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Development of New Generalized Families of Distributions via Multiple Generators with Applications

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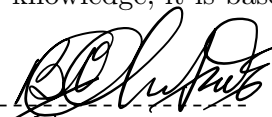


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Abstract

Complex real life phenomena requires robust statistical and mathematical tools that can capture and incorporate many statistical aspects of the processes or events under study with a high degree of accuracy and precision. Classical and other existing probability distributions in literature have some shortcomings when it comes to capturing the characteristics of various data sets, they suffer lack of flexibility hence poor performance when modelling data sets with different types of the hazard rate functions: decreasing, increasing, bathtub, upside-down bathtub and also different levels of skewness and kurtosis as often exhibited by many data sets in real life. In light of the above, we proposed and developed several new generalized families of distributions via new and multiple combinations of generators. We developed the Exponentiated Half Logistic-Weibull-Topp-Leone-G family of distributions, the Weibull-Topp-Leone-G power series class of distributions, the Exponentiated Half Logistic-Type II Topp-Leone-G family of distributions, the Marshall-Olkin-Topp-Leone Half Logistic-G family of distributions, the Marshall-Olkin-Generalized Exponentiated Half Logistic-G family of distributions, the Marshall-Olkin-Odd Exponential Topp-Leone-G family of distributions and the Burr III-Exponentiated Half Logistic-G family of distributions. The use of combinations of multiple generators enabled us to generate new families of distributions which have a strong capacity to handle or capture the important characteristics (skewness and kurtosis) of data thereby adding the much needed flexibility in many applied areas. Mathematical and statistical properties of these generalized distributions including series expansion of probability density functions, hazard rate functions, quantile functions, moments, conditional moments, probability weighted moments, stochastic orders, Rényi entropy and distribution of order statistics are presented. Monte-Carlo simulation algorithms were developed for the new families of distributions. We used the maximum likelihood estimation technique to estimate parameters of our models. We assessed the consistency of our estimators via Monte-Carlo simulations. Applications to real life data have been used to demonstrate the applicability of the new generalized families of distributions. This work will help practitioners in various fields of study to make more accurate decisions.

Dedications

To my wife Miracle C Sengweni, son Lee-Kadesh Keith Sengweni, my parents.



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- (iii) Sengweni, W., Oluyede, B. and Chamunorwa, S., The Type II Half Logistic-Topp-Leone-G Family of Distributions, (Thailand Statistician).
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List of Abbreviations

Abbreviation	Definition
A^*	Anderson-Darling
ABias	Average bias
AIC	Akaike Information Criterion
BIC	Bayesian Information Criterion
CAIC	Corrected Akaike Information Criterion
cdf	Cumulative Distribution Function
CoD	Class of Distributions
CIs	Confidence Intervals
EHL	Exponentiated Half Logistic
Exp-G	Exponentiated-G
FoD	Family of Distributions
GoF	goodness-of-fit
hrf	hazard rate function
i.i.d	Independent and Identically Distributed
K-M	Kaplan-Meier
K-S	Kolmogorov-Smirnov
LR	Likelihood Ratio
MCS	Monte Carlo Simulations
MLEs	Maximum Likelihood Estimates
mgf	moment generating function
pdf	Probability Density Function
PWMs	Probability Weighted Moments
qf	quantile function
RE	Rényi Entropy
RMSE	Root Mean Square Error
rv	Random Variable
rvs	Random Variables
SE	Standard Error
SS	Sum of Squares
TTT	Total Time on Test
Var-Cov	Variance-Covariance
W^*	Cramér-von Mises

1. Introduction

1.1 Background of the Study

In this section, we present the background of the study and a comprehensive literature review.

Probability distributions have a very important role to play when it comes to defining and explaining real-world phenomena. They enable us to quantify the likelihood of certain events happening. They also allow us to estimate the variability of the occurrence of an event. It is a goal of every practitioner to give accurate and precise conclusions about an event or situation under study. In many real life applications probability distributions act as crucial vehicles towards achieving accurate results.

The last years have been characterised by a criticism of classical models in data modelling due to their failure to capture the desired characteristics of the data. The classical models include Half logistic distribution, Weibull distribution, Burr type distributions, Topp-Leone distribution, Exponential distribution, and Rayleigh distribution.

Motivated by the search for more appropriate models which are able to cope with various data sets, we developed some new generalized families that can capture the desired data characteristics and explain tail variations better than conventional models. There is a clear need for flexible models that can assist practitioners to make more precise and accurate decisions in financial modelling, survival analysis, climate modelling, change detection, hydrology, engineering and many other areas of applications. These flexible models can capture various shapes and data sets with different levels of skewness and kurtosis.

In the past two years, researchers developed new models and applied them in the study of the novel Covid-19 pandemic indicating how instrumental probability distributions are in assisting practitioners understand events and their dynamics.

Balakrishnan [20] developed the Half Logistic distribution by folding the standard Logistic distribution, its alternative name is the Folded Logistic distribution. This distribution has found applications in modelling failure times of machine components, it has a uni-modal or reversed-J density function shape, which is a constraint in lifetime data analysis since most real life data shows hazard rate curves that are non-monotonic. These restrictions necessitate the modification and generalization of the half logistic distribution in order to increase its flexibility and versatility in data analysis. Several generalizations of this model have been put forward and these includes: the type I generalized half logistic distribution by Olapade [97], the Kumaraswamy type 1 half logistic family by El-sayed and Mahmoud [55], a power transformation of the half Logistic distribution by Krishnanrani [72], the EHL family by Cordeiro et al. [45], a new extension of the generalized half logistic distribution by Muhammad and Liu [95], the Odd EHL-G family by Afify et al. [1] and the type 1 half logistic family by Cordeiro et al. [44].

Another classical model that is widely used in failure time analysis is the Weibull distribution by Weibull [125]. It was developed by taking the powers of exponentially distributed random variables. The resulting model proved to be more flexible than the classical exponential distribution [85], however, it only captures constant, decreasing and increasing hazard rate which is not enough because a lot of real life phenomena depicts both monotonic and non-monotonic hazard rate functions. There are numerous Weibull distribution generalizations in literature including work done by Tahir et al. [123], Cordeiro et al. [43], Mudholkar et al. ([91], [92]), Bourguignon et al. [30], Oluyede et al. [103], Gusmao et al. [66], Carrasco et al. [33], Famoye et al. [58], Asgharzadeh et al. [12], Percontini et al. [107], Foya et al. [59], Makubate et al. [86], Cordeiro et al. [48], Cordeiro et al. [49] and Alizadeh et al. [10].

The Topp-Leone distribution is a widely used classical model. It was developed by Topp and Leone [124]; it is a one parameter model with a bathtub hazard rate function. This model has a closed form of the cdf and this makes it a better alternative for the Beta distribution since they have the same support $[0,1]$. The Topp-Leone distribution has been widely used in the analysis of many real life scenarios including in biology, ecology, economics. Efforts to improve the performance and general flexibility of the Topp-Leone distribution has seen many researchers extending it, extensions of this model include, the Topp-Leone-G distribution by Rezaei et al. [112], the generalized Topp-Leone-G distribution by Mahdavi [78], the new power Topp-Leone-G distribution by Bantan et al. [23], type II power Topp-Leone-G by Bantan et al. [24], type II Topp-Leone-G distribution by Elgarhy et al. [56], the Weibull-Topp-Leone-G distribution by Karamikabir et al. [71], the Sine Topp-Leone-G distribution by Al-Babtain et al. [4], the Topp-Leone-Gompertz-G distribution by Oluyede et al. [105], the Topp-Leone-Marshall-Olkin-G distribution by Chipepa et al. [37], the Topp-Leone-Nadarajah-Haghighi-G distribution by Yousof and Korkmaz [126], a new extension of the Topp-Leone family by Muhammad et al. [94].

The Burr III distribution is also one of the classical models that was studied and extended in the past years. It was introduced by Burr [32] to model lifetime data or survival data. The Burr III distribution has been widely applied in many scientific fields ranging from income, wealth and wage distributions, forestry, meteorology, survival analysis, environmental studies and reliability theory (Burr and Cislak [31], Mielke [87], Mokhlis [88] and Gove et al. [62]). Kanak and Vinod [80] introduced a new unit Burr III distribution. A lot of work has been done in improving the flexibility of this model and this includes work done by Chipepa et al. [42], Peter et al. [108], Bhatti et al. [28], Moakofi et al. [89].

Another interesting development in the field of distribution theory is the development of compounded CoD which were introduced by Adamidis and Loukas [17] with the main idea being to model the lifetime of a system with N discrete components and positive continuous lifetimes. Power series models have found applications in the fields of reliability analysis, finance, actuarial sciences and engineering. The past few decades have seen researchers extending this CoD and this include the compound class of Weibull and power series distributions by Morais et al. [79], the exponentiated power generalized Weibull power series family of distributions by Aldahlan et al. [13], the odd Weibull-Topp-Leone-G power series family of distributions by Oluyede et al. [101], the exponentiated generalized power series family of distributions by Oluyede et

al. [102], the exponentiated generalized power series family of distributions by Nasiru et al. [96], the complementary extended Weibull power series CoD by Cordeiro et al. [50], the compound class of extended Weibull power series distribution by Silva et al. [117], double bounded Kumaraswamy-power series CoD by Bidram and Nekoukhou [26], generalized exponential power series distribution by Mahmoud et al. [81], compound class of linear failure rate power series distribution by Mahmoud et al. [82], Burr XII power series distribution by Silva et al. [118], generalized linear failure rate power series distribution by Alamatsaz et al. [67].

Numerous techniques for creating or generating new probability distributions have also been developed and studied, these includes the transform-transformer (T-X) by Alzaghal et al. [11], the EHL-G by Cordeiro et al. [53], Weibull-G by Bourguignon et al. [30], the harris-G generator by Pinho et al. [109], gamma transformations by Zografos and Balakrishnan [127], the gamma-G type II generator by Ristić and Balakrishnan [114], the Topp-Leone-G generator by Al-Shomrani et al. [5], and the Odd Burr III-G generator by Alizadeh et al. [7]. According to Barreto-Souza et al. [27], a tried and true method for coming up with flexible and more adaptable FoD is to add extra parameters to existing baseline distributions.

Despite a lot of generalizations in the statistical literature, more can still be done to fill gaps when it comes to the flexibility and versatility of generalized distributions thus, we developed some new generalized classes and families of distributions that can capture the desired data characteristics and explain tail variations better than conventional models. These flexible models have a strong capability of capturing various hazard rate function shapes, modulate tail weight, and data sets with varying levels of the degree of skewness and kurtosis.

1.2 Statement of the Problem

Concerns have been raised over the lack of adequacy of fit by many classical and standard distributions when modelling real life phenomena. Complex real life phenomena requires robust statistical and mathematical tools that can capture and incorporate many statistical aspects of the processes or events under study with a high degree of accuracy and precision. In many applied and practical areas such as lifetime analysis, finance and insurance, there is a clear need for extended and generalized forms of distributions. Therefore, in this research we developed generalized classes and FoD that exhibit a variety of shapes for the pdf including uni-modal, bimodal, right-skewed, left-skewed and the hrf which capture both monotone and non-monotone shapes for example inverted bathtub followed by bathtub, bathtub, inverted bathtub, increasing and decreasing. The new FoD account for tail variations, model extreme values, capture heavy, light, long and short-tailed data characteristics. This study addressed the shortcomings or gaps in the statistical literature on the use of multiple generators to construct and develop new generalized classes and families of distributions that simultaneously cater for varying levels of skewness and kurtosis. This study will significantly assist practitioners in various fields of study to make more accurate decisions.

1.3 Objectives

1.3.1 General Objective

The key aim is to develop new generalized classes and families of distributions using new and multiple combinations of generators and derive their mathematical properties.

1.3.2 Specific Objectives

- To propose and develop new classes and families of distributions via the use of new and multiple generators.
- To characterise the classes and families of distributions via shape, tail variation, skewness and kurtosis.
- To present graphical plots of the pdf and hazard rate functions for the newly developed families of distributions.
- To obtain the mathematical and statistical properties of the proposed models including model comparison via stochastic orders.
- To estimate the parameters of the proposed models using the maximum likelihood estimation technique.
- To develop simulation algorithms and conduct Monte-Carlo simulations for the newly developed models.
- To demonstrate the effectiveness and performance of the proposed models in modelling various types of real life data.
- To use some goodness-of-fit measures to assess the performance of the newly developed models in comparison with other models existing in literature.
- To provide graphical representation of the fits for example fitted densities, Kaplan-Meier survival plots, probability plots, TTT-Transform plots and fitted hazard rate function curves.
- To present profile log-likelihood plots to show that our estimated parameters are global estimates thereby addressing issues of identifiability.

1.4 Motivations and Justifications

Our main motivations for developing these new generalized families of distributions are:

- The need for models that are able to capture and explain various characteristics of data,
- The need to improve the flexibility of the classical models considered when it comes to fitting and modelling real-life data,

- To obtain models with various shapes of the hazard rate functions, that is, models which can capture data with monotonic and non-monotonic failure rates such as bathtub, inverted bathtub, increasing and decreasing shapes,
- To produce models with consistently better fits than other generated distributions under the same underlying distribution,
- The general need for improved models in many scientific applications for example insurance, medicine, biology, engineering, climate modelling.

1.5 Some Generalized Classes of Distributions

Generators are functions or generalizations of classical distributions which one can use to generate or develop new classes and families of distributions. These functions must satisfy certain conditions to guarantee that the end product is really a probability distribution functions. The conditions includes, differentiability, monotonicity and right continuity. Some generators of generalized classes and families of distributions are briefly discussed in this section.

1.5.1 EHL-G Generator

The EHL-G distribution was developed by Cordeiro et al. [53]. The cdf and pdf of the EHL-G distribution are

$$F_{EHL-G}(t; \alpha, \varphi) = \left[\frac{1 - [\bar{G}(t; \varphi)]^\lambda}{1 + [\bar{G}(t; \varphi)]^\lambda} \right]^\alpha \quad (1.1)$$

and

$$f_{EHL-G}(t; \alpha, \varphi) = 2\alpha\lambda g(t; \varphi)[1 - G(t; \varphi)]^{\lambda-1} \frac{[1 - [\bar{G}(t; \varphi)]^\lambda]^{\alpha-1}}{[1 + (\bar{G}(t; \varphi))^\lambda]^{\alpha+1}}, \quad (1.2)$$

respectively, for $t > 0$, $\alpha > 0$ and parameter vector φ , with $\bar{G}(t; \varphi) = 1 - G(t; \varphi)$. If we set $\alpha = 1$ in equation (1.1), we get the half logistic-G generator by Cordeiro et al. [44].

1.5.2 Harris-G Generator

Pinho et al. [109] developed the Harris-G CoD with the cdf and pdf given as

$$F(x; \theta, k, \varphi) = 1 - \left[\frac{\theta \bar{G}^k(x; \varphi)}{1 - \theta \bar{G}^k(x; \varphi)} \right]^{\frac{1}{k}}, \quad (1.3)$$

and

$$f(x; \theta, k, \varphi) = \frac{\theta^{\frac{1}{k}} g(x; \varphi)}{[1 - \theta \bar{G}^k(x; \varphi)]^{1 + \frac{1}{k}}}, \quad (1.4)$$

respectively, for $\theta, k > 0$ and parameter vector φ .

1.5.3 Competing Risk Method

With this method, a new FoD is created by combining the sfs $S_{X_1}(x)$ and $S_{X_2}(x)$ of two continuous independent rvs, X_1 and X_2 , respectively. If X is set to be $\min(X_1, X_2)$, then the cdf of X is given by

$$\begin{aligned} F_X(x; \varphi) &= 1 - [1 - F_1(x; \varphi_1)][1 - F_2(x; \varphi_2)] \\ &= 1 - (S_{X_1}(x; \varphi_1))(S_{X_2}(x; \varphi_2)), \end{aligned} \quad (1.5)$$

where $\varphi = (\varphi_1, \varphi_2)$ is the vector of model parameters. For more on this method, see Codeiro et al. [46], Oluyede et al. [104] and Jiang [70].

1.5.4 Transform-Transformer Method

Alzaghal et al. [11], created the transform-transformer method which is defined as follows. The cdf and pdf are

$$F(x) = \int_a^{W(G(x; \varphi))} r(t) dt, \quad (1.6)$$

and

$$f(x) = \left\{ \frac{d}{dx} W[G(x; \varphi)] \right\} r\{W[G(x; \varphi)]\}, \quad (1.7)$$

respectively, where $r(t)$ is the pdf of a r.v $T \in [a, b]$ for $-\infty \leq a < b \leq \infty$ and the function $W[G(x; \varphi)]$ is the cdf of any rv and satisfies the conditions:

- (1) $W(G(x; \varphi)) \in [a, b]$,
- (2) $W(G(x; \varphi))$ is differentiable and monotonically non-decreasing, and
- (3) $W(G(x; \varphi)) \rightarrow a$ as $x \rightarrow -\infty$ and $W(G(x; \varphi)) \rightarrow b$ as $x \rightarrow \infty$.

1.5.5 Beta-G Generator

If $g(x; \varphi)$ and $G(x; \varphi)$ are assumed to be the pdf and cdf of a continuous rv X respectively; then the cdf and pdf of the beta-G FoD are given by

$$F(x; \alpha, \beta, \varphi) = I_{G(x; \varphi)}(\alpha, \beta) = \frac{B_{G(x; \varphi)}(\alpha, \beta)}{B(\alpha, \beta)}, \quad (1.8)$$

and

$$f(x; \alpha, \beta, \varphi) = \frac{1}{B(\alpha, \beta)} [G(x; \varphi)]^{\alpha-1} [1 - G(x; \varphi)]^{\beta-1} g(x; \varphi), \quad (1.9)$$

respectively, for $\alpha > 0, \beta > 0$, where $B_{G(x; \varphi)}(\alpha, \beta) = \int_0^{G(x; \varphi)} x^{\alpha-1} (1-x)^{\beta-1} dx$ is the incomplete beta function, $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$ is the beta function, $\Gamma(\cdot)$ is the gamma function, and φ is the vector of the baseline cdf G .

1.5.6 Kumaraswamy-G (Kw-G) Generator

The Kw-G generator was developed by Cordeiro and de Castro [52]. Its cdf is given by

$$K(x; \gamma, \theta; \varphi) = 1 - (\Psi(x; \gamma, \varphi))^\theta, \quad (1.10)$$

and its pdf $k(x, \gamma, \theta, \varphi)$ is given by

$$k(x; \gamma, \theta, \varphi) = \gamma \theta g(x; \varphi) (G(x; \varphi))^{\gamma-1} \Psi(x; \gamma, \varphi)^{\theta-1} \quad (1.11)$$

for $x > 0, \gamma > 0, \theta > 0$ and baseline cdf $G(x; \varphi)$, where $\Psi(x; \gamma, \varphi) = (1 - G(x; \varphi)^\gamma)^\theta$ and φ a vector of parameters.

1.5.7 Marshall-Olkin-G Generator

The Marshall-Olkin-G generator was put forward by Marshall and Olkin [85]. The cdf and pdf of new FoD are

$$F_{MO-G}(x, \theta, \varphi) = 1 - \frac{\theta \bar{G}(x; \varphi)}{1 - \theta \bar{G}(x; \varphi)}, \quad (1.12)$$

and

$$f_{MO-G}(x, \theta, \varphi) = \frac{\theta g(x; \varphi)}{(1 - \theta \bar{G}(x; \varphi))^2}, \quad (1.13)$$

respectively, where θ is a tilt parameter and $G(x; \varphi)$ is the baseline cdf.

1.5.8 Compounding Method

This concept was put forward by Adamidis and Loukas [17] and the main idea is modelling of the lifetime of a system with N (discrete components) and the positive continuous rv say, X_i (the lifetime of the i^{th} component), which can be denoted by a non-negative rv $X = X_{(1)} = \min(X_1, \dots, X_N)$ or $X = X_{(n)} = \max(X_1, \dots, X_N)$ based on whether the components of the system are in series or a parallel structure. For the compounding method of generating distributions, there are two scenarios to consider depending on the structure of the components.

Case: 1

Suppose we define the rv $X = X_{(1)} = \min(X_1, \dots, X_N)$, where the cdf of X_i is $G(x; \psi)$, $\psi = (\psi_1, \psi_2, \dots, \psi_k)$, $k \geq 1$, $\psi > 0$ and N is assumed to be discrete rv truncated at zero with a probability mass function (pmf) $P(N = n)$, we obtain the compounded CoD as follows:

Step: 1. Obtain the conditional cdf of $X_{(1)}|N = n$ as

$$G_{X_{(1)}|N=n}(x; \psi) = 1 - [\bar{G}(x; \psi)]^n. \quad (1.14)$$

where $\bar{G}(x; \psi)$ is the survival function of the continuous rv X_i .

Step: 2. Obtain the joint cdf as

$$P(X_{(1)} \leq x, N = n) = P(N = n) (1 - [\bar{G}(x; \psi)]^n). \quad (1.15)$$

Step: 3. The marginal cdf of $X_{(1)}$ is given by

$$F(x; \psi) = \sum_{n=1}^{\infty} P(N = n) (1 - [\bar{G}(x; \psi)]^n), x > 0. \quad (1.16)$$

Case: 2

If we define the rv $X = X_{(n)} = \max(X_1, \dots, X_N)$, where the cdf of X_i is $G(x; \psi)$, $\psi = (\psi_1, \psi_2 \dots \psi_k)$, $k \geq 1$, $\psi > 0$ and N is assumed to be discrete rv truncated at zero with a pmf $P(N = n)$, we obtain the compounded CoD as follows:

Step: 1. Obtain the conditional cdf of $X_{(n)}|N = n$ as

$$G_{X_{(n)}|N=n}(x; \psi) = [G(x; \psi)]^n. \quad (1.17)$$

Step: 2. Obtain the joint cdf as

$$P(X_{(n)} \leq x, N = n) = P(N = n) [G(x; \psi)]^n. \quad (1.18)$$

Step: 3. The marginal cdf of $X_{(n)}$ is given by

$$F(x; \psi) = \sum_{n=1}^{\infty} P(N = n) [G(x; \psi)]^n, x > 0. \quad (1.19)$$

1.5.9 Gamma Transformations

We have the gamma-generator which was developed by Zografos and Balakrishnan [127], and its cdf and pdf are given as:

$$F(x; \delta, \psi, \zeta) = \frac{1}{\Gamma(\delta)\psi^\delta} \int_0^{-\log(1-G(x;\zeta))} t^{\delta-1} e^{-t/\psi} dt = \frac{\gamma(\delta, -\psi^{-1} \log(1-G(x;\zeta)))}{\Gamma(\delta)}, \quad (1.20)$$

and

$$f(x; \delta, \psi, \zeta) = \frac{1}{\Gamma(\delta)\psi^\delta} [-\log(1-G(x;\zeta))]^{\delta-1} (1-G(x;\zeta))^{\frac{1}{\psi}-1} g(x; \zeta), x \in R, \delta > 0, \quad (1.21)$$

respectively, where $\gamma(x, \delta) = \int_0^x t^{\delta-1} e^{-t} dt$ is the incomplete gamma function. We take $\psi = 1$.

The other gamma generator, the RB-G (called the gamma-G type II) by Ristić and Balakrishnan [114] is given as

$$\begin{aligned} F(x; \delta, \zeta) &= 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\ln(G(x;\zeta))} t^{\delta-1} e^{-t} dt \\ &= 1 - \frac{\gamma(\delta, -\log(G(x;\zeta)))}{\Gamma(\delta)}, \end{aligned} \quad (1.22)$$

where $\gamma(x, \delta) = \int_0^x t^{\delta-1} e^{-t} dt$ is the incomplete gamma function.

1.5.10 Topp-Leone-G Generator

The Topp-Leone-G FoD was put forward by Al-Shomrani et al. [5]. Its pdf and cdf are given as

$$f_{TL-G}(t; \gamma, \zeta) = 2\gamma g(t; \zeta) \bar{G}(t; \zeta) \left(1 - \bar{G}^2(t; \zeta)\right)^{\gamma-1} \quad (1.23)$$

and

$$F_{TL-G}(t; \gamma, \zeta) = \left[1 - \bar{G}^2(t; \zeta)\right]^{\gamma}, \quad (1.24)$$

for $\gamma > 0$, $\bar{G}(x; \zeta) = 1 - G(x; \zeta)$ with parameter vector ζ .

1.5.11 Type II Topp-Leone-G Generator

The Type II Topp-Leone-G FoD was developed by Elgarhy et al. [56], its cdf and pdf are given by

$$F_{TIITL-G}(x; \gamma, \zeta) = 1 - \left[1 - G^2(x; \zeta)\right]^{\gamma}, \quad (1.25)$$

and

$$f_{TIITL-G}(x; \gamma, \zeta) = 2\gamma g(x; \zeta) G(x; \zeta) \left[1 - G^2(x; \zeta)\right]^{\gamma-1}, \quad (1.26)$$

respectively, for $\gamma > 0$ and parameter vector ζ .

1.5.12 Burr III-G Generator

The Burr III distribution was introduced by Burr [32], and the cdf and pdf of the Odd Burr III-G FoD as proposed by Alizadeh et al. [7] is given by

$$F(w; a, b, \xi) = \int_0^{\frac{G(w; \xi)}{1 - G(w; \xi)}} abp^{-b-1} (1 + p^{-a})^{-b-1} dp = \left[1 + \left(\frac{1 - G(w; \xi)}{G(w; \xi)}\right)^a\right]^{-b},$$

and

$$f(w; a, b, \xi) = abg(w; \xi) \frac{(1 - G(w; \xi))^{a-1}}{(G(w; \xi))^{a+1}} \left[1 + \left(\frac{1 - G(w; \xi)}{G(w; \xi)}\right)^a\right]^{-b-1} \quad (1.27)$$

for $a > 0, b > 0, w > 0$ and ξ a parameter vector.

1.6 Some Statistical Properties

Some of the statistical properties developed in this thesis include the quantile function, moments, probability weighted moments, Rényi entropy, order statistics.

1.6.1 Quantile Function

Quantile function is obtained by inverting the cumulative distribution function. It is useful for generating random numbers from a model and it helps in simulations, estimating percentiles and outlier detection.

1.6.2 Moments

Moments are quantitative measures that describe specific characteristics of a probability distribution. There are several types of moments which can be calculated and each provides different information about the data set for example central tendencies, dispersion, skewness and kurtosis. They are also useful in parameter estimation.

1.6.3 Probability Weighted Moments (PWMs)

PWMs are expectations of functions of random variables and can be defined for any rv whose mean exists. PWMs were devised by Greenwood et al. [64] as a way of estimating parameters. Estimates obtained via PWMs are considered to be superior compared to standard moment-based estimates. Besides being used for parameter estimation, PWMs are useful in the summarization and description of theoretical probability distributions. PWMs are more robust than the standard moments in the presents of outliers.

1.6.4 Entropy

An entropy is a measure of the predictability or uncertainty of a rv. The Rényi entropy (Rényi [113]) is an extension of the Shannon entropy (Shannon [120]) and these are the well known and commonly used measures of uncertainty. This measure plays an important role many areas such as statistical ecology, statistical mechanics, physics, chemistry and information theory.

1.6.5 Order Statistics

Order statistics are very useful in statistical sciences. They play an important role in reliability, information theory, estimation of parameters of distributions. Order statistics also allow us to summarize data, detect outliers, model extreme events and make inferences.

1.7 Outline of the Study

Work in this thesis is organised into seven chapters. Chapters 2-6 presents full research papers which have been either published, accepted or submitted for consideration for publication in peer-reviewed journals.

In Chapter 2, we present the EHL-Weibull Topp-Leone-G (EHL-WTL-G) FoD. In Chapter 3, we present the EHL-Type II Topp-Leone-G (EHL-TIITL-G) FoD. Its derivation, statistical properties and applications are presented in this chapter. In Chapter 4, we presents the Marshall-Olkin-Topp-Leone Half Logistic-G (MO-TLHL-G) FoD, its derivations, statistical properties, simulations and applications. In Chapter 5, we present the Marshall-Olkin-Generalized EHL-G

(MO-GEHL-G) FoD and the Marshall-Olkin-Odd Exponential Topp-Leone-G (MO-OETL-G) FoD. In Chapter 6, we presented the Weibull Topp-Leone-G Power Series (WTL-GPS) CoD and the Burr III-EHL-G (BIII-EHL-G) FoD. Finally, conclusions and areas for further studies are presented in Chapter 7.

1.8 Summary

In this Chapter, we presented a general introduction to the study and literature review. Some generators of probability distributions including the Exponentiated Half-Logistic-G generator, the Harris-G generator, Competing risks method, T-X method, Beta-G generator, Kumaraswamy-G generator, Marshall-Olkin-G generator, Compounding method, Gamma transformations, Topp-Leone generator, Type II Topp-Leone-G generator and Burr III generator are given. Objectives, motivations and justifications of the study are also presented in this chapter.



2. The Exponentiated Half Logistic-Weibull Topp-Leone-G Family of Distributions

Sengweni, W. and Oluyede, B., The Exponentiated Half Logistic-Weibull-Topp-Leone-G Family of Distributions, (*Accepted: Springer book chapter*).

2.1 Introduction

In this chapter, we developed and studied a new FoD called the EHL-Weibull-Topp-Leone-G (EHL-WTL-G) FoD. The half logistic distribution was developed by Balakrishnan [20] and has application in many scientific areas, however, it has a unimodal or reversed-J density function shape which acts as a limitation in modelling complex data sets. The Weibull distributions is one of the widely used model in survival analysis, engineering, biology, hydrology, economics, finance, to mention a few. Recently, we have seen many authors developing some generalizations of the half logistic distribution so as to improve its performance in data fitting and this includes among others work done by Aldahlan and Afify [18], Chipepa et al. ([39], [40]), Afify et al. [1], Sengweni et al. [119] and Oluyede et al. [98].

Our main motivation for developing this new FoD is to improve the performance of the half logistic, Weibull and the Topp-Leone classical distributions when it comes to modelling lifetime data with monotonic and non-monotonic failure rates. By combining the properties of the aforementioned classical distributions, we managed to develop a new FoD which can capture a variety of shapes making it suitable for modelling complex and highly skewed data sets. The EHL-WTL-G FoD possess a wide range of shapes for the hrfs including inverted bathtub, bathtub, decreasing and increasing shapes. Worth noting is that the new FoD contains some other new FoD.

Results in this chapter are organized as follows. In Section 2.2, we present the EHL-Weibull-Topp-Leone-G (EHL-WTL-G) FoD and its sub-families, hrf and the qf. The moments and PWMs are given in Section 2.3. Section 2.4 contains results on the distribution of order statistics and RE. Some special cases are given in Section 2.5. In Section 2.6, we present results on the estimation of the parameters of the EHL-WTL-G family of distribution via the method of maximum likelihood. To investigate the ABias and RMSE of the maximum likelihood estimators, we performed MCS and the findings are given in Section 2.7. Applications are given in Section 2.8, followed by some concluding remarks in Section 2.9.

2.2 The Generalized Family of Distributions, Sub-Families, Hazard Rate and Quantile Functions

In this part, we present some properties of the EHL-Weibull Topp-Leone-G (EHL-WTL-G) FoD such as special cases, expansion of the density, hrf and qf.

2.2.1 The Generalized Family of Distributions

We consider the Weibull Topp-Leone-G distribution obtained via T-X transformer with $W(G(x; \psi)) = -\log(K_G(x; b, \psi))$ and set $\lambda = 1$ in equation (1.1) to obtain the new family of EHL-WTL-G distributions. The cdf and pdf of the proposed FoD are respectively given by

$$F(x; \alpha, b, \beta, \psi) = \left[\frac{1 - \exp \left\{ - \left(-\log(K_G(x; b, \psi)) \right)^\beta \right\}}{1 + \exp \left\{ - \left(-\log(K_G(x; b, \psi)) \right)^\beta \right\}} \right]^\alpha \quad (2.1)$$

and

$$\begin{aligned} f(x; \alpha, b, \beta, \psi) &= \frac{4b\alpha\beta g(x; \psi) \overline{G}(x; \psi) \left(1 - \overline{G}^2(x; \psi)\right)^{b-1} (-\log[K_G(x; b, \psi)])^{\beta-1}}{K_G(x; b, \psi) (1 + \exp\{-(-\log(K_G(x; b, \psi)))^\beta\})^{\alpha+1}} \\ &\times \left(1 - \exp\{-(-\log(K_G(x; b, \psi)))^\beta\}\right)^{\alpha-1} \\ &\times \exp\{-(-\log[K_G(x; b, \psi)])^\beta\}, \end{aligned} \quad (2.2)$$

where $K_G(x; b, \psi) = 1 - [1 - \overline{G}^2(x; \psi)]^b$, $\overline{G}(x; \psi) = 1 - G(x; \psi)$, $b, \alpha, \beta > 0$ and ψ is a vector of parameters from the baseline cdf G . The hrf is given by

$$\begin{aligned} h(x; \alpha, b, \beta, \psi) &= \frac{4b\alpha\beta g(x; \psi) \overline{G}(x; \psi) \left(1 - \overline{G}^2(x; \psi)\right)^{b-1} (-\log[K_G(x; b, \psi)])^{\beta-1}}{K_G(x; b, \psi) (1 + \exp\{-(-\log[K_G(x; b, \psi)])^\beta\})^{\alpha+1}} \\ &\times \left[1 - \frac{1 - \exp \left\{ - \left(-\log[K_G(x; b, \psi)] \right)^\beta \right\}}{1 + \exp \left\{ - \left(-\log[K_G(x; b, \psi)] \right)^\beta \right\}} \right]^\alpha \\ &\times \left(1 - \exp\{-(-\log(K_G(x; b, \psi)))^\beta\}\right)^{\alpha-1} \\ &\times \exp\{-(-\log[K_G(x; b, \psi)])^\beta\}. \end{aligned}$$

2.2.2 Density Function Expansion

The pdf of the EHL-WTL-G FoD is expressed as an infinite linear combination of Exp-G densities in this section. We applied the generalized binomial series expansion, the exponential series expansion and the logarithmic series expansion given in equation (2.3) to express the the pdf given in equation (2.2) as an infinite linear combination of Exp-G densities given in equation

(2.4). Note that

$$[-\log(1-y)]^{\beta p + \beta - 1} = y^{\beta p + \beta - 1} \sum_{m=0}^{\infty} \binom{\beta p + \beta - 1}{m} y^m \left(\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^m, \quad (2.3)$$

with $y = (1 - \bar{G}(x; \psi))^b$, $0 < y < 1$, and by making use of the result from a power series raised to positive integer with $a_s = (s+2)^{-1}$, that is, $\left[\sum_{s=0}^{\infty} a_s y^s \right]^m = \sum_{s=0}^{\infty} b_{s,m} y^s$, where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^s [m(l+1) - s] a_l b_{s-l,m}$ and $b_{0,m} = a_0^m$ (Gradshteyn and Ryzhik [63]), the pdf of the EHL-WTL-G FoD can be written as

$$\begin{aligned} f(x; \alpha, b, \beta, \psi) &= 4b\alpha\beta g(x; \psi) \bar{G}(x; \psi) \sum_{k,v,p,m,s=0}^{\infty} \frac{(-1)^{k+p}(k+v+1)^p}{p!} \binom{\alpha-1}{k} \\ &\times \binom{-\alpha-1}{v} \binom{\beta p + \beta - 1}{m} b_{s,m} y^{\beta p + \beta + m + s - 1} \\ &\times (1 - \bar{G}^2(x; \psi))^{b-1} (1 - (1 - \bar{G}^2(x; \psi))^b)^{-1} \\ &= 4b\alpha\beta g(x; \psi) \sum_{k,v,p,m,s,n,r=0}^{\infty} \frac{(-1)^{k+p+r}(k+v+1)^p \Gamma(n+1)}{p!n!} \\ &\times \binom{\alpha-1}{k} \binom{-\alpha-1}{v} \binom{\beta p + \beta - 1}{m} \binom{b-1}{r} b_{s,m} \\ &\times (1 - \bar{G}^2(x; \psi))^{b(\beta p + \beta + m + s + n - 1)} \bar{G}^{2r+1}(x; \psi) \\ &= 4b\alpha\beta g(x; \psi) \sum_{k,v,p,m,s,n,r=0}^{\infty} \frac{(-1)^{k+p+r+l}(k+v+1)^p \Gamma(n+1)}{p!n!} \\ &\times \binom{\alpha-1}{k} \binom{-\alpha-1}{v} \binom{\beta p + \beta - 1}{m} \binom{b-1}{r} b_{s,m} \\ &\times \binom{b(\beta p + \beta + m + s + n - 1)}{q} \binom{2(r+q)+1}{l} G^l(x; \psi) \\ &= \sum_{l=0}^{\infty} \omega_{l+1} h_{l+1}(x; \psi), \end{aligned} \quad (2.4)$$

where $h_{l+1}(x; \psi) = (l+1)g(x; \psi)[G(x; \psi)]^l$ is the Exp-G pdf with the power parameter $(l+1)$ and

$$\begin{aligned} \omega_{l+1} &= 4b\alpha\beta \sum_{k,v,p,m,s,n,r,q=0}^{\infty} \frac{(-1)^{k+p+r}(k+v+1)^p \Gamma(n+1)}{p!n!(l+1)} b_{s,m} \\ &\times \binom{\alpha-1}{k} \binom{-\alpha-1}{v} \binom{\beta p + \beta - 1}{m} \binom{b-1}{h} \binom{2(r+q)+1}{l} \\ &\times \binom{b(\beta p + \beta + m + s + n - 1)}{q}. \end{aligned} \quad (2.5)$$

2.2.3 Quantile Function

The qf, $Q(u)$, $0 \leq u \leq 1$, of the EHL-WTL-G FoD is obtained by inverting the cdf as follows

$$\left[\frac{1 - \exp \left\{ - \left(-\log(K_G(x; b, \psi)) \right)^\beta \right\}}{1 + \exp \left\{ - \left(-\log(K_G(x; b, \psi)) \right)^\beta \right\}} \right]^\alpha = u,$$

$0 \leq u \leq 1$. This simplifies to

$$-\log[K_G(x; b, \psi)] = (-\log(z))^{\frac{1}{\beta}},$$

where $z = \frac{1-u^{\frac{1}{\alpha}}}{1+u^{\frac{1}{\alpha}}}$. Thus, the qf for the EHL-WTL-G FoD can be written as

$$Q(u) = G^{-1} \left[1 - \left(1 - \left(1 - \exp[-(-\log(z))^{\frac{1}{\beta}}] \right)^{\frac{1}{b}} \right)^{\frac{1}{2}} \right]. \quad (2.6)$$

The qf is very useful and is used for generating some random numbers from the EHL-WTL-G FoD for a specified baseline distribution G .

2.3 Moments

In this section, ordinary moments, incomplete moments and PWMs of the EHL-WTL-G FoD are presented. To come up with the properties mentioned, we make use of the series expansion of the pdf presented in Section 2.2.2. Let Y_{l+1} be an Exp-G distribution with power parameter $(l+1)$, then the r^{th} ordinary moment of the EHL-WTL-G FoD is given by

$$\mu'_r = E(X^r) = \sum_{l=0}^{\infty} \omega_{l+1} E(Y_{l+1}^r), \quad (2.7)$$

where ω_{l+1} is as given in equation (2.5) and $E(Y_{l+1}^r)$ is the r^{th} moment of the Exp-G distribution. The r^{th} incomplete moment of the EHL-WTL-G FoD is given by

$$\delta_r(y) = \int_{-\infty}^y x^r f(x; \alpha, b, \beta, \psi) dx = \sum_{l=0}^{\infty} \omega_{l+1} \int_{-\infty}^y x^r h_{l+1}(x; \psi) dx,$$

where $\int_{-\infty}^y x^r h_{l+1}(x; \psi) dx$ is the r^{th} incomplete moment of the Exp-G distribution.

From the skewness and kurtosis plots of the EHL-WTL-LLoG given in Figure 2.1, it can be seen that the members of the EHL-WTL-G family can model various data types in terms of skewness and kurtosis by varying individual parameter values of the model, for example, for smaller values of b and large values of α we can model highly skewed data sets.

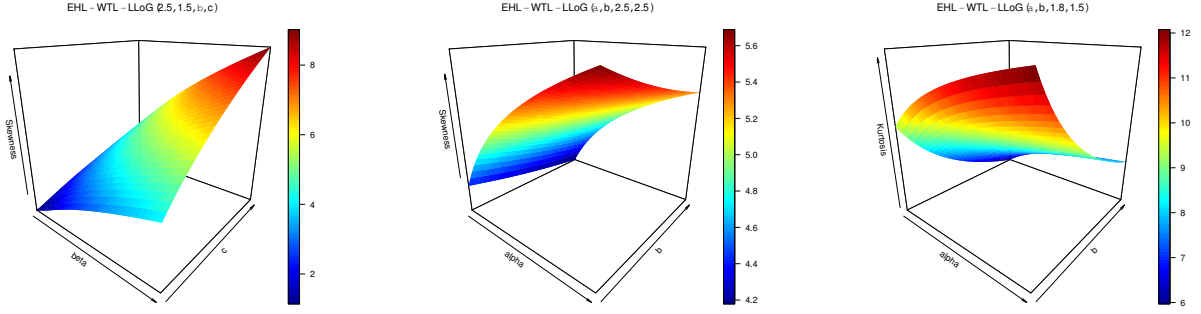


Figure 2.1: Plots of the Skewness and Kurtosis of the EHL-WTL-LLoG

2.3.1 Probability Weighted Moments (PWMs)

The $(i, j)^{th}$ PWMs of a r.v $X \sim EHL - WTL - G$, say $\delta_{i,j}$ is given by

$$\delta_{i,j} = E(X^i (F(X))^j) = \int_{-\infty}^{\infty} x^i f(x) [F(x)]^j dx. \quad (2.8)$$

Note that $f(x)[F(x)]^j$ can be expressed as

$$\begin{aligned} f(x)[F(x)]^j &= 4b\alpha\beta g(x; \psi) \sum_{k,v=0}^{\infty} \sum_{p,m=0}^{\infty} \sum_{s,n,r=0}^{\infty} \sum_{q,l=0}^{\infty} \frac{(-1)^{k+p+q+l} \Gamma(n+1)}{p!n!(k+v+1)^{-p}} b_{s,m} \\ &\times \binom{\alpha(j+1)-1}{k} \binom{-\alpha(j+1)-1}{v} \binom{\beta p + \beta - 1}{m} \binom{b-1}{r} \\ &\times \binom{2(r+q)+1}{l} \binom{b(\beta p + \beta + m + s + n - 1)}{q} G^l(x; \psi). \end{aligned}$$

Therefore,

$$f(x)[F(x)]^j = \sum_{l=0}^{\infty} \Phi_{l+1} h_{l+1}(x; \psi),$$

where

$$\begin{aligned} \Phi_{l+1} &= 4b\alpha\beta \sum_{k,v=0}^{\infty} \sum_{p,m=0}^{\infty} \sum_{s,n=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{k+p+q+l} \Gamma(n+1)}{p!n!(k+v+1)^{-p}(l+1)} b_{s,m} \\ &\times \binom{\alpha(j+1)-1}{k} \binom{-\alpha(j+1)-1}{v} \binom{\beta p + \beta - 1}{m} \binom{b-1}{r} \\ &\times \binom{2(r+q)+1}{l} \binom{b(\beta p + \beta + m + s + n - 1)}{q}. \end{aligned}$$

It follows that the PMW of X reduces to

$$\delta_{i,j} = \sum_{l=0}^{\infty} \Phi_{l+1} \int_{-\infty}^{\infty} x^i h_{l+1}(x; \psi) dx = \sum_{l=0}^{\infty} \Phi_{l+1} E(Y_{l+1}^i), \quad (2.9)$$

where Y_{l+1}^i is the i^{th} power of an Exp-G random variable.

2.4 Order Statistics and Rényi Entropy (RE)

In this section, expressions for the distribution of order statistics and RE are presented.

2.4.1 Order Statistics

Suppose $X_1, X_2, X_3, \dots, X_n$ are i.i.d. EHL-WTL-G r.v.s. Then, the pdf of the i^{th} order statistics $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{j+i-1}, \quad (2.10)$$

where $B(.,.)$ is the beta function. Note that

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= 4b\alpha\beta g(x; \psi) \sum_{k,v=0}^{\infty} (-1)^k \binom{\alpha(j+i)-1}{k} \binom{-\alpha(j+i)-1}{v} \\ &\times \bar{G}(x; \psi) (1 - \bar{G}^2(x; \psi))^{b-1} (-\log(K_G(x; b, \psi)))^{\beta-1} \\ &\times \exp\{-(k+v+1)(-\log[K_G(x; b, \psi)])^\beta\} \\ &\times (K_G(x; b, \psi))^{-1}. \end{aligned}$$

Using the same procedure as presented in the expansion of the density function in Section 2.2.2, the pdf of the i^{th} order statistics, $X_{i:n}$ can be written as

$$\begin{aligned} f_{i:n}(x) &= 4b\alpha\beta g(x; \psi) \sum_{k,v,p,m,s,n,q=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{k+p+q+l+j} (k+v+1)^p \Gamma(n+1)}{B(i, n-i+1) p! n!} b_{s,m} \\ &\times \binom{\alpha(j+i)-1}{k} \binom{-\alpha(j+i)-1}{v} \binom{\beta p + \beta - 1}{m} \binom{b-1}{r} \binom{n-i}{j} \\ &\times \binom{2(r+q)+1}{l} \binom{b(\beta p + \beta + m + s + n - 1)}{q} G^l(x; \psi) \\ &= \sum_{l=0}^{\infty} \psi_{l+1} h_{l+1}(x; \psi), \end{aligned}$$

where

$$\begin{aligned} \psi_{l+1} &= 4b\alpha\beta \sum_{k,v,p,m,s,n,q=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{k+p+q+l+j} (k+v+1)^p \Gamma(n+1)}{B(i, n-i+1) p! n! (l+1)} b_{s,m} \\ &\times \binom{\alpha(j+i)-1}{k} \binom{-\alpha(j+i)-1}{v} \binom{\beta p + \beta - 1}{m} \binom{b-1}{r} \binom{n-i}{j} \\ &\times \binom{2(r+q)+1}{l} \binom{b(\beta p + \beta + m + s + n - 1)}{q}. \end{aligned}$$

2.4.2 Rényi Entropy (RE)

The RE (Rényi [113]) is defined to be

$$I_R(\theta) = \frac{1}{1-\theta} \log \left(\int_0^\infty [f(x)]^\theta dx \right), \theta \neq 1, \theta > 0. \quad (2.11)$$

Using the following expansions $(1-z)^\alpha = \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} z^k$, $(1+z)^\alpha = \sum_{k=0}^\infty \binom{\alpha}{k} z^k$, $e^x = \sum_{k=0}^\infty \frac{z^k}{k!}$, and the series expansion

$$[-\log(1-y)]^{\beta p + \theta(\beta-1)} = y^{\beta p + \theta(\beta-1)} \sum_{m=0}^\infty \binom{\beta p + \theta(\beta-1)}{m} y^m \left(\sum_{s=0}^\infty \frac{y^s}{s+2} \right)^m,$$

with $y = (1 - \overline{G}(x; \psi))^b$, $0 < y < 1$, $f^\theta(x)$ can be written as:

$$\begin{aligned} f^\theta(x) &= (4b\alpha\beta)^\theta g^\theta(x; \psi) \sum_{k,v,p,m,s,n,r,q=0}^\infty \frac{(-1)^{k+p+r+n} (k+v+\theta)^p}{p!} \binom{\theta(\alpha-1)}{k} b_{s,m} \\ &\times \binom{-\theta(\alpha+1)}{v} \binom{\theta(b-1)}{r} \binom{-\theta}{n} \binom{\beta p + \theta(\beta-1)}{m} \binom{2(r+q)+\theta}{l} \\ &\times \binom{b(\beta p + \theta(\beta-1) + m + s + n)}{q} G^l(x; \psi). \end{aligned}$$

Thus, the RE of the EHL-WTL-G FoD can be expressed in terms of the entropy of the Exp-G distribution as follows

$$I_R(\theta) = \frac{1}{1-\theta} \log \left[\sum_{z=0}^\infty \phi_z e^{(1-\theta)I_{REG}} \right], \quad (2.12)$$

where

$$\begin{aligned} \phi_z &= (4b\alpha\beta)^\theta \sum_{k,v,p,m,s,n,r,q=0}^\infty \frac{(-1)^{k+p+r+n} (k+v+\theta)^p}{p!} \binom{\theta(\alpha-1)}{k} b_{s,m} \\ &\times \binom{-\theta(\alpha+1)}{v} \binom{\theta(b-1)}{r} \binom{-\theta}{n} \binom{\beta p + \theta(\beta-1)}{m} \binom{2(r+q)+\theta}{l} \\ &\times \binom{b(\beta p + \theta(\beta-1) + m + s + n)}{q} \left(\frac{z}{\theta} + 1 \right) \end{aligned}$$

and $I_{REG} = \int_0^\infty [(\frac{z}{\theta} + 1)g(x; \psi)G(x; \psi)^{\frac{z}{\theta}}]^\theta dx$ is the RE of the Exp-G distribution with power parameter $\frac{z}{\theta}$.

2.5 Special Cases

In this section, we present two special cases of the EHL-WTL-G FoD. We considered cases where the underlying distribution are Burr XII and Weibull distributions.

2.5.1 EHL-Weibull-Topp-Leone-Burr XII (EHL-W-TL-BXII) Distribution

Taking the Burr XII distribution with cdf $G(x; \theta, \gamma) = 1 - (1 + x^\theta)^{-\gamma}$ and pdf $g(x; \theta, \gamma) = \theta \gamma x^{\theta-1} (1 + x^\theta)^{-\gamma-1}$ for $\theta, \gamma > 0$ and $x > 0$ as a baseline distribution, we obtain the EHL-WTL-BXII distribution with cdf given by

$$F(x; \alpha, b, \beta, \theta, \gamma) = \left[\frac{1 - \exp \left\{ - \left(- \log[K_{BXII}(x; b, \theta, \gamma)] \right)^\beta \right\}}{1 + \exp \left\{ - \left(- \log[K_{BXII}(x; b, \theta, \gamma)] \right)^\beta \right\}} \right]^\alpha,$$

for $\alpha, b, \beta, \theta, \gamma > 0$. Note that, if we set $\gamma = 1$, we obtain the EHL-WTL-LLoG distribution. Also, when $\theta = 1$, we get the EHL-Weibull Topp-Leone-Pareto (EHL-WTL-P) distribution.

Figure 2.2 shows the plots of the pdf and the hrf of the EHL-WTL-BXII distribution for different parameter values. The pdf can take various shapes including the reverse-J, almost symmetric, left and right-skewed. Graphs of the hrf shows bathtub followed by inverted bathtub, bathtub, inverted bathtub, decreasing and increasing shapes.

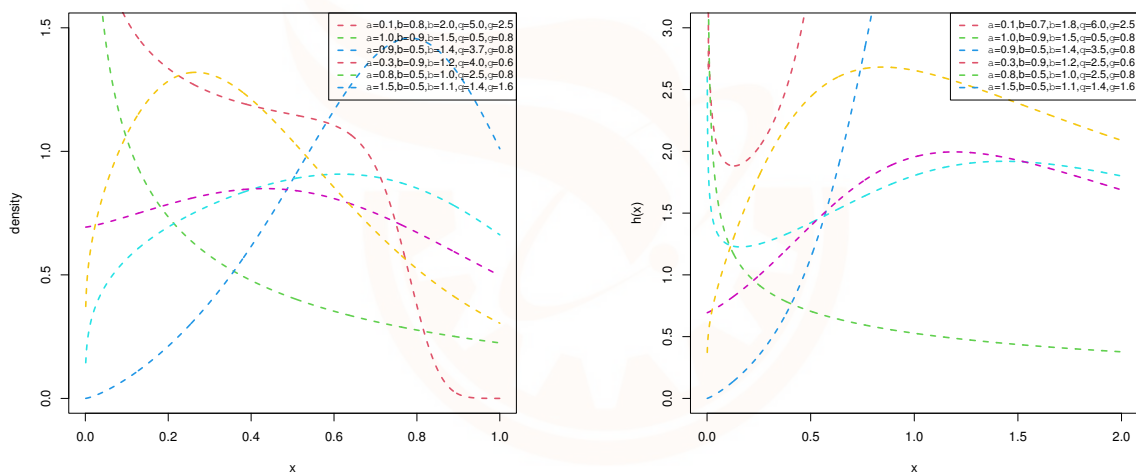


Figure 2.2: Plots of the pdf and hrf of the EHL-WTL-BXII distribution

2.5.2 EHL-Weibull-Topp-Leone-Weibull (EHL-WTL-W) Distribution

If we take the Weibull distribution with cdf and pdf given by $G(x; \gamma) = 1 - \exp(-x^\gamma)$ and $g(x; \gamma) = \gamma x^{\gamma-1} \exp(-x^\gamma)$, respectively, for $\gamma > 0$ and $x > 0$, then, the EHL-WTL-W distribution has cdf given by

$$F(x; \alpha, b, \beta, \gamma) = \left[\frac{1 - \exp \left\{ - \left(- \log(K_W(x; b, \gamma)) \right)^\beta \right\}}{1 + \exp \left\{ - \left(- \log(K_W(x; b, \gamma)) \right)^\beta \right\}} \right]^\alpha,$$

for $\alpha, b, \beta, \gamma > 0$.

Figure (2.3) shows the plots of the pdf and the hrf of the EHL-WTL-W distribution for different parameter values. The pdf can take various shapes including the reverse-J, almost symmetric,

left and right-skewed. Graphs of the hrf shows the upside-down bathtub, increasing, decreasing, and bathtub shapes.

