a very crucial role in statistical analysis especially in applications. Moments are crucial in studying the characteristics of a distribution such as dispersion, central tendency, kurtosis and skewness.

1.3.4 Conditional Moments

The r^{th} conditional moment for a continuous distribution with pdf $f(x)$ is given by

$$E(X^r|X > t) = \frac{1}{S(x)} \int_t^{\infty} x^r f(x)dx, \tag{1.17}$$

where $S(x)$ is the survival function.

1.3.5 Mean Deviations

The mean deviation about the mean and about the median for a random variable X with pdf $f(x)$ are defined by

$$\sigma_1(x) = \int_0^{\infty} \mid x - \mu \mid f(x)dx, \tag{1.18}$$

and

$$\sigma_1(x) = \int_0^{\infty} \mid x - M \mid f(x)dx, \tag{1.19}$$

respectively, where $\mu = E(X)$ and $M = \text{median}(X)$.

1.3.6 Bonferroni and Lorenz Curves

The Bonferroni curve was introduced by Bonferroni [12]. Let X be a non-negative continuous random variable with mean μ and a cdf $F(x)$. Then the Bonferroni curve is defined as

$$\begin{aligned} B(p) &= \frac{1}{p\mu}\int_0^q xf(x)dx \\ &= \frac{1}{p\mu}[\mu - T(q)], \end{aligned} \tag{1.20}$$

where $T(q) = \int_q^\infty xf(x)dx$.

Gastwirth [32] defined the Lorenz curve as

$$\begin{aligned} L(p) &= \mu^{-1}\int_0^q xf(x)dx \\ &= \frac{1}{\mu}[\mu - T(q)] \end{aligned} \tag{1.21}$$

where μ is the mean of the underlying income distribution, $q = F^{-1}(p)$ and $0 \leq p \leq 1$. Bonferroni and Lorenz curves are widely used as a graphical representation of wealth and income distributions.

1.3.7 Entropy

Entropy is a measure of variation of uncertainty for a random variable X with pdf $f(x)$. There are two popular measures of entropy, namely Shannon entropy and Rényi entropy. Rényi entropy (Rényi [59]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty [f(x)]^v dx\right), v \neq 1, v > 0. \tag{1.22}$$

Rényi entropy tends to Shannon entropy as $v \to 1$.

1.3.8 Order Statistics

The order statistics of a random sample $X_1, ..., X_n$ are the sample values placed in ascending order, denoted as $X_{(1)}, ..., X_{(n)}$ and they are the random variables that satisfy $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq .... \leq X_{(n)}$. The cdf and the pdf of the i^{th} order statistic, for $i = 1, ..., n$ are defined by

$$F_{i:n}(x) = \sum_{i=1}^{n}\binom{n}{i}F(x)^i[1-F(x)]^{n-i}, \tag{1.23}$$

and

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} [1 - F(x)]^{n-i}, \tag{1.24}$$

respectively.

1.3.9 Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ have a joint density denoted as

$$f_\theta(x_1, x_2, x_3.....x_n) = f(x_1, x_2, x_3....x_n|\theta), \tag{1.25}$$

with respect to some parameter $\theta \in R$.
Then the likelihood of θ is a function written as,

$$L(\theta) = f(x_1, x_2, x_3....x_n|\theta). \tag{1.26}$$

The maximum likelihood estimate of the parameter θ is the value that maximizes $L(\theta)$ if available. If the $X_i's$ are independent and identically distributed, then the likelihood simplifies to

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta). \tag{1.27}$$

Since the logarithm is an increasing function, maximizing $L(\theta)$ is equivalent to maximizing its log function denoted as

$$\begin{aligned} \ell(\theta) &= log(L(\theta)) \\ &= \sum_{i=1}^{n} log(f(x_i|\theta)). \end{aligned} \tag{1.28}$$

1.3.10 Goodness-of-fit Statistics

Goodness-of-fit statistics tests how well the observed data corresponds to the fitted model by testing the hypothesis that
H_0 : the model M_0 fits
vs
H_a : the model M_0 does not fit.
The goodness-of-fit statistics include Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information

Criterion (BIC), the -2Log-likelihood statistics, Cramer-von-Misses (W^*), Anderson-Darling (A^*) and Kolmogorov-Smirnov test (K-S) statistics among others, they are respectively denoted as

$$AIC = 2p - 2plog(L), \quad CAIC = AIC + \frac{2p(p+1)}{n-p-1}$$
$$BIC = p \times log(n) - 2log(L), \quad K - S = sup_x \mid F_n(x) - F(x) \mid,$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \leq x}$ is the empirical distribution function, $F(x)$ is the cdf, p is the number of parameters in the statistical model, n is the sample size and $L = L(\hat{\theta})$ denote the value of the likelihood function under the considered model. The Anderson-Darling and Cramer-von-Misses statistics are given by

$$A^* = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[ln(F(x_{(i)})) + ln(1 - F(x_{(n-i+1)}))] \tag{1.29}$$

and

$$W^* = \frac{1}{12n} + \sum_{i=1}^{n}\Big[\frac{2i-1}{2n} - F(x_{(i)})\Big], \tag{1.30}$$

respectively, where n is the sample size, $F(x_{(i)})$ is the cdf of the specified distribution and $x_{(i)}$ is the i^{th} sample when the data is sorted in ascending order.

1.4 Aims and Objectives

The key aim is to develop new generalized distributions and derive their structural properties and to introduce models that could offer a better fit when modelling lifetime data. The specific objectives are:

1. To propose and develop three new class of distributions.
2. To obtain mathematical properties of the proposed distributions.
3. To estimate the parameters of the GLLLoG, NWG-G and EHLOL-G distributions using maximum likelihood method.
4. To illustrate the performance of the proposed models.

1.5 Outline of Thesis

The outline of this thesis is as follows: In Chapter 2, the generalized distribution, namely new Gamma Lindley log-logistic distribution including the hazard and reverse hazard functions, quantile function and various sub-models, moments, conditional moments, mean deviations, Bonferroni and Lorenz curves, Rényi entropy and order statistics are presented. Estimates of model parameters comparisons with other models, simulation study to examine the performance of the maximum likelihood estimators of the GLLLoG distribution and applications of the proposed model to real data are also given in Chapter 2. We introduce the new Weibull generalized G family of distributions and its properties including the hazard and reverse hazard functions, quantile function and sub-models, moments, conditional moments, mean deviations, Bonferroni and Lorenz curve, Rényi entropy and distribution of order statistics in Chapter 3. Estimation of parameters, simulation study and some real data examples to illustrate the usefulness and applicability of the NWG-G distribution are also given in Chapter 3. In Chapter 4, we introduce the Exponentiated Half-Logistic Odd Lindley-G family of distributions and its properties including the hazard and reverse hazard functions, quantile function and sub-models, moments, conditional moments, mean deviations, Bonferroni and Lorenz curve, Rényi entropy and distribution of order statistics. Estimation of parameters, simulation study and some real data examples to illustrate the usefulness and applicability of the EHLOL-G family of distributions are also given in Chapter 4. Summary and areas of further studies are presented in Chapter 5.

Chapter 2

The New Gamma Lindley Log-Logistic Distribution

2.1 Introduction

There are very useful and important generalizations of the Lindley distribution in the literature that are suitable for modeling data with different types of hazard rate functions: increasing, decreasing, bathtub and unimodal.

The one parameter cdf of the Lindley (L) distribution is given by

$$G_L(x;\lambda) = 1 - \frac{\lambda + 1 + \lambda x}{\lambda + 1} e^{-\lambda x}, \quad \text{for } x > 0, \text{ and } \lambda > 0. \tag{2.1}$$

The corresponding Lindley pdf is given by

$$g_L(x;\lambda) = \frac{\lambda^2(1+x)}{1+\lambda} e^{-\lambda x}, \quad \text{for } x > 0, \text{ and } \lambda > 0. \tag{2.2}$$

Lindley distribution is a mixture of exponential and gamma distributions, that is $f(x;\lambda) = (1-p)f_G(x;\lambda) + pf_E(x;\lambda)$ with $p = \frac{1}{1+\lambda}$, where $f_G(x;\lambda) \equiv GAM(2,\lambda)$, and $f_E(x;\lambda) \equiv EXP(\lambda)$.

Nadarajah et al. [51] presented the mathematical and statistical properties of the exponentiated or generalized Lindley (GL) distribution. The cumulative distribution function (cdf) of the GL distribution is given by

$$G_{GL}(x;\alpha,\lambda) = \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha}, \tag{2.3}$$

and the corresponding GL probability density function (pdf) is given by

$$g_{GL}(x;\alpha,\lambda) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha-1}\exp(-\lambda x), \qquad (2.4)$$

for $x > 0$, $\lambda > 0$, $\alpha > 0$.

We use the competing risk transformation as well as the Zografos and Balakrishnan [69] transformation to obtain a new distribution involving both the Lindley and log-logistic distributions. The new distribution is quite useful and generalized the Lindley and log-logistic distributions, and is a more flexible distribution for the description of reliability and lifetime data. The combined distribution of Lindley and log-logistic is obtained from the product of the reliability or survival functions of the Lindley and log-logistic distributions via competing risk model, followed by Zografos and Balakrishnan transformation to obtain the gamma Lindley Log-logistic distribution. A motivation for developing this model is the advantages presented by this extended distribution with respect to having a hazard function that exhibits increasing, decreasing and bathtub shapes, as well as the versatility and flexibility of generalized distributions in general, the Lindley and log-logistic distributions in modelling lifetime data.

This chapter is organized as follows. In section 2.2, some basic results of the GLLLoG distribution and its sub-models, hazard function and the quantile function are presented. The moments and moment generating function, mean and median deviations are given in section 2.3. Section 2.4 contains some additional useful results on the distribution of order statistics and Rényi entropy. In section 2.5, results on the estimation of the parameters of the GLLLoG distribution via the method of maximum likelihood are presented. A Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators in section 2.6. Applications are given in section 2.7, followed by some concluding remarks in section 2.8.

2.2 The Model, Sub-models, Hazard and Quantile Functions

In this section, we develop some properties of the gamma Lindley log-Logistic (GLLLoG) distribution including expansion of the density, hazard function, quantile function and sub-models, moments, conditional moments and maximum likelihood estimation of model parameters.

2.2.1 Zografos and Balakrishnan Model

Consider the family of distributions with the pdf $f(x)$ and cdf $F(x)$ given by:

$$f(x) = \frac{1}{\Gamma(\delta)\psi^{\delta}} \left[-log(1-G(x))\right]^{\delta-1} (1-G(x))^{\frac{1}{\psi}-1} g(x), x \in R, \delta > 0, \tag{2.5}$$

and

$$F(x) = \frac{1}{\Gamma(\delta)\psi^{\delta}} \int_0^{-log(1-G(x))} t^{\delta-1} e^{-t/\psi} dt = \frac{\gamma\left(\delta, -\psi^{-1} log(1-G(x))\right)}{\Gamma(\delta)}, \tag{2.6}$$

respectively, where $\gamma(x,\delta) = \int_0^x t^{\delta-1} e^{-t} dt$ is the incomplete gamma function. The corresponding hazard function is

$$h_F(x) = \frac{\left[-log(1-G(x))\right]^{\delta-1} f(x)(1-G(x))^{1/\psi-1}}{\psi^{\delta}\left(\Gamma(\delta) - \gamma\left(-\psi^{-1} log(1-G(x)), \delta\right)\right)}.$$

When ψ=1, this distribution is referred to as the Zografos and Balakrishnan-G (ZB-G) family of distributions [69].

2.2.2 The Model

The cdf and pdf of the Lindley log-logistic (LLLoG) distribution (Oluyede et al. [53]) are given by

$$G(x;\lambda,c) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}, \tag{2.7}$$

and

$$g(x;\lambda,c) \quad = \quad \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right]$$

for $\lambda, c > 0$. If a random variable X has the LLLoG distribution, we write $X \sim LLLoG(\lambda, c)$.

The cdf and pdf of the proposed gamma Lindley log-logistic (GLLLoG) distribution are given by

$$\begin{aligned} F(x;\lambda,c,\delta) &= \frac{1}{\Gamma(\delta)}\int_0^{-log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)\right)} t^{\delta-1}e^{-t}dt \\ &= \frac{\gamma(-log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)\right),\delta)}{\Gamma(\delta)}, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} f(x;\lambda,c,\delta) &= \frac{1}{\Gamma(\delta)}\left[-log\left[1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)\right]\right]^{\delta-1} \\ &\times \frac{(1+x^c)^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right] \end{aligned} \tag{2.9}$$

for $\lambda, c, \delta > 0$. If a random variable X has the GLLLoG distribution, we write $X \sim GLLLoG(\lambda, c, \delta)$. Plots of the GLLLoG pdf are given in figure 2.1. The plots reveal different shapes including right-skewed, almost symmetric, J-shape among several others.

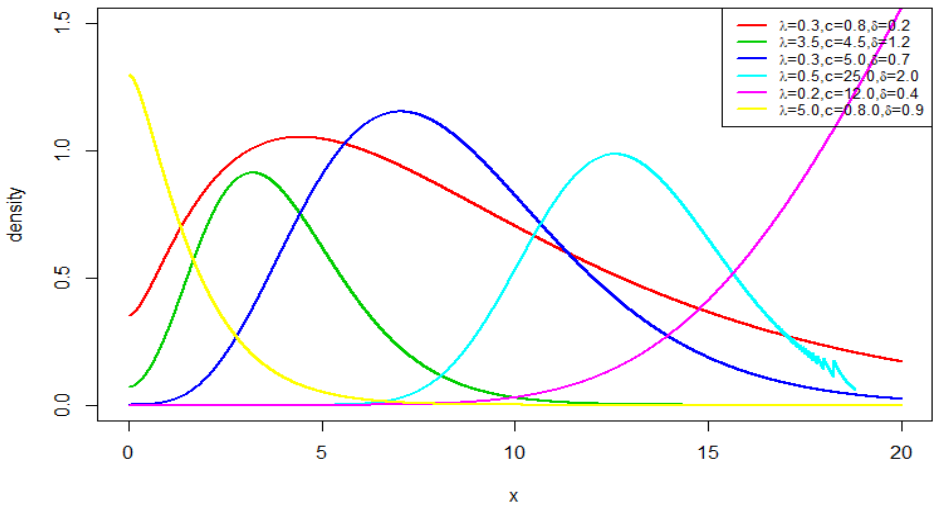


Figure 2.1: Plots of GLLLoG density function

2.2.3 Series Expansion of the Density Function

A series expansion of the GLLLoG pdf is presented below. Let

$$y = \left[1-\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)}\right],$$

$0 < y < 1$. Using the series representation

$$-log(1-y) = \sum_{i=1}^{\infty} \frac{y^{i+1}}{i+1}, \tag{2.10}$$

we have

$$\Big[-log(1-y) \Big]^{\delta-1} = y^{\delta-1} \sum_{m=0}^{\infty} \binom{\delta-1}{m} y^m \sum_{s=0}^{\infty} \Big(\frac{y^s}{s+2} \Big)^m. \tag{2.11}$$

Applying the result on a power series raised to a positive integer with $a_s = (s+2)^{-1}$, that is,

$$\Big(\sum_{s=0}^{\infty} a_s y^s \Big)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \tag{2.12}$$

where

$$b_{s,m} = (sa_0)^{-1} \sum_{i=1}^{s} [m(l+i) - s] a_l b_{s-l,m} \quad \text{and} \quad b_{0,m} = a_0^m, \tag{2.13}$$

(Gradshteyn and Ryzhik [36]), the pdf of the GLLLoG distribution can be expressed as

$$\begin{aligned} f(x;\lambda,c,\delta) &= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} y^{m+s+\delta-1} \frac{(1+x^c)^{-1}}{\Gamma(\delta)(1+\lambda)} e^{-\lambda x} \\ &\times \Big[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \Big] \\ &= \frac{1}{\Gamma(\delta)} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \Big[1 - \frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)} \Big]^{m+s+\delta-1} \\ &\times \frac{(1+x^c)^{-1}}{(1+\lambda)} e^{-\lambda x} \Big[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \Big]. \end{aligned}$$

Consequently the GLLLoG pdf can be written as

$$f(x;\lambda,c,\delta) = \frac{1}{\Gamma(\delta)} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} \frac{b_{s,m}}{m+s+\delta} f_{ELLLoG}(x;\lambda,c,m+s+\delta), \tag{2.14}$$

where $f_{ELLLoG}(x;\lambda,c,m+s+\delta)$ is the exponentiated Lindley-Log-Logistic (ELLLoG) pdf with exponentiation parameter $m+s+\delta > 0$. Therefore the mathematical and statistical properties of the GLLLoG distribution follows from those of the ELLLoG distribution.

2.2.4 Sub-models of GLLLoG Distribution

In this subsection, some sub-models are presented.

- When $\lambda \to 0^+$, we obtain the gamma log-logistic (GLLoG) distribution.
- When $\delta = 1$, we obtain the Lindley log-logistic (LLLoG) distribution.
- When $\lambda \to 0^+$, and $\delta = 1$, we obtain the log-logistic (LLoG) distribution.
- If $\delta = c = 1$, we obtain the one parameter distribution denoted by $GLLLoG(\lambda, 1, 1)$, with cdf

$$F(x;\lambda) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x)}, \quad \lambda > 0. \tag{2.15}$$

- If $c = 1$, we obtain the two-parameter distribution with pdf

$$\begin{aligned} f(x;\lambda,\delta) &= \frac{1}{\Gamma(\delta)}\left[-log\left[1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x)}\right)\right]\right]^{\delta-1} \\ &\times \frac{(1+x)^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)}{1+x}\right] \end{aligned} \tag{2.16}$$

  for $\lambda, \delta > 0$.

2.2.5 Hazard and Quantile Functions

In this section, we present the hazard and quantile functions of the GLLLoG distribution. The hazard rate function of the GLLLoG distribution is given by

$$\begin{aligned} h_F(x) &= \frac{f(x)}{\overline{F}(x)} = \frac{\left[-log\left[1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)\right]\right]^{\delta-1}}{\Gamma(\delta)-\gamma\Big(-log\Big(1-\Big(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\Big)\Big),\delta\Big)} \\ &\times \frac{(1+x^c)^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right]. \end{aligned} \tag{2.17}$$

Plots of the GLLLoG hazard shows different shapes including decreasing, increasing, bathtub followed by upside down, as well as upside down bathtub shapes.

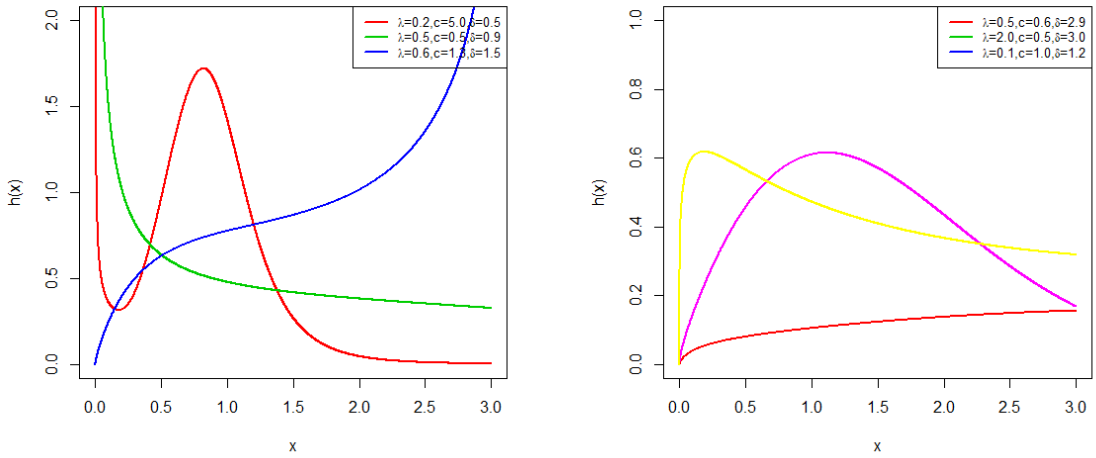


Figure 2.2: Plots of GLLLoG hazard function

Let X be a random variable with a cdf as given in Eqn (2.8). Then, the quantile function $Q_X(u)$ is defined by $F_X(Q_X(u)) = u$, $0 \leq u \leq 1$, that is, we are solving the non-linear equation:

$$\frac{1}{\Gamma(\delta)}\gamma\left(-log\left(1-\left(1-\frac{1+\lambda+\lambda Q_X(u)}{1+\lambda}\frac{e^{-\lambda Q_X(u)}}{(1+(Q_X(u))^c)}\right)\right),\delta\right)=u. \tag{2.18}$$

That is,

$$\frac{(1+\lambda+\lambda Q_X(u))e^{-\lambda Q_X(u)}}{(1+\lambda)(1+(Q_X(u))^c)}-e^{(-\gamma^{-1}(u\Gamma(\delta),\delta))}=0. \tag{2.19}$$

Thus, random numbers can be readily generated from the GLLLoG distribution by numerically solving the non-linear equation (2.19).

Table 2.1: **Table of quantiles for selected parameter values of the GLLLoG Distribution**

	(λ, c, δ)				
u	$(0.3, 1.5, 2)$	$(2.0, 3.5, 0.9)$	$(0.5, 4.0, 1.5)$	$(0.9, 5.0, 1.8)$	$(0.5, 0.9, 2)$
0.1	0.7118	0.0619	0.6266	0.6388	0.5082
0.2	1.0453	0.1389	0.7686	0.7882	0.8859
0.3	1.3752	0.2268	0.8732	0.8886	1.2835
0.4	1.7394	0.3246	0.9663	0.9736	1.7248
0.5	2.1686	0.4319	1.0574	1.0548	2.2323
0.6	2.7057	0.5508	1.1532	1.1388	2.8383
0.7	3.4268	0.6889	1.2605	1.2328	3.6008
0.8	4.5015	0.8720	1.3903	1.3478	4.6471
0.9	6.4758	1.2524	1.5642	1.5091	6.3781

2.3 Moments, Conditional Moments and Mean Deviations

In this section, we present the moments, conditional moments and mean deviations of the GLLLoG distribution.

2.3.1 Moments

Moments are useful because some features and important characteristics of a distribution such as skewness and kurtosis can be obtained through them. The r^{th} moment of the GLLLoG distribution, denoted by μ_r is

$$\begin{aligned}\mu_r = E(X^r) &= \frac{1}{\Gamma(\delta)}\sum_{m,s,k=0}^{\infty}\frac{(-1)^k\Gamma(m+s+\delta)b_{s,m}}{\Gamma(m+s+\delta-k)(1+\lambda)^{k+1}k!}\binom{\delta-1}{m} \\ &\times \int_0^{\infty} x^r\frac{(1+\lambda+\lambda x)^k}{(1+x^c)^{k+1}}e^{-(k+1)\lambda x}\Big[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\Big]dx,\end{aligned}$$

where $b_{s,m}$ is defined as in Eqn (2.13).

Lemma 1. Let

$$L_1(\lambda,c,r,k) = \int_0^{\infty} x^r\frac{(1+\lambda+\lambda x)^k}{(1+x^c)^{k+1}}e^{-(k+1)\lambda x}\Big[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\Big]dx.$$

Then

$$\begin{aligned}L_1(\lambda,c,r,k) &= \sum_{p=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^p(k+1)^p(1+\lambda)^k\lambda^{k+p-j+2}}{p!} \\ &\times \Big[B\Big(k-\frac{p+r+k}{c}+2,\frac{p+r+k+j}{c}+1\Big) \\ &+ B\Big(\frac{p+r+k-j+1}{c},\frac{p+r+k+j}{c}+1\Big)\Big] \\ &+ \sum_{p=0}^{\infty}\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(-1)^p(k+1)^p(1+\lambda)^{k+1}\lambda^{k+p-j+1}}{p!} \\ &\times B\Big(k-\frac{p+r+k+1}{c}+2,\frac{p+r+c+k-j+1}{c}\Big). \qquad (2.20)\end{aligned}$$

Proof:

Using the series expansion

$$e^{-(k+1)\lambda x} = \sum_{p=0}^{\infty}\frac{(-1)^p(k+1)^p\lambda^p x^p}{p!},$$

we have

$$\begin{aligned} L_1(\lambda,c,r,k) &= \sum_{p=0}^{\infty}\frac{(-1)^p(k+1)^p\lambda^p}{p!}\int_0^{\infty}x^{p+r}(1+\lambda+\lambda x)^k(1+x^c)^{-(k+1)} \\ &\times \Big[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\Big]dx \\ &= \sum_{p=0}^{\infty}\frac{(-1)^p(k+1)^p\lambda^p}{p!}\Bigg[\int_0^{\infty}x^{p+r}(1+\lambda+\lambda x)^k\lambda^2(1+x)(1+x^c)^{-(k+1)}dx \\ &+ \int_0^{\infty}x^{p+r+c-1}(1+\lambda+\lambda x)^{k+1}(1+x^c)^{-(k+2)}dx\Bigg]. \end{aligned} \tag{2.21}$$

$$\begin{aligned} \text{Let}\quad M_1 &= \int_0^{\infty}x^{p+r}(1+\lambda+\lambda x)^k\lambda^2(1+x)(1+x^c)^{-(k+1)}dx \\ \text{and}\quad M_2 &= \int_0^{\infty}x^{p+r+c-1}(1+\lambda+\lambda x)^{k+1}(1+x^c)^{-(k+2)}dx. \end{aligned}$$

Using the binomial expansion

$$(a+b)^n=\sum_{r=0}^{n}\binom{n}{r}a^rb^{n-r}, \tag{2.22}$$

it follows that,

$$\begin{aligned} M_1 &= \sum_{j=0}^{k}\binom{k}{j}(1+\lambda)^k\lambda^{k-j+2}\int_0^{\infty}x^{p+r+c+k-j-1}(1+x)(1+x^c)^{-(k+1)}dx \\ &= \sum_{j=0}^{k}\binom{k}{j}(1+\lambda)^k\lambda^{k-j+2}\Bigg[\int_0^{\infty}x^{p+r+c+k-j-1}(1+x^c)^{-(k+1)}dx \\ &+ \int_0^{\infty}x^{p+r+c+k-j}(1+x^c)^{-(k+1)}dx\Bigg] \end{aligned}$$

and

$$M_2 = \sum_{j=0}^{k+1}\binom{k+1}{j}(1+\lambda)^{k+1}\lambda^{k-j+1}\int_0^{\infty}x^{p+r+c+k-j}(1+x^c)^{-(k+2)}dx.$$

$$\text{Let}\quad t=(1+x^c)^{-1}\quad \text{then}\quad x=\Big(\frac{1-t}{t}\Big)^{\frac{1}{c}}\quad \text{and}\quad dx=\frac{-dt}{t^2\Big(\frac{1-t}{t}\Big)^{\frac{1}{c}}}.$$

Then,

$$M_1 = \sum_{j=0}^{k} \binom{k}{j}(1+\lambda)^k \lambda^{k-j+2}\Big[B\Big(k-\frac{p+r+k}{c}+2, \frac{p+r+k+j}{c}+1\Big) + B\Big(\frac{p+r+c+k-j+1}{c}, \frac{p+r+k+j}{c}+1\Big)\Big] \quad (2.23)$$

and

$$M_2 = \sum_{j=0}^{k+1} \binom{k+1}{j}(1+\lambda)^{k+1}\lambda^{k-j+1} \times B\Big(k-\frac{p+r+k-j+1}{c}+2, \frac{p+r+c+k-j+1}{c}\Big). \quad (2.24)$$

Substituting Eqn (2.23) and Eqn (2.24) in Eqn (2.21) gives

$$\begin{aligned} L_1(\lambda,c,r,k) &= \sum_{p=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^p(k+1)^p(1+\lambda)^k\lambda^{k+p-j+2}}{p!} \\ &\times \Big[B\Big(k-\frac{p+r+k}{c}+2, \frac{p+r+k+j}{c}+1\Big) \\ &+ B\Big(\frac{p+r+k-j+1}{c}, \frac{p+r+k+j}{c}+1\Big)\Big] \\ &+ \sum_{p=0}^{\infty}\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(-1)^p(k+1)^p(1+\lambda)^{k+1}\lambda^{k+p-j+1}}{p!} \\ &\times B\Big(k-\frac{p+r+k+1}{c}+2, \frac{p+r+c+k-j+1}{c}\Big), \end{aligned}$$

where $B(a,b)=\int_0^1 y^{a-1}(1-y)^{b-1}dy$ is the complete beta function.□

From Lemma 1, the r^{th} moment of the GLLLoG distribution is given by

$$\mu_r = \frac{1}{\Gamma(\delta)}\sum_{m,s,k=0}^{\infty}\frac{(-1)^k\Gamma(m+s+\delta)b_{s,m}}{\Gamma(m+s+\delta-k)(1+\lambda)^{k+1}k!}\binom{\delta-1}{m}L_1(\lambda,c,r,k). \quad (2.25)$$

Table 2.2: **Moments of GLLLoG distribution for some parameter values**

	(λ, c, δ)					
	$(5.0, 1.5, 0.5)$	$(5.0, 0.8, 0.9)$	$(0.3, 5.0, 0.7)$	$(0.2, 8.0, 0.7)$	$(0.5, 15.0, 0.8)$	$(0.5, 15.0, 0.8)$
$E(X)$	0.10464	0.16128	0.44698	0.50390	0.49524	0.21549
$E(X^2)$	0.02987	0.05506	0.33383	0.41135	0.41741	0.20494
$E(X^3)$	0.01299	0.02685	0.26439	0.34692	0.36776	0.19613
$E(X^4)$	0.00719	0.01606	0.21734	0.29879	0.33082	0.18839
$E(X^5)$	0.00461	0.01091	0.18354	0.26145	0.30134	0.18143
$E(X^6)$	0.00327	0.00806	0.15823	0.23171	0.27689	0.17507
SD	0.00248	0.00630	0.13867	0.20753	0.25611	0.16919
CV	0.00198	0.00513	0.12316	0.18754	0.23815	0.16373
CS	0.00163	0.00431	0.11060	0.17081	0.22246	0.15864
CK	0.00138	0.00370	0.10025	0.15659	0.20858	0.15386

The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD=σ), CV, CS and CK are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively.

2.3.2 Conditional Moments

It is very important to study the conditional moments for lifetime distributions, because through them the mean residual life function can be readily calculated. In this subsection, the r^{th} conditional moment is presented. The mean residual life function as well as income inequality measures such as Bonferroni and Lorenz curves as well as mean deviation about the mean and mean deviation about the median can be readily obtained from the conditional moments. The r^{th} conditional moment is given by

$$\begin{aligned} E(X^r|X>x) &= \frac{1}{\overline{F}(x)\Gamma(\delta)} \sum_{m,s,k=0}^{\infty} \frac{(-1)^k \Gamma(m+s+\delta) b_{s,m}}{\Gamma(m+s+\delta-k)(1+\lambda)^{k+1} k!} \binom{\delta-1}{m} \\ &\times \int_t^{\infty} x^r \frac{(1+\lambda+\lambda x)^k}{(1+x^c)^{k+1}} e^{-(k+1)\lambda x} \Big[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\Big] dx, \end{aligned}$$

where $b_{s,m}$ is defined as in Eqn (2.13).

Lemma 2.1. *Let*

$$L_2(\lambda, c, r, t) = \int_t^{\infty} x^r \frac{(1+\lambda+\lambda x)^k}{(1+x^c)^{k+1}} e^{-(k+1)\lambda x} \Big[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\Big] dx.$$

Then

$$\begin{aligned} L_2(\lambda,c,r,t) &= \sum_{p=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^p(k+1)^p(1+\lambda)^k\lambda^{k+p-j+2}}{p!} \\ &\times \Big[B_{(1+t^c)^{-1}}\Big(k-\frac{p+r+k}{c}+2,\frac{p+r+k+j}{c}+1\Big) \\ &+ B_{(1+t^c)^{-1}}\Big(\frac{p+r+k-j+1}{c},\frac{p+r+k+j}{c}+1\Big)\Big] \\ &+ \sum_{p=0}^{\infty}\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(-1)^p(k+1)^p(1+\lambda)^{k+1}\lambda^{k+p-j+1}}{p!} \\ &\times B_{(1+t^c)^{-1}}\Big(k-\frac{p+r+k+1}{c}+2,\frac{p+r+c+k-j+1}{c}\Big), \end{aligned}$$

where $B_y(a,b)=\int_0^y t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function.