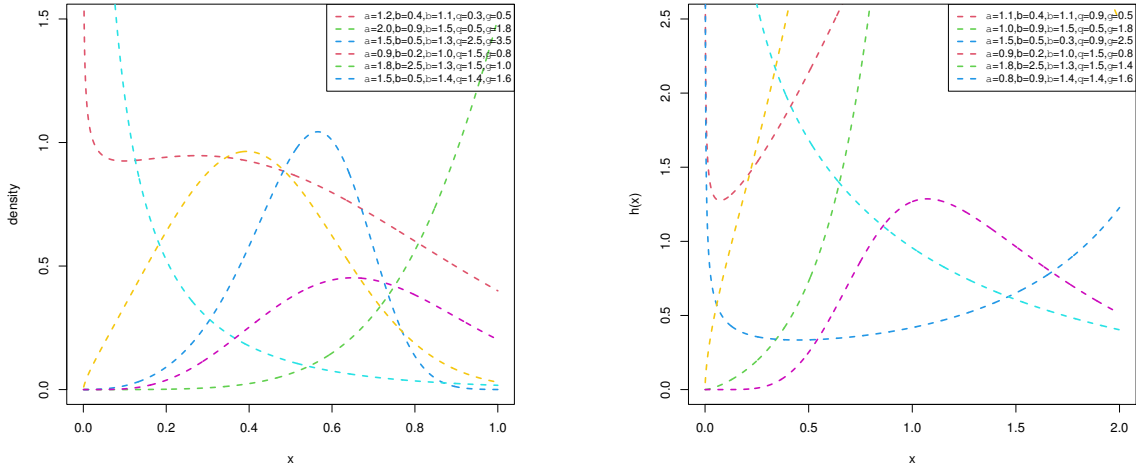


Figure 2.3: Plots of the pdf and hrf of the EHL-WTL-W distribution

2.6 Estimation

A maximum likelihood method of parameter estimation is presented in this section. If $X \sim EHL - WTL - G(\alpha, b, \beta, \psi)$ and $\Delta = (\alpha, b, \beta, \psi)^T$ a vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ from a random sample of size n of the EHL-WTL-G FoD is given as

$$\begin{aligned}
 \ell_n(\Delta) = & n \log(4b) + n \log(\alpha) + n \log(\beta) + \sum_{i=1}^n \log g(x_i; \psi) + \sum_{i=1}^n \log \bar{G}(x_i; \psi) \\
 & + (b-1) \sum_{i=1}^n \log \left(1 - \bar{G}^2(x_i; \psi) \right) + (\beta-1) \sum_{i=1}^n \log \left(-\log(K_G(x_i; b, \psi)) \right) \\
 & + (\alpha-1) \sum_{i=1}^n \log \left(1 - \exp \left(- \left(-\log(K_G(x_i; b, \psi)) \right)^\beta \right) \right) \\
 & - \sum_{i=1}^n \left(-\log(K_G(x_i; b, \psi)) \right)^\beta - \sum_{i=1}^n \log(K_G(x_i; b, \psi)) \\
 & - (\alpha+1) \sum_{i=1}^n \log \left(1 - \exp \left(- \left(-\log[K_G(x_i; b, \psi)] \right)^\beta \right) \right). \tag{2.13}
 \end{aligned}$$

The first derivative of the log-likelihood function with respect to each component of the parameter vector $\Delta = (\alpha, b, \beta, \zeta)^T$, that is, $U(\Delta)$ are given by

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^n \log(1 - \exp(-(-\log(K_G(x_i; b, \psi)))^\beta)) - \sum_{i=1}^n \log(1 + \exp(-(-\log(K_G(x_i; b, \psi)))^\beta)), \quad (2.14)$$

$$\begin{aligned} U_b &= \frac{n}{b} + \sum_{i=1}^n \log(1 - \bar{G}^2(x_i; \zeta)) - (\beta - 1) \sum_{i=1}^n y \frac{(1 - \bar{G}^2(x_i; \zeta))^b \log(1 - \bar{G}^2(x_i; \zeta))}{K_G(x_i; b, \psi) \log(K_G(x_i; b, \psi))} \\ &- \beta \sum_{i=1}^n \frac{(1 - \bar{G}^2(x_i; \zeta))^b (-\log(1 - (1 - \bar{G}^2(x_i; \zeta))^b))^{\beta-1} \log(1 - \bar{G}^2(x_i; \zeta))}{K_G(x_i; b, \psi)} \\ &+ \beta(\alpha - 1) \sum_{i=1}^n \frac{(1 - \bar{G}^2(x_i; \zeta))^b (-\log(K_G(x_i; b, \psi)))^{\beta-1} \log(1 - \bar{G}^2(x_i; \zeta))}{(1 - \exp(-(-\log(K_G(x_i; b, \psi)))^\beta))(K_G(x_i; b, \psi))} \\ &\times \exp(-(-\log(K_G(x_i; b, \psi)))^\beta) + \beta(\alpha + 1) \sum_{i=1}^n \frac{\log(1 - \bar{G}^2(x_i; \zeta))}{(K_G(x_i; b, \psi))} \\ &\times \frac{(1 - \bar{G}^2(x_i; \zeta))^b (-\log(K_G(x_i; b, \psi)))^{\beta-1}}{(1 + \exp(-(-\log(K_G(x_i; b, \psi)))^\beta))} - \sum_{i=1}^n \frac{(1 - \bar{G}^2(x_i; \zeta))^b \log(1 - \bar{G}^2(x_i; \zeta))}{(K_G(x_i; b, \psi))}, \end{aligned}$$

$$\begin{aligned} U_\beta &= \frac{n}{\beta} + \sum_{i=1}^n \log(-\log(K_G(x_i; b, \psi))) - \sum_{i=1}^n (-\log(K_G(x_i; b, \psi)))^\beta \\ &\times \log(-\log(K_G(x_i; b, \psi))) + (\alpha - 1) \sum_{i=1}^n \frac{(-\log(K_G(x_i; b, \psi)))^\beta}{(1 - \exp(-(-\log(K_G(x_i; b, \psi)))^\beta))} \\ &\times \exp(-(-\log(K_G(x_i; b, \psi)))^\beta) \log(-\log(K_G(x_i; b, \psi))) + (\alpha + 1) \\ &\times \sum_{i=1}^n \frac{(-\log(K_G(x_i; b, \psi)))^\beta \exp(-(-\log(K_G(x_i; b, \psi)))^\beta)}{(1 + \exp(-(-\log(K_G(x_i; b, \psi)))^\beta))} \\ &\times \log(-\log(K_G(x_i; b, \psi))) \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} U_{\zeta_k} &= \sum_{i=1}^n \frac{\partial g(x_i; \zeta)/\partial \zeta_k}{g(x_i; \zeta)} + \sum_{i=1}^n \frac{\partial \bar{G}(x_i; \zeta)/\partial \zeta_k}{\bar{G}(x_i; \zeta)} + (b - 1) \sum_{i=1}^n \frac{\partial(1 - \bar{G}^2(x_i; \zeta))/\partial \zeta_k}{(1 - \bar{G}^2(x_i; \zeta))} \\ &+ (\beta - 1) \sum_{i=1}^n \frac{\partial(-\log(K_G(x_i; b, \psi)))/\partial \zeta_k}{(K_G(x_i; b, \psi))(-\log(K_G(x_i; b, \psi)))} \\ &+ (\alpha - 1) \sum_{i=1}^n \frac{\partial(1 - \exp(-(-\log(K_G(x_i; b, \psi)))^\beta))/\partial \zeta_k}{(1 - \exp(-(-\log(K_G(x_i; b, \psi)))^\beta))} \\ &- \sum_{i=1}^n \frac{\partial(-\log(K_G(x_i; b, \psi)))^\beta/\partial \zeta_k}{(K_G(x_i; b, \psi))} - \sum_{i=1}^n \frac{\partial(K_G(x_i; b, \psi))/\partial \zeta_k}{(K_G(x_i; b, \psi))} \\ &- (\alpha + 1) \sum_{i=1}^n \frac{\partial(1 - \exp(-(-\log(K_G(x_i; b, \psi)))^\beta))/\partial \zeta_k}{(1 - \exp(-(-\log(K_G(x_i; b, \psi)))^\beta))}, \end{aligned} \quad (2.16)$$

respectively. The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ are obtained by solving the non-linear equation $(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \zeta_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure via R software. The packages *bbmle*, *numDeriv* in R were employed for the procedure. The multivariate normal distribution $N_{p+3}(\mathbf{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\mathbf{0} = (0, 0, 0, \mathbf{0})^T$ and $J(\hat{\Delta})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$, can be used to construct CIs and confidence regions for the individual model parameters.

2.7 Simulation Study

Here, we evaluate the effectiveness of the EHL-WTL-LLoG distribution through conducting simulations for various sets of initial parameter values: A: (0.7, 0.7, 2.5, 1.5), B: (0.7, 0.8, 2.5, 1.5), C: (0.7, 2.5, 2.5, 1.5) and D: (0.8, 0.8, 2.5, 1.5) and for different sample sizes ($n=50, 100, 200, 400, 800$) via the R package. Table 2.1 and 2.2 gives the mean MLEs of the parameters together with the respective ABias and RMSEs. The ABias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

		A : (0.7, 0.7, 2.5, 1.5)			B : (0.7, 0.8, 2.5, 1.5)		
parameter	Sample Size	Mean	RMSE	ABias	Mean	RMSE	ABias
α	50	1.1471	1.1237	0.4471	1.2183	1.3534	0.5183
	100	0.9605	0.6330	0.2605	0.9909	0.7406	0.2909
	200	0.8996	0.4969	0.1996	0.9292	0.6022	0.2292
	400	0.8541	0.4072	0.1541	0.8679	0.4673	0.1679
	800	0.7918	0.2974	0.0918	0.7815	0.2420	0.0815
b	50	0.8018	0.4469	0.1018	0.8605	0.4434	0.0605
	100	0.8120	0.4257	0.1120	0.8731	0.4097	0.0731
	200	0.7910	0.3922	0.0910	0.8520	0.3816	0.0520
	400	0.7742	0.3642	0.0742	0.8393	0.3485	0.0393
	800	0.7658	0.3109	0.0658	0.8498	0.2921	0.0498
β	50	4.5108	3.6897	2.0108	4.3772	3.5758	1.8772
	100	4.2190	3.1278	1.7190	4.0655	2.9672	1.5655
	200	3.9026	2.7353	1.4026	3.7880	2.6341	1.2880
	400	3.6439	2.4417	1.1439	3.5162	2.2777	1.0162
	800	3.4018	2.1447	0.9018	3.3706	2.0386	0.8706
c	50	1.6677	1.4951	0.1677	1.7171	1.5669	0.2171
	100	1.5340	1.2147	0.0340	1.5980	1.3078	0.0980
	200	1.5279	1.1293	0.0279	1.5993	1.2249	0.0993
	400	1.5214	1.0167	0.0214	1.5727	1.0680	0.0727
	800	1.4625	0.7935	-0.0375	1.4634	0.8022	-0.0366

Table 2.1: Simulation Results

		C : (0.7, 2.5, 2.5, 1.5)			D : (0.8, 0.8, 2.5, 1.5)		
parameter	Sample Size	Mean	RMSE	ABias	Mean	RMSE	ABias
α	50	1.1645	2.4918	0.4645	1.4479	1.7940	0.6479
	100	0.8627	1.0687	0.1627	1.1705	0.9011	0.3705
	200	0.7664	0.4845	0.0664	1.0604	0.6869	0.2604
	400	0.7612	0.3784	0.0612	0.9953	0.5299	0.1953
	800	0.7526	0.3078	0.0526	0.9113	0.3375	0.1113
b	50	3.0726	2.1041	0.5726	0.8565	0.4408	0.0565
	100	3.2169	1.9422	0.7169	0.8608	0.4197	0.0608
	200	3.0399	1.4928	0.5399	0.8559	0.3880	0.0559
	400	2.8235	1.1376	0.3235	0.8407	0.3529	0.0407
	800	2.6532	0.7775	0.1532	0.8532	0.3056	0.0532
β	50	9.0974	16.6219	6.5974	4.4223	3.7109	1.9223
	100	6.4444	10.0044	3.9444	4.0941	3.0867	1.5941
	200	4.7860	6.0918	2.2860	3.8572	2.7352	1.3572
	400	4.3542	4.7792	1.8542	3.5381	2.2904	1.0381
	800	3.4471	2.7815	0.9471	3.4529	2.1734	0.9529
c	50	1.2831	0.9331	-0.2169	1.7298	1.5846	0.2298
	100	1.3756	0.8355	-0.1244	1.6336	1.3667	0.1336
	200	1.4203	0.7182	-0.0797	1.6085	1.2642	0.1085
	400	1.3818	0.6565	-0.1182	1.5783	1.1014	0.0783
	800	1.4003	0.5280	-0.0997	1.4700	0.8661	-0.0300

Table 2.2: Simulation Results

The results in Tables 2.1 and 2.2 reveals that the mean estimates of the parameters are approaching the initial parameter values as we increase the sample size n . Also, the RMSEs and ABias are decaying toward zero with increasing sample size, which suggest strong consistency of the MLE of the model parameters. Thus, we can apply the new model to data and be sure of obtaining reliable, consistent and efficient parameter estimates.

2.8 Applications

We provide some examples to highlight the potentiality and flexibility of the EHL-WTL-LLoG distribution for data modeling. The EHL-WTL-LLoG distribution is fitted to real data sets, (the plasma concentration data set and the survival times (in years) of a group of patients given chemotherapy treatment) and the fits of the EHL-WTL-LLoG distribution are compared to the fits of the beta odd Lindley-exponential (BOL-E) and beta odd Lindley-uniform (BOL-U) by Chipepa et al. [41], the Odd EHL Burr XII (OEHLBXII) by Aldahlam et al. [15], and the Weibull BXII (WBXII) by Afify et al. [3]. For purposes of comparison, we used some well known GoF statistics and these includes: $-2 \ln(L)$, AIC, CAIC, BIC, W^* , A^* , K-S statistic and its p-value, as described by Chen and Balakrishnan [36] as well as the SS from the probability plots. Generally, the model with the lowest values of all the GoF statistics and a large value of the p-value of the K-S statistic performs better than the others. The pdfs of these distributions are given in the appendix.

In Figure 2.4 and Figure 2.8 we present fitted densities, data histograms and probability plots (Chambers et al. [34]). Probability plots were created by plotting $F(x_{(j)})$ versus $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, with $x_{(j)}$ being ordered values of the observed data. The SS measures the closeness of the fitted model to the empirical and is given by

$$SS = \sum_{j=1}^n \left[F(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

Plots of the estimated cdf, K-M and the estimated hrf are presented in Figure 2.5 and Figure 2.9. Presented in Figure 2.6 and Figure 2.10 are profile plots of the log-likelihood function.

The MLEs of the parameters with their SEs in parenthesis and the values of the statistics; $-2 \log L$, AIC, CAIC, BIC, W^* and A^* , K-S and its p-value and the (SS) are given in Table 2.3 and Table 2.4.

2.8.1 Plasma Concentrations Data

The EHL-WTL-LLoG was fitted to the plasma concentration data set which was taken from R base package in the Indometh object.

Distribution	Estimates				statistics								
	α	b	β	c	-2logL	AIC	CAIC	BIC	W*	A*	K-S	P-value	SS
EHL-WTL-LLoG	313.68 (5.8940×10^{-8})	1.4406 (0.0022)	27.035 (1.2369×10^{-5})	0.0064 (5.8577×10^{-4})	55.7584	63.7590	64.4148	72.5176	0.2254	1.4030	0.1290	0.2222	0.2354
HL-WTL-LLoG	-	1.0318 (0.3417)	2.7036 (1.8159)	0.3853 (0.3093)	63.3423	69.3423	69.7294	75.9113	0.2580	1.6465	0.1392	0.1544	0.2394
EHL-WTL-LLoG(1,1, β ,c)	-	-	2.5463 (0.3124)	0.4142 (0.0617)	63.3511	67.3511	67.5416	71.7304	0.2580	1.6466	0.1390	0.1557	0.2382
	a	b	λ	θ									
BOL-E	0.9765 (0.1492)	30.2540 (9.7968×10^{-7})	38.5620 (7.3602×10^{-7})	0.0015 (0.0003)	62.7418	70.7418	71.3975	79.5004	0.2537	1.6264	0.1461	0.1193	0.2683
	a	b	λ	θ									
BOL-U	0.9773 (0.1495)	14.2040 (2.7436)	1.0252×10^5 (0.0004)	8.8118×10^5 (4.4180×10^{-5})	62.7359	70.7359	71.3917	79.4946	0.2537	1.6260	0.1462	.1186	0.2686
	α	λ	a	b									
OEHLBXII	0.0777 (0.0275)	0.2749 (0.3008)	8.0387 (2.9358)	0.3080 (0.1267)	73.4574	81.4574	82.1132	90.2160	0.3220	2.0260	0.1408	0.1459	0.3733
	a	b	α	β									
WBXII	533.5800 (0.0005)	2.1081 (1.2610)	0.5659 (0.3393)	0.0922 (0.1444)	60.8423	68.8423	69.4980	77.6009	0.2391	1.5301	0.1389	0.1563	0.2203

Table 2.3: Estimates and GoF Statistics for Plasma Concentrations Data

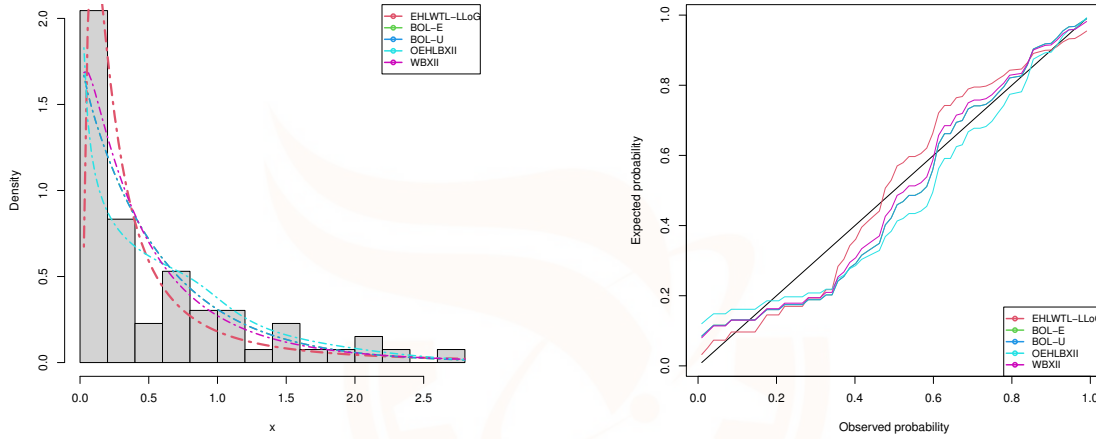


Figure 2.4: Fitted densities and probability plots for plasma concentrations data

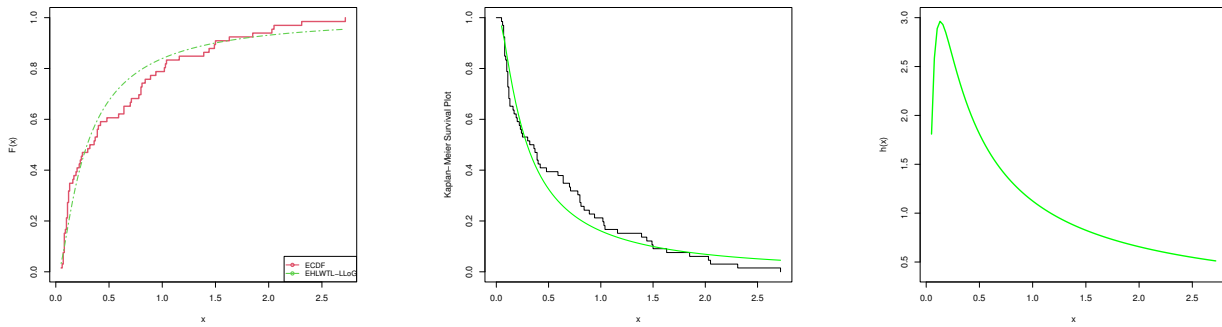


Figure 2.5: Estimated cdf, K-M survival plots and estimated hazard rate plot of the EHL-WTL-LLoG distribution for plasma concentrations data

To assess the efficacy of the EHL-WTL-LLoG distribution in fitting the plasma concentration data, we used the goodness-of-fit statistics presented in Table 2.3, plots of the fitted densities and probability plots displayed in Figure 2.4. Based on those GoF statistics and plots, we conclude that the EHL-WTL-LLoG distribution provides the best fit to the plasma concentration data compared to the other models presented.

The estimated var-cov matrix for the plasma concentrations data is given by,

$$\begin{bmatrix} 3.4739 \times 10^{-15} & 1.2878 \times 10^{-10} & -7.2874 \times 10^{-13} & -2.5832 \times 10^{-11} \\ 1.2878 \times 10^{-10} & 4.7759 \times 10^{-6} & -2.7029 \times 10^{-8} & -9.7380 \times 10^{-7} \\ -7.2874 \times 10^{-13} & -2.7029 \times 10^{-8} & 1.5298 \times 10^{-10} & 5.5496 \times 10^{-9} \\ -2.5832 \times 10^{-11} & -9.7380 \times 10^{-7} & 5.5496 \times 10^{-9} & 3.4313 \times 10^{-7} \end{bmatrix}$$

and the 95% asymptotic CIs for the parameters α, b, β and c are: $313.68 \pm 1.1552 \times 10^{-7}$, 1.4406 ± 0.0043 , $27.035 \pm 2.4243 \times 10^{-5}$ and 0.0064 ± 0.0011 , respectively.

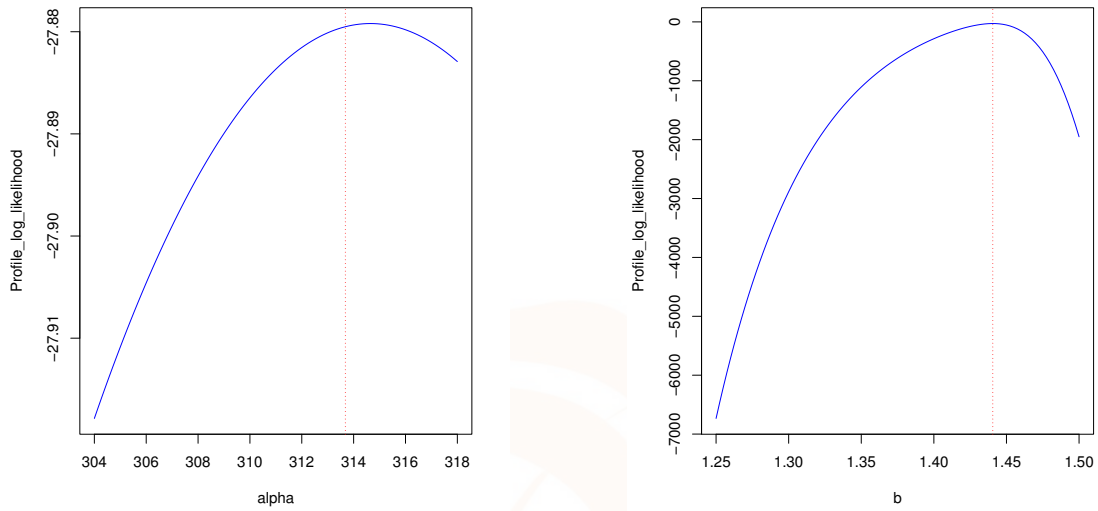


Figure 2.6: Profile plots for the plasma concentrations data

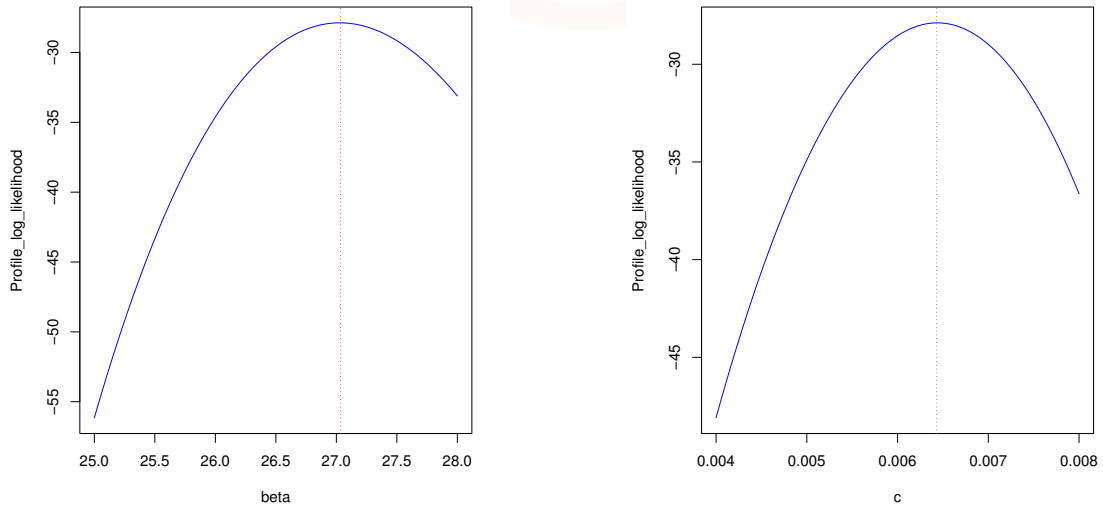


Figure 2.7: Profile plots for the plasma concentrations data

The profile plots in Figure 2.6 and Figure 2.7 shows that the estimates of the EHL-WTL-LLoG distribution are global maximums for the plasma concentrations data set.

2.8.2 Survival Times Data

The second real data set is a subset of data reported by Bekker et al. [22] which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. Table 2.4 and Figure 2.8 displays the GoF statistics, the fitted densities and probability plots, respectively. We note that the EHL-WTL-LLoG model provides the best fit compared to other competing models presented.

The estimated var-cov matrix for the survival times data is given by,

$$\begin{bmatrix} 0.0165 & -0.2894 & -0.1483 & -0.0125 \\ -0.2894 & 6.4322 & 1.3584 & 0.3705 \\ -0.1483 & 1.3584 & 2.8371 & -0.0535 \\ -0.0125 & 0.3705 & -0.0535 & 0.0316 \end{bmatrix}$$

and the 95% asymptotic CIs for the parameters α, b, β and c are: 0.1424 ± 0.2517 , 4.8234 ± 4.9709 , 3.2621 ± 3.3013 and 0.7358 ± 0.3485 , respectively.

	Estimates				statistics								
Distribution	α	b	β	c	-2logL	AIC	CAIC	BIC	W*	A*	K-S	P-value	SS
EHL-WTL-LLoG	0.1424 (0.1284)	4.8234 (2.5362)	3.2622 (1.6844)	0.7358 (0.1778)	112.0029	120.003	121.003	127.2296	0.0536	0.3767	0.0815	0.9022	0.0410
HL-WTL-LLoG	-	1.4999 (0.1830)	2.8045 (3.4129)	0.3622 (0.4611)	116.8202	122.8202	123.4056	128.2402	0.0781	0.5240	0.1104	0.6035	0.0821
EHL-WTL-LLoG(1,1, β ,c)	-	-	0.9814 (0.2391)	1.2016 (0.3695)	119.6150	123.6150	123.9007	127.2283	0.0788	0.5342	0.1149	0.5528	0.1188
BOL-E	a	b	λ	θ	115.9622	123.9622	124.9622	131.1889	0.0859	0.5722	0.1229	0.4677	0.1059
	1.0442 (0.2623)	0.3782 (0.4825)	22.3842 (27.4203)	0.0829 (0.0804)									
BOL-U	a	b	λ	θ	115.8709	123.8709	124.8709	131.0975	0.0936	0.6198	0.1195	0.5027	0.1019
	0.9574 (0.3187)	3.3672 (7.7617)	3.8385 (6.8110)	17.1583 (30.4901)									
OEHLBXII	α	λ	a	b	117.8487	125.8487	126.8487	133.0754	0.0862	0.5758	0.1058	0.6552	0.0710
	0.1230 (0.0442)	0.0191 (0.0401)	6.5252 (2.3417)	0.5074 (0.1674)									
WBXII	a	b	α	β	116.0615	124.0615	125.0615	131.2882	0.0733	0.4949	0.1052	0.6619	0.0759
	18.4009 (65.4891)	3.0282 (4.0329)	0.4176 (0.6145)	0.4282 (0.5739)									

Table 2.4: Estimates and GoF Statistics for the Survival Times Data

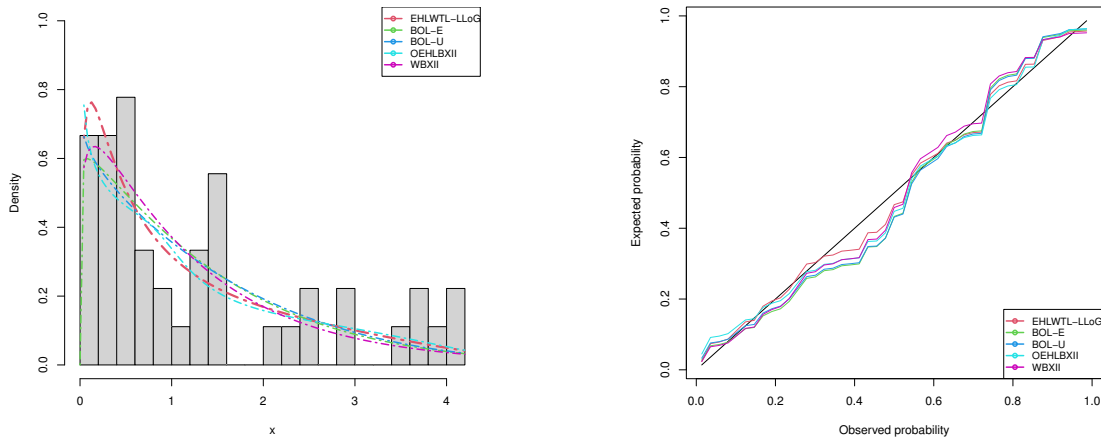


Figure 2.8: Fitted densities and probability plots for survival times data

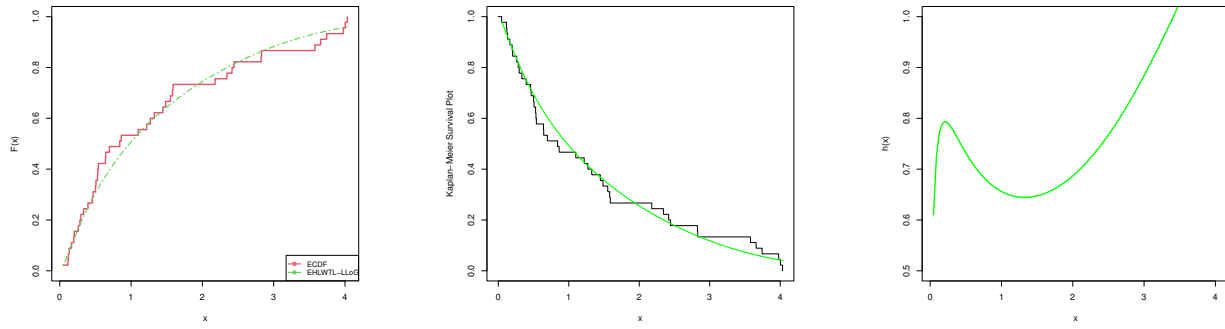


Figure 2.9: Estimated cdf, K-M survival plots and estimated hazard rate plot of the EHL-WTL-LLoG distribution for the survival times data.

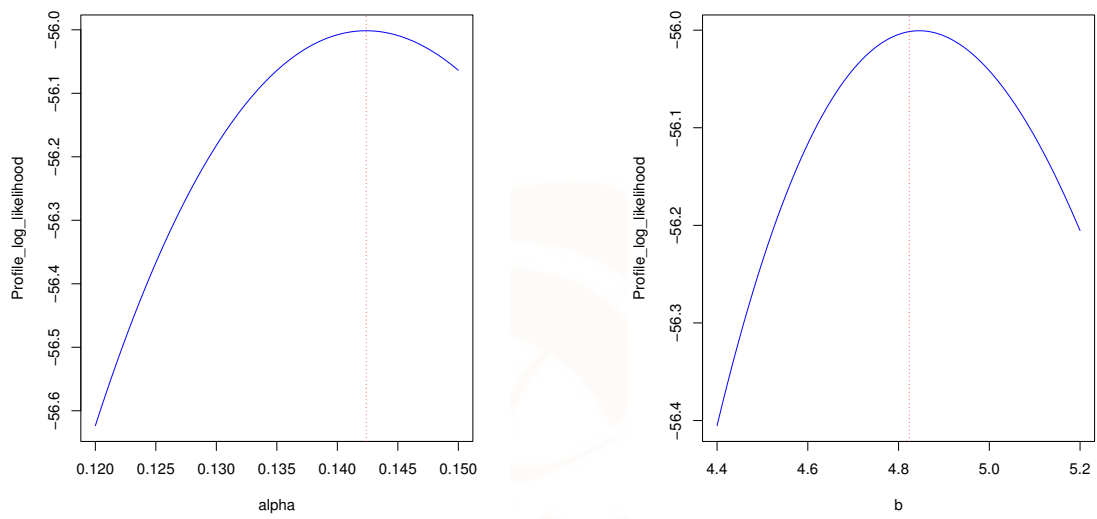


Figure 2.10: Profile plots for the survival times (in years) of a group of patients data

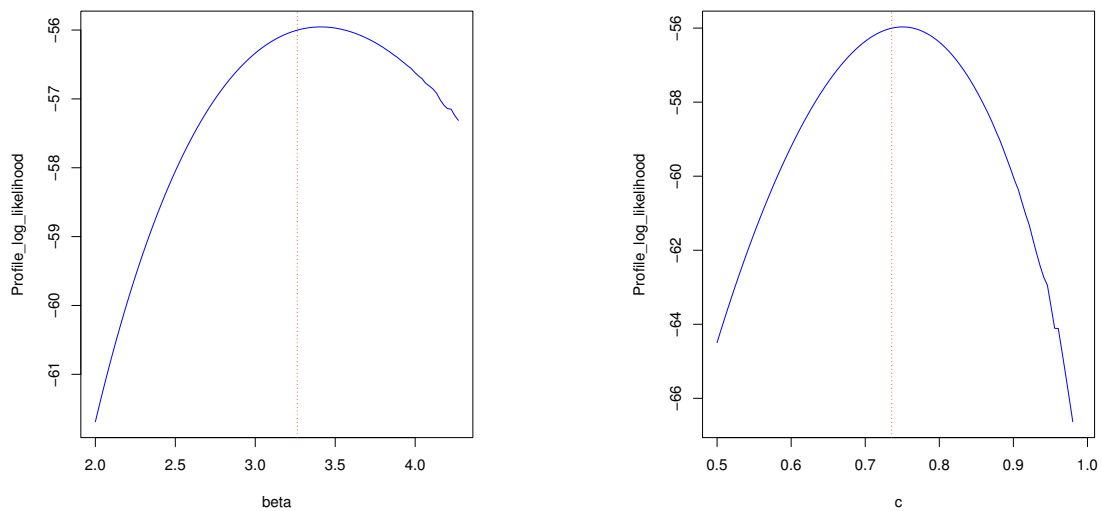


Figure 2.11: Profile plots for the survival times (in years) of a group of patients data

The profile plots in Figures 2.10 and 2.11 shows that the estimates of the EHL-WTL-LLoG distribution are global maximums for the survival times data.

Model	df	Plasma Concentration Data		Survival Times Data	
		Chi-Square	p-value	Chi-Square	p-value
EHL-WTL-LLoG($1, b, \beta, c$)	1	7.5839	0.005889	4.8173	0.028175
EHL-WTL-LLoG($1, 1, \beta, c$)	2	7.5927	0.022453	7.6121	0.022236

Table 2.5: **Likelihood ratio test results**

The likelihood ratio (LR) test are presented in Table 2.5. The results show that there is a significant difference between the EHL-WTL-LLoG distribution and its nested models: (EHL-WTL-LLoG($1, b, \beta, c$), EHL-WTL-LLoG($1, 1, \beta, c$)) for both data sets

2.9 Concluding Remarks

A new generalized FoD called the EHL-WTL-G distribution was developed and presented. The hrf possesses a variety of shapes which makes it suitable for modelling a variety of data sets. Moreover, expressions for the moments, incomplete moments, distribution of order statistics and RE were also obtained. The model parameters were estimated using the MLE approach. The performance of EHL-WTL-LLoG distribution special cases was investigated using simulations with varied sample sizes. To show the applicability of the proposed FoD, a special case of the EHL-WTL-LLoG distribution was fitted to real life data. The new model has proven to be flexible and may be used in a variety of scientific fields such as survival analysis, engineering, economics to mention a few.

3. The EHL-Type II Topp-Leone-G Family of Distributions with Applications

Sengweni, W., Oluyede, B. and Kethamile Rannona, K., The Exponentiated Half-Logistic-Type II Topp-Leone-G Family of Distributions with Applications,
Accepted: Journal of Statistics and Management Systems

3.1 Introduction

In this chapter, we develop and study a new generalized FoD called the EHL-Type II Topp-Leone-G (EHL-TIITL-G) FoD. The extension of CoD to obtain new and flexible FoD has attracted many researchers in recent years. Generalized FoD play a crucial role in survival and reliability analysis when it comes to modeling of non-monotone and monotone hrf. These extended and generalized distributions are important in explaining some desirable characteristics of the underlying data generating process or event. The new models assists practitioners in various scientific fields to make informed decisions and inferences about the process or event they will be studying, for example, Dey et al. [54] developed a two parameter exponentiated Gumbel distribution and applied it to flood data. Al-Dubaicy and Yassin [14] illustrated the usefulness and applicability of the Exponentiated Gumbel distribution to censored data. There are a number of newly developed FoD in the statistical literature and these includes the Weibull-G FoD by Bourguignon et al. [30], the Odd power generalized Weibull-G FoD by Moakofi et al. [90], the Generalized beta-generated distributions by Alexander et al.[6] and a new power generalized Weibull-G FoD by Oluyede et al. [106]. Sengweni et al. [119] developed the EHL-Odd Lindley-G FoD in an effort to improve the performance of the half-logistic distribution in data modelling. This chapter employed the transformer (T-X) by Alzaghal et al. [11] to come up with the EHL-Type II Topp-Leone-G FoD.

Our main motivation for developing this new FoD is the flexibility gained when it comes to modelling lifetime data with monotonic and non-monotonic failure rates. The new model produced various shapes of the density function as well as the hrf. We noted that the new model captures a variety of real data sets and provide consistently better fits than other generated distributions with the same underlying model. Worth noting is that the new FoD contains some other new FoD.

The structure of this chapter is as follows. Several fundamental results of the EHL-Type II Topp-Leone-G (EHL-TIITL-G) FoD, its sub-models, hrf and the qf are provided in Section 3.2. Section 3.3 contains information on moments, mgf and PMWs. Further insightful results on the distribution of order statistics and RE are presented in Section 3.4. Some special cases are given in Section 3.5. Results on the estimation of parameters of the EHL-TIITL-G FoD via the method of maximum likelihood are presented in Section 3.6. Section 3.7 provides results on the Monte

Carlo simulation study performed to evaluate the ABias and RMSE of the maximum likelihood estimators. Section 3.8 presents some examples on the applicability of the EHL-TIITL-G to real life data. Finally, Section 3.9 gives some concluding remarks.

3.2 The New Generalized Model

The model synopsis and some useful statistical properties are given in this section. Here we used the EHL FoD by Cordeiro et al. [53] as given in equation (1.1) and the Type II Topp-Leone distribution (equation (1.25)) to obtain the new family of EHL-TIITL-G distributions. The cdf and pdf of the proposed FoD are given by

$$F(x; \alpha, b, \varphi) = \left[\frac{F_{TIITL-G}(x; b, \varphi)}{1 + (1 - G^2(x; \varphi))^b} \right]^\alpha \quad (3.1)$$

and

$$f(x; \alpha, b, \varphi) = \frac{4b\alpha g(x; \varphi)G(x; \varphi)(1 - G^2(x; \varphi))^{b-1}(F_{TIITL-G}(x; b, \varphi))^{\alpha-1}}{(1 + (1 - G^2(x; \varphi))^b)^{\alpha+1}}, \quad (3.2)$$

where $F_{TIITL-G}(x; b, \varphi) = 1 - (1 - G^2(x; \varphi))^b$, $b, \alpha > 0$ and φ is a vector of parameters of the baseline cdf G . The hrf is given by

$$\begin{aligned} h(x; \alpha, b, \varphi) &= \frac{4b\alpha g(x; \varphi)G(x; \varphi)(1 - G^2(x; \varphi))^{b-1}(F_{TIITL-G}(x; b, \varphi))^{\alpha-1}}{(1 + (1 - G^2(x; \varphi))^b)^{\alpha+1}} \\ &\times \left\{ 1 - \left[\frac{F_{TIITL-G}(x; b, \varphi)}{1 + (1 - G^2(x; \varphi))^b} \right]^\alpha \right\}^{-1}. \end{aligned} \quad (3.3)$$

3.2.1 Linear Representation

In this section, we expressed the pdf of the EHL-TIITL-G FoD as an infinite linear combination of Exp-G densities. This is an important result since it will assist us in the derivation of other

statistical properties of this new model. Note that the pdf can be written as:

$$\begin{aligned}
 f(x; \alpha, b, \varphi) &= 4b\alpha g(x; \varphi)G(x; \varphi)(1 - G^2(x; \varphi))^{b-1}(F_{TIITL-G}(x; b, \varphi))^{\alpha-1} \\
 &\times (1 + (1 - G^2(x; \varphi))^b)^{-\alpha-1} \\
 &= 4b\alpha g(x; \varphi)G(x; \varphi) \sum_{i,j=0}^{\infty} \binom{-\alpha-1}{i} \binom{\alpha-1}{j} (-1)^j \\
 &\times (1 - G^2(x; \varphi))^{b(j+i+1)-1} \\
 &= 4b\alpha g(x; \varphi) \sum_{i,j,n=0}^{\infty} \binom{-\alpha-1}{i} \binom{\alpha-1}{j} (-1)^{j+n} \\
 &\times \binom{b(j+i+1)-1}{n} (1 - \bar{G}(x; \varphi))^{2n+1} \\
 &= 4b\alpha g(x; \varphi) \sum_{i,j,n,k,l=0}^{\infty} (-1)^{j+n+k+l} \binom{-\alpha-1}{i} \binom{\alpha-1}{j} \\
 &\times \binom{b(j+i+1)-1}{n} \binom{2n+1}{k} \binom{k}{l} G^l(x; \varphi) \\
 &= \sum_{l=0}^{\infty} a_{l+1}^* h_{l+1}(x; \varphi), \tag{3.4}
 \end{aligned}$$

where

$$a_{l+1}^* = 2 \sum_{i,j,n,k=0}^{\infty} \frac{(-1)^j b \alpha}{l+1} \binom{-\alpha-1}{i} \binom{\alpha-1}{j} \binom{b(j+i+1)-1}{n} \binom{2n+1}{k} \binom{k}{l}, \tag{3.5}$$

and $h_{l+1}(x; \varphi) = (l+1)g(x; \varphi)G^l(x; \varphi)$ is the Exp-G density with power parameter $(l+1)$. Thus, the EHL-TIITL-G density can be expressed as an infinite linear combination of Exp-G densities.

3.2.2 Sub-Families

Some sub-families of the EHL-TIITL-G FoD are given in Table 3.1.

	cdf	Model
If $\alpha = 1$	$\frac{F_{TIITL-G}(x; b, \varphi)}{1 + (1 - G^2(x; \varphi))^b}$	Half Logistic Type II Topp-Leone-G FoD
If $b = 1$	$\left[\frac{G^2(x; \varphi)}{1 + G^2(x; \varphi)} \right]^\alpha$	New FoD
If $\alpha = b = 1$	$\frac{G^2(x; \varphi)}{1 + G^2(x; \varphi)}$	New FoD

Table 3.1: Some Sub-Families of the EHL-TIITL-G Distribution

3.2.3 Quantile Function

The qf, $Q(u)$, $0 \leq u \leq 1$, of the EHL-TIITL-G FoD is obtained by inverting the cdf as follows

$$X_q = G^{-1} \left[1 - \left(\frac{1 - u^{\frac{1}{\alpha}}}{1 + u^{\frac{1}{\alpha}}} \right)^{\frac{1}{b}} \right]^{\frac{1}{2}}. \tag{3.6}$$

This function is useful in generating some random numbers from the EHL-TIITL-G distribution for a specified baseline cdf $G(x; \varphi)$, and will also help in the simulations when assessing the performance of estimators.

3.3 Moments

In this section, we assume a rv Y_{l+1} to be an Exp-G distribution with power parameter $(l+1)$, and make use of the linear representation results to derive the statistical properties of this new model. Thus, the r^{th} ordinary moment of the EHL-TIITL-G FoD is given by

$$\mu'_r = E(X^r) = \sum_{l=0}^{\infty} a_{l+1}^* E(Y_{l+1}^r), \quad (3.7)$$

where a_{l+1}^* is as given in equation (3.5) and $E(Y_{l+1}^r)$ is the r^{th} moment of the Exp-G distribution. The r^{th} incomplete moment of the EHL-TIITL-G FoD is given by

$$\delta_r(y) = \int_{-\infty}^y x^r f(x; \alpha, b, \varphi) dx = \sum_{l=0}^{\infty} a_{l+1}^* \int_{-\infty}^y x^r h_{l+1}(x; \varphi) dx,$$

where $\int_{-\infty}^y x^r h_{l+1}(x; \varphi) dx$ is the r^{th} incomplete moment of the Exp-G distribution. The conditional moments can also be obtained from the incomplete moments. The q^{th} central moment of the EHL-TIITL-G FoD is given

$$\mu_q = \sum_{r=0}^q \binom{q}{r} (-\mu'_1)^{q-r} E(X^r) = \sum_{r=0}^q \sum_{l=0}^{\infty} a_{l+1}^* \binom{q}{r} (-\mu'_1)^{q-r} E(Y_{l+1}^r).$$

The mgf $M_X(t) = E(e^{tX})$ of the EHL-TIITL-G FoD can be obtained from equation (3.4) and is given as

$$M_X(t) = \sum_{l=0}^{\infty} a_{l+1}^* M_{l+1}(t),$$

with $M_{l+1}(t)$ being the mgf of Y_{l+1} .

The skewness and kurtosis plots of the EHL-TIITL-W are given in Figure 3.1 and 3.2. From the plots, it can be seen that the members of the EHL-TIITL-G family can model various data types in terms of skewness and kurtosis.

3.3.1 Probability Weighted Moments (PWMs)

The $(i, j)^{th}$ PWMs of a rv $X \sim EHL - TIITL - G$, say $\wp_{i,j}$ is given by

$$\wp_{i,j} = E(X^i (F(X))^j) = \int_{-\infty}^{\infty} x^i f(x) (F(x))^j dx. \quad (3.8)$$

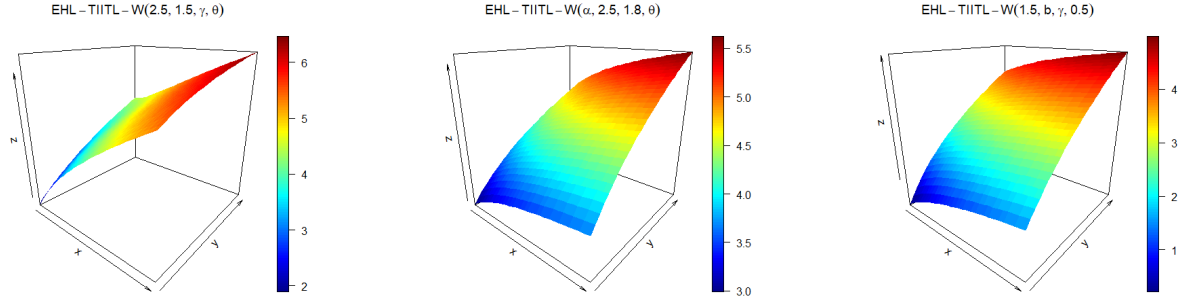


Figure 3.1: Skewness plots for the EHL-TIITL-W distribution

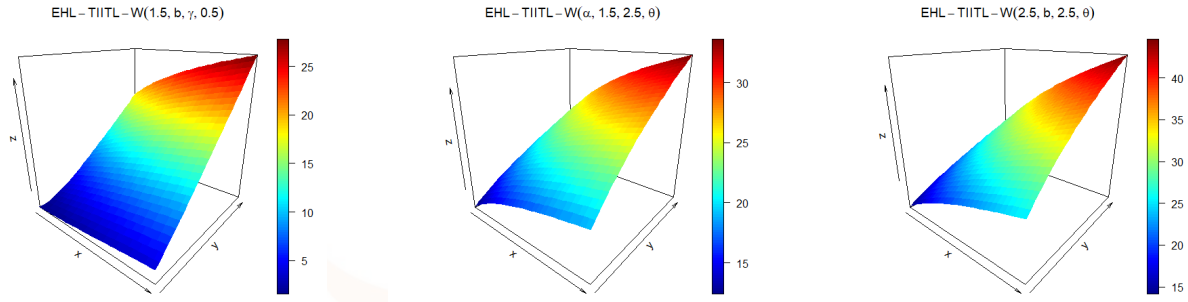


Figure 3.2: Kurtosis plots for the EHL-TIITL-W distribution.

Note that $f(x)(F(x))^j$ can be expressed as follows:

$$\begin{aligned}
 f(x)(F(x))^j &= 4b\alpha g(x; \varphi)G(x; \varphi)(1 - G^2(x; \varphi))^{b-1}(F_{TIITL-G}(x; b, \varphi))^{\alpha(j+1)-1} \\
 &\times (1 + (1 - G^2(x; \varphi))^b)^{-\alpha(j+1)-1} \\
 &= 4b\alpha g(x; \varphi)G(x; \varphi) \sum_{i,q=0}^{\infty} (-1)^q \binom{-\alpha(j+1)-1}{i} \binom{\alpha(j+1)-1}{q} \\
 &\times (1 - G^2(x; \varphi))^{b(i+q+1)-1} \\
 &= 4b\alpha g(x; \varphi) \sum_{i,q,n,k,l=0}^{\infty} (-1)^{q+l} \binom{-\alpha(j+1)-1}{i} \binom{\alpha(j+1)-1}{q} \\
 &\times \binom{b(i+q+1)-1}{n} \binom{2n+1}{k} \binom{k}{l} G^l(x; \varphi) \\
 &= \sum_{l=0}^{\infty} \varphi_{l+1}^* h_{l+1}(x; \varphi),
 \end{aligned}$$

where $h_{l+1}(x; \varphi) = (l+1)g(x; \varphi)G^l(x; \varphi)$ and

$$\varphi_{l+1}^* = 2b\alpha \sum_{i,q=0}^{\infty} \frac{(-1)^{q+l}}{l+1} \binom{-\alpha(j+1)-1}{i} \binom{\alpha(j+1)-1}{q} \binom{b(i+q+1)-1}{n} \binom{2n+1}{k} \binom{k}{l}.$$

Consequently, the PWM of X can be written as

$$\varrho_{j,i} = \sum_{l=0}^{\infty} \varphi_{l+1}^* \int_{-\infty}^{\infty} x^j h_{l+1}(x; \varphi) dx = \sum_{l=0}^{\infty} \varphi_{l+1}^* E(Y_{l+1}^j). \quad (3.9)$$

3.4 Order Statistics and Rényi Entropy(RE)

In this subsection, expressions for the distribution of order statistics and RE are presented.

3.4.1 Order Statistics

Suppose $X_1, X_2, X_3, \dots, X_n$ are i.i.d. EHL-TIITL-G rvs. Then, the pdf of the i^{th} order statistics ($X_{i:n}$) can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} (F(x))^{j+i-1}, \quad (3.10)$$

where $B(.,.)$ is the beta function. Note that

$$\begin{aligned} f(x)(F(x))^{j+i-1} &= 4b\alpha g(x; \varphi) G(x; \varphi) (1 - G^2(x; \varphi))^{b-1} (F_{TIITL-G}(x; b, \varphi))^{\alpha(j+i-1)} \\ &\times (1 + (1 - G^2(x; \varphi))^b)^{-\alpha(j+i-1)} \\ &= 4b\alpha g(x; \varphi) G(x; \varphi) \sum_{q,p=0}^{\infty} (-1)^q \binom{\alpha(j+i-1)}{q} \binom{-\alpha(j+i-1)}{p} \\ &\times (1 - G^2(x; \varphi))^{b(q+p+1)-1} \\ &= 4b\alpha g(x; \varphi) \sum_{q,p,n,k,l=0}^{\infty} (-1)^{q+l} \binom{\alpha(j+i-1)}{q} \binom{-\alpha(j+i-1)}{p} \\ &\times \binom{b(q+p+1)-1}{n} \binom{2n+1}{k} \binom{k}{l} G^l(x; \varphi). \end{aligned} \quad (3.11)$$

Substituting equation (3.11) into equation (3.10), we obtain

$$f_{i:n}(x) = \sum_{l=0}^{\infty} \omega_{l+1}^* h_{l+1}(x; \varphi),$$

where $h_{l+1}(x; \varphi) = (l+1)g(x; \varphi)G^{l+1}(x; \varphi)$ is the Exp-G density with power parameter $(l+1)$ and

$$\begin{aligned} \omega_{l+1}^* &= 2b\alpha \sum_{q,p,n,k=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{q+l+j}}{B(i, n-i+1)(l+1)} \binom{n-i}{j} \binom{\alpha(j+i-1)}{q} \\ &\times \binom{-\alpha(j+i-1)}{p} \binom{b(q+p+1)-1}{n} \binom{2n+1}{k} \binom{k}{l}. \end{aligned}$$

3.4.2 Rényi Entropy (RE)

The RE (Rényi [113]) is defined to be

$$I_R(\nu) = \frac{1}{1-\nu} \log \left(\int_0^\infty [f(x)]^\nu dx \right), \nu \neq 1, \nu > 0. \quad (3.12)$$

From the EHL-TIITL-G pdf, $f^\nu(x)$ can be expressed as

$$\begin{aligned} f^\nu(x) &= (4b\alpha)^\nu g^\nu(x, \varphi) G^\nu(x; \varphi) (1 - G^2(x; \varphi))^{\nu(b-1)} (F_{TIITL-G}(x; b, \varphi))^{\nu(\alpha-1)} \\ &\times (1 + (1 - G^2(x; \varphi))^b)^{-\nu(\alpha+1)} \\ &= (4b\alpha)^\nu g^\nu(x; \varphi) G^\nu(x, \varphi) \sum_{i,j=0}^{\infty} \binom{-\nu(\alpha+1)}{i} \binom{\nu(\alpha-1)}{j} (-1)^j \\ &\times (1 - G^2(x, \varphi))^{b(j+i+\nu)-\nu} \\ &= (4b\alpha)^\nu g^\nu(x, \varphi) \sum_{i,j=0}^{\infty} (-1)^{j+l} \binom{-\nu(\alpha+1)}{i} \binom{\nu(\alpha-1)}{j} \\ &\times \binom{b(j+i+\nu)-\nu}{n} \binom{2n+\nu}{k} \binom{k}{l} G^l(x; \varphi). \end{aligned}$$

Thus, the RE for EHL-TIITL-G FoD in terms of the RE of the Exp-G distribution is as follows

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{l=0}^{\infty} c_l^* e^{(1-\nu)I_{REG}} \right],$$

where

$$\begin{aligned} c_l^* &= (4b\alpha)^\nu \sum_{i,j=0}^{\infty} (-1)^{j+l} \binom{-\nu(\alpha+1)}{i} \binom{\nu(\alpha-1)}{j} \binom{b(j+i+\nu)-\nu}{n} \\ &\times \binom{2n+\nu}{k} \binom{k}{l} \left(\frac{l}{\nu} + 1 \right) \end{aligned}$$

and $I_{REG} = \int_0^\infty [(\frac{l}{\nu} + 1)g(x; \varphi)G(x; \varphi)^{\frac{l}{\nu}}]^\nu dx$ is the RE of the Exp-G distribution with power parameter $\frac{l}{\nu}$.

3.5 Selected Special Cases

In this subsection, we present three special cases of the EHL-TIITL-G FoD. We looked at cases where the underlying distribution is Burr XII, Weibull and Uniform distributions.

3.5.1 Exponentiated Half-Logistic Type II Topp-Leone-Weibull (EHL-TIITL-W) Distribution

If we take $G(x; \gamma, \theta) = 1 - \exp(-\theta x^\gamma)$ and $g(x; \gamma, \theta) = \gamma \theta x^{\gamma-1} \exp(-\theta x^\gamma)$ as the baseline distribution for $\theta, \gamma > 0$ and $x > 0$, then, the EHL-TIITL-W distribution has cdf and pdf given

by

$$F(x; \alpha, b, \gamma, \theta) = \left[\frac{1 - (1 - (1 - e^{-\theta x^\gamma})^2)^b}{1 + (1 - (1 - e^{-\theta x^\gamma})^2)^b} \right]^\alpha$$

and

$$\begin{aligned} f(x; \alpha, b, \gamma, \theta) &= \frac{4b\alpha\gamma\theta x^{\gamma-1} e^{-\theta x^\gamma} (1 - e^{-\theta x^\gamma}) (1 - (1 - e^{-\theta x^\gamma})^2)^{b-1}}{(1 + (1 - (1 - e^{-\theta x^\gamma})^2)^b)^{\alpha+1}} \\ &\times (1 - (1 - (1 - e^{-\theta x^\gamma})^2)^b)^{\alpha-1}, \end{aligned}$$

for $\alpha, b, \gamma, \theta > 0$.

The versatility and flexibility of the EHL-TIITL-W distribution can be seen in Figure 3.3. From the plots of the pdf, the EHL-TIITL-W distribution can handle a variety of data sets including skewed to the left and skewed to the right. In terms of the hrf, it can be seen that the shapes include increasing, inverted bathtub decreasing and bathtub shapes.

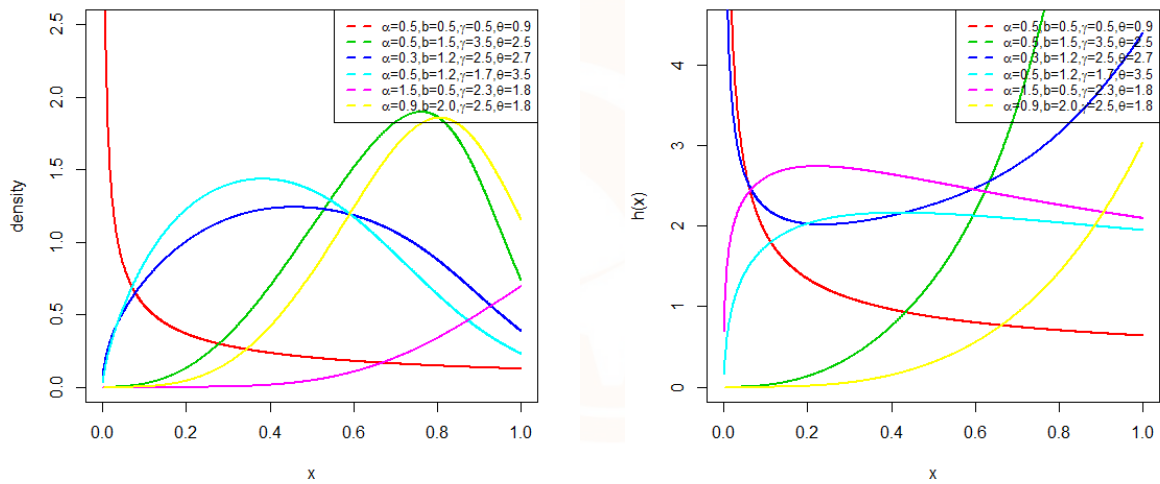


Figure 3.3: Plots of the pdf and hrf of the EHL-TIITL-W distribution

3.5.2 EHL-Type II Topp-Leone-Burr XII (EHL-TIITL-BXII) Distribution

Taking the Burr XII distribution with cdf $G(x; \theta, \gamma) = 1 - (1 + x^\theta)^{-\gamma}$ and pdf $g(x; \theta, \gamma) = \theta\gamma x^{\theta-1} (1 + x^\theta)^{-\gamma-1}$ for $\theta, \gamma > 0$ and $x > 0$ as a baseline distribution, we obtain the EHL-TIITL-BXII distribution with cdf and pdf given by

$$F(x; \alpha, b, \theta, \gamma) = \left[\frac{1 - (1 - (1 - (1 + x^\theta)^{-\gamma})^2)^b}{1 + (1 - (1 - (1 + x^\theta)^{-\gamma})^2)^b} \right]^\alpha$$

and

$$f(x; \alpha, b, \theta, \gamma) = \frac{4b\alpha\theta\gamma x^{\theta-1}(1+x^\theta)^{-\gamma-1}(1-(1+x^\theta)^{-\gamma})(1-(1-(1+x^\theta)^{-\gamma})^2)^{b-1}}{(1+(1-(1-(1+x^\theta)^{-\gamma})^2)^b)^{\alpha+1}} \\ \times (1-(1-(1-(1+x^\theta)^{-\gamma})^2)^b)^{\alpha-1},$$

for $\alpha, b, \theta, \gamma > 0$. Note that, if we set $\gamma = 1$ we obtain the EHL-Type II Topp Leone-Log-Logistic (EHL-TIITL-LLoG) distribution. Also, when $\theta = 1$, we obtain the EHL-Type II Topp Leone-Lomax (EHL-TIITL-L) distribution.

Presented in Figure 3.4 are the graphs of the pdf and the hrf of the EHL-TIITL-BXII distribution. It can be seen that the pdf can take various shapes including almost symmetric, the reverse-J, left and right-skewed. Graphs of the hrf shows decreasing, increasing, bathtub and inverted bathtub shapes.

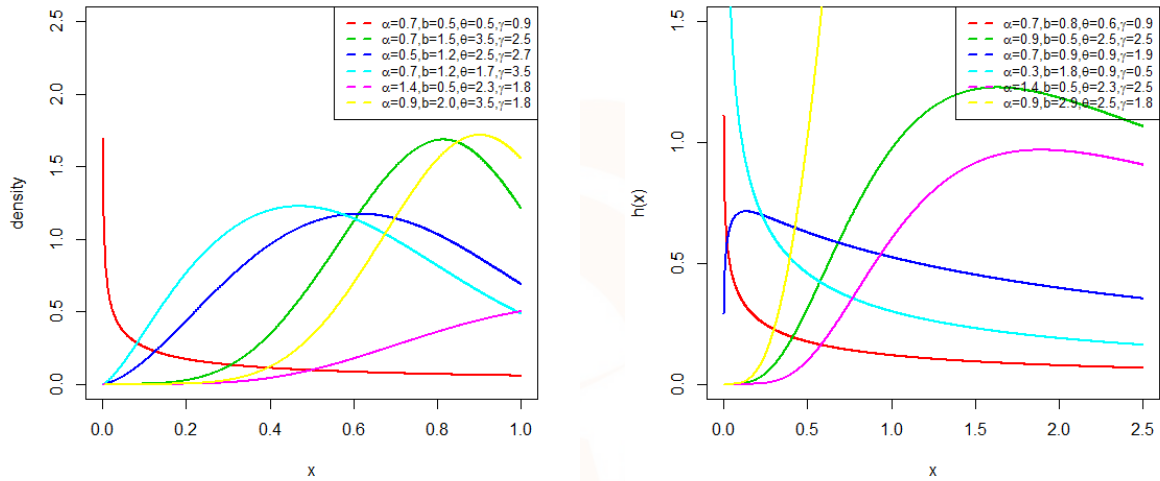


Figure 3.4: Plots of the pdf and hrf of the EHL-TIITL-BXII distribution

3.5.3 Exponentiated Half-Logistic Type II-Topp-Leone-Uniform (EHL-TIITL-U) Distribution

Suppose we take a uniform distribution with cdf and pdf stated as $G(x; \gamma) = \frac{x}{\gamma}$ and $g(x; \gamma) = \frac{1}{\gamma}$ respectively, for $0 < x < \gamma$, as the baseline model, we obtain the EHL-TIITL-U distribution with cdf and pdf given by

$$F(x; \alpha, b, \gamma) = \left[\frac{1 - (1 - (\frac{x}{\gamma})^2)^b}{1 + (1 - (\frac{x}{\gamma})^2)^b} \right]^\alpha$$

and

$$f(x; \alpha, b, \gamma) = \frac{4b\alpha \frac{x}{\gamma^2} (1 - (\frac{x}{\gamma})^2)^{b-1} (1 - (1 - (\frac{x}{\gamma})^2)^b)^{\alpha-1}}{(1 + (1 - (\frac{x}{\gamma})^2)^b)^{\alpha+1}},$$

for $\alpha, b, \gamma > 0$.

Figure 3.5 displays the graphs of the pdf and the hrf of the EHL-TIITL-U distribution for different parameter values. The pdf can take various shapes including almost symmetric, the reverse-J, left and right-skewed, U-shaped. Plots of the hrf reveals bathtub and increasing shapes.

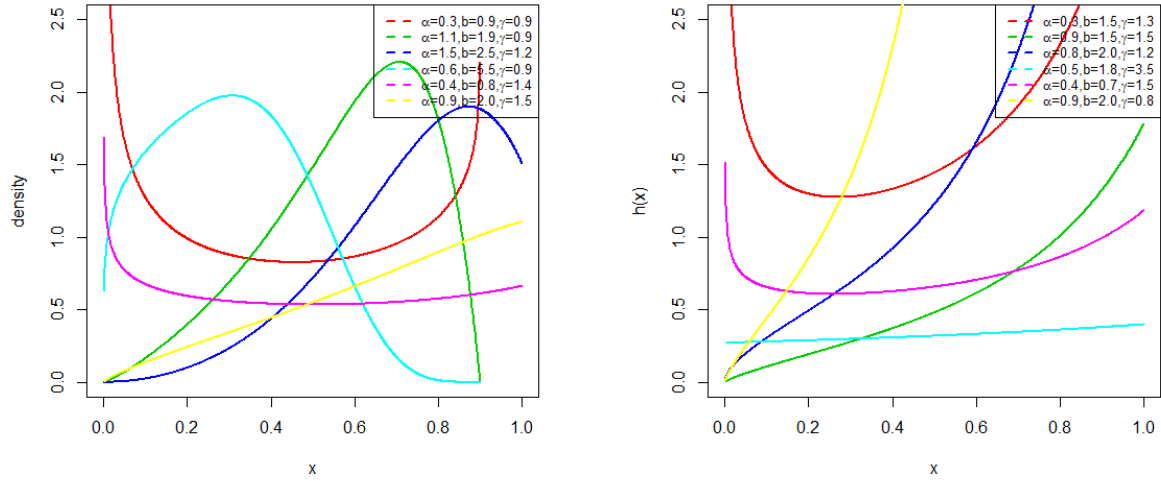


Figure 3.5: Plots of the pdf and hrf of the EHL-TIITL-U distribution

3.6 Estimation and Inferences

In this section, we used the method of maximum likelihood to obtain estimates of the parameters of the EHL-TIITL-G FoD. Suppose a random variable $X \sim EHL - TIITL - G(\alpha, b, \varphi)$ and $\Delta = (\alpha, b, \varphi)^T$ the parameter vector for the model. The log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n is

$$\begin{aligned}
 \ell_n(\Delta) &= n \ln(4b) + n \ln \alpha + \sum_{i=1}^n \ln(g(x_i; \varphi)) + \sum_{i=1}^n \ln(G(x_i; \varphi)) \\
 &+ (b-1) \sum_{i=1}^n \ln(1 - G^2(x_i; \varphi)) + (\alpha-1) \sum_{i=1}^n \ln(F_{TIITL-G}(x_i; b, \varphi)) \\
 &- (\alpha-1) \sum_{i=1}^n \ln(1 + (1 - G^2(x_i; \varphi))^b).
 \end{aligned} \tag{3.13}$$

By differentiating $\ell_n(\Delta)$ with respect to (w.r.t) each component of $\Delta = (\alpha, b, \varphi)^T$ we obtain the elements of the score vector $U(\Delta)$ which are given by

$$U_\alpha(\Delta) = \frac{n}{\alpha} + \sum_{i=1}^n \ln(F_{TII TL-G}(x_i; b, \varphi))^b - \sum_{i=1}^n \ln(1 + (1 - G^2(x_i; \varphi))^b), \quad (3.14)$$

$$\begin{aligned} U_b(\Delta) &= \frac{n}{b} - (\alpha - 1) \sum_{i=1}^n \frac{(1 - G^2(x_i; \varphi))^b \ln(1 - G^2(x_i; \varphi))}{(F_{TII TL-G}(x_i; b, \varphi))} \\ &+ \sum_{i=1}^n \ln(1 - G^2(x_i; \varphi)) - (\alpha + 1) \sum_{i=1}^n \frac{(1 - G^2(x_i; \varphi))^b \ln(1 - G^2(x_i; \varphi))}{(1 + (1 - G^2(x_i; \varphi))^b)} \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} U_{\varphi_k}(\Delta) &= \sum_{i=1}^n \frac{\partial g(x_i; \varphi)/\partial \varphi_k}{g(x_i; \varphi)} + (b - 1) \sum_{i=1}^n \frac{\partial(1 - G^2(x_i; \varphi))/\partial \varphi_k}{(1 - G^2(x_i; \varphi))} \\ &+ \sum_{i=1}^n \frac{\partial G(x_i; \varphi)/\partial \varphi_k}{G(x_i; \varphi)} + (\alpha - 1) \sum_{i=1}^n \frac{\partial(F_{TII TL-G}(x_i; b, \varphi))/\partial \varphi_k}{(F_{TII TL-G}(x_i; b, \varphi))} \\ &- (\alpha + 1) \sum_{i=1}^n \frac{\partial(1 + (1 - G^2(x_i; \varphi))^b)/\partial \varphi_k}{(1 + (1 - G^2(x_i; \varphi))^b)}. \end{aligned} \quad (3.16)$$

By solving the non-linear equation $(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \varphi_k})^T = \mathbf{0}$, we obtain the estimates of the parameters $\hat{\Delta}$. Since we can not obtain explicit solutions of the likelihood equations, we make use of some numerical techniques for example Newton-Raphson procedure to optimize the likelihood function. For purposes of inferences to construct CIs and confidence regions of the individual parameters, we assumed that under standard regularity conditions the the MLEs have approximately a multivariate normal distribution $N_{p+2}(\mathbf{0}, J(\hat{\Delta})^{-1})$, where $\mathbf{0} = (0, 0, \mathbf{0})^T$ is the mean vector and $J(\hat{\Delta})^{-1}$ is the observed FIM evaluated at $\hat{\Delta}$.

3.7 Simulations

A simulation study was conducted to evaluate the performance of the EHL-TIITL-W distribution. We simulated $N = 1000$ samples for various sizes ($n=50, 100, 200, 400, 800, 1000$), this was done using the R package. Tables 3.2 and 3.3 shows the simulation results and it can be seen that the mean estimates of the parameters are approaching the initialised parameter values with an increase in the sample size. The ABias and the RMSEs are also decreasing towards zero as we increase the sample size. This shows that the MLE of the model parameters are strongly consistent and the results from the estimation can be trusted in making inferences about model parameters. For the estimated parameter, say $\hat{\kappa}$, ABias and RMSE are given by:

$$ABias(\hat{\kappa}) = \frac{\sum_{i=1}^N \hat{\kappa}_i}{N} - \kappa, \quad \text{and} \quad RMSE(\hat{\kappa}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\kappa}_i - \kappa)^2}{N}},$$

respectively.

parameter	Sample Size	(0.7, 0.3, 0.3, 0.9)			(0.5, 0.3, 0.3, 0.2)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
α	50	2.0716	3.8409	1.3716	1.1240	1.1035	0.6240
	100	1.3069	1.0641	0.6069	0.9022	0.7422	0.4022
	200	0.9583	0.4057	0.2583	0.6936	0.3234	0.1936
	400	0.8467	0.2423	0.1467	0.6046	0.1903	0.1046
	800	0.7768	0.1583	0.0768	0.5735	0.1111	0.0735
	1000	0.7520	0.1050	0.0520	0.5042	0.0741	0.0042
b	50	4.0168	6.7624	3.7168	5.8827	10.3558	5.5827
	100	2.1943	4.0595	1.8943	3.3136	6.8296	3.0136
	200	1.0892	1.8621	0.7892	1.6606	3.2189	1.3606
	400	0.7795	1.2219	0.4795	0.6840	1.0101	0.3840
	800	0.4029	0.3358	0.1029	0.4368	0.3029	0.1368
	1000	0.3204	0.1183	0.0204	0.3366	0.1470	0.0366
γ	50	0.1927	0.1237	-0.1073	0.2142	0.1309	-0.0858
	100	0.2156	0.1013	-0.0844	0.2395	0.1110	-0.0605
	200	0.2484	0.0692	-0.0516	0.2538	0.0823	-0.0462
	400	0.2642	0.0544	-0.0358	0.2721	0.0542	-0.0279
	800	0.2800	0.0335	-0.0200	0.2755	0.0391	-0.0245
	1000	0.2892	0.0246	-0.0108	0.3051	0.0382	0.0051
θ	50	1.3875	1.5774	0.4875	0.5084	0.8340	0.3084
	100	1.2561	1.1734	0.3561	0.4502	0.6683	0.2502
	200	1.2186	0.9386	0.3186	0.3248	0.3835	0.1248
	400	1.0177	0.5698	0.1177	0.2719	0.2475	0.0719
	800	1.0332	0.4644	0.1332	0.2473	0.1311	0.0473
	1000	0.9987	0.2908	0.0987	0.1863	0.0662	-0.0137

Table 3.2: Simulation Results

parameter	Sample Size	(0.7, 0.5, 0.5, 0.2)			(1.0, 1.5, 0.6, 1.8)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
α	50	1.5667	2.0830	0.8667	2.0523	4.7999	1.0523
	100	1.1225	0.9514	0.4225	1.3621	0.9659	0.3621
	200	0.8952	0.4028	0.1952	1.1338	0.4464	0.1338
	400	0.8245	0.2416	0.1245	1.0917	0.3004	0.0917
	800	0.7722	0.1677	0.0722	1.0545	0.2114	0.0545
	1000	0.7584	0.1399	0.0584	1.0492	0.1972	0.0492
b	50	6.1019	12.2763	5.6019	8.6312	17.8288	7.1312
	100	4.4174	8.2893	3.9174	7.5343	13.2828	6.0343
	200	2.9775	5.5220	2.4775	4.9808	7.9097	3.4808
	400	2.5918	4.2024	2.0918	4.1267	5.7617	2.6267
	800	1.8521	3.0269	1.3521	3.6179	4.7746	2.1179
	1000	1.4822	2.2386	0.9822	3.3955	4.1966	1.8955
γ	50	0.4019	0.2112	-0.0981	0.6053	0.2945	0.0053
	100	0.4314	0.1901	-0.0686	0.6083	0.2406	0.0083
	200	0.4452	0.1455	-0.0548	0.6266	0.1940	0.0266
	400	0.4400	0.1148	-0.0600	0.6095	0.1524	0.0095
	800	0.4581	0.0956	-0.0419	0.6024	0.1253	0.0024
	1000	0.4618	0.0851	-0.0382	0.6012	0.1236	0.0012
θ	50	0.6323	1.0010	0.4323	4.5573	5.1293	2.7573
	100	0.3934	0.6376	0.1934	3.6014	4.0646	1.8014
	200	0.2745	0.3283	0.0745	3.3119	3.4814	1.5119
	400	0.2150	0.1768	0.0150	3.0342	3.0279	1.2342
	800	0.1897	0.0988	-0.0103	2.6131	2.3250	0.8131
	1000	0.1916	0.0899	-0.0084	2.5160	2.2084	0.7160

Table 3.3: Simulation Results

3.8 Applicability of the EHL-TIITL-G Distribution

Here we demonstrate the applicability of the EHL-TIITL-G FoD by fitting the EHL-TIITL-W distribution to two real data sets in comparison to the fits of the EHL distribution, the Half Logistic distribution (Johnson et al. [69]), the Type II Topp-Leone Power Lomax distribution by Al-Marzouki et al. [16], the Type II Topp-Leone Dagum (TIITLD) distribution (Sakthivel and Dhivakar [115]), the Odd Exponentiated Half-Logistic Burr XII (OEHLBXII) by Aldahlan et al. [15]. To assess the performance of these models, the GoF statistics $-2\ln(L)$, AIC, CAIC, BIC, CVM (W^*), AD (A^*), K-S statistic and its p-value, as described by Chen and Balakrishnan [36] and the SS are presented in Table 3.4 and Table 3.5. These GoF statistics are used to show which distribution fits the data better. The general rule is the smaller the values of $(-2\log L)$, AIC, CAIC, BIC, W^* , A^* and SS, the better the fit. The pdfs of these distributions are given in the appendix.

We presented plots of the histograms, probability plots (Chambers et al. [34]), fitted densities and box plots of the data in Figure 3.6 and Figure 3.10. Figure 3.9 and Figure 3.11 shows the estimated cumulative density function, K-M survival plots and the estimated hazard rate plot for the EHL-TIITL-W distribution. We plotted $F(x_{(j)})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, to obtain our probability plots, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness was obtained and is given by the sum of squares (SS):

$$SS = \sum_{j=1}^n \left[F(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

3.8.1 Glass Fibre Data

The EHL-TIITL-W was fitted to the strength of glass fibre data, it consists of 63 observations of the strengths of 1.5 cm glass fibres. Smith and Naylor [121] also analysed this data. The data are as follows: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Model	Estimates				Statistics								
	α	b	γ	θ	-2logL	AIC	CAIC	BIC	W^*	A^*	K-S	P-value	SS
EHL-TIITL-W	0.7920 (0.2552)	235.8400 (0.0004)	3.0064 (0.6392)	0.0173 (0.0078)	27.3583	35.3583	36.0480	43.9309	0.1688	0.9393	0.1396	0.1717	0.1633
EHL-TIITL-W(1,1, γ,θ)	-	-	3.5577 (0.3619)	0.3808 (0.0778)	31.7561	35.7561	35.9561	40.0423	0.2742	1.5030	0.1592	0.0820	0.2392
EHL-TIITL-W($\alpha,1,\gamma,1$)	1.9855 (0.2657)	-	2.2211 (0.1184)	-	39.9583	43.9583	44.1583	48.2445	0.4300	2.3622	0.2097	0.0079	0.4962
EHL	-	-	17.1769 (4.9707)	2.6559 (0.2297)	61.2453	65.2453	65.4453	69.5316	0.7577	4.1418	0.2237	0.0036	0.7530
HL	1.0127 (0.0989)	-	-	-	155.0062	157.0062	157.0718	159.1493	0.5209	2.8596	0.4137	8.59×10^{-10}	3.6496
W	5.7807 (0.5761)	1.6281 (0.0371)	-	-	30.4137	34.4137	34.6137	38.7000	0.2372	1.3037	0.1522	0.1078	0.2111
TIITLPL	2200.3 (0.0070)	42.3960 (7.2957)	2.9003 (0.2893)	25756 (0.0120)	30.4442	38.4442	39.1339	47.0168	0.2381	1.3081	0.1524	0.1073	0.2115
TIITLD	16255 (6.7509×10^{-6})	10815 (1.3584×10^{-5})	4.2938 (0.3219)	0.6737 (0.0185)	30.4194	38.4194	39.1090	46.9919	0.2374	1.3047	0.1523	0.1076	0.2113
HLBXII-LL	0.3225 (0.0670)	0.0030 (0.0036)	11.8172 (0.0075)	0.8356 (0.1347)	50.3244	58.3244	59.0141	66.8969	0.2417	1.3747	0.1423	0.1558	0.3093

Table 3.4: MLEs of the EHL-TIITL-W Model for Glass Fibre Data

To assess the efficacy of the EHL-TIITL-W distribution in fitting the glass fibre data, we used the GoF statistics presented in Table 3.4, plots of the fitted densities as well as the probability plots presented in Figure 3.6. Based on those statistics and plots, we conclude that the EHL-TIITL-W distribution provides the best fit to the glass fibre data compared to the other models presented.

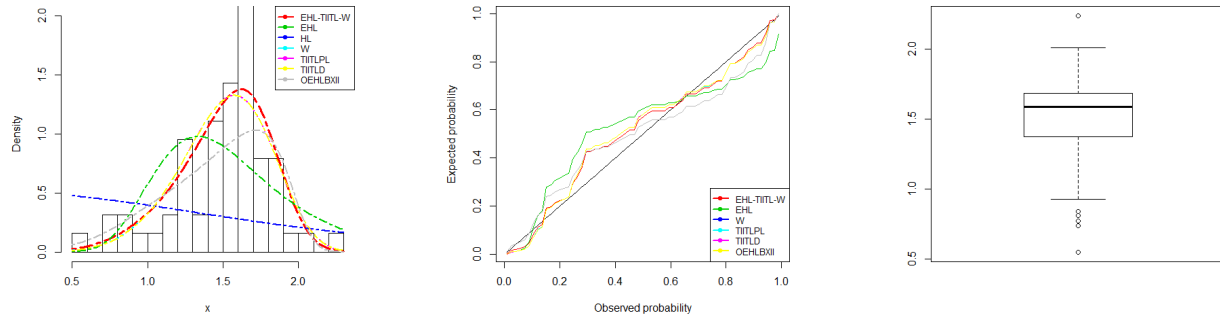


Figure 3.6: Fitted densities, probability plots and box plot for glass fibre data

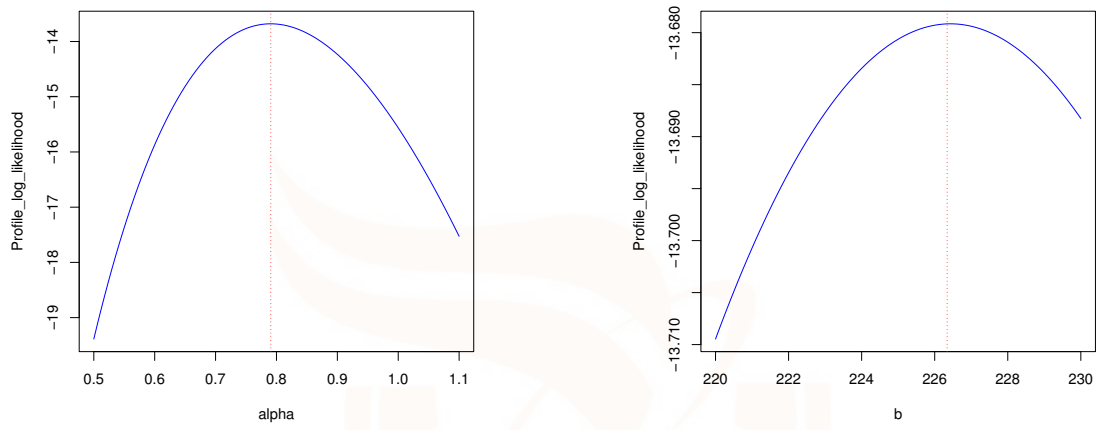


Figure 3.7: Profile plots for the glass fibre data

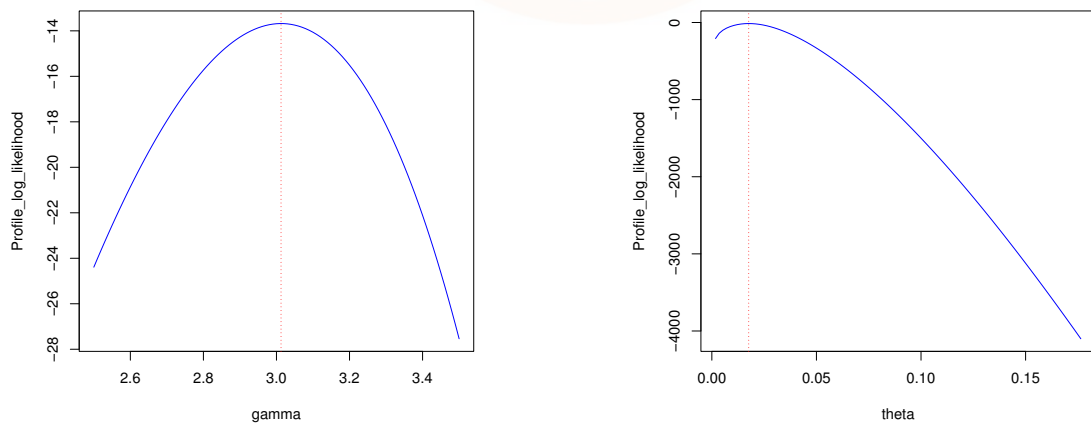


Figure 3.8: Profile plots for the glass fibre data

The profile plots in Figures 3.7 and 3.8 shows that the estimates of the EHL-TIITL-W distribution are global maximums and can be uniquely determined for the glass fibre data. The

estimated var-cov matrix for the glass fibre data is given by,

$$\begin{bmatrix} 0.0651 & -8.8430 \times 10^{-5} & -0.1431 & 0.0018 \\ -8.8430 \times 10^{-5} & 1.4924 \times 10^{-7} & 0.0002 & -2.9727 \times 10^{-6} \\ -0.1431 & 0.0002 & 0.4086 & -0.0049 \\ 0.0018 & -2.9727 \times 10^{-6} & -0.0049 & 6.0502 \times 10^{-5} \end{bmatrix}$$

and the 95% asymptotic CIs for the parameters α, b, γ and θ are: 0.7920 ± 0.5002 , 235.8400 ± 0.0008 , 3.0064 ± 1.2528 and 0.0173 ± 0.0153 , respectively.

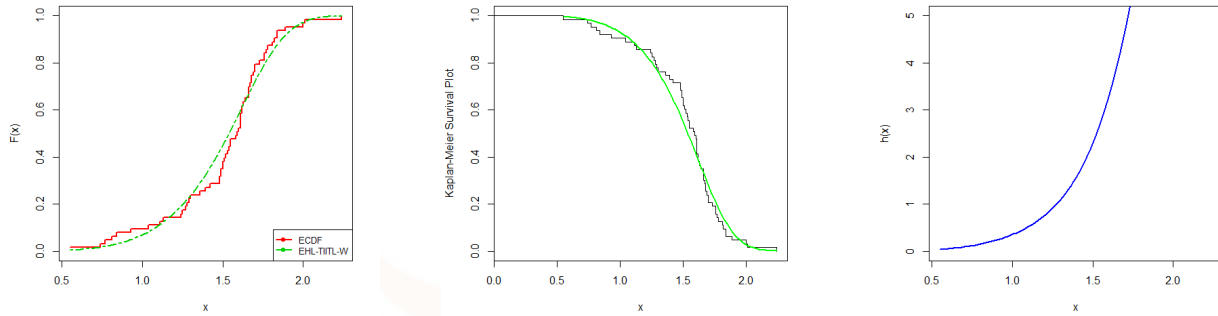


Figure 3.9: Estimated cdf, Kaplan-Meier survival plots and estimated hazard rate plot of the EHL-TIITL-W distribution for glass fibre data

3.8.2 Plasma Concentration

The EHL-TIITL-W was fitted to the plasma concentration data set which was taken from R base package in the Indometh object. The data are as follows: 1.50, 0.94, 0.78, 0.48, 0.37, 0.19, 0.12, 0.11, 0.08, 0.07, 0.05, 2.03, 1.63, 0.71, 0.70, 0.64, 0.36, 0.32, 0.20, 0.25, 0.12, 0.08, 2.72, 1.49, 1.16, 0.80, 0.80, 0.39, 0.22, 0.12, 0.11, 0.08, 0.08, 1.85, 1.39, 1.02, 0.89, 0.59, 0.40, 0.16, 0.11, 0.10, 0.07, 0.07, 2.05, 1.04, 0.81, 0.39, 0.30, 0.23, 0.13, 0.11, 0.08, 0.10, 0.06, 2.31, 1.44, 1.03, 0.84, 0.64, 0.42, 0.24, 0.17, 0.13, 0.10, 0.09.

Table 3.5 and Figure 3.10 displays the goodness-of-fit statistics, the fitted densities and probability plots, respectively. We note that the EHL-TIITL-W model provides the best fit based on the GoF statistics and the plots compared to other competing models presented.

Model	Estimates				Statistics								
	α	b	γ	θ	-2logL	AIC	CAIC	BIC	W*	A*	K-S	P-value	SS
EHL-TIITL-W	104.3400 (7.7125×10^{-06})	0.4636 (0.0163)	0.1703 (0.1504)	15.9850 (0.0004)	55.7905	63.7905	64.4462	72.5491	0.2237	1.3963	0.1317	0.2024	0.2347
EHL-TIITL-W($1, 1, \gamma, \theta$)	-	-	0.5971 (0.0572)	2.9935 (0.2480)	62.2080	66.2080	66.3984	70.5873	0.2504	1.5989	0.140	0.1505	0.2341
EHL-TIITL-W($\alpha, 1, \gamma, 1$)	0.2161 (0.0333)	-	1.5769 (0.1879)	-	68.4827	72.4827	72.6732	76.8620	0.2867	1.8619	0.1426	0.1364	0.4043
EHL	-	-	0.7653 (0.1156)	1.9197 (0.2789)	67.2545	71.2545	71.4450	75.6340	0.2927	1.8667	0.1459	0.1203	0.3292
HL	2.2304 (0.2377)	-	-	-	70.5479	72.5479	72.6104	74.7375	0.2942	1.8723	0.2045	0.0080	0.7038
W	0.9546 (0.0904)	0.5787 (0.0791)	-	-	62.5121	66.5121	66.7026	70.8914	0.2509	1.6080	0.1348	0.1817	0.2263
TIITLPL	0.5368 (0.0664)	1.6471×10^5 (2.5932×10^{-7})	0.7243 (0.0625)	4.1860×10^4 (1.0203×10^{-6})	59.1377	67.1376	67.7934	75.8962	0.2279	1.4584	0.1414	0.1430	0.2198
TIITLD	3.4768 (0.0032)	0.0054 (0.0006)	0.5442 (0.0487)	83.4750 (2.1108×10^{-6})	56.3737	64.3737	65.0295	73.1324	0.2258	1.4147	0.1345	0.1833	0.2302
HLBXII-LL	0.0777 (0.0276)	0.2749 (0.3009)	8.0388 (2.9359)	0.3081 (0.1267)	73.4574	81.4574	82.1132	90.2160	0.3220	2.0260	0.1408	0.1459	0.3733

Table 3.5: Estimation of the EHL-TIITL-W Model for Plasma Concentration Data

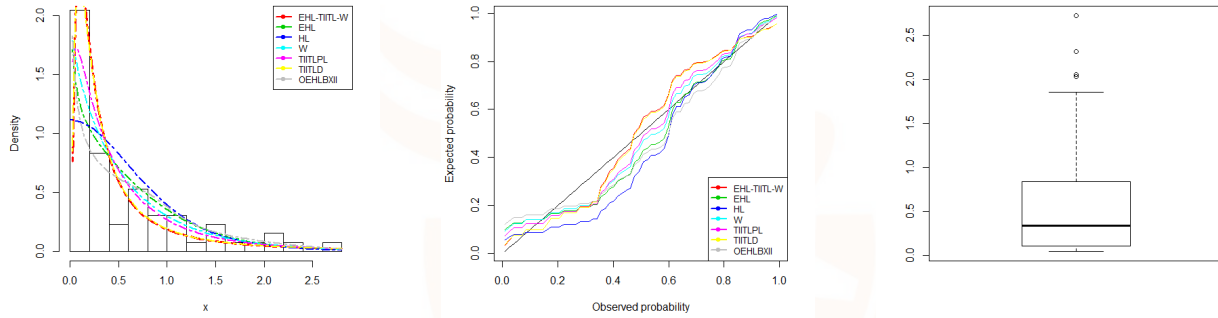


Figure 3.10: Fitted densities, probability plots and box plot for plasma concentration data

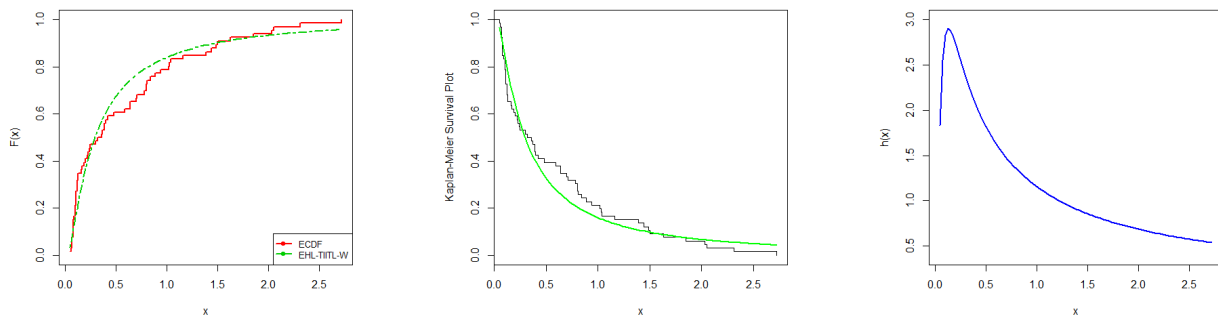


Figure 3.11: Estimated cdf, Kaplan-Meier survival plots and estimated hazard rate plot of the EHL-TIITL-W distribution for plasma concentration data set.

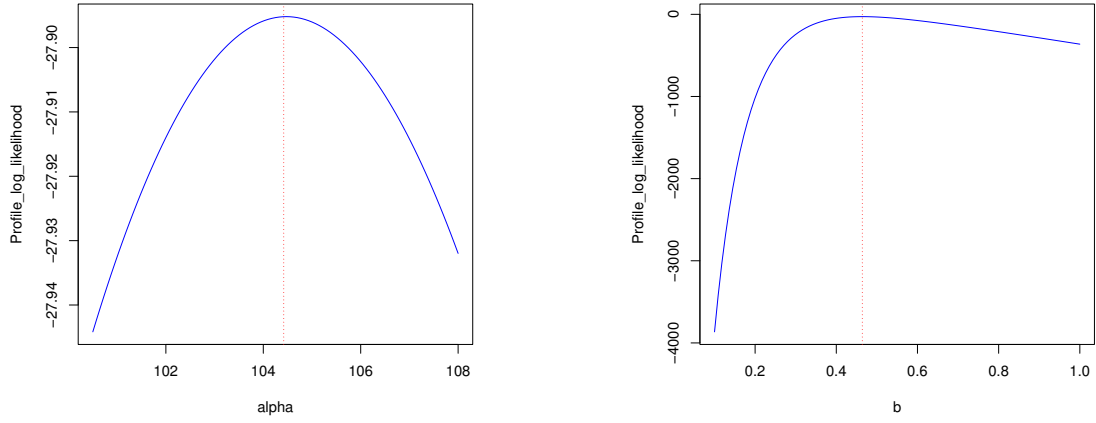


Figure 3.12: Profile plots for the plasma concentration data

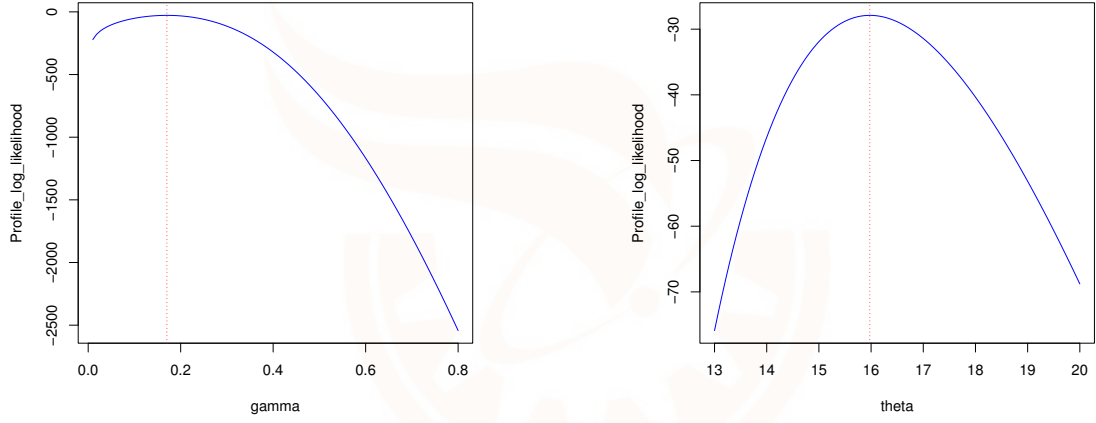


Figure 3.13: Profile plots for the plasma concentration data

The profile plots in Figures 3.12 and 3.13 shows that the estimates of the EHL-TIITL-W distribution can be uniquely identified. The estimated var-cov matrix for the plasma concentrations data is given by,

$$\begin{bmatrix} 5.9483 \times 10^{-11} & -1.1668 \times 10^{-7} & -4.7845 \times 10^{-8} & -3.5097 \times 10^{-9} \\ -1.1668 \times 10^{-7} & 0.0003 & 0.0002 & 7.9935 \times 10^{-6} \\ -4.7845 \times 10^{-8} & 0.0002 & 0.0002 & 5.2805 \times 10^{-6} \\ -3.5097 \times 10^{-9} & 7.9935 \times 10^{-6} & 5.2805 \times 10^{-6} & 2.3927 \times 10^{-7} \end{bmatrix}$$

and the 95% asymptotic CIs for the parameters α, b, γ and θ are: $104.3400 \pm 1.5117 \times 10^{-5}$, 0.4636 ± 0.0319 , 0.1703 ± 0.2948 and 15.9850 ± 0.0008 , respectively.

Model	df	Glass fibre data		Plasma concentration data	
		Chi-Square	p-value	Chi-Square	p-value
EHL-TIITL-W($1, 1, \gamma, \theta$)	2	4.3592	0.03681	6.4175	0.0113
EHL-TIITL-W($\alpha, 1, \gamma, 1$)	2	12.600	0.000386	12.6922	0.000367
EHL	2	33.887	< 0.00001	11.4640	0.00071
HL	1	127.6479	< 0.00001	14.7574	0.000122

Table 3.6: **Likelihood ratio test results**

The likelihood ratio (LR) test are presented in Table 3.6. The results show that there is a significant difference between the EHL-TIITL-W distribution and its nested models: (EHL-TIITL-W($1, 1, \gamma, \theta$), EHL-TIITL-W($\alpha, 1, \gamma, 1$), EHL, HL) for both data sets. Also, based on the lowest values of other GoF statistics, we can infer that the EHL-TIITL-W distribution provides the best fit to the glass fibre data and the plasma concentration data compared to the its nested models.

3.9 Conclusions

We developed a new FoD called EHL-TIITL-G distribution. We derived some mathematical and statistical properties of the new model and assessed the effectiveness of a particular case of the EHL-TIITL-W distribution by carrying out a number of simulations for various sample sizes. The EHL-TIITL-W distribution was applied to two real data sets to demonstrate the potentiality of the new FoD.

4. The Marshall-Olkin-Topp-Leone Half Logistic-G Family of Distributions with Applications

Sengweni, W., Oluyede, B. and Makubate, B., The Marshall-Olkin-Topp-Leone Half Logistic-G Family of Distributions.

Journal of Statistics, Optimisation and Information Computing, Vol. x, March 2021, pp 0-25,
https://drive.google.com/file/d/16cZGce_pBAIUtnZTnf6V5R3l1IxIkQRJ/view?usp=sharing

4.1 Introduction

In this chapter, we develop and study a new generalized FoD called the Marshall-Olkin-Topp-Leone Half Logistic-G (MO-TLHL-G) FoD. The need for more flexible models that can explain and provide better fit to real life data sets has motivated many researchers to develop new generalized distributions by extending the classical ones. Marshall and Olkin [85] introduced a new method of adding a parameter into a FoD. Addition of parameters to an existing baseline distribution has proven to be an effective technique for improving the flexibility of new FoD (Barreto-Souza et al. [27]). The Marshall-Olkin technique has been applied to build several well known distributions which include papers by Barreto-Souza et al. [27], Cordeiro et al. [52], Santos-Neo et al. [122], Chakraborty et al. [35], Lazhar et al. [74], Kumar et al. [73], Lepetu et al. [76] and Ghitany et al. [61]).

We employed the Marshall-Olkin transformation in this chapter to come up with a new FoD called the Marshall-Olkin-Topp-Leone Half Logistic-G (MO-TLHL-G) FoD. The desirable properties and flexibility obtained from this new extended FoD with regards to the shapes of the density and hrf which makes the model useful in lifetime analysis has motivated us to develop this model.

The findings in this chapter are arranged as follows. The MO-TLHL-G FoD, its sub-models, hrf, qf, stochastic ordering, series expansion of the pdf and some special cases are all presented in Section 4.2. Moments, mgf and incomplete moments are provided in Section 4.3. The RE and distribution of order statistics are presented in Section 4.4. A maximum likelihood approach for parameter estimation is presented in Section 4.5. In Section 4.6 results on MCS carried out to evaluate the behaviour of ABias and RMSE of maximum likelihood estimators are presented. Examples on the applicability of the new model to real data are given in Section 4.7. In Section 4.8 we provided some concluding remarks.

4.2 The New Generalized Model and Properties

Some statistical properties of the Marshall-Olkin Topp-Leone Half Logistic-G (MO-TLHL-G) FoD including special cases, density expansion, stochastic ordering, hrf and qf are provided in this section.

4.2.1 The New Generalized Model

Following the Marshall-Olkin extension [85], the cdf and pdf of the Marshall-Olkin-Topp-Leone Half Logistic-G FoD are given by

$$F(x; b, \delta, \varphi) = \frac{F_{TLHL-G}(x; b, \varphi)}{1 - \bar{\delta} \left(1 - F_{TLHL-G}(x; b, \varphi)\right)} \quad (4.1)$$

and

$$f(x; b, \delta, \varphi) = \frac{4b\delta g(x; \varphi) \bar{G}(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{b-1}}{(1 + (1 - [1 - \bar{G}^2(x; \varphi)]^b))^2 \left\{1 - \bar{\delta} \left(1 - F_{TLHL-G}(x; b, \varphi)\right)\right\}^2}, \quad (4.2)$$

where $F_{TLHL-G}(x; b, \varphi) = \frac{[1 - \bar{G}^2(x; \varphi)]^b}{(1 + (1 - [1 - \bar{G}^2(x; \varphi)]^b))}$, $\delta, b > 0$, $\bar{\delta} = 1 - \delta$ and φ is a parameter vector for the baseline distribution $G(\cdot)$.

4.2.2 Hazard Rate and Quantile Functions

In this subsection, we provide the hrf and qf of the MO-TLHL-G FoD. The hrf of the MO-TLHL-G FoD is given by

$$h(x; b, \delta, \varphi) = \frac{2bg(x; \varphi) \bar{G}(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{b-1} (1 - (1 - \bar{G}^2(x; \varphi))^b)^{-1}}{(1 + (1 - [1 - \bar{G}^2(x; \varphi)]^b))^2 \left\{1 - \bar{\delta} \left(1 - F_{TLHL-G}(x; b, \varphi)\right)\right\}^2}. \quad (4.3)$$

The qf of the MO-TLHL-G FoD is given by

$$Q(u) = G^{-1} \left[1 - \left(1 - \left[\frac{2\delta u}{1 + u - 2u\bar{\delta}} \right]^{\frac{1}{b}} \right)^{\frac{1}{2}} \right]. \quad (4.4)$$

4.2.3 Stochastic Order

Stochastic ordering is an important tool in the comparison of probability models in areas such as reliability, survival analysis, finance, risks and economics. In this section, we present a stochastic ordering result from the MO-TLHL-G FoD.

Suppose X and Y be two rvs with cumulative distribution functions (cdfs) $F_X(t)$ and $F_Y(t)$, respectively. Then X is stochastically smaller than Y if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all t , where $\bar{F}_X(t) = 1 - F_X(t)$ is the survival function. Stochastic ordering of X and Y is denoted by $X <_{st} Y$. A stronger ordering is the LR order denoted by $X <_{lr} Y$, and given by $\frac{f_X(t)}{f_Y(t)}$

decreasing in t . Hazard rate order is stronger than stochastic order but weaker than LR order. Hazard rate order is given by $h_X(t) \geq h_Y(t)$ for all t , and denoted by $X <_{hr} Y$. Thus, $X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_{st} Y$ (see Shaked and Shanthikumar [116]).

Theorem 2.1. Suppose $X_1 \sim MO-TLHL-G(\delta_1, b, \varphi)$ and $X_2 \sim MO-TLHL-G(\delta_2, b, \varphi)$. If $\delta_1 < \delta_2$, then $\frac{f(x; \delta_1, b, \varphi)}{g(x; \delta_2, b, \varphi)}$ is decreasing in x .

Proof:

After some simplification, we have

$$\frac{f(x; \delta_1, b, \varphi)}{g(x; \delta_2, b, \varphi)} = \frac{\delta_1}{\delta_2} \left[\frac{1 - \bar{\delta}_2 (1 - F_{TLHL-G}(x; b, \varphi))}{1 - \bar{\delta}_1 (1 - F_{TLHL-G}(x; b, \varphi))} \right]^2, \quad (4.5)$$

where $\bar{\delta}_1 = 1 - \delta_1$, and $\bar{\delta}_2 = 1 - \delta_2$.

Differentiating equation (4.5) with respect to x , we have

$$\frac{\partial}{\partial x} \left(\frac{f(x; \delta_1, b, \varphi)}{g(x; \delta_2, b, \varphi)} \right) = \frac{2\delta_1}{\delta_2} (\bar{\delta}_2 - \bar{\delta}_1) \frac{(1 - \bar{\delta}_2 (1 - F_{TLHL-G}(x; b, \varphi)))}{(1 - \bar{\delta}_1 (1 - F_{TLHL-G}(x; b, \varphi)))^3} (F_{TLHL-G}(x; b, \varphi))',$$

where $F_{TLHL-G}(x; b, \varphi) = \frac{[1 - \bar{G}^2(x; \varphi)]^b}{1 + (1 - [1 - \bar{G}^2(x; \varphi)]^b)}$ and $(F_{TLHL-G}(x; b, \varphi))' = \frac{dF_{TLHL-G}(x; b, \varphi)}{dx}$.

If $\delta_1 < \delta_2$, then $\frac{\partial}{\partial x} \left(\frac{f(x; \delta_1, b, \varphi)}{g(x; \delta_2, b, \varphi)} \right) < 0$ which implies that the likelihood ratio order exist among X_1 and X_2 , that is $X_2 <_{lr} X_1$. Consequently, X_1 and X_2 are stochastically ordered, that is $X_1 <_{st} X_2$.

4.2.4 Linear Representation

In this subsection, we provide a linear representation for the pdf, $f(x)$ of the MO-TLHL-G distribution. The pdf in equation (4.2) can be written as

$$\begin{aligned} f(x) &= 4b\delta g(x; \varphi) \bar{G}(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{b-1} (1 + (1 - [1 - \bar{G}^2(x; \varphi)]^b))^{-2} \\ &\times \underbrace{\left\{ 1 - \bar{\delta} (1 - F_{TLHL-G}(x; b, \varphi)) \right\}^{-2}}_A. \end{aligned} \quad (4.6)$$

Using the generalized binomial series expansion, we have

$$A = \sum_{l,k=0}^{\infty} (-1)^{l+k} \binom{-2}{l} \binom{l}{k} \bar{\delta}^l F_{TLHL-G}^k(x; b, \varphi).$$

Substituting A in equation (4.6), we have

$$\begin{aligned}
 f(x) &= 4b\delta g(x; \varphi) \overline{G}(x; \varphi) \sum_{l,k=0}^{\infty} (-1)^{l+k} \binom{-2}{l} \binom{l}{k} \overline{\delta}^l (1 - \overline{G}^2(x; \varphi))^{b(k+1)-1} \\
 &\times \underbrace{(1 + (1 - [1 - \overline{G}^2(x; \varphi)]^b))^{-(k+2)}}_B.
 \end{aligned} \tag{4.7}$$

Applying the generalized binomial series expansion on B, we obtain

$$B = \sum_{i,j=0}^{\infty} (-1)^i \binom{-(k+2)}{j} \binom{j}{i} [1 - \overline{G}^2(x; \varphi)]^{bi}.$$

Substituting B in equation (4.7), we have

$$\begin{aligned}
 f(x) &= 4b\delta g(x; \varphi) \overline{G}(x; \varphi) \sum_{l,k,i,j=0}^{\infty} (-1)^{l+k+i} \binom{-2}{l} \binom{-(k+2)}{j} \binom{j}{i} \binom{l}{k} \overline{\delta}^l \\
 &\times \underbrace{(1 - \overline{G}^2(x; \varphi))^{b(k+i+1)-1}}_C.
 \end{aligned} \tag{4.8}$$

Using the binomial series expansion, we have

$$C = \sum_{q=0}^{\infty} (-1)^q \binom{b(k+i+1)-1}{q} \overline{G}^{2q}(x; \varphi).$$

Substituting C in equation (4.8) we obtain

$$\begin{aligned}
 f(x) &= 4b\delta g(x; \varphi) \sum_{l,k,i,j,q=0}^{\infty} (-1)^{l+k+i+q} \binom{-2}{l} \binom{-(k+2)}{j} \binom{j}{i} \binom{l}{k} \\
 &\times \binom{b(k+i+1)-1}{q} \overline{\delta}^l \overline{G}^{2q+1}(x; \varphi).
 \end{aligned} \tag{4.9}$$

Using the binomial series expansion, we have

$$(1 - G(x; \varphi))^{2q+1} = \sum_{p=0}^{\infty} (-1)^p \binom{2q+1}{p} G^p(x; \varphi).$$

Thus, the pdf of the MO-TLHL-G FoD can be expressed as

$$\begin{aligned}
 f(x) &= 4b\delta g(x; \varphi) \sum_{l,k,i,j,q,p=0}^{\infty} (-1)^{l+k+i+q+p} \binom{-2}{l} \binom{-(k+2)}{j} \binom{j}{i} \binom{l}{k} \\
 &\times \binom{b(k+i+1)-1}{q} \binom{2q+1}{p} \overline{\delta}^l G^p(x; \varphi).
 \end{aligned} \tag{4.10}$$

Or equivalently, we can write

$$f(x) = \sum_{p=0}^{\infty} c_{p+1} h_{p+1}(x; \varphi), \tag{4.11}$$

where

$$c_{p+1} = 4b\delta \sum_{l,k,i,j,q=0}^{\infty} \frac{(-1)^{l+k+i+q+p}}{p+1} \binom{-2}{l} \binom{-(k+2)}{j} \binom{j}{i} \binom{l}{k} \times \binom{b(k+i+1)-1}{q} \binom{2q+1}{p} \bar{\delta}^l, \quad (4.12)$$

and $h_{p+1}(x; \varphi) = (p+1)g(x; \varphi)G(x; \varphi)^p$ is the exp-G density with power parameter $(p+1)$. Thus, the MO-TLHL-G FoD can be written as an infinite linear combination of Exp-G densities.

Sub-Families

Some sub-families of the MO-TLHL-G FoD are given in Table 4.1.

	pdf	Model
If $\delta = 1$	$2bg(x; \varphi)(1 - \bar{G}^2(x; \varphi))^{b-1} \exp \left[- \left(\frac{[1 - \bar{G}^2(x; \varphi)]^b}{1 - [1 - \bar{G}^2(x; \varphi)]^b} \right) \right]$	Topp-Leone-Half Logistic-G FoD (New Model)
If $b = 1$	$\frac{2\delta g(x; \varphi) \exp \left[- \left(\frac{[1 - \bar{G}^2(x; \varphi)]}{\bar{G}^2(x; \varphi)} \right) \right]}{\left\{ 1 - \bar{\delta} \exp \left[- \left(\frac{1 - \bar{G}^2(x; \varphi)}{\bar{G}^2(x; \varphi)} \right) \right] \right\}^2}$	New FoD
If $\delta = b = 1$	$2g(x; \varphi) \exp \left[- \left(\frac{[1 - \bar{G}^2(x; \varphi)]}{\bar{G}^2(x; \varphi)} \right) \right]$	New FoD

Table 4.1: Some Sub-Families of the MO-TLHL-G Distribution

4.2.5 Special Cases

In this subsection, we consider some examples of the MO-TLHL-G FoD, when $G(x; \varphi)$, the baseline distribution is Burr XII, Weibull, Kumaraswamy, Burr III and Uniform distributions.

Marshall-Olkin-Topp-Leone Half Logistic-Burr XII Distribution

Consider, $G(x; \theta, \gamma) = 1 - (1 + x^\theta)^{-\gamma}$ and $g(x; \theta, \gamma) = \theta\gamma x^{\theta-1}(1 + x^\theta)^{-\gamma-1}$ for $\theta, \gamma > 0$ and $x > 0$. Then, the MO-TLHL-Burr XII (MO-TLHL-BXII) distribution has cdf and pdf given by

$$F(x; b, \delta, \theta, \gamma) = \frac{\frac{[1 - (1 + x^\theta)^{-2\gamma}]^b}{(1 + (1 - [1 - (1 + x^\theta)^{-2\gamma}]^b))}}{1 - \bar{\delta} \left(1 - \left[\frac{[1 - (1 + x^\theta)^{-2\gamma}]^b}{1 + (1 - [1 - (1 + x^\theta)^{-2\gamma}]^b)} \right] \right)}$$

and

$$f(x; b, \delta, \theta, \gamma) = \frac{4b\delta\theta\gamma x^{\theta-1}(1 + x^\theta)^{-2\gamma-1}(1 - (1 + x^\theta)^{-2\gamma})^{b-1}}{(1 + (1 - [1 - (1 + x^\theta)^{-2\gamma}]^b))^2 \left\{ 1 - \bar{\delta} \left(1 - \left[\frac{[1 - (1 + x^\theta)^{-2\gamma}]^b}{1 + (1 - [1 - (1 + x^\theta)^{-2\gamma}]^b)} \right] \right) \right\}^2},$$

respectively for $b, \delta, \theta, \gamma > 0$. Figure 4.1 displays the graphs of the MO-TLHL-BXII distribution for various selected parameter values. The pdf assumes a variety of shapes such as left-skewed, right-skewed, uni-modal, reverse-J and almost symmetric.

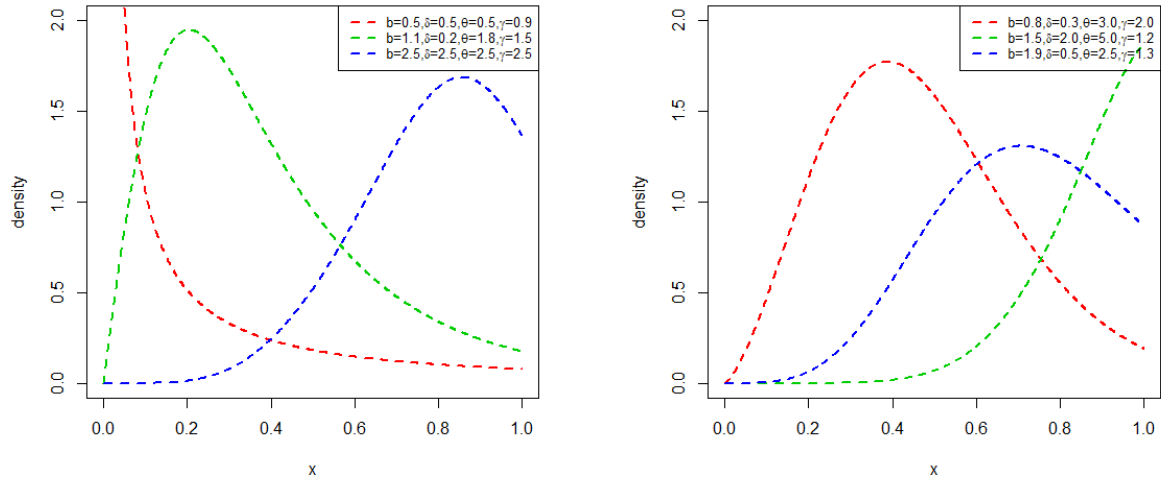


Figure 4.1: Plots of MO-TLHL-BXII density function

Figure 4.2 displays the graphs of the hrf, the graphs exhibits increasing, decreasing, inverted bathtub and bathtub followed by inverted bathtub shapes.