Proof:

Using the series expansion $e^z=\sum_{p=0}^{\infty}\frac{z^p}{p!}$, we have

$$\begin{aligned} L_2(\lambda,c,r,t) &= \sum_{p=0}^{\infty}\frac{(-1)^p(k+1)^p\lambda^p}{p!}\int_t^{\infty}x^{p+r}(1+\lambda+\lambda x)^k(1+x^c)^{-(k+1)} \\ &\times \Big[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\Big]dx \\ &= \sum_{p=0}^{\infty}\frac{(-1)^p(k+1)^p\lambda^p}{p!}\Bigg[\int_t^{\infty}x^{p+r}(1+\lambda+\lambda x)^k\lambda^2(1+x)(1+x^c)^{-(k+1)}dx \\ &+ \int_t^{\infty}x^{p+r+c-1}(1+\lambda+\lambda x)^{k+1}(1+x^c)^{-(k+2)}dx\Bigg]. \end{aligned} \tag{2.26}$$

Let

$$W_1=\int_t^{\infty}x^{p+r}(1+\lambda+\lambda x)^k\lambda^2(1+x)(1+x^c)^{-(k+1)}dx \tag{2.27}$$

and

$$W_2=\int_t^{\infty}x^{p+r+c-1}(1+\lambda+\lambda x)^{k+1}(1+x^c)^{-(k+2)}dx. \tag{2.28}$$

Using the binomial expansion in Eqn (2.22) it follows that,

$$\begin{aligned} W_1 &= \sum_{j=0}^{k}\binom{k}{j}(1+\lambda)^k\lambda^{k-j+2}\int_t^\infty x^{p+r+c+k-j-1}(1+x)(1+x^c)^{-(k+1)}dx \\ &= \sum_{j=0}^{k}\binom{k}{j}(1+\lambda)^k\lambda^{k-j+2}\Bigg[\int_t^\infty x^{p+r+c+k-j-1}(1+x^c)^{-(k+1)}dx \\ &+ \int_t^\infty x^{p+r+c+k-j}(1+x^c)^{-(k+1)}dx\Bigg] \end{aligned}$$

and

$$W_2 = \sum_{j=0}^{k+1}\binom{k+1}{j}(1+\lambda)^{k+1}\lambda^{k-j+1}\int_t^\infty x^{p+r+c+k-j}(1+x^c)^{-(k+2)}dx.$$

Now, let $y=(1+x^c)^{-1}$ then $x=\left(\frac{1-y}{y}\right)^{\frac{1}{c}}$ and $dx=\dfrac{-dy}{y^2\left(\frac{1-y}{y}\right)^{\frac{1}{c}}}$,

then,

$$\begin{aligned} W_1 &= \sum_{j=0}^{k}\binom{k}{j}(1+\lambda)^k\lambda^{k-j+2}\Big[B_{(1+t^c)^{-1}}\Big(k-\frac{p+r+k}{c}+2,\frac{p+r+k+j}{c}+1\Big) \\ &+ B_{(1+t^c)^{-1}}\Big(\frac{p+r+c+k-j+1}{c},\frac{p+r+k+j}{c}+1\Big)\Big] \qquad (2.29) \end{aligned}$$

and

$$\begin{aligned} W_2 &= \sum_{j=0}^{k+1}\binom{k+1}{j}(1+\lambda)^{k+1}\lambda^{k-j+1} \\ &\times B_{(1+t^c)^{-1}}\Big(k-\frac{p+r+k-j+1}{c}+2,\frac{p+r+c+k-j+1}{c}\Big). \qquad (2.30) \end{aligned}$$

Substituting Eqn (2.29) and Eqn (2.30) in Eqn (2.26) gives

$$\begin{aligned} L_2(\lambda,c,r,t) &= \sum_{p=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^p(k+1)^p(1+\lambda)^k\lambda^{k+p-j+2}}{p!} \\ &\times \Big[B_{(1+t^c)^{-1}}\Big[\Big(k-\frac{p+r+k}{c}+2,\frac{p+r+k+j}{c}+1\Big) \\ &+ B_{(1+t^c)^{-1}}\Big(\frac{p+r+k-j+1}{c},\frac{p+r+k+j}{c}+1\Big)\Big] \\ &+ \sum_{p=0}^{\infty}\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(-1)^p(k+1)^p(1+\lambda)^{k+1}\lambda^{k+p-j+1}}{p!} \\ &\times B_{(1+t^c)^{-1}}\Big(k-\frac{p+r+k+1}{c}+2,\frac{p+r+c+k-j+1}{c}\Big) \end{aligned}$$

□

From Lemma 2, the r^{th} conditional moment of the GLLLoG distribution is given by

$$E(X^r|X>x) = \frac{1}{\overline{F}(x)\Gamma(\delta)}\sum_{m,s,k=0}^{\infty}\frac{(-1)^k\Gamma(m+s+\delta)b_{s,m}}{\Gamma(m+s+\delta-k)(1+\lambda)^{k+1}k!} \times \binom{\delta-1}{m}L_2(\lambda,c,r,t). \tag{2.31}$$

2.3.3 Mean Deviation, Lorenz and Bonferroni Curves

Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the GLLLoG distribution are presented in this subsection. The application of Lorenz and Bonferroni curves is not limited to economics for the study of income and poverty, but they can also be used in other fields like reliability, demography, insurance and medicine.

2.3.4 Mean Deviations

The mean deviation about the mean and the mean deviation about the median are defined by

$$\sigma_1(x) = \int_0^{\infty} \mid x-\mu \mid f_{GLLLoG}(x;\lambda,c,\delta)dx,$$

and

$$\sigma_2(x) = \int_0^{\infty} \mid x-M \mid f_{GLLLoG}(x;\lambda,c,\delta)dx,$$

respectively, where $\mu = E(X)$ and $M = Median(X)$. The deviations can also be expressed as

$$\sigma_1(x) = 2\mu F_{GLLLoG}(\mu) - 2\int_0^{\mu} f_{GLLLoG}(x)dx$$

and

$$\sigma_2(x) = \mu - 2\int_0^{M} x f_{GLLLoG}(x)dx.$$

Bonferroni and Lorenz curves for the GLLLoG didtribution are given as

$$B(p) = \frac{1}{p\mu}\int_0^{1-p} x f_{GLLLoG}(x;\lambda,c,\delta)dx \text{ and } L(p) = \frac{1}{p}\int_0^{1-p} x f_{GLLLoG}(x;\lambda,c,\delta)dx,$$

respectively, where $\mu = E(X)$.

2.4 Order Statistics and Rényi Entropy

In this section, we present the pdf of the i^{th} order statistic as well as Rényi entropy.

2.4.1 Order Statistics

Suppose that $X_1, \cdots, X_n$ is a random sample of size n from a continuous pdf, $g(x)$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. If $X_1, \cdots, X_n$ is a random sample from the GLLLoG distribution, it follows from the equations (2.8) and (2.9) that the pdf of the i^{th} order statistic from the GLLLoG distribution is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} f_{GLLLoG}(x)[F_{GLLLoG}(x)]^{i-1}[1-F_{GLLLoG}(x)]^{n-i} \qquad (2.32) \\ &= \frac{n!}{(i-1)!(n-i)!} f_{GLLLoG}(x)[F_{GLLLoG}(x)]^{i-1} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j [F_{GLLLoG}(x)]^j \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j [F_{GLLLoG}(x)]^{i+j-1} f_{GLLLoG}(x) \\ &= \frac{n!}{(i-1)!(n-i)!} \frac{(-1)^j}{(i+j)} \binom{n-i}{j} (i+j)[F_{GLLLoG}(x)]^{i+j-1} f_{GLLLoG}(x) \\ &= \sum_{j=0}^{n-i} \omega(i,j) f_{EGLLLoG}(x) \end{aligned}$$

where $\omega(i,j) = \frac{n!}{(i-1)!(n-i)!} \frac{(-1)^j}{(i+j)} \binom{n-i}{j}$ and $f_{EGLLLoG}(x) = (i+j)[F_{GLLLoG}(x)]^{i+j-1} f_{GLLLoG}(x)$ is the Exponentiated gamma Lindley log-logistic (EGLLLoG) distribution.

2.4.2 Rényi Entropy

Rényi entropy is a measure of variation of uncertainty for a random variable X. Rényi Entropy is defined by,

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f_{GLLLoG}(x; \lambda, c, \delta)]^v dx \right), v \neq 1, v > 0. \qquad (2.33)$$

Note that,

$$\begin{aligned}\int_0^\infty [f_{GLLLoG}(x;\lambda,c,\delta)]^v dx &= \frac{1}{[\Gamma(\delta)(1+\lambda)]^v} \\ &\times \int_0^\infty \Big[-log\Big(1-\Big(1-\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)}\Big)\Big)\Big]^{\delta v-v} \\ &\times (1+x^c)^{-v} e^{-\lambda v x}\Big(\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\Big)^v dx. \end{aligned} \tag{2.34}$$

Using the series representations in equations (2.10), (2.11) and (2.12), we have,

$$\begin{aligned}\Big[-log\Big(1-\Big(1-\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)}\Big)\Big)\Big]^{\delta v-v} &= \sum_{m,s=0}^\infty \binom{\delta v-v}{m} b_{s,m} \\ &\times \Big[1-\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)}\Big]^{m+s+\delta v-v}. \end{aligned}$$

Now using the generalized binomial theorem, that is,

$$(1-z)^{b-1} = \sum_{k=0}^\infty \frac{(-1)^k \Gamma(b)}{\Gamma(b-k)k!} z^k, \tag{2.35}$$

for a positive real integer b and $\mid z \mid < 1$, it follows that,

$$\begin{aligned}\Big[1-\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)}\Big]^{m+s+\delta v-v} &= \sum_{k=0}^\infty \frac{(-1)^k \Gamma(m+s+\delta v-v+1)}{\Gamma(m+s+\delta v-v+1-k)(1+\lambda)^k k!} \\ &\times \Big[\frac{(1+\lambda+\lambda x)}{(1+x^c)} e^{-\lambda x}\Big]^k. \end{aligned} \tag{2.36}$$

Therefore,

$$\begin{aligned}\int_0^\infty [f_{GLLLoG}(x)]^v dx &= \frac{1}{[\Gamma(\delta)(1+\lambda)]^v} \sum_{m,s,k=0}^\infty \binom{\delta v-v}{m} \\ &\times \frac{b_{s,m}(-1)^k \Gamma(m+s+\delta v-v+1)}{\Gamma(m+s+\delta v-v+1-k)(1+\lambda)^k k!} \int_0^\infty (1+x^c)^{-v} e^{-\lambda v x} \\ &\times \Big(\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\Big)^v \\ &\times \Big[\frac{(1+\lambda+\lambda x)}{(1+x^c)} e^{-\lambda x}\Big]^k dx. \end{aligned} \tag{2.37}$$

Let,

$$\begin{aligned}\vartheta &= \int_0^\infty (1+x^c)^{-v} e^{-\lambda v x}\Big(\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\Big)^v \Big[\frac{(1+\lambda+\lambda x)}{(1+x^c)} e^{-\lambda x}\Big]^k dx \\ &= \int_0^\infty (1+x^c)^{-2v-k} e^{-\lambda(v+k)x} (1+\lambda+\lambda x)^k \Big(\lambda^2(1+x)(1+x^c)+(1+\lambda+\lambda x)cx^{c-1}\Big)^v dx. \end{aligned}$$

Applying the binomial expansion in Eqn (2.22), we have

$$\begin{aligned}\left(\lambda^2(1+x)(1+x^c)+(1+\lambda+\lambda x)cx^{c-1}\right)^v &= \sum_{j=0}^{v}\binom{v}{j}[(1+\lambda+\lambda x)cx^{c-1}]^v \\ &\times\ [\lambda^2(1+x)(1+x^c)]^{v-j}.\end{aligned}$$

This gives,

$$\vartheta = \sum_{j=0}^{v}\binom{v}{j}\lambda^{2(v-j)}c^v\int_0^\infty(1+x^c)^{-(v+k+j)}e^{-\lambda(v+k)x}(1+\lambda+\lambda x)^{v+k}(1+x)^{v-j}x^{cv-v}dx.$$

Using the series expansion in Eqn (2.3.1), that is,

$$e^{-\lambda(v+k)x}=\sum_{p=0}^{\infty}\frac{(-1)^p(v+k)^p\lambda^p x^p}{p!},$$

we have,

$$\begin{aligned}\vartheta &= \sum_{j=0}^{v}\sum_{p=0}^{\infty}\binom{v}{j}\frac{(-1)^p\lambda^{2(v-j)+p}c^v(v+k)^p}{p!} \\ &\times\ \int_0^\infty(1+x^c)^{-(v+k+j)}(1+\lambda+\lambda x)^{v+k}(1+x)^{v-j}x^{cv-v+p}dx.\end{aligned}$$

Applying the binomial expansion in Eqn (2.22), we have

$$\begin{aligned}\vartheta &= \sum_{p=0}^{\infty}\sum_{j=0}^{v}\sum_{q=0}^{v+k}\binom{v}{j}\binom{v+k}{q}\frac{(-1)^p\lambda^{2(v-j)+p+v+k-q}c^v(v+k)^p}{p!} \\ &\times\ \int_0^\infty(1+x^c)^{-(v+k+j)}(1+x)^{v-j}x^{cv+k+p-q}dx.\end{aligned}$$

Applying the series expansion,

$$(1+x)^{v-j} = \sum_{w=0}^{v-j}\binom{v-j}{w}x^w,$$

we obtain

$$\begin{aligned}\vartheta &= \sum_{p=0}^{\infty}\sum_{j=0}^{v}\sum_{q=0}^{v+k}\sum_{w=0}^{v-j}\binom{v}{j}\binom{v+k}{q}\binom{v-j}{w}\frac{(-1)^p\lambda^{2(v-j)+p+v+k-q}c^v(v+k)^p}{p!} \\ &\times\ \int_0^\infty(1+x^c)^{-(v+k+j)}x^{cv+k+p+w-q}dx.\end{aligned}$$

Let

$$t=(1+x^c)^{-1}\ \text{ then }\ x=\left(\frac{1-t}{t}\right)^{\frac{1}{c}}\ \text{ and }\ dx=\frac{-dt}{t^2\left(\frac{1-t}{t}\right)^{\frac{1}{c}}},\ \text{ so that}$$

$$\int_0^{\infty}(1+x^c)^{-(v+k+j)}x^{cv+k+p+w-q}dx \;=\; \int_0^1 t^{v+k+j-2-\frac{cv+k+p+w-q-1}{c}}(1-t)^{\frac{cv+k+p+w-q-1}{c}}dt.$$

This reduces to,

$$B\Big(v+k+j-\frac{cv+k+p+w-q-1}{c}-1,\frac{cv+k+p+w-q-1}{c}+1\Big)$$

$$= B(v+k+j-a^*-1,a^*+1),$$

Thus,

$$\begin{aligned}\vartheta &= \sum_{p=0}^{\infty}\sum_{j=0}^{v}\sum_{q=0}^{v+k}\sum_{w=0}^{v-j}\binom{v}{j}\binom{v+k}{q}\binom{v-j}{w}\frac{(-1)^p\lambda^{2(v-j)+p+v+k-q}c^v(v+k)^p}{p!}\\ &\times B\Big(v+k+j-a^*-1,a^*+1\Big).\end{aligned} \tag{2.38}$$

Substituting Eqn (2.38) in Eqn (2.37) gives,

$$\begin{aligned}\int_0^{\infty}[f_{GLLLoG}(x)]^v dx &= \frac{1}{[\Gamma(\delta)(1+\lambda)]^v}\sum_{m,s,k,p=0}^{\infty}\sum_{j=0}^{v}\sum_{q=0}^{v+k}\sum_{w=0}^{v-j}\binom{v}{j}\binom{v+k}{q}\binom{v-j}{w}\binom{\delta v-v}{m}\\ &\times \frac{b_{s,m}(-1)^k\Gamma(m+s+\delta v-v+1)(-1)^p\lambda^{2(v-j)+p+v+k-q}c^v(v+k)^p}{\Gamma(m+s+\delta v-v+1-k)(1+\lambda)^k k!p!}\\ &\times B\Big(v+k+j-a^*-1,a^*+1\Big).\end{aligned} \tag{2.39}$$

Therefore, the Rényi entropy of the GLLLoG distribution reduces to

$$\begin{aligned}I_R(v) &= \frac{1}{1-v}log\Big[\frac{1}{[\Gamma(\delta)(1+\lambda)]^v}\sum_{m,s,k,p=0}^{\infty}\sum_{j=0}^{v}\sum_{q=0}^{v+k}\sum_{w=0}^{v-j}\binom{v}{j}\binom{v+k}{q}\binom{v-j}{w}\binom{\delta v-v}{m}\\ &\times \frac{b_{s,m}(-1)^k\Gamma(m+s+\delta v-v+1)(-1)^p\lambda^{2(v-j)+p+v+k-q}c^v(v+k)^p}{\Gamma(m+s+\delta v-v+1-k)(1+\lambda)^k k!p!}\\ &\times B\Big(v+k+j-a^*-1,a^*+1\Big)\Big],\end{aligned}$$

where $B(a,b)=\int_0^1 y^{a-1}(1-y)^{b-1}dy$ is the complete beta function,
and $a^*=\frac{cv+k+p+w-q-1}{c}$.

2.5 Estimation and Inference

In this section, the maximum likelihood estimates (MLE's) of the parameters, Fisher Information Matrix and the asymptotic confidence intervals of the GLLLoG parameters are presented.

2.5.1 Maximum Likelihood Estimation

Let $X \sim GLLLoG(\lambda, c, \delta)$ and $\Delta = (\lambda, c, \delta)^T$ be the parameter vector. The log-likelihood function for a single observation x of X is given by

$$\begin{aligned}\ell &= (\delta-1)log\Big[-log\Big(1-\Big(1-\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)}\Big)\Big)\Big] - log(1+x^c) \\ &- log(1+\lambda) - \lambda x + log\Big[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{(1+x^c)}\Big] - log(\Gamma(\delta)).\end{aligned} \tag{2.40}$$

The elements of the score vector, that is, the first derivatives of the log-likelihood function with respect to the parameters $\Delta = (\lambda, c, \delta)^T$, are given by

$$\begin{aligned}\frac{d\ell}{d\lambda} &= \frac{(\delta-1)}{-log\Big(1-\Big(1-\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)}\Big)\Big)} \\ &\times \frac{(1+x)(-1-\lambda)+x(1+\lambda)(1+\lambda+\lambda x)+(1+\lambda+\lambda x)}{(1+\lambda+\lambda x)(1+\lambda)} \\ &- \frac{1}{1+\lambda} - x + \frac{(1+x)[2\lambda(1+x^c)+cx^{c-1}]}{\lambda^2(1+x)(1+x^c)+(1+\lambda+\lambda x)cx^{c-1}},\end{aligned} \tag{2.41}$$

$$\begin{aligned}\frac{d\ell}{dc} &= \frac{(\delta-1)}{-log\Big(1-\Big(1-\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)}\Big)\Big)} \times \frac{x^c \ln x}{(1+x^c)} - \frac{x^c \ln x}{(1+x^c)} \\ &- \frac{\frac{(1+\lambda+\lambda x)}{(1+x^c)^2}\Big((x^{c-1}+cx^{c-1}\ln x)(1+x^c) - cx^{c-1}x^c \ln x\Big)}{\Big[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{(1+x^c)}\Big]},\end{aligned} \tag{2.42}$$

and

$$\frac{d\ell}{d\delta} = log\Big[-log\Big(1-\Big(1-\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)}\Big)\Big)\Big] - \frac{\Gamma'(\delta)}{\Gamma(\delta)}. \tag{2.43}$$

The equations that are obtained by setting the elements of the score vector to zero are not in closed form and the MLEś of the parameters λ, c and δ must be

obtained by iterative methods.

2.5.2 Fisher Information Matrix and Asymptotic Confidence Intervals

The Fisher Information Matrix (FIM) of the GLLLoG distribution is given by

$$I(\Delta) = -E\begin{pmatrix} I_{\lambda\lambda} & I_{\lambda c} & I_{\lambda\delta} \\ I_{c\lambda} & I_{cc} & I_{c\delta} \\ I_{\delta\lambda} & I_{\delta c} & I_{\delta\delta} \end{pmatrix}, \text{where } I_{\lambda\lambda} = \frac{\partial^2 \ell}{\partial \lambda^2}, \ldots, I_{\delta\delta} = \frac{\partial^2 \ell}{\partial \delta^2}. \tag{2.44}$$

The elements of the FIM are obtained by considering the second order partial derivatives of Eqn (2.40) with respect to parameters $\Delta = (\lambda, c, \delta)^T$.The asymptotic distribution of $\sqrt{n}(\Delta - \hat{\Delta})$ is $N_3(\mathbf{0}, I^{-1}(\Delta))$ under usual or standard regularity conditions (Lehmann & Casella [42] page 461-463). The asymptotic behavior remains valid if $I(\Delta)$ is replaced by the observed information matrix, $J(\Delta)$ multiplied by n^{-1} with Δ approximated by $(\hat{\Delta})$. A large sample $100(1-\alpha)\%$ confidence intervals for the parameters $\lambda, c,$ and δ are given by

$$\lambda \mp z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\lambda\lambda}(\hat{\Delta})}, \quad c \mp z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{cc}(\hat{\Delta})}, \quad \text{and} \quad \delta \mp z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\delta\delta}(\hat{\Delta})},$$

respectively, where $z_{\frac{\alpha}{2}}$ is the upper $(\frac{\alpha}{2})^{th}$ percentile of the standard normal distribution.

2.6 Monte Carlo Simulations

In this section, we examine the performance of the GLLLoG distribution by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800, 1200) via the R package. We simulate $N = 1000$ samples for the true parameters values given by $I = (1.5, 0.5, 2.0)$, $II = (2.5, 2.5, 2.0)$, $III = (2.0, 2.0, 2.0)$ and $IV = (2.0, 2.5, 2.0)$ with the results presented in Table 2.3 and Table 2.4. The Average Bias and Root Mean Square Error (RMSE) were computed. The average bias and RMSE for the estimated parameter $\hat{\theta}$, say, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

The tables list the mean MLEs of the parameters, average bias (ABias) along with the respective root mean squared errors (RMSEs). From the results, we can see that the mean estimates of the parameters tend to be closer to the true parameter values, since the RMSEs decay towards zero for all parameters, as the sample size n increases, thus the estimators are consistent. We can observe that for all the parametric values, the biases decrease as the sample size n increases.

Table 2.3: **Monte Carlo Simulation Results**

		(1.5, 0.5, 2.0)			(2.5, 2.5, 2.0)		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias
λ	25	1.4973	0.6039	-0.0026	3.3184	1.8094	0.8185
	50	1.4608	0.4717	-0.0392	2.9454	1.2375	0.4454
	100	1.4642	0.3315	-0.0357	2.8765	0.9389	0.3764
	200	1.4843	0.2342	-0.0156	2.7275	0.6816	0.2275
	400	1.4908	0.1618	-0.0091	2.6200	0.5114	0.1201
	800	1.4981	0.1102	-0.0019	2.5481	0.2954	0.0481
	1200	1.5008	0.0888	0.0009	2.5152	0.2332	0.0152
c	25	0.7052	0.5043	0.2052	2.9742	3.3020	0.4742
	50	0.6426	0.4379	0.1427	2.5706	1.5359	0.0707
	100	0.5941	0.2969	0.0941	2.3633	1.1314	-0.1367
	200	0.5511	0.2034	0.0511	2.3996	0.8306	-0.1003
	400	0.5193	0.1224	0.0193	2.4067	0.6476	-0.0932
	800	0.5082	0.0778	0.0083	2.4704	0.3928	-0.0295
	1200	0.5021	0.0628	0.0022	2.5028	0.3126	0.0028
δ	25	1.9904	0.5971	-0.0096	2.5969	1.2854	0.5969
	50	1.9636	0.4749	-0.0363	2.3366	0.8982	0.3366
	100	1.9604	0.3342	-0.0395	2.2873	0.6897	0.2873
	200	1.9811	0.2440	-0.0188	2.1736	0.5056	0.1736
	400	1.9936	0.1778	-0.0063	2.0984	0.3880	0.0984
	800	1.9975	0.1181	-0.0025	2.0362	0.2135	0.0362
	1200	2.0004	0.0955	0.0004	2.0116	0.1679	0.0116

Table 2.4: **Monte Carlo Simulation Results**

		(2.0, 2.0, 2.0)			(2.0, 2.5, 2.0)		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias
λ	25	2.6690	1.5029	0.6691	2.8678	1.8059	0.8678
	50	2.3689	1.0982	0.3689	2.4657	1.2672	0.4657
	100	2.3420	0.9119	0.3420	2.4111	1.0292	0.4112
	200	2.2543	0.7322	0.2543	2.2623	0.7811	0.2623
	400	2.1610	0.5683	0.1610	2.1347	0.5696	0.1347
	800	2.1178	0.4424	0.1178	2.0592	0.3446	0.0592
	1200	2.0699	0.3636	0.0699	2.0199	0.2552	0.0199
c	25	2.0146	1.5377	0.0146	2.4104	1.8431	-0.0896
	50	1.9236	1.0647	-0.0763	1.9237	1.0647	-0.0763
	100	1.7992	0.8596	-0.2008	2.2237	1.0174	-0.2763
	200	1.8267	0.7284	-0.1733	2.3098	0.7998	-0.1902
	400	1.8575	0.6177	-0.1425	2.3728	0.6321	-0.1272
	800	1.8915	0.4766	-0.1085	2.4492	0.3835	-0.0507
	1200	1.9354	0.3976	-0.0645	2.4864	0.2861	-0.0136
δ	25	2.5643	1.2211	0.5644	2.7351	1.4976	0.7351
	50	2.3243	0.8974	0.3243	2.4043	1.0433	0.4043
	100	2.3009	0.7537	0.3009	2.3552	0.8515	0.3552
	200	2.2239	0.6038	0.2239	2.2262	0.6452	0.2262
	400	2.1479	0.4803	0.1479	2.1211	0.4757	0.1211
	800	2.1037	0.3728	0.1038	2.0499	0.2800	0.0499
	1200	2.0625	0.3093	0.0625	2.0170	0.2047	0.0171

2.7 Applications

In this section, we present examples to illustrate the flexibility and usefulness of the GLLLoG distribution and its sub-models for data modeling. Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [15]) are given in Figure 2.3 and Figure 2.4. For the probability plot, we plotted $F_{GLLLoG}(x_{(j)}; \hat{\lambda}, \hat{c}, \hat{\delta})$ against $\dfrac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[F_{GLLLoG}(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2 .$$

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan [16] are also presented in Table 2.5 and Table 2.6. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit. The GLLLoG distribution was compared with some non-nested distributions, namely the Weibull Lindley (WL) distribution and the Exponentiated Power Lindley (EPL) distribution. The

pdf of the Weibull Lindley distribution (Asgharzadeh et al. [8]) is

$$f(x;\lambda,\alpha,\beta)=\frac{1}{1+\lambda}\Big[\alpha\lambda(\beta x)^{\alpha}+\alpha\beta(1+\lambda)(\beta x)^{\alpha-1}+\lambda^{2}(1+x)\Big]e^{-\lambda x-(\beta x)^{\alpha}}, \quad (2.45)$$

for $\lambda,\alpha,\beta > 0$ and the pdf of the exponentiated power Lindley distribution (Samir et al. [60]) is given as follows,

$$g(x;\theta,\alpha,\beta)=\frac{\alpha\theta^{2}\beta x^{\beta-1}}{\theta+1}(1+x^{\beta})e^{-\theta x^{\beta}}\Big(1-\Big(1+\frac{\theta x^{\beta}}{1+\theta}\Big)e^{-\theta x^{\beta}}\Big)^{\alpha-1}, \quad (2.46)$$

for $\theta,\alpha,\beta>0$.

2.7.1 Exceedances of Wheaton river flood Data

This data set consists of 72 exceedances of flood peaks in (m^3/s) of the Wheaton river near Carcross in Yukon Territory, Canada for the years 1958-1984. The data was analysed by Choulakian and Stephens [17].

Table 2.5: **Estimation of the GLLLoG model for Wheaton River Flooding data set**

			Estimates								
Distribution	λ	c	δ	-2logL	AIC	CAIC	BIC	W^*	A^*	K–S	P–value
GLLLoG	0.1065	0.6826	2.224	498.3623	504.3623	504.7153	511.1923	0.1643	0.9533	0.1328	0.1579
	(0.0209)	(0.1180)	(0.1884)								
GLLoG	0	1.0456	2.2313	524.3853	528.3853	528.5593	532.9387	0.4846	2.7768	0.2129	0.0029
	-	(0.0990)	(0.1858)								
LLLoG	0.0797	0.4410	1	561.3467	565.3467	565.5207	569.9001	0.0742	0.4643	0.4070	8.7430×10^{-11}
	(0.0143)	(0.0810)	-								
LLoG	0	0.7162	1	589.1207	591.1207	591.1778	593.3974	0.3244	1.8304	0.4133	4.1400×10^{-11}
	-	(0.0679)	-								
	λ	α	β								
WL	0.0468	0.1176	0.8027	502.2245	508.2245	508.5775	515.0545	0.1604	0.9907	0.3126	1.547×10^{-06}
	(0.0399)	(0.0342)	(0.1557)								
	θ	α	β								
EPL	1.3251	0.0301	0.5102	665.2675	671.2675	671.6205	678.0975	0.4971	3.1751	0.1843	0.01504
	(0.0939)	(0.0091)	(0.0417)								

The covariance matrix for the exceedence of the Wheaton river flood data is given by,

$$\begin{pmatrix} 0.0004369303 & -0.001250869 & 0.0008525526 \\ -0.0012508687 & 0.013922296 & 0.0073393955 \\ 0.0008525526 & 0.007339396 & 0.0354814521 \end{pmatrix}, \quad (2.47)$$

and the 95% confidence intervals for the parameters λ, c and δ are: 0.1065 ± 0.040964, 0.6283 ± 0.2312604 and 2.2224 ± 0.36926, respectively.

It can be seen that the GLLLoG distribution provides a better fit for the Wheaton river flooding data set since it has the smallest values of all the goodness of fit statistics. We used the likelihood ratio (LR) test to test if the the GLLLoG distribution is significantly better than the GLLoG, LLLoG and the LLoG distributions and the results are as follows $H_0 : GLLoG$ against $H_a : GLLLoG$ $(26.023, p - value < 0.00001)$, $H_0 : LLLoG$ against $H_a : GLLLoG$ $(62.9844, p - value < 0.00001)$ and $H_0 : LLoG$ against $H_a : GLLLoG$ $(90.7584, p - value < 0.00001)$. We can conclude that the GLLLoG distribution is also better than the non-nested WL and the EPL distribution on the Wheaton river flooding data set, based on the goodness-of-fit statistics presented in Table 2.5.

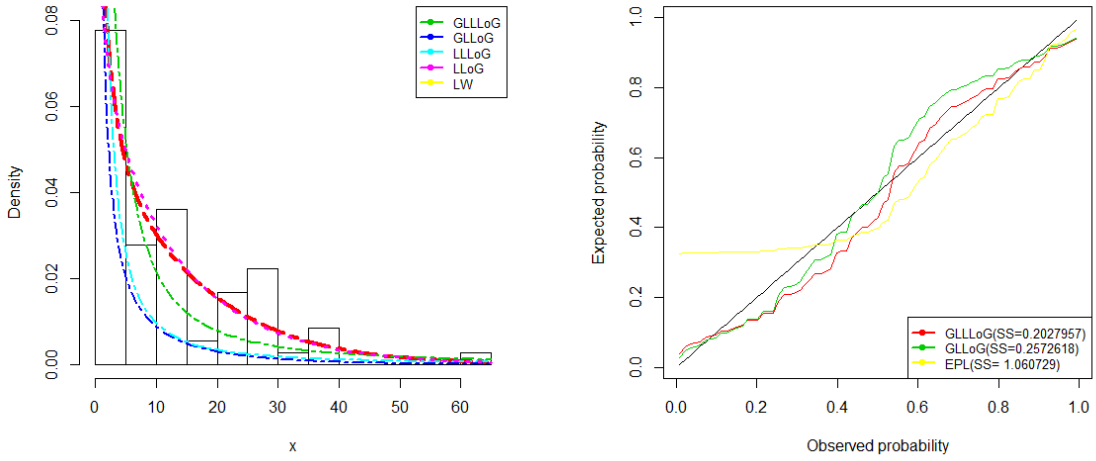


Figure 2.3: Fitted densities and probability plots for the Wheaton River flooding data set.

2.7.2 Quarterly earnings per Johnson and Johnson Share 1960-1980 data set

This data set consist of quarterly earnings (dollars) per Johnson & Johnson share 1960-80, Shumway and Stoffer [62].