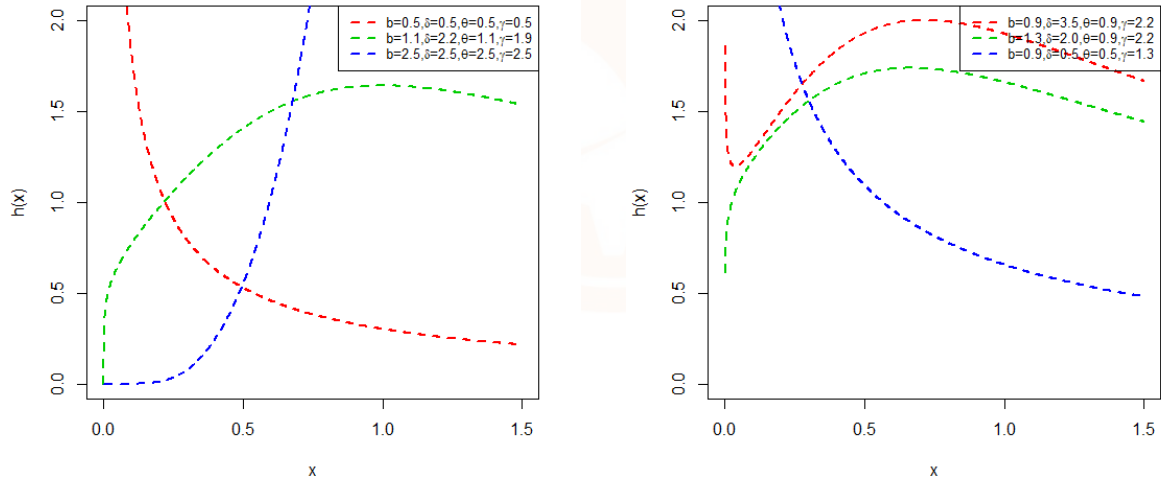


Figure 4.2: Plots of MO-TLHL-BXII hrf

Marshall-Olkin-Topp-Leone Half Logistic-Weibull Distribution

Consider the Weibull distribution with $G(x; \theta, \gamma) = 1 - \exp(-\theta x^\gamma)$ and $g(x; \theta, \gamma) = \gamma \theta x^{\gamma-1} \exp(-\theta x^\gamma)$ for $x > 0, \theta, \gamma > 0$. Then, the MO-TLHL-Weibull (MO-TLHL-W) distribution has cdf and pdf given by

$$F(x; b, \delta, \theta, \gamma) = \frac{\frac{[1 - \exp(-2\theta x^\gamma)]^b}{(1 + (1 - [1 - \exp(-2\theta x^\gamma)]^b))^b}}{1 - \delta \left(1 - \left[\frac{[1 - \exp(-2\theta x^\gamma)]^b}{1 + (1 - [1 - \exp(-2\theta x^\gamma)]^b)} \right] \right)} \quad (4.13)$$

and

$$f(x; b, \delta, \theta, \gamma) = \frac{4b\delta\gamma\theta x^{\gamma-1} \exp(-2\theta x^\gamma)(1 - \exp(-2\theta x^\gamma))^{b-1}}{(1 + (1 - [1 - \exp(-2\theta x^\gamma)]^b))^2 \left\{ 1 - \bar{\delta} \left(1 - \left[\frac{[1 - \exp(-2\theta x^\gamma)]^b}{1 + (1 - [1 - \exp(-2\theta x^\gamma)]^b)} \right] \right) \right\}^2}. \quad (4.14)$$

The pdf of the MO-TLHL-W distribution can take a variety of shapes including right and left-skewed and reverse-J as shown in Figure 4.3.

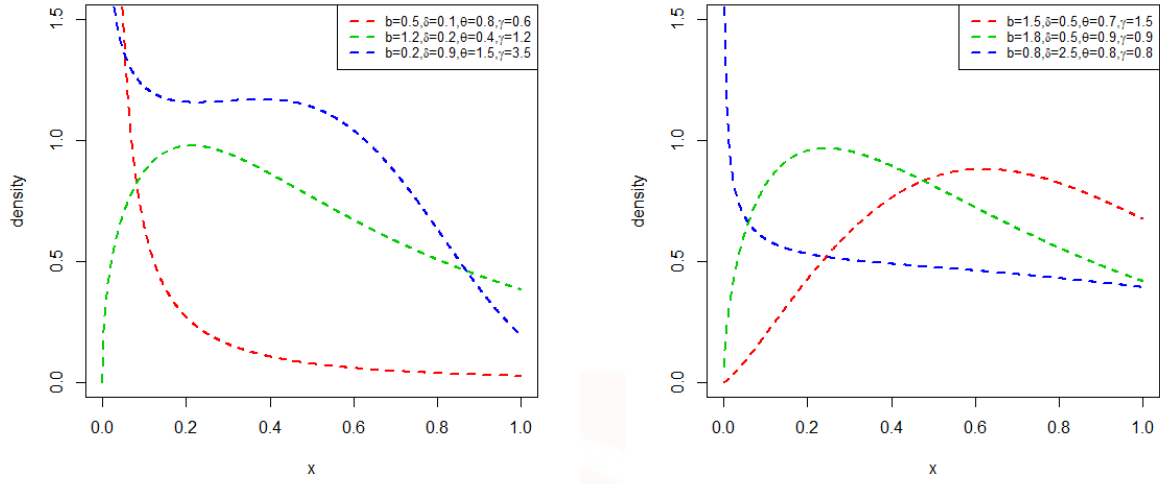


Figure 4.3: Plots of MO-TLHL-W density function

Figure 4.4 has the graphs of the hrf and the graphs show increasing, decreasing, bathtub and inverted bathtub shapes.

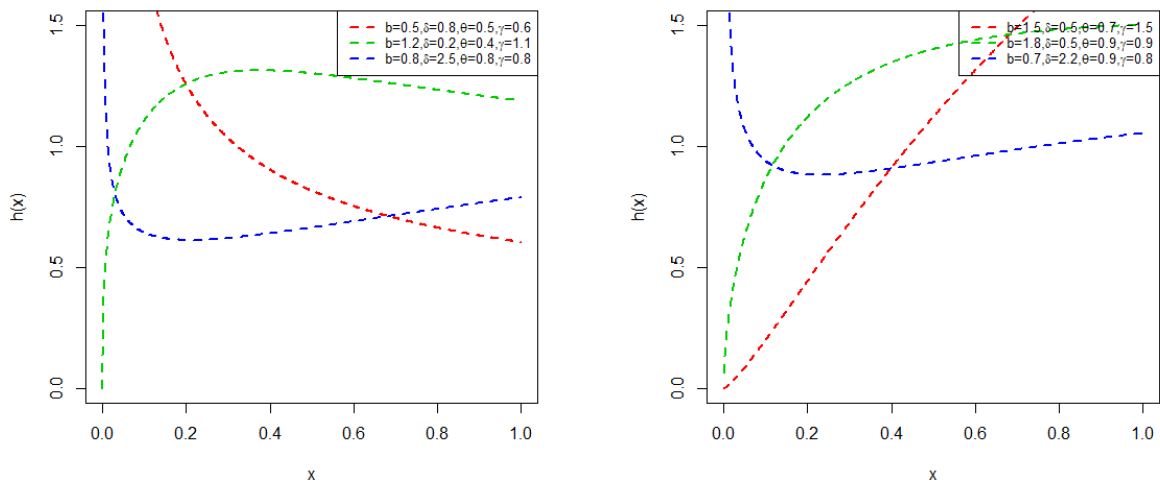


Figure 4.4: Plots of MO-TLHL-W hrf

Marshall-Olkin-Topp-Leone Half Logistic-Kumaraswamy Distribution

If we consider the Kumaraswamy distribution with $G(x; \theta, \gamma) = 1 - (1 - x^\theta)^\gamma$ and $g(x; \theta, \gamma) = \theta\gamma x^{\theta-1}(1 - x^\theta)^{\gamma-1}$ for $x > 0$, $\theta, \gamma > 0$, then, the MO-TLHL-Kumaraswamy (MO-TLHL-Kum) distribution has cdf and pdf given by

$$F(x; b, \delta, \theta, \gamma) = \frac{\frac{[1 - (1 - x^\theta)^{2\gamma}]^b}{(1 + (1 - [1 - (1 - x^\theta)^{2\gamma}]^b))}}{1 - \bar{\delta} \left(1 - \left[\frac{[1 - (1 - x^\theta)^{2\gamma}]^b}{1 + (1 - [1 - (1 - x^\theta)^{2\gamma}]^b)} \right] \right)}$$

and

$$f(x; b, \delta, \theta, \gamma) = \frac{4b\delta\theta\gamma x^{\theta-1}(1 - x^\theta)^{2\gamma-1}(1 - (1 - x^\theta)^{2\gamma})^{b-1}}{(1 + (1 - [1 - (1 - x^\theta)^{2\gamma}]^b))^2 \left\{ 1 - \bar{\delta} \left(1 - \left[\frac{[1 - (1 - x^\theta)^{2\gamma}]^b}{1 + (1 - [1 - (1 - x^\theta)^{2\gamma}]^b)} \right] \right) \right\}^2}.$$

Figure 4.5 presents the graphs of the MO-TLHL-Kum distribution for various selected parameter values. The pdf assumes a variety of shapes such as left-skewed and right-skewed, reverse-J, almost symmetric and uni-modal.

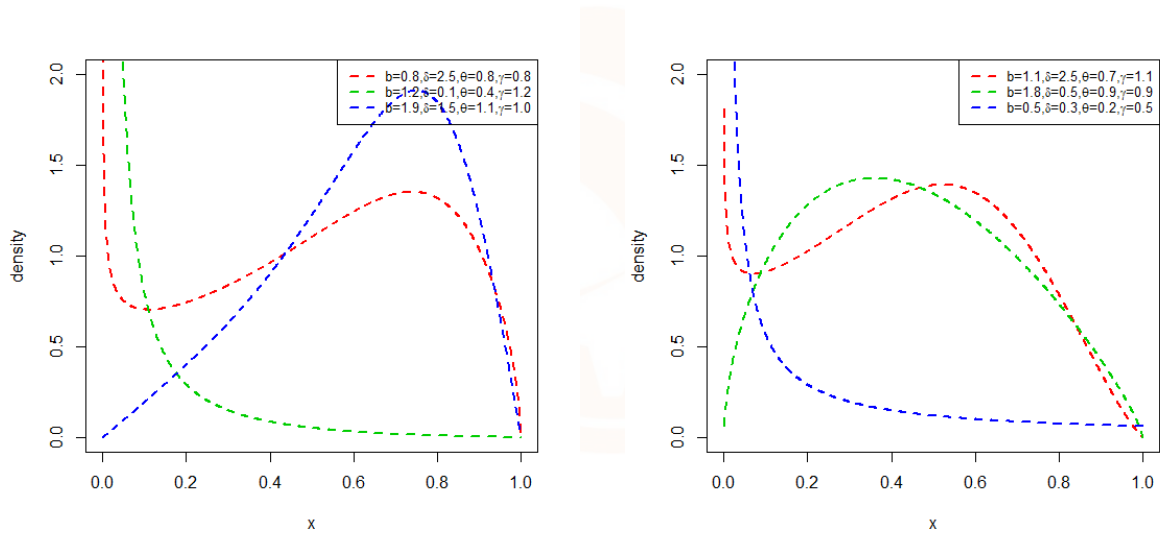


Figure 4.5: Plots of MO-TLHL-Kum density function

Figure 4.6 presents the graphs of the hrf. We have bathtub and increasing shapes.

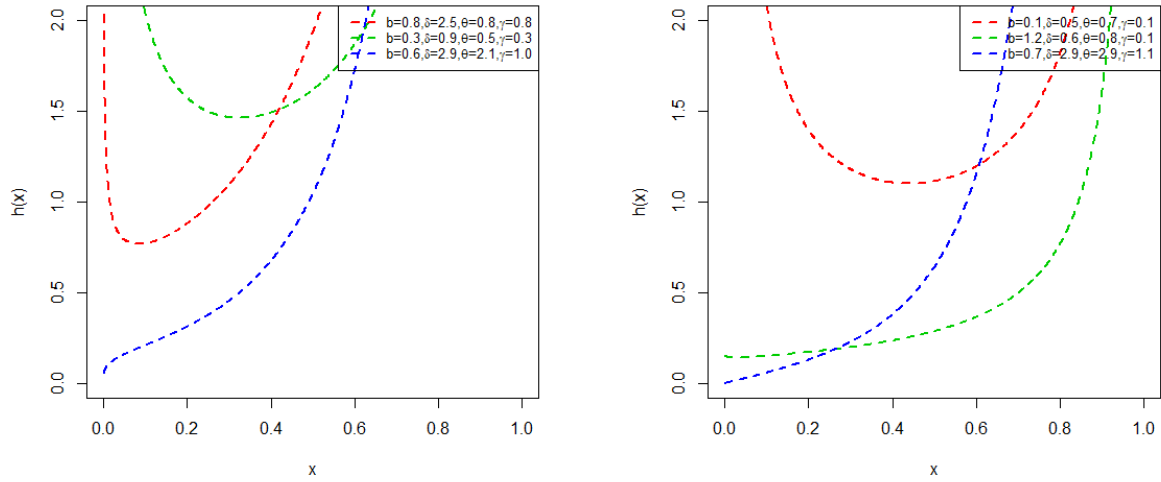


Figure 4.6: Plots of MO-TLHL-Kum hrf

Marshall-Olkin-Topp-Leone Half Logistic-Burr III Distribution

If $G(x; \theta, \gamma) = (1 + x^{-\theta})^{-\gamma}$ and $g(x; \theta, \gamma) = \theta\gamma x^{-\theta-1}(1 + x^{-\theta})^{-\gamma-1}$ for $x > 0$, $\theta, \gamma > 0$, then, the MO-TLHL-Burr III (MO-TLHL-BIII) distribution has cdf and pdf given by

$$F(x; b, \delta, \theta, \gamma) = \frac{\frac{[1 - (1 + x^{-\theta})^{-\gamma}]^b}{(1 + (1 - [1 - (1 + x^{-\theta})^{-\gamma}]^b))}}{1 - \bar{\delta} \left(1 - \left[\frac{[1 - (1 + x^{-\theta})^{-\gamma}]^b}{1 + (1 - [1 - (1 + x^{-\theta})^{-\gamma}]^b)} \right] \right)}$$

and

$$f(x; b, \delta, \theta, \gamma) = \frac{4b\delta\theta\gamma x^{-\theta-1}(1 + x^{-\theta})^{-\gamma-1}(1 - (1 + x^{-\theta})^{-\gamma})(1 - (1 - (1 + x^{-\theta})^{-\gamma})^2)^{b-1}}{(1 + (1 - [1 - (1 - (1 + x^{-\theta})^{-\gamma})^2]^b))^2 \left\{ 1 - \bar{\delta} \left(1 - \left[\frac{[1 - (1 + x^{-\theta})^{-\gamma}]^b}{1 + (1 - [1 - (1 + x^{-\theta})^{-\gamma}]^b)} \right] \right) \right\}^2}.$$

Figure 4.7 shows the graphs of the MO-TLHL-BIII distribution for various parameter values. The pdf can have a variety shapes such as left and right-skewed, the reverse-J and uni-modal.

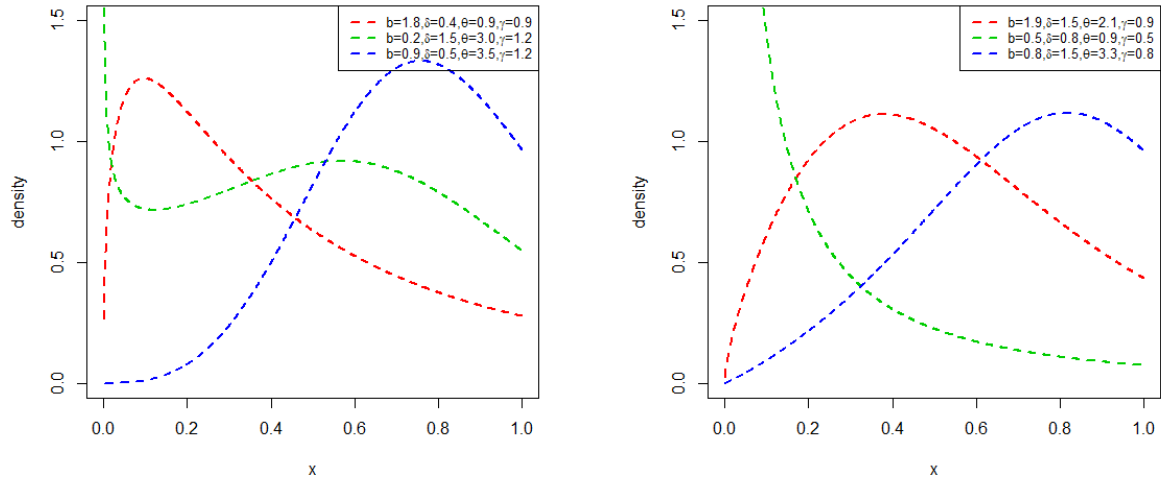


Figure 4.7: Plots of MO-TLHL-BIII density function

Figure 4.8 shows the plots of the hrf of the MO-TLHL-BIII distribution. Graphs exhibits decreasing, increasing, inverted bathtub and bathtub followed by inverted bathtub shapes.

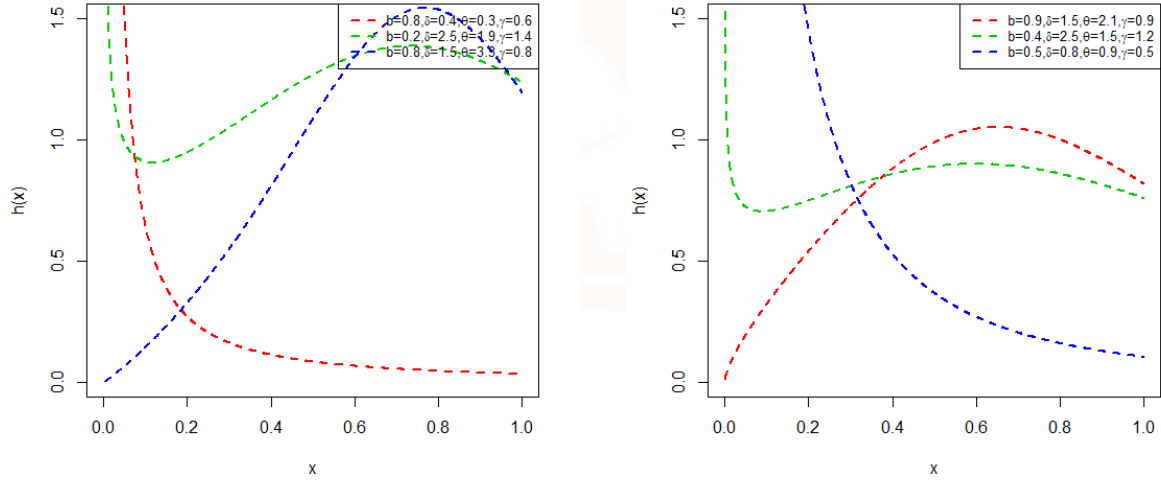


Figure 4.8: Plots of MO-TLHL-BIII hrf

Marshall-Olkin-Topp-Leone Half Logistic-Uniform Distribution

If we take $G(x; \theta) = \frac{x}{\theta}$ and $g(x; \theta) = \frac{1}{\theta}$ for $0 < x < \theta < \infty$. Then, the MO-TLHL-Uniform (MO-TLHL-U) distribution has cdf and pdf given by

$$F(x; b, \delta, \theta) = \frac{\frac{[1-(1-\frac{x}{\theta})^2]^b}{(1+(1-[1-(1-\frac{x}{\theta})^2]^b))}}{1 - \delta \left(1 - \left[\frac{[1-(1-\frac{x}{\theta})^2]^b}{1+(1-[1-(1-\frac{x}{\theta})^2]^b)} \right] \right)}$$

and

$$f(x; b, \delta, \theta) = \frac{4b\delta\frac{1}{\theta}(1 - \frac{x}{\theta})(1 - (1 - \frac{x}{\theta})^2)^{b-1}}{(1 + (1 - [1 - (1 - \frac{x}{\theta})^2]^b))^2 \left\{ 1 - \delta \left(1 - \left[\frac{[1 - (1 - \frac{x}{\theta})^2]^b}{1 + (1 - [1 - (1 - \frac{x}{\theta})^2]^b)} \right] \right) \right\}^2}.$$

The pdf of the MO-TLHL-U distribution can assume a variety of shapes such as decreasing-increasing-decreasing, uni-modal, reverse-J, almost symmetric and left-skewed as shown in Figure 4.9.

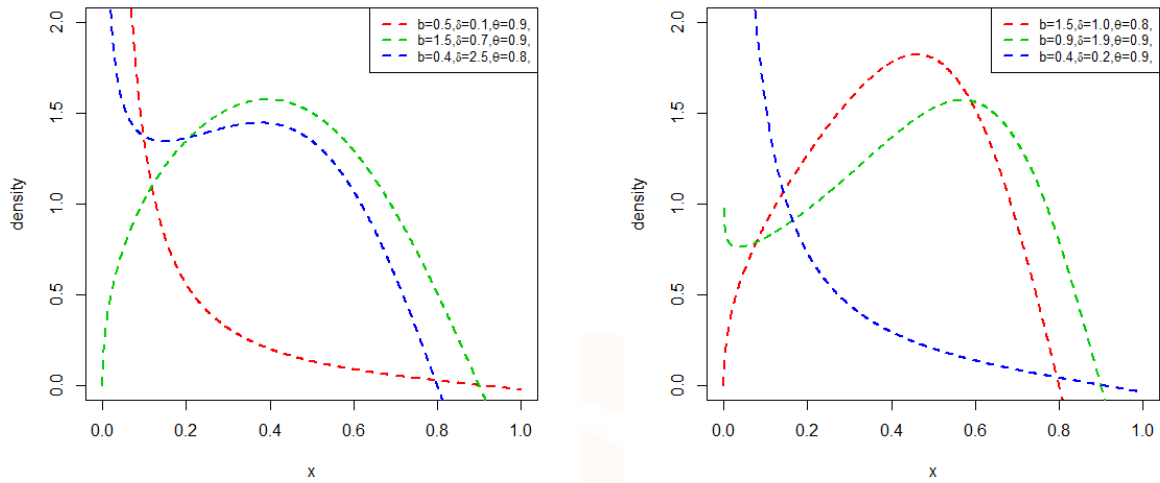


Figure 4.9: Plots of MO-TLHL-U density function

Figure 4.10 displays the graphs of the hrf of the MO-TLHL-U distribution. The graphs exhibit bathtub and increasing shapes.

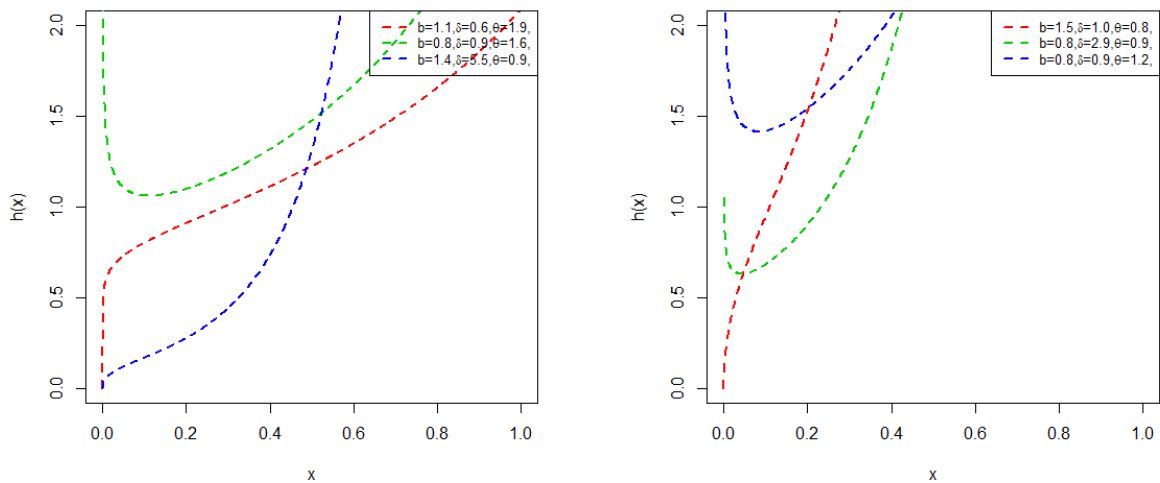


Figure 4.10: Plots of MO-TLHL-U hrf

Table 4.2 presents random numbers generated from the MO-TLHL-W distribution for various

selected parameter values.

u	$(b, \delta, \theta, \gamma)$				
	(0.9, 0.9, 0.9, 0.9)	(2.0, 1.5, 0.9, 0.5)	(0.5, 0.9, 1.5, 1.5)	(0.9, 0.8, 1.8, 0.5)	(0.8, 0.5, 0.6, 0.4)
0.1	0.06175	0.14829	0.04451	0.00131	0.00052
0.2	0.14579	0.34892	0.10445	0.00631	0.00496
0.3	0.24409	0.59315	0.17000	0.01625	0.02000
0.4	0.35749	0.88728	0.24021	0.03292	0.05750
0.5	0.48982	1.24708	0.31611	0.05905	0.13932
0.6	0.64882	1.70300	0.40048	0.09966	0.30978
0.7	0.84978	2.31676	0.49872	0.16490	0.67010
0.8	1.12786	3.23657	0.62263	0.27974	1.50459
0.9	1.59681	4.97860	0.80860	0.53480	4.01451

Table 4.2: Table of Quantiles

4.3 Moments and Incomplete Moments

Moments, mgf and incomplete moments are presented in this section.

4.3.1 Moments and Generating Function

Let T_{p+1} be an Exp-G rv with power parameter $(p+1)$. Then the r^{th} raw moment of X , say μ'_r follows from equation (4.2) as

$$\mu'_r = E(X^r) = \sum_{p=0}^{\infty} c_{p+1} E(T_{p+1}^r), \quad (4.15)$$

where c_{p+1} is as provided in equation (4.12).

The mgf $M_X(t) = E(e^{tX})$ of X can be obtained from equation (4.2) and can be written as

$$M_X(t) = \sum_{p=0}^{\infty} c_{p+1} M_{p+1}(t),$$

where $M_{p+1}(t)$ is the mgf of the rv T_{p+1} .

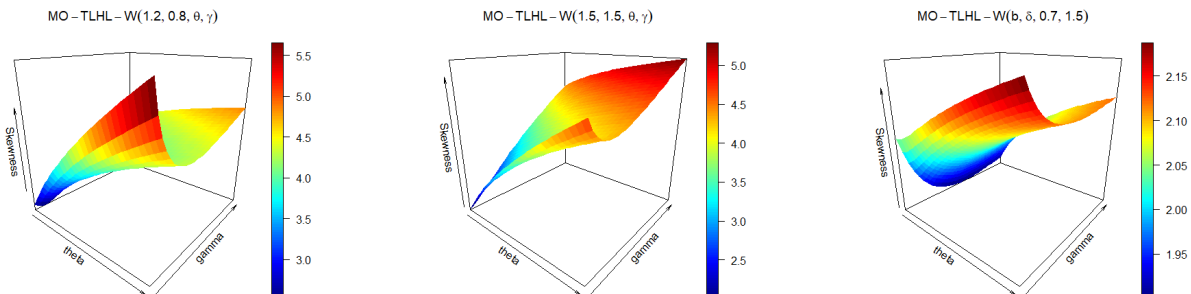


Figure 4.11: Plots of the skewness of the MOTLHL-W distribution.

Figures 4.11 and 4.12 shows the skewness and kurtosis plots of the MO-TLHL-W distribution, it can be seen that the members of the MO-TLHL-G family are capable of modelling various data sets with different levels of skewness and kurtosis.

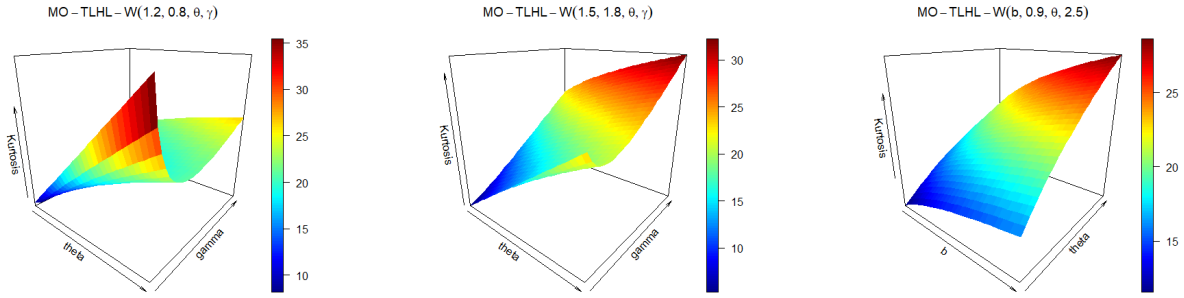


Figure 4.12: Plots of the kurtosis of the MO-TLHL-W distribution.

4.3.2 Incomplete Moments

Conditional and incomplete moments are very instrumental when it comes to reliability analysis. The r^{th} incomplete moment of the MO-TLHL-G FoD is written as

$$\begin{aligned} m_r(t) &= \int_{-\infty}^t x^r f(x; b, \delta, \varphi) dx \\ &= \sum_{p=0}^{\infty} c_{p+1} \int_{-\infty}^t x^r h_{p+1}(x; \varphi) dx. \end{aligned} \quad (4.16)$$

Note that $m_1(t)$ can be used in the construction of Bonferroni and Lorenz curves. These curves vital in demography, insurance, economics and reliability, just to mention a few.

A general equation for $m_1(t)$ can be obtained from equation (4.16) as

$$m_1(t) = \sum_{p=0}^{\infty} c_{p+1} K_{p+1}(t; \varphi),$$

where $K_{p+1}(t; \varphi) = \int_{-\infty}^t x h_{p+1}(x; \varphi) dx$ is the first incomplete moment of the Exp-G distribution.

4.4 Order Statistics and Rényi Entropy (RE)

In probability and statistics, notably in reliability, lifetime data analysis and information theory, order statistics and entropy play crucial roles. This part presents the distribution of the i^{th} order statistics and RE for the MO-TLHL-G FoD.

4.4.1 Order Statistics

Let $X_1, X_2, \dots, X_n$ be i.i.d. MO-TLHL-G rvs. The pdf of $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{j+i-1}, \quad (4.17)$$

where $B(.,.)$ is the beta function. Note that

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= 4b\delta g(x; \varphi)(1 - \bar{G}^2(x; \varphi))^{b(j+i)-1}(1 + (1 - (1 - \bar{G}^2(x; \varphi))))^{-(j+i+1)} \\ &\times \bar{G}(x; \xi) \left\{ 1 - \bar{\delta} \left(1 - F_{TLHL-G}(x; b, \varphi) \right) \right\}^{-(j+i+1)}. \end{aligned}$$

Using the generalized binomial series expansion, we obtain

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= 4b\delta g(x; \varphi) \bar{G}(x; \varphi) \sum_{k=0}^{\infty} (-1)^k \binom{-(j+i+1)}{k} \bar{\delta}^k \\ &\times (1 - \bar{G}^2(x; \varphi))^{b(j+i)-1} (1 + (1 - (1 - \bar{G}^2(x; \varphi))))^{-(j+i+1)} \\ &\times \left(1 - F_{TLHL-G}(x; b, \varphi) \right)^k \\ &= 4b\delta g(x; \varphi) \bar{G}(x; \varphi) \sum_{k,q=0}^{\infty} (-1)^{k+q} \binom{-(j+i+1)}{k} \binom{k}{q} \bar{\delta}^k \\ &\times (1 - \bar{G}^2(x; \varphi))^{b(j+i+q)-1} (1 + (1 - (1 - \bar{G}^2(x; \varphi))))^{-(j+i+q+1)} \\ &= 4b\delta g(x; \varphi) \bar{G}(x; \varphi) \sum_{k,q,i,p=0}^{\infty} (-1)^{k+q+p} \binom{-(j+i+1)}{k} \binom{k}{q} \binom{i}{p} \\ &\times \binom{-(j+i+q+1)}{i} \bar{\delta}^k (1 - \bar{G}^2(x; \varphi))^{b(j+i+q+p)-1} \\ &= 4b\delta g(x; \varphi) \sum_{k,q,i,p,w,s=0}^{\infty} (-1)^{k+q+p+w+s} \binom{-(j+i+1)}{k} \binom{k}{q} \\ &\times \binom{i}{p} \binom{-(j+i+q+1)}{i} \binom{b(j+i+q+p)-1}{w} \binom{2w+1}{s} \\ &\times \bar{\delta}^k G(x; \varphi)^s. \end{aligned} \tag{4.18}$$

Substituting equation (4.18) into equation (4.17), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{s=0}^{\infty} c_{s+1} h_{s+1}(x; \varphi), \tag{4.19}$$

where $h_{s+1}(x; \varphi)$ is the Exp-G density function with power parameter $(s+1)$ and

$$\begin{aligned} c_{s+1} &= 4b\delta \sum_{k,q,i,p,w=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{k+q+p+w+s+j}}{B(i, n-i+1)(s+1)} \binom{n-i}{j} \binom{-(j+i+1)}{k} \binom{k}{q} \\ &\times \binom{i}{p} \binom{-(j+i+q+1)}{i} \binom{b(j+i+q+p)-1}{w} \binom{2w+1}{s} \bar{\delta}^k. \end{aligned}$$

It follows that the pdf of the i^{th} order statistics can be expressed as an infinite linear combination of exponentiated-G densities.

4.4.2 Rényi Entropy (RE)

The RE (Rényi [113]) is an extension of Shannon entropy. RE is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f(x; b, \delta, \varphi)]^v dx \right), v \neq 1, v > 0. \quad (4.20)$$

RE tends to Shannon entropy as $v \rightarrow 1$. Set $[f(x; b, \delta, \varphi)]^v = [f(x)]^v$. From equation (4.2),

$$\begin{aligned} [f(x)]^v &= (4b\delta g(x; \varphi) \bar{G}(x; \varphi))^v (1 - \bar{G}^2(x; \varphi))^{v(b-1)} (1 + (1 - [1 - \bar{G}^2(x; \varphi)]^b))^{-2v} \\ &\times \left\{ 1 - \bar{\delta} \left(1 - F_{TLHL-G}(x; b, \varphi) \right) \right\}^{-2v}. \end{aligned}$$

Using the generalized binomial series expansion, we have

$$\begin{aligned} [f(x)]^v &= (4b\delta g(x; \varphi) \bar{G}(x; \varphi))^v \sum_{l,k=0}^{\infty} (-1)^{l+k} \binom{-2v}{l} \binom{l}{k} \bar{\delta}^l (1 - \bar{G}^2(x; \varphi))^{b(v+k)-v} \\ &\times (1 + (1 - [1 - \bar{G}^2(x; \varphi)]^b))^{-2v-k} \\ &= (4b\delta g(x; \varphi) \bar{G}(x; \varphi))^v \sum_{l,k,i,j=0}^{\infty} (-1)^{l+k+i} \binom{-2v}{l} \binom{l}{k} \binom{-(2v+k)}{j} \binom{j}{i} \\ &\times \bar{\delta}^l (1 - \bar{G}^2(x; \varphi))^{b(v+k+i)-v} \\ &= (4b\delta)^v \sum_{l,k,i,j,q,p=0}^{\infty} (-1)^{l+k+i+q+p} \binom{-2v}{l} \binom{l}{k} \binom{-(2v+k)}{j} \\ &\times \binom{j}{i} \binom{b(k+i+v)-v}{q} \bar{\delta}^l \bar{G}^{2q+v}(x; \varphi) \\ &= \sum_{l,k,i,j,q,p=0}^{\infty} (-1)^{l+k+i+q+p} (4b\delta)^v \binom{-2v}{l} \binom{l}{k} \binom{-(2v+k)}{j} \\ &\times \binom{j}{i} \binom{b(k+i+v)-v}{q} \binom{2q+v}{p} \bar{\delta}^l [g(x; \varphi)]^v G(x; \varphi)^p. \end{aligned}$$

Thus, the RE of the MO-TLHL-G FoD can be expressed as

$$I_R(v) = \frac{1}{1-v} \log \left[\sum_{p=0}^{\infty} b_{p+1} \left(\int_0^\infty [g(x; \varphi)]^v G(x, \varphi)^p dx \right) \right], \quad (4.21)$$

where

$$\begin{aligned} b_{p+1} &= \sum_{l,k,i,j,q=0}^{\infty} (-1)^{l+k+i+q+p} (4b\delta)^v \binom{-2v}{l} \binom{l}{k} \binom{-(2v+k)}{j} \\ &\times \binom{j}{i} \binom{b(k+i+v)-v}{q} \binom{2q+v}{p} \bar{\delta}^l. \end{aligned}$$

Note that, $\int_0^\infty [g(x; \varphi)]^v G(x, \varphi)^p dx$ can be obtained numerically. The Exp-G distribution's RE can be used to calculate the entropy of the MO-TLHL-G FoD as follows

$$I_R(v) = \frac{1}{1-v} \log \left[\sum_{p=0}^{\infty} \Phi_{p+1} e^{(1-v)I_{REG}} \right], \quad (4.22)$$

where

$$\begin{aligned} \Phi_{p+1} &= \sum_{l,k,i,j,q=0}^{\infty} (-1)^{l+k+i+q+p} (4b\delta)^v \binom{-2v}{l} \binom{l}{k} \binom{-(2v+k)}{j} \\ &\times \binom{j}{i} \binom{b(k+i+v)-v}{q} \binom{2q+v}{p} \left(\frac{p}{v} + 1\right) \bar{\delta}^l \end{aligned}$$

and $I_{REG} = \int_0^\infty [(\frac{p}{v} + 1)g(x; \varphi)G(x; \varphi)^{\frac{p}{v}}]^v dx$ is the RE of the Exp-G distribution with power parameter $\frac{p}{v}$.

4.5 Estimation and Inferences

We provide the maximum likelihood technique for estimating parameters of the MO-TLHL-G FoD. Let $X \sim MO-TLHL-G(b, \delta, \varphi)$ and $\Delta = (b, \delta, \varphi)^T$ be a vector of parameters. The log-likelihood function, $\ell_n = \ell_n(\Delta)$, based on a random sample of size n from the MO-TLHL-G distribution is given by

$$\begin{aligned} \ell_n(\Delta) &= n \ln(4b) + n \ln(\delta) + \sum_{i=1}^n \ln(g(x_i; \varphi)) + \sum_{i=1}^n \ln(\bar{G}(x_i; \varphi)) + (b-1) \\ &\times \sum_{i=1}^n \ln(1 - \bar{G}^2(x_i; \varphi)) - 2 \sum_{i=1}^n \ln(1 + (1 - (1 - \bar{G}^2(x_i; \varphi))^b)) \\ &- 2 \sum_{i=1}^n \ln\left(1 - \bar{\delta} \left(1 - F_{TLHL-G}(x; b, \varphi)\right)\right). \end{aligned} \quad (4.23)$$

The first derivative of $\ell_n = \ell_n(\Delta)$ w.r.t each of component of the parameters $\Delta = (b, \delta, \varphi)^T$ are given by

$$\begin{aligned} \frac{\partial \ell_n(\Delta)}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \ln(1 - \bar{G}^2(x_i; \varphi)) + 2 \sum_{i=1}^n \frac{(1 - \bar{G}^2(x_i; \varphi))^b \ln(1 - \bar{G}^2(x_i; \varphi))}{1 + (1 - [1 - \bar{G}^2(x; \varphi)]^b)} \\ &+ 4\bar{\delta} \sum_{i=1}^n \frac{(1 - \bar{G}^2(x_i; \varphi))^b \ln(1 - \bar{G}^2(x_i; \varphi))}{(1 + (1 - [1 - \bar{G}^2(x; \varphi)]^b))^2 (1 - \bar{\delta} (1 - F_{TLHL-G}(x; b, \varphi)))}, \\ \frac{\partial \ell_n(\Delta)}{\partial \delta} &= \frac{n}{\delta} - 2 \sum_{i=1}^n \frac{(1 - F_{TLHL-G}(x; b, \varphi))}{1 - \bar{\delta} (1 - F_{TLHL-G}(x; b, \varphi))} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ell_n(\Delta)}{\partial \varphi_k} &= \sum_{i=1}^n \frac{g'(x_i; \varphi)}{g(x_i; \varphi)} + \sum_{i=1}^n \frac{\overline{G}'(x_i; \varphi)}{\overline{G}(x_i; \varphi)} + 2(b-1) \sum_{i=1}^n \frac{\overline{G}(x_i; \varphi) \overline{G}'(x_i; \varphi)}{1 - \overline{G}^2(x_i; \varphi)} \\ &+ 2b \sum_{i=1}^n \frac{\overline{G}(x_i; \varphi) (1 - \overline{G}^2(x_i; \varphi))^{b-1} \overline{G}'(x_i; \varphi)}{1 + (1 - [1 - \overline{G}^2(x; \varphi)]^b)} \\ &- 8b\bar{\delta} \sum_{i=1}^n \frac{\overline{G}(x_i; \varphi) (1 - \overline{G}^2(x_i; \varphi))^{b-1} \overline{G}'(x_i; \varphi)}{(1 + (1 - [1 - \overline{G}^2(x; \varphi)]^b))^2 \left(1 - \bar{\delta} \left(1 - F_{TLHL-G}(x; b, \varphi)\right)\right)}, \end{aligned}$$

where $g'(x_i; \varphi) = \frac{\partial g(x_i; \varphi)}{\partial \varphi_k}$ and $\overline{G}'(x_i; \varphi) = \frac{\partial \overline{G}(x_i; \varphi)}{\partial \varphi_k}$.

The MLEs of the parameters, denoted by $\hat{\Delta}$ are obtained by solving the non-linear equations $(\frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \varphi_k})^T = \mathbf{0}$. We applied numerical methods specifically the Newton-Raphson procedure to get solutions of the likelihood equations since the equations can not be solved explicitly.

4.6 Simulation Study

Here, we evaluate the behaviour of the ML estimators of the MO-TLHL-W distribution by carrying out simulations for various sample sizes ($n=35, 70, 140, 280, 560, 1120$) via the R package. We generated $N = 1000$ samples for the initial true parameters. Results provided in Tables 4.3 and 4.4 shows the mean MLEs of the model parameters along with the respective RMSEs and ABias. The ABias and RMSE for the calculated parameters, $\hat{\theta}$, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

parameter	Sample Size	(1.0, 0.02, 0.02, 2.5)			(1.1, 0.02, 0.7, 2.5)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
b	35	1.6053	3.4577	0.5053	3.1300	9.3935	2.0300
	70	1.5432	2.7637	0.4432	1.9672	6.3920	0.8672
	140	1.3546	1.2978	0.2546	1.4503	1.6396	0.3503
	280	1.2463	1.4873	0.1463	1.2514	1.0963	0.1514
	560	1.1752	0.7361	0.0752	1.1512	0.4241	0.0512
	1120	1.1158	0.1775	0.0158	1.1308	0.3042	0.0308
δ	35	1.2490	8.4814	1.2290	0.3040	1.6737	0.2840
	70	0.4390	3.2529	0.4190	0.2851	2.7249	0.2651
	140	0.2196	1.5353	0.1996	0.1184	0.6326	0.0984
	280	0.0528	0.2760	0.0328	0.0536	0.2470	0.0336
	560	0.0359	0.1469	0.0159	0.0284	0.0368	0.0084
	1120	0.0248	0.0146	0.0048	0.0253	0.0233	0.0053
θ	35	0.3165	0.7680	0.2965	3.2153	13.7223	2.5153
	70	0.1306	0.4106	0.1106	1.5605	5.2816	0.8605
	140	0.0997	0.3245	0.0797	1.0748	1.1403	0.3748
	280	0.0516	0.1806	0.0316	0.8964	0.5911	0.1964
	560	0.0390	0.1278	0.0190	0.8177	0.3515	0.1177
	1120	0.0265	0.0370	0.0065	0.7888	0.2640	0.0888
γ	35	2.4130	0.9596	-0.0870	3.0604	2.5888	0.5604
	70	2.4991	0.6519	-0.0009	2.8395	1.8071	0.3395
	140	2.4208	0.4515	-0.0792	2.5923	1.1554	0.0923
	280	2.4532	0.2922	-0.0468	2.5751	0.9044	0.0751
	560	2.4592	0.2291	-0.0408	2.5024	0.4382	0.0024
	1120	2.4807	0.1365	-0.0193	2.4900	0.2929	-0.0100

Table 4.3: Simulations

parameter	Sample Size	(0.6, 0.03, 0.03, 2.3)			(0.6, 0.6, 1.2, 2.3)		
		Mean	RMSE	Bias	Mean	RMSE	Bias
b	35	0.7733	0.8171	0.1733	1.1633	6.6970	0.5633
	70	0.7947	0.9815	0.1947	0.6500	0.6908	0.0500
	140	0.8149	1.0174	0.2149	0.5904	0.3583	-0.0096
	280	0.7350	0.7175	0.1350	0.5826	0.2273	-0.0174
	560	0.7128	0.5948	0.1128	0.5915	0.1577	-0.0085
	1120	0.6438	0.2526	0.0438	0.5874	0.1092	-0.0126
δ	35	0.6950	3.6399	0.6650	4.1928	12.1237	3.5928
	70	0.3995	2.0308	0.3695	2.5914	7.4818	1.9914
	140	0.2219	0.9081	0.1919	1.4276	6.6402	0.8276
	280	0.0958	0.3359	0.0658	0.8436	1.1266	0.2436
	560	0.0583	0.1451	0.0283	0.6975	0.5683	0.0975
	1120	0.0397	0.0507	0.0097	0.6410	0.3183	0.0410
θ	35	0.2768	0.7176	0.2468	1.5077	1.0855	0.3077
	70	0.2013	0.5964	0.1713	1.3462	0.8986	0.1462
	140	0.1845	0.5469	0.1545	1.2007	0.6686	0.0007
	280	0.1217	0.3775	0.0917	1.1652	0.5233	-0.0348
	560	0.0961	0.2717	0.0661	1.1777	0.3828	-0.0223
	1120	0.0595	0.1397	0.0295	1.1793	0.2780	-0.0207
γ	35	2.4825	1.0870	0.1825	3.7828	3.3829	1.4828
	70	2.4440	0.9155	0.1440	3.3699	2.6276	1.0699
	140	2.3637	0.7804	0.0637	3.0936	1.9246	0.7936
	280	2.3118	0.6016	0.0118	2.7996	1.3650	0.4996
	560	2.2344	0.4810	-0.0656	2.5424	0.8203	0.2424
	1120	2.2589	0.3606	-0.0411	2.4501	0.5733	0.1501

Table 4.4: **Simulations**

According to the results provided in Tables 4.3 and 4.4 the mean parameter estimations gets closer to the initial parameters values, ABias and RMSEs are decreasing towards zero with an increase in the sample size showing a significant consistency of the MLEs of the model parameters.

4.7 Applications

This section gives some examples to show the applicability of the MO-TLHL-W distribution in modeling real life data. We compared the fits of the MO-TLHL-W distribution is to the Kumaraswamy Weibull (KumW) dustribution (Cordeiro et al. [48]), Odd Log-Logistic Exponentiated Weibull (OLLE-W) distribution (Afify et al. [2]), the Beta generalized Lindley (BGL) distribution (Oluyede and Yang [99]), Weibull Lomax (WL) distribution (Tahir et al. [123]) and the Generalized Weibull Log-Logistic (GWLL) distribution (Cordeiro et al. [43]. The pdfs of the used models are given below in the appendix.

Figure 4.13 and Figure 4.19 show graphs of the fitted densities, the data's histogram and probability plots (Chambers et al. [34]). The probability plots were created by plotting $F(x_{(j)}; \hat{b}, \hat{\delta}, \hat{\varphi})$ versus $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The sum of squares measures the closeness of the fitted model to the empirical and is given as

$$SS = \sum_{j=1}^n \left[F(x_{(j)}; \hat{b}, \hat{\delta}, \hat{\varphi}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

Tables 4.5 and 4.6 provides MLEs of the parameters with SEs in parenthesis and GoF statistics (-2ln(L), AIC, CAIC, BIC, W^* , A^* , K-S, p-value of the K-S statistic). These GoF statistics can be used to evaluate and compare the performance of models so as to determine the distribution which best describes the data being modelled. Generally, the model with the least values of GoF statistics performs better than the other comparative models.

4.7.1 Failure Times of 50 Components

The failure times of 50 components (per 1000 hours) data set were taken from Murthy et al. [93]. The data are as follows: 0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183, 0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904, 0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228, 3.931, 9.337, 0.086, 0.379, 1.600, 4.073, 10.940, 0.102, 0.381, 2.006, 4.393, 11.020, 0.103, 0.538, 2.054, 4.534, 13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116, 0.574, 3.058, 6.274, 15.08. The estimated parameters (SEs of estimates in parenthesis) and the GoF statistics are presented in Table 4.5.

Model	Estimates				statistics								
	b	δ	θ	γ	-2ln L	AIC	CAIC	BIC	W^*	A^*	K-S	P-value	SS
MO-TLHL-W	0.1731 (0.0164)	0.1405 (0.0616)	1.8835×10^{-5} (1.1390×10^{-5})	3.8215 (0.0007)	200.2998	208.2998	209.1887	215.9479	0.1148	0.7461	0.1162	0.4739	0.1166
BGL	α 0.1186 (0.0077)	θ 1.6897×10^{-7} (2.5943×10^{-6})	a 0.3015 (0.0007)	b 10.1000 (2.2357×10^{-5})	361.3658	369.5053	370.3942	377.1534	0.1592	0.9857	0.4576	4.3100×10^{-10}	3.2251
OLLE-W	α 44.9931 (0.1415)	β 3.4902 (4.4068)	θ 2.2405 (0.2619)	γ 0.0575 (0.0728)	205.8275	213.8275	214.7164	218.0930	0.1444	0.8940	0.1169	0.4662	0.1393
Kum-W	a 3.8120 (3.1860)	b 0.1245 (0.0916)	c 0.5299 (0.0981)	λ 29.7109 (28.9529)	202.4449	210.4449	211.3338	218.0930	0.1444	0.8940	0.1169	0.4662	0.1895
WL	a 0.0944 (0.1692)	b 1.5271 (1.1335)	α 0.3356 (0.3456)	β 0.0129 (0.0316)	202.6376	210.6376	211.5265	218.2857	0.1461	0.9038	0.1182	0.4523	0.1606
GWLL	α 0.0064 (0.0364)	β 9.9661 (0.0006)	a 4.1023×10^{-5} (0.0002)	γ 0.1320 (0.0469)	204.4580	212.4579	213.3468	220.1060	0.1538	0.9582	0.1166	0.4701	0.1578

Table 4.5: MLEs and GoF of the MO-TLHL-W model for the failure times data

Figure 4.13 provides fitted densities, histogram and probability plots, based on the plots, GoF statistics, p-value of the K-S statistic we can infer that the MO-TLHL-W distribution provides a superior fit when compared to other models in Table 4.5.

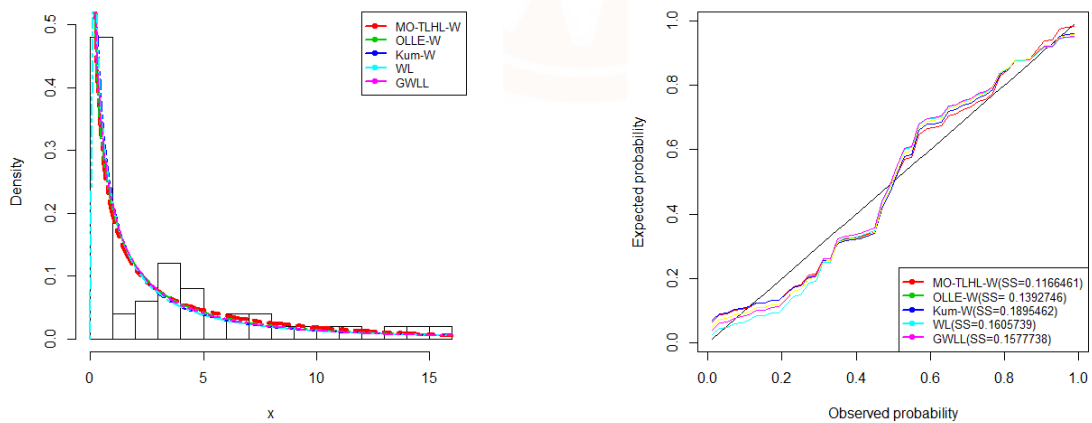


Figure 4.13: Fitted densities and probability plots for the failure times data

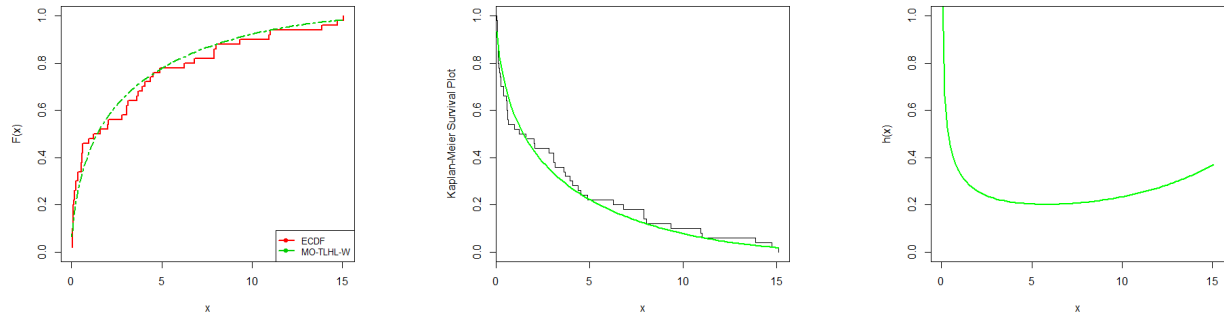


Figure 4.14: Estimated cdf, K-M survival plots and estimated hazard rate plot of the MO-TLHL-W distribution for the failure times data

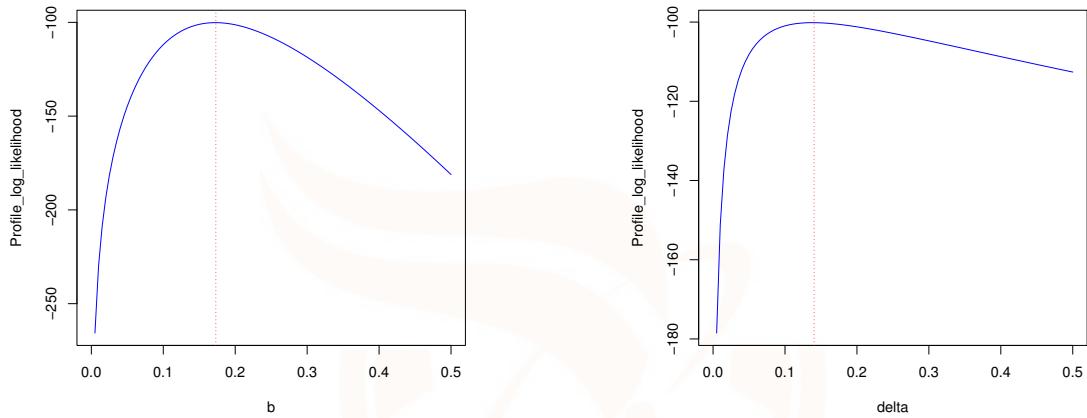


Figure 4.15: Profile plots for the failure times data

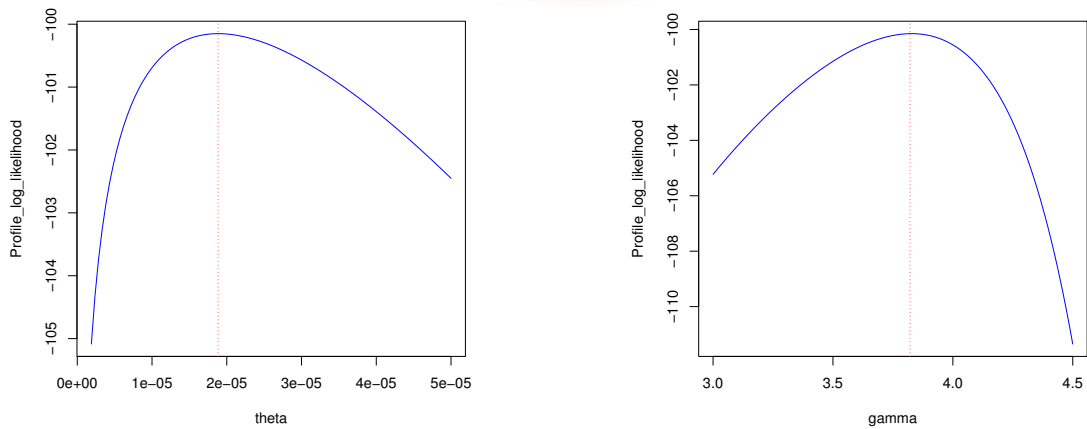


Figure 4.16: Profile plots for the failure times data

The profile plots in Figure 4.17 and Figure 4.18 shows that the estimates of the MO-TLHL-W distribution can be uniquely identified for the failure times data. The estimated var-cov matrix

for the failure times of 50 components data is given by,

$$\begin{bmatrix} 0.00027 & 0.00010 & 7.6689 \times 10^{-8} & 0.00001 \\ 0.00010 & 0.00004 & 2.8741 \times 10^{-8} & 4.7879 \times 10^{-6} \\ 7.6689 \times 10^{-8} & 2.8741 \times 10^{-8} & 1.2973 \times 10^{-10} & 3.6216 \times 10^{-9} \\ 0.00001 & 4.7879 \times 10^{-6} & 3.6216 \times 10^{-9} & 6.0331 \times 10^{-7} \end{bmatrix}$$

and the 95% asymptotic CIs for the parameters b, δ, θ and γ are: 0.1731 ± 0.32144 , 0.1405 ± 0.12074 , $1.8835 \times 10^{-5} \pm 0.00002$ and 3.8215 ± 0.00137 , respectively.