Table 2.6: **Estimation of the GLLLoG model for Quarterly earnings per Johnson and Johnson Share 1960-1980**

	Estimates										
Distribution	λ	c	δ	-2logL	AIC	CAIC	BIC	W^*	A^*	K−S	P−value
GLLLoG	0.2323	1.2283	2.0716	366.7620	372.7620	373.0908	379.7935	0.2238	1.3839	0.1090	0.3194
	(0.0747)	(0.2029)	(0.1758)								
GLLoG	0	1.5943	1.9749	375.5134	379.5134	379.6756	384.2010	0.2505	1.5727	0.1225	0.1978
	-	(0.1478)	(0.1565)								
LLLoG	0.1268	1.0162	1	423.5794	427.5794	427.7416	432.2670	0.2070	1.2593	0.3538	8.514×10^{-09}
	(0.0454)	(0.1471)	-								
LLoG	0	1.2096	1	428.6017	430.6017	430.6551	432.9455	0.2486	1.5544	0.3247	1.777×10^{-07}
	-	(0.1137)	-								
	λ	α	β								
WL	7.7173×10^{-7}	0.1865	1.1292	374.1194	380.1194	380.4481	387.1508	0.2702	1.6311	0.1196	0.2210
	(0.3410)	(0.0411)	(0.1004)								
	θ	α	β								
EPL	1.3194	0.1232	0.5103	634.6917	640.6917	641.0205	647.7231	0.8918	5.1703	0.3249	1.746×10^{-07}
	(0.0904)	(0.0232)	(0.0403)								

The LR test statistic results for the GLLLoG model for the quarterly earnings per Johnson and Johnson share data set are as follows; $H_0 : GLLoG$ against $H_a : GLLLoG$ $(8.7514, p - value = 0.003094)$, $H_0 : LLLoG$ against $H_a : GLLLoG$ $(56.8174, p - value < 0.00001)$ and $H_0 : LLoG$ against $H_a : GLLLoG$ $(61.8397, p - value < 0.00001)$. We can conclude that there GLLLoG distribution is also better than the non-nested WL and the EPL distribution on the quarterly earnings per Johnson and Johnson share data set, based on all the statistics presented in Table 2.6. The covariance matrix for the quarterly earnings per Johnson and Johnson share data set is given by

$$\begin{pmatrix} 0.005585173 & -0.009841170 & 0.005275748 \\ -0.009841170 & 0.041167293 & 0.001483935 \\ 0.005275748 & 0.001483935 & 0.030901687 \end{pmatrix}, \tag{2.48}$$

and the 95% confidence intervals for the parameters λ, c and δ are: 0.2323 ± 0.146412, 1.2283 ± 0.3977 and 2.0716 ± 0.3446, respectively.

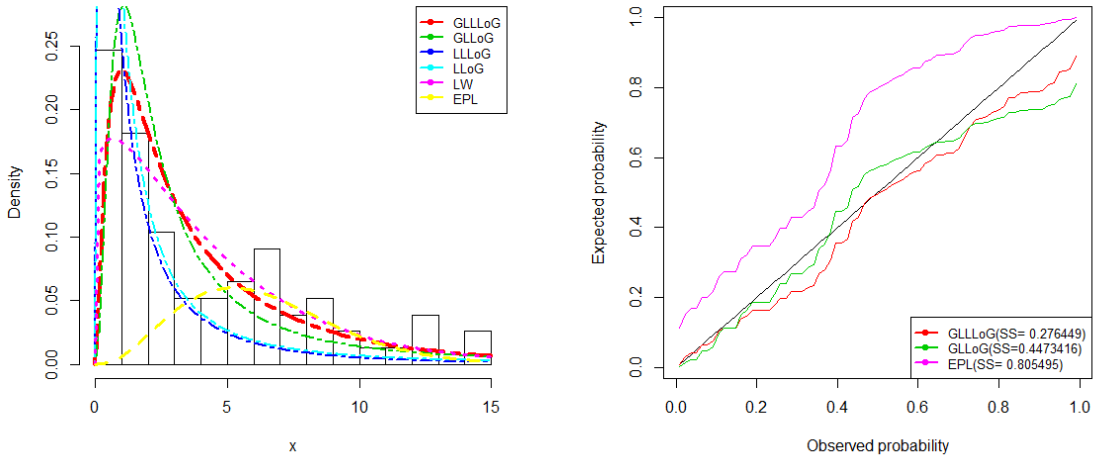


Figure 2.4: Fitted densities and probability plots for Quarterly earnings per Johnson and Johnson Share 1960-1980.

2.8 Concluding Remarks

A new three parameter distribution that generalizes both the Lindley and the log-logistic distributions called the Gamma Lindley log-logistic (GLLLoG) distribution is presented. The distribution has several new and known distributions as special cases or sub-models. The density of this new distribution can be expressed as a linear combination of ELLLoG density functions. The hazard function of the GLLLoG distribution has a flexible behavior. We also obtain closed form expressions for the moments, mean and median deviations, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. The performance of the new distribution was examined by conducting various simulations for different sample sizes. Finally, the model is fitted to real data to illustrate the applicability and usefulness of the proposed distribution.

Chapter 3

The New Weibull Generalised-G Family of Distributions

3.1 Introduction

The need for new generalized and extended distributions cannot be overemphasized due to their importance in statistical modeling. In the statistical literature, the need for extended and generalized distributions for the modeling of monotone and non-monotone hazard rates functions, particularly in reliability and survival analysis are very crucial. Generalized distributions and related distributions including modified Weibull distributions (see Oluyede et al. [57] for details) play a very prominent role in the modelling of monotone and non-monotone hazard rate functions, exploring tail variation and improved goodness-of-fit tests.

Let X be a random variable with cumulative distribution function (cdf) $G(x;\xi)$ depending on the parameter vector ξ. The cdf and probability density function (pdf) of the generalized-G (G-G) distribution is given by

$$K(x;\xi) = 1-(1-G(x;\xi))^{\alpha} \quad \text{and} \quad k(x;\xi) = \alpha g(x;\xi)(1-G(x;\xi))^{\alpha-1}, \tag{3.1}$$

for $\alpha > 0$, $x \in \mathcal{R}$. The corresponding hazard rate function (hrf) is given by

$$h_{_K}(x;\xi) = \alpha g(x;\xi)(1-G(x;\xi))^{-1}. \tag{3.2}$$

By making use of the $T-X$ family of distributions by Alzaghal et al. [7], we consider the transformation $W(G(x;\xi)) = -\ln(1-\overline{G}^{\alpha}(x;\xi))$ for the baseline cumulative distribution function (cdf) $G(x;\xi)$. In fact, a new generalized family of distributions is developed via the generalized-G family of distributions and the Weibull distribution, by introducing additional shape parameters to obtain a new generalized family of distributions called the new Weibull generalized-G (NWG-G) family of distributions. Comparison of a new member of the family of distributions with several other distributions in the literature via several goodness-of-fit statistics is conducted in order to establish the effectiveness and usefulness of this new distribution. It turns out that the new Weibull-G (NW-G) family of distributions given by Tahir et al. [65] is a special case of the NWG-G family of distributions.

The results in this Chapter are organized in the following manner. Section 3.2 contain the new NWG-G family of distributions and its sub-models, hazard function and the quantile function, series expansion of the density function and some special cases of the NWG-G family of distributions. In section 3.3, moments, generating function and incomplete moments are given. Section 3.4 contain results on the distribution of order statistics and Rényi entropy. Presented in section 3.5 is the the estimation of the parameters of the NWG-G family of distributions via the method of maximum likelihood and other methods, followed by a Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimators in section 3.6. Some applications to real data sets are given in section 3.7, followed by some concluding remarks in section 3.8.

3.2 The Model, Sub-models, Hazard and Quantile Functions, Shape, Series Expansion of Density Function, and Some Special Cases

The derivation of some of the statistical properties of the new Weibull generalized-G (NWG-G) family of distributions including special cases, expansion of the density, hazard function and quantile function are presented in this section.

3.2.1 The Model

The cdf and pdf of the proposed new Weibull generalized-G (NWG-G) family of distributions are given by

$$\begin{aligned} F(x;\lambda,\beta,\alpha,\xi) &= 1-\int_0^{-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)} \lambda\beta t^{\beta-1}e^{-\lambda t^{\beta}}dt \\ &= \exp\left(-\lambda\left(-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)\right)^{\beta}\right) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} f(x;\lambda,\beta,\alpha,\xi) &= \lambda\alpha\beta\frac{g(x;\xi)}{\left[1-\overline{G}^{\alpha}(x;\xi)\right]}[\overline{G}(x;\xi)]^{\alpha-1}\left(-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)\right)^{\beta-1} \\ &\times \quad e^{\left(-\lambda\left[-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)\right]\right)^{\beta}}, \end{aligned} \tag{3.4}$$

respectively, for $\lambda,\alpha,\beta>0$ and parameter vector ξ. If a random variable X has the NWG-G family of distributions, we write $X\sim NWG-G(\lambda,\alpha,\beta,\xi)$. Note that when $\alpha=1$, we obtain the new Weibull-G (NW-G) family of distributions given by Tahir et al. [65] with the pdf

$$f(x;\lambda,\beta,\xi) = \lambda\beta\frac{g(x;\xi)}{[G(x;\xi)]}\left(-\log\left(G(x;\xi)\right)\right)^{\beta-1}e^{\left(-\lambda[-\log(G(x;\xi))]\right)^{\beta}} \tag{3.5}$$

for $\lambda,\beta>0$, and parameter vector ξ.

We also note that different baseline distribution function $G(x;\xi)$, lead to several new generalized family of distributions as sub-models.

3.2.2 Hazard and Quantile Functions

In this section, we present the hazard and quantile functions of the NWG-G family of distributions. The hazard rate function of the NWG-G family of distributions is given by

$$\begin{aligned} h_F(x;\lambda,\alpha,\beta,\xi) &= \lambda\alpha\beta\frac{g(x;\xi)}{\left[1-\overline{G}^{\alpha}(x;\xi)\right]}[\overline{G}(x;\xi)]^{\alpha-1}\left(-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)\right)^{\beta-1} \\ &\times e^{\left(-\lambda\left[-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)\right]\right)^{\beta}}\left(1-e^{\left(-\lambda\left(-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)\right)^{\beta}\right)}\right)^{-1}. \end{aligned} \tag{3.6}$$

The quantile function of the NWG-G class of distributions is obtained by solving the nonlinear equation:

$$F(Q(u)) = \exp\left(-\lambda\left(-\log\left(1-\overline{G}^{\alpha}(Q(u);\xi)\right)\right)^{\beta}\right) = u, \tag{3.7}$$

for $0 \leq u \leq 1$, that is,

$$\overline{G}^{\alpha}(Q(u);\xi) = 1 - e^{-\left[-\frac{\ln(u)}{\lambda}\right]^{1/\beta}}. \tag{3.8}$$

Consequently, the quantile function of the NWG-G family of distributions is given by

$$Q(u) = G^{-1}\left[1-\left(1-e^{-\left[-\frac{\ln(u)}{\lambda}\right]^{1/\beta}}\right)^{1/\alpha}\right]. \tag{3.9}$$

It follows therefore that random numbers can be generated based on equation (3.9).

3.2.3 Shapes

The derivatives of the logarithm of the density and hazard functions of the NWG-G family of distributions allows one to analytically describe the shapes of these functions. The critical points of the NWG-G density function $f(x;\lambda,\alpha,\beta,\xi)$ are the roots of the equation given by

$$\begin{aligned} \frac{\partial\log(f(x;\lambda,\alpha,\beta,\xi)}{\partial x} &= \frac{g'(x,\xi)}{g(x;\xi)} - \frac{\alpha\overline{G}^{\alpha-1}(x;\xi)g(x;\xi)}{\left(1-\overline{G}^{\alpha}(x;\xi)\right)} - (\alpha-1)\frac{g(x;\xi)}{\overline{G}(x;\xi)} \\ &- \frac{\alpha(\beta-1)g(x;\xi)\overline{G}^{\alpha-1}(x,\xi)}{\left(1-\overline{G}^{\alpha}(x;\xi)\right)\log(1-\overline{G}^{\alpha}(x;\xi))} \\ &+ \frac{\lambda\alpha\overline{G}^{\alpha-1}(x;\xi)g(x;\xi)}{1-[G(x;\xi)]^{\alpha}} = 0. \end{aligned} \tag{3.10}$$

Similarly, the critical points of the hazard rate function $h_F(x;\lambda,\alpha,\beta,\xi)$ are the zeros of the equation given by

$$\begin{aligned}\frac{\partial \log(h_F(x;\lambda,\alpha,\beta,\xi)}{\partial x} &= \frac{g'(x,\xi)}{g(x;\xi)} - \frac{\alpha\overline{G}^{\alpha-1}(x;\xi)g(x;\xi)}{\big(1-\overline{G}^{\alpha}(x;\xi)\big)} - (\alpha-1)\frac{g(x;\xi)}{\overline{G}(x;\xi)} \\ &- \frac{\alpha(\beta-1)g(x;\xi)\overline{G}^{\alpha-1}(x,\xi)}{\big(1-\overline{G}^{\alpha}(x;\xi)\big)\log(1-\overline{G}^{\alpha}(x;\xi))} \\ &+ \frac{\lambda\alpha\beta\overline{G}^{\alpha-1}(x;\xi)g(x;\xi)\frac{g(x;\xi)}{1-\overline{G}^{\alpha}(x;\xi)}}{1-\exp[-\lambda(-\log[1-\overline{G}^{\alpha}(x;\xi)])^{\beta}]} \\ &\times \exp[-\lambda(-\log[1-\overline{G}^{\alpha}(x;\xi)])^{\beta}] = 0. \end{aligned} \tag{3.11}$$

Accordingly, the density function $f(x;\alpha,\beta,\lambda,\xi)$ could attain a maximum, a minimum or a point of inflection according to whether

$$\frac{\mathrm{d}^2\ \log\ f(x;\alpha,\beta,\lambda,\xi)}{\mathrm{d}x^2} < 0, \frac{\mathrm{d}^2\ \log\ f(x;\alpha,\beta,\lambda,\xi)}{\mathrm{d}x^2} > 0, \text{ or } \frac{\mathrm{d}^2\ \log\ f(x;\alpha,\beta,\lambda,\xi)}{\mathrm{d}x^2} = 0.$$

Similarly, for the hazard rate function. The points of inflection and extrema of equations (3.10) and (3.11) can be readily examined numerically.

3.2.4 Series Expansion

In this section, we present the series expansion of the NWG-G cdf and density function. Note that

$$\begin{aligned} F(x;\lambda,\alpha,\beta,\xi) &= \exp[-\lambda(-\log[1-\overline{G}^{\alpha}(x;\xi)])^{\beta}] \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\lambda^i(-\log[1-\overline{G}^{\alpha}(x;\xi)])^{i\beta}. \end{aligned} \tag{3.12}$$

For $i \geq 1$,

$$\begin{aligned}(-\log[1-\overline{G}^{\alpha}(x;\xi)])^{i\beta} &= \sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+l}i\beta}{(i\beta-l)}\binom{k-i\beta}{k}\binom{k}{j} \\ &\times \binom{i\beta+k}{l}P_{j,k}[1-\overline{G}^{\alpha}(x;\xi)]^l \\ &= \sum_{k,l,p=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+l+p}i\beta}{(i\beta-l)}\binom{k-i\beta}{k}\binom{k}{j} \\ &\times \binom{i\beta+k}{l}\binom{l}{p}P_{j,k}\overline{G}^{p\alpha}(x;\xi)\end{aligned}$$

$$= \sum_{k,l,p,q=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j+k+l+p} i\beta}{(i\beta - l)} \binom{k - i\beta}{k} \binom{k}{j} \times \binom{i\beta + k}{l} \binom{l}{p} \binom{p\alpha}{q} P_{j,k} [G(x;\xi)]^q,$$

where (for $j \geq 0$), $P_{j,0} = 1$ and for ($k = 1, 2, ...$),

$$P_{j,k} = \frac{1}{k} \sum_{m=1}^{k} \frac{(-1)^m [m(j+1) - k]}{(m+1)} P_{j,k-m}, \tag{3.13}$$

(see Gradshteyn and Ryzhik [36]). Now, using these results in the NWG-G cdf we have

$$F(x;\lambda,\alpha,\beta,\xi) = \sum_{q=0}^{\infty} c_q [G(x;\xi)]^q, \tag{3.14}$$

where

$$c_q = \sum_{i,k,l,p=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j+k+l+p} i\beta}{(i\beta - l)} \binom{k - i\beta}{k} \binom{k}{j} \times \binom{i\beta + k}{l} \binom{l}{p} \binom{p\alpha}{q} P_{j,k}. \tag{3.15}$$

Consequently, NWG-G density function can be written as a mixture of exponentiated-G (exp-G) density functions with power parameter $q + 1$, that is,

$$f(x;\lambda,\alpha,\beta,\xi) = \sum_{q=0}^{\infty} c_{q+1} h_{q+1}(x;\xi), \tag{3.16}$$

where $h_{q+1}(x;\xi) = (q+1)[G(x;\xi)]^q g(x;\xi)$ is the exp-G density with power parameter $q + 1$. The mathematical and statistical properties of the NWG-G family of distributions follow directly from those of the exponentiated-G distribution.

Some Sub-Families of NWG-G Family of Distributions

In this section, we present some sub-classes of the NWG-G family of distributions.

- When $\alpha = 1$, we obtain the New Weibull-G (NW-G) family of distributions with pdf given by

  $$f(x;\lambda,\beta,\xi) = \lambda\beta \frac{g(x;\xi)}{[G(x;\xi)]} \left(-\log\left(G(x;\xi)\right)\right)^{\beta-1} e^{(-\lambda[-\log(G(x;\xi))])^{\beta}}$$

  for $\lambda, \beta,$ and parameter vector ξ, (see Tahir et al. [65]).

- When $\beta = 1$, we get the New exponential generalized-G (NEG-G) family of distributions with the pdf

$$f(x;\lambda,\alpha,\xi) = \lambda\alpha\frac{g(x;\xi)}{\left[1-\overline{G}^{\alpha}(x;\xi)\right]}[\overline{G}(x;\xi)]^{\alpha-1}e^{\left(-\lambda\left[-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)\right]\right)}$$

for $\lambda, \alpha > 0$ and parameter vector ξ. This is a new family of distributions.

- When $\beta = 2$, we get the New Rayleigh generalized-G (NRG-G) family of distributions with the pdf

$$\begin{aligned} f(x;\lambda,\alpha,\xi) &= 2\lambda\alpha\frac{g(x;\xi)}{\left[1-\overline{G}^{\alpha}(x;\xi)\right]}[\overline{G}(x;\xi)]^{\alpha-1}\left(-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)\right) \\ &\times e^{\left(-\lambda\left[-\log\left(1-\overline{G}^{\alpha}(x;\xi)\right)\right]\right)^2} \end{aligned}$$

for $\lambda, \alpha > 0$ and parameter vector ξ. This is a new family of distributions.

- When $\alpha = \beta = 1$, we get the New exponential-G (NE-G) family of distributions with the pdf

$$f(x;\lambda,\xi) = \lambda\frac{g(x;\xi)}{[G(x;\xi)]}e^{(-\lambda[-\log(G(x;\xi))])}$$

for $\lambda > 0$ and parameter vector ξ.

- When $\alpha = 1$, and $\beta = 2$, we get the New Rayleigh-G (NR-G) family of distributions with the pdf

$$f(x;\lambda,\xi) = 2\lambda\frac{g(x;\xi)}{[G(x;\xi)]}\left(-\log\left(G(x;\xi)\right)\right)e^{(-\lambda[-\log(G(x;\xi))])^2}$$

for $\lambda > 0$ and parameter vector ξ.

3.2.5 Some Special Cases

In this section, we consider some special cases of the NWG-G family of distributions, specifically when the baseline distribution function $G(x;\xi)$ is uniform, log-logistic, normal, half-logistic and Weibull distributions, respectively. Plots of the pdf, and hazard functions are also presented.

NWG-Uniform Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta) = \frac{x}{\theta}$ and $g(x;\theta) = \frac{1}{\theta}$ for $0 < x < \theta < \infty$. Then, the new NWG-Uniform (NWG-U) distribution has cdf and pdf given by

$$F(x;\alpha,\beta,\lambda,\theta) = \exp\left(-\lambda\left[-\log\left(1-\left(1-\frac{x}{\theta}\right)^{\alpha}\right)\right]^{\beta}\right), \tag{3.17}$$

and

$$\begin{aligned} f(x;\alpha,\beta,\lambda,\theta) &= \frac{\lambda\alpha\beta}{\theta}\frac{\left(1-\frac{x}{\theta}\right)^{\alpha-1}}{1-\left(1-\frac{x}{\theta}\right)^{\alpha}}\left[-\log\left(1-\left(1-\frac{x}{\theta}\right)^{\alpha}\right)\right]^{\beta-1} \\ &\times \exp\left(-\lambda\left[-\log\left(1-\left(1-\frac{x}{\theta}\right)^{\alpha}\right)\right]^{\beta}\right), \end{aligned} \tag{3.18}$$

respectively, for $\alpha,\beta,\lambda > 0$, and $0 < x < \theta < \infty$. The hazard rate function is given by

$$\begin{aligned} h_F(x;\alpha,\beta,\lambda,\theta) &= \frac{\lambda\alpha\beta}{\theta}\frac{\left(1-\frac{x}{\theta}\right)^{\alpha-1}}{1-\left(1-\frac{x}{\theta}\right)^{\alpha}}\left[-\log\left(1-\left(1-\frac{x}{\theta}\right)^{\alpha}\right)\right]^{\beta-1} \\ &\times \exp\left(-\lambda\left[-\log\left(1-\left(1-\frac{x}{\theta}\right)^{\alpha}\right)\right]^{\beta}\right) \\ &\times \left(1-\exp\left(-\lambda\left[-\log\left(1-\left(1-\frac{x}{\theta}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{-1}. \end{aligned} \tag{3.19}$$

Figure (3.1) shows the plots of the NWG-U distribution for different parameter values. The pdf can take various shapes including the reverse J, uni-modal, almost symmetric, left skewed and right skewed.

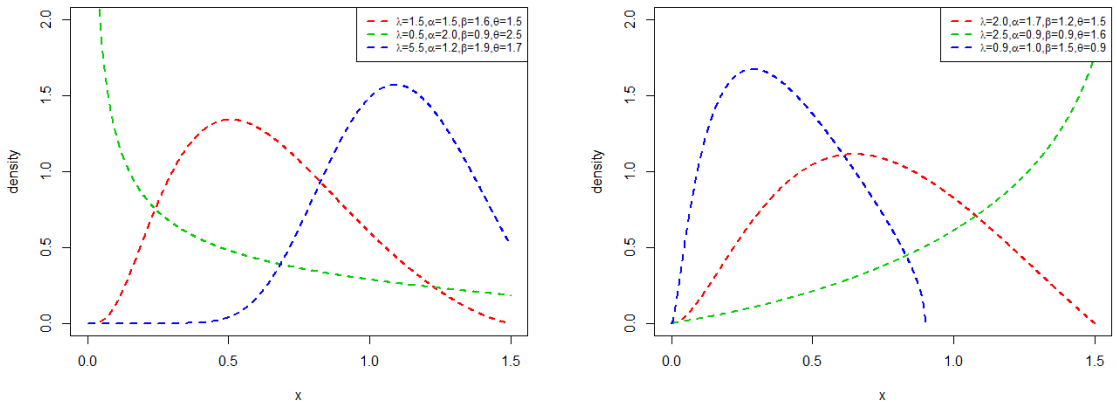


Figure 3.1: Plots of NWG-U density function

Figure (3.2) shows the plots of the hazard rate function of the NWG-U distribution for different parameter values. Graphs of the hazard rate function exhibits increasing, decreasing and bathtub shapes.

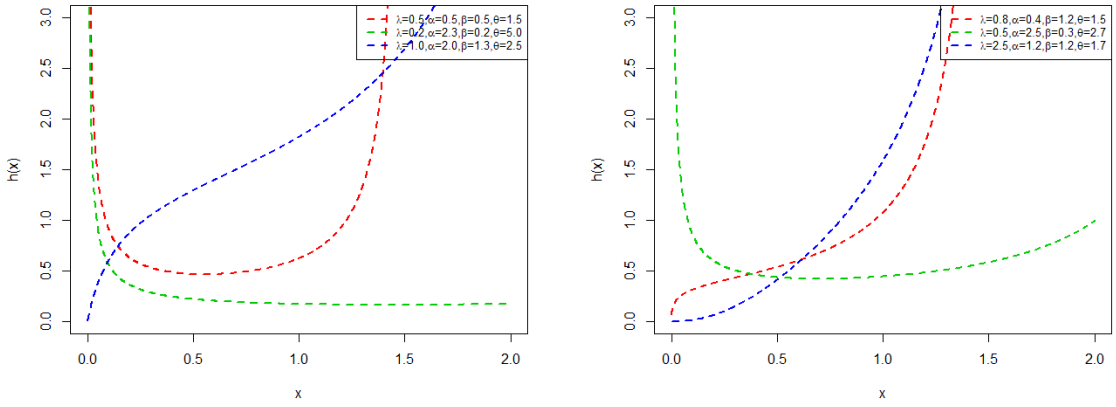


Figure 3.2: Plots of the NWG-U hazard function

NWG-Log-Logistic Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;c) = 1 - (1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$ for $c > 0$. Then, the NWG log-logistic (NWG-LLoG) distribution has cdf and pdf given by

$$F(x;\alpha,\beta,\lambda,c) = e^{-\lambda\left(-\log\left[1-(1+x^c)^{-\alpha}\right]\right)^{\beta}},$$

and

$$\begin{aligned} f(x;\alpha,\beta,\lambda,c) &= c\lambda\alpha\beta\frac{x^{c-1}(1+x^c)^{-2}}{1-(1+x^c)^{-\alpha}}\left(-\log\left[1-(1+x^c)^{-\alpha}\right]\right)^{\beta-1} \\ &\times e^{-\lambda\left(-\log\left[1-(1+x^c)^{-\alpha}\right]\right)^{\beta}}, \end{aligned}$$

respectively, for $\alpha,\beta,\lambda,c > 0$. The hazard rate function is given by

$$\begin{aligned} h_F(x;\alpha,\beta,\lambda,c) &= c\lambda\alpha\beta\frac{x^{c-1}(1+x^c)^{-2}}{1-(1+x^c)^{-\alpha}}\left(-\log\left[1-(1+x^c)^{-\alpha}\right]\right)^{\beta-1} \\ &\times e^{-\lambda\left(-\log\left[1-(1+x^c)^{-\alpha}\right]\right)^{\beta}}\left(1-e^{-\lambda\left(-\log\left[1-(1+x^c)^{-\alpha}\right]\right)^{\beta}}\right)^{-1}. \end{aligned}$$

Figure (3.3) shows the plots of the NWG-LLoG distribution for different parameter values. The pdf can take various shapes including the reverse J, uni-modal, almost symmetric, left skewed and right skewed.

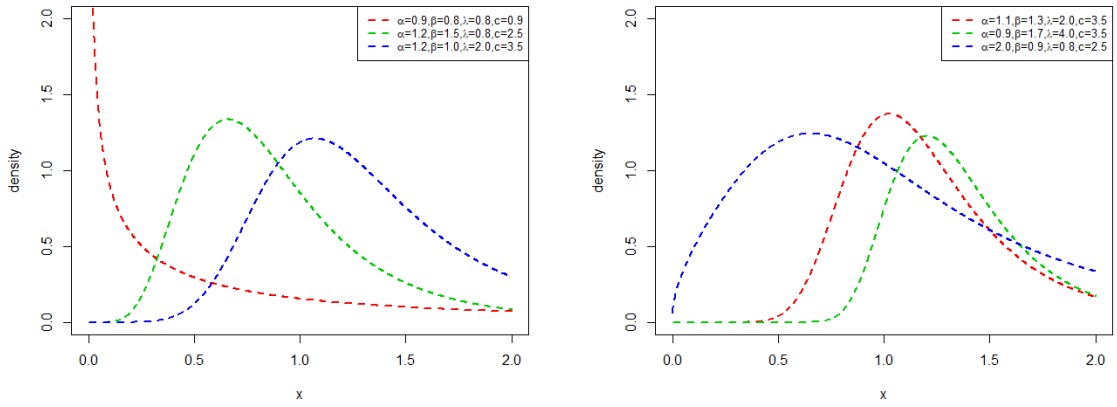


Figure 3.3: Plots of NWG-LLoG density function

Figure (3.4) shows the plots of the hazard rate function of the NWG-LLoG distribution for different parameter values. Graphs of the hazard rate function exhibits bathtub, decreasing, increasing, bathtub followed by upside-down bathtub and upside-down bathtub shapes.

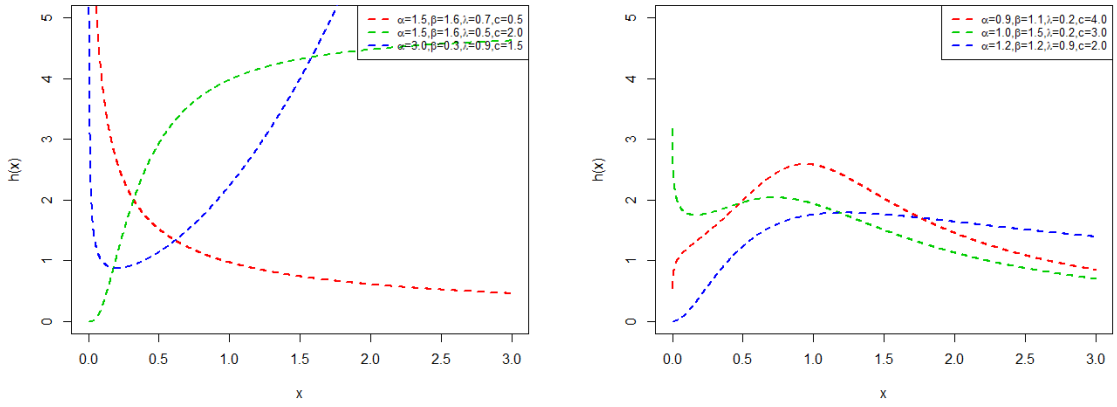


Figure 3.4: Plots of NWG-LLoG hazard function

NWG-Normal Distribution

Suppose the cdf and pdf of the baseline distribution are given by $N(\mu, \sigma^2)$, with mean μ and variance σ^2. Then the new NWG-Normal (NWG-N) distribution has cdf and pdf given by

$$F(x;\lambda,\beta,\alpha,\mu,\sigma) \quad = \quad \exp\left(-\lambda\left(-\log\left(1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right)\right)^{\beta}\right),$$

and

$$
\begin{aligned}
f(x;\lambda,\beta,\alpha,\mu,\sigma) &= \lambda\alpha\beta\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\left[1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right]}\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha-1} \\
&\times \left(-\log\left(1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right)\right)^{\beta-1} \\
&\times e^{\left(-\lambda\left[-\log\left(1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right)\right]\right)^{\beta}},
\end{aligned}
$$

respectively, for $\alpha, \beta, \lambda > 0$, $-\infty < \mu < \infty$ and $\sigma > 0$. The hazard rate function is given by

$$
\begin{aligned}
h_F(x;\lambda,\alpha,\beta,\mu,\sigma) &= \lambda\alpha\beta\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\left[1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right]}\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha-1} \\
&\times \left(-\log\left(1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right)\right)^{\beta-1} \\
&\times e^{\left(-\lambda\left[-\log\left(1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right)\right]\right)^{\beta}} \\
&\times \left(1-e^{\left(-\lambda\left(-\log\left(1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right)\right)^{\beta}\right)}\right)^{-1}.
\end{aligned}
$$

Figure (3.5) shows the plots of the NWG-N distribution for different parameter values. The pdf can take various shapes including the uni-modal, almost symmetric, left skewed and right skewed.

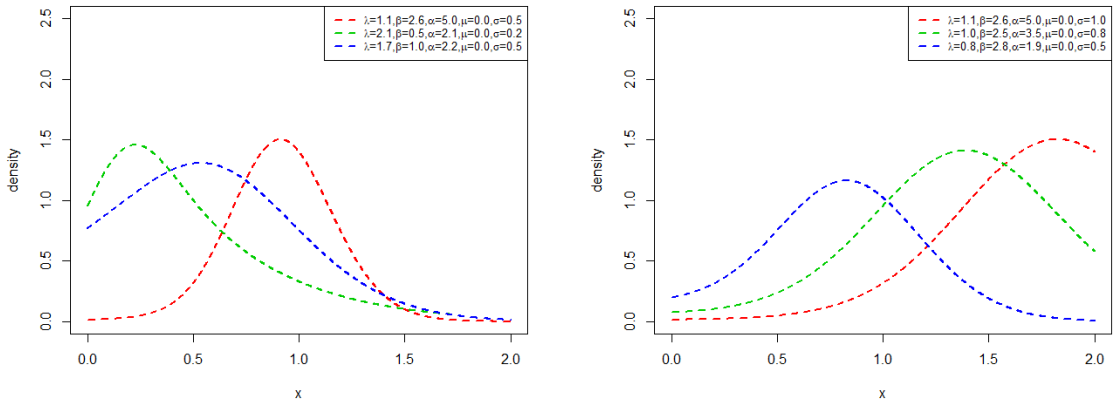


Figure 3.5: Plots of NWG-N density function

Figure (3.6) shows the plots of the hazard rate function of the NWG-N distribution for different parameter values. Graphs of the hazard rate function

exhibits upside-down bathtub followed by bathtub, upside-down bathtub and increasing shapes.