4.7.2 Kevlar 49/Epoxy Strands Failure at 90% Stress Level

The data represent the stress-rupture life of kevlar 49/epoxy strands, (see Andrews and Herzberg [19]; Barlow et al. [21] for additional details). The data are as follows: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

Model	Estimates				statistics								
	b	δ	θ	γ	-2ln (L)	AIC	CAIC	BIC	W*	A*	K-S	P-value	SS
MO-TLHL-W	1.0922 (0.6325)	2.0489 (2.0543)	1.0613 (0.5921)	0.6613 (0.2560)	203.66	211.6600	212.0767	222.1205	0.1260	0.7663	0.0724	0.6657	0.1189
BGL	α 0.1233 (0.00559)	θ 1.6897×10^{-6} (2.8310×10^{-6})	a 0.3020 (0.0005)	b 10.100 (1.7373×10^{-5})	596.9422	604.9524	605.3691	615.4129	0.4136	2.2013	0.4450	2.2000×10^{-16}	7.0925
OLLE-W	α 1.6254 (2.6056)	β 1.1122 (0.4411)	θ 1.1688 (0.9425)	γ 0.6079 (0.8972)	205.5335	213.5335	213.9501	223.9939	0.1616	0.9432	0.0798	0.5407	0.1548
Kum-W	a 0.7546 (0.2978)	b 0.3338 (0.7513)	c 1.0338 (0.1645)	λ 2.5736 (6.0408)	205.2936	213.2936	213.7103	223.7541	0.1489	0.8846	0.0792	0.5505	0.1463
WL	a 0.2506 (0.4173)	b 0.7860 (0.1804)	α 1.3581 (0.4580)	β 0.3303 (0.6282)	205.1976	213.1976	213.6143	223.6581	0.1440	0.8627	0.0787	0.5587	0.1426
GWLL	α 9.8528×10^6 (7.4924×10^{-7})	β 8.1611 (2.6690)	a 6.3927×10^6 (9.5495×10^{-7})	γ 0.1213 (0.4427)	206.0739	214.0739	214.4906	224.5344	0.2059	1.1450	0.0918	0.3622	0.2019

Table 4.6: MLEs and GoF of the MO-TLHL-W model for the Kevlar 49/Epoxy Strands Failure Data

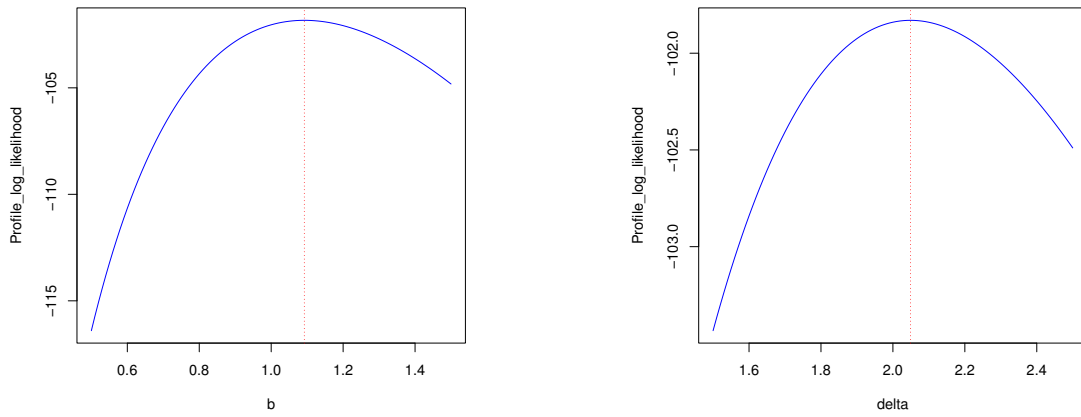


Figure 4.17: Profile plots for the failure times data

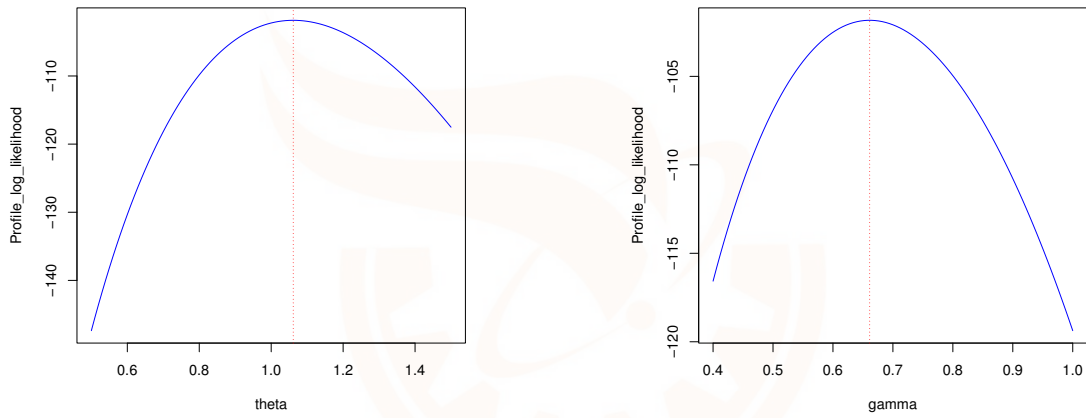


Figure 4.18: Profile plots for the failure times data

The profile plots in Figures ?? and ?? shows that the estimates of the MO-TLHL-W distribution are global maximums. Figure 4.19 gives fitted densities, data's histogram and probability plots, based on these plots and GoF statistics, we can infer that the MO-TLHL-W distribution performs better compared to other models in Table 4.6 in modelling the stress-rupture life data.

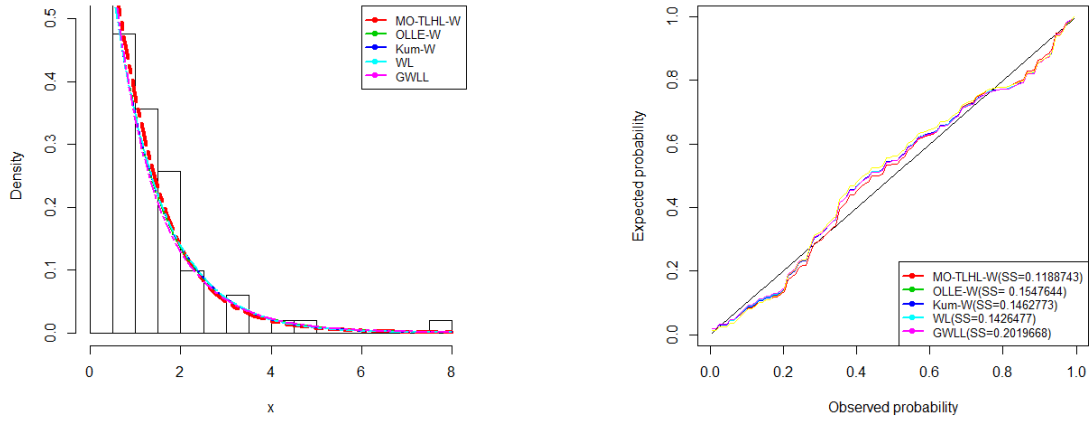


Figure 4.19: Fitted densities and probability plots for the Kevlar 49/Epoxy strands failure data

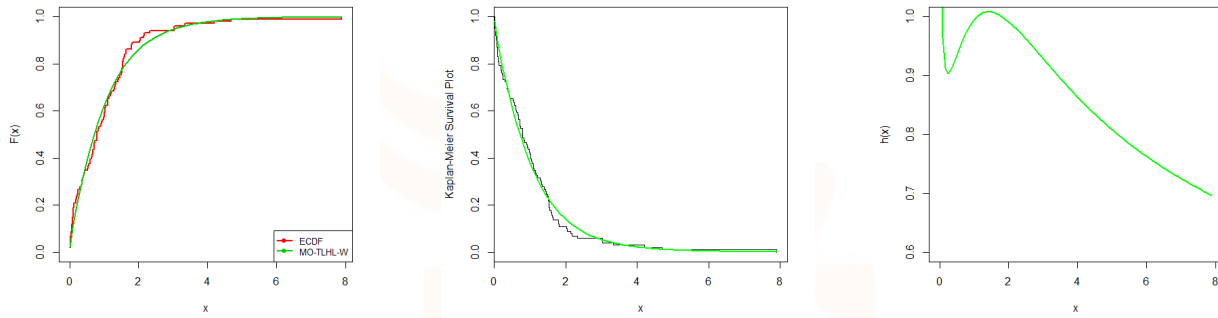


Figure 4.20: Estimated cdf, K-M survival plots and estimated hazard rate plot of the MO-TLHL-W distribution for the Kevlar 49/ Epoxy strands failure data

The estimated var-cov matrix for the Kevlar 49/ Epoxy strands failure at 90% stress level data is given by,

$$\begin{bmatrix} 0.40008 & 0.41146 & 0.27047 & -0.13001 \\ 0.41146 & 4.22002 & 1.06270 & -0.40941 \\ 0.27047 & 1.06270 & 0.35061 & -0.14638 \\ -0.13001 & -0.40941 & -0.14638 & 0.06555 \end{bmatrix}$$

and the 95% CIs for the parameters b, δ, θ and γ are: 1.0922 ± 1.2397 , 2.0489 ± 4.02643 , 1.0613 ± 1.16052 and 0.6613 ± 0.50176 , respectively.

4.8 Concluding Remarks

A new generalized FoD called the Marshall-Olkin-Topp-Leone Half Logistic-G (MO-TLHL-G) distribution was developed and presented. The MO-TLHL-G distribution possesses hrf with monotone and non-monotone shapes. We also provided explicit expressions for the moments, incomplete moments, distribution of order statistics and RE. We used the MLE technique to estimate the parameters of the MO-TLHL-G distribution. We also assessed the effectiveness of

the new generalized model by carrying out some simulations for various sample sizes. Finally, we applied the MO-TLHL-W distribution to real data to demonstrate its flexibility and applicability in data analysis.



5. Some New Marshall-Olkin Generalizations with Applications

5.1 The Marshall-Olkin-Generalized Exponentiated Half Logistic-G Family of Distributions with Applications

Sengweni W, Oluyede B and Gabanakgosi M, The Marshall-Olkin-Generalized Exponentiated Half Logistic-G Family of Distributions with Applications

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5.1.1 Introduction

In this chapter, we develop and study a new generalized FoD called the Marshall-Olkin-Generalized Exponentiated Half Logistic-G (MO-GEHL-G) FoD. Even though creating a model that captures the exact characteristics of a data generating remains a utopian idea, its of paramount importance to have a model that mimic the properties of the situation or the process being studied as close as possible. Classical distributions, for example the exponential, weibull, gamma, half logistic, log-logistic distributions to mention a few have proved to fail to capture and provide better fits when it comes to modelling various real life scientific and engineering phenomena. In an effort to come up with new and improved models for data modelling, Marshall and Olkin [85] introduced a new way of adding a tilt parameter to classical distributions. Several well known distributions have been put forward using this technique and these include papers by ([27], [52],[122], [35], [74], [73], [76], [61]). In this paper, we employ the Marshall-Olkin transformation to come up with a new FoD called the Marshall-Olkin-Generalized EHL-G (MO-GEHL-G) generalized distributions.

Our main motivations for developing this new FoD are to:

- improve the flexibility of the classical models considered when it comes to fitting and modelling real-life data.
- obtain hrfs with a variety of shapes (increasing, decreasing, bathtub and inverted bathtub shapes) which can give a true reflection of the characteristics of many systems and processes in scientific areas.
- produce models with consistently better fits than other generated distributions under the same underlying distribution.

Findings in this section are arranged as follows. In subsection 5.1.2 we presented the MO-GEHL-G FoD, its sub-families, hrf, qf, stochastic ordering, linear representation of the pdf and some selected special cases. Moments, incomplete moments and mgf are provided in subsection 5.1.3. RE and distribution of order statistics are presented in subsection 5.1.4. A maximum

likelihood approach for parameter estimation is presented in subsection 5.1.5. In subsection 5.1.6 results on MCS carried out to evaluate the behaviour of ABias and RMSE of maximum likelihood estimators for the MO-GEHL-G distribution are presented. In subsection 5.1.7 we provided some examples to show the usefulness of the proposed model. subsection 5.1.8 gives some concluding remarks.

5.1.2 The Generalized Model, Properties and Some Special Cases

In this subsection, we provide derivations of some statistical properties of the Marshall-Olkin-Generalized Exponentiated Half Logistic-G (MO-GEHL-G) FoD.

The New Generalized Model

Marshall and Olkin [85] developed a new strategy for the addition of a tilt parameter to a distribution and this leads to a more flexible model. To enjoy the flexibility brought about by this parameter, we develop a new model with a tilt parameter by making use of this Marshall-Olkin method (see equation (1.12)). We used the EHL-G by Cordeiro et al. [53] given in equation (1.1) to come up with the generalised EHL-G FoD with cdf given by

$$F_{GEHL-G}(x; \alpha, \gamma, \varphi) = 1 - \left[1 - (H_G(x; \varphi))^\alpha \right]^\gamma, \quad (5.1)$$

where $H_G(x; \varphi) = \left(\frac{G(x; \varphi)}{1 + G(x; \varphi)} \right)$, $\alpha, \gamma > 0$ and baseline parameter vector φ . Now, using equation (1.12) and equation (5.1), we obtain the new Marshall-Olkin-Generalized EHL-G FoD with cdf and pdf given by

$$F(x; \delta, \alpha, \gamma, \varphi) = 1 - \frac{\delta \left[1 - (H_G(x; \varphi))^\alpha \right]^\gamma}{1 - \bar{\delta} \left[1 - (H_G(x; \varphi))^\alpha \right]^\gamma} \quad (5.2)$$

and

$$f(x; \delta, \alpha, \gamma, \varphi) = \frac{2\gamma\alpha\delta g(x; \varphi)(H_G(x; \varphi))^{\alpha-1} \left[1 - (H_G(x; \varphi))^\alpha \right]^{\gamma-1}}{(1 + \bar{G}(x; \varphi))^2 \left(1 - \bar{\delta} \left[1 - (H_G(x; \varphi))^\alpha \right]^\gamma \right)^2}, \quad (5.3)$$

where $\bar{\delta} = 1 - \delta$, $\delta > 0$, $\alpha, \gamma > 0$ and φ is the parameter vector for the baseline cdf $G(x; \varphi)$.

Sub-Families

In this subsection, we give sub-families of the MO-GEHL-G FoD.

- When $\delta = 1$, we obtain the Generalized EHL-G (GEHL-G) FoD with cdf

$$F(x; \alpha, \gamma, \varphi) = 1 - \left[1 - (H_G(x; \varphi))^\alpha \right]^\gamma \quad (5.4)$$

for $\alpha, \gamma > 0$ and parameter vector φ . This is a new FoD.

- If $\gamma = 1$, we obtain the Marshall-Olkin-EHL-G (MO-EHL-G) FoD with cdf

$$F(x; \delta, \alpha, \varphi) = 1 - \frac{\delta \left[1 - (H_G(x; \varphi))^\alpha \right]}{1 - \bar{\delta} \left[1 - (H_G(x; \varphi))^\alpha \right]} \quad (5.5)$$

for $\alpha, \delta > 0$, $\bar{\delta} = 1 - \delta$ and parameter vector φ .

- If $\alpha = 1$ and $\gamma = 1$, we obtain Marshall-Olkin-Half Logistic-G (MO-HL-G) FoD with cdf given by

$$F(x; \delta, \varphi) = 1 - \frac{\delta \left[1 - H_G(x; \varphi) \right]}{1 - \bar{\delta} \left[1 - (H_G(x; \varphi))^\alpha \right]} \quad (5.6)$$

for $\delta > 0$ and parameter vector φ .

- If $\alpha = \delta = 1$, we obtain the Generalised Half logistic-G (GHL-G) FoD with cdf given by

$$F(x; \gamma, \varphi) = 1 - \left[1 - H_G(x; \varphi) \right]^\gamma \quad (5.7)$$

for $\gamma > 0$ and parameter vector φ .

- If $\alpha = \delta = \gamma = 1$, we have the Half logistic-G (HL-G) FoD.

Hazard Rate and Quantile Functions

In this subsection, we present the hrf and qf of the MO-GEHL-G FoD. The hrf is given by

$$h_F(x; \delta, \alpha, \gamma, \varphi) = \frac{2\gamma\alpha g(x; \varphi)(H_G(x; \varphi))^{\alpha-1} \left[1 - (H_G(x; \varphi))^\alpha \right]^{-1}}{(1 + \bar{G}(x; \varphi))^2 \left(1 - \bar{\delta} \left[1 - (H_G(x; \varphi))^\alpha \right]^\gamma \right)}. \quad (5.8)$$

The qf of the MO-GEHL-G family of distributions is obtained by solving the non-linear equation, $F(Q(u)) = u$, $0 < u < 1$, that is,

$$F(Q(u)) = 1 - \frac{\delta \left[1 - \left(\frac{G(Q(u))}{1 + \bar{G}(Q(u))} \right)^\alpha \right]^\gamma}{1 - \bar{\delta} \left[1 - \left(\frac{G(Q(u))}{1 + \bar{G}(Q(u))} \right)^\alpha \right]^\gamma} = u. \quad (5.9)$$

Making $Q(u)$ the subject in the equation above, we obtain the qf as:

$$Q_F(u) = G^{-1} \left[\frac{2 \left[1 - \left\{ \frac{\left(\frac{1-u}{\delta} \right)}{1 + \bar{\delta} \left(\frac{1-u}{\delta} \right)} \right\}^{\frac{1}{\gamma}} \right]^{\frac{1}{\alpha}}}{1 + \left[1 - \left\{ \frac{\left(\frac{1-u}{\delta} \right)}{1 + \bar{\delta} \left(\frac{1-u}{\delta} \right)} \right\}^{\frac{1}{\gamma}} \right]^{\frac{1}{\alpha}}} \right]. \quad (5.10)$$

Consequently, we can generate or simulate from the MO-GEHL-G family for a specified baseline

cdf G .

Stochastic Ordering

Stochastic ordering plays a crucial role in the assessment of probabilistic models in different areas such as reliability, survival analysis, finance, risks and economics. In this section, we present a stochastic ordering result from the MO-GEHL-G FoD.

Let X and Y be two rvs with distribution functions $F_X(t)$ and $F_Y(t)$, respectively. Then X is stochastically smaller than Y if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all t , where $\bar{F}_X(t) = 1 - F_X(t)$ is the survival function. Stochastic ordering of X and Y is denoted by $X <_{st} Y$. A stronger ordering is the likelihood ratio order denoted by $X <_{lr} Y$, and given by $\frac{f_X(t)}{f_Y(t)}$ decreasing in t . Hazard rate order is stronger than stochastic order but weaker than likelihood ratio order. Hazard rate order is given by $h_X(t) \geq h_Y(t)$ for all t , and denoted by $X <_{hr} Y$. Thus, $X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_{st} Y$ (see Shaked and Shanthikumar [116]).

Theorem 2.1. Suppose $X_1 \sim MO - GEHL - G(\delta_1, \alpha, \gamma, \varphi)$ and $X_2 \sim MO - GEHL - G(\delta_2, \alpha, \gamma, \varphi)$. If $\delta_1 < \delta_2$, then $\frac{f(x; \delta_1, \alpha, \gamma, \varphi)}{f(x; \delta_2, \alpha, \gamma, \varphi)}$ is decreasing in x .



Proof:

After some simplification, we have

$$\frac{f(x; \delta_1, \alpha, \gamma, \varphi)}{f(x; \delta_2, \alpha, \gamma, \varphi)} = \frac{\delta_1}{\delta_2} \left(\frac{1 - \bar{\delta}_2 \left(1 - (H_G(x; \varphi))^\alpha \right)^\gamma}{1 - \bar{\delta}_1 \left(1 - (H_G(x; \varphi))^\alpha \right)^\gamma} \right)^2. \quad (5.11)$$

Differentiating equation (5.11) with respect to x , we have

$$\frac{\partial}{\partial x} \left(\frac{f(x; \delta_1, \alpha, \gamma, \varphi)}{f(x; \delta_2, \alpha, \gamma, \varphi)} \right) = \frac{4\delta_1}{\delta_2} (\delta_1 - \delta_2) \frac{\alpha \gamma g(x; \varphi) (1 - \bar{\delta}_2 Z^\gamma) Z^{\gamma-1} (H_G(x; \varphi))^{\alpha-1}}{(1 + \bar{G}(x; \varphi))^2 (1 - \bar{\delta}_1 Z^\gamma)^3},$$

where $Z = [1 - (H_G(x; \varphi))^\alpha]$.

If $\delta_1 < \delta_2$, then $\frac{\partial}{\partial x} \left(\frac{f(x; \delta_1, \alpha, \gamma, \varphi)}{f(x; \delta_2, \alpha, \gamma, \varphi)} \right) < 0$, which implies that likelihood ratio order exist among X_1 and X_2 , that is $X_2 <_{lr} X_1$. Consequently, X_1 and X_2 are stochastically ordered, that is, $X_1 <_{st} X_2$.

Density Function Expansion

In this subsection, series expansion for the MO-GEHL-G pdf is presented. Let $f(x; \delta, \alpha, \gamma, \varphi)$ be written as $f(x)$, then the pdf in equation (5.3) can be expressed as

$$\begin{aligned} f(x) &= 2\gamma\alpha\delta g(x; \varphi) (H_G(x; \varphi))^{\alpha-1} \left[1 - (H_G(x; \varphi))^\alpha \right]^{\gamma-1} (1 + \bar{G}(x; \varphi))^{-2} \\ &\times \left(1 - \bar{\delta} \left[1 - (H_G(x; \varphi))^\alpha \right]^\gamma \right)^{-2}. \end{aligned}$$

Using the generalized binomial series expansions, $(1 - z)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k$, $(1 + z)^{-\alpha} = \sum_{p=0}^{\infty} (-1)^p \binom{\alpha+p-1}{p} z^p$ and $(1 - z)^{-\alpha} = \sum_{q=0}^{\infty} \frac{\Gamma(\alpha+q)}{\Gamma(\alpha)q!} z^q$, for $|z| < 1$, we have

$$\begin{aligned}
 f(x) &= 2\gamma\alpha\delta g(x; \varphi) \sum_{q=0}^{\infty} \frac{\Gamma(q+2)\bar{\delta}^q}{q!} (H_G(x; \varphi))^{\alpha-1} \\
 &\times \left[1 - (H_G(x; \varphi))^\alpha \right]^{\gamma(q+1)-1} (1 + \bar{G}(x; \varphi))^{-2} \\
 &= 2\gamma\alpha\delta g(x; \varphi) \sum_{q,k=0}^{\infty} \frac{(-1)^k \Gamma(q+2)\bar{\delta}^q}{q!} \binom{\gamma(q+1)-1}{k} [G(x; \varphi)]^{\alpha(k+1)-1} \\
 &\times (1 + \bar{G}(x; \varphi))^{-(\alpha(k+1)+1)} \\
 &= 2\gamma\alpha\delta g(x; \varphi) \sum_{q,k,m=0}^{\infty} \frac{(-1)^{k+m} \Gamma(q+2)\bar{\delta}^q}{q!} \binom{\gamma(q+1)-1}{k} \\
 &\times \binom{\alpha(k+1)+m}{m} \bar{G}(x; \varphi)^m [1 - \bar{G}(x; \varphi)]^{\alpha(k+1)-1} \\
 &= 2\gamma\alpha\delta g(x; \varphi) \sum_{q,k,m,n=0}^{\infty} \frac{(-1)^{k+m+n} \Gamma(q+2)\bar{\delta}^q}{q!} \binom{\gamma(q+1)-1}{k} \\
 &\times \binom{\alpha(k+1)+m}{m} \binom{\alpha(k+1)-1}{n} \bar{G}(x; \varphi)^{m+n} \\
 &= 2\gamma\alpha\delta g(x; \varphi) \sum_{q,k,m,n,\tau=0}^{\infty} \frac{(-1)^{k+m+n+\tau} \Gamma(q+2)\bar{\delta}^q}{q!} \binom{\gamma(q+1)-1}{k} \\
 &\times \binom{\alpha(k+1)+m}{m} \binom{\alpha(k+1)-1}{n} \binom{m+n}{\tau} G^\tau(x; \varphi)
 \end{aligned}$$

which can be expressed as

$$f(x) = \sum_{\tau=0}^{\infty} C_{\tau+1} g_{\tau+1}^*(x; \varphi), \quad (5.12)$$

where

$$\begin{aligned}
 C_{\tau+1} &= 2\gamma\alpha\delta \sum_{q,k,m,n=0}^{\infty} \frac{(-1)^{k+m+n+\tau} \Gamma(q+2)\bar{\delta}^q}{(\tau+1)q!} \binom{\gamma(q+1)-1}{k} \\
 &\times \binom{\alpha(k+1)+m}{m} \binom{\alpha(k+1)-1}{n} \binom{m+n}{\tau}, \quad (5.13)
 \end{aligned}$$

and $g_{\tau+1}^*(x; \varphi) = (\tau+1)g(x; \varphi)G^\tau(x; \varphi)$ is the Exp-G density with $(\tau+1)$ as a power parameter. The statistical properties of the MO-GEHL-G family can be obtained directly from those of the Exp-G FoD.

Some Special Cases

In this subsection, we present three special cases of the MO-GEHL-G FoD, specifically when the baseline distribution function $G(x; \varphi)$ are Weibull, log-logistic, and Burr XII distributions.

Marshall-Olkin-Generalized EHL-Weibull (MO-GEHL-W) Distribution

Let $G(x; \theta) = 1 - \exp(-x^\theta)$ and $g(x; \theta) = \theta x^{\theta-1} \exp(-x^\theta)$ be the baseline distribution for $\theta > 0$ and $x > 0$. Then, the MO-GEHL-Weibull (MO-GEHL-W) distribution is given by

$$F(x; \delta, \alpha, \gamma, \theta) = 1 - \frac{\delta \left[1 - \left(\frac{1 - \exp(-x^\theta)}{1 + \exp(-x^\theta)} \right)^\alpha \right]^\gamma}{1 - \bar{\delta} \left[1 - \left(\frac{1 - \exp(-x^\theta)}{1 + \exp(-x^\theta)} \right)^\alpha \right]^\gamma}$$

and

$$f(x; \delta, \alpha, \gamma, \theta) = \frac{2\gamma\alpha\delta\theta x^{\theta-1} \exp(-x^\theta) \left(\frac{1 - \exp(-x^\theta)}{1 + \exp(-x^\theta)} \right)^{\alpha-1} \left[1 - \left(\frac{1 - \exp(-x^\theta)}{1 + \exp(-x^\theta)} \right)^\alpha \right]^{\gamma-1}}{(1 + \exp(-x^\theta))^2 \left(1 - \bar{\delta} \left[1 - \left(\frac{1 - \exp(-x^\theta)}{1 + \exp(-x^\theta)} \right)^\alpha \right]^\gamma \right)^2},$$

where $\bar{\delta} = 1 - \delta$, $\delta > 0, \alpha, \gamma, \theta > 0$.

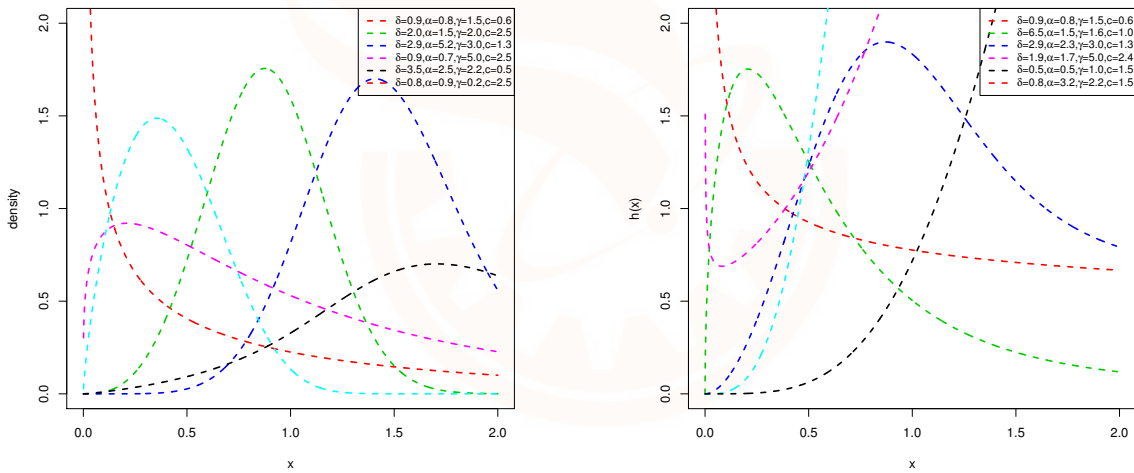


Figure 5.1: Plots of pdf and hrf for the MO-GEHL-W distribution

As shown in Figure 5.1, the density function of the MO-GEHL-W distribution reveals a variety of shapes such as the uni-modal, reverse-J, left-skewed and right-skewed, and almost symmetric. The graphs of the hrf of the MO-GEHL-W distribution on the other hand exhibits increasing, inverted bathtub, bathtub, decreasing shapes.

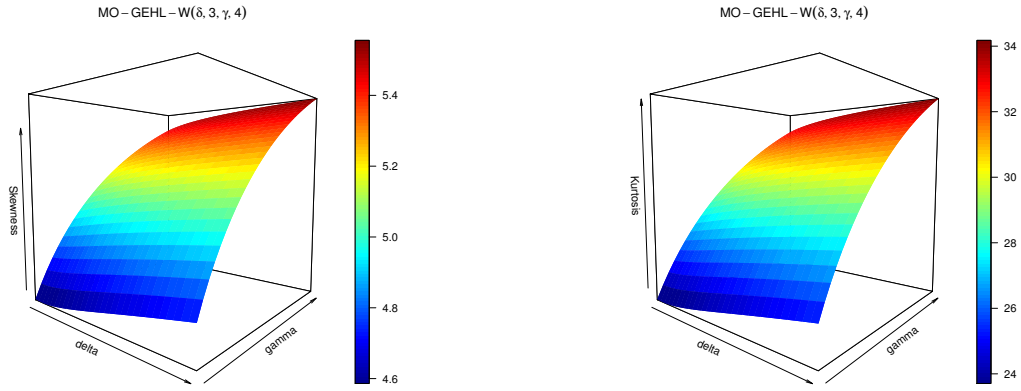


Figure 5.2: Plots of Skewness and Kurtosis for MO-GEHL-W distribution

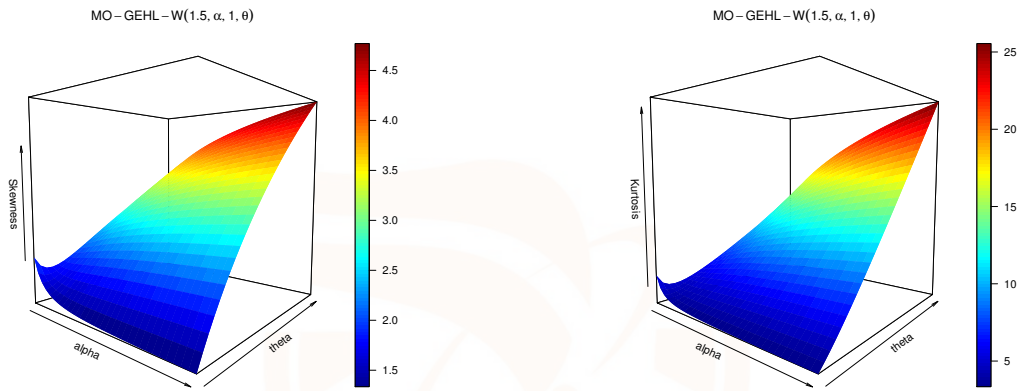


Figure 5.3: Plots of Skewness and Kurtosis for MO-GEHL-W distribution

The plots in Figures 5.2 and 5.3 indicate that skewness and kurtosis for MO-GEHL-W distribution

- increases when holding α and θ constant and as δ and γ increases.
- increase when holding δ and γ constant while α and θ increases.

Marshall-Olkin-Generalized EHL-Log-Logistic (MO-GEHL-LLoG) Distribution

Assume the log-logistic distribution to be the baseline distribution. The log-logistic distribution has pdf and cdf given by $g(x; c) = cx^{c-1}(1+x^c)^{-2}$ and $G(x; c) = 1 - (1+x^c)^{-1}$, for $c > 0$, respectively. Therefore, the MO-GEHL-LLoG distribution have the cdf and pdf given by

$$F(x; \delta, \alpha, \gamma, c) = 1 - \frac{\delta \left[1 - \left(\frac{1 - (1+x^c)^{-1}}{1 + (1+x^c)^{-1}} \right)^\alpha \right]^\gamma}{1 - \delta \left[1 - \left(\frac{1 - (1+x^c)^{-1}}{1 + (1+x^c)^{-1}} \right)^\alpha \right]^\gamma}$$

and

$$f(x; \delta, \alpha, \gamma, c) = \frac{2\gamma\alpha\delta c x^{c-1} (1+x^c)^{-2} \left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}} \right)^{\alpha-1} \left[1 - \left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}} \right)^{\alpha} \right]^{\gamma-1}}{(1 + (1+x^c)^{-1})^2 \left(1 - \bar{\delta} \left[1 - \left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}} \right)^{\alpha} \right]^{\gamma} \right)^2},$$

where $\bar{\delta} = 1 - \delta$, $\delta > 0, \alpha, \gamma, c > 0$.

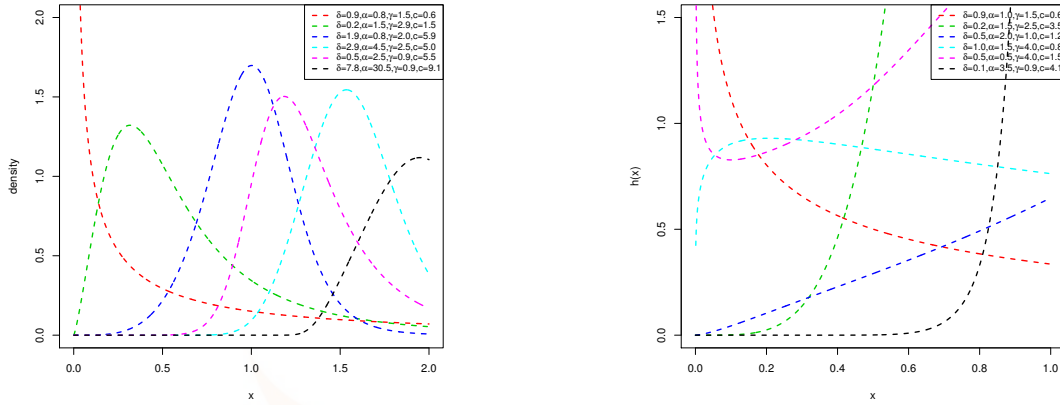


Figure 5.4: Plots of pdf and hrf for the MO-GEHL-LLoG distribution

Figure 5.4 displays the graphs of the pdf and hrf for MO-GEHL-LLoG distribution. The density can assume a variety of shapes such as left-skewed, right-skewed, reverse-J, uni-modal and almost symmetric, whereas hrf shows bathtub, increasing, inverted bathtub and decreasing shapes.

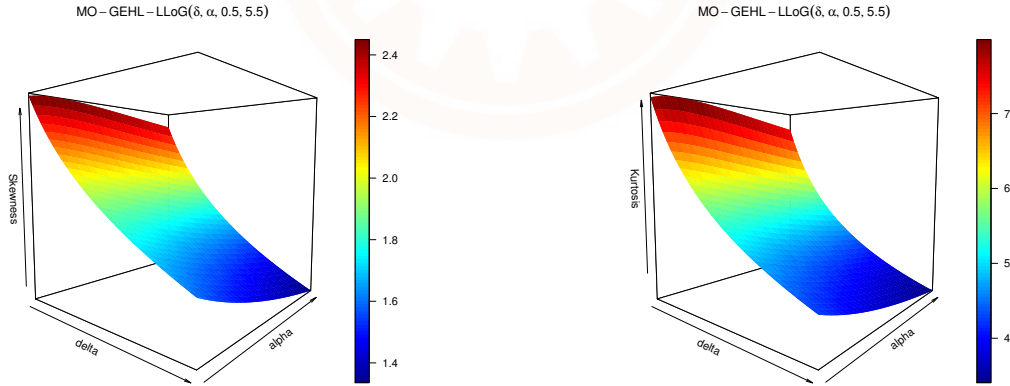


Figure 5.5: Plots of Skewness and Kurtosis for MO-GEHL-LLoG distribution

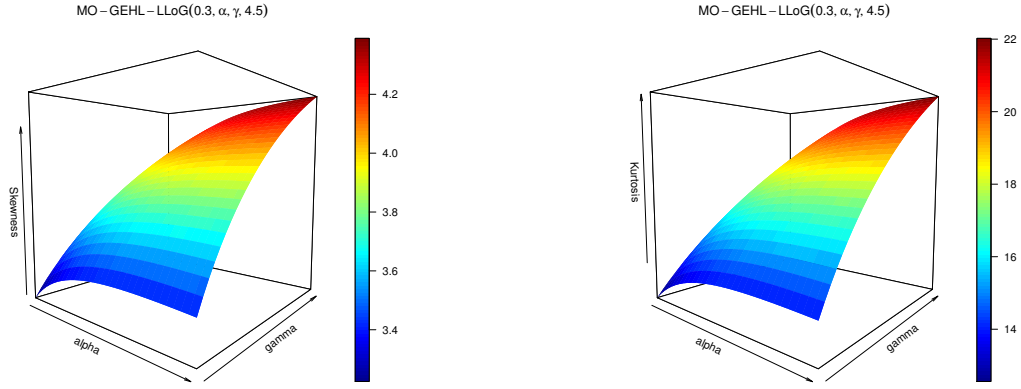


Figure 5.6: Plots of Skewness and Kurtosis for MO-GEHL-LLoG distribution

Figures 5.5 and 5.6 gives the plots of skewness and kurtosis for MO-GEHL-LLoG distribution. The plots shows that skewness and kurtosis

- decrease when holding γ and c constant and as δ and α increases.
- increase when holding δ and c constant while α and γ increases.

5.1.3 Moments and Incomplete Moments

In this subsection, moments, mgf and incomplete moments for the MO-GEHL-G FoD are presented.

Moments and Generating Function

Let $Y_{\tau+1}$, be an Exp-G rv with power parameter $(\tau + 1)$. Then, the r^{th} raw moment of X , say μ'_r follows from equation (5.12) as

$$\mu'_r = E(X^r) = \sum_{\tau=0}^{\infty} C_{\tau+1} E(Y_{\tau+1}^r), \quad (5.14)$$

and the mgf $M_X(t) = E(e^{tX})$ of X can be obtained from equation (5.3) and is given as

$$M_X(t) = \sum_{\tau=0}^{\infty} C_{\tau+1} M_{\tau+1}(t),$$

where $C_{\tau+1}$ is as given in equation (5.13) and $M_{\tau+1}(t)$ is the mgf of $Y_{\tau+1}$. Thus, $M_X(t)$ can be calculated from the Exp-G generating function.

Incomplete Moments

Incomplete moments are crucial in the analysis of income distribution, lifetime data, reliability and in measuring inequality. The r^{th} incomplete moment of the MO-GEHL-G FoD is given by

$$m_r(t) = \int_{-\infty}^t x^r f(x; \delta, \alpha, \gamma, \varphi) dx = \sum_{\tau=0}^{\infty} C_{\tau+1} \int_{-\infty}^t x^r g_{\tau+1}^*(x; \varphi) dx. \quad (5.15)$$

A general equation for $m_1(t)$ can be obtained from equation (5.15) as

$$m_1(t) = \sum_{\tau=0}^{\infty} C_{\tau+1} T_{\tau+1}(t),$$

where $T_{\tau+1}(t) = \int_{-\infty}^t x g_{\tau+1}^*(x; \varphi) dx$ is the first incomplete moment of the Exp-G distribution and $g_{\tau+1}^*(x; \varphi) = (\tau + 1)g(x; \varphi)G^\tau(x; \varphi)$.

5.1.4 Order Statistics and Rényi Entropy (RE)

Order statistics and RE are crucial in many scientific fields, particularly in information theory and reliability analysis. This subsection provides the distribution of the i^{th} order statistics and the RE for the MO-GEHL-G FoD.

Order Statistics

Suppose $X_1, X_2, \dots, X_n$ are i.i.d. MO-GEHL-G rvs, such that the pdf of the i^{th} order statistic, $X_{i:n}$ can be given as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} (F(x))^{j+i-1}, \quad (5.16)$$

where $B(.,.)$ is the beta function. Note that

$$\begin{aligned} f(x)(F(x))^{j+i-1} &= 2\gamma\alpha\delta g(x; \varphi)(H_G(x; \varphi))^{\alpha-1} \left[1 - (H_G(x; \varphi))^\alpha\right]^{\gamma-1} \\ &\times (1 + \overline{G}(x; \varphi))^{-2} \left(1 - \overline{\delta} \left[1 - (H_G(x; \varphi))^\alpha\right]^\gamma\right)^{-2} \\ &\times \left[1 - \frac{\delta \left[1 - (H_G(x; \varphi))^\alpha\right]^\gamma}{1 - \overline{\delta} \left[1 - (H_G(x; \varphi))^\alpha\right]^\gamma}\right]^{j+i-1}. \end{aligned}$$

Using the generalized binomial series expansions, $(1 - z)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k$, $(1 + z)^{-\alpha} = \sum_{p=0}^{\infty} (-1)^p \binom{\alpha+p-1}{p} z^p$ and $(1 - z)^{-\alpha} = \sum_{q=0}^{\infty} \frac{\Gamma(\alpha+q)}{\Gamma(\alpha)q!} z^q$, for $|z| < 1$, we have

$$\begin{aligned}
f(x)(F(x))^{j+i-1} &= 2\gamma\alpha g(x; \varphi) \sum_{p=0}^{\infty} \delta^{p+1} (-1)^p \binom{j+i-1}{p} (H_G(x; \varphi))^{\alpha-1} \\
&\times \left[1 - (H_G(x; \varphi))^\alpha\right]^{\gamma(p+1)-1} (1 + \bar{G}(x; \varphi))^{-2} \\
&\times \left(1 - \bar{\delta} \left[1 - (H_G(x; \varphi))^\alpha\right]^\gamma\right)^{-(\gamma p+2)} \\
&= 2\gamma\alpha g(x; \varphi) \sum_{p,q=0}^{\infty} (-1)^p \binom{j+i-1}{p} \frac{\Gamma(\gamma p + q + 2) \delta^{p+1} \bar{\delta}^q}{\Gamma(\gamma p + 2) q!} \\
&\times (H_G(x; \varphi))^{\alpha-1} \left[1 - (H_G(x; \varphi))^\alpha\right]^{\gamma(p+q+1)-1} \\
&\times (1 + \bar{G}(x; \varphi))^{-2} \\
&= 2\gamma\alpha g(x; \varphi) \sum_{p,q,k=0}^{\infty} (-1)^{p+k} \binom{j+i-1}{p} \frac{\Gamma(\gamma p + q + 2) \delta^{p+1} \bar{\delta}^q}{\Gamma(\gamma p + 2) q!} \\
&\times \binom{\gamma(p+q+1)-1}{k} [G(x; \varphi)]^{\alpha(k+1)-1} (1 + \bar{G}(x; \varphi))^{-(\alpha(k+1)+1)} \\
&= 2\gamma\alpha g(x; \varphi) \sum_{p,q,k,m=0}^{\infty} (-1)^{p+k+m} \binom{j+i-1}{p} \frac{\Gamma(\gamma p + q + 2) \delta^{p+1} \bar{\delta}^q}{\Gamma(\gamma p + 2) q!} \\
&\times \binom{\gamma(p+q+1)-1}{k} \binom{\alpha(k+1)+m}{m} \bar{G}(x; \varphi)^m [1 - \bar{G}(x; \varphi)]^{\alpha(k+1)-1} \\
&= 2\gamma\alpha g(x; \varphi) \sum_{p,q,k,m,n=0}^{\infty} (-1)^{p+k+m+n} \binom{j+i-1}{p} \frac{\Gamma(\gamma p + q + 2) \delta^{p+1} \bar{\delta}^q}{\Gamma(\gamma p + 2) q!} \\
&\times \binom{\gamma(p+q+1)-1}{k} \binom{\alpha(k+1)+m}{m} \binom{\alpha(k+1)-1}{n} \bar{G}(x; \varphi)^{m+n} \\
&= 2\gamma\alpha g(x; \varphi) \sum_{p,q,k,m,n,\tau=0}^{\infty} (-1)^{p+k+m+n+\tau} \binom{j+i-1}{p} \frac{\Gamma(\gamma p + q + 2) \delta^{p+1} \bar{\delta}^q}{\Gamma(\gamma p + 2) q!} \\
&\times \binom{\gamma(p+q+1)-1}{k} \binom{\alpha(k+1)+m}{m} \binom{\alpha(k+1)-1}{n} \binom{m+n}{\tau} G^\tau(x; \varphi).
\end{aligned} \tag{5.17}$$

Substituting equation (5.17) into equation (5.16), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{\tau=0}^{\infty} b_{\tau+1} g_{\tau+1}^*(x; \varphi), \tag{5.18}$$

where $g_{\tau+1}^*(x; \varphi) = (\tau + 1)g(x; \varphi)G^\tau(x; \varphi)$ is the Exp-G density with power parameter $\tau + 1$ and

$$\begin{aligned} b_{\tau+1} &= 2\gamma\alpha \sum_{p,q,k,m,n,\tau=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{j+p+k+m+n+\tau}}{(\tau+1)} \binom{j+i-1}{p} \\ &\times \frac{\Gamma(\gamma p + q + 2)\delta^{p+1}\bar{\delta}^q}{B(i, n-i+1)\Gamma(\gamma p + 2)q!} \binom{\gamma(p+q+1)-1}{k} \binom{n-i}{j} \\ &\times \binom{\alpha(k+1)+m}{m} \binom{\alpha(k+1)-1}{n} \binom{m+n}{\tau}. \end{aligned}$$

Consequently, the distribution of the i^{th} order statistics from the MO-GEHL-G FoD can be expressed as an infinite linear combination of Exp-G densities.

Rényi Entropy (RE)

The RE (Rényi [113]) is defined to be

$$I_R(\nu) = \frac{1}{1-\nu} \log \left(\int_0^\infty [f(x; \delta, \alpha, \gamma, \varphi)]^\nu dx \right), \nu \neq 1, \nu > 0. \quad (5.19)$$

Let $[f(x; \delta, \alpha, \gamma, \varphi)]^\nu = [f(x)]^\nu$, then from equation (5.3), we have

$$\begin{aligned} [f(x)]^\nu &= (2\gamma\alpha\delta)^\nu g^\nu(x; \varphi) (H_G(x; \varphi))^{\nu(\alpha-1)} \left[1 - (H_G(x; \varphi))^\alpha \right]^{\nu(\gamma-1)} \\ &\times (1 + \bar{G}(x; \varphi))^{-2\nu} \left(1 - \bar{\delta} \left[1 - (H_G(x; \varphi))^\alpha \right]^\gamma \right)^{-2\nu}. \end{aligned}$$

Using the generalized binomial series expansions, $(1 - z)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k$, $(1 + z)^{-\alpha} = \sum_{p=0}^{\infty} (-1)^p \binom{\alpha+p-1}{p} z^p$ and $(1 - z)^{-\alpha} = \sum_{q=0}^{\infty} \frac{\Gamma(\alpha+q)}{\Gamma(\alpha)q!} z^q$, for $|z| < 1$, we have

$$\begin{aligned}
[f(x)]^\nu &= (2\gamma\alpha\delta)^\nu g^\nu(x; \varphi) \sum_{q=0}^{\infty} \frac{\Gamma(q+2\nu)\bar{\delta}^q}{\Gamma(2\nu)q!} (H_G(x; \varphi))^{\nu(\alpha-1)} \\
&\times \left[1 - (H_G(x; \varphi))^\alpha\right]^{\gamma(q+\nu)-\nu} (1 + \bar{G}(x; \varphi))^{-2\nu} \\
&= (2\gamma\alpha\delta)^\nu g^\nu(x; \varphi) \sum_{q,k=0}^{\infty} \frac{\Gamma(q+2\nu)\bar{\delta}^q (-1)^k}{\Gamma(2\nu)q!} \binom{\gamma(q+\nu)-\nu}{k} [G(x; \varphi)]^{\alpha(k+\nu)-\nu} \\
&\times (1 + \bar{G}(x; \varphi))^{-(\alpha(k+\nu)+\nu)} \\
&= (2\gamma\alpha\delta)^\nu g^\nu(x; \varphi) \sum_{q,k,m=0}^{\infty} \frac{\Gamma(q+2\nu)\bar{\delta}^q (-1)^{k+m}}{\Gamma(2\nu)q!} \binom{\gamma(q+\nu)-\nu}{k} \\
&\times \binom{\alpha(k+\nu)+\nu+m-1}{m} \bar{G}(x; \varphi)^m [1 - \bar{G}(x; \varphi)]^{\alpha(k+\nu)-\nu} \\
&= (2\gamma\alpha\delta)^\nu g^\nu(x; \varphi) \sum_{q,k,m,n=0}^{\infty} \frac{\Gamma(q+2\nu)\bar{\delta}^q (-1)^{k+m+n}}{\Gamma(2\nu)q!} \binom{\gamma(q+\nu)-\nu}{k} \\
&\times \binom{\alpha(k+\nu)+\nu+m-1}{m} \binom{\alpha(k+\nu)-\nu}{n} \bar{G}(x; \varphi)^{m+n} \\
&= (2\gamma\alpha\delta)^\nu g^\nu(x; \varphi) \sum_{q,k,m,n=0}^{\infty} \frac{\Gamma(q+2\nu)\bar{\delta}^q (-1)^{k+m+n}}{\Gamma(2\nu)q!} \binom{\gamma(q+\nu)-\nu}{k} \\
&\times \binom{\alpha(k+\nu)+\nu+m-1}{m} \bar{G}(x; \varphi)^m [1 - \bar{G}(x; \varphi)]^{\alpha(k+\nu)-\nu} \\
&= (2\gamma\alpha\delta)^\nu g^\nu(x; \varphi) \sum_{q,k,m,n,\tau=0}^{\infty} \frac{\Gamma(q+2\nu)\bar{\delta}^q (-1)^{k+m+n+\tau}}{\Gamma(2\nu)q!} \binom{\gamma(q+\nu)-\nu}{k} \\
&\times \binom{\alpha(k+\nu)+\nu+m-1}{m} \binom{\alpha(k+\nu)-\nu}{n} \binom{m+n}{\tau} G^\tau(x; \varphi).
\end{aligned}$$

Thus, the RE of the MO-GEHL-G FoD can be expressed as

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{\tau=0}^{\infty} \varphi_{\tau+1}^* \left(\int_0^\infty [g(x; \varphi)]^\nu G(x, \varphi)^\tau dx \right) \right], \quad (5.20)$$

where

$$\begin{aligned}
\varphi_{\tau+1}^* &= (2\gamma\alpha\delta)^\nu \sum_{q,k,m,n=0}^{\infty} \frac{\Gamma(q+2\nu)\bar{\delta}^q (-1)^{k+m+n+\tau}}{\Gamma(2\nu)q!} \binom{\gamma(q+\nu)-\nu}{k} \\
&\times \binom{\alpha(k+\nu)+\nu+m-1}{m} \binom{\alpha(k+\nu)-\nu}{n} \binom{m+n}{\tau}.
\end{aligned}$$

RE of the MO-GEHL-G FoD can be readily obtained directly from that of the Exp-G distribution as follows:

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{\tau=0}^{\infty} \phi_{\tau+1}^* e^{(1-\nu)I_{REG}} \right], \quad (5.21)$$

where

$$\begin{aligned} \phi_{\tau+1}^* &= (2\gamma\alpha\delta)^\nu \sum_{q,k,m,n=0}^{\infty} \frac{\Gamma(q+2\nu)\bar{\delta}^q(-1)^{k+m+n+\tau}}{\Gamma(2\nu)q!} \binom{\gamma(q+\nu)-\nu}{k} \\ &\times \binom{\alpha(k+\nu)+\nu+m-1}{m} \binom{\alpha(k+\nu)-\nu}{n} \binom{m+n}{\tau} \left(\frac{\tau}{\nu}+1\right)^{-1}, \end{aligned}$$

and $I_{REG} = \int_0^\infty [(\frac{\tau}{\nu}+1)g(x;\varphi)G(x;\varphi)^{\frac{\tau}{\nu}}]^\nu dx$ is the RE of the Exp-G distribution with power parameter $\frac{\tau}{\nu}+1$.

5.1.5 Estimation and Inferences

In this subsection, we present maximum likelihood technique for estimating the parameters of the MO-GEHL-G FoD. If we let $\Delta = (\delta, \alpha, \gamma, \varphi)$ to be the vector of parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ is given by

$$\begin{aligned} \ell_n(\Delta) &= n \ln(2\gamma) + n \ln(\alpha) + n \ln(\delta) + \sum_{i=1}^n \ln(g(x; \varphi)) + (\alpha-1) \sum_{i=1}^n \ln(G(x; \varphi)) \\ &- (\alpha-1) \sum_{i=1}^n \ln(1 + \bar{G}(x; \varphi)) + (\gamma-1) \sum_{i=1}^n \ln \left[1 - \left(H_G(x; \varphi) \right)^\alpha \right] \\ &- 2 \sum_{i=1}^n \ln(1 + \bar{G}(x; \varphi)) - 2 \sum_{i=1}^n \ln \left(1 - \bar{\delta} \left[1 - \left(H_G(x; \varphi) \right)^\alpha \right]^\gamma \right). \end{aligned}$$

The likelihood function can be maximised via the use of the nlm function in R and NLMixed in SAS for specified baseline cdf $G(x; \varphi)$. In order to build CIs for the model parameters $(\delta, \alpha, \gamma, \varphi)$ and test hypothesis concerning the parameters, the observed information matrix is required and is given by

$$J(\Delta) = \begin{bmatrix} J_{\delta\delta}(\Delta) & J_{\delta\alpha}(\Delta) & J_{\delta\gamma}(\Delta) & J_{\delta\varphi}(\Delta) \\ J_{\alpha\delta}(\Delta) & J_{\alpha\alpha}(\Delta) & J_{\alpha\gamma}(\Delta) & J_{\alpha\varphi}(\Delta) \\ J_{\gamma\delta}(\Delta) & J_{\gamma\alpha}(\Delta) & J_{\gamma\gamma}(\Delta) & J_{\gamma\varphi}(\Delta) \\ J_{\varphi\delta}(\Delta) & J_{\varphi\alpha}(\Delta) & J_{\varphi\gamma}(\Delta) & J_{\varphi\varphi}(\Delta) \end{bmatrix}, \quad (5.22)$$

where $J_{i,j}(\Delta) = -\frac{\partial^2 \ell_n(\Delta)}{\partial i \partial j}$, $i, j = \delta, \alpha, \gamma, \varphi$, and φ is a p component vector of baseline parameters.

Under the usual regularity conditions, $\hat{\Delta}$ is asymptotically normally distributed, (Ferguson [60]), that is, $\hat{\Delta} \sim N_{3+p}(\underline{0}, I^{-1}(\Delta))$ as $n \rightarrow \infty$, where $I(\Delta)$ is the expected information matrix.

The asymptotic behaviour remains valid if $I(\Delta)$ is replaced by $J(\hat{\Delta})$, the observed information matrix evaluated at $\hat{\Delta}$.

5.1.6 Simulation Study

The effectiveness of the MO-GEHL-LLoG distribution is assessed by carrying out simulations for various sample sizes ($n=100, 200, 400, 800, 1000$) using the R package. We generated $N = 3000$ samples for the initialised parameters values as given in Tables 5.1 and 5.2. The ABias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

		$\delta = 1.0, \alpha = 1.0, \gamma = 0.6, c = 2.5$			$\delta = 0.5, \alpha = 2.0, \gamma = 0.5, c = 3.0$		
Parameter	n	Mean	RMSE	ABias	Mean	RMSE	ABias
δ	100	1.4679	1.4219	0.4679	0.8040	0.8189	0.3040
	200	1.3651	1.1880	0.3651	0.7451	0.6527	0.2451
	400	1.3891	1.0868	0.3891	0.6950	0.5589	0.1950
	800	1.3709	0.9908	0.3709	0.6496	0.4496	0.1496
	1000	1.3238	0.9064	0.3238	0.6281	0.4259	0.1281
α	100	1.7318	1.5266	0.7318	2.0601	1.1829	0.0601
	200	1.5170	1.1879	0.5170	1.9884	1.0185	-0.0116
	400	1.3518	0.9741	0.3518	1.9897	0.9635	-0.0103
	800	1.2797	0.8564	0.2797	1.9859	0.8665	-0.0140
	1000	1.1378	0.6399	0.1378	2.0047	0.8311	0.0047
γ	100	1.1468	2.8655	0.5468	0.6895	1.0375	0.1895
	200	0.7258	0.5882	0.1258	0.5236	0.3438	0.0236
	400	0.6145	0.2887	0.0145	0.4926	0.1897	-0.0074
	800	0.6052	0.2568	0.0052	0.4937	0.1597	-0.0063
	1000	0.6049	0.2357	0.0049	0.4983	0.1521	-0.0017
c	100	3.1172	1.9554	0.6172	3.8486	2.3654	0.8486
	200	3.0026	1.7385	0.5026	3.7448	2.0039	0.7448
	400	3.0169	1.4880	0.5169	3.5773	1.6656	0.5773
	800	2.9987	1.4018	0.4987	3.4419	1.3817	0.4419
	1000	2.9158	1.2398	0.4158	3.3751	1.3122	0.3751

Table 5.1: Simulation Results for MO-GEHL-LLoG Distribution

$\delta = 0.5, \alpha = 0.5, \gamma = 0.3, c = 3.0$					$\delta = 0.4, \alpha = 2.0, \gamma = 0.6, c = 1.5$		
Parameter	n	Mean	RMSE	ABias	Mean	RMSE	ABias
δ	100	0.7986	0.7862	0.2986	0.3799	0.3559	-0.0200
	200	0.7534	0.6551	0.2534	0.3519	0.2219	-0.0481
	400	0.6769	0.5454	0.1769	0.3724	0.1652	-0.0276
	800	0.6485	0.4421	0.1485	0.3862	0.1112	-0.0138
	1000	0.6250	0.4287	0.1250	0.3936	0.0954	-0.0064
α	100	1.4737	1.8311	0.9737	2.2658	1.4159	0.2658
	200	1.2155	1.4126	0.7155	2.0638	1.1389	0.0638
	400	0.9944	1.0178	0.4944	1.9698	1.0025	-0.0302
	800	0.8162	0.7315	0.3162	2.0799	1.0153	0.0799
	1000	0.7944	0.6753	0.2945	1.9605	0.9326	-0.0395
γ	100	0.5381	0.8646	0.2381	1.3652	4.9635	0.7652
	200	0.4273	0.4093	0.1273	0.7597	1.0659	0.1597
	400	0.3521	0.1769	0.0521	0.6173	0.3298	0.0173
	800	0.3477	0.1058	0.0477	0.6359	0.2899	0.0359
	1000	0.3287	0.0819	0.0287	0.6026	0.2554	0.0026
c	100	3.8819	2.3497	0.8819	1.2926	1.0454	-0.2074
	200	3.8177	2.1040	0.8177	1.2949	0.7910	-0.2051
	400	3.5406	1.6641	0.5406	1.4049	0.6019	-0.0950
	800	3.4371	1.3467	0.4371	1.4267	0.4108	-0.0733
	1000	3.3699	1.3270	0.3699	1.4612	0.3727	-0.0388

Table 5.2: Simulation Results for MO-GEHL-LLoG Distribution

From Tables 5.1 and 5.2, we can see that the MLEs are strongly consistent since the mean estimates of the parameters are approaching the initial parameter values as the sample size increases. The RMSEs and the ABias is decreasing towards zero as we increase the sample size.

5.1.7 Applications

In this subsection, the importance and flexibility of the MO-GEHL-LLoG distribution is illustrated by means of application to two data sets. The fit of the MO-GEHL-LLoG distribution is compared to the fits of the exponentiated half logistic odd Lindley-Weibull (EHLLOL-W) distribution (Sengweni et al. [119]), the Marshall-Olkin-Gompertz-Weibull (MO-Gom-W) distribution by Chipepa and Oluyede [38], Marshall-Olkin log-logistic Weibull (MOLLW) distribution (Lepetu et al. [76]), Marshall-Olkin exponential Weibull (MOEW) distribution by Pogány et al. [110], Weibull Lomax (WL) distribution (Tahir et al. [123]) and the odd log-logistic exponentiated Weibull (OLLEW) distribution by Afify et al. [2]. The pdfs of the listed distributions are given in the appendix.

Figures 5.9 and 5.13 presents the fitted densities, data's histograms and probability plots (Chambers et al. [34]) are given in . Probability plot were created by plotting $F(x_{(j)}; \hat{\delta}, \hat{\alpha}, \hat{\gamma}, \hat{c})$ versus $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, with $x_{(j)}$ being ordered values of the observed data. The sum of squares (SS) measures the accuracy of the model in capturing the observed data and is given by

$$SS = \sum_{j=1}^n \left[F(x_{(j)}; \hat{\delta}, \hat{\alpha}, \hat{\gamma}, \hat{c}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

Tables 5.3 and 5.4 presents the MLEs of the parameters with their SEs in parenthesis and GoF statistics, $-2\ln(L)$, AIC, CAIC, BIC, W^* , A^* (Chen and Balakrishnan [36]). These GoF statistics are used for comparing models so as to ascertain which one fits the data better. Generally, the model with the least values of the GoF statistics better fits the data.

Fatigue Time of 101 6061-T6 Aluminium Coupons

The first data set represent the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps) (Birnbaum and Saunders [25]). The data are: 70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212.

The estimated var-cov matrix for the fatigue time of aluminium coupons data is

$$\begin{bmatrix} 0.0032 & 0.3543 & 0.0639 & 0.0025 \\ 0.3543 & 39.5410 & 7.1355 & 0.2805 \\ 0.0639 & 7.1355 & 1.2877 & 0.0506 \\ 0.0025 & 0.2805 & 0.0506 & 0.0020 \end{bmatrix}$$

and the 95% CIs are given by $\delta \in [961.9379, 962.1588]$, $\alpha \in [16.5940, 41.2436]$, $\gamma \in [74.5087, 78.9569]$ and $c \in [0.5493, 0.7243]$.

Model	Estimates				Statistics							
	δ	α	γ	c	$-2\ln L$	AIC	$CAIC$	BIC	W^*	A^*	KS	P-value
MO-GEHL-LLoG	962.0483 (0.0563)	28.9187 (6.2882)	76.7328 (1.1347)	0.6368 (0.0445)	911.1443	919.1443	919.561	929.6048	0.0378	0.2711	0.0473	0.9776
EHLOL-W	λ	α	a	b	911.47	919.47	919.89	929.94	0.0481	0.3049	0.0623	0.8285
	0.0032 (0.0096)	4.5436 (0.1224)	0.2860 (0.1194)	0.13405 (0.5831)								
MO-Gom-W	δ	θ	λ	γ	918.710	926.710	927.120	937.170	0.0930	0.6093	0.0702	0.7022
	179.040 (1.0494×10^{-4})	0.02942 (0.0091)	0.3627 (0.0346)	0.8501 (0.2071)								
MOLLW	c	α	β	δ	915.26	923.26	923.68	933.72	0.0709	0.4826	0.0512	0.9536
	0.5037 (3.1076×10^{-9})	3.6114×10^{-4} (1.1169×10^{-5})	1.9572 (2.0086×10^{-8})	2.4242×10^3 (2.7994×10^{-13})								
MOEW	α	β	λ	k	913.44	921.44	921.85	931.89	0.0562	0.3889	0.0601	0.8596
	1.8989×10^3 (4.7904×10^{-8})	9.2736×10^{-5} (2.7708×10^{-5})	0.0360 (6.4499×10^{-3})	2.1002 (6.8113×10^{-5})								
OLLEW	α	β	γ	θ	911.44	919.44	919.86	929.90	0.0594	0.3612	0.0613	0.8419
	0.4802 (0.5317)	0.1582 (0.0189)	7.5475 (1.5178)	19.0869 (1.2688)								
WL	a	b	α	β	922.05	930.05	930.47	940.51	0.1212	0.7956	0.0944	0.3294
	0.4454 (2.5089)	12.9512 (1.4122)	0.2597 (0.0777)	9.3330 (0.4927)								

Table 5.3: Estimates of models and GoF statistics for fatigue time of aluminium coupons data

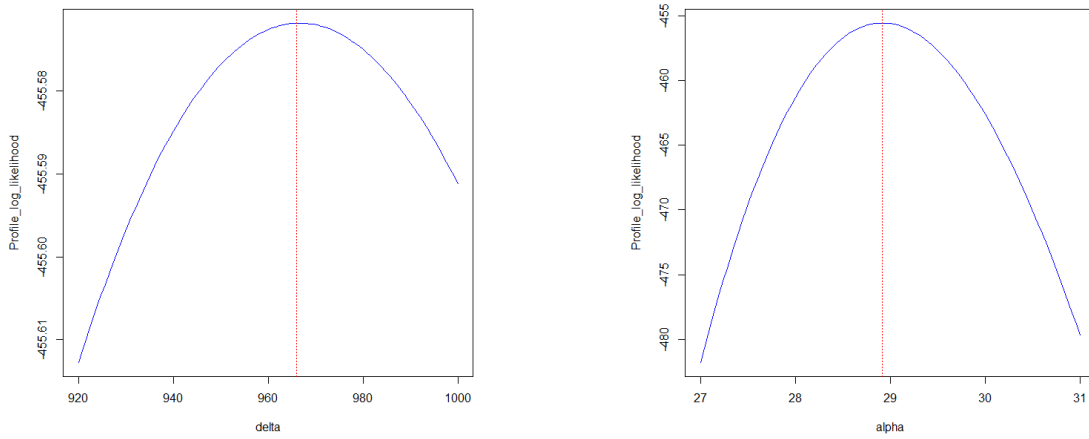


Figure 5.7: Profile log-likelihood plots for the aluminium coupons data

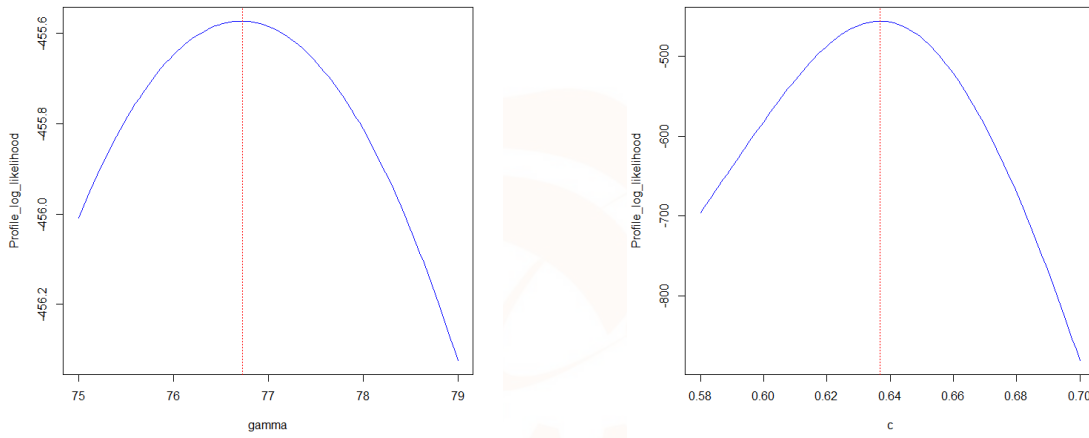


Figure 5.8: Profile log-likelihood plots for the aluminium coupons data

From Table 5.3, the GoF statistics: AIC, CAIC, BIC, W^* , A^* , and KS are smallest for the MO-GEHL-LLoG distribution as compared to the non-nested models listed in the table. This shows that MO-GEHL-LLoG distribution fits the aluminium coupons data better. The p-value of the KS statistic is the largest and the SS value from the probability plots in Figure 5.9 is also the smallest for the MO-GEHL-LLoG distribution. From Figures 5.7 and 5.8 we can see that the log-likelihood profile plots, the estimates are attaining global solutions for all parameters for the aluminium coupons data set.

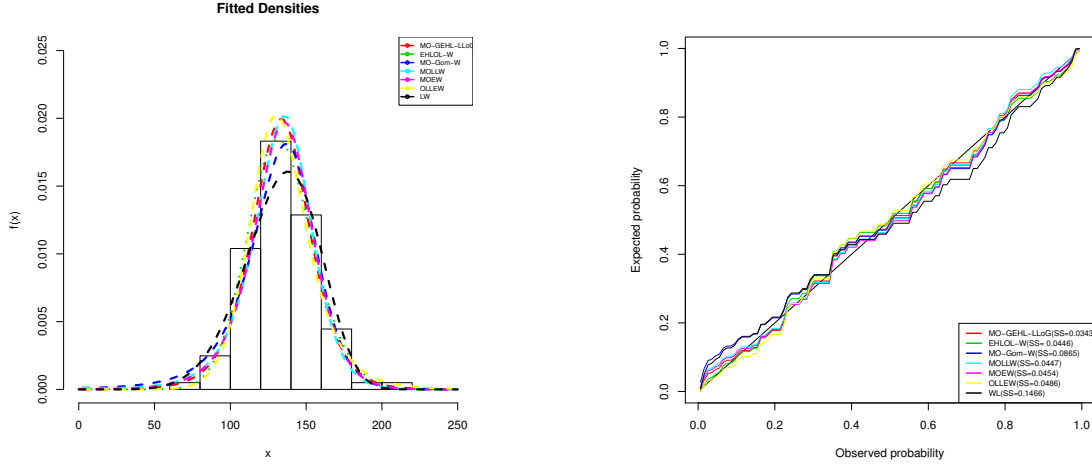


Figure 5.9: Fitted densities and probability plots for aluminium coupons data

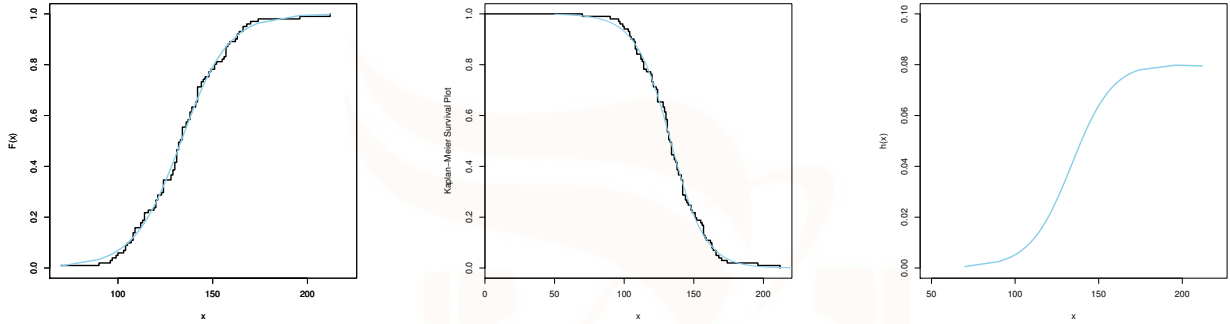


Figure 5.10: Estimated cdf, Kaplan-Meier survival and estimated hrf plots for the MO-GEHL-LLoG distribution for aluminium coupons data

In Figure 5.10, the fitted cdf is closely following the empirical cdf and also the fitted survival function is estimating the survival probabilities quite well since it is very close to the Kaplan-Meier survival plot. The estimated hrf plots for aluminium coupons data is exhibiting an increasing shape.

Kevlar 49/Epoxy Strands Failure at 90% Stress Level

The second data set corresponds to the stress-rupture life of kevlar 49/epoxy strands, (Andrews and Herzberg [19]; Barlow et al. [21]). The observations are given as: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

Model	Estimates				Statistics							
	δ	α	γ	c	$-2 \ln L$	AIC	$CAIC$	BIC	W^*	A^*	KS	P-value
MO-GEHL-LLoG	0.9063 (0.6453)	0.1115 (0.0551)	0.4111 (0.2445)	5.1983 (2.1403)	200.11	208.11	208.53	218.57	0.0721	0.5056	0.0683	0.7336
EHLOL-W	λ	α	a	b	204.08	212.080	212.490	222.540	0.1269	0.7754	0.0722	0.6683
	906.56 (3.1861×10^{-5})	0.8812 (0.1238)	0.8712 (0.0175)	1.7856×10^3 (1.4142×10^{-5})								
MO-Gom-W	δ	θ	λ	γ	203.69	211.69	212.10	222.15	0.1184	0.7343	0.0706	0.6952
	3.9320 (3.7292)	2.0036 (0.8115)	0.6943 (0.1663)	1.6012×10^{-8} (0.0346)								
MOLLW	c	α	β	δ	203.14	211.140	211.560	221.60	0.0734	0.5415	0.0689	0.7237
	2.1576 (0.4913)	1.0904 (0.6103)	0.4982 (0.1081)	3.1094 (2.4212)								
MOEW	α	β	λ	k	203.68	211.68	212.10	222.14	0.1189	0.7369	0.0704	0.6985
	3.9163 (3.9460)	1.9955 (1.6096)	0.0001 (1.1589)	0.6932 (0.2201)								
OLLEW	α	β	γ	θ	205.64	213.64	214.06	224.10	0.1599	0.9381	0.0755	0.6118
	2.8888 (4.5164)	1.3029 (0.7359)	0.3593 (0.5164)	1.5253 (1.0846)								
WL	a	b	α	β	205.190	213.190	213.610	223.660	0.1440	0.8627	0.0787	0.5587
	0.2506 (0.4173)	0.7860 (0.1804)	1.3581 (0.4579)	0.3303 (0.6282)								

Table 5.4: Estimates of models and GoF statistics for epoxy strands failure data

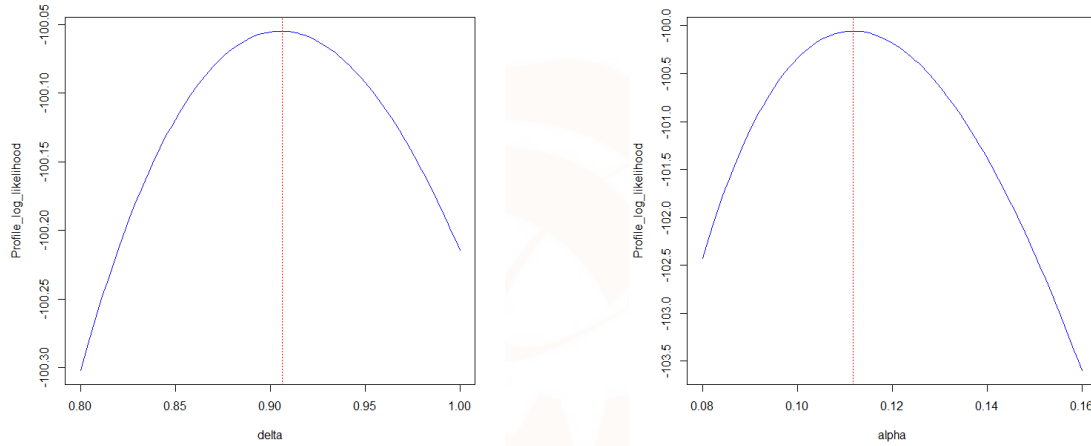


Figure 5.11: Profile log-likelihood plots for the epoxy strands failure data.

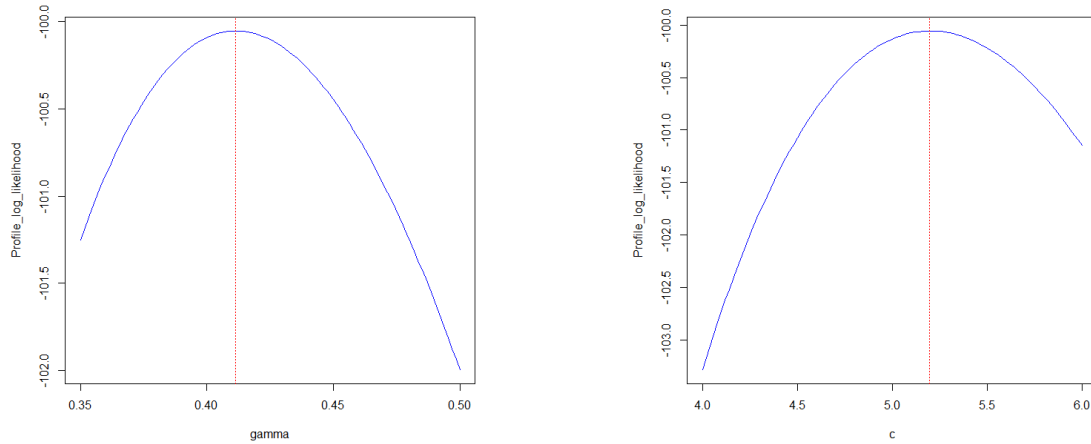


Figure 5.12: Profile log-likelihood plots for the epoxy strands failure data.

The log likelihood profile plots presented in Figure 5.11 and Figure 5.12 shows that the estimates are identifiable for the epoxy strands failure data set.

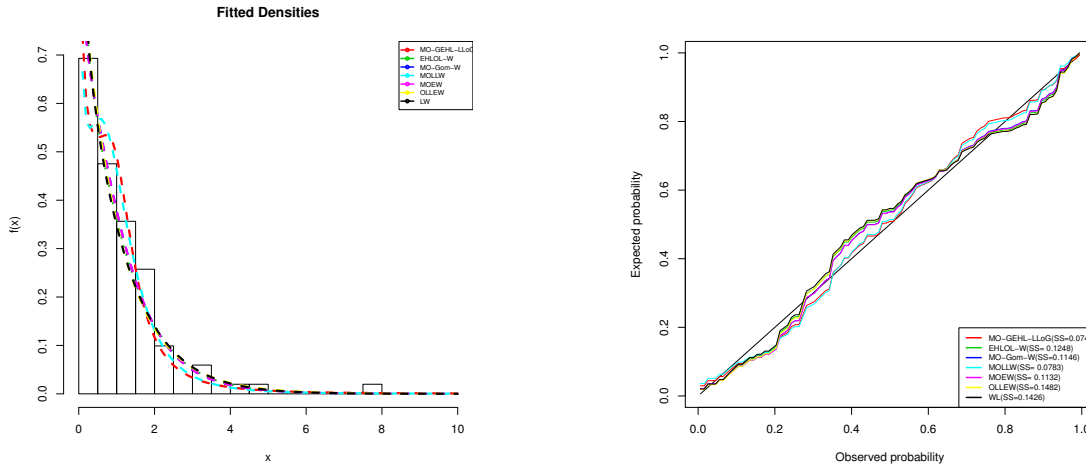


Figure 5.13: Fitted densities and probability plots for epoxy strands failure data

Again from Table 5.4, we see that the GoF statistics: AIC, CAIC and BIC are the smallest for the MO-GEHL-LLoG distribution. The values of the goodness-of-fit statistics W^* , A^* and KS indicate that the MO-GEHL-LLoG distribution is a better fit for the epoxy strands failure data as compared to EHLWL-W, MO-Gom-W, MOLLW, MOEW, OLLEW and WL distributions. The p-value of the KS statistic is the largest and the SS value from the probability plots in Figure 5.13 is the smallest for MO-GEHL-LLoG distribution. The estimated var-cov matrix for epoxy strands failure data is given by,

$$\begin{bmatrix} 0.4165 & 0.0158 & 0.1417 & -1.0183 \\ 0.0158 & 0.0030 & 0.0102 & -0.1047 \\ 0.1417 & 0.0102 & 0.0598 & -0.4856 \\ -1.0183 & -0.1047 & -0.4856 & 4.5809 \end{bmatrix}$$

and the 95% CIs are given by $\delta \in [-0.3585, 2.1712]$, $\alpha \in [0.0035, 0.2194]$, $\gamma \in [-0.0682, 0.8904]$ and $c \in [1.0033, 9.3934]$.

Presented in Figure 5.14 are the fitted cdf, Kaplan-Meier survival function and estimated hazard rate function plots for the epoxy strands failure data. It can be seen that the model is capturing the characteristics of the data and can be used to make inferences about the epoxy strands failure data set. The fitted cdf is closely following the empirical cdf, the same applies with the fitted survival function on the Kaplan-Meier survival plot. The flexibility of the model is also shown in its ability to capture non-monotonic hazard rate as shown by the bathtub followed by upside-down shape in Figure 5.14.

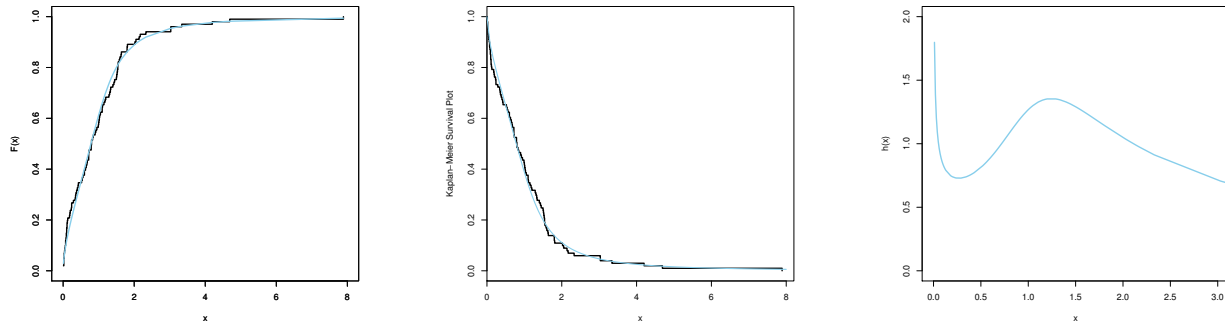


Figure 5.14: Estimated cdf, Kaplan-Meier survival and estimated hrf plots for the MO-GEHL-LLoG distribution for epoxy strands failure data

5.1.8 Conclusions

We developed a new generalized distribution called Marshall-Olkin-Generalized EHL-G (MO-GEHL-G) FoD. We studied the properties of this new FoD. We noted that the hrf takes various desirable shapes and the density function can capture data sets with different levels of skewness. The maximum likelihood estimation method was used to estimate the model parameters. The performance of the maximum likelihood estimators of the MO-GEHL-LLoG distribution was evaluated by performing various simulations for various sample sizes. Finally, the special case of the MO-GEHL-LLoG distribution was fitted to aluminium coupons and epoxy strands failure data sets to illustrate its importance and flexibility.

5.2 The Marshall-Olkin-Odd Exponential Topp-Leone-G FoD with Applications

Sengweni, W. and Oluyede, B., The Marshall-Olkin-Odd Exponential Topp-Leone-G Family of Distributions.

(Submitted: *International Journal of Mathematics in Operations Research (IJMOR)*)

5.2.1 Introduction

In this chapter, we develop and study a new generalized FoD called the Marshall-Olkin-Odd Exponential Topp-Leone-G (MO-OETL-G) distribution. The resulting generalizations have proved to give much needed flexibility and better fits when it comes to lifetime data analysis. Marshall and Olkin [85] introduced a new way of adding a parameter to a FoD so as to improve its flexibility. This technique has been applied to several well known distributions which include some papers by ([27], [52], [122], [35], [74], [73], [76], [61]). In this chapter, we employ the Marshall-Olkin transformation to develop a new FoD called the Marshall-Olkin-Odd Exponential Topp-Leone-G (MO-OETL-G) distribution.

The Topp-Leone distribution has a simple bounded J-shaped density function which is very useful in the modelling of lifetime data. This distribution has attracted many statisticians as an alternative to the Beta distribution. However, it does not provide a reasonable fit in the event

of non-monotonic hrf such as unimodal, bathtub shapes which are common in real life situations and phenomena.

The desirable properties obtained from this new extended FoD in terms of the shapes of the hrf and also the incorporation of the *Topp-Leone Strategy* which is a well known technique for improving the flexibility of a model in lifetime analysis has motivated us to develop this model.

The findings in this section are arranged as follows. Subsection 5.2.2 presents the MO-OETL-G FoD and its sub-families, hrf, qf, stochastic ordering, linear representation of the pdf and some special cases. Moments, mgf and incomplete moments are given subsection 5.2.3. Results on the Rényi entropy and distribution of order statistics are given in subsection 5.2.4. A maximum likelihood estimation technique is presented in subsection 5.2.5. Subsection 5.2.6 has some results on MCS to assess the ABias and RMSE of the maximum likelihood estimators. Real life examples on the applicability of the MO-OETL-G FoD are given in subsection 5.2.7. Finally, some concluding remarks are provided in subsection 5.2.8.

5.2.2 The Generalized Distribution and Properties

This subsection presents various statistical properties of the Marshall-Olkin-Odd Exponential Topp-Leone-G (MO-OETL-G) FoD including hrf and qf, particular cases and expansion of the density.

The New Generalized Distribution

The addition of a tilt parameter to a distribution leads to a more flexible model which can model various types of hazard rate functions than its parent model. To enjoy the flexibility brought about by this parameter, we develop a new model with a tilt parameter by making use of the Marshall-Olkin technique, we developed called the new MO-OETL-G FoD with cdf and pdf given by

$$F(x; \delta, b, \varphi) = \frac{1 - \exp(-D_G(x; \varphi))}{1 - \bar{\delta} \exp(-D_G(x; \varphi))} \quad (5.23)$$

and

$$f(x; \delta, b, \varphi) = \frac{2b\delta g(x; \varphi) \bar{G}(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{b-1} \exp(-D_G(x; \varphi))}{(1 - [1 - \bar{G}^2(x; \varphi)]^b)^2 \left\{ 1 - \bar{\delta} \exp(-D_G(x; \varphi)) \right\}^2}, \quad (5.24)$$

respectively, where $D_G(x; \varphi) = \left(\frac{[1 - \bar{G}^2(x; \varphi)]^b}{1 - [1 - \bar{G}^2(x; \varphi)]^b} \right)$, $\bar{\delta} = 1 - \delta$, $\delta > 0$, $b > 0$ and parameter vector φ for the baseline cdf $G(x; \varphi)$.

Sub-Families of the Distribution

We give some sub-families of the MO-OETL-G FoD in this subsection.

- When $\delta = 1$, we obtain the Odd Exponentiated Topp-Leone-G (OETL-G) FoD with cdf

$$F(x; b, \varphi) = 1 - \exp(-D_G(x; \varphi)),$$

for $b > 0$ and parameter vector φ .

- If $b = 1$, we obtain a new FoD with cdf

$$F(x; \delta, \varphi) = \frac{1 - \exp \left[- \left(\frac{[1 - \bar{G}^2(x; \varphi)]}{1 - [1 - \bar{G}^2(x; \varphi)]} \right) \right]}{1 - \bar{\delta} \exp \left[- \left(\frac{[1 - \bar{G}^2(x; \varphi)]}{1 - [1 - \bar{G}^2(x; \varphi)]} \right) \right]},$$

for $\delta > 0$, $\bar{\delta} = 1 - \delta$ and parameter vector φ .

- If $b = 1$ and $\delta = 1$, the resulting FoD has cdf given by

$$F(x; \varphi) = 1 - \exp \left[- \left(\frac{[1 - \bar{G}^2(x; \varphi)]}{1 - [1 - \bar{G}^2(x; \varphi)]} \right) \right],$$

for parameter vector φ .

Hazard Rate and Quantile Functions

In this subsection, we present the hrf and qf of the MO-OETL-G FoD. The hrf is given by

$$h(x; \delta, b, \varphi) = \frac{2bg(x; \varphi)\bar{G}(x; \varphi)(1 - \bar{G}^2(x; \varphi))^{b-1}}{(K_G(x; b, \varphi))^2(1 - \bar{\delta} \exp(-D_G(x; \varphi)))}. \quad (5.25)$$

The qf of the MO-OETL-G family of distributions is obtained by solving the non-linear equation, $F(Q(u)) = u$, $0 < u < 1$, that is,

$$F(Q(u)) = \frac{1 - \exp \left[- \left(\frac{[1 - \bar{G}^2(Q(u))]^b}{1 - [1 - \bar{G}^2(Q(u))]^b} \right) \right]}{1 - \bar{\delta} \exp \left[- \left(\frac{[1 - \bar{G}^2(Q(u))]^b}{1 - [1 - \bar{G}^2(Q(u))]^b} \right) \right]} = u. \quad (5.26)$$

Making $Q(u)$ the subject in the equation above, we obtain the qf as:

$$Q(u) = G^{-1} \left[1 - \left\{ 1 - \left(\frac{\ln(\frac{1-u}{\delta})}{\ln(\frac{1-u}{\delta}) - 1} \right)^{\frac{1}{b}} \right\}^{\frac{1}{2}} \right]. \quad (5.27)$$

Stochastic Ordering

Stochastic ordering plays a very important role in the assessment of probabilistic models in different areas such as reliability, survival analysis, finance, risks and economics. In this section, we present a stochastic ordering result from the MO-OETL-G FoD.

Let X and Y be two rvs with distribution functions $F_X(t)$ and $F_Y(t)$, respectively. Then X is

stochastically smaller than Y if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all t , where $\bar{F}_X(t) = 1 - F(t)$ is the survival function. Stochastic ordering of X and Y is denoted by $X <_{st} Y$. A stronger ordering is the likelihood ratio order denoted by $X <_{lr} Y$, and given by $\frac{f_X(t)}{f_Y(t)}$ decreasing in t . Hazard rate order is stronger than stochastic order but weaker than likelihood ratio order. Hazard rate order is given by $h_X(t) \geq h_Y(t)$ for all t , and denoted by $X <_{hr} Y$. Thus, $X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_{st} Y$ (see Shaked and Shanthikumar [116]).

Theorem 2.1. Suppose $X_1 \sim MOOETL - G(\delta_1, b, \varphi)$ and $X_2 \sim MOOETL - G(\delta_2, b, \varphi)$. If $\delta_1 < \delta_2$, then $\frac{f(x; \delta_1, b, \varphi)}{g(x; \delta_2, b, \varphi)}$ is decreasing in x .

Proof:

After some simplification, we have

$$\frac{f(x; \delta_1, b, \varphi)}{g(x; \delta_2, b, \varphi)} = \frac{\delta_1}{\delta_2} \left[\frac{1 - \bar{\delta}_2 \exp(-D_G(x; \varphi))}{1 - \bar{\delta}_1 \exp(-D_G(x; \varphi))} \right]^2. \quad (5.28)$$

Differentiating equation (5.28) with respect to x , we have

$$\frac{\partial}{\partial x} \left(\frac{f(x; \delta_1, b, \varphi)}{g(x; \delta_2, b, \varphi)} \right) = \frac{2\delta_1}{\delta_2} (\bar{\delta}_2 - \bar{\delta}_1) \frac{(1 - \bar{\delta}_2 \exp(-D_G(x; \varphi))) D'_G(x; \varphi) \exp(-D_G(x; \varphi))}{(1 - \bar{\delta}_1 \exp(-D_G(x; \varphi)))^3},$$

where $D'_G(x; \varphi) = \frac{dD_G(x; \varphi)}{dx}$.

If $\delta_1 < \delta_2$, then $\frac{\partial}{\partial x} \left(\frac{f(x; \delta_1, b, \varphi)}{g(x; \delta_2, b, \varphi)} \right) < 0$ which implies that the likelihood ratio order exist among X_1 and X_2 , that is $X_2 <_{lr} X_1$. Consequently, X_1 and X_2 are stochastically ordered, that is $X_1 <_{st} X_2$.

Density Function Expansion

In this subsection, a useful series expansion for the MOOETL-G pdf is presented. The pdf (5.24) can be expressed as

$$\begin{aligned} f(x) &= 2b\delta g(x; \varphi) \bar{G}(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{b-1} \exp(-D_G(x; \varphi)) \\ &\times \underbrace{(K_G(x; b, \varphi))^{-2} \left\{ 1 - \bar{\delta} \exp(-D_G(x; \varphi)) \right\}^{-2}}_A. \end{aligned}$$

Using the generalized binomial series expansion, we have

$$A = \sum_{k=0}^{\infty} \frac{\Gamma(k+2) \bar{\delta}^k}{k!} \exp \left[-k D_G(x; \varphi) \right] \quad (5.29)$$

and substituting for A, we obtain

$$f(x) = 2b\delta g(x; \varphi) \overline{G}(x; \varphi) \sum_{k=0}^{\infty} \frac{\Gamma(k+2) \bar{\delta}^k}{k!} (1 - \overline{G}^2(x; \varphi))^{b-1} \times \underbrace{(K_G(x; b, \varphi))^{-2} \exp \left[- (k+1) D_G(x; \varphi) \right]}_B.$$

Applying the exponential series expansion, we have

$$B = \sum_{l=0}^{\infty} \frac{(-1)^l (k+1)^l}{l!} [D_G(x; \varphi)]^l.$$

Substituting for B, we obtain

$$\begin{aligned} f(x) &= 2b\delta g(x; \varphi) \overline{G}(x; \varphi) \sum_{k,l=0}^{\infty} \frac{(-1)^l \Gamma(k+2) (k+1)^l}{k! l!} \bar{\delta}^k (1 - \overline{G}^2(x; \varphi))^{b(l+1)-1} \\ &\times (K_G(x; b, \varphi))^{-(l+2)} \\ &= 2b\delta g(x; \varphi) \overline{G}(x; \varphi) \sum_{k,l,j=0}^{\infty} \frac{(-1)^{l+j} \Gamma(k+2) (k+1)^l}{k! l!} \binom{-(l+2)}{j} \bar{\delta}^k \\ &\times (1 - \overline{G}^2(x; \varphi))^{b(l+j+1)-1} \\ &= 2b\delta g(x; \varphi) \sum_{k,l,j,i=0}^{\infty} \frac{(-1)^{l+j+i} \Gamma(k+2) (k+1)^l}{k! l!} \binom{-(l+2)}{j} \\ &\times \binom{b(l+j+1)-1}{i} \bar{\delta}^k \overline{G}^{2i+1}(x; \varphi) \\ &= 2b\delta g(x; \varphi) \sum_{k,l,j,i,q=0}^{\infty} \frac{(-1)^{l+j+i+q} \Gamma(k+2) (k+1)^l}{k! l!} \binom{-(l+2)}{j} \\ &\times \binom{b(l+j+1)-1}{i} \binom{2i+1}{q} \bar{\delta}^k G(x; \varphi)^q \end{aligned}$$

which can be expressed as

$$f(x) = \sum_{q=0}^{\infty} c_{q+1} h_{q+1}(x; \varphi), \quad (5.30)$$

where

$$\begin{aligned} c_{q+1} &= 2b\delta \sum_{k,l,j,i=0}^{\infty} \frac{(-1)^{l+j+i+q} \Gamma(k+2) (k+1)^l}{k! l! (q+1)} \binom{-(l+2)}{j} \\ &\times \binom{b(l+j+1)-1}{i} \binom{2i+1}{q} \bar{\delta}^k, \end{aligned} \quad (5.31)$$

and $h_{q+1}(x; \varphi) = (q+1)g(x; \varphi)G(x; \varphi)^q$ is the Exp-G density with power parameter $(q+1)$. The mathematical properties of the MO-OETL-G family can be obtained from those of the Exp-G FoD.

Some Special Cases

In this subsection, we present four special cases of the MO-OETL-G FoD, specifically when the baseline distribution function $G(x; \varphi)$ are Weibull, Burr XII, Burr III and Kumaraswamy distributions.

Marshall-Olkin-Odd Exponential Topp-Leone-Weibull (MO-OETL-W) Distribution

Suppose we have $G(x; \gamma) = 1 - \exp(-x^\gamma)$ and $g(x; \gamma) = \gamma x^{\gamma-1} \exp(-x^\gamma)$ as the cdf and pdf of the baseline distribution for $\gamma > 0$ and $x > 0$. Then, the MO-OETL-Weibull (MO-OETL-W) distribution has cdf and pdf given by

$$F(x; \delta, b, \gamma) = \frac{1 - \exp \left[- \left(\frac{[1 - \exp(-2x^\gamma)]^b}{1 - [1 - \exp(-2x^\gamma)]^b} \right) \right]}{1 - \bar{\delta} \exp \left[- \left(\frac{[1 - \exp(-2x^\gamma)]^b}{1 - [1 - \exp(-2x^\gamma)]^b} \right) \right]}$$

and

$$f(x; \delta, b, \gamma) = \frac{2b\delta\gamma x^{\gamma-1} \exp(-2x^\gamma) (1 - \exp(-2x^\gamma))^{b-1} \exp \left[- \left(\frac{[1 - \exp(-2x^\gamma)]^b}{1 - [1 - \exp(-2x^\gamma)]^b} \right) \right]}{(1 - [1 - \exp(-2x^\gamma)]^b)^2 \left\{ 1 - \bar{\delta} \exp \left[- \left(\frac{[1 - \exp(-2x^\gamma)]^b}{1 - [1 - \exp(-2x^\gamma)]^b} \right) \right] \right\}^2},$$

for $b, \delta, \gamma > 0$, and $\bar{\delta} = 1 - \delta$.

As shown in Figure 5.15, the density function of the MO-OETL-W distribution reveals a variety of shapes including uni-modal, reverse-J, almost symmetric, right-skewed and left-skewed.

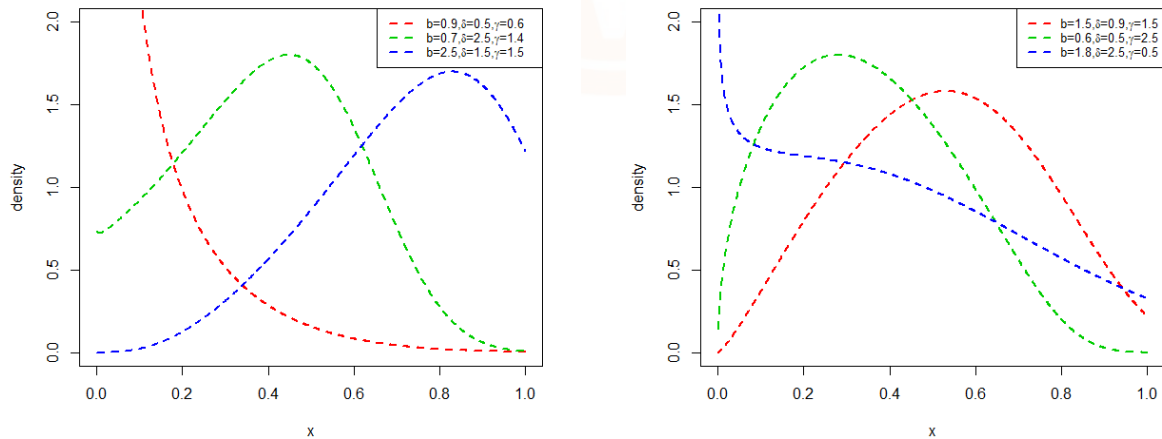


Figure 5.15: Plots of MO-OETL-W density function

The graphs of the hrf of the MO-OETL-W distribution in Figure 5.16 reveals decreasing, increasing and bathtub shapes.

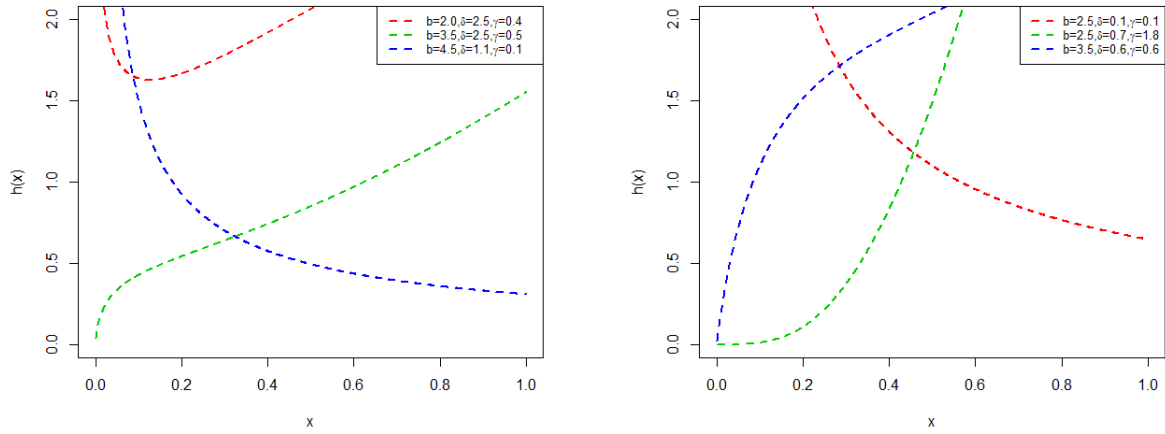


Figure 5.16: Plots of MOOETL-W hrf

Marshall-Olkin-Odd Exponential Topp-Leone-Burr XII (MO-OETL-BXII) Distribution

Taking the Burr XII distribution with cdf $G(x; \theta, \gamma) = 1 - (1 + x^\theta)^{-\gamma}$ and pdf $g(x; \theta, \gamma) = \theta\gamma x^{\theta-1}(1 + x^\theta)^{-\gamma-1}$ for $\theta, \gamma > 0$ and $x > 0$ as the baseline distribution, we obtain the MO-OETL-Burr XII (MO-OETL-BXII) distribution with the following cdf and pdf

$$F(x; \delta, b, \theta, \gamma) = \frac{1 - \exp \left[- \left(\frac{[1 - (1 + x^\theta)^{-2\gamma}]^b}{1 - [1 - (1 + x^\theta)^{-2\gamma}]^b} \right) \right]}{1 - \bar{\delta} \exp \left[- \left(\frac{[1 - (1 + x^\theta)^{-2\gamma}]^b}{1 - [1 - (1 + x^\theta)^{-2\gamma}]^b} \right) \right]}$$

and

$$f(x; \delta, b, \theta, \gamma) = \frac{2b\delta\theta\gamma x^{\theta-1}(1 + x^\theta)^{-2\gamma-1}(1 - (1 + x^\theta)^{-2\gamma})^{b-1} \exp \left[- \left(\frac{[1 - (1 + x^\theta)^{-2\gamma}]^b}{1 - [1 - (1 + x^\theta)^{-2\gamma}]^b} \right) \right]}{(1 - [1 - (1 + x^\theta)^{-2\gamma}]^b)^2 \left\{ 1 - \bar{\delta} \exp \left[- \left(\frac{[1 - (1 + x^\theta)^{-2\gamma}]^b}{1 - [1 - (1 + x^\theta)^{-2\gamma}]^b} \right) \right] \right\}^2},$$

for $b, \delta, \theta, \gamma > 0$, and $\bar{\delta} = 1 - \delta$.

Figure 5.17 displays the graphs of the MO-OETL-BXII distribution. The density can be unimodal, reverse-J, right-skewed, almost symmetric and left-skewed.

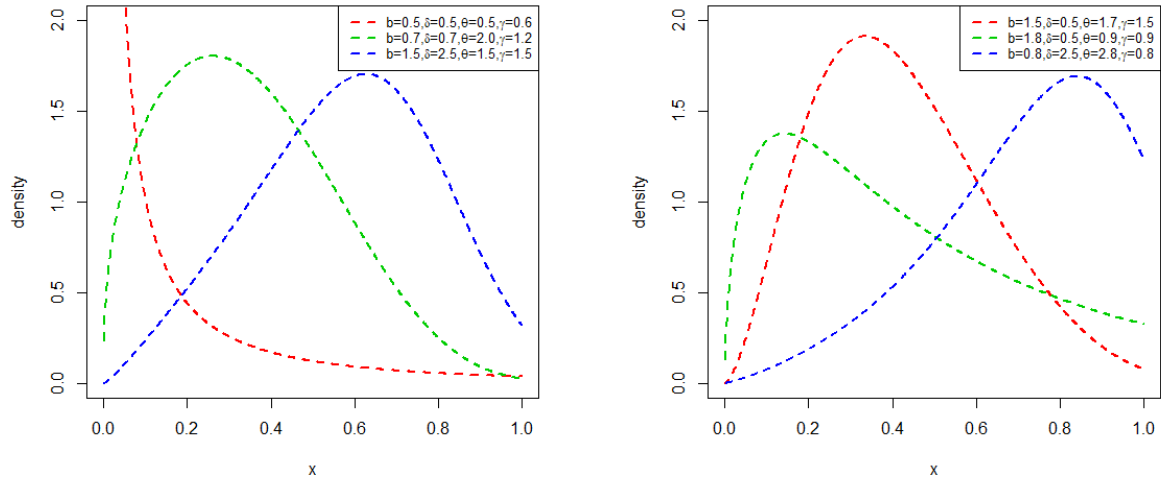


Figure 5.17: Plots of MO-OETL-BXII density function

Figure 5.18 has the graphs of the hrf of the MO-OETL-BXII distribution. The hrf exhibits inverted bathtub followed by bathtub, decreasing, bathtub and increasing shapes.

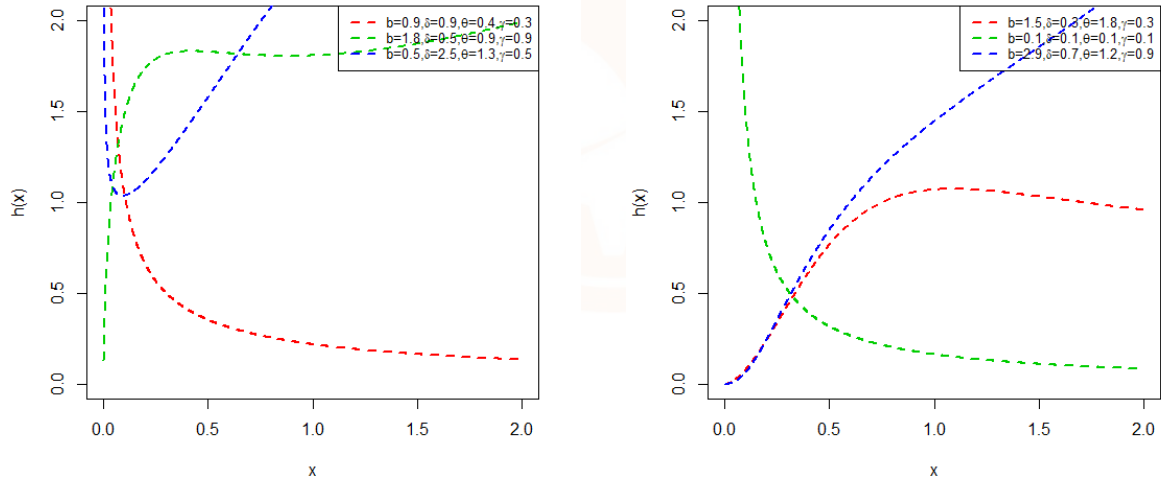


Figure 5.18: Graphs of the MO-OETL-BXII hrf

Marshall-Olkin-Odd Exponential Topp-Leone-Burr III (MOOETL-BIII) Distribution

Consider the Burr III with cdf and pdf $G(x; \theta, \gamma) = (1 + x^{-\theta})^{-\gamma}$ and $g(x; \theta, \gamma) = \theta\gamma x^{-\theta-1}(1 + x^{-\theta})^{-\gamma-1}$ as a baseline distribution for $\theta, \gamma > 0$ and $x > 0$, then the cdf and pdf of the MO-OETL-Burr III (MO-OETL-BIII) distribution are given by

$$F(x; \delta, b, \theta, \gamma) = \frac{1 - \exp \left[- \left(\frac{[1 - (1 - (1 + x^{-\theta})^{-\gamma})^2]^b}{1 - [1 - (1 - (1 + x^{-\theta})^{-\gamma})^2]^b} \right) \right]}{1 - \bar{\delta} \exp \left[- \left(\frac{[1 - (1 - (1 + x^{-\theta})^{-\gamma})^2]^b}{1 - [1 - (1 - (1 + x^{-\theta})^{-\gamma})^2]^b} \right) \right]}$$

and

$$f(x; \delta, b, \theta, \gamma) = 2b\delta\theta\gamma x^{-\theta-1}(1+x^{-\theta})^{-\gamma-1}(1-(1+x^{-\theta})^{-\gamma}) \\ \times \frac{(1-(1-(1+x^{-\theta})^{-\gamma})^2)^{b-1} \exp\left[-\left(\frac{[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b}{1-[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b}\right)\right]}{(1-[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b)^2 \left\{1 - \bar{\delta} \exp\left[-\left(\frac{[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b}{1-[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b}\right)\right]\right\}^2},$$

for $b, \delta, \theta, \gamma > 0$, and $\bar{\delta} = 1 - \delta$.

From Figure 5.19, plots of the MO-OETL-BIII density function reveals left-skewed and right-skewed, uni-modal, reverse-J and almost symmetric shapes.

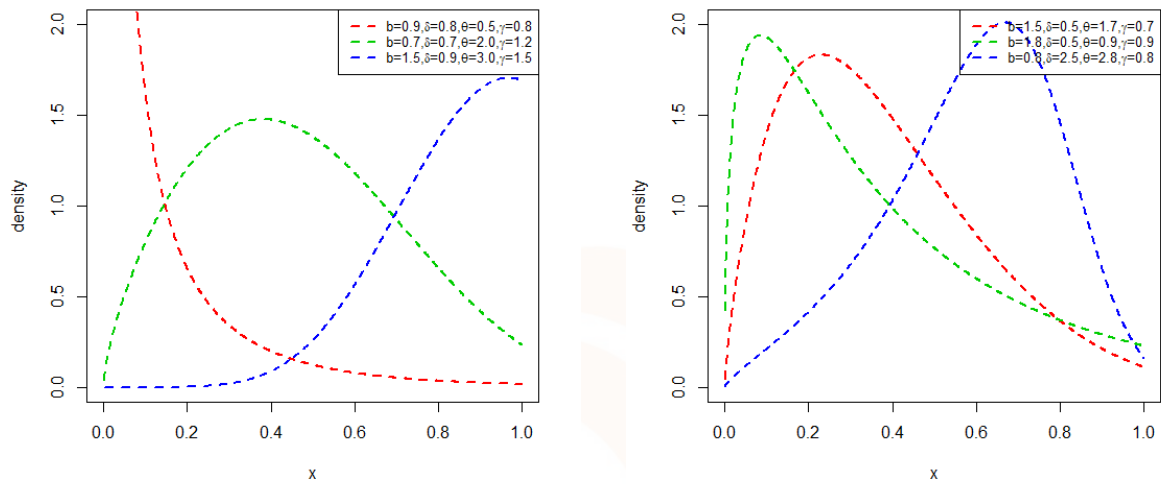


Figure 5.19: Plots of MO-OETL-BIII density function

From Figure 5.20, the graphs of the hrf of the MO-OETL-BIII distribution shows inverted bathtub, decreasing and increasing shapes.

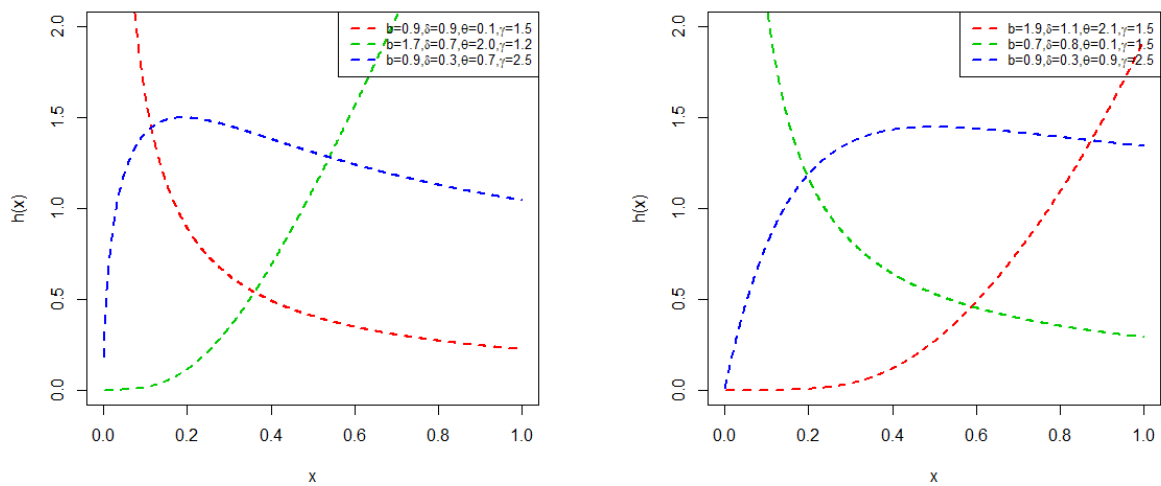


Figure 5.20: Graphs of the MO-OETL-BIII hrf

Marshall-Olkin-Odd Exponential Topp Leone-Kumaraswamy (MO-OETL-Kum) Distribution

Suppose we take $G(x; \theta, \gamma) = 1 - (1 - x^\theta)^\gamma$ and $g(x; \theta, \gamma) = \theta\gamma x^{\theta-1}(1 - x^\theta)^{\gamma-1}$ as a baseline model for $\theta, \gamma > 0$ and $x > 0$, then the cdf and pdf of MO-OETL-Kumaraswamy (MO-OETL-Kum) distribution are given by

$$F(x; \delta, b, \theta, \gamma) = \frac{1 - \exp \left[- \left(\frac{[1 - (1 - x^\theta)^{2\gamma}]^b}{1 - [1 - (1 - x^\theta)^{2\gamma}]^b} \right) \right]}{1 - \bar{\delta} \exp \left[- \left(\frac{[1 - (1 - x^\theta)^{2\gamma}]^b}{1 - [1 - (1 - x^\theta)^{2\gamma}]^b} \right) \right]}$$

and

$$f(x; \delta, b, \theta, \gamma) = \frac{2b\delta\theta\gamma x^{\theta-1}(1 - x^\theta)^{2\gamma-1}(1 - (1 - x^\theta)^{2\gamma})^{b-1} \exp \left[- \left(\frac{[1 - (1 - x^\theta)^{2\gamma}]^b}{1 - [1 - (1 - x^\theta)^{2\gamma}]^b} \right) \right]}{(1 - [1 - (1 - x^\theta)^{2\gamma}]^b)^2 \left\{ 1 - \bar{\delta} \exp \left[- \left(\frac{[1 - (1 - x^\theta)^{2\gamma}]^b}{1 - [1 - (1 - x^\theta)^{2\gamma}]^b} \right) \right] \right\}^2},$$

for $b, \delta, \theta, \gamma > 0$, and $\bar{\delta} = 1 - \delta$.