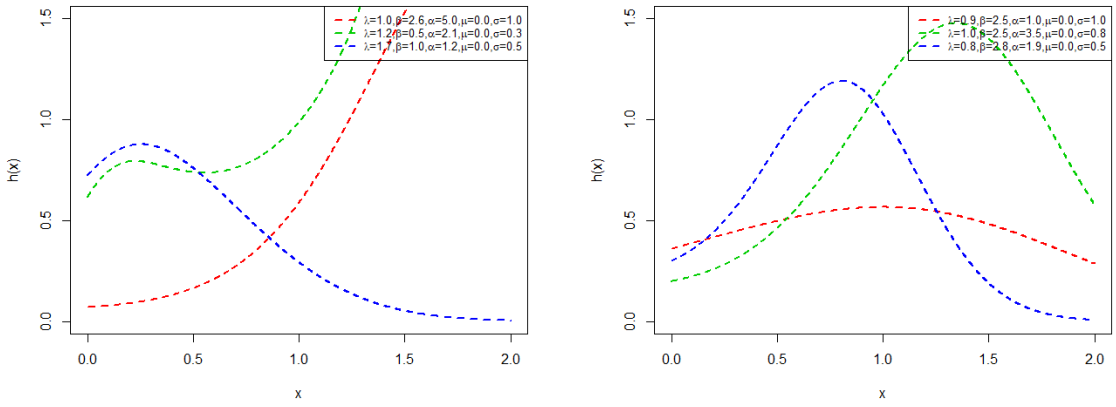


Figure 3.6: Plots of NWG-N hazard function

NWG-Half-logistic Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta) = \frac{1-e^{-\theta x}}{1+e^{-\theta x}}$ and $g(x;\theta) = \frac{2\theta e^{-\theta x}}{(1+e^{-\theta x})^2}$ for $\theta > 0$ and $x > 0$. Then, the NWG-Half-logistic (NWG-HL) distribution has cdf and pdf given by

$$F(x;\lambda,\beta,\alpha,\theta) = \exp\left(-\lambda\left(-\log\left(1-\left(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^{\alpha}\right)\right)^{\beta}\right),$$

and

$$\begin{aligned} f(x;\lambda,\beta,\alpha,\theta) &= \lambda\alpha\beta\frac{\frac{2\theta e^{-\theta x}}{(1+e^{-\theta x})^2}(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}})^{\alpha-1}}{\left[1-(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}})^{\alpha}\right]}\left(-\log\left(1-\left(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^{\alpha}\right)\right)^{\beta-1} \\ &\times e^{\left(-\lambda\left[-\log\left(1-(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}})^{\alpha}\right)\right]\right)^{\beta}}, \end{aligned}$$

respectively, for α, β, λ and $\theta > 0$. The hazard rate function is given by

$$\begin{aligned} h_F(x;\lambda,\alpha,\beta,\theta) &= \lambda\alpha\beta\frac{\frac{2\theta e^{-\theta x}}{(1+e^{-\theta x})^2}\left(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^{\alpha-1}}{\left[1-(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}})^{\alpha}\right]}\left(-\log\left(1-\left(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^{\alpha}\right)\right)^{\beta-1} \\ &\times e^{\left(-\lambda\left[-\log\left(1-(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}})^{\alpha}\right)\right]\right)^{\beta}}\left(1-e^{\left(-\lambda\left(-\log\left(1-(1-\frac{1-e^{-\theta x}}{1+e^{-\theta x}})^{\alpha}\right)\right)^{\beta}\right)}\right)^{-1}. \end{aligned}$$

Figure (3.7) shows the plots of the NWG-HL distribution for different parameter values. The pdf can take various shapes including left skewed, right skewed and almost symmetric.

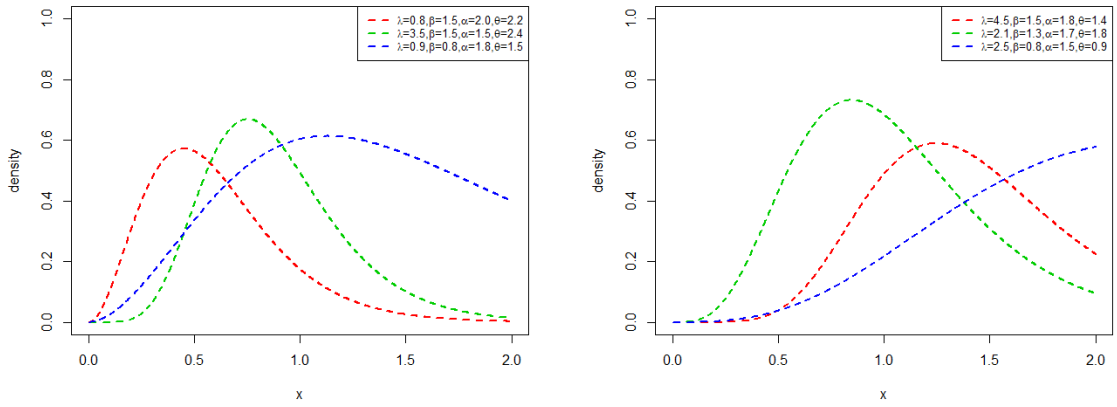


Figure 3.7: Plots of NWG-HL density function

Figure (3.8) shows the plots of the hazard rate function of the NWG-HL distribution for different parameter values. Graphs of the hazard rate function exhibits decreasing, increasing, bathtub, upside-down bathtub and bathtub followed by upside-down bathtub shapes.

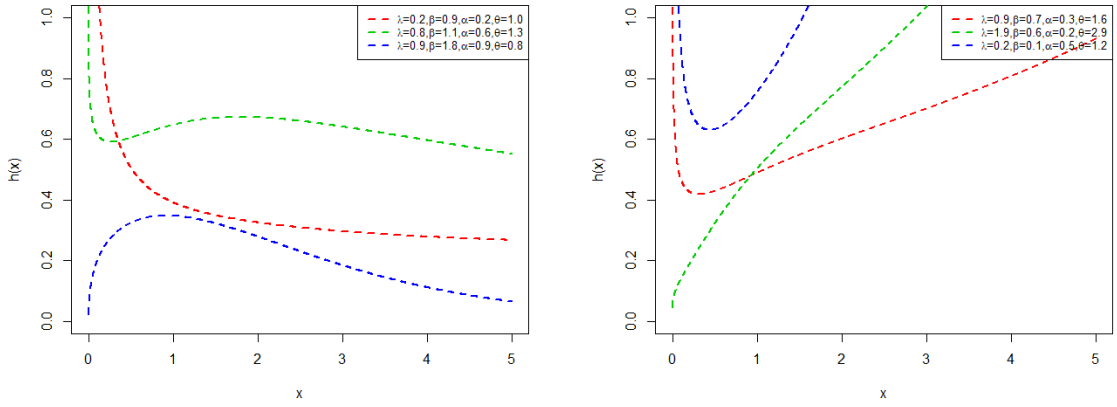


Figure 3.8: Plots of the NWG-HL hazard function

NWG-Gamma Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta) = \frac{\Gamma_x(\theta)}{\Gamma(\theta)}$ and $g(x;\theta) = \frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)}$ for $\theta > 0$ and $x > 0$. Then, the NWG-Gamma (NWG-Ga) distribution has cdf and pdf given by

$$F(x;\lambda,\beta,\alpha,\theta) = \exp\left(-\lambda\left(-\log\left(1-\left(1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)}\right)^{\alpha}\right)\right)^{\beta}\right),$$

and

$$f(x;\lambda,\beta,\alpha,\theta) = \lambda\alpha\beta\frac{\frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)}(1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)})^{\alpha-1}}{\left[1-(1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)})^{\alpha}\right]}\left(-\log\left(1-\left(1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)}\right)^{\alpha}\right)\right)^{\beta-1} \times e^{\left(-\lambda\left[-\log\left(1-(1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)})^{\alpha}\right)\right]\right)^{\beta}},$$

respectively, for α, β, λ and $\theta > 0$. The hazard rate function is given by

$$h_F(x;\lambda,\alpha,\beta,\theta) = \lambda\alpha\beta\frac{\frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)}[1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)}]^{\alpha-1}}{\left[1-(1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)})^{\alpha}\right]}\left(-\log\left(1-\left(1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)}\right)^{\alpha}\right)\right)^{\beta-1} \times \left(1-\exp\left(-\lambda\left(-\log\left(1-\left(1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)}\right)^{\alpha}\right)\right)^{\beta}\right)\right)^{-1} \times e^{\left(-\lambda\left[-\log\left(1-\left(1-\frac{\Gamma_x(\theta)}{\Gamma(\theta)}\right)^{\alpha}\right)\right]\right)^{\beta}}.$$

Figure (3.9) shows the plots of the NWG-Ga distribution for different parameter values. The pdf can take various shapes including the reverse J, almost symmetric, left and right-skewed.

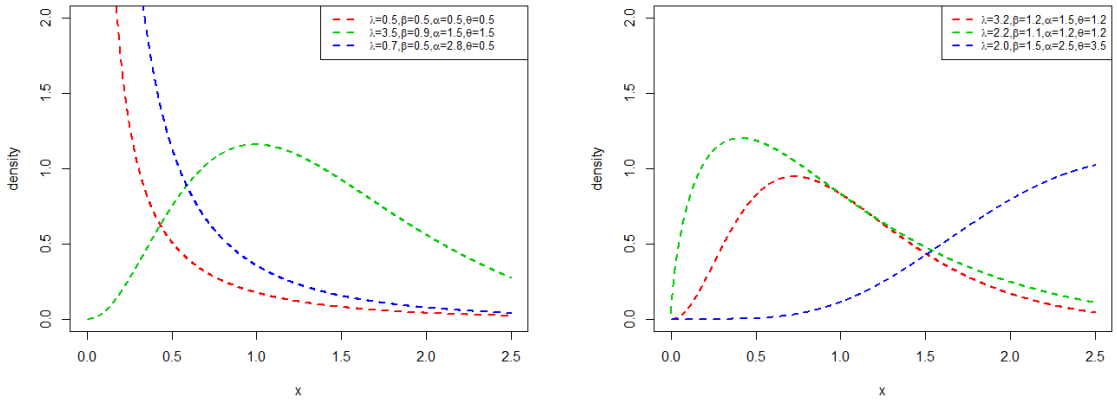


Figure 3.9: Plots of NWG-Gamma density function

Figure (3.10) shows the plots of the hazard rate function of the NWG-Ga distribution for different parameter values. Graphs of the hazard rate function exhibits decreasing,bathtub, upside-down bathtub and upside-down followed by bathtub shapes.

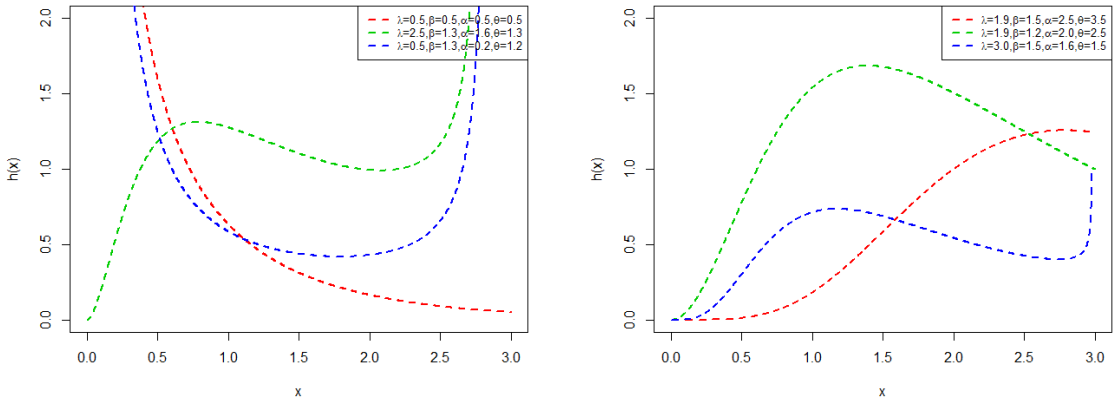


Figure 3.10: Plots of the NWG-Gamma hazard function

NWG-Weibull Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta,\gamma) = 1-\exp(-\theta x^{\gamma})$ and $g(x;\theta,\gamma) = \gamma\theta x^{\gamma-1}\exp(-\theta x^{\gamma})$ for $\theta,\gamma > 0$ and $x > 0$. Then, the NWG-Weibull (NWG-W) distribution has cdf and pdf given by

$$F(x;\alpha,\beta,\lambda,\theta,\gamma) = e^{-\lambda\left[-\log\left(1-e^{-\gamma\alpha x^{\theta}}\right)\right]^{\beta}}, \tag{3.20}$$

and

$$\begin{aligned} f(x;\alpha,\beta,\lambda,\theta,\gamma) &= \frac{\lambda\alpha\beta\theta\gamma x^{\theta-1}e^{-\gamma x^{\theta}}e^{-(\alpha-1)\gamma x^{\theta}}}{1-e^{-\alpha\gamma x^{\theta}}}\left[-\log\left(1-e^{-\gamma\alpha x^{\theta}}\right)\right]^{\beta-1} \\ &\times e^{-\lambda\left[-\log\left(1-e^{-\gamma\alpha x^{\theta}}\right)\right]^{\beta}}, \end{aligned}$$

respectively, for $\alpha,\beta,\lambda,\theta,\gamma > 0$, and $x > 0$. When $\gamma = 1$ and $\gamma = 2$, we obtain the NWG-exponential (NWG-E) and NWG-Rayleigh (NWG-R) distributions, respectively. The NWG-W hazard rate function is given by

$$\begin{aligned} h_F(x;\alpha,\beta,\lambda,\theta,\gamma) &= \frac{\lambda\alpha\beta\theta\gamma x^{\theta-1}e^{-\gamma x^{\theta}}e^{-(\alpha-1)\gamma x^{\theta}}}{1-e^{-\alpha\gamma x^{\theta}}}\left[-\log\left(1-e^{-\gamma\alpha x^{\theta}}\right)\right]^{\beta-1} \\ &\times e^{-\lambda\left[-\log\left(1-e^{-\gamma\alpha x^{\theta}}\right)\right]^{\beta}}\left(1-e^{-\lambda\left[-\log\left(1-e^{-\gamma\alpha x^{\theta}}\right)\right]^{\beta}}\right)^{-1}. \end{aligned}$$

Figure (3.11) shows the plots of the NWG-W distribution for different parameter values. The pdf can take various shapes including the reverse J, uni-modal, almost symmetric, left skewed and right skewed.

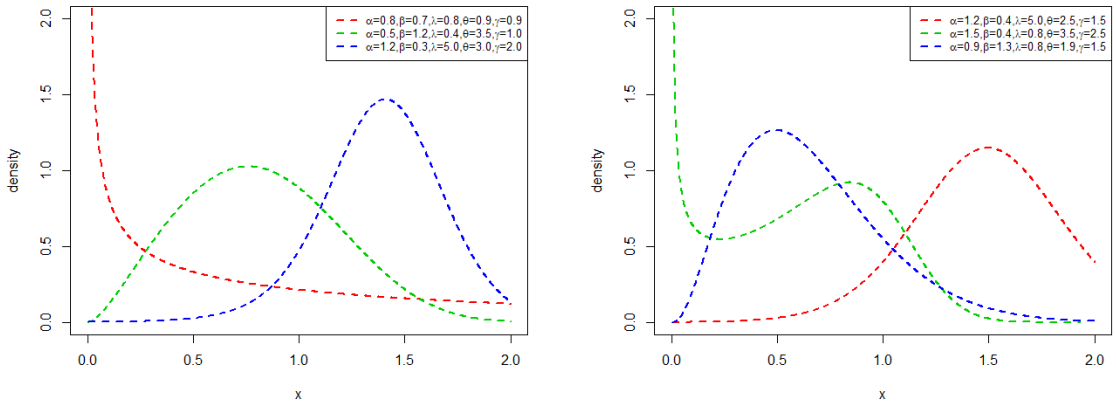


Figure 3.11: Plots of NWG-W density function

Figure (3.12) shows the plots of the hazard rate function of the NWG-W distribution for different parameter values. Graphs of the hazard rate function exhibits bathtub, upside-down bathtub, increasing and decreasing shapes.

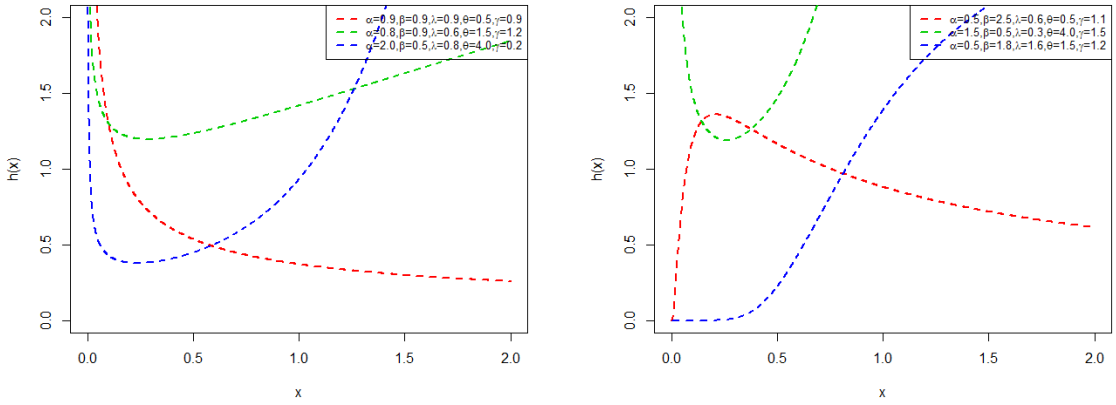


Figure 3.12: Plots of NWG-W hazard function

Table 3.1 shows some random numbers that are generated from the NWG-W distribution for selected parameter values.

Table 3.1: **Table of quantiles for selected parameter values of the NWG-W distribution**

	$(\alpha, \beta, \lambda, \theta, \gamma)$				
u	$(1.9, 3.4, 1, 0.5, 1)$	$(2, 3.5, 0.9, 0.5, 1.2)$	$(0.5, 4, 1.5, 1.5, 0.5)$	$(0.9, 5, 1.8, 0.5, 0.7)$	$(0.8, 4.5, 1.6, 0.4, 0.9)$
0.1	0.02955	0.01722	1.36411	0.46678	0.24877
0.2	0.04012	0.02336	1.47642	0.56078	0.32257
0.3	0.05063	0.02939	1.56864	0.64653	0.39412
0.4	0.06217	0.03605	1.65629	0.73547	0.47217
0.5	0.07588	0.04393	1.74663	0.83503	0.56378
0.6	0.09319	0.05391	1.84605	0.83504	0.56379
0.7	0.11695	0.06756	1.96383	1.11002	0.83669
0.8	0.15403	0.08885	2.11844	1.33848	1.08252
0.9	0.22874	0.13187	2.36686	1.76809	1.58332

3.3 Moments and Incomplete Moments

In this section, moments, moment generating function and incomplete moments for the NWG-G family of distributions are presented. These measures (moments, moment generating function and incomplete moments) can be readily obtained for the sub-families and sub-models given in section 2.

3.3.1 Moments and Generating Function

Let $Y \sim Exponentiated - G(q+1, \xi)$, then the r^{th} raw moment, μ'_r of the NWG-G family of distributions is given by:

$$\begin{aligned} \mu'_r = E(X^r) &= \sum_{q=0}^{\infty} c_{q+1} E(Y_q^r) \\ &= \sum_{q=0}^{\infty} c_{q+1} \int_0^{\infty} y^r g^*_{q+1}(y;\xi) dy \\ &= \sum_{q=0}^{\infty} c_{q+1}(q+1) \int_0^1 [Q_G(u;\xi)]^r u^q du, \end{aligned} \tag{3.21}$$

where $g^*_{q+1}(y;\xi)$ is the pdf of the exponentiated distribution with power parameter $q+1$. The moment generating function (MGF), for $|t| < 1$, is given by:

$$M_X(t) = \sum_{q=0}^{\infty} c_{q+1} M_Y(t) = \sum_{q=0}^{\infty} c_{q+1} \frac{t^i}{i!} E(Y^i).$$

The variance (σ^2), Standard deviation (SD=σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively.

Note that the r^{th} cumulant of the random variable X can be readily obtained from the recursive relationship: $\kappa_r = \mu_r' - \sum_{s=1}^{r-1} \binom{r-1}{s-1} \mu_{r-s}' \kappa_s$, where $\mu_r' = E(X - \mu_1')^r$, so that the CS and CK are given by $\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}$ and $\gamma_2 = \frac{\kappa_4}{\kappa_2^2}$.

A table of moments for some selected parameters values of the special case of the NWG-G family of distribution namely NWG-Weibull distribution is given in Table 3.2.

Table 3.2: **Table of Moments for the NWG-W distribution for selected parameter values**

	(1.9,3.4,1.0,0.5,1.0)	(2.0,3.5,0.9,0.5,1.2)	(0.5,4.0,1.5,1.5,0.5)	(0.9,5.0,1.8,0.5,0.7)	(0.8,4.5,1.6,0.4,0.9)
$E(X)$	0.10959	0.06463	0.00041	0.41989	0.38213
$E(X^2)$	0.02359	0.00878	0.00039	0.29914	0.22629
$E(X^3)$	0.00884	0.00239	0.00038	0.22560	0.15239
$E(X^4)$	0.00469	0.00104	0.00037	0.17797	0.11213
$E(X^5)$	0.00303	0.00060	0.00036	0.14543	0.08772
$E(X^6)$	0.00219	0.00041	0.00035	0.12220	0.07164
SD	0.00170	0.00030	0.00034	0.10497	0.06037
CV	0.00139	0.00024	0.00033	0.09176	0.05208
CS	0.00117	0.00019	0.00032	0.08139	0.04576
CK	0.00101	0.00017	0.00031	0.07305	0.04078

3.3.2 Incomplete Moments

Conditional and incomplete moments are particularly useful for the analysis of lifetime models, reliability and in measuring inequality. The r^{th} incomplete

moment of the NWG-G family of distributions is given by

$$\begin{aligned} m_r(t) &= \int_t^{\infty} x^r f(x;\lambda,\alpha,\beta,\xi)dx \\ &= \sum_{q=0}^{\infty} c_{q+1} E\left(Y^r I_{\{Y^r \geq t\}}\right), \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} E\left(Y^r I_{\{Y^r \geq t\}}\right) &= \int_t^{\infty} y^r g^*_{q+1}(y;\xi)dy \\ &= (q+1)\int_{G(u;\xi)}^{1} [Q_G(u;\xi)]^r u^q du, \end{aligned} \tag{3.23}$$

for $\alpha, \beta, \lambda > 0$, and parameter vector ξ. The mean residual life function is given by $E(X - t|X > t) = E(X|X > t) - t = V_{_F}(t) - t$, where $V_{_F}(t)$ is referred to as the vitality function of the distribution function F. Also, the mean deviations, Bonferroni and Lorenz curves can be readily computed from the first incomplete moment.

3.4 Order Statistics and Rényi Entropy

Order statistics and entropy play important roles in probability and statistics, particularly in reliability, lifetime data analysis and information theory. In this section, we present the distribution of the i^{th} order statistics and Rényi entropy for the NWG-G family of distributions.

3.4.1 Order Statistics

In this subsection, the pdf of the i^{th} order statistic is presented. Let $X_1, X_2, ..., X_n$ be independent and identically distributed NWG-G random variables. Using the binomial expansion

$$(1 - F(x))^{n-i} = \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j [F(x)]^j,$$

the pdf of the i^{th} order statistic from the NWG-G pdf $f(x;\lambda,\alpha,\beta,\xi) = f(x)$ can be written as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!f(x)}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i} \\ &= \frac{n!f(x)}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}(-1)^m\binom{n-i}{m}[F(x)]^{m+i-1} \\ &= \sum_{m=0}^{n-i}\frac{n!}{(i-1)!(n-i)!(m+i)}(-1)^m\binom{n-i}{m}(m+i)[F(x)]^{m+i-1}f(x) \\ &= \sum_{m=0}^{n-i}\omega(i,m)f_{m+i}(x;\lambda,\alpha,\beta,\xi), \end{aligned} \tag{3.24}$$

where $f_{m+i}(x;\lambda,\alpha,\beta,\xi)$ is the exponentiated new Weibull generalized-G pdf with exponentiated parameter $m+i>0$ and weights $\omega(i,m)$ given by

$$\begin{aligned} \omega(i,m) &= \frac{1}{B(i,n-i+1)}\frac{(-1)^m}{m+i}\binom{n-i}{m} \\ &= (-1)^m\binom{n}{m+i}\binom{m+i-1}{m}. \end{aligned} \tag{3.25}$$

Note that we can also write the pdf of the i^{th} order statistic as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^{j+i-1}f(x) \\ &= \sum_{r,k=0}^{\infty}m_{r,k}h_{r+k+1}(x), \end{aligned} \tag{3.26}$$

where $h_{r+k+1}(x)$ is the exp-G density function with power parameter $r+k+1$,

$$m_{r,k} = \frac{n!(r+1)(i-1)!c_{r+1}}{r+k+1}\sum_{j=0}^{n-i}\frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!j!}, \tag{3.27}$$

and c_r is defined by equation (3.15). The quantities $f_{j+i-1,k}$ are given recursively by

$$f_{j+i-1,k} = (kc_0)^{-1}\sum_{m=1}^{k}[m(j+i)-k]c_m f_{j+i-1,k-m}. \tag{3.28}$$

The s^{th} moment of the distribution of the i^{th} order statistics is given by

$$E(X_{i:n}^s) = \sum_{r,k=0}^{\infty}m_{r,k}E(Y_{r+k+1}^s), \tag{3.29}$$

where Y_{r+k+1} is distributed as Exp-G with power parameter $r+k+1>0$.

3.4.2 Rényi Entropy

Rényi entropy (Rényi [59]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty [f(x;\lambda,\alpha,\beta,\xi)]^v dx\right), v \neq 1, v > 0. \tag{3.30}$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Set $[f(x;\lambda,\alpha,\beta,\xi)]^v = (f(x))^v$. Note that

$$\begin{aligned}
[f(x)]^v &= \frac{(\lambda\alpha\beta)^v (g(x;\xi))^v (\overline{G}(x;\xi))^{v\alpha-v}}{(1-\overline{G}^\alpha(x;\xi))^v}\left[-\log(1-\overline{G}^\alpha(x;\xi))\right]^{v\beta-v} e^{-v\lambda\left[-\log(1-\overline{G}^\alpha(x;\xi))\right]^\beta} \\
&= \frac{(\lambda\alpha\beta)^v (g(x;\xi))^v (\overline{G}(x;\xi))^{v\alpha-v}}{(1-\overline{G}^\alpha(x;\xi))^v}\sum_{i=0}^\infty \frac{(-1)^i (v\lambda)^i}{i!}\left[-\log(1-\overline{G}^\alpha(x;\xi))\right]^{i\beta+v\beta-v} \\
&= \sum_{i=0}^\infty \frac{(\lambda\alpha\beta)^v (g(x;\xi))^v (\overline{G}(x;\xi))^{v\alpha-v}}{(1-\overline{G}^\alpha(x;\xi))^v}\frac{(-1)^i (v\lambda)^i}{i!}\left[-\log(1-\overline{G}^\alpha(x;\xi))\right]^{i\beta+v\beta-v} \\
&= \sum_{i=0}^\infty \frac{(\lambda\alpha\beta)^v (g(x;\xi))^v (\overline{G}(x;\xi))^{v\alpha-v}}{(1-\overline{G}^\alpha(x;\xi))^v}\frac{(-1)^i (v\lambda)^i}{i!}\sum_{k,l=0}^\infty\sum_{j=0}^k \frac{(-1)^{j+k+l}}{(i\beta+v\beta-v-l)} \\
&\times \binom{k-(i\beta+v\beta-v)}{k}\binom{k}{j}\binom{i\beta+v\beta-v+k}{l} P_{j,k}(1-\overline{G}^\alpha(x;\xi))^l
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^\infty\sum_{k,l=0}^\infty\sum_{j=0}^k (\lambda\alpha\beta)^v (g(x;\xi))^v (\overline{G}(x;\xi))^{v\alpha-v}\frac{(v\lambda)^i}{i!}\frac{(-1)^{i+j+k+l}}{(i\beta+v\beta-v-l)} \\
&\times \binom{k-(i\beta+v\beta-v)}{k}\binom{k}{j}\binom{i\beta+v\beta-v+k}{l} P_{j,k}(1-\overline{G}^\alpha(x;\xi))^{l-v} \\
&= \sum_{i=0}^\infty\sum_{k,l=0}^\infty\sum_{j=0}^k (\lambda\alpha\beta)^v \frac{(v\lambda)^i}{i!}\frac{(-1)^{i+j+k+l}}{(i\beta+v\beta-v-l)} \\
&\times \binom{k-(i\beta+v\beta-v)}{k}\binom{k}{j}\binom{i\beta+v\beta-v+k}{l} P_{j,k} \\
&\times \sum_{s=0}^\infty (-1)^s \binom{l-v}{s}\overline{G}^{s\alpha}(x;\xi)(\overline{G}(x;\xi))^{v\alpha-v}(g(x;\xi))^v \\
&= \sum_{i=0}^\infty\sum_{k,l,s=0}^\infty\sum_{j=0}^k (\lambda\alpha\beta)^v \frac{(v\lambda)^i}{i!}\frac{(-1)^{i+j+k+l+s}}{(i\beta+v\beta-v-l)} \\
&\times \binom{k-(i\beta+v\beta-v)}{k}\binom{k}{j}\binom{i\beta+v\beta-v+k}{l} P_{j,k} \\
&\times \binom{l-v}{s}\sum_{q=0}^\infty (-1)^q \binom{\alpha(s+v)-v}{q}[G(x;\xi)]^q (g(x;\xi))^v
\end{aligned}$$

$$= \sum_{i=0}^{\infty}\sum_{k,l,s,q=0}^{\infty}\sum_{j=0}^{k}(\lambda\alpha\beta)^{v}\frac{(v\lambda)^{i}}{i!}\frac{(-1)^{i+j+k+l+s+q}}{(i\beta+v\beta-v-l)}\binom{k-(i\beta+v\beta-v)}{k}\binom{k}{j}$$
$$\times \binom{i\beta+v\beta-v+k}{l}P_{j,k}\binom{l-v}{s}\binom{\alpha(s+v)-v}{q}[G(x;\xi)]^{q}(g(x;\xi))^{v},$$

where $v>0, v\neq 1$, and for $k=1,2,...$

$$P_{j,k}=\frac{1}{k}\sum_{m=1}^{k}(-1)^{m}\frac{[m(j+1)-k]}{m+1}P_{j,k-m} \quad \text{and } P_{j,0}=1 \text{ for } j\geq 0. \tag{3.31}$$

Now,

$$\begin{aligned}\int_{0}^{\infty} f^{v}(x)dx &= \sum_{i=0}^{\infty}\sum_{k,l,s,q=0}^{\infty}\sum_{j=0}^{k}(\lambda\alpha\beta)^{v}\frac{(v\lambda)^{i}}{i!}\frac{(-1)^{i+j+k+l+s+q}}{(i\beta+v\beta-v-l)}\binom{k-(i\beta+v\beta-v)}{k}\\ &\times \binom{k}{j}\binom{i\beta+v\beta-v+k}{l}P_{j,k}\binom{l-v}{s}\binom{\alpha(s+v)-v}{q}\\ &\times \int_{0}^{\infty}[G(x;\xi)]^{q}(g(x;\xi))^{v}dx.\end{aligned}$$

Consequently, Rényi entropy for the NWG-G family of distributions is given by

$$\begin{aligned}I_{R}(v) &= \frac{1}{1-v}\log\Bigg[\sum_{i=0}^{\infty}\sum_{k,l,s,q=0}^{\infty}\sum_{j=0}^{k}(\lambda\alpha\beta)^{v}\frac{(v\lambda)^{i}}{i!}\frac{(-1)^{i+j+k+l+s+q}}{(i\beta+v\beta-v-l)}\\ &\times \binom{k-(i\beta+v\beta-v)}{k}\binom{k}{j}\binom{i\beta+v\beta-v+k}{l}P_{j,k}\binom{l-v}{s}\binom{\alpha(s+v)-v}{q}\\ &\times \int_{0}^{\infty}[G(x;\xi)]^{q}(g(x;\xi))^{v}dx\Bigg],\end{aligned} \tag{3.32}$$

for $v>0$, $v\neq 1$, where the integral $\int_{0}^{\infty}[G(x;\xi)]^{q}\,g(x;\xi))^{v}dx$ can be numerically evaluated.