Figure 5.21 reveals that the pdf of the MO-OETL-Kum distribution is flexible and can be left-skewed, right-skewed reverse-J, uni-modal and almost symmetric.

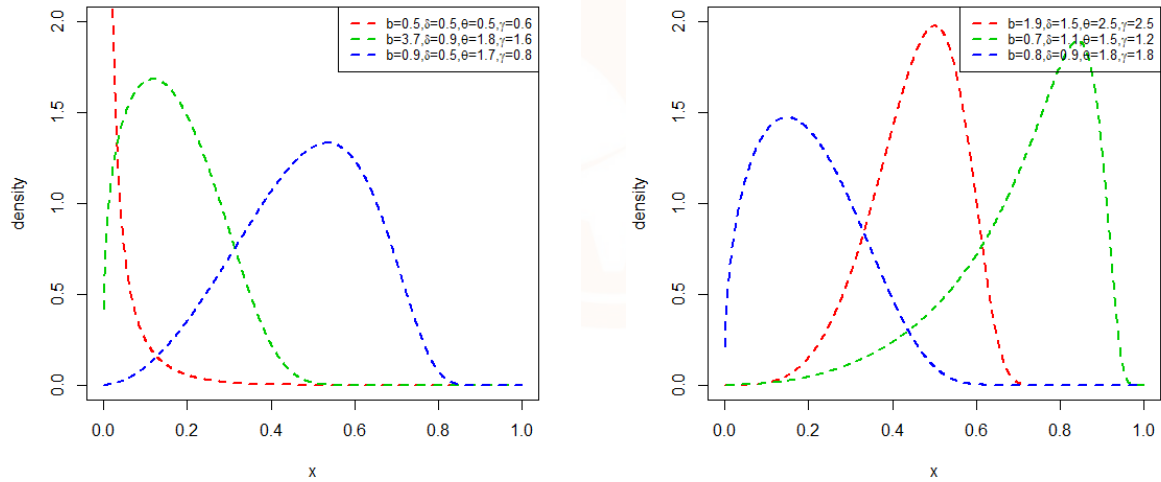


Figure 5.21: Plots of MO-OETL-Kum density function

In Figure 5.22 are the plots of the hrf of the MO-OETL-Kum distribution and the hrf takes increasing and bathtub shapes.

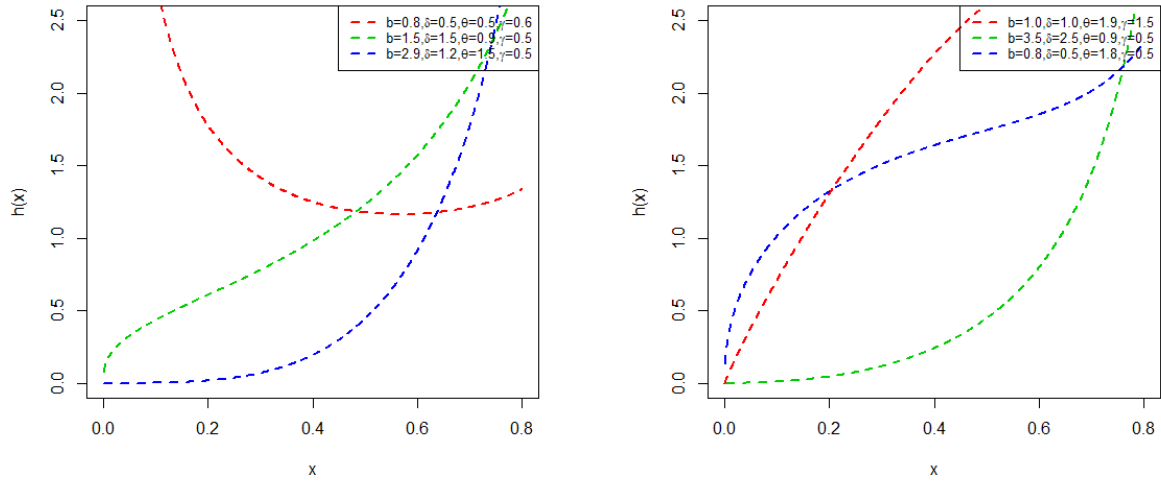


Figure 5.22: Plots of MO-OETL-Kum hrf

Random numbers generated from the MO-OETL-W distribution for selected parameter values are given Table 5.5.

u	(δ, b, γ)				
	(0.9, 0.9, 0.9)	(2.0, 1.5, 0.5)	(0.5, 0.9, 1.5)	(0.9, 0.8, 0.5)	(1.8, 0.5, 0.9)
0.1	0.0236	0.0517	0.0243	0.0009	0.0831
0.2	0.0558	0.1143	0.0593	0.0045	0.1385
0.3	0.0932	0.1840	0.0991	0.0115	0.1922
0.4	0.1354	0.2607	0.1427	0.0231	0.2483
0.5	0.1832	0.3457	0.1902	0.0405	0.3093
0.6	0.2380	0.4422	0.2424	0.0663	0.3786
0.7	0.3029	0.5563	0.3014	0.1046	0.4608
0.8	0.3848	0.7016	0.3722	0.1646	0.5654
0.9	0.5042	0.9199	0.4688	0.2743	0.7191

Table 5.5: Table of Quantiles for the MO-OETL-W Distribution

5.2.3 Moments and Incomplete Moments

Moments, mgf and incomplete moments for the MO-OETL-G FoD are provided in this subsection.

Moments and Generating Function

Let Y_{q+1} , denote the Exp-G rv with $(q+1)$ as its power parameter. Then the r^{th} raw moment of X , say μ'_r can be obtained from equation (5.30) as

$$\mu'_r = E(X^r) = \sum_{q=0}^{\infty} c_{q+1} E(Y_{q+1}^r). \quad (5.32)$$

The mgf $M_X(t) = E(e^{tX})$ of X can be obtained using equation (5.24) and is given by

$$M_X(t) = \sum_{q=0}^{\infty} c_{q+1} M_{q+1}(t),$$

where c_{q+1} is as given in equation (5.31) and $M_{q+1}(t)$ is the mgf of Y_{q+1} .

Table 5.6 shows moments for some selected parameter values of the special case of the MO-OETL-G family of distribution namely MO-OETL-Weibull distribution.

	(0.9, 0.9, 0.9)	(2.0, 1.5, 0.5)	(0.5, 0.9, 1.5)	(0.9, 0.8, 0.5)	(1.8, 0.5, 0.9)
$E(X)$	0.2282	0.3330	0.2221	0.0963	0.3413
$E(X^2)$	0.0879	0.1839	0.0783	0.0274	0.1702
$E(X^3)$	0.0431	0.1216	0.0339	0.0118	0.1031
$E(X^4)$	0.0245	0.0890	0.0167	0.0064	0.0704
$E(X^5)$	0.0154	0.0694	0.0090	0.0040	0.0519
$E(X^6)$	0.0105	0.0565	0.0052	0.0027	0.0404
SD	0.0076	0.0474	0.0031	0.0020	0.0327
CV	0.0057	0.0408	0.0020	0.0016	0.0273
CS	0.0045	0.0357	0.0013	0.0013	0.0233
CK	0.0036	0.0317	0.0009	0.0011	0.0203

Table 5.6: Table of Moments for the MO-OETL-W Distribution

Incomplete Moments

Incomplete moments are important in numerous scientific fields such as reliability and income distribution analysis. The r^{th} incomplete moment of the MO-OETL-G FoD is given by

$$m_r(t) = \int_{-\infty}^t x^r f(x; \delta, b, \varphi) dx = \sum_{q=0}^{\infty} c_{q+1} \int_{-\infty}^t x^r h_{q+1}(x; \varphi) dx. \quad (5.33)$$

Note that $m_1(t)$ can be used to obtain Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = m_1(q^*)/(\pi\mu'_1)$ and $L(\pi) = m_1(q^*)/\mu'_1$, respectively, where μ'_1 is given by equation (5.32) with $r = 1$ and $q^* = Q(\pi)$ is the qf of X at π .

The first incomplete moment $m_1(t)$ can be obtained from equation (5.33) as

$$m_1(t) = \sum_{q=0}^{\infty} c_{q+1} T_{q+1}(t),$$

where $T_{q+1}(t) = \int_{-\infty}^t x h_{q+1}(x; \varphi) dx$ is the first incomplete moment of the Exp-G distribution.

5.2.4 Order Statistics and Rényi Entropy

In this subsection, we provide the distribution of the i^{th} order statistics and RE for the MO-OETL-G FoD.

Order Statistics

Suppose $X_1, X_2, \dots, X_n$ are i.i.d. MO-OETL-G rvs. Then, the pdf of $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{j+i-1}, \quad (5.34)$$

where $B(.,.)$ is the beta function. Note that

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= 2b\delta g(x; \varphi) \bar{G}(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{b-1} \exp(-D_G(x; \varphi)) \\ &\times (K_G(x; b, \varphi))^{-2} \left[1 - \exp(-D_G(x; \varphi)) \right]^{j+i-1} \\ &\times \left\{ 1 - \bar{\delta} \exp(-D_G(x; \varphi)) \right\}^{-j-i-1}. \end{aligned}$$

Using the generalized binomial series expansion, we have

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= 2b\delta g(x; \varphi) \bar{G}(x; \varphi) \sum_{s=0}^{\infty} (-1)^s \binom{j+i-1}{s} (1 - \bar{G}^2(x; \varphi))^{b-1} \\ &\times (K_G(x; b, \varphi))^{-2} \exp(-(s+1)D_G(x; \varphi)) \\ &\times \left\{ 1 - \bar{\delta} \exp(-D_G(x; \varphi)) \right\}^{-j-i-1} \\ &= 2b\delta g(x; \varphi) \bar{G}(x; \varphi) \sum_{s,k=0}^{\infty} (-1)^{s+k} \binom{j+i-1}{s} \binom{-(j+i+1)}{k} \bar{\delta}^k \\ &\times (1 - \bar{G}^2(x; \varphi))^{b-1} (K_G(x; b, \varphi))^{-2} \\ &\times \exp(-(s+k+1)D_G(x; \varphi)). \end{aligned} \quad (5.35)$$

Applying the exponential series expansion to equation (5.35), we obtain

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= 2b\delta g(x; \varphi) \bar{G}(x; \varphi) \sum_{s,k,l=0}^{\infty} \frac{(-1)^{s+k+l} \bar{\delta}^k}{l!(s+k+1)^{-l}} \binom{-(j+i+1)}{k} \\ &\times \binom{j+i-1}{s} (1 - \bar{G}^2(x; \varphi))^{b(l+1)-1} (K_G(x; b, \varphi))^{-l-2}. \end{aligned} \quad (5.36)$$

Applying the generalized binomial series expansion to equation (5.36), we obtain

$$\begin{aligned}
 f(x)[F(x)]^{j+i-1} &= 2b\delta g(x; \varphi) \bar{G}(x; \varphi) \sum_{s,k,l,q=0}^{\infty} \frac{(-1)^{s+k+l+q} \bar{\delta}^k}{l!(s+k+1)^{-l}} \binom{-(j+i+1)}{k} \\
 &\times \binom{j+i-1}{s} \binom{-(l+2)}{q} (1 - \bar{G}^2(x; \varphi))^{b(l+q+1)-1} \\
 &= 2b\delta g(x; \varphi) \sum_{s,k,l,q,n=0}^{\infty} \frac{(-1)^{s+k+l+q+n} \bar{\delta}^k}{l!(s+k+1)^{-l}} \binom{-(j+i+1)}{k} \\
 &\times \binom{j+i-1}{s} \binom{-(l+2)}{q} \binom{b(l+q+1)-1}{n} \bar{G}(x; \varphi)^{2n+1} \\
 &= 2b\delta g(x; \varphi) \sum_{s,k,l,q,n,p=0}^{\infty} \frac{(-1)^{s+k+l+q+n+p} \bar{\delta}^k}{l!(s+k+1)^{-l}} \binom{-(j+i+1)}{k} \\
 &\times \binom{j+i-1}{s} \binom{-(l+2)}{q} \binom{b(l+q+1)-1}{n} \binom{2n+1}{p} G(x; \varphi)^p.
 \end{aligned} \tag{5.37}$$

Substituting equation (5.37) into equation (5.34), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{p=0}^{\infty} b_{p+1} h_{p+1}(x; \varphi), \tag{5.38}$$

where $h_{p+1}(x; \varphi)$ is the Exp-G density function with power parameter $(p+1)$ and

$$\begin{aligned}
 b_{p+1} &= 2b\delta \sum_{s,k,l,q,n=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{s+k+l+q+n+p+j} \bar{\delta}^k}{B(i, n-i+1) l!(s+k+1)^{-l} (p+1)} \binom{-(j+i+1)}{k} \binom{n-i}{j} \\
 &\times \binom{j+i-1}{s} \binom{-(l+2)}{q} \binom{b(l+q+1)-1}{n} \binom{2n+1}{p}.
 \end{aligned}$$

Rényi Entropy

RE (Rényi [113]) is given by

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^{\infty} [f(x; \delta, b, \varphi)]^v dx \right), v \neq 1, v > 0. \tag{5.39}$$

Let $[f(x; \delta, b, \varphi)]^v = [f(x)]^v$, then from equation (5.24), we have

$$\begin{aligned}
 [f(x)]^v &= (2b\delta g(x; \varphi) \bar{G}(x; \varphi))^v (1 - \bar{G}^2(x; \varphi))^{v(b-1)} \exp(-vD_G(x; \varphi)) \\
 &\times (K_G(x; b, \varphi))^{-2v} \left\{ 1 - \bar{\delta} \exp(-D_G(x; \varphi)) \right\}^{-2v}.
 \end{aligned}$$

Using the generalized binomial series expansion, we have

$$\begin{aligned}
 [f(x)]^v &= (2b\delta g(x; \varphi) \bar{G}(x; \varphi))^v \sum_{k=0}^{\infty} (-1)^k \binom{-2v}{k} \bar{\delta}^k (1 - \bar{G}^2(x; \xi))^{v(b-1)} \\
 &\times (K_G(x; b, \varphi))^{-2v} \exp(-(k+v)D_G(x; \varphi)).
 \end{aligned} \tag{5.40}$$

Applying the exponential series expansion to equation (5.40), we obtain

$$\begin{aligned}
 [f(x)]^v &= (2b\delta g(x; \varphi) \overline{G}(x; \varphi))^v \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l}}{(k+v)^{-l} l!} \binom{-2v}{k} \overline{\delta}^k (1 - \overline{G}^2(x; \varphi))^{b(v+l)-v} \\
 &\times (K_G(x; b, \varphi))^{-(2v+l)}.
 \end{aligned} \tag{5.41}$$

Using the binomial series expansion in equation (5.41), we have

$$\begin{aligned}
 [f(x)]^v &= (2b\delta g(x; \varphi) \overline{G}(x; \varphi))^v \sum_{k,l,j=0}^{\infty} \frac{(-1)^{k+l+j}}{(k+v)^{-l} l!} \binom{-2v}{k} \binom{-(2v+l)}{j} \overline{\delta}^k \\
 &\times (1 - \overline{G}^2(x; \varphi))^{b(v+l+j)-v} \\
 &= (2b\delta g(x; \varphi))^v \sum_{k,l,j,i=0}^{\infty} \frac{(-1)^{k+l+j+i}}{(k+v)^{-l} l!} \binom{-2v}{k} \binom{-(2v+l)}{j} \\
 &\times \binom{b(v+l+j)-v}{i} \overline{\delta}^k \overline{G}(x; \varphi)^{2i+v} \\
 &= \sum_{k,l,j,i,q=0}^{\infty} \frac{(2b\delta)^v (-1)^{k+l+j+i+q}}{(k+v)^{-l} l!} \binom{-2v}{k} \binom{-(2v+l)}{j} \\
 &\times \binom{b(v+l+j)-v}{i} \binom{2i+v}{q} \overline{\delta}^k g(x; \varphi)^v G(x, \varphi)^q.
 \end{aligned}$$

Thus, the RE of the MO-OETL-G family of distribution can be expressed as

$$I_R(v) = \frac{1}{1-v} \log \left[\sum_{q=0}^{\infty} \varphi_q \left(\int_0^{\infty} [g(x; \varphi)]^v G(x, \varphi)^q dx \right) \right], \tag{5.42}$$

where

$$\begin{aligned}
 \varphi_q &= \sum_{k,l,j,i=0}^{\infty} \frac{(2b\delta)^v (-1)^{k+l+j+i+q}}{(k+v)^{-l} l!} \binom{-2v}{k} \binom{-(2v+l)}{j} \\
 &\times \binom{b(v+l+j)-v}{i} \binom{2i+v}{q} \overline{\delta}^k, v \neq 1, v > 0.
 \end{aligned}$$

RE of the MO-OETL-G FoD can be readily obtained from that of the Exp-G distribution as follows:

$$I_R(v) = \frac{1}{1-v} \log \left[\sum_{q=0}^{\infty} \phi_q e^{(1-v)I_{REG}} \right], \tag{5.43}$$

where

$$\begin{aligned}
 \phi_q &= \sum_{k,l,j,i=0}^{\infty} \frac{(2b\delta)^v (-1)^{k+l+j+i+q}}{(k+v)^{-l} l!} \binom{-2v}{k} \binom{-(2v+l)}{j} \\
 &\times \binom{b(v+l+j)-v}{i} \binom{2i+v}{q} \left(\frac{q}{v} + 1 \right) \overline{\delta}^k,
 \end{aligned}$$

and $I_{REG} = \int_0^\infty [(\frac{q}{v} + 1)g(x; \varphi)G(x; \varphi)^{\frac{q}{v}}]^v dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{q}{v}$.

5.2.5 Estimation and Inferences

In this subsection, we provide maximum likelihood method for estimating the parameters of the MO-OETL-G FoD. Let $\Delta = (\delta, b, \varphi)$ be the vector of parameters. The log-likelihood function $\ell_n = \ell_n(\Delta)$ is given by

$$\begin{aligned} \ell_n(\Delta) &= n \ln(2b) + n \ln(\delta) + \sum_{i=1}^n \ln(g(x_i; \varphi)) + \sum_{i=1}^n \ln(\bar{G}(x_i; \varphi)) \\ &+ (b-1) \sum_{i=1}^n \ln(1 - \bar{G}^2(x_i; \varphi)) - \sum_{i=1}^n D_G(x; \varphi) \\ &- 2 \sum_{i=1}^n \ln(1 - \bar{\delta} \exp(-D_G(x; \varphi))) - 2 \sum_{i=1}^n \ln(1 - (1 - \bar{G}^2(x_i; \varphi))^b). \end{aligned} \quad (5.44)$$

The first derivatives of $\ell_n = \ell_n(\Delta)$ with respect to each of component of the parameter vector $\Delta = (b, \delta, \varphi)^T$ are given by

$$\begin{aligned} \frac{\partial \ell_n(\Delta)}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \ln(1 - \bar{G}^2(x_i; \varphi)) - \sum_{i=1}^n \frac{(1 - \bar{G}^2(x_i; \varphi))^b \ln(1 - \bar{G}^2(x_i; \varphi))}{(1 - (1 - \bar{G}^2(x_i; \varphi))^b)^2} \\ &- 2 \sum_{i=1}^n \frac{\bar{\delta}(1 - \bar{G}^2(x_i; \varphi))^b \ln(1 - \bar{G}^2(x_i; \varphi)) \exp(-D_G(x; \varphi))}{(1 - (1 - \bar{G}^2(x_i; \varphi))^b)^2 (1 - \bar{\delta} \exp(-D_G(x; \varphi)))} \\ &+ 2 \sum_{i=1}^n \frac{(1 - \bar{G}^2(x_i; \varphi))^b \ln(1 - \bar{G}^2(x_i; \varphi))}{1 - (1 - \bar{G}^2(x_i; \varphi))^b}, \end{aligned} \quad (5.45)$$

$$\frac{\partial \ell_n(\Delta)}{\partial \delta} = \frac{n}{\delta} - 2 \sum_{i=1}^n \frac{\exp(-D_G(x; \varphi))}{(1 - \bar{\delta} \exp(-D_G(x; \varphi)))} \quad (5.46)$$

and

$$\begin{aligned} \frac{\partial \ell_n(\Delta)}{\partial \xi_k^*} &= \sum_{i=1}^n \frac{g'(x_i; \varphi)}{g(x_i; \varphi)} + \sum_{i=1}^n \frac{\bar{G}'(x_i; \varphi)}{\bar{G}(x_i; \varphi)} + 2(b-1) \sum_{i=1}^n \frac{\bar{G}(x_i; \varphi) \bar{G}'(x_i; \varphi)}{1 - \bar{G}^2(x_i; \varphi)} \\ &+ 4b \sum_{i=1}^n \frac{\bar{\delta} \bar{G}(x_i; \varphi) (1 - \bar{G}^2(x_i; \varphi))^{b-1} \exp(-D_G(x; \varphi)) \bar{G}'(x_i; \varphi)}{(1 - (1 - \bar{G}^2(x_i; \varphi))^b)^2 (1 - \bar{\delta} \exp(-D_G(x; \varphi)))} \\ &- 2b \sum_{i=1}^n \frac{\bar{G}(x_i; \varphi) (1 - \bar{G}^2(x_i; \varphi))^{b-1} \bar{G}'(x_i; \varphi)}{(1 - (1 - \bar{G}^2(x_i; \varphi))^b)^2} \\ &+ 4b \sum_{i=1}^n \frac{\bar{G}(x_i; \varphi) (1 - \bar{G}^2(x_i; \varphi))^{b-1} \bar{G}'(x_i; \varphi)}{(1 - (1 - \bar{G}^2(x_i; \varphi))^b)}, \end{aligned} \quad (5.47)$$

where $g'(x_i; \varphi) = \frac{\partial g(x_i; \varphi)}{\partial \xi_k^*}$ and $\bar{G}'(x_i; \varphi) = \frac{\partial \bar{G}(x_i; \varphi)}{\partial \xi_k^*}$.

The likelihood function can be maximised via the use of the nlm function in R and NLMixed in

SAS for specified baseline cdf $G(x; \varphi)$.

In order to build CIs for the model parameters (δ, b, φ) and test hypothesis concerning the parameters, the observed information matrix is required and is given by

$$J(\Delta) = \begin{bmatrix} J_{bb}(\Delta) & J_{b\delta}(\Delta) & J_{b\varphi}(\Delta) \\ J_{\delta b}(\Delta) & J_{\delta\delta}(\Delta) & J_{\delta\varphi}(\Delta) \\ J_{\varphi b}(\Delta) & J_{\varphi\delta}(\Delta) & J_{\varphi\varphi}(\Delta) \end{bmatrix}, \quad (5.48)$$

where $J_{i,j}(\Delta) = -\frac{\partial^2 \ell_n(\Delta)}{\partial i \partial j}$, $i, j = \delta, b, \varphi$, and φ is a p component vector of baseline parameters.

Under the usual regularity conditions, $\hat{\Delta}$ is asymptotically normally distributed, (Ferguson [60]) that is $\hat{\Delta} \sim N_{2+p}(\mathbf{0}, I^{-1}(\hat{\Delta}))$ as $n \rightarrow \infty$, where $I(\Delta)$ is the expected information matrix. The asymptotic behaviour remains valid if $I(\Delta)$ is replaced by $J(\hat{\Delta})$, the observed information matrix evaluated at $\hat{\Delta}$.

5.2.6 Simulation Study

The performance of the MO-OETL-W distribution is examined by conducting various simulations for different sizes ($n=25, 50, 100, 200, 400, 800, 1000$) via the R package. We simulate $N = 1000$ samples for the true parameters values given in Table 5.7 and Table 5.8. The table lists the mean MLEs of the model parameters along with the respective ABias and RMSEs. The ABias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

parameter	Sample Size	(0.05, 1.0, 1.0)			(0.05, 1.0, 1.2)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
b	25	0.0567	0.0582	0.0067	0.0899	0.7220	0.0399
	50	0.0490	0.0322	-0.0010	0.0515	0.0443	0.0015
	100	0.0460	0.0226	-0.0040	0.0474	0.0236	-0.0026
	200	0.0459	0.0184	-0.0041	0.0477	0.0182	-0.0023
	400	0.0461	0.0129	-0.0039	0.0468	0.0132	-0.0032
	800	0.0467	0.0099	-0.0033	0.0477	0.0102	-0.0023
	1000	0.0463	0.0096	-0.0037	0.0479	0.0096	-0.0021
δ	25	0.9850	0.8283	-0.0150	0.9848	0.8287	-0.0152
	50	1.0075	0.7057	0.0075	1.0220	0.6907	0.0220
	100	0.9483	0.5484	-0.0517	0.9277	0.5325	-0.0723
	200	0.9279	0.4651	-0.0721	0.9335	0.4532	-0.0665
	400	0.9462	0.3781	-0.0538	0.9288	0.3680	-0.0712
	800	0.9490	0.2957	-0.0510	0.9297	0.2926	-0.0703
	1000	0.9641	0.2715	-0.0359	0.9485	0.2686	-0.0515
γ	25	1.5654	1.0112	0.5654	1.8522	1.2112	0.6522
	50	1.5356	1.0630	0.5356	1.7636	1.1758	0.5636
	100	1.5078	0.9971	0.5078	1.8180	1.1995	0.6180
	200	1.4661	0.9567	0.4661	1.6987	1.0601	0.4987
	400	1.3436	0.8276	0.3436	1.5994	0.9327	0.3994
	800	1.2278	0.6324	0.2278	1.4795	0.7240	0.2795
	1000	1.1947	0.5887	0.1947	1.4281	0.6606	0.2281

Table 5.7: Simulation Results

parameter	Sample Size	(0.5, 0.5, 1.5)			(0.5, 1.5, 0.5)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
b	25	1.9510	8.0754	1.4510	1.6368	8.8877	1.1368
	50	1.6960	6.9020	1.1960	0.8132	5.1297	0.3132
	100	0.8119	2.5904	0.3119	0.4911	0.2606	-0.0089
	200	0.5616	0.6003	0.0616	0.4850	0.1608	-0.0150
	400	0.5210	1.0386	0.0210	0.4697	0.1070	-0.0303
	800	0.4621	0.1248	-0.0379	0.4594	0.0886	-0.0406
	1000	0.4597	0.1201	-0.0403	0.4557	0.0724	-0.0443
δ	25	0.4668	0.3387	-0.0332	1.3188	0.7272	-0.1812
	50	0.4964	0.2629	-0.0036	1.4684	0.4926	-0.0316
	100	0.4857	0.2125	-0.0143	1.5045	0.3348	0.0045
	200	0.4766	0.1795	-0.0234	1.5152	0.2434	0.0152
	400	0.4718	0.1437	-0.0282	1.5317	0.1651	0.0317
	800	0.4624	0.1261	-0.0376	1.5350	0.1187	0.0350
	1000	0.4659	0.1121	-0.0341	1.5313	0.0960	0.0313
γ	25	2.2605	1.5006	0.7605	0.9298	1.0642	0.4298
	50	2.0096	1.2348	0.5096	0.6553	0.5570	0.1553
	100	1.9433	1.0523	0.4433	0.5567	0.2470	0.0567
	200	1.8686	0.8999	0.3686	0.5240	0.0798	0.0240
	400	1.8166	0.7407	0.3166	0.5134	0.0469	0.0134
	800	1.8032	0.6476	0.3032	0.5105	0.0300	0.0105
	1000	1.7674	0.5724	0.2674	0.5117	0.0308	0.0117

Table 5.8: Simulation Results

The results in Tables 5.7 and 5.8 reveals that the mean estimates of the parameters are approaching the true parameter values with an increase in the sample size. Also, the RMSEs and the ABias are decaying towards zero with increasing sample size, which suggest strong consistency of the MLEs of the model.

5.2.7 Applications

In this subsection, we provide examples to demonstrate the applicability and usefulness of the MO-OETL-W distribution and its sub-models for data modeling. The MO-OETL-W distribution is fitted to data sets and these fits are compared to the fits of the Kumaraswamy Weibull (Kum-W) distribution (Codeiro et al [48]), Odd Log-Logistic Exponentiated Weibull (OLLE-W) distribution (Afify et al. [2]), Exponentiated Weibull Lindley (EWL) distribution (Cordeiro et al. [49]), the Beta Generalized Lindley (BGL) distribution (Oluyede and Yang [99]) and Weibull Lomax (WL) distribution (Tahir et al. [123]). The pdfs of the used models are given in the appendix.

Figure 5.23 and Figure 5.28 shows plots of fitted densities, the histograms and probability plots (Chambers et al. [34]). We plotted $F(x_{(j)}; \hat{b}, \hat{\delta}, \hat{\xi})$ versus $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, to get the probability plots, where $x_{(j)}$ are the ordered values of the observed data. The sum of squares (SS) measures the closeness of the model to the empirical:

$$SS = \sum_{j=1}^n \left[F(x_{(j)}; \hat{b}, \hat{\delta}, \hat{\xi}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The MLEs of the parameters with SEs in parenthesis and the values of the GoF statistics, $-2\ln(L)$, AIC, CAIC and BIC) are given in Table 5.9 and Table 5.10.

Kevlar 49/Epoxy Strands Failure at 90% Stress Level

The data represent the stress-rupture life of kevlar 49/epoxy strands, (Andrews and Herzberg [19]; Barlow et al. [21]).

Distribution	Estimates				-2logL	AIC	CAIC	BIC	W*	A*	K-S	P-value	SS
	b	δ	θ	γ									
MO-OETL-W	306.67 (0.004)	5.378 (3.456)	3.370 (0.102)	0.077 (0.011)	202.144	210.144	210.561	220.605	0.143	0.820	0.073	0.648	0.130
OETL-W	4.547 (3.892)	1 (-)	0.978 (0.412)	0.299 (0.109)	206.248	212.248	212.495	220.093	0.198	1.114	0.097	0.294	0.204
MO-OETL-W($1, \delta, \theta, \gamma$)	1 (-)	0.401 (0.558)	0.199 (0.168)	0.823 (0.319)	211.309	217.309	217.557	225.155	0.202	1.180	0.102	0.240	0.214
MO-OETL-W($1, 1, \theta, \gamma$)	1 (-)	1 (-)	0.326 (0.025)	0.626 (0.047)	211.293	215.293	215.415	220.523	0.130	0.881	0.099	0.266	0.173
BGL	α 0.123 (0.005)	θ 1.690×10^{-6} (2.831×10^{-6})	a 0.3036 (4.697×10^{-4})	b 10.100 (1.7373×10^{-5})	596.942	604.952	605.369	615.413	0.414	2.201	0.445	2.200×10^{-16}	7.092
OLLE-W	α 1.625 (2.606)	β 1.112 (0.441)	θ 1.169 (0.943)	γ 0.608 (0.897)	205.534	213.534	213.950	223.994	0.162	0.943	0.0798	0.541	0.157
Kum-W	a 0.755 (0.298)	b 0.334 (0.751)	c 1.034 (0.165)	λ 2.574 (6.041)	205.294	213.294	213.710	223.754	0.149	0.885	0.079	0.551	0.146
WL	a 0.251 (0.417)	b 0.786 (0.180)	α 1.358 (0.458)	β 0.330 (0.628)	205.198	213.198	213.614	223.658	0.144	0.863	0.079	0.558	0.143
EWL	α 0.039 (0.021)	v 1.654 (0.798)	a 41.315 (0.001)	b 0.552 (0.144)	204.748	212.749	213.165	223.209	0.159	0.930	0.083	0.494	0.156

Table 5.9: Estimation of the MO-OETL-W model for the Kevlar 49/Epoxy Strands Failure at 90% Stress Level data

The likelihood ratio (LR) test statistic results for the $MOOETL - W$ model for the kevlar 49/epoxy strands at the 90% stress level data set are as follows; $H_0 : OETL - W$ against $H_a : MO - OETL - W$ (4.104, p-value = 0.042782), $H_0 : MO - OETL - W(1, \delta, \theta, \gamma)$ against $H_a : MO - OETL - W$ (9.165, p-value = 0.002467) and $H_0 : MO - OETL - W(1, 1, \theta, \gamma)$ against $H_a : MO - OETL - W$ (9.149, p-value = 0.002489). Thus, the $MO - OETL - W$ model performed significantly better than its nested models. Also, from the goodness-of-fit statistics presented in Table 5.9, we can conclude that the $MOOETL - W$ model is better than the other non-nested models. The histogram and fitted densities as well as the probability plots are given in Figure 5.23.

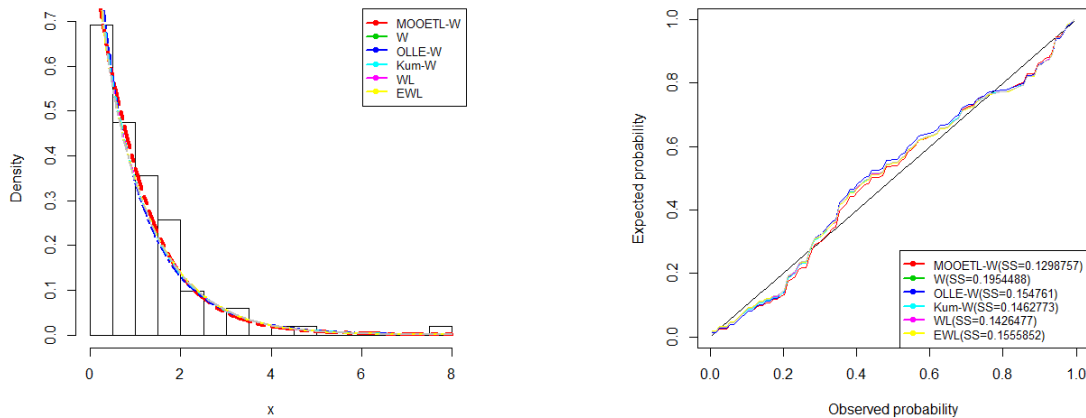


Figure 5.23: Fitted densities and probability plots for the Kevlar 49/Epoxy Strands Failure at 90% Stress Level data

The estimated var-cov matrix for the Kevlar 49/Epoxy strands failure at 90% stress level data

is

$$\begin{bmatrix} 0.00002 & -0.01493 & -0.00042 & 0.00004 \\ -0.01493 & 11.94448 & 0.33800 & -0.03128 \\ -0.00042 & 0.33800 & 0.01033 & -0.00094 \\ 0.00004 & -0.03128 & -0.00094 & 0.00012 \end{bmatrix}$$

and the 95% CIs for the parameters b, δ, θ and γ are: (306.67 ± 0.00784) , (5.378 ± 6.77376) , (3.370 ± 0.19992) and (0.077 ± 0.02156) , respectively.

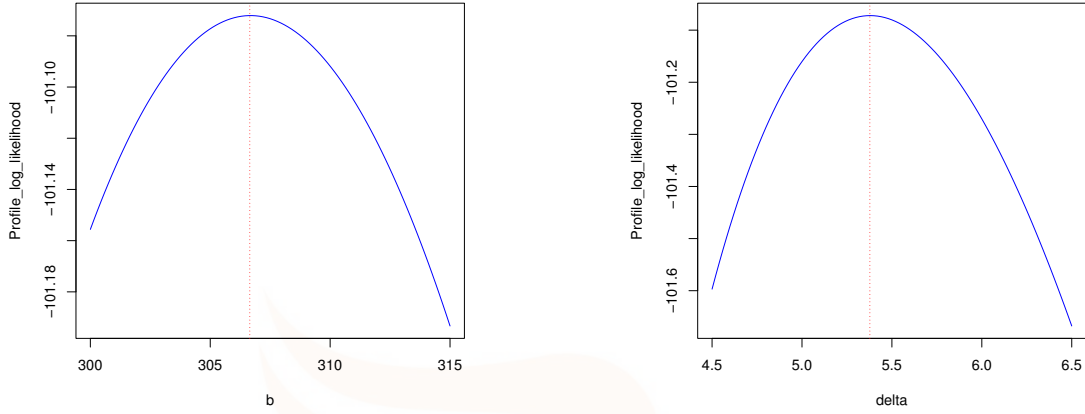


Figure 5.24: Profile plots for the stress-rupture life of kevlar 49/epoxy strands data

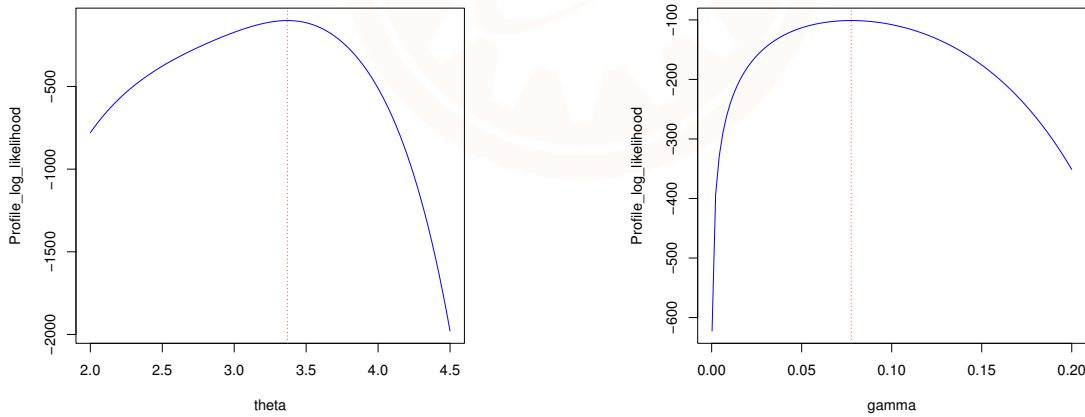


Figure 5.25: Profile plots for the stress-rupture life of kevlar 49/epoxy strands data

The profile plots in Figure 5.24 and Figure 5.25 shows that the estimates of the MO-OETL-W distribution are global solutions for the stress-rupture life of kevlar 49/epoxy strands data.

Relief Times of Patients

The data set was taken from Gross and Clark [65] and consists relief times of patients receiving an analgesic. Estimated parameters, SEs (in parenthesis) of estimates and the GoF statistics are presented in Table 5.10.

Model	Estimates				-2logL	AIC	CAIC	BIC	W*	A*	K-S	P-value	SS
	b	δ	θ	γ									
MO-OETL-W	273.820 (2.236×10^{-4})	0.045 (0.061)	1.798 (0.124)	0.401 (0.085)	30.704	38.704	41.371	42.687	0.025	0.149	0.090	0.997	0.018
OETL-W	2.132×10^6 (2.966×10^{-8})	1 (-)	6.704 (0.127)	0.144 (0.020)	36.173	42.173	43.673	45.160	0.116	0.688	0.199	0.405	0.151
MO-OETL-W($1, \delta, \theta, \gamma$)	1 (-)	0.002 (0.009)	4.072×10^{-5} (1.731×10^{-4})	5.844 (3.116×10^{-5})	32.908	38.908	40.408	41.895	0.056	0.333	0.112	0.964	0.030
MO-OETL-W($1, 1, \theta, \gamma$)	1 (-)	1 (-)	0.086 (0.032)	1.722 (0.286)	47.092	51.092	51.797	53.084	0.262	1.534	0.217	0.303	6.622
BGL	α 0.120 (0.013)	θ 1.520×10^{-7} (4.342×10^{-6})	a 0.302 (0.001)	b 10.100 (3.494×10^{-5})	162.523	170.538	173.205	174.521	0.089	0.523	0.512	5.526×10^{-5}	1.571
OLLE-W	α 0.027 (0.021)	β 0.034 (0.007)	θ 90.217 (5.119×10^{-5})	γ 1.821 (0.037)	32.987	40.987	43.654	44.970	0.052	0.313	0.120	0.969	0.029
Kum-W	a 23108.000 (9.797×10^{-6})	b 1.465 (0.536)	c 0.309 (0.004)	λ 989.690 (2.177×10^{-4})	31.319	39.319	41.986	43.302	0.036	0.207	0.121	0.931	0.032
WL	a 1753.0 (0.018)	b 5.843 (12.087)	a 0.124 (0.129)	β 0.334 (2.583)	38.901	46.901	49.568	50.884	0.151	0.892	0.178	0.554	0.138
EWL	α 0.003 (0.003)	v 982.470 (2.881×10^{-6})	a 348.730 (1.466×10^{-6})	b 0.374 (0.060)	31.657	39.657	42.323	43.640	0.042	0.244	0.123	0.924	0.034

Table 5.10: MLEs of the MO-OETL-W model for relief times of 20 patients data

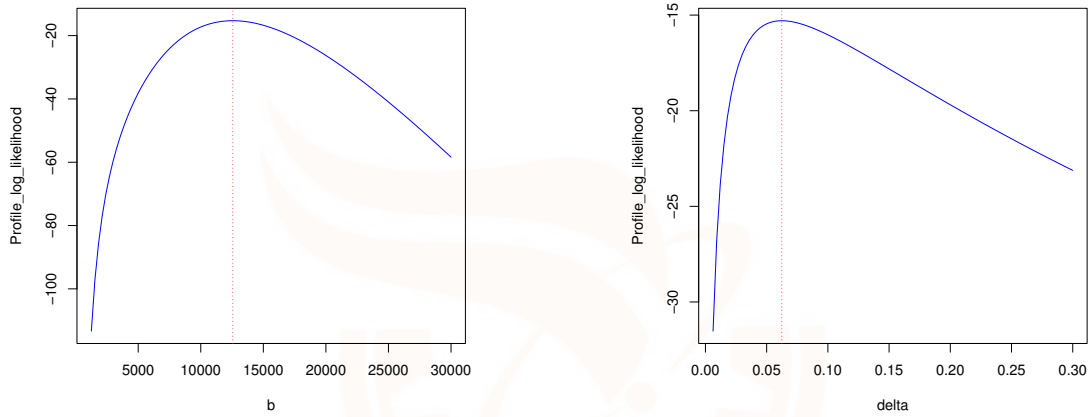


Figure 5.26: Profile plots for the relief times data

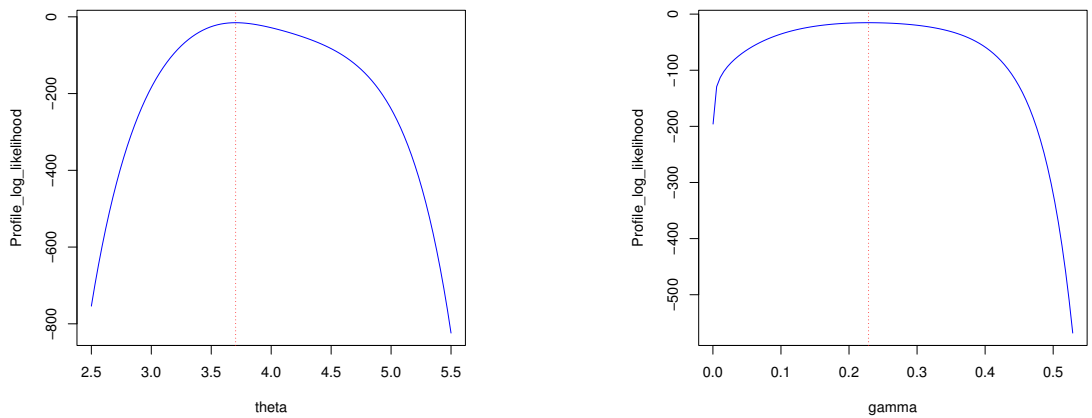


Figure 5.27: Profile plots for the relief times data

The profile plots in Figures 5.26 and 5.27 shows that the estimates of the MO-OETL-W distribution are attaining maximum solutions for the relief times data.

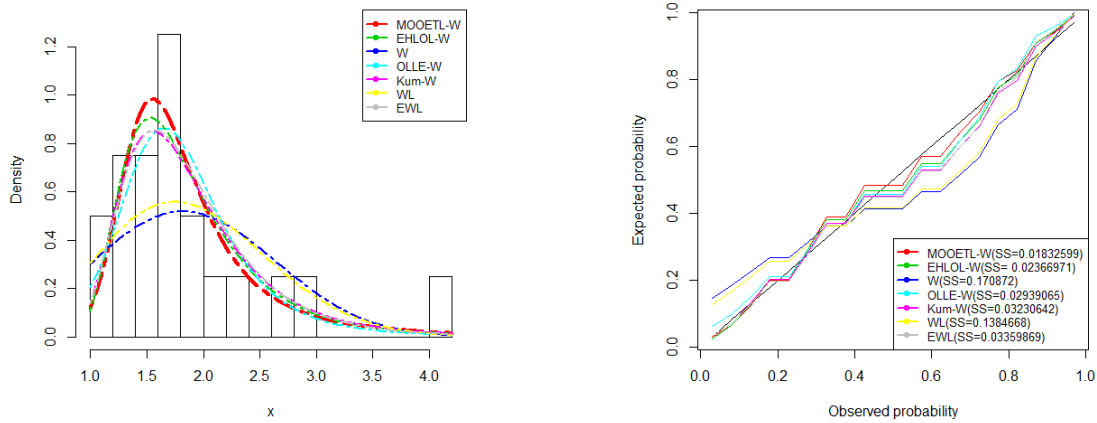


Figure 5.28: Fitted densities and probability plots for relief times of 20 patients data

The LR test statistic results for the $MO - OETL - W$ model for the relief times of 20 patients data set are as follows; $H_0 : OETL - W$ against $H_a : MO - OETL - W$ (5.469, p-value = 0.019357), $H_0 : MO - OETL - W(1, \delta, \theta, \gamma)$ against $H_a : MO - OETL - W$ (2.204, p-value = 0.137653) and $H_0 : MO - OETL - W(1, 1, \theta, \gamma)$ against $H_a : MO - OETL - W$ (16.388, p-value = 0.000052). The LR test show that there is a significant difference between the $MO - OETL - W$ and the nested models: $OETL - W$ and $MO - OETL - W(1, 1, \theta, \gamma)$. There is no significant difference between and $MO - OETL - W(1, \delta, \theta, \gamma)$ and $MO - OETL - W$ distributions based on the LR test. However, the values of the GoF statistics W^* , A^* and K-S clearly shows that $MO - OETL - W$ distribution is the better fit for the data. Presented in Figure 5.28 is the histogram, fitted densities and the probability plots.

The estimated var-cov matrix for the relief times of 20 patients data is given by,

$$\begin{bmatrix} 4.9987 \times 10^{-8} & -0.00001 & -2.77102 \times 10^{-5} & 6.95519 \times 10^{-9} \\ -0.00001 & 0.00376 & 0.00632 & 0.00302 \\ -2.77102 \times 10^{-5} & 0.00632 & 0.01555 & 0.00118 \\ 6.95519 \times 10^{-9} & 0.00302 & 0.00118 & 0.00718 \end{bmatrix}$$

and the 95% CIs for the parameters b, δ, θ and γ are: (273.820 ± 0.0004) , (0.045 ± 0.119) , (1.798 ± 0.243) and (0.401 ± 0.166) , respectively.

5.2.8 Conclusions

A new generalized FoD called the Marshall-Olkin-Odd Exponentiated Topp-Leone-G (MO-OETL-G) distribution was developed and presented. The MO-OETL-G FoD possesses hrf with various desirable shapes. We obtained closed form expressions for useful statistical properties such as the distribution of order statistics and RE, moments, incomplete moments. The maximum likelihood estimation technique was used to estimate the model parameters. The performance of the special case of the MO-OETL-W distribution was evaluated by carrying out

simulations for various sample sizes. The MO-OETL-W distribution was fitted to real data sets to demonstrate the applicability of the proposed FoD.



6. The Weibull-Topp-Leone-G Power Series Class of Distributions and Burr III-Exponentiated Half Logistic-G Family of Distributions with Applications

6.1 The Weibull-Topp-Leone-G Power Series Class of Distributions with Applications

Sengweni, W., and Oluyede, B. The Weibull-Topp-Leone -G Power Series Class of Distributions, (*Submitted: Pakistan Journal of Statistics and Operations Research*)

6.1.1 Introduction

In this chapter, we develop and study a new family of distributions (FoD) called the Weibull-Topp-Leone-G Power Series (WTL-GPS) class of distributions (CoD). More often than not systems, components or devices may fail due to the presence of an unknown number, say N , of initial defects of the same kind that can only be identified after causing the failure and getting repaired. This scenario necessitates the development of a special type of models called the power series CoD. Power series models have also found applications in the fields of medicine, disease modelling, finance, actuarial sciences and engineering. A number of power series CoD have been developed in the past few decades and these include the compound class of Weibull and power series distribution by Morais et al. [79], the complementary extended Weibull power series CoD by Cordeiro et al. [50], the compound class of extended Weibull power series distribution by Silva et al. [117], double bounded Kumaraswamy-power series CoD by Bidram and Nekoukhou [26], generalized exponential power series distribution by Mahmoud et al. [81], compound class of linear failure rate power series distributions by Mahmoud et al. [82], Burr XII power series distribution by Silva et al. [118], generalized linear failure rate power series distribution by Alamatsaz et al. [67], the exponentiated power generalized Weibull power series FoD by Aldahlan et al. [13], the odd Weibull-Topp-Leone-G power series FoD by Oluyede et al. [101], the exponentiated generalized power series FoD by Oluyede et al. [102], the exponentiated generalized power series FoD by Nasiru et al. [96], to mention a few.

Our main motivation for developing this new CoD is to come up with models that can capture data with monotonic and non-monotonic failure rates ranging from bathtub, upside-down bathtub, increasing and decreasing shapes. Also, we seek to improve the performance of the Weibull-Topp-Leone-G FoD when it comes to modelling reliability, biomedical and survival data sets.

This chapter is organized as follows. In Section 6.1.2, some basic results, the WTL-GPS CoD,

its hrf and qf are presented. The density expansion and some special cases are presented in Section 6.1.3. The moments, moment generating function, incomplete moments, distribution of order statistics and Rényi entropy are given in Section 6.1.4. In Section 6.1.5, results on the estimation of the parameters of the WTL-GPS CoD via the method of maximum likelihood are presented. A Monte Carlo simulation study is carried out to evaluate the ABias and RMSE of the maximum likelihood estimators in Section 6.1.6. Some examples on the applicability of the model are given in Section 6.1.7, followed by some concluding remarks in Section 6.1.8.

6.1.2 The WTL-GPS Class, Sub-Classes, Hazard Rate and Quantile Functions

Karamikabir et al. [71] developed the Weibull-Topp-Leone generated FoD via odds of the Topp-Leone-G distribution. In this chapter, we developed the Weibull-Topp-Leone-G (WTL-G) FoD via T-X method and compound it with the power series distribution to obtain the Weibull-Topp-Leone-G Power Series CoD. The cdf of the Weibull Topp-Leone-G (WTL-G) FoD we used is obtained as follows

$$\begin{aligned} F_{WTL-G}(x; b, \beta, \varphi) &= \int_{-\infty}^{-\log(K_G(x; b, \varphi))} \beta t^{\beta-1} \exp(-t^\beta) dt \\ &= 1 - \exp\{-[-\log(K_G(x; b, \varphi))]^\beta\}, \end{aligned} \quad (6.1)$$

where $\bar{G}(x; \varphi) = 1 - G(x; \varphi)$, $b, \beta > 0$, and φ a vector of parameters.

Suppose N is a discrete rv following a power series which is assumed to be truncated at zero with a probability mass function (pmf) given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, n = 1, 2, \dots, \quad (6.2)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n \geq 1}$ a sequence of real positive numbers. The power series FoD includes the binomial, geometric, Poisson and logarithmic distributions. We denote the cdf of the WTL-G distribution $F_{WTL-G}(x; b, \beta, \varphi)$ by $F_{WTL-G}(x)$. Let $X_1, X_2, \dots, X_N$ be i.i.d. rvs having the WTL-G distribution. We assume that the rv N has a power series distribution given by equation (6.2). Suppose $X = X_{(1)} = \min(X_1, \dots, X_N)$, then the conditional distribution of X given $N = n$ is

$$F_{X|N=n}(x) = 1 - \prod_{i=1}^n (1 - F_{WTL-G}(x)) = 1 - S_{WTL-G}^n(x), \quad (6.3)$$

where $S_{WTL-G}(x) = 1 - F_{WTL-G}(x)$ is the survival function of the WTL-G FoD. The cdf of the life of the whole system, X , say F_θ , is given by

$$F_\theta(x) = 1 - \frac{C(\theta S_{WTL-G}(x))}{C(\theta)}. \quad (6.4)$$

The corresponding pdf is given by

$$f_{\theta}(x) = \frac{dF_{\theta}(x)}{dx} = \frac{\theta f_{WTL-G}(x) C'(\theta S_{WTL-G}(x))}{C(\theta)}. \quad (6.5)$$

The hrf is given by

$$h_{\theta}(x) = \frac{f_{\theta}(x)}{S_{\theta}(x)} = \theta f_{WTL-G}(x) \frac{C'(\theta S_{WTL-G}(x))}{C(\theta S_{WTL-G}(x))}.$$

Now, using equations (6.1), (6.4) and (6.5), the cdf and pdf of the proposed WTL-GPS CoD are respectively, given by

$$F(x; b, \beta, \theta, \varphi) = 1 - \frac{C\left(\theta \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^{\beta} \right\} \right)\right)}{C(\theta)} \quad (6.6)$$

and

$$\begin{aligned} f(x; b, \beta, \theta, \varphi) &= \frac{2b\theta\beta g(x; \varphi) \bar{G}(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{b-1} (-\log(K_G(x; b, \varphi)))^{\beta-1}}{C(\theta)(K_G(x; b, \varphi))} \\ &\times C'\left(\theta \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^{\beta} \right\} \right)\right) \\ &\times \exp\{-(-\log(K_G(x; b, \varphi)))^{\beta}\}, \end{aligned} \quad (6.7)$$

for $b, \beta, \theta > 0$, where $G(x; \varphi)$ is the baseline distribution with parameter vector φ and $\bar{G}(x; \varphi) = 1 - G(x; \varphi)$. Table 6.1 present some special families of the WTL-GPS CoD when $C(\theta)$ is specified.

Distribution	a_n	$C(\theta)$	cdf
W-TL-G Binomial	$\binom{m}{n}$	$(1 + \theta)^m - 1$	$1 - \frac{\left(1 + \theta \left[\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^{\beta} \right\} \right]\right)^m}{(1 + \theta)^m - 1} - 1$
W-TL-G Poisson	$(n!)^{-1}$	$e^{\theta} - 1$	$1 - \frac{\exp\left(\theta \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^{\beta} \right\} \right)\right)}{\exp(\theta) - 1} - 1$
W-TL-G Geometric	1	$\theta(1 - \theta)^{-1}$	$1 - \frac{(1 - \theta) \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^{\beta} \right\} \right)}{1 - \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^{\beta} \right\} \right)}$
W-TL-G Logarithmic	n^{-1}	$-\log(1 - \theta)$	$1 - \frac{\log \left(1 - \theta \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^{\beta} \right\} \right)\right)}{\log(1 - \theta)}$

Table 6.1: **Some Special Families of the WTL-GPS CoD**

The hrf for the WTL-GPS CoD is given by

$$\begin{aligned} h_{\theta}(x) &= \frac{2b\theta\beta g(x; \varphi) \bar{G}(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{b-1} (-\log(K_G(x; b, \varphi)))^{\beta-1}}{C\left(\theta \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^{\beta} \right\} \right)\right) (K_G(x; b, \varphi))} \\ &\times C'\left(\theta \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^{\beta} \right\} \right)\right) \\ &\times \exp\{-(-\log(K_G(x; b, \varphi)))^{\beta}\}, \end{aligned} \quad (6.8)$$

for for $b, \beta, \theta > 0$ and parameter vector φ . The qf, $Q(u)$, of the WTL-GPS CoD can be obtained by solving the equation $F(Q(u)) = u$, $0 \leq u \leq 1$, as follows

$$F(Q(u)) = 1 - \frac{C\left(\theta\left(\exp\left\{-\left(-\log(1 - [1 - \overline{G}^2(Q(u))]^b)\right)^{\beta}\right\}\right)\right)}{C(\theta)} = u,$$

so that

$$Q(u) = G^{-1}\left[1 - \left\{1 - \left(1 - \exp\left\{-\left[-\ln\left(\frac{C^{-1}[C(\theta)(1-u)]}{\theta}\right)\right]^{\frac{1}{\beta}}\right\}\right)^{\frac{1}{b}}\right\}^{\frac{1}{2}}\right].$$

This function is crucial in generating some random numbers from the WTL-GPS distribution for a specified baseline distribution G , and the function $C(\theta)$.