3.5 Maximum Likelihood Estimation and Other Estimation Techniques

In this section, we present several methods of estimation for the parameters of the NWG-G family of distributions including maximum likelihood, ordinary least squares, weighted least squares, minimum distance and maximum product of spacings. The method of maximum likelihood is presented in detail. Let $X\sim NWG-G(\lambda,\alpha,\beta,\xi)$ and $\mathbf{\Delta}=(\lambda,\alpha,\beta,\xi)^{T}$ be the vector of model parameters.

3.5.1 Maximum Likelihood Estimation

The log-likelihood function $\ell_n = \ell_n(\boldsymbol{\Delta})$ based on a random sample of size n from the NWG-G family of distributions is given by

$$\begin{aligned}\ell_n(\boldsymbol{\Delta}) &= n\ln(\lambda) + n\ln(\alpha) + n\ln(\beta) + \sum_{i=1}^{n}\ln(g(x_i;\xi)) + (\alpha-1)\sum_{i=1}^{n}\ln(\overline{G}(x_i;\xi)) \\ &+ (\beta-1)\sum_{i=1}^{n}\ln\left[-\ln(1-\overline{G}^{\alpha}(x;\xi))\right] - \lambda\sum_{i=1}^{n}\left[-\ln(1-\overline{G}^{\alpha}(x_i;\xi))\right]^{\beta}.\end{aligned} \tag{3.33}$$

The first derivative of the log-likelihood function with respect to each component of the parameter vector $\boldsymbol{\Delta} = (\lambda, \alpha, \beta, \xi)^T$, that is, elements of the score vector $U(\boldsymbol{\Delta})$ are given by

$$\frac{\partial \ell_n}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left[-\ln(1-\overline{G}^{\alpha}(x_i;\xi))\right]^{\beta}, \tag{3.34}$$

$$\begin{aligned}\frac{\partial \ell_n}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(\overline{G}(x_i;\xi)) + \sum_{i=1}^{n}\frac{\overline{G}^{\alpha}(x_i;\xi)\ln(\overline{G}(x_i;\xi))}{1-\overline{G}^{\alpha}(x_i;\xi)} \\ &- (\beta-1)\sum_{i=1}^{n}\frac{\overline{G}^{\alpha}(x_i;\xi)\ln(\overline{G}(x_i;\xi))}{(1-\overline{G}^{\alpha}(x_i;\xi))\ln(1-\overline{G}^{\alpha}(x_i;\xi))} \\ &- \lambda\sum_{i=1}^{n}\frac{\beta\left[-\ln(1-\overline{G}^{\alpha}(x_i;\xi))\right]^{\beta-1}\overline{G}^{\alpha}(x_i;\xi)\ln(\overline{G}(x_i;\xi))}{1-\overline{G}^{\alpha}(x_i;\xi)},\end{aligned} \tag{3.35}$$

$$\begin{aligned}\frac{\partial \ell_n}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n}\ln\left[-\ln(1-\overline{G}^{\alpha}(x_i;\xi))\right] \\ &- \lambda\sum_{i=1}^{n}\left[-\ln(1-\overline{G}^{\alpha}(x_i;\xi))\right]^{\beta}\ln\left[-\ln(1-\overline{G}^{\alpha}(x_i;\xi))\right],\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}\frac{\partial \ell_n}{\partial \xi_k} &= +\sum_{i=1}^{n}\frac{\frac{\partial g(x_i;\xi)}{\partial \xi_k}}{g(x_i;\xi)} + (\alpha-1)\sum_{i=1}^{n}\frac{\frac{\partial \overline{G}(x_i;\xi)}{\partial \xi_k}}{\overline{G}(x_i;\xi)} + \alpha\sum_{i=1}^{n}\frac{(\overline{G}(x_i;\xi))^{\alpha-1}\frac{\partial \overline{G}(x_i;\xi)}{\partial \xi_k}}{1-\overline{G}^{\alpha}(x;\xi)} \\ &- (\beta-1)\sum_{i=1}^{n}\frac{\alpha(\overline{G}(x_i;\xi))^{\alpha-1}\frac{\partial \overline{G}(x_i;\xi)}{\partial \xi_k}}{(1-\overline{G}^{\alpha}(x;\xi))\left[-\ln(1-\overline{G}^{\alpha}(x;\xi))\right]} \\ &- \lambda\sum_{i=1}^{n}\frac{\beta\left[-\ln(1-\overline{G}^{\alpha}(x;\xi))\right]^{\beta-1}\alpha(\overline{G}(x_i;\xi))^{\alpha-1}\frac{\partial \overline{G}(x_i;\xi)}{\partial \xi_k}}{(1-\overline{G}^{\alpha}(x;\xi))}.\end{aligned} \tag{3.37}$$

The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial \lambda}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \xi_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{p+3}(\underline{0}, J^{-1}(\hat{\mathbf{\Delta}}))$, where the mean vector $\underline{\mathbf{0}} = (0, 0, 0, \underline{0})^T$ and $J^{-1}(\hat{\mathbf{\Delta}})$ is the observed Fisher information matrix evaluated at $\hat{\mathbf{\Delta}}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

3.5.2 Ordinary Least Squares

In this subsection, we discuss the method of ordinary least squares. Let $y_{1:n} < y_{2:n} < ... < y_{n:n}$ denote the order statistics based on a random sample of size n from the distribution with cdf $F(y)$, then

$$E[F(y_{i:n})] = \frac{i}{n+1} \quad \text{and} \quad Var[F(y_{i:n})] = \frac{i(n-i+1)}{(n+1)^2(n+2)}. \tag{3.38}$$

The ordinary least-square (OLS) estimates of the NWG-G parameters $\mathbf{\Delta} = (\lambda, \alpha, \beta, \xi_k)^T$, say, $(\hat{\lambda}_{_{OLS}}, \hat{\alpha}_{_{OLS}}, \hat{\beta}_{_{OLS}}, \hat{\xi}_{k_{_{OLS}}})^T$ are obtained by minimizing the function

$$Q(\mathbf{\Delta}|\mathbf{y}) = \sum_{i=1}^{n} \left(F(y_{i:n}|\mathbf{\Delta}) - \frac{i}{n+1} \right)^2. \tag{3.39}$$

The OLS estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}_{\mathbf{OLS}}$ is obtained by solving the nonlinear equations

$$\left(\frac{\partial Q(\mathbf{\Delta}|\mathbf{y})}{\partial \lambda}, \frac{\partial Q(\mathbf{\Delta}|\mathbf{y})}{\partial \alpha}, \frac{\partial Q(\mathbf{\Delta}|\mathbf{y})}{\partial \beta}, \frac{\partial Q(\mathbf{\Delta}|\mathbf{y})}{\partial \xi_k} \right)^T = \mathbf{0},$$

using numerical methods.

3.5.3 Weighted Least Squares

The weighted least square (WLS) estimates of the NWG-G parameters $\mathbf{\Delta} = (\lambda, \alpha, \beta, \xi)^T$, say, $(\hat{\lambda}_{_{WLS}}, \hat{\alpha}_{_{WLS}}, \hat{\beta}_{_{WLS}}, \hat{\xi}_{k_{_{WLS}}})^T$ are obtained by minimizing the function

$$\begin{aligned} W(\mathbf{\Delta}|\mathbf{y}) &= \sum_{i=1}^{n} \frac{1}{Var[F(y_{i:n})]} \left(F(y_{i:n}|\mathbf{\Delta}) - \frac{i}{n+1} \right)^2 \\ &= \sum_{i=1}^{n} w_i \left(F(y_{i:n}|\mathbf{\Delta}) - \frac{i}{n+1} \right)^2, \end{aligned} \tag{3.40}$$

where $w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$. The WLS estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}_{\mathbf{WLS}}$ is obtained by solving the nonlinear equations

$$\left(\frac{\partial W(\boldsymbol{\Delta}|\mathbf{y})}{\partial \lambda}, \frac{\partial W(\boldsymbol{\Delta}|\mathbf{y})}{\partial \alpha}, \frac{\partial W(\boldsymbol{\Delta}|\mathbf{y})}{\partial \beta}, \frac{\partial W(\boldsymbol{\Delta}|\mathbf{y})}{\partial \xi_k}\right)^T = \mathbf{0},$$

using a numerical method.

3.5.4 Minimum Distance Methods

The estimates of the NWG-G parameters can be obtained via the minimization of the well known Anderson-Darling and Cramer-von Mises goodness-of-fit statistics. This class of goodness-of-fit statistics is based on the difference between the estimates of the NWG-G cdf and the corresponding empirical distribution function.

Anderson-Darling Method

The Anderson-Darling (AD) estimates of the NWG-G model parameters $\boldsymbol{\Delta}_{AD}$, say $\hat{\boldsymbol{\Delta}}_{AD}$ are obtained by minimizing the function

$$AD(\boldsymbol{\Delta}|\mathbf{y}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(F(y_{i:n}|\boldsymbol{\Delta})[1-F(y_{n+1-i:n}|\boldsymbol{\Delta})]\right). \tag{3.41}$$

with respect to parameters. The AD estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}_{\mathbf{AD}}$ is obtained by solving the nonlinear equations

$$\left(\frac{\partial AD(\boldsymbol{\Delta}|\mathbf{y})}{\partial \lambda}, \frac{\partial AD(\boldsymbol{\Delta}|\mathbf{y})}{\partial \alpha}, \frac{\partial AD(\boldsymbol{\Delta}|\mathbf{y})}{\partial \beta}, \frac{\partial AD(\boldsymbol{\Delta}|\mathbf{y})}{\partial \xi_k}\right)^T = \mathbf{0},$$

using a numerical method.

Cramer-von Mises Method

The Cramer-von Mises (CVM) estimates of the NWG-G parameters $\boldsymbol{\Delta}_{CVM}$, say $\hat{\boldsymbol{\Delta}}_{CVM}$ are obtained by minimizing the function

$$CVM(\boldsymbol{\Delta}|\mathbf{y}) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(y_{i:n}|\boldsymbol{\Delta}) - \frac{2i-1}{2n}\right)^2, \tag{3.42}$$

with respect to the parameters $\boldsymbol{\Delta} = (\lambda, \alpha, \beta, \xi_k)^T$.

3.5.5 Maximum Product of Spacings Method

The $(n+1)$ uniform spacings of the first order of the sample are given by $D_1 = F(y_{1:n}|\boldsymbol{\Delta})$, $D_{n+1} = 1 - F(y_{n:n}|\boldsymbol{\Delta})$ and $D_i = F(y_{i:n}|\boldsymbol{\Delta}) - F(y_{(i-1):n}|\boldsymbol{\Delta})$, $i = 1, 2, ..., n$. The maximum product of spacings (MPS) method consist of finding the values of $\boldsymbol{\Delta}$ which maximizes the geometric mean of the spacings given by

$$GM(\boldsymbol{\Delta}|\mathbf{y}) = \left(\prod_{i=1}^{n} D_i\right)^{\frac{1}{(n+1)}}, \tag{3.43}$$

or equivalently

$$\log\left(GM(\boldsymbol{\Delta}|\mathbf{y})\right) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log(D_i), \tag{3.44}$$

by taking $0 = F(y_{0:n}|\boldsymbol{\Delta}) < F(y_{1:n}|\boldsymbol{\Delta}) < ... < F(y_{n:n}|\boldsymbol{\Delta}) < F(y_{n+1:n}|\boldsymbol{\Delta}) = 1$. The MPS estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}_{\mathbf{MPS}}$ is obtained by solving the nonlinear equations

$$\left(\frac{GM(\boldsymbol{\Delta}|\mathbf{y})}{\partial\lambda}, \frac{\partial GM(\boldsymbol{\Delta}|\mathbf{y})}{\partial\alpha}, \frac{\partial GM(\boldsymbol{\Delta}|\mathbf{y})}{\partial\beta}, \frac{\partial GM(\boldsymbol{\Delta}|\mathbf{y})}{\partial\xi_k}\right)^T = \mathbf{0},$$

using a numerical method. See Chen and Amin [18] for additional details.

The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic $(-2\ln(L))$, Akaike Information Criterion $(AIC = 2p - 2\ln(L))$, Bayesian Information Criterion $(BIC = p\ln(n) - 2\ln(L))$ and Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\boldsymbol{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented. In order to compare the models, we use the criteria stated above. We also obtain the goodness-of-fit statistics W^* and A^* described by Chen and Balakrishnan [16], as well as Kolmogorov-Smirnov (KS) statistic and its P-value. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred, and for AIC, AICC, BIC, and the goodness-of-fit statistics W^*, A^* and $K - S$, smaller values are preferred.

3.6 Simulation Study

The performance of the NWG-W distribution is examined by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800) via the R package.

We simulate $N = 1000$ samples for the true parameter values given by $I = (0.8, 0.8, 2.5, 2.5, 0.8)$, $II = (0.8, 0.8, 2.5, 1.6, 0.8)$, $III = (0.5, 0.5, 1.5, 4.0, 1.0)$ and $IV = (0.5, 0.5, 4.0, 4.0, 1.0)$ and the results are presented in Table 3.3 and 3.4. The table lists the mean MLEs of the model parameters along with the respective average bias and root mean squared errors (RMSEs). The average bias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

Table 3.3: **Monte Carlo Simulation Results**

		$(0.8, 0.8, 2.5, 2.5, 0.8)$			$(0.8, 0.8, 2.5, 1.6, 0.8)$		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	0.6229	0.7098	-0.1770	0.6087	0.6909	-0.1913
	50	0.7246	0.6781	-0.0753	0.6932	0.6967	-0.1067
	100	0.7092	0.6291	-0.0907	0.6794	0.6338	-0.1205
	200	0.7161	0.5254	-0.0838	0.6867	0.5370	-0.1132
	400	0.7268	0.4340	-0.0731	0.7032	0.4407	-0.0968
	800	0.7349	0.3398	-0.0650	0.7649	0.3314	-0.0350
β	25	1.6526	4.6449	0.8526	1.6393	4.4257	0.8393
	50	1.0607	0.7256	0.2607	1.0908	0.7318	0.2908
	100	1.0365	0.5992	0.2365	0.6795	0.6338	-0.1205
	200	0.9794	0.4804	0.1794	1.0157	0.5156	0.2157
	400	0.9414	0.4078	0.1415	0.9657	0.4345	0.1657
	800	0.8995	0.2975	0.0995	0.8731	0.2841	0.0731
λ	25	2.6227	6.5402	0.1227	2.4094	5.0913	-0.0905
	50	2.6730	3.9564	0.1730	2.4130	3.1029	-0.0869
	100	2.2991	2.4111	-0.2008	2.2146	2.4525	-0.2854
	200	2.2776	1.9648	-0.2223	2.1864	2.0152	-0.3136
	400	2.2389	1.5997	-0.2611	2.1595	1.5949	-0.3404
	800	2.2585	1.3167	-0.2414	2.3759	1.2879	-0.1240
θ	25	5.1505	4.5671	2.6505	3.2639	2.8789	1.6639
	50	4.3087	3.4881	1.8087	2.9254	2.3762	1.3254
	100	3.9611	2.8111	1.4611	2.6266	1.9287	1.0266
	200	3.3749	1.8784	0.8749	2.2227	1.2826	0.6227
	400	3.0325	1.2644	0.5325	1.9899	0.8786	0.3899
	800	2.7961	0.7252	0.2962	1.7389	0.4298	0.1389
γ	25	0.6229	0.7098	-0.1770	0.6087	0.6909	-0.1913
	50	0.7246	0.6781	-0.0753	0.6933	0.6967	-0.1067
	100	0.7092	0.6291	-0.0907	0.6794	0.6338	-0.1205
	200	0.7161	0.5254	-0.0838	0.6867	0.5370	-0.1132
	400	0.7268	0.4340	-0.0732	0.7031	0.4407	-0.0968
	800	0.7349	0.3398	-0.0650	0.7650	0.3314	-0.0350

Table 3.4: **Monte Carlo Simulation Results**

		(0.5, 0.5, 1.5, 4.0, 1.0)			(0.5, 0.5, 4.0, 4.0, 1.0)		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	0.5343	0.5508	0.0343	0.3934	0.4597	-0.1066
	50	0.5229	0.4887	0.0229	0.4252	0.4598	-0.0748
	100	0.4606	0.3805	-0.0394	0.4235	0.4345	-0.0764
	200	0.4469	0.2962	-0.0530	0.4334	0.3430	-0.0665
	400	0.4505	0.2296	-0.0494	0.4528	0.2874	-0.0471
	800	0.4693	0.1622	-0.0307	0.4649	0.2470	-0.0350
β	25	0.6857	1.1416	0.1857	1.2037	1.2262	0.7038
	50	0.5823	0.2697	0.0823	0.8663	0.6859	0.3663
	100	0.5898	0.2195	0.0898	0.7963	0.5539	0.2963
	200	0.5645	0.1554	0.0645	0.6616	0.3596	0.1617
	400	0.5439	0.1199	0.0439	0.6042	0.2691	0.1043
	800	0.5232	0.0725	0.0231	0.5652	0.1938	0.0652
λ	25	1.4468	1.1881	-0.0532	3.1944	2.9681	-0.8056
	50	1.4461	0.8862	-0.0539	3.2169	3.0343	-0.7831
	100	1.3629	0.7037	-0.1371	3.2284	2.5389	-0.7716
	200	1.3798	0.5446	-0.1201	3.3895	1.7991	-0.6105
	400	1.4065	0.4090	-0.0935	3.5687	1.3993	-0.4313
	800	1.4526	0.2739	-0.0474	3.6583	1.0884	-0.3417
θ	25	5.1667	2.4905	1.1667	5.2147	2.5153	1.2147
	50	4.8931	2.0917	0.8931	5.2902	2.3838	1.2902
	100	4.7314	1.7084	0.7314	5.0049	2.0226	1.0048
	200	4.4616	1.1783	0.4616	4.6393	1.4502	0.6393
	400	4.2714	0.7517	0.2714	4.3521	0.9200	0.3521
	800	4.1158	0.4475	0.1158	4.2206	0.6018	0.2206
γ	25	0.9506	0.4172	-0.0494	0.8558	0.5050	-0.1442
	50	0.9689	0.3513	-0.0311	0.8657	0.5081	-0.1343
	100	0.9482	0.2624	-0.0518	0.9062	0.4377	-0.0938
	200	0.9517	0.1918	-0.0483	0.9349	0.2773	-0.0651
	400	0.9690	0.1258	-0.0309	0.9695	0.2059	-0.0304
	800	0.9825	0.0826	-0.0174	0.9740	0.1431	-0.0259

The results in Tables 3.3 and 3.4 clearly show that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

3.7 Applications

In this section, we present examples to illustrate the flexibility and usefulness of the NWG-W distribution for data modeling. The NWG-W distribution is fitted to data sets and these fits are compared to the fits of the Gamma Dagum (GD) distribution (Oluyede et al. [56]), Exponentiated Kumaraswamy Dagum (EKD) distribution (Huang and Oluyede [40]), Exponentiated Kumaraswamy Weibull (EKW) distribution (Eissa and Abdulazi [27]), the Beta Weibull Poisson (BWP) distribution (Percontini et al. [58]), the Gamma log-logistic Weibull (GLLoGW) distribution (Foya et al. [31]) and Beta Weibull log-logistic (BWLLoG) distribution (Makubate et al. [46]). The pdfs of the listed distributions are given

below

$$\begin{aligned} f_{GD}(x) &= \frac{\lambda\beta\delta x^{-\delta-1}}{\Gamma(\alpha)\theta^{\alpha}}(1+\lambda x^{-\delta})^{-\beta-1}\Big(-\log[1-(1+\lambda x^{-\delta})^{-\beta}]\Big)^{\alpha-1} \\ &\times [1-(1+\lambda x^{-\delta})^{-\beta}]^{(\frac{1}{\theta})-1}, \end{aligned} \tag{3.45}$$

for λ, α, β, δ, θ and $x > 0$,

$$\begin{aligned} f_{EKD}(x) &= \alpha\lambda\delta\phi\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\alpha-1}[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi-1} \\ &\times \{1-[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi}\}^{\theta-1}, \end{aligned} \tag{3.46}$$

for α, λ, δ, ϕ, $\theta > 0$, and $x > 0$,

$$\begin{aligned} f_{EKW}(x) &= \lambda\beta ab\theta(\lambda x)^{\beta-1}e^{-(\lambda x)^{\beta}}(1-e^{-(\lambda x)^{\beta}})^{a-1}[1-((1-e^{-(\lambda x)^{\beta}})^{a})]^{b-1} \\ &\times \{1-[1-((1-e^{-(\lambda x)^{\beta}})^{a})]^{b}\}^{\theta-1}, \end{aligned} \tag{3.47}$$

for a, b, c, θ, $\lambda > 0$ and $x > 0$,

$$f_{BWP}(x) = cux^{\alpha-1}e^{\lambda u}(e^{\lambda}-e^{\lambda u})^{a-1}(e^{\lambda u}-1)^{b-1}, \tag{3.48}$$

where

$$c = \frac{\alpha\beta\lambda e^{-\lambda}(e^{\lambda}-1)^{2-a-b}}{B(a,b)(1-e^{-\lambda})}, \quad \text{and} \quad u = e^{-\beta x^{\alpha}},$$

for a, b, α, θ, $\lambda > 0$ and $x > 0$,

$$\begin{aligned} f_{GLLoGW}(x) &= \frac{1}{\Gamma(\delta)\theta^{\delta}}(1+x^{c})^{-1}e^{-\alpha x^{\beta}}[(1+x^{c})^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}] \\ &\times \big(-log[1-(1+x^{c})^{-1}e^{-\alpha x^{\beta}}]\big)^{\delta-1} \\ &\times [1-(1+x^{c})^{-1}e^{-\alpha x^{\beta}}]^{(\frac{1}{\theta})-1}, \end{aligned} \tag{3.49}$$

for α, β, θ, δ, $c > 0$ and $x > 0$,

$$\begin{aligned} f_{BWLLoG}(x) &= \frac{\alpha\beta c}{B(a,b)}x^{c-1}(1+x^{c})^{-2}\frac{[1-(1+x^{c})^{-1}]^{\beta-1}}{[(1+x^{c})^{-1}]^{\beta+1}} \\ &\times \exp\{-\alpha b[(1+x^{c})-1]^{\beta}\}[1-\exp\{-\alpha[(1+x^{c})-1]^{\beta}\}]^{a-1}, \end{aligned}$$

for c, a, b, α, $\beta > 0$ and $x > 0$.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [15]) are given in Figure 3.13 and Figure 3.14. For the

probability plot, we plotted $F(x_{(j)}; \hat{\lambda}, \hat{\alpha}, \hat{\beta}, \hat{\xi_k})$ against $\frac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness is given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[F(x_{(j)}; \hat{\lambda}, \hat{\alpha}, \hat{\beta}, \hat{\xi_k}) - \left(\frac{j-0.375}{n+0.25} \right) \right]^2. \tag{3.50}$$

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan [16] are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit.

The MLEs of the parameters with standard errors in parenthesis and the values of the statistics (-2$\log(L)$, AIC, AICC and BIC) are given in Table 5 and Table 6. From the statistics presented, it can be seen that the NWG-W distribution performed better than all the models since it has the lowest values for the statistics considered including Kolmogorov-Smirnov (K-S), W^*, A^* and Sum of Squares (SS).

3.7.1 Failure times of 50 components

The data set on the failure times of 50 components (per 1000 hours) were taken from Murthy et al. [50]. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics are presented in Table 3.5.

Table 3.5: **Estimation of the NWG-W model for failure times of 50 components data set**

			Estimates										
Distribution	α	β	λ	θ	γ	-2logL	AIC	CAIC	BIC	W^*	A^*	K–S	P–value
NWG-W	0.0048	1.4564	0.0279	4.1307	0.0013	163.1958	173.1959	175.1313	181.2504	0.0647	0.4998	0.1152	0.6675
	(0.0464)	(0.3457)	(0.0402)	(3.2647)	(0.0126)								
	λ	β	δ	α	θ								
GD	32.6781	3.6782	0.9746	0.1413	0.0105	165.8551	175.8551	177.7906	183.9097	0.0747	0.5876	0.1299	0.5176
	(0.1560)	(4.6221)	(0.7881)	(0.2695)	(0.0588)								
	α	λ	δ	ϕ	θ								
EKD	2.1541	489.72	1.5156	2.1300	0.1419	165.9092	175.9092	177.8447	183.9638	0.0788	0.62146	0.1294	0.5237
	(7.3212)	(0.027510)	(2.7049)	(0.0010)	(0.7.9982)								
	a	b	c	θ	λ								
EKW	3.9111	3.0070	1.2737	0.0985	0.0883	165.8597	175.8597	177.7952	183.9143	0.0744	0.5818	0.1217	0.6002
	(3.2074)	(4.2722)	(0.7687)	(0.0879)	(0.0461)								
	a	b	α	β	λ								
BWP	0.1258	0.3451	3.5855	0.0004	2.0571	165.4581	175.4581	177.3936	183.5127	0.0802	0.6325	0.1292	0.5249
	(0.0231)	(0.1862)	(0.0015)	(0.0003)	(0.0127)								
	α	β	θ	δ	c								
GLLoGW	3.0036	0.7724	1.3798	0.1150	0.9273	167.5602	177.5602	179.4957	185.6148	0.1857	1.2939	0.8841	2.2×10^{-16}
	(4.82361)	(0.25532)	(1.33115)	(0.12502)	(0.67184)								
	c	a	b	α	β								
BWLLoG	0.8576	0.1070	0.6325	5.2565×10^{-5}	4.7466	165.7697	175.7821	177.7176	183.8367	0.0819	0.6496	0.1265	0.5525
	(0.0209)	(0.1180)	(0.1884)	(0.0209)	(0.0209)								

The covariance matrix for the failure times of 50 components data is given by,

$$\begin{bmatrix} 0.00216 & 0.00050 & 0.00071 & -0.07500 & -0.00030 \\ 0.00050 & 0.11951 & -0.00827 & -0.19922 & 0.00015 \\ 0.00071 & -0.00827 & 0.00162 & -0.08890 & 0.00019 \\ -0.07500 & -0.19922 & -0.0889 & 10.65877 & -0.02050 \\ -0.00030 & 0.00015 & 0.00019 & -0.02050 & 0.00016 \end{bmatrix}$$

and the 95% confidence intervals for the parameters $\alpha, \beta, \lambda, \theta$ and γ are: 0.0048 $\pm$ 0.0909, 1.4564 $\pm$ 0.6776, 0.0279 $\pm$ 0.0788, 4.1307 $\pm$ 6.3988, and 0.0013 $\pm$ 0.0247, respectively.

The histogram and fitted densities as well as the probability plots are given in Figure (3.13). Clearly, from the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figure (3.13), we can conclude that the NWG-W distribution provide a better fit compared to the other models in Table 3.5.

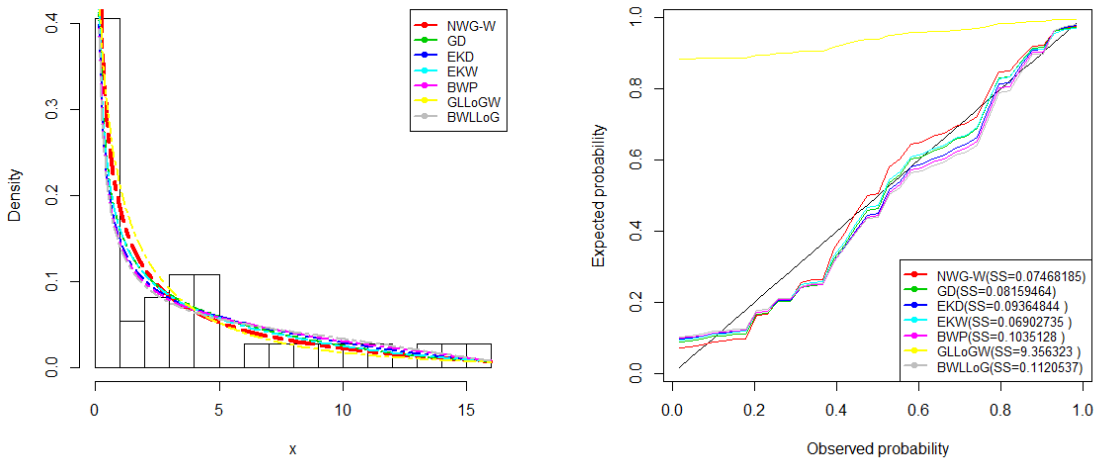


Figure 3.13: Fitted densities and probability plots for the failure times data set.

3.7.2 Strengths of the glass fibres data set

The data consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. The units of measurement are not given in the paper. Smith and Naylor [63] also analysed this data set. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics for this data set are presented in Table 3.6.

Table 3.6: **Estimation of the NWG-W model for strengths of the glass fibres data set**

	Estimates												
Distribution	α	β	λ	θ	γ	-2logL	AIC	CAIC	BIC	W^*	A^*	K–S	P–value
NWG-W	0.6052	0.3648	2.5904	4.9862	0.6592	25.6864	35.6889	36.7416	46.4046	0.0982	0.5736	0.1068	0.4685
	(5.5007)	(0.20761)	(0.9917)	(1.0494)	(5.9915)								
	λ	β	δ	α	θ								
GD	9.4111	5.4265	3.0547	0.3485	0.0089	29.7299	39.7299	40.7825	50.4455	0.2197	1.2120	0.1608	0.0769
	(9.1549)	(2.7663)	(2.1075)	(0.2073)	(0.0329)								
	α	λ	δ	ϕ	θ								
EKD	3.6279	23.7956	4.8758	218.1505	0.3092	29.4887	39.4887	40.5414	50.2044	0.2191	1.2058	0.1606	0.0774
	(1.9949)	(24.9209)	(1.7764)	(21.2447)	(0.1938)								
	a	b	c	θ	λ								
EKW	4.6030	2.2958	4.8396	0.2435	0.5904	29.8237	39.8237	40.8764	50.5394	0.2179	1.2027	0.1585	0.0843
	(8.2292)	(9.0921)	(7.8274)	(0.2804)	(0.0702)								
	a	b	α	β	λ								
BWP	1.1771	0.3241	4.9416	0.3067	6.9639×10^{-5}	32.4558	42.4550	43.5085	53.1716	0.2863	1.5683	0.1793	0.0348
	(0.1958)	(0.1862)	(0.0015)	(0.0003)	(0.0127)								
	c	a	b	α	β								
BWLLoG	2.9026	0.4493	0.0496	0.5568	2.4159	26.0882	36.0882	37.1408	46.8039	2.0396	11.1272	0.1133	0.3942
	(0.1507)	(0.3067)	(0.2785)	(0.3261)	(0.0002)								

The covariance matrix for the strength of the glass fibre data set is given by,

$$\begin{bmatrix} 30.2581 & -0.0631 & 0.3275 & -0.3235 & -32.6856 \\ -0.0631 & 0.0431 & -0.1596 & 0.1121 & -0.0687 \\ 0.3275 & -0.1596 & 0.9835 & -0.8537 & 0.3567 \\ -0.3235 & 0.1121 & -0.8537 & 1.1012 & -0.3524 \\ -32.6856 & -0.0687 & 0.3567 & -0.3524 & 35.8981 \end{bmatrix}$$

and the 95% confidence intervals for the parameters $\alpha, \beta, \lambda, \theta$ and γ are: 0.6052 $\pm$10.7814, 0.3648 $\pm$ 0.4069, 2.5904 $\pm$ 1.9437, 4.9862 $\pm$ 2.0568, and 0.6592 $\pm$ 11.7433, respectively.

Figure (3.14) shows the histogram, fitted densities as well as the probability plots for the strength of the glass fibre data set. Clearly, from the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figure (3.14), we can conclude that the NWG-W distribution provide a better fit compared to the other models in Table 3.6.

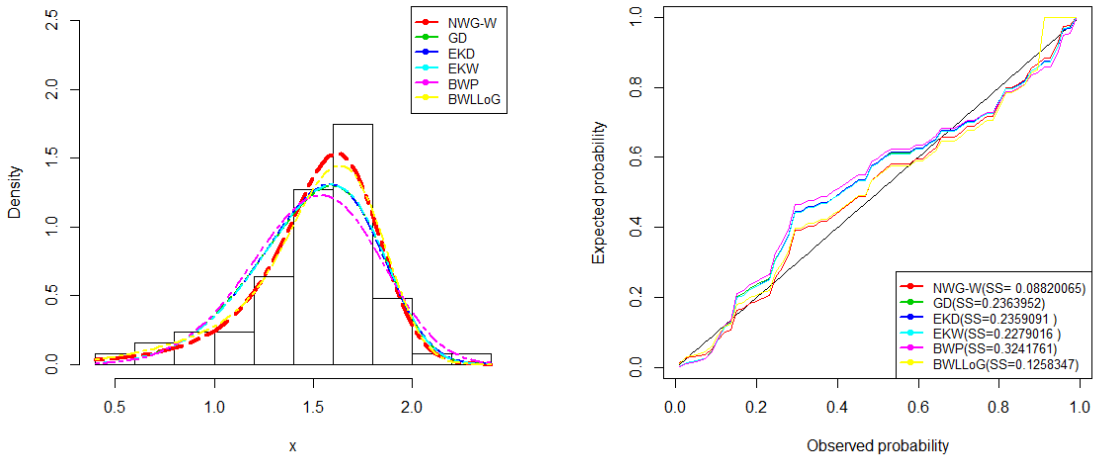


Figure 3.14: Fitted densities and probability plots for strength of the glass fibre data set.

3.8 Concluding Remarks

A new generalized family of distributions called the New Weibull generalized-G (NWG-G) family of distributions is developed and presented. The NWG-G distribution has several new and known sub-families of distributions as special cases or sub-models. The NWG-G family of distributions possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, incomplete moments, distribution of order statistics and Rényi entropy. Maximum likelihood estimation (MLE) method and other techniques are presented and MLE was used to estimate the model parameters. The performance of the special case of the NWG-G family of distributions was examined by conducting various simulations for different sizes. Finally, the special case of the NWG-G family of distributions was fitted to real data sets to illustrate the applicability and usefulness of the proposed family of distributions.

Chapter 4

Exponentiated Half Logistic Odd Lindley-G Family of Distributions

4.1 Introduction

The density function of the Half Logistic (HL) distribution does not possess many desirable properties for data modelling (Afify et al. [1]). The fact that most real data exhibits non-monotonic hazard rate function shapes act as a limitation to the applicability of the HL distribution in data modelling. In the past few years, some researchers have generalized the HL distribution and this include work by Afify et al. [1], Cordeiro et al. ([20], [19]) and El-sayed and Mahmoud [28]. In this chapter, we present a new generalization of the HL distribution with our goal being to study and propose a new flexible family of distributions called the Exponentiated Half Logistic Odd Lindley-G (EHLOL-G) family of distributions using the HL distribution as the generator and study its mathematical properties. By so doing, we will improve the flexibility of the exponentiated HL distribution and of the baseline distribution.

The results in this chapter are organized in the following manner. Section 4.2 contain the EHLOL-G family of distributions and its sub-models, hazard function and the quantile function, series expansion of the density function

and some special cases of the EHLOL-G family of distributions. In section 4.3, moments, generating function and incomplete moments are given. Section 4.4 contain results on the distribution of order statistics and Rényi entropy. Presented in section 4.5 is the estimation of the parameters of the EHLOL-G family of distributions via the method of maximum likelihood, followed by a Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimators in section 4.6. Some applications to real data sets are given in section 4.7, followed by some concluding remarks in section 4.8.

4.2 The Model, Sub-models, Hazard and Quantile Functions, Series Expansion of Density Function, and Some Special Cases

The derivation of some of the statistical properties of the Exponentiated Half-Logistic Odd Lindley-G (EHLOL-G) family of distributions including special cases, expansion of the density, hazard function and quantile function are presented in this section.

4.2.1 The Model

Gomes-Silva et al [35] developed the Odd-Lindley-G family of distributions. Its cdf is given by

$$\begin{aligned} F_{OL-G}(x;\lambda,\xi) &= \int_0^{\frac{G(x;\xi)}{(1-G(x;\xi))}} \frac{\lambda^2}{1+\lambda}(1+t)\exp(-\lambda t)dt \\ &= 1-\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\Big[-\lambda\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big] \end{aligned} \tag{4.1}$$

and the corresponding pdf is given by

$$f_{OL-G}(x;\lambda,\xi) = \frac{\lambda^2 g(x;\xi)}{(1+\lambda)\overline{G}^3(x;\xi)}\exp\Big[-\lambda\frac{G(x;\xi)}{\overline{G}(x;\xi)}\Big], \tag{4.2}$$

where $g(x;\xi)$ is the baseline distribution.

The cumulative distribution function (cdf) of the type I half-logistic (TIHL) family of distributions (Cordeiro et al. [19]) is given by

$$\begin{aligned} F(x;\lambda,\xi) &= \int_0^{-ln(1-G(x;\xi))} \frac{2\lambda\exp(-\lambda t)}{(1+\exp(-\lambda t))^2}dt \\ &= \frac{1-(1-G(x;\xi))^\lambda}{1+(1-G(x;\xi))^\lambda}, \end{aligned} \tag{4.3}$$

where $\lambda > 0$, is the shape parameter. If $\lambda = 1$, we have

$$F(x;\xi) = \frac{G(x;\xi)}{1+\overline{G}(x;\xi)}, \quad \text{where} \quad \overline{G}(x;\xi) = 1 - G(x;\xi).$$

The cdf and pdf of the exponentiated half-logistic Odd Lindley-G (EHLOL-G) family of distributions are, respectively given by

$$F_{EHLOL-G}(x;\lambda,\alpha,\xi) = \left\{ \frac{1-\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)} \exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]}{1+\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)} \exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]} \right\}^{\alpha} \tag{4.4}$$

and

$$\begin{aligned} f_{EHLOL-G}(x;\lambda,\alpha,\xi) &= 2\alpha\lambda^2 g(x;\xi) \\ &\times \frac{\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]\left\{1-\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^{\alpha-1}}{(1+\lambda)\overline{G}^3(x;\xi)\left\{1+\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^{\alpha+1}}, \end{aligned} \tag{4.5}$$

where $g(x;\xi) = \frac{dG(x;\xi)}{dx}$, $\lambda > 0$, $\alpha > 0$ and ξ is the vector of parameters for the baseline cdf G. If a random variable X has the EHLOL-G family of distributions, we write $X \sim EHLOL - G(\lambda,\alpha,\xi)$.

4.2.2 Hazard and Quantile Functions

In this section, we present the hazard and quantile functions of the EHLOL-G family of distributions. The hazard rate function of the EHLOL-G family of distributions is given by

$$\begin{aligned} h(x) &= 2\alpha\lambda^2 g(x;\xi)\frac{\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]\left\{1-\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^{\alpha-1}}{(1+\lambda)\overline{G}^3(x;\xi)\left\{1+\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^{\alpha+1}} \\ &\times \left[1-\left\{\frac{1-\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]}{1+\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]}\right\}^{\alpha}\right]^{-1}. \end{aligned} \tag{4.6}$$

The quantile function $Q(u)$ of the EHLOL-G family of distributions is obtained by solving the non-linear equation

$$F_{EHLOL-G}(Q(u)) = \left\{ \frac{1-\frac{\lambda+1-G(Q(u))}{(1+\lambda)(1-G(Q(u)))}\exp\left[\frac{-\lambda G(Q(u))}{1-G(Q(u))}\right]}{1+\frac{\lambda+1-G(Q(u))}{(1+\lambda)(1-G(Q(u)))}\exp\left[\frac{-\lambda G(Q(u))}{1-G(Q(u))}\right]} \right\}^{\alpha} = u.$$

so that,

$$-\frac{\lambda+1-G(Q(u))}{(1+\lambda)(1-G(Q(u)))}\exp\Big[-\lambda\frac{G(Q(u))}{1-G(Q(u))}\Big]=\frac{u^{\frac{1}{\alpha}}-1}{u^{\frac{1}{\alpha}}+1},$$

By multiplying both sides of the above equation by $\exp[-(1+\lambda)]$ and $(1+\lambda)$, we have

$$-\frac{\lambda+1-G(Q(u))}{1-G(Q(u))}\exp\Big[-\frac{\lambda+1-G(Q(u))}{1-G(Q(u))}\Big]=\frac{(u^{\frac{1}{\alpha}}-1)(1+\lambda)}{u^{\frac{1}{\alpha}}+1}\exp[-(1+\lambda)].$$

We noted that $-\frac{\lambda+1-G(Q(u))}{1-G(Q(u))}$ is a Lambert W function of the real argument $\frac{(u^{\frac{1}{\alpha}}-1)(1+\lambda)}{u^{\frac{1}{\alpha}}+1}\exp[-(1+\lambda)]$. The Lambert function say W is defined as

$$W(x)e^{W(x)}=x,$$

this has 2 real branches with a branching point located at $(-e^{-1},1)$. The lower branch, $W_{-1}(x)$, is defined in the closed interval $[-e^{-1},1]$ and it has a negative singularity as $x\to 0^-$. The upper branch, $W_0(x)$, is defined for $x\in[-e^{-1},\infty]$. This implies that,

$$W\left[\frac{(u^{\frac{1}{\alpha}}-1)(1+\lambda)}{u^{\frac{1}{\alpha}}+1}\exp[-(1+\lambda)]\right]=-\Big[1+\frac{\lambda}{1-G(Q(u))}\Big]. \tag{4.7}$$

For any $\lambda>0$ and $u\in(0,1)$, we have $\Big[1+\frac{\lambda}{1-G(Q(u))}\Big]>1$ and $\frac{(u^{\frac{1}{\alpha}}-1)(1+\lambda)}{u^{\frac{1}{\alpha}}+1}\exp[-(1+\lambda)]<0$, we can write equation (4.7) as

$$W_{-1}\left[\frac{(u^{\frac{1}{\alpha}}-1)(1+\lambda)}{u^{\frac{1}{\alpha}}+1}\exp[-(1+\lambda)]\right]=-\Big[1+\frac{\lambda}{1-G(Q(u))}\Big].$$

Hence, the quantile function for the EHLOL-G family of distributions is given by

$$Q(u)=G^{-1}\left\{1+\lambda\left[1+W_{-1}\left[\frac{(u^{\frac{1}{\alpha}}-1)(1+\lambda)}{u^{\frac{1}{\alpha}}+1}\exp[-(1+\lambda)]\right]\right]^{-1}\right\}. \tag{4.8}$$

4.2.3 Series Expansion of the Density function

In this section, a useful representation for the EHLOL-G pdf is presented. The pdf (4.5) can be expressed as

$$\begin{aligned} f(x) &= \frac{2\alpha\lambda^2 g(x;\xi)}{(1+\lambda)\overline{G}^3(x;\xi)}\exp\Big[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\Big]\underbrace{\Big\{1-\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\Big[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\Big]\Big\}^{\alpha-1}}_{Y} \\ &\times \underbrace{\Big\{1+\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\Big[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\Big]\Big\}^{-\alpha-1}}_{Z}. \end{aligned} \tag{4.9}$$

By making use of the generalised binomial series, we have

$$Y = \sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\Big[\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\Big]^j \exp\Big[\frac{-\lambda j G(x;\xi)}{\overline{G}(x;\xi)}\Big], \tag{4.10}$$

and

$$Z = \sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\Big[\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\Big]^i \exp\Big[\frac{-\lambda i G(x;\xi)}{\overline{G}(x;\xi)}\Big]. \tag{4.11}$$

Putting together equations (4.9), (4.10) and (4.11), we obtain

$$\begin{aligned} f(x) &= \frac{2\alpha\lambda^2 g(x;\xi)}{(1+\lambda)\overline{G}^3(x;\xi)}\sum_{j,i=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\binom{-\alpha-1}{i}\Big[\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\Big]^{j+i} \\ &\times \exp\Big[\frac{-\lambda(j+i+1)G(x;\xi)}{\overline{G}(x;\xi)}\Big]. \end{aligned}$$

and

$$(\lambda+\overline{G}(x;\xi))^{j+i} = \sum_{q=0}^{j+i}\binom{j+i}{q}\lambda^{j+i}\overline{G}(x;\xi)^{j+i-q}. \tag{4.12}$$

The exponential series can be applied to obtain the following result,

$$\begin{aligned} f(x) &= 2\alpha g(x;\xi)\sum_{j,i,k=0}^{\infty}\sum_{q=0}^{j+i}\frac{(-1)^{j+k}\lambda^{j+i+k+2}(j+i+1)^k}{k!(1+\lambda)^{j+i+1}}\binom{\alpha-1}{j}\binom{-\alpha-1}{i} \\ &\times \binom{j+i}{q}G(x;\xi)^k\overline{G}(x;\xi)^{-(k+q+3)}. \end{aligned} \tag{4.13}$$

Consider the generalized binomial series

$$(1-z)^{-p} = \sum_{l=0}^{\infty}(-1)^l\binom{-p}{l}z^l \quad \text{for} \quad |\,z\,|<1 \quad \text{and} \quad p>0. \tag{4.14}$$

By making use of the generalized binomial series in equation (4.14), equation (4.13) can be expressed as

$$\begin{aligned} f(x) &= 2\alpha g(x;\xi)\sum_{j,i,k,l=0}^{\infty}\sum_{q=0}^{j+i}\frac{(-1)^{j+k+l}\lambda^{j+i+k+2}(j+i+1)^k}{k!(1+\lambda)^{j+i+1}}\binom{\alpha-1}{j}\binom{-\alpha-1}{i} \\ &\times \binom{j+i}{q}\binom{-(k+q+3)}{l}G(x;\xi)^{k+l}. \end{aligned} \tag{4.15}$$

which can be similarly expressed as

$$f(x) = \sum_{k,l=0}^{\infty}a_{k,l}h_{k+l+1}(x), \tag{4.16}$$

where

$$a_{k,l} = 2\alpha \sum_{j,i=0}^{\infty} \sum_{q=0}^{j+i} \frac{(-1)^{j+k+l}\lambda^{j+i+k+2}(j+i+1)^k}{k!(1+\lambda)^{j+i+1}(k+l+1)} \binom{\alpha-1}{j} \binom{-\alpha-1}{i} \times \binom{j+i}{q} \binom{-(k+q+3)}{l}, \tag{4.17}$$

and $h_{k+l+1}(x) = (k+l+1)g(x;\xi)G(x,\xi)^{k+l}$ is the Exp-G density with power parameter $(k+l+1)$. Several mathematical properties of the EHLOL-G family can be obtained from the properties of the Exp-G family.

The cdf of the EHLOL-G family can also be expressed as a mixture of Exp-G cdfs by integrating equation (4.16) such that,

$$F(x) = \sum_{k,l=0}^{\infty} a_{k,l} H_{k+l+1}(x), \tag{4.18}$$

where $H_{k+l+1}(x)$is the cdf of the Exp-G family with power parameter $(k+l+1)$.

4.2.4 Some Sub-Families of EHLOL-G Family of Distributions

In this section, we consider some sub-families of the EHLOL-G family of distributions.

- If $\alpha = 1$, we obtain a new family of distributions with pdf given by

$$f_{HLOL-G}(x) = \frac{2\lambda^2 g(x;\xi)\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]}{(1+\lambda)\overline{G}^3(x;\xi)\left\{1+\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^2},$$

- If $\lambda = 1$, we obtain a new family of distributions with pdf given by

$$f_{OEHL-G}(x) = \frac{\alpha g(x;\xi)\exp\left[\frac{-G(x;\xi)}{\overline{G}(x;\xi)}\right]\left\{1-\frac{G(x;\xi)}{2\overline{G}(x;\xi)}\exp\left[\frac{-G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^{\alpha-1}}{\overline{G}^3(x;\xi)\left\{1+\frac{G(x;\xi)}{2\overline{G}(x;\xi)}\exp\left[\frac{-G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^{\alpha+1}},$$

- If $\alpha = 1$ and $\lambda = 1$ we obtain a new family of distributions with pdf given by

$$f(x) = \frac{g(x;\xi)\exp\left[\frac{-G(x;\xi)}{\overline{G}(x;\xi)}\right]}{\overline{G}^3(x;\xi)\left\{1+\frac{1+\overline{G}(x;\xi)}{2\overline{G}(x;\xi)}\exp\left[\frac{-G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^2}.$$

4.2.5 Some Special Cases

In this section, we present some special cases of the EHLOL-G family of distributions, specifically when the baseline distribution function $G(x;\xi)$ is Weibull, Uniform, Kumaraswamy and Fréchet distributions, respectively. Plots of the pdf, and hazard functions are also presented.

EHLOL-Weibull Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;a,b) = 1 - e^{-(x/b)^a}$ and $g(x;a,b) = ab^{-a}x^{a-1}e^{-(x/b)^a}$, respectively, where $a > 0$ is a shape parameter, $b > 0$ is a scale parameter and $x > 0$. Then, the EHLOL-Weibull (EHLOL-W) distribution has cdf and pdf given by

$$F_{EHLOL-W}(x) = \left\{ \frac{1-\frac{\lambda+e^{-(x/b)^a}}{(1+\lambda)e^{-(x/b)^a}} \exp\left[\frac{-\lambda(1-e^{-(x/b)^a})}{e^{-(x/b)^a}}\right]}{1+\frac{\lambda+e^{-(x/b)^a}}{(1+\lambda)e^{-(x/b)^a}} \exp\left[\frac{-\lambda(1-e^{-(x/b)^a})}{e^{-(x/b)^a}}\right]} \right\}^{\alpha}$$

and

$$\begin{aligned} f_{EHLOL-W}(x) &= 2\alpha\lambda^2 ab^{-a}x^{a-1}e^{-(x/b)^a} \\ &\times \frac{\exp\left[\frac{-\lambda(1-e^{-(x/b)^a})}{e^{-(x/b)^a}}\right]\left\{1-\frac{\lambda+e^{-(x/b)^a}}{(1+\lambda)e^{-(x/b)^a}}\exp\left[\frac{-\lambda(1-e^{-(x/b)^a})}{e^{-(x/b)^a}}\right]\right\}^{\alpha-1}}{(1+\lambda)(e^{-(x/b)^a})^3\left\{1+\frac{\lambda+e^{-(x/b)^a}}{(1+\lambda)e^{-(x/b)^a}}\exp\left[\frac{-\lambda(1-e^{-(x/b)^a})}{e^{-(x/b)^a}}\right]\right\}^{\alpha+1}}. \end{aligned}$$

Figure 4.1 shows the plots of the EHLOL-W distribution for different parameter values. The pdf can take various shapes including the reverse J, uni-modal, almost symmetric, left skewed and right skewed.

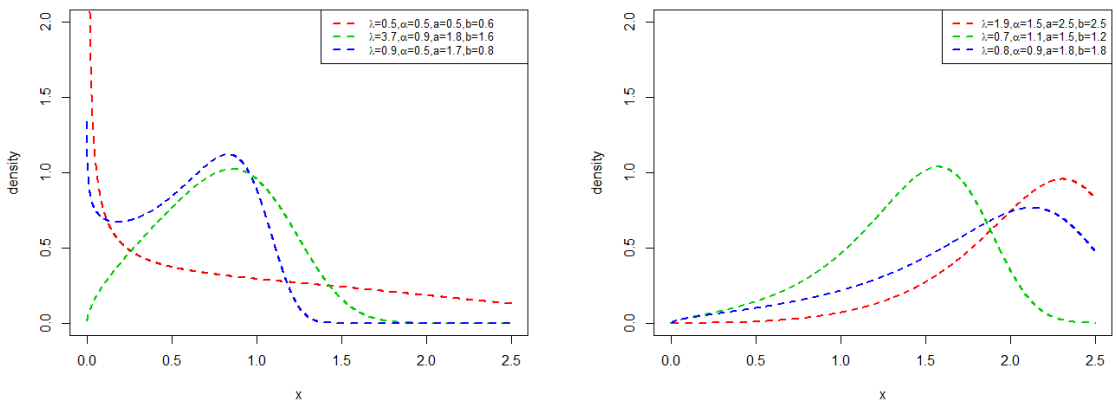


Figure 4.1: Plots of EHLOL-W density function

Figure 4.2 shows the plots of the hazard rate function of the EHLOL-W distribution for different parameter values. Graphs of the hazard rate function exhibits decreasing, increasing and bathtub shapes.

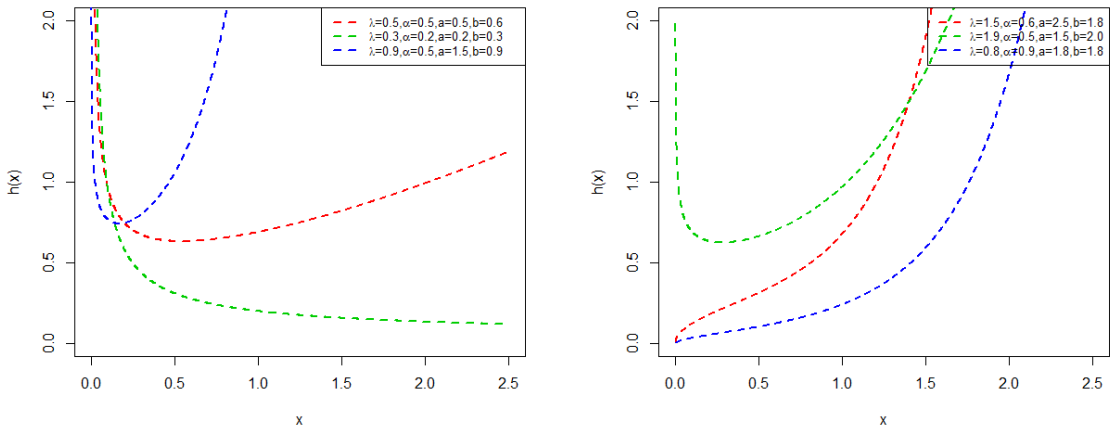


Figure 4.2: Plots of EHLOL-W hazard rate function

EHLOL-Uniform Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta) = \frac{x}{\theta}$ and $g(x;\theta) = \frac{1}{\theta}$ for $0 < x < \theta < \infty$. Then, the EHLOL-Uniform (EHLOL-U) distribution has cdf and pdf given by

$$F_{EHLOL-U}(x) = \left\{ \frac{1-\frac{\lambda+(1-\frac{x}{\theta})}{(1+\lambda)(1-\frac{x}{\theta})} \exp\left[\frac{-\lambda x}{\theta(1-\frac{x}{\theta})}\right]}{1+\frac{\lambda+(1-\frac{x}{\theta})}{(1+\lambda)(1-\frac{x}{\theta})} \exp\left[\frac{-\lambda x}{\theta(1-\frac{x}{\theta})}\right]} \right\}^{\alpha}$$

and

$$f_{EHLOL-U}(x) = \frac{2\alpha\lambda^2}{\theta} \frac{\exp\left[\frac{-\lambda x}{\theta(1-\frac{x}{\theta})}\right]\left\{1-\frac{\lambda+(1-\frac{x}{\theta})}{(1+\lambda)(1-\frac{x}{\theta})} \exp\left[\frac{-\lambda x}{\theta(1-\frac{x}{\theta})}\right]\right\}^{\alpha-1}}{(1+\lambda)(1-\frac{x}{\theta})^3\left\{1+\frac{\lambda+(1-\frac{x}{\theta})}{(1+\lambda)(1-\frac{x}{\theta})} \exp\left[\frac{-\lambda x}{\theta(1-\frac{x}{\theta})}\right]\right\}^{\alpha+1}}.$$

Figure 4.3 shows the plots of the EHLOL-U distribution for different parameter values. The pdf can take various shapes including the uni-modal, left skewed and right skewed.

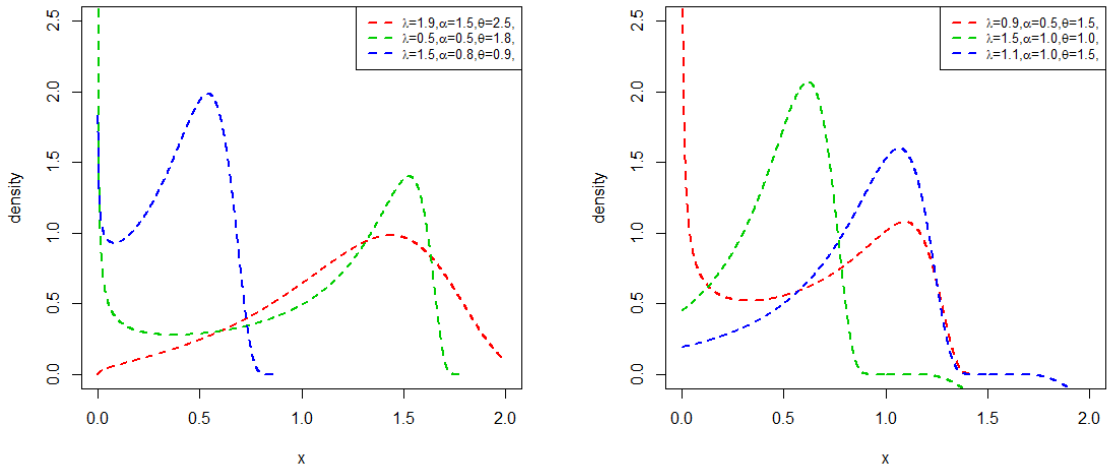


Figure 4.3: Plots of EHLOL-U density function

Figure 4.4 shows the plots of the hazard rate function of the EHLOL-U distribution for different parameter values. Graphs of the hazard rate function exhibits increasing and bathtub shapes.

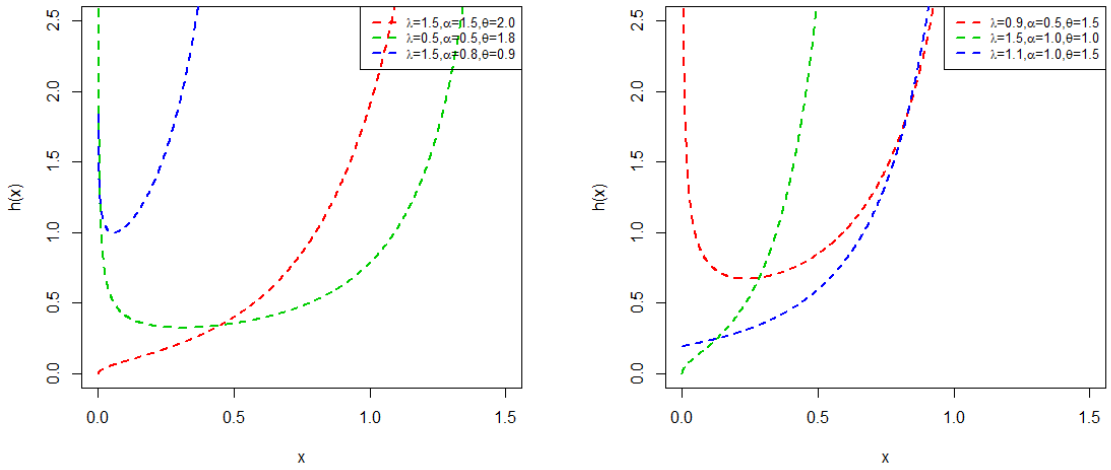


Figure 4.4: Plots of EHLOL-U hazard rate function

EHLOL-Kumaraswamy Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;a,b) = 1-(1-x^a)^b$ and $g(x;a,b) = abx^{a-1}(1-x^a)^{b-1}$ for $a,b>0$ and $x>0$. Then, the

EHLOL-Kumaraswamy (EHLOL-Kum) distribution has cdf and pdf given by

$$F_{EHLOL-Kum}(x) = \left\{ \frac{1-\frac{\lambda+(1-x^a)^b}{(1+\lambda)(1-x^a)^b}\exp\left[\frac{-\lambda(1-(1-x^a)^b)}{(1-x^a)^b}\right]}{1+\frac{\lambda+(1-x^a)^b}{(1+\lambda)(1-x^a)^b}\exp\left[\frac{-\lambda(1-(1-x^a)^b)}{(1-x^a)^b}\right]} \right\}^{\alpha}$$

and

$$\begin{aligned} f_{EHLOL-Kum}(x) &= 2\alpha\lambda^2 abx^{a-1}(1-x^a)^{b-1} \\ &\times \frac{\exp\left[\frac{-\lambda(1-(1-x^a)^b)}{(1-x^a)^b}\right]\left\{1-\frac{\lambda+(1-x^a)^b}{(1+\lambda)(1-x^a)^b}\exp\left[\frac{-\lambda(1-(1-x^a)^b)}{(1-x^a)^b}\right]\right\}^{\alpha-1}}{(1+\lambda)(1-x^a)^{3b}\left\{1+\frac{\lambda+(1-x^a)^b}{(1+\lambda)(1-x^a)^b}\exp\left[\frac{-\lambda(1-(1-x^a)^b)}{(1-x^a)^b}\right]\right\}^{\alpha+1}}. \end{aligned}$$

Figure 4.5 shows the plots of the EHLOL-Kum distribution for different parameter values. The pdf can take various shapes including the reverse J, uni-modal, left skewed and right skewed.

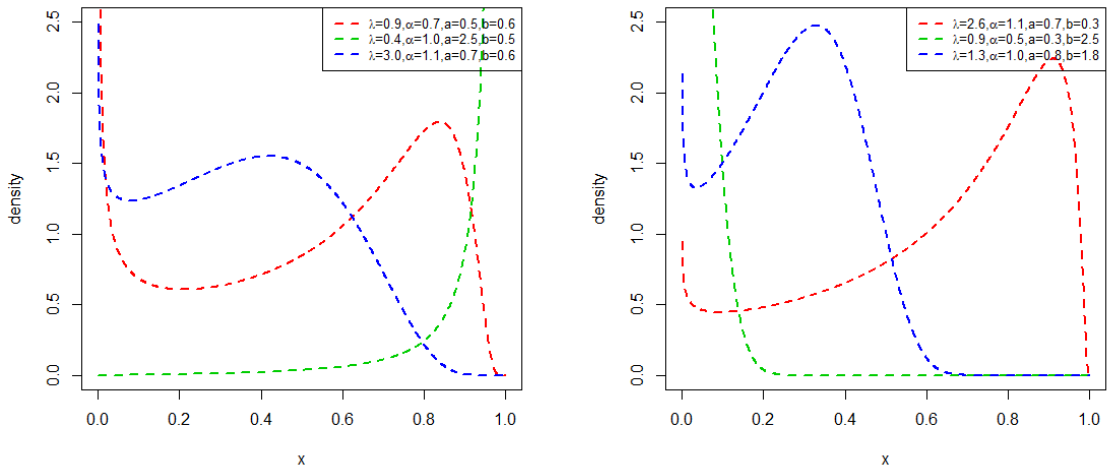


Figure 4.5: Plots of EHLOL-Kum density function

Figure 4.6 shows the plots of the hazard rate function of the EHLOL-Kum distribution for different parameter values. Graphs of the hazard rate function exhibits increasing and bathtub shapes.

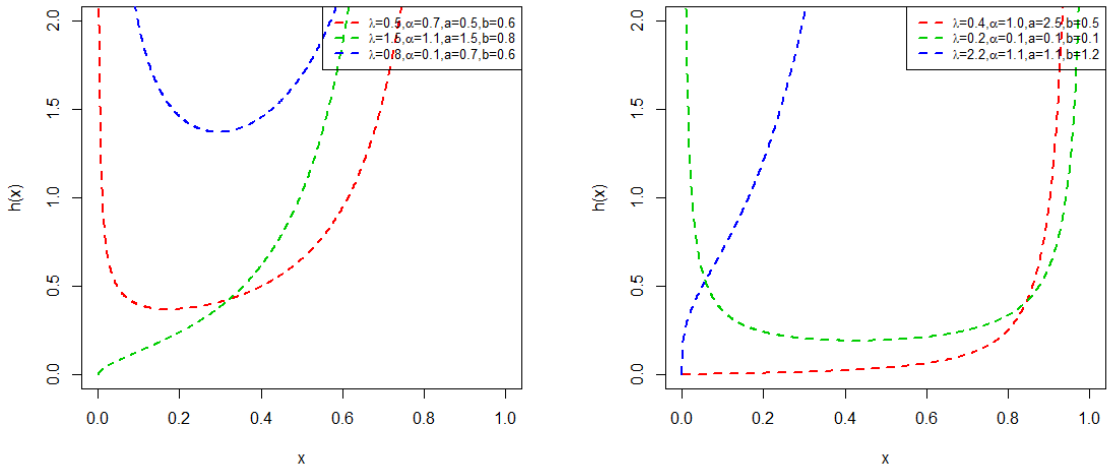


Figure 4.6: Plots of EHLOL-Kum hazard rate function

EHLOL-Fréchet Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;a,b) = \exp[-(\frac{a}{x})^b]$ and $g(x;a,b) = ba^b x^{-(b+1)} \exp[-(\frac{a}{x})^b]$ for $a, b > 0$. Then, the EHLOL-Fréchet (EHLOL-Fr) distribution has cdf and pdf given by

$$F_{EHLOL-Fr}(x) = \left\{ \frac{1-\frac{\lambda+(1-\exp[-(\frac{a}{x})^b])}{(1+\lambda)(1-\exp[-(\frac{a}{x})^b])} \exp\left[\frac{-\lambda \exp[-(\frac{a}{x})^b]}{(1-\exp[-(\frac{a}{x})^b])}\right]}{1+\frac{\lambda+(1-\exp[-(\frac{a}{x})^b])}{(1+\lambda)(1-\exp[-(\frac{a}{x})^b])} \exp\left[\frac{-\lambda \exp[-(\frac{a}{x})^b]}{(1-\exp[-(\frac{a}{x})^b])}\right]} \right\}^{\alpha}$$

and

$$f_{EHLOL-Fr}(x) = \frac{2\alpha\lambda^2 b a^b x^{-(b+1)} \exp[-(\frac{a}{x})^b] \exp\left[\frac{-\lambda \exp[-(\frac{a}{x})^b]}{(1-\exp[-(\frac{a}{x})^b])}\right]}{(1+\lambda)(1-\exp[-(\frac{a}{x})^b])^3} \times \frac{\left\{1-\frac{\lambda+(1-\exp[-(\frac{a}{x})^b])}{(1+\lambda)(1-\exp[-(\frac{a}{x})^b])} \exp\left[\frac{-\lambda \exp[-(\frac{a}{x})^b]}{(1-\exp[-(\frac{a}{x})^b])}\right]\right\}^{\alpha-1}}{\left\{1+\frac{\lambda+(1-\exp[-(\frac{a}{x})^b])}{(1+\lambda)(1-\exp[-(\frac{a}{x})^b])} \exp\left[\frac{-\lambda \exp[-(\frac{a}{x})^b]}{(1-\exp[-(\frac{a}{x})^b])}\right]\right\}^{\alpha+1}}.$$

Figure 4.7 shows the plots of the EHLOL-Fr distribution for different parameter values. The pdf can take various shapes including the reverse J, uni-modal, left skewed and right skewed.

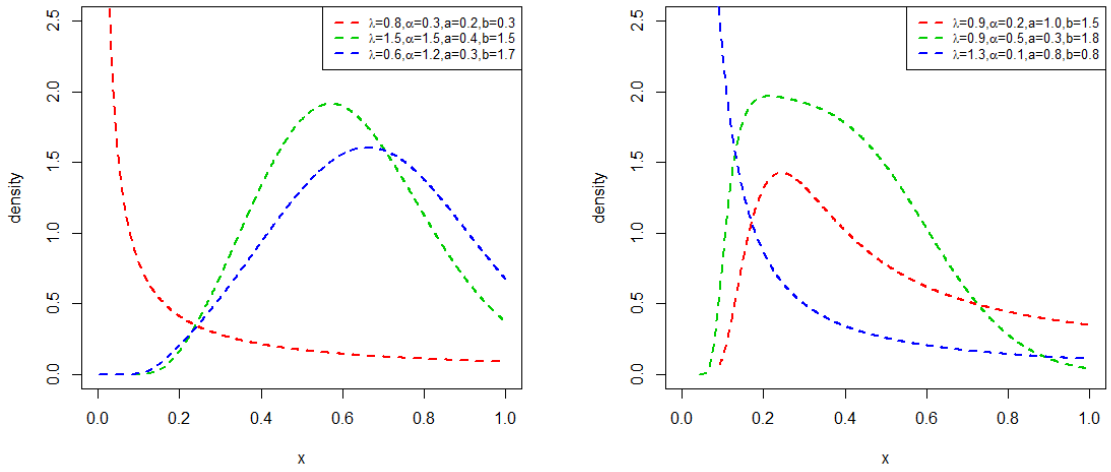


Figure 4.7: Plots of EHLOL-Fr density function

Figure 4.8 shows the plots of the hazard rate function of the EHLOL-Fr distribution for different parameter values. Graphs of the hazard rate function exhibits decreasing, increasing and upside-down bathtub shapes.

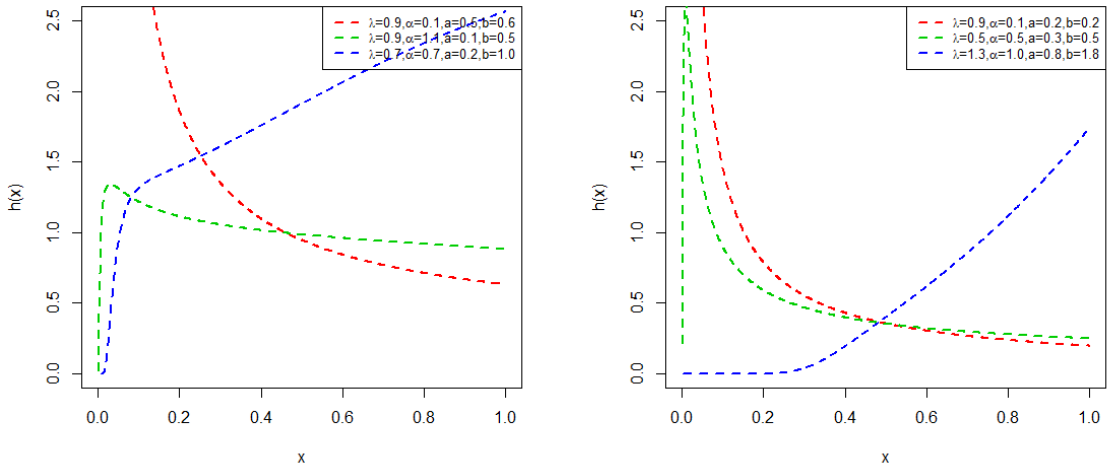


Figure 4.8: Plots of EHLOL-Fr hazard rate function

Table 4.1 shows some random numbers that are generated from the EHLOL-W distribution for selected parameter values.

Table 4.1: **Table of quantiles for selected parameter values of the EHLOL-W distribution**

	(λ, α, a, b)				
u	(0.8, 0.8, 0.7, 0.5)	(2.0 ,2.5, 0.9, 0.5)	(0.5, 2.0, 1.5, 1.5)	(0.9, 5.0, 1.8, 0.5)	(0.8, 4.5, 1.6, 0.4)
0.1	0.0722	0.2117	1.7522	0.5685	0.4744
0.2	0.1882	0.2761	1.9609	0.6011	0.5060
0.3	0.3081	0.3250	2.0959	0.6236	0.5276
0.4	0.4256	0.3679	2.2024	0.6422	0.5455
0.5	0.5417	0.4090	2.2957	0.6593	0.5619
0.6	0.6597	0.4508	2.3838	0.6759	0.5779
0.7	0.7853	0.4962	2.4732	0.6936	0.5948
0.8	0.9298	0.5504	2.5729	0.7138	0.6142
0.9	1.1249	0.6272	2.7036	0.7416	0.6408

4.3 Moments and Incomplete Moments

In this section, moments, moment generating function and incomplete moments for the EHLOL family of distributions are presented. These measures (moments, moment generating function and incomplete moments) can be readily obtained for the sub-families and sub-models given in section 4.2.4.

4.3.1 Moments and Generating Function

Let T_{k+l+1}, denote the Exp-G random variable with power parameter $k+l+1$. Then the r^{th} raw moment of X, say μ'_r follows from equation (4.16) as

$$\mu'_r = E(X^r) \quad = \quad \sum_{k,l=0}^{\infty} a_{k,l} E(T^r_{k+l+1}), \tag{4.19}$$

where $a_{k,l}$ is as given in equation (4.17).

The n^{th} central moment of X is given by

$$\mu_n = \sum_{r=0}^{n} \binom{n}{r} (-\mu'_r)^{n-r} E(X^r) = \sum_{r=0}^{n} \sum_{k,l=0}^{\infty} a_{k,l} \binom{n}{r} (-\mu'_r)^{n-r} E(T^r_{k+l+1}).$$

The variance (σ^2), Standard deviation (SD=σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively.

The cumulants (κ_n) of the random variable X can be readily obtained from the recursive relationship:

$$\kappa_n = \mu'_n - \sum_{r=0}^{n-1} \binom{n-1}{r-1} \kappa_r \mu'_{n-r},$$

where $\mu'_n = E(X - \mu'_1)^n$, so that the CS and CK are given by $\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}$ and $\gamma_2 = \frac{\kappa_4}{\kappa_2^2}$.

The moment generating function (mgf) $M_X(t) = E(e^{tX})$ of X can be derived from equation (4.16) and is given by

$$M_X(t) = \sum_{k,l=0}^{\infty} a_{k,l} M_{k+l+1}(t),$$

where $M_{k+l+1}(t)$ is the mgf of T_{k+l+1}. Hence, $M_X(t)$ can be determined from the Exp-G generating function.

A table of moments for some selected parameters values of the special case of the EHLOL-G family of distribution namely EHLOL-Weibull distribution is given in Table 4.2.

Table 4.2: **Table of Moments for the EHLOL-W distribution for selected parameter values**

	(0.8, 0.8, 0.7, 0.5)	(2.0 ,2.5, 0.9, 0.5)	(0.5, 2.0, 1.5, 1.5)	(0.9, 5.0, 1.8, 0.5)	(0.8, 4.5, 1.6, 0.4)
$E(X)$	0.3828	0.4155	0.0043	0.6567	0.5592
$E(X^2)$	0.2450	0.1982	0.0036	0.4358	0.3170
$E(X^3)$	0.1778	0.1047	0.0031	0.2922	0.1820
$E(X^4)$	0.1384	0.0599	0.0027	0.1978	0.1058
$E(X^5)$	0.1128	0.0367	0.0024	0.1351	0.0622
$E(X^6)$	0.0948	0.0238	0.0022	0.0931	0.0369
SD	0.0817	0.0162	0.0020	0.0647	0.0221
CV	0.0716	0.0115	0.0018	0.0453	0.0134
CS	0.0636	0.0084	0.0017	0.0319	0.0081
CK	0.0573	0.0064	0.0016	0.0227	0.0051

4.3.2 Incomplete Moments

Conditional and incomplete moments are particularly useful for the analysis of lifetime models, reliability and in measuring inequality. The r^{th} incomplete moment of the EHLOL-G family of distributions is given by

$$\begin{aligned} \vartheta_r(t) &= \int_{-\infty}^{t} x^r f(x;\lambda,\alpha,\xi)dx \\ &= \sum_{k,l=0}^{\infty} a_{k,l} \int_{-\infty}^{t} x^r h_{k+l+1}(x)dx. \end{aligned} \tag{4.20}$$

Note that $\vartheta_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \vartheta_1(q)/(\pi\mu_1')$ and $L(\pi) = \vartheta_1(q)/\mu_1'$, respectively, where μ_1' is given by equation (4.19) with $r = 1$ and $q = Q(\pi)$ is the quantile function of X at π. These curves are of great importance in insurance, demography, reliability, medicine and economics.

A general equation for $\vartheta_1(t)$ can be obtained from equation (4.20) as

$$\vartheta_1(t) = \sum_{k,l=0}^{\infty} a_{k,l} P_{k+l+1}(t),$$

where $P_{k+l+1}(t) = \int_{-\infty}^{t} x h_{k+l+1}(x)dx$ is the first incomplete moment of the Exp-G distribution.

4.4 Order Statistics, Probability Weighted Moments and Rényi Entropy

Order statistics and entropy play important roles in probability and statistics, particularly in reliability, lifetime data analysis and information theory. In this section, we present the distribution of the i^{th} order statistics and Rényi entropy for the EHLOL-G family of distributions.

4.4.1 Order Statistics

In this subsection, the pdf of the i^{th} order statistic is presented. Let $X_1, X_2, ..., X_n$ be independent and identically distributed EHLOL-G random variables. The pdf of $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}, \tag{4.21}$$

where $B(.,.)$ is the beta function.

Using equation (4.4) and equation (4.5), we have

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \frac{2\alpha\lambda^2 g(x;\xi)}{(1+\lambda)\overline{G}^3(x;\xi)} \exp\Big[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\Big] \\ &\times \Big\{1 - \frac{\lambda + \overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)} \exp\Big[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\Big]\Big\}^{\alpha(j+i)-1} \\ &\times \Big\{1 + \frac{\lambda + \overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)} \exp\Big[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\Big]\Big\}^{-\alpha(j+i)-1}. \end{aligned}$$

After applying the generalized binomial and the exponential series, we obtain

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{w,s,k,l=0}^{\infty}\sum_{q=0}^{w+s}\frac{2\alpha g(x;\xi)(-1)^{s+k+l}\lambda^{k+w+s+2}}{k!(1+\lambda)^{w+s+1}(w+s+1)^{-k}}\binom{w+s}{q} \\ &\times \binom{\alpha(j+i)-1}{s}\binom{-\alpha(j+i)-1}{w}\binom{-(k+q+3)}{l}G(x;\xi)^{k+l}. \end{aligned} \tag{4.22}$$

Substituting equation (4.22) in equation (4.21), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{k,l=0}^{\infty} b_{k,l}h_{k+l+1}(x),$$

where $h_{k+l+1}(x)$ is the Exp-G density with power parameter $(k+l+1)$ and

$$\begin{aligned} b_{k,l} &= \sum_{w,s=0}^{\infty}\sum_{j=0}^{n-i}\sum_{q=0}^{w+s}\frac{2\alpha(-1)^{j+s+k+l}\lambda^{k+w+s+2}(w+s+1)^{k}}{k!(1+\lambda)^{w+s+1}B(i,n-i+1)}\binom{n-i}{j} \\ &\times \binom{\alpha(j+i)-1}{s}\binom{-\alpha(j+i)-1}{w}\binom{w+s}{q}\binom{-(k+q+3)}{l}. \end{aligned}$$

Note that, the density function of the EHLOL-G order statistics is a mixture of Exp-G densities. From the last equation, we note that the properties of $X_{i:n}$ follows from those of T_{k+l+1}. Thus, the q^{th} moments of $X_{i:n}$ can be expressed as

$$E(X_{i:n}^{q}) = \sum_{k,l=0}^{\infty} b_{k,l}E(T_{k+l+1}^{q}). \tag{4.23}$$

4.4.2 Probability Weighted Moments

The probability weighted moment (PWM) is the expectation of a function of a random variable given that the mean of the variable exists. PWMs can be used as a method for estimating the parameters of a distribution which has an inexplicit inverse form. The $(j,i)^{th}$ PWM of a random variable $X \sim EHLOL-G(x)$, say $\wp_{j,i}$ is given by

$$\wp_{j,i} = E(X^{j}F(X)^{i}) = \int_{-\infty}^{\infty} x^{j}f(x)F(X)^{i}dx \tag{4.24}$$

From equation (4.22),

$$\begin{aligned} f(x)F(X)^{i} &= \sum_{w,s,k,l=0}^{\infty}\sum_{q=0}^{w+s}\frac{2\alpha g(x;\xi)(-1)^{s+k+l}\lambda^{k+w+s+2}}{k!(1+\lambda)^{w+s+1}(w+s+1)^{-k}}\binom{w+s}{q} \\ &\times \binom{\alpha(i+1)-1}{s}\binom{-\alpha(i+1)-1}{w}\binom{-(k+q+3)}{l}G(x;\xi)^{k+l}. \end{aligned}$$

which can be written as

$$f(x)F(X)^i = \sum_{k,l=0}^{\infty} d_{k,l} h_{k+l+1}(x),$$

where

$$\begin{aligned} d_{k,l} &= \sum_{w,s,k,l=0}^{\infty} \sum_{q=0}^{w+s} \frac{2\alpha(-1)^{s+k+l}\lambda^{k+w+s+2}}{k!(1+\lambda)^{w+s+1}(w+s+1)^{-k}} \binom{w+s}{q} \\ &\times \binom{\alpha(i+1)-1}{s} \binom{-\alpha(i+1)-1}{w} \binom{-(k+q+3)}{l}. \end{aligned}$$

Such that the PWM of X is given by

$$\wp_{j,i} = \sum_{k,l=0}^{\infty} d_{k,l} \int_{-\infty}^{\infty} x^j h_{k+l+1}(x) dx = \sum_{k,l=0}^{\infty} d_{k,l} E(T_{k+l+1}^j). \tag{4.25}$$

4.4.3 Rényi Entropy

Rényi entropy (Rényi [59]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^{\infty} [f(x;\lambda,\alpha,\xi)]^v dx \right), v \neq 1, v > 0. \tag{4.26}$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Set $[f(x;\lambda,\alpha,\xi)]^v = [f(x)]^v$. Using the pdf (4.5), we can write

$$[f(x)]^v = (2\alpha\lambda^2)^v g(x;\xi)^v \frac{\exp\left[\frac{-\lambda v G(x;\xi)}{\overline{G}(x;\xi)}\right] \left\{1 - \frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)} \exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^{v(\alpha-1)}}{(1+\lambda)^v \overline{G}^{3v}(x;\xi) \left\{1 + \frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)} \exp\left[\frac{-\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right]\right\}^{v(\alpha+1)}}.$$

By applying the binomial series, the last equation reduces to

$$\begin{aligned} [f(x)]^v &= \frac{(2\alpha\lambda^2)^v g(x;\xi)^v}{(1+\lambda)^v \overline{G}^{3v}(x;\xi)} \sum_{j,i=0}^{\infty} (-1)^j \binom{v(\alpha-1)}{j} \binom{-v(\alpha+1)}{i} \Big[\frac{\lambda+\overline{G}(x;\xi)}{(1+\lambda)\overline{G}(x;\xi)}\Big]^{j+i} \\ &\times \exp\Big[\frac{-\lambda(j+i+v)G(x;\xi)}{\overline{G}(x;\xi)}\Big]. \end{aligned}$$

From the exponential series, we can write

$$\begin{aligned} [f(x)]^v &= (2\alpha)^v g(x;\xi)^v \sum_{j,i,k=0}^{\infty} \sum_{q=0}^{j+i} \frac{(-1)^{j+k}\lambda^{j+i+k+2v}(j+i+v)^k}{k!(1+\lambda)^{j+i+v}} \binom{v(\alpha-1)}{j} \\ &\times \binom{-v(\alpha+1)}{i} \binom{j+i}{q} G(x;\xi)^k \overline{G}(x;\xi)^{-(k+q+3v)}. \end{aligned}$$

Using the power series in equation (4.14), we have

$$\begin{aligned}[f(x)]^v &= \sum_{j,i,k,l=0}^{\infty}\sum_{q=0}^{j+i}\frac{(2\alpha)^v(-1)^{j+k+1}\lambda^{j+i+k+2v}(j+i+v)^k}{k!(1+\lambda)^{j+i+v}}\binom{v(\alpha-1)}{j} \\ &\times \binom{-v(\alpha+1)}{i}\binom{j+i}{q}\binom{-(k+q+3v)}{l}g(x;\xi)^v G(x;\xi)^{k+l}.\end{aligned}$$

Then, the Rényi entropy of the EHLOL-G family of distributions is given by

$$I_R(v) = \frac{1}{1-v}\log\left[\sum_{k,l=0}^{\infty}c_{k,l}\left(\int_0^{\infty}[g(x;\xi)^v G(x;\xi)^{k+l}]dx\right)\right], \tag{4.27}$$

where

$$\begin{aligned}c_{k,l} &= \sum_{j,i=0}^{\infty}\sum_{q=0}^{j+i}\frac{(2\alpha)^v(-1)^{j+k+l}\lambda^{j+i+k+2v}(j+i+v)^k}{k!(1+\lambda)^{j+i+v}}\binom{v(\alpha-1)}{j} \\ &\times \binom{-v(\alpha+1)}{i}\binom{j+i}{q}\binom{-(k+q+3v)}{l}.\end{aligned}$$

and $\int_0^{\infty} g(x;\xi)^v G(x;\xi)^{k+l}dx$ can be numerically obtained.

Alternatively, the Rényi entropy of the EHLOL-G family of distributions can be expressed as

$$I_R(v) = \frac{1}{1-v}\log\left[\sum_{k,l=0}^{\infty}d_{k,l}e^{(1-v)I_{REG}}\right], \tag{4.28}$$

where

$$\begin{aligned}d_{k,l} &= \sum_{j,i=0}^{\infty}\sum_{q=0}^{j+i}\frac{(2\alpha)^v(-1)^{j+k+l}\lambda^{j+i+k+2v}(j+i+v)^k}{k!(1+\lambda)^{j+i+v}}\binom{v(\alpha-1)}{j} \\ &\times \binom{-v(\alpha+1)}{i}\binom{j+i}{q}\binom{-(k+q+3v)}{l}\left(\frac{k+l}{v}+1\right)\end{aligned}$$

and $I_{REG} = \int_0^{\infty}[(\frac{k+l}{v}+1)g(x;\xi)G(x;\xi)^{\frac{k+l}{v}}]^v dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{k+l}{v}$. Thus, the Rényi entropy of the EHLOL-G family of distributions can be obtained directly from that of the Exp-G distribution.

4.5 Maximum Likelihood Estimation

In this section, we present the method of maximum likelihood for estimating the parameters of the EHLOL-G family of distributions. Let $X \sim EHLOL{-}G(\lambda,\alpha,\xi)$

and $\boldsymbol{\Delta} = (\lambda, \alpha, \xi)^T$ be the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\boldsymbol{\Delta})$ based on a random sample of size n from the EHLOL-G family of distributions is given by

$$\begin{aligned}
\ell_n(\boldsymbol{\Delta}) &= nlog(2\alpha) + 2nlog(\lambda) - nlog(1+\lambda) + \sum_{i=1}^{n} log(g(x_i;\xi)) - 3\sum_{i=1}^{n} log(\overline{G}(x_i;\xi)) \\
&- \lambda\sum_{i=1}^{n}\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)} + (\alpha-1)\sum_{i=1}^{n} log\Big\{1 - \frac{\lambda+\overline{G}(x_i;\xi)}{(1+\lambda)\overline{G}(x_i;\xi)}\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]\Big\} \\
&- (\alpha+1)\sum_{i=1}^{n} log\Big\{1 + \frac{\lambda+\overline{G}(x_i;\xi)}{(1+\lambda)\overline{G}(x_i;\xi)}\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]\Big\}. \qquad (4.29)
\end{aligned}$$

The first derivative of the log-likelihood function with respect to each component of the parameter vector $\boldsymbol{\Delta} = (\lambda, \alpha, \xi)^T$, that is, elements of the score vector $U(\boldsymbol{\Delta})$ are given by

$$\begin{aligned}
U_\alpha(\boldsymbol{\Delta}) &= \frac{n}{\alpha} + \sum_{i=1}^{n} log\Big\{1 - \frac{\lambda+\overline{G}(x_i;\xi)}{(1+\lambda)\overline{G}(x_i;\xi)}\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]\Big\} \\
&- \sum_{i=1}^{n} log\Big\{1 + \frac{\lambda+\overline{G}(x_i;\xi)}{(1+\lambda)\overline{G}(x_i;\xi)}\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]\Big\}, \qquad (4.30) \\
U_\lambda(\boldsymbol{\Delta}) &= \frac{2n}{\lambda} - \frac{n}{(1+\lambda)} - \sum_{i=1}^{n}\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)} + (\alpha-1) \\
&\times \sum_{i=1}^{n}\frac{\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]\Big\{(\lambda+\overline{G}x_i;\xi) - (1+\lambda)\Big[1 - \frac{G(x_i;\xi)(\lambda+\overline{G}(x_i;\xi))}{\overline{G}(x_i;\xi)}\Big]\Big\}}{(1+\lambda)^2\overline{G}(x_i;\xi)\Big\{1 - \frac{\lambda+\overline{G}(x_i;\xi)}{(1+\lambda)\overline{G}(x_i;\xi)}\Big\}} \\
&+ (\alpha+1)\sum_{i=1}^{n}\frac{\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]\Big\{(1+\lambda)\Big[1 - \frac{G(x_i;\xi)(\lambda+\overline{G}(x_i;\xi))}{\overline{G}(x_i;\xi)}\Big] - (\lambda+\overline{G}x_i;\xi)\Big\}}{(1+\lambda)^2\overline{G}(x_i;\xi)\Big\{1 + \frac{\lambda+\overline{G}(x_i;\xi)}{(1+\lambda)\overline{G}(x_i;\xi)}\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]\Big\}}
\end{aligned}$$

and

$$\begin{aligned}
U_{\xi_k}(\boldsymbol{\Delta}) &= \sum_{i=1}^{n}\frac{g'(x_i;\xi)}{g(x_i;\xi)} - 3\sum_{i=1}^{n}\frac{\overline{G}'(x_i;\xi)}{\overline{G}(x_i;\xi)} - \lambda\sum_{i=1}^{n}\frac{\Psi}{\overline{G}^2(x_i;\xi)} \qquad (4.31) \\
&+ \lambda(\alpha-1)\sum_{i=1}^{n}\frac{[(\lambda+\overline{G}(x_i;\xi))\Psi + \overline{G}'(x_i;\xi)\overline{G}(x_i;\xi)]\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]}{(1+\lambda)\overline{G}^3(x_i;\xi)\Big\{1 - \frac{\lambda+\overline{G}(x_i;\xi)}{(1+\lambda)\overline{G}(x_i;\xi)}\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]\Big\}} \\
&+ \lambda(\alpha+1)\sum_{i=1}^{n}\frac{[(\lambda+\overline{G}(x_i;\xi))\Psi + \overline{G}'(x_i;\xi)\overline{G}(x_i;\xi)]\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]}{(1+\lambda)\overline{G}^3(x_i;\xi)\Big\{1 + \frac{\lambda+\overline{G}(x_i;\xi)}{(1+\lambda)\overline{G}(x_i;\xi)}\exp\Big[\frac{-\lambda G(x_i;\xi)}{\overline{G}(x_i;\xi)}\Big]\Big\}} \qquad (4.32)
\end{aligned}$$

where $\Psi = [\overline{G}(x_i;\xi)G'(x_i;\xi) - G(x_i;\xi)\overline{G}'(x_i;\xi)]$, $g'(x_i;\xi) = \frac{\partial g(x_i;\xi)}{\partial \xi_k}$, $G'(x_i;\xi) = \frac{\partial G(x_i;\xi)}{\partial \xi_k}$ and $\overline{G}'(x_i;\xi) = \frac{\partial \overline{G}(x_i;\xi)}{\partial \xi_k}$.

The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial \lambda}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \xi_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{p+2}(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0, 0, \underline{0})^T$ and $J(\hat{\mathbf{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\mathbf{\Delta}}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

4.6 Simulation Study

We simulate $N = 1000$ samples for the true parameter values given by $I = (0.5, 0.5, 0.9, 0.8)$, $II = (0.5, 0.7, 0.5, 0.8)$, $III = (0.2, 0.5, 0.7, 0.4)$ and $IV = (0.5, 0.7, 0.7, 0.8)$ with the results given in Table 4.3 and Table 4.4. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

Table 4.3: **Monte Carlo Simulation Results**

		$(0.5, 0.5, 0.9, 0.8)$			$(0.5, 0.7, 0.5, 0.8)$		
parameter	**Sample Size**	**Mean**	**RMSE**	**Bias**	**Mean**	**RMSE**	**Bias**
λ	25	3.4098	13.5816	2.9098	5.5524	21.8681	5.0525
	50	1.6430	8.2837	1.1430	5.7617	29.4288	5.2617
	100	1.3057	5.4868	0.8057	2.4042	7.7581	1.9042
	200	0.7384	0.5207	0.2384	1.3829	3.2974	0.8829
	400	0.6821	0.3055	0.1821	1.0200	0.6966	0.5200
	800	0.6270	0.2043	0.1270	0.8646	0.4600	0.3646
α	25	0.9196	11.9527	0.4196	6.9310	19.3710	6.2310
	50	0.4505	0.1984	-0.0495	0.6332	0.3021	-0.0668
	100	0.4564	0.1412	-0.0436	0.6437	0.2196	-0.0563
	200	0.4869	0.0996	-0.0131	0.6966	0.1455	-0.0034
	400	0.4912	0.0687	-0.0088	0.6947	0.1018	-0.0053
	800	0.5042	0.0407	0.00422	0.7040	0.0560	0.0039
a	25	1.8740	1.7941	0.9740	1.1787	1.2676	0.6787
	50	1.4348	1.1274	0.5348	0.9539	0.9003	0.4539
	100	1.3055	0.8158	0.4055	0.8504	0.6353	0.3504
	200	1.1045	0.4432	0.2045	0.6916	0.3575	0.1916
	400	1.0506	0.2852	0.1506	0.6740	0.2556	0.1740
	800	0.9822	0.1564	0.0822	0.6231	0.1577	0.1231
b	25	2.1505	6.0822	1.3505	7.7567	46.2418	6.9567
	50	1.2024	0.9284	0.4024	4.8867	18.5952	4.0867
	100	1.1668	0.7777	0.3668	3.1189	4.5684	2.3189
	200	1.0364	0.4479	0.2364	2.1833	2.3932	1.3833
	400	1.0135	0.3510	0.2135	2.0223	1.5740	1.2223
	800	0.9466	0.2457	0.1466	1.6828	1.1068	0.8828

Table 4.4: **Monte Carlo Simulation Results**

		$(0.2, 0.5, 0.7, 0.4)$			$(0.5, 0.7, 0.7, 0.8)$		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias
λ	25	2.8938	10.0060	2.6938	4.7467	16.1085	4.2467
	50	0.8839	2.0789	0.6839	3.2587	12.2905	2.7587
	100	0.8111	3.2274	0.6111	2.6873	11.4352	2.1873
	200	0.4426	0.5011	0.2426	1.0587	2.4737	0.5587
	400	0.3887	0.2455	0.1887	0.7260	0.4543	0.2260
	800	0.3409	0.1650	0.1409	0.6433	0.2660	0.1433
α	25	1.7306	22.6125	1.2306	0.7736	2.6632	0.0736
	50	0.5008	0.2096	0.0008	0.5963	0.3149	-0.1036
	100	0.5089	0.1366	0.0089	0.6259	0.2114	-0.0741
	200	0.5339	0.1014	0.0339	0.6757	0.1473	-0.0243
	400	0.5358	0.0728	0.0358	0.6816	0.0925	-0.0184
	800	0.5344	0.0503	0.0344	0.6972	0.0582	-0.0027
a	25	1.5559	1.5273	0.8559	1.8099	2.0653	1.1099
	50	1.2039	1.0579	0.5039	1.3711	1.3420	0.6711
	100	1.0639	0.7010	0.3660	1.1018	0.8451	0.4018
	200	0.8925	0.3998	0.1925	0.8851	0.4692	0.1851
	400	0.8592	0.2303	0.1593	0.8210	0.2540	0.1210
	800	0.8209	0.1470	0.1209	0.7711	0.1397	0.0711
b	25	2.2257	15.8486	1.8257	2.4989	3.2731	1.6989
	50	0.9814	0.9640	0.5814	1.9460	2.5301	1.1460
	100	0.9043	0.8034	0.5043	1.6719	1.8813	0.8720
	200	0.7237	0.4966	0.3237	1.2543	0.9739	0.4543
	400	0.6971	0.3766	0.2971	1.1344	0.6190	0.3344
	800	0.6318	0.2698	0.2318	1.0264	0.4047	0.2264

The results in Tables 4.3 and 4.4 clearly show that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

4.7 Applications

The flexibility and usefulness of the EHLOL-W distribution and its sub-models for data modeling is presented in this section. We fitted the EHLOL-W distribution to real data sets and these fits are compared to the fits of the Odd Log-Logistic Exponentiated Weibull (OLLE-W) distribution (Afify et al. [2]), Kumaraswamy Weibull (KumW) dustribution (Cordeiro etal [24]), Exponentiated Weibull Lindley (EWL) distribution (Cordeiro et al. [22]),Weibull Lomax (WL) distribution (Tahir et al. [64]) and the Beta Generalized Lindley (BGL) distribution (Oluyede and Yang [55]). The pdfs of the listed distributions are given below

$$f_{OLLE-W}(x) = \frac{\theta\beta\gamma x^{\beta-1}e^{-(\frac{x}{\alpha})^{\beta}}[1-e^{-(\frac{x}{\alpha})^{\beta}}]^{\gamma\theta-1}\{1-[1-e^{-(\frac{x}{\alpha})^{\beta}}]^{\alpha}\}^{\theta-1}}{\alpha^{\beta}\{[1-e^{-(\frac{x}{\alpha})^{\beta}}]^{\gamma\theta}+\{1-[1-e^{-(\frac{x}{\alpha})^{\beta}}]^{\gamma}\}^{\theta}\}^{2}}, \tag{4.33}$$

for $\alpha, \beta, \theta, \gamma > 0$ and $x > 0$,

$$f_{Kum-W}(x) = ab\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}(1-e^{-(\alpha x)^{\beta}})^{a-1}\{1-[1-e^{-(\alpha x)^{\beta}}]^{a}\}^{b-1}, \tag{4.34}$$

for $\alpha, \beta, a, b > 0$ and $x > 0$,

$$\begin{aligned} f_{EWL}(x) &= \frac{abv(1+\alpha)\alpha^2(1+x)\exp(\alpha x)}{(1+\alpha+\alpha x)^2} \exp\left\{ -a\left[\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right]^b \right\} \\ &\times \left(1 - \exp\left\{ -a\left[\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right]^b \right\}\right)^{v-1} \\ &\times \left\{\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right\}^{b-1}, \end{aligned} \tag{4.35}$$

for $\alpha, v, a, b > 0$ and $x > 0$,

$$\begin{aligned} f_{WL}(x) &= \frac{ab\alpha}{\beta}\Big[1 + (\frac{x}{\beta})\Big]^{b\alpha-1}\Big\{1 - \Big[1 + (\frac{x}{\beta})\Big]^{-\alpha}\Big\}^{b-1} \\ &\times \exp\Big\{ -a\Big\{\Big[1 + (\frac{x}{\beta})\Big]^{\alpha} - 1\Big\}\Big\}, \end{aligned} \tag{4.36}$$

for $\alpha, \beta, a, b > 0$ and $x > 0$,

$$\begin{aligned} f_{BGL}(x) &= \frac{\alpha\theta^2(1+x)e^{-\theta x}}{B(a,b)(1+\theta)}\Big[1 - \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\Big]^{a\alpha-1} \\ &\times \Big[1 - \Big(1 - \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\Big)^{a}\Big]^{b-1}, \end{aligned} \tag{4.37}$$

for $\alpha, \theta, a, b > 0$ and $x > 0$.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [15]) are given in Figure 4.9 and Figure 4.10. For the probability plot, we plotted $F(x_{(j)}; \hat{\lambda}, \hat{\alpha}, \hat{\xi})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness is given by the sum of squares

$$SS = \sum_{j=1}^{n}\left[F(x_{(j)}; \hat{\lambda}, \hat{\alpha}, \hat{\xi}) - \left(\frac{j - 0.375}{n + 0.25}\right)\right]^2.$$

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan [16] are also presented in the tables. We used these statistics to decide which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit.

The MLEs of the parameters, their standard errors in parenthesis and the values of the statistics (-2$\log(L)$, AIC, AICC, BIC, W^*, A^*, K-S and its p-value) are given in Table 4.5 and Table 4.6.

4.7.1 Relief times of 20 patients

This data set is taken from Gross and Clark [37], which gives the relief times of 20 patients receiving an analgesic. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics are presented in Table 4.5.

Table 4.5: **Estimation of the EHLOL-W model for relief times of 20 patients data set**

	Estimates												
Distribution	λ	α	a	b	-2logL	AIC	CAIC	BIC	W^*	A^*	K–S	P–value	SS
EHLOL-W	24.355	8265.200	0.335	37.842	31.065	39.065	41.731	43.048	0.033	0.188	0.103	0.984	0.023
	(5.714)	(0.001)	(0.069)	(2.028)									
	α	θ	a	b									
BGL	0.120	1.520×10^{-07}	0.302	10.100	162.523	170.538	173.205	174.521	0.089	0.523	0.512	5.526×10^{-05}	1.571
	(0.013)	(4.342×10^{-06})	(0.001)	(3.494×10^{-05})									
	α	β	θ	γ									
OLLE-W	0.027	0.034	90.217	1.821	32.987	40.987	43.654	44.970	0.052	0.313	0.120	0.969	0.029
	(0.021)	(0.007)	(5.119×10^{-05})	(0.037)									
	a	b	c	λ									
Kum-W	23108.000	1.465	0.309	989.690	31.319	39.319	41.986	43.302	0.036	0.207	0.121	0.931	0.032
	(9.797×10^{-06})	(0.536)	(0.004)	(2.177×10^{-04})									
	a	b	α	β									
WL	1753.0	5.843	0.124	0.334	38.901	46.901	49.568	50.884	0.151	0.892	0.178	0.554	0.138
	(0.018)	(12.087)	(0.129)	(2.583)									
	α	v	a	b									
EWL	0.003	982.470	348.730	0.374	31.657	39.657	42.323	43.640	0.042	0.244	0.123	0.924	0.034
	(0.003)	(2.881×10^{-06})	(1.466×10^{-06})	(0.060)									
			a	b									
W			0.122	2.787	41.173	45.173	45.879	47.164	0.186	1.093	0.185	0.501	0.171
			(0.056)	(0.427)									

The covariance matrix for the relief times of 20 patients data is given by,

$$\begin{bmatrix} 32.6503 & 0.0046 & 0.3914 & -11.5889 \\ 0.0046 & 6.4167 \times 10^{-07} & 5.4864 \times 10^{-05} & -0.0016 \\ 0.3914 & 5.4867 \times 10^{-05} & 0.0047 & -0.1389 \\ -11.5889 & -0.0016 & -0.1389 & 4.1133 \end{bmatrix}$$

and the 95% confidence intervals for the parameters λ, α, a and b are: 24.355 $\pm$11.199 , 8265.200 $\pm$ 0.002 ,0.335 $\pm$ 0.135 and 37.842 $\pm$ 3.975, respectively.

From the results presented in Table 4.5, it can be seen clearly that the EHLOL-W distribution has smallest values of the goodness-of-fit statistics (AIC, AICC, BIC, W^*, A^*), also the biggest p-value for the KS statistic and the smallest Sum of Squares value (SS) indicating that the EHLOL-W distribution provides a better fit compared to the other models in Table 4.5. The plots of histogram, fitted densities and probability plots are given in Figure 4.9.

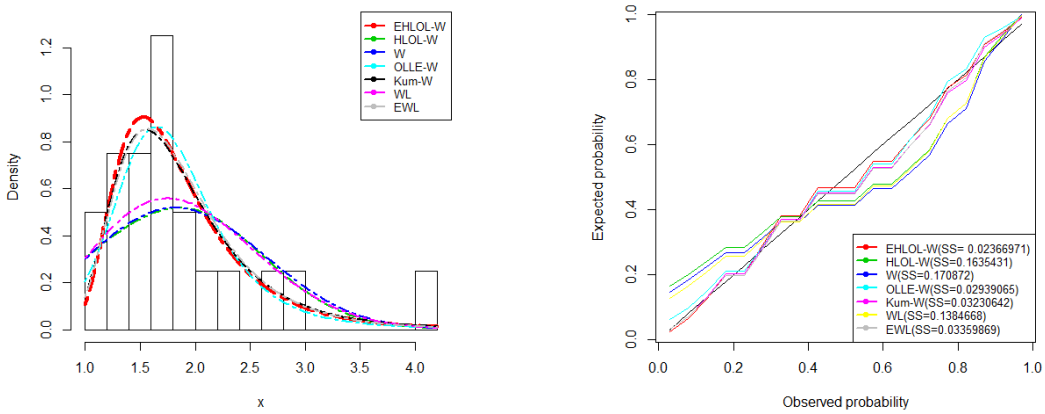


Figure 4.9: Fitted densities and probability plots for relief times of 20 patients data set.

4.7.2 Strengths of the glass fibres data set

The data consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. The units of measurement are not given in the paper. Smith and Naylor [63] also analysed this data set. The estimated parameters, standard error (in parenthesis) of estimates and the goodness-of-fit statistics for this data set are presented in Table 4.6.

Table 4.6: **Estimation of the EHLOL-W model for strengths of the glass fibres data set**

			Estimates									
Distribution	λ	α	a	b	-2logL	AIC	CAIC	BIC	W^*	A^*	K–S	P–value	SS
EHLOL-W	0.331	1.436	1.327	0.872	26.763	34.763	35.453	43.336	0.134	0.762	0.123	0.301	0.124
	(1.007)	(0.770)	(1.176)	(1.272)									
	α	θ	a	b									
BGL	0.121	2.290×10^{-06}	0.302	0.101	492.265	500.270	500.960	508.843	0.638	3.492	0.521	2.776×10^{-15}	5.151
	(0.007)	(2.983×10^{-06})	(4.513×10^{-04})	(2.218×10^{-05})									
	α	β	θ	γ									
OLLE-W	1.992	8.749	1.687	0.302	28.050	36.050	36.739	44.622	0.186	1.031	0.132	0.222	0.153
	(0.297)	(3.940)	(0.744)	(0.267)									
	a	b	α	β									
WL	0.013	2.840	0.010	95.349	28.792	36.792	37.481	45.365	0.173	0.974	0.136	0.193	0.167
	(0.006)	(0.270)	(0.005)	(0.005)									
	α	v	a	b									
EWL	4.195	1.808	0.020	0.717	28.219	36.219	36.909	44.792	0.160	0.899	0.131	0.230	0.150
	(5.940)	(0.981)	(0.028)	(1.146)									
			a	b									
W			0.060	5.781	30.414	34.414	34.614	38.700	0.237	1.304	0.152	0.108	0.211
			(0.021)	(0.576)									

The covariance matrix for the strengths of the glass fibres data set is given by,

$$\begin{bmatrix} 1.0132 & 0.6607 & 1.1548 & 1.2726 \\ 0.6607 & 0.5925 & 0.6657 & 0.7803 \\ 1.1548 & 0.6658 & 1.3835 & 1.4868 \\ 1.2726 & 0.7803 & 1.4868 & 1.6184 \end{bmatrix}$$

and the 95% confidence intervals for the parameters λ, α, a and b are: 0.331 ± 1.974 , 1.436 ± 1.509 ,1.327 ± 2.305 and 0.872 ± 2.489 , respectively.

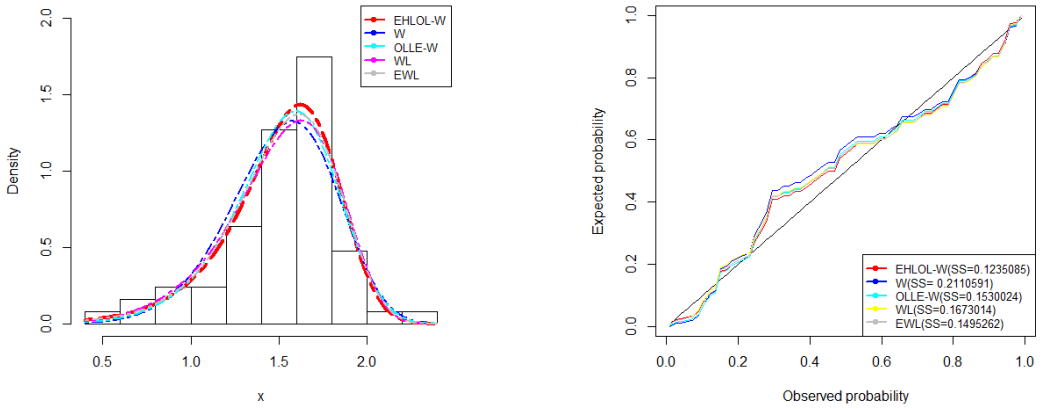


Figure 4.10: Fitted densities and probability plots for strengths of the glass fibres data set.

Figure 4.10 shows the histogram, fitted densities as well as the probability plots for the strength of glass fibre data set. Clearly, the EHLOL-W distribution has the smallest values of the goodness-of-fit statistics, the largest p-value for the K-S statistic and the smallest SS value, hence we can conclude that the EHLOL-W distribution provides a better fit compared to the other models presented in Table 4.6.

4.8 Concluding Remarks

A new generalized family of distributions called the exponentiated Half-Logistic Odd Lindley-G (EHLOL-G) family of distributions is developed and presented. The EHLOL-G family of distributions possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, incomplete

moments, distribution of order statistics and Rényi entropy. Maximum likelihood estimation (MLE) was used to estimate the model parameters. The performance of the special case of the EHLOL-G family of distributions was examined by conducting various simulations for different sizes. Finally, the special case of the EHLOL-G family of distributions was fitted to real data sets to illustrate the applicability and usefulness of the proposed family of distributions.

Chapter 5

Conclusions and Suggestions for Further Studies

5.1 Conclusions

Three new generalized distributions called the new Gamma Lindley Log-Logistic (GLLLoG) distribution, the New Weibull Generalized-G (NWG-G) family of distributions and the exponentiated Half-Logistic Odd Lindley-G (EHLOL-G) family of distributions were proposed and developed. Properties of these distributions including the series expansion of pdfs, cdfs, moments, hazard functions, reverse hazard functions, quantile functions and sub-models, moments, conditional moments, mean deviations, Bonferroni and Lorenz curves, Rényi entropy and distribution of order statistics are derived. Estimation of the parameters of the models and applications to real dataset are also given.

5.2 Suggestions for Further Studies

We present some suggestions for further studies that may be conducted. In this thesis, we introduced and presented the three generalised models, their properties and applications. Other possible research topic could include the application of the proposed models to regression analysis and also the application of Bayesian techniques to the proposed models, as well as estimation of the model parameters under various censoring schemes. Consider different

parameter estimation techniques on the proposed models.

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Appendix

PART 1A

```
#define pdf GLLLoG
f1=function(lambda,c,delta,x)
{
y=(1/gamma(delta))*(-log(1-(1-((1+lambda+lambda*x)*exp(-lambda*x)
/((1+lambda)*(1+x^c))))))^(delta-1)*((1+x^c)^(-1)/(1+lambda))
*exp(-lambda*x)*((lambda^2*(1+x)+(1+lambda+lambda*x)
*c*x^(c-1)/(1+x^c)))
}
x=seq(0,20.0,by=0.001)
y1=f1(x,0.3,0.8,0.2)
plot(x,y1,ylim=c(0,1.5),col=2,'l',lwd=2,xlab="x",ylab="density")
y2=f1(x,3.5,4.5,1.2)
lines(x,y2,col=3,lwd=2)
y3=f1(x,0.3,5.0,0.7)
lines(x,y3,col=4,lwd=2)
y4=f1(x,0.5,25.0,2.0)
lines(x,y4,col=5,lwd=2)
y5=f1(x,0.2,12.0,0.4)
lines(x,y5,col=6,lwd=2)
y6=f1(x,5.0,0.8,0.9)
lines(x,y6,col=7,lwd=2)
legend("topright",c(
  expression(paste(lambda,'=0.3,',c,'=0.8,',delta,'=0.2')),
  expression(paste(lambda,'=3.5,',c,'=4.5,',delta,'=1.2')),
  expression(paste(lambda,'=0.3,',c,'=5.0,',delta,'=0.7')),
  expression(paste(lambda,'=0.5,',c,'=25.0,',delta,'=2.0')),
  expression(paste(lambda,'=0.2,',c,'=12.0,',delta,'=0.4')),
  expression(paste(lambda,'=5.0,',c,'=0.8.0,',delta,'=0.9')))
  ,col=c(2,3,4,5,6,7),lwd=c(2,2,2,2,2,2),cex = 0.8)

#define Hazard Function
f1=function(x,lambda, c,delta)
{
y=(1/gamma(delta))*(-log(1-(1-((1+lambda+lambda*x*exp(lambda*x)
/((1+lambda)*(1+x^c))))))^(delta-1)*((1+x^c)^(-1)/(1+lambda))
*exp(-lambda*x)*((lambda^2*(1+x)+(1+lambda+lambda*x)*c*x^(c-1)
/(1+x^c)))/(1-pgamma(-log(1-(1-((1+lambda +lambda*x)
*exp(-lambda*x))/((1+lambda)*(1+x^c)))),delta)/gamma(delta))
}
x=seq(0,3.0,by=0.001)
y1=f1(x,0.2,5.0,0.5)
plot(x,y1,ylim=c(0,2.0),col=2,'l',lwd=2,xlab="x",ylab="h(x)")
y2=f1(x,0.5,0.5,0.9)
lines(x,y2,col=3,lwd=2)
y3=f1(x,0.6,1.3,1.5)
lines(x,y3,col=4,lwd=2)
y4=f1(x,0.5,0.6,2.9)
```

```
lines(x,y4,col=5,lwd=2)
y5=f1(x,2.0,0.5,3.0)
lines(x,y5,col=6,lwd=2)
y6=f1(x,0.10,1.0,1.2)
lines(x,y6,col=7,lwd=2)
legend("topright",c(
  expression(paste(lambda,'=0.2,',c,'=5.0,',delta,'=0.5')),
  expression(paste(lambda,'=0.5,',c,'=0.5,',delta,'=0.9')),
  expression(paste(lambda,'=0.6,',c,'=1.3,',delta,'=1.5')),
  expression(paste(lambda,'=0.5,',c,'=0.6,',delta,'=2.9')),
  expression(paste(lambda,'=2.0,',c,'=0.5,',delta,'=3.0')),
  expression(paste(lambda,'=0.1,',c,'=1.0,',delta,'=1.2')))
  ,col=c(2,3,4,5,6,7),lwd=c(2,2,2,2,2,2),cex = 0.8)
# Simulation Algorithm
library(numDeriv)
library(Matrix)
library(zipfR)

lambda =1.15
c=0.5
delta=1.25
#Define the quantile of  GLLLoG
GLLLoG_quantile =function(lambda,c,delta,u)
{
f=function(x)
{
pgamma((-log(1-(1-((1+lambda +lambda*x)/(1+lambda)
*exp(-lambda*x)/(1+x^c)))))),delta)-u*gamma(delta)
}
x=uniroot(f,c(0,100))$root
return(x)
}
GLLLoG_LL<-function(par)
{
-sum(log((-log(1-(1-((1+par[1]+par[1]*x)/(1+par[1])*exp(-par[1]
*x)/(1+x^par[2])))))^(par[3]-1)*((1+x^par[2])^(-1)*exp(-par[1]*x)
/(1+par[1]))*((par[1]^2*(1+x)+((1+par[1]+par[1]*x)
*par[2]*x^(par[2]-1))/(1+x^par[2])))*(1/gamma(par[3]))))
}
n1<-c(800)
# If you want to check one sample at a time then use
n1<-c(sample size)
for (i in 1: length(n1)){
  n=n1[i]
  N=1000
  mle_lambda<-c(rep(0,N))
  mle_c<-c(rep(0,N))
  mle_delta<-c(rep(0,N))


  LC_lambda<-c(rep(0,N))
```

```
UC_lambda<-(rep(0,N))
LC_c<-c(rep(0,N))
UC_c<-c(rep(0,N))
LC_delta<-c(rep(0,N))
UC_delta<-c(rep(0,N))
count_lambda=0
count_c=0
count_delta=0
temp=1
HH1<-matrix(c(rep(3,9)), nrow=3, ncol=3)
HH2<-matrix(c(rep(3,9)), nrow=3, ncol=3)
for (i in 1:N)
{
  #print (i)
  #flush.console()
  repeat{
    x<-c(rep(0,n))
    #Generate a random variable from log-logistic distribution
    u<-0
    u<-runif(n,min=0,max=1)

    for (k in 1:n){
      x[k]<-GLLLoG_quantile(lambda,c,delta,u[k])
    }

    #Maximum likelihood estimation
    mle.result<-nlminb(c( lambda,c,delta), GLLLoG_LL,
                       lower = 0, upper = Inf)

    temp=mle.result$convergence
    if (temp==0){
      temp_lambda<-mle.result$par[1]
      temp_c<-mle.result$par[2]
      temp_delta<-mle.result$par[3]

      HH1<-hessian(GLLLoG_LL, c(temp_lambda,temp_c, temp_delta))
      if (sum(is.nan(HH1))==0 & (diag(HH1)[1]>0)
          & (diag(HH1)[2]>0)
          & (diag(HH1)[3]>0)){
        HH2<-solve(HH1)
        #print(det(hh1))
      }
      else{
        temp=1}
    }

    if ((temp==0) & (diag(HH2)[1]>0) & (diag(HH2)[2]>0)
        & (diag(HH2)[3]>0)  & (sum( is.nan(HH2))==0)){
      break
    }
  }
```

```
#print(temp)
temp=1
mle_lambda[i]<-mle.result$par[1]
mle_c[i]<-mle.result$par[2]
mle_delta[i]<-mle.result$par[3]


HH<-hessian(GLLLoG_LL,c(mle_lambda[i],mle_c[i],mle_delta[i]))
H<-solve(HH)

LC_lambda[i]<-mle_lambda[i]-1.96*sqrt(diag(H)[1])
UC_lambda[i]<-mle_lambda[i]+1.96*sqrt(diag(H)[1])
if ((LC_lambda[i]<=lambda) & (lambda<=UC_lambda[i])){
  count_lambda=count_lambda+1
}

LC_c[i]<-mle_c[i]-1.96*sqrt(diag(H)[2])
UC_c[i]<-mle_c[i]+1.96*sqrt(diag(H)[2])
if ((LC_c[i]<=c) & (c<=UC_c[i])){
  count_c=count_c+1
}

LC_delta[i]<-mle_delta[i]-1.96*sqrt(diag(H)[3])
UC_delta[i]<-mle_delta[i]+1.96*sqrt(diag(H)[3])
if ((LC_delta[i]<=delta) & (delta<=UC_delta[i])){
  count_delta=count_delta+1
}

#Calculate Mean
Mean_lambda<-sum(mle_lambda)/N
Mean_c<-sum(mle_c)/N
Mean_delta<-sum(mle_delta)/N

print(cbind(Mean_lambda,Mean_c, Mean_delta))

#Calculate Average Bias
Bias_lambda<-sum(mle_lambda-lambda)/N
Bias_c<-sum(mle_c-c)/N
Bias_delta<-sum(mle_delta-delta)/N

print(cbind(Bias_lambda,Bias_c, Bias_delta))

#Calculate RMSE
RMSE_lambda<-sqrt(sum((lambda-mle_lambda)^2)/N)
RMSE_c<-sqrt(sum((c-mle_c)^2)/N)
RMSE_delta<-sqrt(sum((delta-mle_delta)^2)/N)


print(cbind(RMSE_lambda,RMSE_c, RMSE_delta))

#Converge Probability
```

```
      CP_lambda<-count_lambda/N
      CP_c<-count_c/N
      CP_delta<-count_delta/N


      print(cbind(CP_lambda,CP_c, CP_delta))

      # Average Width
      AW_lambda<-sum(abs(UC_lambda-LC_lambda))/N
      AW_c<-sum(abs(UC_c-LC_c))/N
      AW_delta<-sum(abs(UC_delta-LC_delta))/N
      print(cbind(AW_lambda,AW_c, AW_delta))
    }
}
#Goodness-of-fit
rm(list=ls())
library(stats4)
library(AdequacyModel)
library(bbmle)
library(stats)
air<-c(0.71, 0.63, 0.85, 0.44, 0.61, 0.69, 0.92, 0.55, 0.72,
0.77, 0.92, 0.60, 0.83, 0.80, 1.00, 0.77, 0.92, 1.00,
1.24,1.00,1.16, 1.30 ,1.45, 1.25, 1.26, 1.38,1.86,1.56,
1.53,1.59 ,1.83, 1.86 ,1.53, 2.07, 2.34 ,2.25,2.16,2.43
,2.70,2.25, 2.79,3.42,3.69 ,3.60, 3.60 ,4.32,4.05,4.86,
5.04 ,4.41,5.58, 5.85 ,6.5, 5.31 ,6.03,6.39 ,6.93,5.85,
6.93,7.74 ,7.83,6.12, 7.74 ,8.91 ,8.28,6.84, 9.54, 10.26,
9.54,8.73 ,11.88, 12.06 ,12.15 ,8.91, 14.04 ,12.96,14.85)
# define GLLLoG pdf
GLLLoG_pdf<-function(parameter,x){
  lambda=parameter[1]
  c=parameter[2]
  delta=parameter[3]

(1/gamma(delta))*(-log(1-(1-((1+lambda +lambda*x)/(1+lambda)
*(exp(-lambda*x)/(1+x^c))))))^(delta -1)*((1+x^c)^(-1)/(1+lambda))
*(exp(-lambda*x))*(lambda^2*(1+x)+(((1+lambda +lambda*x)
/(1+x^c))*c*x^(c-1)))
}
# define GLLLoG cdf
GLLLoG_cdf<-function(parameter,x)
{
  lambda=parameter[1]
  c=parameter[2]
  delta=parameter[3]
  (pgamma(-log(1-((1-((1+lambda+lambda*x)/(1+lambda))
  *(exp(-lambda*x)/(1+x^c))))),delta))/gamma(delta)
}
goodness.fit(pdf= GLLLoG_pdf,cdf= GLLLoG_cdf,mle=c(0.232310,
1.228278, 2.071583 ),data=air,method = "BFGS",domain = c(0,100)
,lim_inf = c(0,0,0,0), lim_sup = c(100,100,100,100))
```

```
#Define GLLoG pdf
GLLoG_pdf<-function(parameter,x)
{
  c =parameter[1]
  delta =parameter[2]

(1/gamma(delta))*(((-log(1-(1-(1/((1+x^c)))))))^(delta-1))
*((1+x^c)^-1)*((c*x^(c-1))/(1+x^c))
}
# define GLLoG cdf
GLLoG_cdf<-function(parameter,x){
  c=parameter[1]
  delta=parameter[2]
  (pgamma(-log(1-(1-(1/(1+x^c)))),delta))/gamma(delta)
}
goodness.fit(pdf= GLLoG_pdf,cdf= GLLoG_cdf,mle=c(1.59427
, 1.97489 ),data=air,method = "BFGS",domain = c(0,100)
,lim_inf = c(0,0,0,0),lim_sup = c(100,100,100,100))


# Lindley Log-logistic
LLLoG_pdf<-function(par,x){

  lambda=par[1]
  c=par[2]

((1+x^c)^(-1))/(1+lambda)*exp(-lambda*x)*(lambda^2*(1+x)
+((1+lambda+lambda*x)*c*x^(c-1))/(1+x^c))
}

LLLoG_cdf<-function(par,x){

  lambda=par[1]
  c=par[2]

  1-((1+lambda+lambda*x)/(1+lambda))*(exp(-lambda*x))/((1+x^c))
}

goodness.fit(pdf=LLLoG_pdf, cdf=LLLoG_cdf,mle=c( 0.126751,1.016227)
,data = air, method = "BFGS",domain = c(0,100), lim_inf = c(0,0),
lim_sup = c(100,100))

#Log-logistic

LLoG_pdf<-function(par,x){

  c=par[1]

  c*x^(c-1)*(1+x^c)^(-2)
}
```

```
LLoG_cdf<-function(par,x){

  c=par[1]

  1-(1+x^c)^(-1)
}
goodness.fit(pdf=LLoG_pdf, cdf=LLoG_cdf,mle=c( 1.20962)
,data = air, method = "BFGS",domain = c(0,100),
lim_inf = c(0),lim_sup = c(100))

# Lindley Weibull
LW_pdf<-function(par,x)
{
lambda=par[1]
alpha=par[2]
beta=par[3]

(exp((-lambda*x)-(alpha*x^beta)))/(1+lambda)*(lambda^2*(1+x)
+(1+lambda+lambda*x)*alpha*beta*x^(beta-1))
}

LW_cdf<-function(par,x){

  lambda=par[1]
  alpha=par[2]
  beta=par[3]

1-(exp((-lambda*x)-(alpha*x^beta)))/((1+lambda)*(1+lambda+lambda*x))
}

goodness.fit(pdf=LW_pdf, cdf=LW_cdf,mle=c( 7.7173e-07,1.8646e-01
,1.1292e+00),data = air,method = "BFGS", domain = c(0,100),
lim_inf = c(0,0,0),lim_sup = c(100,100,100))

# Exponentiated power Lindley
EPL_pdf<-function(par,x)
{
  alpha=par[1]
  beta=par[2]
  delta=par[3]

alpha*beta*delta*x^(alpha-1)*(1+x^alpha)*exp(-beta*x^alpha)
*(1-(1+(beta*x^alpha)/(1+beta)*exp(-beta*x^alpha))^(delta-1))
}

EPL_cdf<-function(par,x){

  alpha=par[1]
  beta=par[2]
  delta=par[3]
```

```
(1-((1+beta*x^alpha)/(1+beta))*exp(-beta*x^alpha))^delta
}

goodness.fit(pdf=EPL_pdf, cdf=EPL_cdf,mle=c(1.319356 ,0.123189
,0.510267),data = air,method = "BFGS", domain = c(0,100),
lim_inf = c(0,0,0),lim_sup = c(100,100,100))


#Fitted Probability Plots

data=c(0.71, 0.63, 0.85, 0.44, 0.61, 0.69, 0.92, 0.55, 0.72,
0.77, 0.92, 0.60, 0.83, 0.80, 1.00, 0.77, 0.92, 1.00,
1.24, 1.00, 1.16, 1.30 ,1.45, 1.25, 1.26, 1.38, 1.86,
1.56, 1.53, 1.59 ,1.83, 1.86 ,1.53, 2.07, 2.34 ,2.25,
2.16, 2.43 ,2.70 ,2.25, 2.79 ,3.42, 3.69 ,3.60, 3.60,
4.32, 4.05, 4.86 ,5.04 ,4.41, 5.58, 5.85 ,6.5, 5.31,
6.03 ,6.39 ,6.93, 5.85, 6.93, 7.74 ,7.83, 6.12, 7.74,
8.91 ,8.28, 6.84, 9.54, 10.26, 9.54, 8.73 ,11.88,
12.06 ,12.15 ,8.91, 14.04 ,12.96, 14.85 )
data<-sort(data)
n=length(data)
###
lambda1= 0.232310
c1= 1.228278
delta1=2.071583
###
lambda2= 0
c2=1.59427
delta2=1.97489
###
lambda3= 0.126751
c3=1.016227
delta3=1
###
lambda4= 0
c4=1.20962
delta4=1
###
lambda5= 7.7173e-07
alpha5= 1.8646e-01
beta5=1.1292e+00
###
alpha6= 1.319356
beta6=0.123189
delta6=0.510267
###
require(gsl)
require(stats)

F_observed<-c(rep(0,n))
```

```
m=n+0.25
for (i in 1:n)
{
  F_observed[i]<-(i-0.375)/m
}

GLLLoG_cdf <- function(data,lambda,c,delta,n) {
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]

F[i]<- pgamma(-log(1-(1-(((1+lambda +lambda*x)*exp(-lambda*x))
/((1+lambda)*(1+x^c)))),delta)/gamma(delta)

  }
  return(F)
}
###

LW_cdf <- function(data,lambda,alpha,beta,n) {
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
F[i]<- 1-(exp((-lambda*x)-(alpha*x^beta)))
/((1+lambda)*(1+lambda+lambda*x))
  }
  return(F)
}
###

EPL_cdf <- function(data,alpha,beta,delta,n) {
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
F[i]<- (1-((1+beta*x^alpha)/(1+beta))*exp(-beta*x^alpha))^delta
  }
  return(F)
}
###

F1<-GLLLoG_cdf(data,lambda1,c1,delta1,n)
F1
F2<-GLLLoG_cdf(data,lambda2,c2,delta2,n)
F2
F3<-GLLLoG_cdf(data,lambda3,c3,delta3,n)
F3
F4<-GLLLoG_cdf(data,lambda4,c4,delta4,n)
F4
```

```
F5<-LW_cdf(data,lambda5,alpha5,beta5,n)
F5
F6<-EPL_cdf(data,alpha6,beta6,delta6,n)
F6

SS<-function(F,F_line,n){
  SS<-0
  for (i in 1:n)
  {
    SS<-SS+(F[i]-F_line[i])^2
  }
  return(SS)
}

###

SS1<-SS(F1,F_observed,n)
SS1
SS2<-SS(F2,F_observed,n)
SS2
SS3<-SS(F3,F_observed,n)
SS3
SS4<-SS(F4,F_observed,n)
SS4
SS5<-SS(F5,F_observed,n)
SS5
SS6<-SS(F6,F_observed,n)
SS6
op <- par(mfrow=c(1,1))
plot(F_observed,F_observed,type='l',xlab="Observed probability"
,ylab="Expected probability")
lines(F_observed,F1,col=2)
lines(F_observed,F2,col=3)
lines(F_observed,F3,col=4)
lines(F_observed,F4,col=5)
lines(F_observed,F5,col=6)
lines(F_observed,F6,col=7)

legend("bottomright",pch=1,cex=0.8,c("GLLLoG(SS=0.2027957)"
,"GLLoG(SS=0.2572618)","LLLoG(SS= 4.327514)",
"LLoG(SS=4.25635)","LW(SS= 0.2505857)","EPL(SS= 1.060729)")
,col=c(2,3,4,5,6,7),lwd=c(2,2,2,2,2,2))
```