6.1.3 Special Classes of the WTL-GPS CoD

In this subsection, we present some special cases of the WTL-GPS CoD when the baseline distribution is the log-logistic distribution with cdf, $G(x; \gamma) = 1 - (1 + x^{\gamma})^{-1}$ and pdf, $g(x; \gamma) = \gamma x^{\gamma-1}(1 + x^{\gamma})^{-2}$, for $x > 0, \gamma > 0$.

The Weibull-Topp-Leone-Log-Logistic Power Series CoD

Here we present the Weibull-Topp-Leone-Log-Logistic Power Series (WTL-LLoGPS) CoD. Sub-models of the WTL-LLoGPS distribution, namely the Weibull-Topp-Leone-Log-Logistic-Poisson (WTL-LLoG-P), Weibull-Topp-Leone-Log-Logistic-Logarithmic (WTL-LLoG-L) and Weibull-Topp-Leone-Log-Logistic-Geometric (WTL-LLoG-G) distributions are presented in this part. The cdf and pdf of WTL-LLoGPS CoD are given by

$$F(x; b, \beta, \theta, \gamma) = 1 - \frac{C\left(\theta\left(\exp\left\{-\left(-\log(1 - [1 - (1 + x^{\gamma})^{-2}]^b)\right)^{\beta}\right\}\right)\right)}{C(\theta)}$$

and

$$\begin{aligned} f(x; b, \beta, \theta, \gamma) &= \frac{2b\beta\theta\gamma x^{\gamma-1}(1 + x^{\gamma})^{-2}(1 + x^{\gamma})^{-1}(1 - (1 + x^{\gamma})^{-2})^{b-1}}{C(\theta)(1 - (1 - (1 + x^{\gamma})^{-2})^b)} \\ &\times C'\left(\theta\left(\exp\left\{-\left(-\log(1 - [1 - (1 + x^{\gamma})^{-2}]^b)\right)^{\beta}\right\}\right)\right) \\ &\times \exp\{-(-\log(1 - [1 - (1 + x^{\gamma})^{-2}]^b))^{\beta}\} \\ &\times (-\log(1 - (1 - (1 + x^{\gamma})^{-2})^b))^{\beta-1}, \end{aligned}$$

for $b, \beta, \theta, \gamma > 0$.

The WTL-LLoG-Poisson Distribution

The cdf and pdf of the WTL-LLoG-P distribution are given by

$$F(x; b, \beta, \gamma, \theta) = 1 - \frac{\exp\left(\theta\left(\exp\left\{-\left(-\log(1 - [1 - (1 + x^{\gamma})^{-2}]^b)\right)^{\beta}\right\}\right)\right) - 1}{\exp(\theta) - 1}$$

and

$$\begin{aligned}
 f(x; b, \beta, \theta, \gamma) &= \frac{2b\beta\theta\gamma x^{\gamma-1}(1+x^\gamma)^{-3}(1-(1+x^\gamma)^{-2})^{b-1}}{(\exp(\theta)-1)(1-(1-(1+x^\gamma)^{-2})^b)} \\
 &\times \exp\left(\theta\left(\exp\left\{-\left(-\log(1-[1-(1+x^\gamma)^{-2}]^b)\right)^\beta\right\}\right)\right) \\
 &\times \exp\{-(-\log(1-[1-(1+x^\gamma)^{-2}]^b))^\beta\} \\
 &\times (-\log(1-(1-(1+x^\gamma)^{-2})^b))^{\beta-1},
 \end{aligned}$$

respectively, for $b, \beta, \theta, \gamma > 0$.

Plots of the pdf and hrf of the Weibull-Topp-Leone-Log-Logistic-Poisson distribution are presented in Figure 6.1. The pdf assumes various shapes including left-skewed, right-skewed and reverse-J. The hrf shows bathtub followed by inverted bathtub, decreasing, increasing and inverted bathtub shapes.

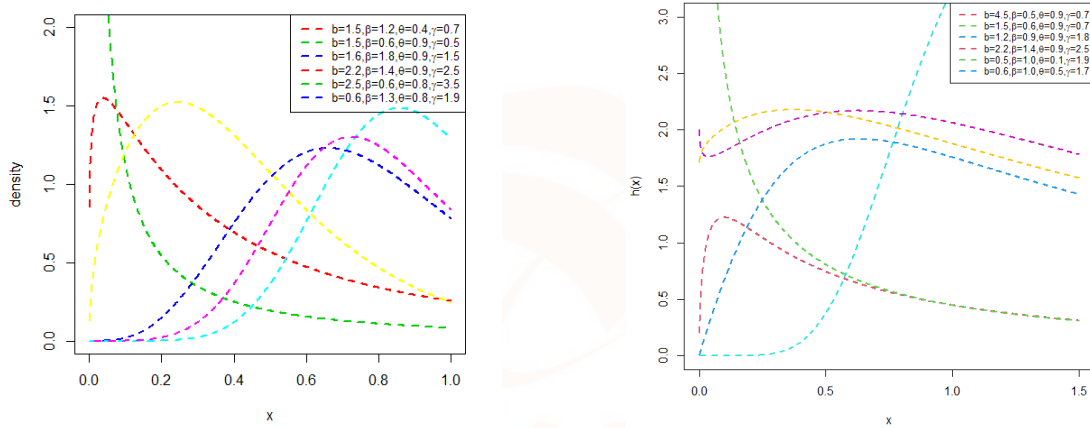


Figure 6.1: Plots of the pdf and hrf of the Weibull-Topp-Leone-Log-Logistic-Poisson distribution

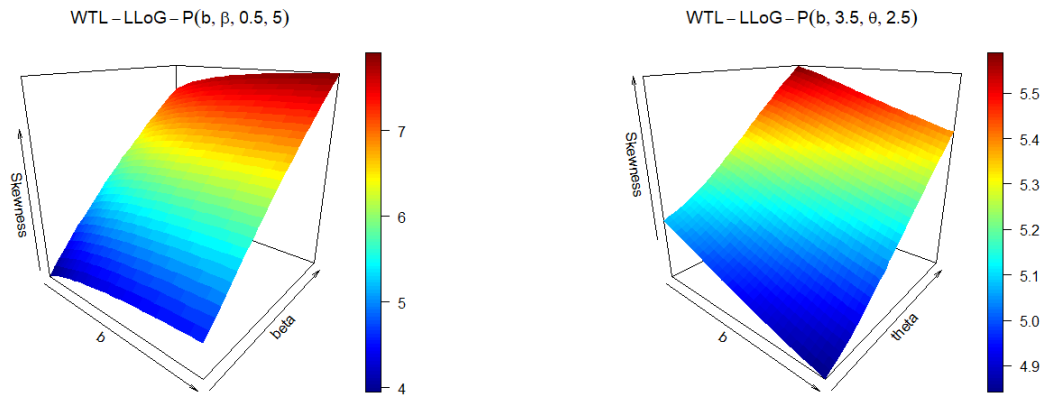


Figure 6.2: Plots of the skewness of the WTL-LLoG-Poisson distribution

The skewness and kurtosis plots of the WTL-LLoG-Poisson distribution are presented in Figures

6.2 and 6.3. The WTL-LLoG-Poisson class of distribution is capable of modelling various data sets with different levels of skewness and kurtosis. It can be seen from Figures 6.2 and 6.3 that if we vary the parameters of the WTL-LLoG-Poisson distribution, we obtain different levels of skewness and kurtosis.

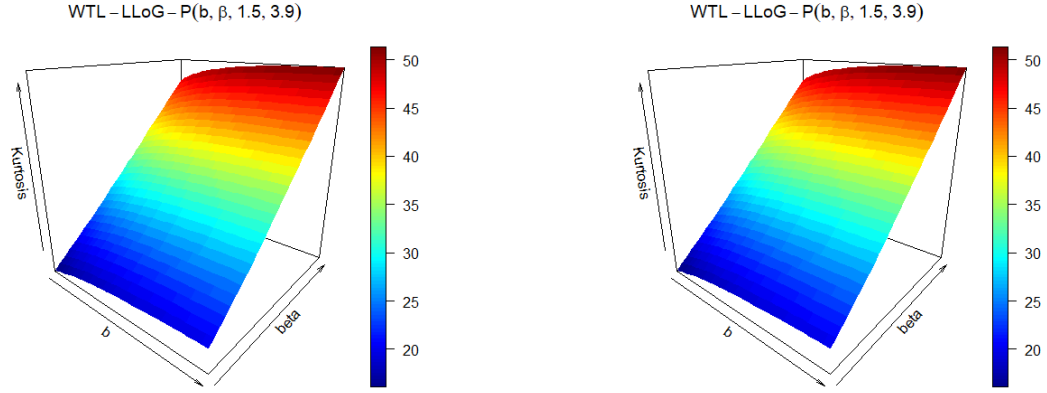


Figure 6.3: Plots of the kurtosis of the WTL-LLoG-Poisson distribution

The WTL-LLoG-Logarithmic Distribution

The cdf and pdf of the Weibull-Topp-Leone-Log-Logistic-Logarithmic distribution are respectively, given by

$$F(x; b, \beta, \theta, \gamma) = 1 - \frac{\log \left(1 - \theta \left(\exp \left\{ - \left(-\log(1 - [1 - (1 + x^\gamma)^{-2}]^b \right)^\beta \right\} \right) \right)}{\log(1 - \theta)}$$

and

$$\begin{aligned} f(x; b, \beta, \theta, \gamma) &= \frac{2b\beta\theta\gamma x^{\gamma-1}(1+x^\gamma)^{-3}(1-(1+x^\gamma)^{-2})^{b-1}}{-\log(1-\theta)(1-(1-(1+x^\gamma)^{-2})^b)} \\ &\times \left(1 - \left(\theta \left(\exp \left\{ - \left(-\log(1 - [1 - (1 + x^\gamma)^{-2}]^b \right)^\beta \right\} \right) \right) \right)^{-1} \\ &\times \exp \{ -(-\log(1 - [1 - (1 + x^\gamma)^{-2}]^b))^\beta \} \\ &\times (-\log(1 - (1 - (1 + x^\gamma)^{-2})^b))^{\beta-1}, \end{aligned}$$

for $b, \beta, \theta, \gamma > 0$.

Plots of the pdf and hrf of the Weibull-Topp-Leone-Log-Logistic-Logarithmic distribution are presented in Figure 6.4. The pdf assumes a variety of shapes such as left-skewed, right-skewed and reverse-J. The hrf reveals decreasing, increasing, inverted bathtub and bathtub followed by inverted bathtub shapes.

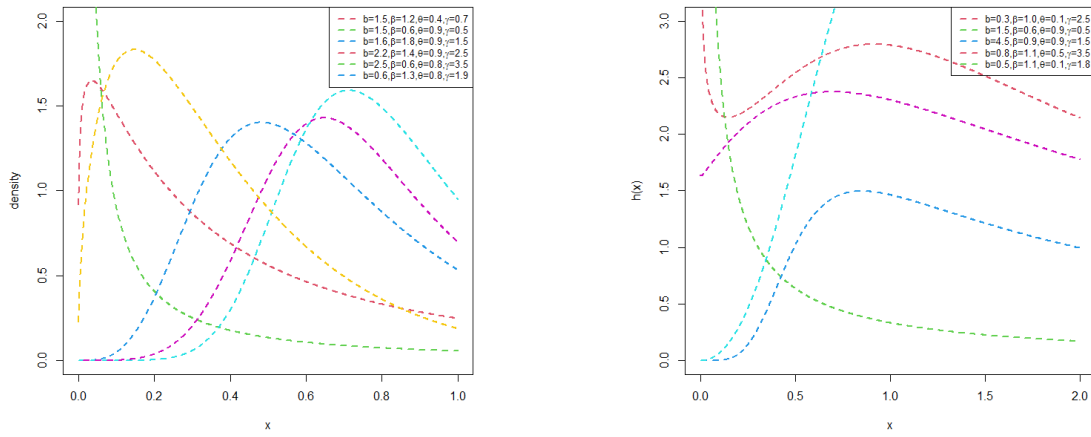


Figure 6.4: Plots of the pdf and hrf of the Weibull-Topp-Leone-Log-Logistic Logarithmic distribution

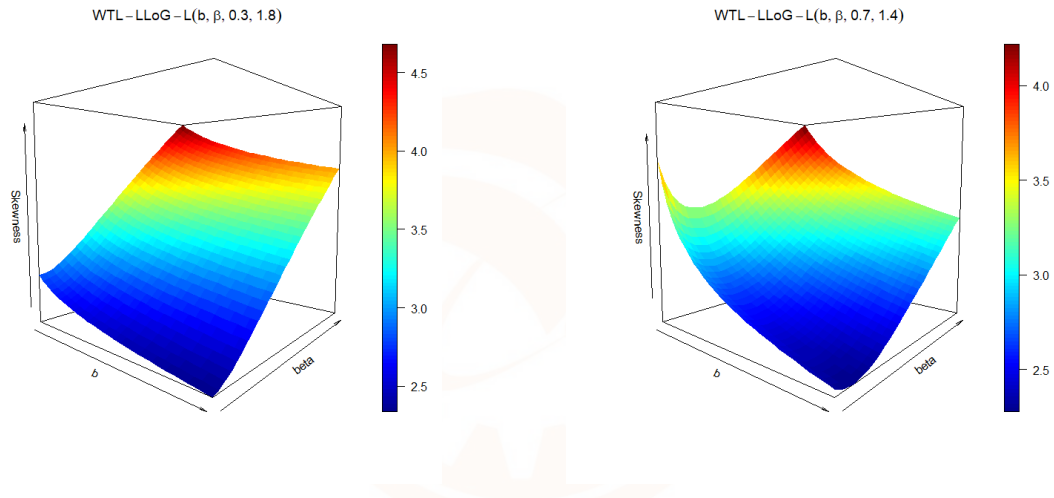


Figure 6.5: Plots of the skewness of the WTL-LLoG-Logarithmic distribution

Figure 6.5 presents some skewness plots of WTL-LLoG-Logarithmic distribution. It can be noted that as we increase the values of b and β , the model can capture highly skewed data sets.

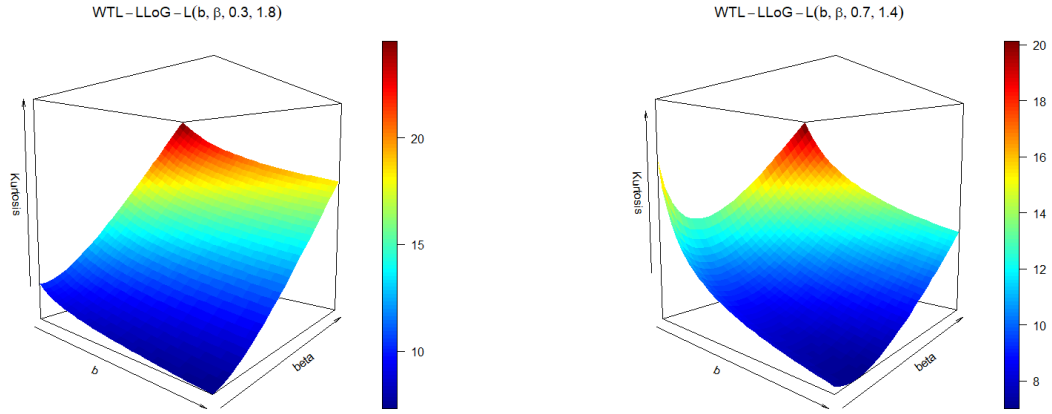


Figure 6.6: Plots of the kurtosis of the WTL-LLoG-Logarithmic distribution

Figure 6.6 shows some 3D plots for kurtosis of the WTL-LLoG-Logarithmic distribution. It can be seen that the model is flexible and can capture data sets with different levels of kurtosis.

WTL-LLoG-Geometric Distribution

The cdf and pdf of the Weibull-Topp-Leone-Log-Logistic-Geometric distribution are respectively, given by

$$F(x; b, \beta, \theta, \gamma) = 1 - \frac{(1 - \theta) \left(\exp \left\{ - \left(- \log(1 - [1 - (1 + x^\gamma)^{-2}]^b) \right)^\beta \right\} \right)}{1 - \left(\exp \left\{ - \left(- \log(1 - [1 - (1 + x^\gamma)^{-2}]^b) \right)^\beta \right\} \right)}$$

and

$$\begin{aligned} f(x; b, \beta, \theta, \gamma) &= \frac{2b\beta\gamma x^{\gamma-1} (1 + x^\gamma)^{-3} (1 - (1 + x^\gamma)^{-2})^{b-1}}{(1 - \theta)^{-1} (1 - (1 - (1 + x^\gamma)^{-2})^b)} \\ &\times \left(1 - \left(\theta \left(\exp \left\{ - \left(- \log(1 - [1 - (1 + x^\gamma)^{-2}]^b) \right)^\beta \right\} \right) \right) \right)^{-2} \\ &\times \exp \left\{ - \left(- \log(1 - [1 - (1 + x^\gamma)^{-2}]^b) \right)^\beta \right\} \\ &\times \left(- \log(1 - (1 - (1 + x^\gamma)^{-2})^b) \right)^{\beta-1}, \end{aligned}$$

for $b, \beta, \theta, \gamma > 0$.

Plots of the pdf and hrf of the Weibull-Topp-Leone-Log-Logistic-Geometric distribution are presented in Figure 6.7. The pdf assumes a variety of shapes such as right-skewed, reverse-J, almost symmetric and left-skewed. The hrf exhibits decreasing, increasing, inverted bathtub and bathtub followed by inverted shapes.

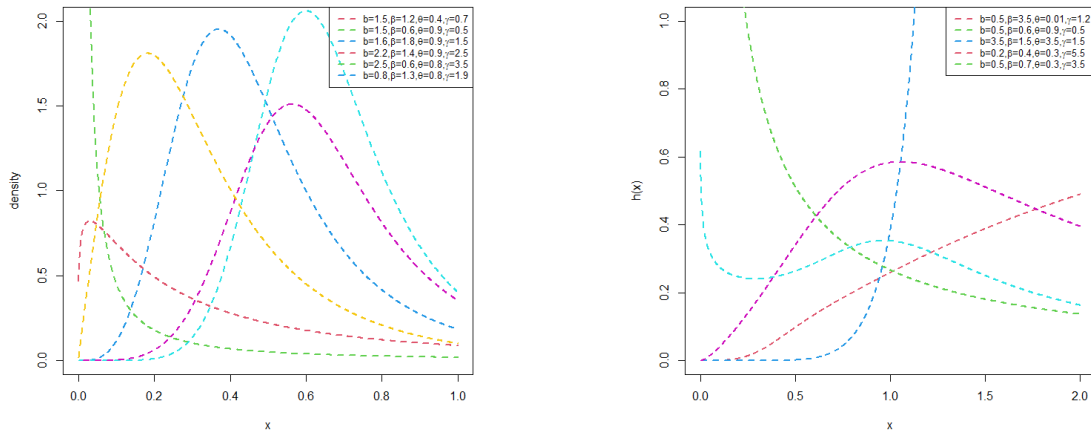


Figure 6.7: Plots of the pdf and hrf of the Weibull-Topp-Leone-Log-Logistic Geometric distribution

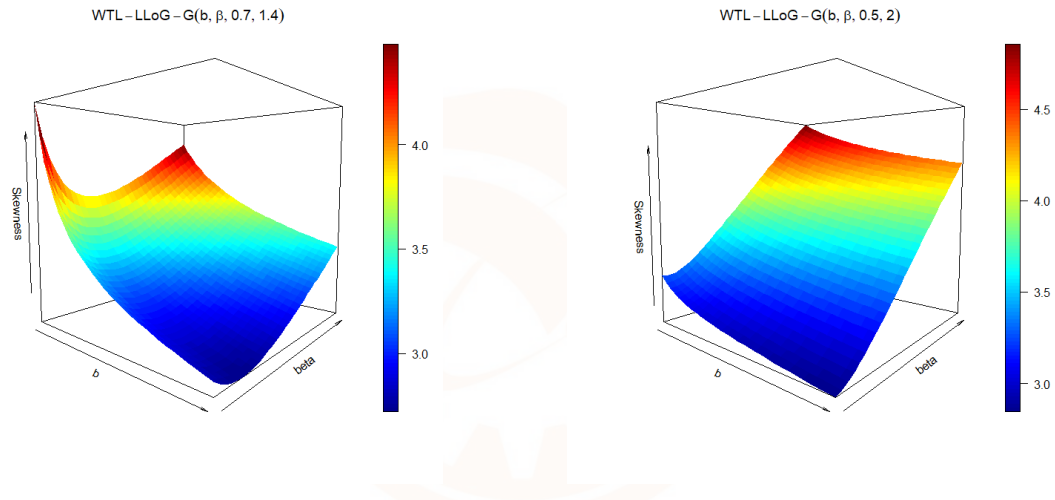


Figure 6.8: Plots of the skewness of the WTL-LLoG-Geometric distribution

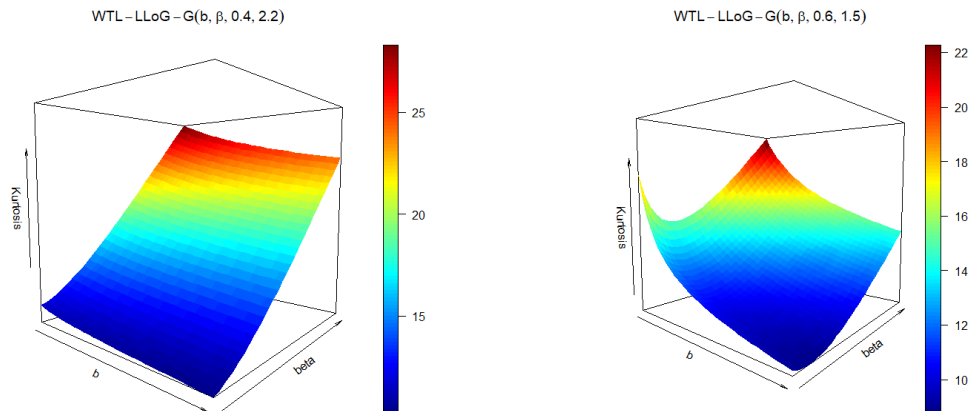


Figure 6.9: Plots of the kurtosis of the WTL-LLoG-Geometric distribution

Figure 6.8 and Figure 6.9 shows plots of skewness and kurtosis for the WTL-LLoG-Geometric distribution. We can see the flexibility of this new model in capturing different skewness and kurtosis levels by varying the model parameters.

Density Expansion

In this subsection, we provide the pdf of the WTL-GPS CoD as an infinite linear combination of exponentiated-G densities. Equation (6.7) can be written as

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{na^n\theta^n}{C(\theta)} \frac{2b\beta g(x; \varphi) \overline{G}(x; \varphi) (-\log(K_G(x; b, \varphi)))^{\beta-1}}{(K_G(x; b, \varphi))} \\ &\times (1 - \overline{G}^2(x; \varphi))^{b-1} \exp\{-n(-\log(K_G(x; b, \varphi)))^\beta\}. \end{aligned} \quad (6.9)$$

By considering the following expansions,

$$(K_G(x; b, \varphi))^{-1} = \sum_{h=0}^{\infty} (-1)^h \Gamma(h+1) (1 - \overline{G}^2(x; \varphi))^{bh}.$$

and

$$\begin{aligned} \exp\{-n(-\log(K_G(x; b, \varphi)))^\beta\} &= \sum_{k=0}^{\infty} \frac{(-1)^k n^k}{k!} \\ &\times (-\log(K_G(x; b, \varphi)))^{\beta k}, \end{aligned}$$

equation (6.9) can be expressed as

$$\begin{aligned} f(x) &= \sum_{h,k=0}^{\infty} \sum_{n=1}^{\infty} 2b\beta \frac{n^{k+1} a^n \theta^n \Gamma(k+1) (-1)^{h+k}}{C(\theta) k!} g(x; \varphi) \overline{G}(x; \varphi) \\ &\times (1 - \overline{G}^2(x; \varphi))^{b(h+1)-1} (-\log(K_G(x; b, \varphi)))^{\beta(k+1)-1}. \end{aligned}$$

Let $y = [1 - \overline{G}^2(x; \varphi)]^b$, $0 < y < 1$. Using the series representation

$$-\log(1 - y) = \sum_{i=1}^{\infty} \frac{y^{i+1}}{i+1}, \quad (6.10)$$

we get

$$[-\log(1 - y)]^{\beta(k+1)-1} = y^{\beta(k+1)-1} \sum_{q=0}^{\infty} \binom{\beta(k+1)-1}{q} y^q \sum_{s=0}^{\infty} \left(\frac{y^s}{s+2}\right)^q.$$

Applying the result on a power series raised to a positive integer with $a_s = (s+2)^{-1}$ (see, Gradshteyn and Ryzhik [63]), that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s\right)^q = \sum_{s=0}^{\infty} b_{s,q} y^s,$$

where

$$b_{s,q} = (sa_0)^{-1} \sum_{i=1}^s [q(l+i) - s] a_l b_{s-l,q} \quad \text{and} \quad b_{0,q} = a_0^m, \quad (6.11)$$

the pdf of the WTL-GPS CoD can be written as

$$\begin{aligned} f(x) &= \sum_{h,k,q,s=0}^{\infty} \sum_{n=1}^{\infty} 2b\beta \frac{n^{k+1} a^n \theta^n \Gamma(k+1) (-1)^{h+k}}{C(\theta) k!} b_{s,q} \binom{\beta(k+1)-1}{q} \\ &\times g(x; \varphi) \bar{G}(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{b(\beta(k+1)+q+s+h)-1} \\ &= \sum_{h,k,q,s,l=0}^{\infty} \sum_{n=1}^{\infty} 2b\beta \frac{n^{k+1} a^n \theta^n \Gamma(k+1) (-1)^{h+k+l}}{C(\theta) k!} b_{s,q} \binom{\beta(k+1)-1}{q} \\ &\times \binom{b(\beta(k+1)+q+s+h)-1}{l} g(x; \varphi) \bar{G}^{2l+1}(x; \varphi) \\ &= \sum_{h,k,q,s,l,w=0}^{\infty} \sum_{n=1}^{\infty} 2b\beta \frac{n^{k+1} a^n \theta^n \Gamma(k+1) (-1)^{h+k+l+w}}{C(\theta) k!} b_{s,q} \binom{\beta(k+1)-1}{q} \\ &\times \binom{b(\beta(k+1)+q+s+h)-1}{l} \binom{2l+1}{w} g(x; \varphi) G^w(x; \varphi) \\ &= \sum_{w=0}^{\infty} \phi_{w+1} h_{w+1}(x; \varphi), \end{aligned} \quad (6.12)$$

where

$$\begin{aligned} \phi_{w+1} &= \sum_{h,k,q,s,l=0}^{\infty} \sum_{n=1}^{\infty} 2b\beta \frac{n^{k+1} a^n \theta^n \Gamma(k+1) (-1)^{h+k+l+w}}{C(\theta) k! (w+1)} b_{s,q} \binom{\beta(k+1)-1}{q} \\ &\times \binom{b(\beta(k+1)+q+s+h)-1}{l} \binom{2l+1}{w}, \end{aligned} \quad (6.13)$$

and $h_{w+1}(x; \varphi) = (w+1)g(x; \varphi)G^w(x; \varphi)$ is the exponentiated-G (Exp-G) density with power parameter $(w+1)$. Consequently, the statistical properties of the WTL-GPS CoD follows from those of the Exp-G distribution.

6.1.4 Moments, Moment Generating Function and Incomplete Moments, Order Statistics and Rényi Entropy (RE)

In this subsection, some statistical properties of the WTL-GPS CoD including moments, mgf, incomplete moments, distribution of order statistics and RE.

Moments, Moment Generating Function and Incomplete Moments

Suppose the rv Y_{w+1} has an Exp-G distribution with power parameter $(w+1)$, and making use of equation (6.12), the r^{th} moment of the WTL-GPS CoD can be obtained as follows:

$$\mu'_r = E(X^r) = \sum_{w=0}^{\infty} \phi_{w+1} E(Y_{w+1}^r), \quad (6.14)$$

where ϕ_{w+1} is given in equation (6.13). The mgf $M_X(t) = E(e^{tX})$ of X can be obtained using equation (6.12) and is given by

$$M_X(t) = \sum_{w=0}^{\infty} \phi_{w+1} M_w(t), \quad (6.15)$$

where $M_w(t)$ is the mgf of Y_{w+1} . Consequently, $M_X(t)$ can be determined from the mgf of the Exp-G distribution.

The r^{th} incomplete moment of the WTL-GPS CoD is given by

$$\delta_r(y) = \int_{-\infty}^y x^r f(x; b, \beta, \theta, \varphi) dx = \sum_{w=0}^{\infty} \phi_{w+1} \int_{-\infty}^y x^r h_{w+1}(x; \varphi) dx,$$

where $\int_{-\infty}^y x^r h_{w+1}(x; \varphi) dx$ is the r^{th} incomplete moment of the Exp-G distribution.

Order Statistics

In this subsection, we present expression for the distribution of order statistics. Let $X_1, X_2, X_3, \dots, X_n$ are i.i.d. WTL-GPS rvs. Then, the pdf of the i^{th} order statistics $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} (F(x))^{j+i-1}, \quad (6.16)$$

where $B(.,.)$ is the beta function. Now, note that

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= \sum_{n=1}^{\infty} \sum_{r=0}^{j+i-1} (-1)^r \binom{j+i-1}{r} \frac{na^n \theta^n}{[C(\theta)]^{r+1}} \frac{2b\beta g(x; \varphi) \bar{G}(x; \varphi)}{(K_G(x; b, \varphi))} \\ &\times (1 - \bar{G}^2(x; \varphi))^{b-1} \exp\{-n(-\log(K_G(x; b, \varphi)))^\beta\} \\ &\times \left[C\left(\theta \left(\exp\left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^\beta \right\} \right) \right) \right]^r \\ &\times (-\log(K_G(x; b, \varphi)))^{\beta-1}. \end{aligned}$$

This can be expressed as

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= \sum_{n=1}^{\infty} \sum_{r=0}^{j+i-1} \sum_{z,q=0}^{\infty} (-1)^{r+z} \binom{j+i-1}{r} \frac{na^n \theta^{n+z} \Gamma(q+1) d_{z,r}}{[C(\theta)]^{r+1} q!} \\ &\times \underbrace{\exp\{-(n+z)(-\log(K_G(x; b, \varphi)))^\beta\}}_A \\ &\times 2b\beta g(x; \varphi) \bar{G}(x; \varphi) (-\log(K_G(x; b, \varphi)))^{\beta-1} \\ &\times (1 - \bar{G}^2(x; \varphi))^{b(k+1)-1}, \end{aligned}$$

by applying the power series raised to a positive integer (Gradshteyn and Ryzhik [63]), where $d_{z,r} = (zb_0)^{-1} \sum_{m=1}^z [r(m+1)-z]b_m d_{z-m,r}$ and $d_{0,r} = b_0^r$. Using the exponential series expansion,

$$A = \sum_{p=0}^{\infty} \frac{(-1)^p (n+z)^p}{p!} \left(-\log(K_G(x; b, \varphi)) \right)^{\beta p},$$

and substituting for A, we obtain

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= \sum_{n=1}^{\infty} \sum_{r=0}^{j+i-1} \sum_{z,q=0}^{\infty} \frac{2b\beta(-1)^{r+z+p} n a^n \theta^{n+z} (n+z)^p \Gamma(q+1) d_{z,r}}{[C(\theta)]^{r+1} q! p!} \\ &\times \binom{j+i-1}{r} (-\log(K_G(x; b, \varphi)))^{\beta p + \beta - 1} \\ &\times g(x; \varphi) \overline{G}(x; \varphi) (1 - \overline{G}^2(x; \varphi))^{b(k+1)-1}. \end{aligned}$$

Let $y = [1 - \overline{G}^2(x; \varphi)]^b$, $0 < y < 1$ and making use of the series expansion in equation (6.10), we get

$$[-\log(1-y)]^{\beta p + \beta - 1} = y^{\beta p + \beta - 1} \sum_{m=0}^{\infty} \binom{\beta p + \beta - 1}{m} y^m \left(\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^m, \quad (6.17)$$

and by making use of the result from a power series raised to positive integer with $a_s = (s+2)^{-1}$, that is, $\left(\sum_{s=0}^{\infty} a_s y^s \right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s$, where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^s [m(l+1)-s] a_l b_{s-l,m}$ and $b_{0,m} = a_0^m$ (Gradshteyn and Ryzhik [63]), we can write

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= \sum_{n=1}^{\infty} \sum_{r=0}^{j+i-1} \sum_{z,q,k,l=0}^{\infty} \frac{2b\beta(-1)^{r+z+p+k+l} n a^n \theta^{n+z} (n+z)^p \Gamma(q+1) d_{z,r}}{[C(\theta)]^{r+1} q! p!} \\ &\times \binom{j+i-1}{r} \binom{\beta p + \beta - 1}{m} \binom{b(\beta p + \beta + m + s + q) - 1}{k} \\ &\times \binom{2k+1}{l} b_{s,m} g(x; \varphi) (G(x; \varphi))^l. \end{aligned} \quad (6.18)$$

Substituting equation (6.18) into equation (6.16), we get

$$\begin{aligned} f_{i:n}(x) &= \sum_{j=0}^{n-i} \sum_{n=1}^{\infty} \sum_{r=0}^{j+i-1} \sum_{z,q,s,m,k,l=0}^{\infty} \frac{2b\beta(-1)^{j+r+z+p+k+l} n a^n \theta^{n+z} (n+z)^p \Gamma(q+1)}{B(i, n-i+1) [C(\theta)]^{r+1} q! p!} \\ &\times \binom{j+i-1}{r} \binom{\beta p + \beta - 1}{m} \binom{b(\beta p + \beta + m + s + q) - 1}{k} \\ &\times \binom{2k+1}{l} \binom{n-i}{j} d_{z,r} b_{s,m} g(x; \varphi) (G(x; \varphi))^l. \\ &= \sum_{l=0}^{\infty} \phi_{l+1}^* h_{l+1}^*(x; \varphi), \end{aligned} \quad (6.19)$$

where

$$\begin{aligned}
 \phi_{l+1}^* &= \sum_{j=0}^{n-i} \sum_{n=1}^{\infty} \sum_{r=0}^{j+i-1} \sum_{z,q,s,m,k=0}^{\infty} \frac{2b\beta(-1)^{j+r+z+p+k+l} n a^n \theta^{n+z} (n+z)^p \Gamma(q+1)}{B(i, n-i+1) [C(\theta)]^{r+1} q! p! (l+1)} \\
 &\times \binom{j+i-1}{r} \binom{\beta p + \beta - 1}{m} \binom{b(\beta p + \beta + m + s + q) - 1}{k} \\
 &\times \binom{2k+1}{l} \binom{n-i}{j} d_{z,r} b_{s,m}, \tag{6.20}
 \end{aligned}$$

and $h_{l+1}^*(x; \varphi) = (l+1)g(x; \varphi)G^l(x; \varphi)$ is the Exp-G density with power parameter $(l+1)$.

Rényi Entropy (RE)

The RE (Rényi [113]) is an extension of Shannon entropy. RE is defined to be

$$I_R(\nu) = \frac{1}{1-\nu} \log \left(\int_0^\infty [f(x)]^\nu dx \right), \nu \neq 1, \nu > 0. \tag{6.21}$$

From the WTL-GPS pdf, $f^\nu(x)$ can be expressed as

$$\begin{aligned}
 f^\nu(x) &= \frac{(2b\theta\beta)^\nu g^\nu(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{\nu(b-1)} (-\log(K_G(x; b, \varphi)))^{\nu(\beta-1)}}{C^\nu(\theta) (K_G(x; b, \varphi))^\nu} \\
 &\times \bar{G}^\nu(x; \varphi) \left[C' \left(\theta \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^\beta \right\} \right) \right) \right]^\nu \\
 &\times \exp \{ -\nu (-\log(K_G(x; b, \varphi)))^\beta \}. \tag{6.22}
 \end{aligned}$$

By making use of the power series raised to a positive integer (Gradshteyn and Ryzhik [63]), where $d_{z,\nu} = (zb_0)^{-1} \sum_{m=1}^z [\nu(m+1) - z] b_m d_{z-m,\nu}$ and $d_{0,\nu} = b_0^\nu$, we have

$$\begin{aligned}
 f^\nu(x) &= \frac{(2b\theta\beta)^\nu g^\nu(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{\nu(b-1)} (-\log(K_G(x; b, \varphi)))^{\nu(\beta-1)}}{C^\nu(\theta) (K_G(x; b, \varphi))^\nu} \\
 &\times \bar{G}^\nu(x; \varphi) \sum_{z=0}^{\infty} d_{z,\nu} \theta^z \exp \left\{ - (z + \nu) \left(-\log(K_G(x; b, \varphi)) \right)^\beta \right\} \\
 &= \sum_{z,p=0}^{\infty} \frac{(2b\theta\beta)^\nu g^\nu(x; \varphi) (1 - \bar{G}^2(x; \varphi))^{\nu(b-1)} \bar{G}^\nu(x; \varphi) (-1)^p d_{z,\nu} (z + \nu)^p \theta^z}{C^\nu(\theta) p! (K_G(x; b, \varphi))^\nu} \\
 &\times (-\log(K_G(x; b, \varphi)))^{\beta(p+\nu)-\nu} \\
 &= \sum_{z,p,k=0}^{\infty} \frac{(2b\theta\beta)^\nu (-1)^{p+k} d_{z,\nu} (z + \nu)^p \theta^z}{C^\nu(\theta) p!} \binom{-\nu}{k} g^\nu(x; \varphi) \bar{G}^\nu(x; \varphi) \\
 &\times (1 - \bar{G}^2(x; \varphi))^{b(k+\nu)-\nu} (-\log(K_G(x; b, \varphi)))^{\beta(p+\nu)-\nu}.
 \end{aligned}$$

Let $y = [1 - \bar{G}^2(x; \varphi)]^b$, $0 < y < 1$. Using the series representation given in equation (6.10), we have

$$[-\log(1-y)]^{\beta(p+\nu)-\nu} = y^{\beta(p+\nu)-\nu} \sum_{q=0}^{\infty} \binom{\beta(p+\nu)-\nu}{q} y^q \sum_{s=0}^{\infty} \left(\frac{y^s}{s+2} \right)^q.$$

Applying the result on a power series raised to a positive integer (see, Gradshteyn and Ryzhik [63]) with $a_s = (s + 2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s \right)^q = \sum_{s=0}^{\infty} b_{s,q} y^s,$$

where

$$b_{s,q} = (sa_0)^{-1} \sum_{i=1}^s [q(l+i) - s] a_l b_{s-l,q} \quad \text{and} \quad b_{0,q} = a_0^m, \quad (6.23)$$

we have,

$$\begin{aligned} f^\nu(x) &= \sum_{z,p,k,q,s=0}^{\infty} \frac{(2b\theta\beta)^\nu (-1)^{p+k} (z+\nu)^p \theta^z}{C^\nu(\theta) p!} d_{z,\nu} b_{s,q} \binom{\beta(p+\nu) - \nu}{q} \\ &\times \binom{-\nu}{k} g^\nu(x; \varphi) \overline{G}^\nu(x; \varphi) (1 - \overline{G}^2(x; \varphi))^{b\beta(p+\nu) + b(q+s+k) - \nu} \\ &= \sum_{z,p,k,q,s,l,w=0}^{\infty} \frac{(2b\theta\beta)^\nu (-1)^{p+k+l+w} (z+\nu)^p \theta^z}{C^\nu(\theta) p!} d_{z,\nu} b_{s,q} \binom{\beta(p+\nu) - \nu}{q} \\ &\times \binom{-\nu}{k} \binom{b\beta(p+\nu) + b(q+s+k) - \nu}{l} \binom{2l+\nu}{w} g^\nu(x; \varphi) G^w(x; \varphi). \end{aligned}$$

Thus, RE of the WTL-GPS CoD is given by

$$I_R(\nu) = \frac{1}{1-\nu} \log \left[\sum_{w=0}^{\infty} \varphi_w^* e^{(1-\nu)I_{REG}} \right], \quad (6.24)$$

where

$$\begin{aligned} \varphi_w^* &= \sum_{z,p,k,q,s,l=0}^{\infty} \frac{(2b\theta\beta)^\nu (-1)^{p+k+l+w} (z+\nu)^p \theta^z}{C^\nu(\theta) p!} d_{z,\nu} b_{s,q} \binom{\beta(p+\nu) - \nu}{q} \\ &\times \binom{-\nu}{k} \binom{b\beta(p+\nu) + b(q+s+k) - \nu}{l} \binom{2l+\nu}{w} \left(\frac{1}{(w/\nu + 1)^\nu} \right) \end{aligned}$$

and $I_{REG} = (1-\nu)^{-1} \log \left(\int_0^\infty [(\frac{w}{\nu} + 1) g(x; \varphi) G(x; \varphi)^{\frac{w}{\nu}}]^\nu dx \right)$ is the RE of the Exp-G distribution with power parameter $\frac{w}{\nu}$.

6.1.5 Maximum Likelihood Estimation

In this subsection, the method of maximum likelihood for estimating the parameters of the WTL-GPS CoD is presented. Suppose $X \sim WTL-GPS(b, \beta, \theta, \varphi)$ and let $\Delta = (b, \beta, \theta, \varphi)^T$ be the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ of the WTL-GPS

class of distributions is given by

$$\begin{aligned}
\ell_n(\Delta) = & n \log(2b\theta\beta) + \sum_{i=1}^n \log(g(x; \varphi)) + (b-1) \sum_{i=1}^n \log((1 - \overline{G}^2(x; \varphi))) \\
& + \sum_{i=1}^n \log(\overline{G}(x; \varphi)) + (\beta-1) \sum_{i=1}^n \log(-\log(K_G(x; b, \varphi))) \\
& - \sum_{i=1}^n \log(C(\theta)) - \sum_{i=1}^n \log(K_G(x; b, \varphi)) \\
& + \sum_{i=1}^n \log \left(C' \left(\theta \left(\exp \left\{ - \left(-\log(K_G(x; b, \varphi)) \right)^\beta \right\} \right) \right) \right) \\
& + \sum_{i=1}^n (-(-\log(K_G(x; b, \varphi)))^\beta). \tag{6.25}
\end{aligned}$$

The maximum likelihood estimates can be obtained by differentiating the $\ell_n = \ell_n(\Delta)$ w.r.t each component of the parameter vector $\Delta = (b, \beta, \theta, \varphi)^T$ and equating to zero, however, due to the complexities of the equations, we make use of some numerical methods like the Newton-Raphson technique in R. The multivariate normal distribution $N_{p+3}(\underline{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\underline{0} = (0, 0, 0, \underline{0})^T$ and $J(\hat{\Delta})^{-1}$ is the observed FIM evaluated at $\hat{\Delta}$, can be used to construct CIs and confidence regions for the individual model parameters.

6.1.6 Simulation Study

To assess the performance of the WTL-LLoG-Poisson distribution, we simulate $N = 3000$ samples of sizes $n = 25, 50, 100, 200, 400$ and 800 for the true parameters values given in Table 6.2 and Table 6.3. The tables lists the mean MLEs of the model parameters along with the respective ABias and RMSEs. The ABias and RMSE for the estimated parameter, say $\hat{\theta}$, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

parameter	Sample Size	(1.2, 1.7, 2.5, 1.5)			(1.4, 1.4, 2.5, 1.5)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
b	25	1.8071	1.8292	0.6071	1.8071	0.9706	0.4071
	50	1.7687	1.8500	0.5687	1.6994	0.7766	0.2994
	100	1.5336	1.3108	0.3336	1.5899	0.6759	0.1899
	200	1.4427	1.2175	0.2427	1.5628	0.7252	0.1628
	400	1.2920	0.7165	0.0920	1.5617	0.7683	0.1617
	800	1.2218	0.3021	0.0218	1.4482	0.3313	0.0482
β	25	3.0352	3.3091	1.3352	2.3418	1.9166	0.9418
	50	2.7774	2.8775	1.0774	2.3749	2.2642	0.9749
	100	2.7203	2.4492	1.0203	2.2322	1.7051	0.8322
	200	2.4715	1.9754	0.7715	2.1143	1.6090	0.7143
	400	2.3923	1.6445	0.6923	1.9064	0.9887	0.5064
	800	2.1227	1.0917	0.4227	1.8303	0.9808	0.4303
θ	25	2.9714	2.9791	0.4714	2.1360	1.3856	0.7360
	50	2.9627	2.0320	0.4627	2.0834	1.2806	0.6834
	100	2.9565	2.2019	0.4565	1.8902	1.1835	0.4902
	200	2.9375	2.3731	0.4375	1.8744	1.4382	0.4744
	400	2.7696	1.1891	0.2696	1.8396	1.2709	0.4396
	800	2.6403	0.9262	0.1403	1.6719	0.9231	0.2719
γ	25	1.5265	1.5345	0.0265	1.3546	1.3987	-0.1454
	50	1.5799	1.3636	0.0799	1.4633	1.3278	-0.0367
	100	1.6494	1.7267	0.1494	1.4200	1.1547	-0.0800
	200	1.6164	1.2946	0.1164	1.4104	0.9228	-0.0896
	400	1.6641	1.3160	0.1641	1.3835	0.8103	-0.1165
	800	1.5373	0.7324	0.0373	1.4145	0.6305	-0.0855

Table 6.2: Simulation Results

parameter	Sample Size	(1.2, 1.7, 1.7, 1.5)			(1.4, 1.4, 1.4, 1.5)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
b	25	1.7256	1.4377	0.5256	1.9137	1.6038	0.5137
	50	1.5553	1.1293	0.3553	1.8357	1.4718	0.4357
	100	1.4405	0.8864	0.2405	1.6858	1.2135	0.2858
	200	1.3471	0.7412	0.1471	1.5711	0.9788	0.1711
	400	1.2730	0.5651	0.0730	1.6166	0.9775	0.2166
	800	1.2054	0.2924	0.0054	1.4724	0.3538	0.0724
β	25	2.9253	2.8371	1.2253	2.6406	3.0570	1.2406
	50	2.9109	3.1791	1.2109	2.5365	3.0880	1.1365
	100	3.0194	3.1529	1.3194	2.3417	2.2501	0.9417
	200	2.5782	2.0977	0.8782	2.1406	1.8240	0.7406
	400	2.4199	1.5742	0.7199	1.8509	0.9593	0.4509
	800	2.4103	1.8267	0.7103	1.7357	0.8952	0.3357
θ	25	2.4611	1.7876	0.7611	2.7401	1.7892	0.2401
	50	2.3908	1.9793	0.6908	2.8919	1.9940	0.3919
	100	2.1938	1.4321	0.4938	2.8163	1.7707	0.3163
	200	2.1029	1.5699	0.4029	2.7981	1.6590	0.2981
	400	1.8952	1.0177	0.1952	2.9987	1.4771	0.4987
	800	1.7761	0.7009	0.0761	2.7885	1.0780	0.2885
γ	25	1.4075	1.4386	-0.0925	1.4075	1.3277	-0.0925
	50	1.6038	1.4410	0.1038	1.5058	1.4161	0.0058
	100	1.5131	1.5216	0.0131	1.5051	1.5272	0.0051
	200	1.5779	1.2290	0.0779	1.4777	1.1434	-0.0223
	400	1.5440	1.0784	0.0440	1.4763	1.0494	-0.0237
	800	1.5195	0.8347	0.0195	1.4900	0.7221	-0.0100

Table 6.3: Simulation Results

Table 6.2 and Table 6.3 shows the simulation results for some randomly selected parameter values of the WTL-LLoG-Poisson distribution. We noted that the mean estimates of the parameters are approaching the true parameter values with an increase in the sample size. Also, the RMSEs and the ABias are decreasing with increasing sample size, which suggest strong consistency of the MLE of the model parameters.

6.1.7 Applications

In this subsection, we present examples to illustrate the importance and flexibility of the WTL-LLoG-Poisson distribution in data modeling. The WTL-LLoG-Poisson distribution is applied to two data sets and these fits are compared to the fits of the Weibull-Lomax (WL) distribution (Tahir et al. [123]) and the Exponentiated Weibull-Lindley (EWL) distribution (Cordeiro et al. [49]), the Topp-Leone-Weibull-Poisson (TL-WP) distribution by Chipepa et al. [84], Log-logistic-Weibull-Poisson (LLoGWP) distribution by Oluyede et al. [100] and the Lindley-Burr XII-Poisson (LBXIIP) distribution by Makubate et al. [83], The non-nested pdfs are given in

the appendix:

The selection of the best statistical model is based on some GoF statistics measures such as the $-2\log(L)$, AIC, CAIC, BIC, W^* , A^* , see (Chen and Balakrishnan [36]), the K-S and its p-value, and the SS from the probability plots. Also, plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [34]) are given in Figure 6.17 and Figure 6.22. Plots of the estimated cdf, Kaplan-Meier (K-M) survival function are given in Figure 6.18 and Figure 6.23. Figures 6.19 and Figure 6.24 shows the estimated hrfs and the TTT-Transform plots. The probability plot were created by plotting $F(x_{(j)}; \hat{b}, \hat{\beta}, \hat{\theta}, \hat{\varphi})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness was obtained and is given by the sum of squares

$$SS = \sum_{j=1}^n \left[F(x_{(j)}; \hat{b}, \hat{\beta}, \hat{\theta}, \hat{\varphi}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

Note that, the smaller the values of the GoF statistics measures: $-2\log(L)$, AIC, CAIC, BIC, W^* , A^* , $K - S$ and a large p-value, the better the fit. These measures together with the MLEs of the parameters with SEs in parenthesis are given in Table 6.4, Table 6.5 and Table 6.6 .

COVID-19 Survival Times Data

The first data set refers to 53 COVID-19 survival times of patients in critical conditions in China in the first two months of 2020. Liu et al. [77] also analysed the data. The times measured from the admission to the hospital until death are: 0.054, 0.064, 0.087, 0.087, 0.235, 0.352, 0.364, 0.421, 0.437, 0.458, 0.479, 0.548, 0.568, 0.704, 0.787, 0.796, 0.816, 0.865, 0.976, 0.976, 0.978, 1.756, 1.978, 2.089, 2.643, 2.869, 3.079, 3.348, 3.543, 3.646, 3.867, 3.890, 4.092, 4.093, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.058, 7.274, 8.273, 9.324, 10.827, 11.282, 13.324, 14.278, 15.287, 16.978, 17.209, 19.092, 20.083.

The var-cov matrix for WTL-LLoG-Poisson distribution on the COVID-19 survival times data is given by

$$\begin{bmatrix} 0.00424 & -0.0005 & 0.08496 & 0.00038 \\ -0.0005 & 0.00006 & -0.01103 & -0.00004 \\ 0.08496 & -0.01103 & 1.97079 & 0.00690 \\ 0.00038 & -0.00004 & 0.00690 & 0.00005 \end{bmatrix}$$

and the 95% CIs for the model parameters are given by

$$b \in [1.7282 \pm 0.1276], \beta \in [19.3044 \pm 0.0154], \theta \in [0.3136 \pm 2.7515] \text{ and } \gamma \in [0.0448 \pm 0.0133].$$

Distribution	Estimates				statistics							
	b	β	θ	γ	$-2 \ln(L)$	AIC	CAIC	BIC	W^*	A^*	K-S	P-value
WTL-LLoGP	1.7282 (0.0651)	19.3044 (0.0079)	0.3136 (1.4038)	0.0448 (0.0068)	266.6818	274.6818	275.5152	282.563	0.0759	0.4689	0.1221	0.4083
WTL-LLoGP (β, θ, γ)	1	0.5724 (0.1039)	3.3849×10^{-8} (0.0191)	1.3649 (0.3762)	324.5035	330.5035	330.9933	336.4144	0.2523	1.5053	0.4041	6.1040×10^{-8}
WTL-LLoGP (b, θ, γ)	3.5962 (0.3518)	1	6.1910×10^{-9} (0.0142)	0.5715 (0.1009)	274.887	280.887	281.3768	286.7979	0.1630	0.9913	0.1465	0.2053
WTL-LLoGP (β, θ)	1	0.6974 (0.0597)	9.1364×10^{-9} (0.0101)	1	326.6509	330.6509	330.8909	334.5915	0.1953	1.1674	0.4105	3.4980×10^{-8}
WL	0.0214 (0.064906)	1.6095 (1.746235)	0.4354 (0.550872)	0.0137 (0.059001)	266.0294	274.0294	274.8627	281.9106	0.0720	0.4403	0.1166	0.4673
EWL	α 0.0008 (0.0005)	v 4.5968 (0.0005)	a 43.6210 (2.1194×10^{-7})	b 0.2460 (0.0246)	267.8402	275.8402	276.6735	283.7214	0.0842	0.5204	0.1248	0.3812
TL-WP	α 0.1670 (0.2576)	β 0.7783 (0.4650)	b 1.0246 (1.0143)	θ 6.4704×10^{-9} (0.0084)	266.7983	274.7983	275.6316	282.6794	0.0757	0.4762	0.1250	0.3793
LBXHP	λ 0.3304 (0.0751)	c 0.5064 (0.1393)	k 1.8498 (0.3939)	θ 3.8140×10^{-9} (0.0302)	382.6111	390.6116	391.445	398.4928	0.0841	0.5654	0.4948	1.07×10^{-11}

Table 6.4: Estimates and GoF Statistics for the COVID-19 Survival Times Data

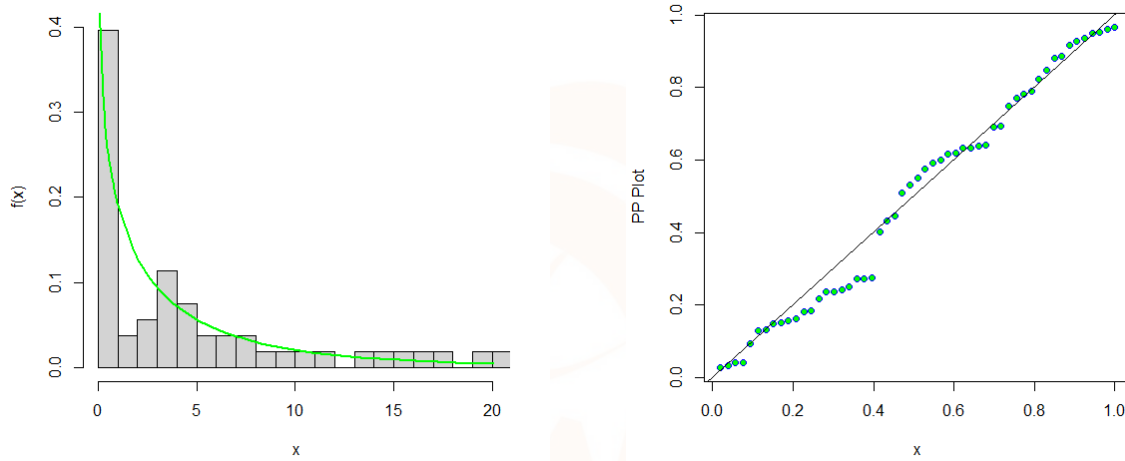


Figure 6.10: Fitted densities and probability plots for COVID-19 survival times data

Table 6.4 shows the parameter estimates (SE in parenthesis) and some GoF statistics. The WTL-LLoG-Poisson model has the lowest values of $-2\log(L)$, AIC, CAIC, BIC, W^* , A^* and K-S statistics. Also, the WTL-LLoG-Poisson distribution has the largest p-value of the K-S statistic. We can conclude that the WTL-LLoG-Poisson model performs better than the nested and non-nested models considered.

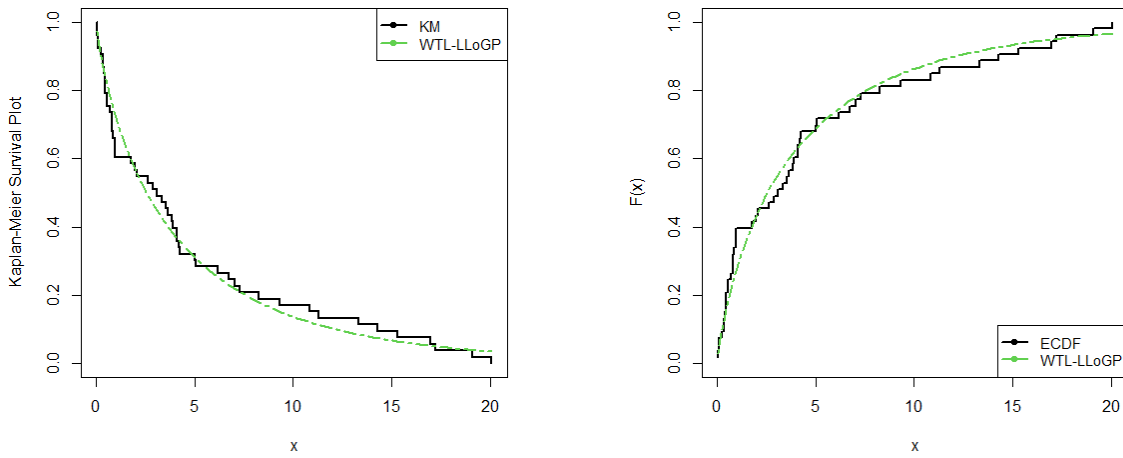


Figure 6.11: Fitted K-M Survival and CDF plots for COVID-19 survival times data

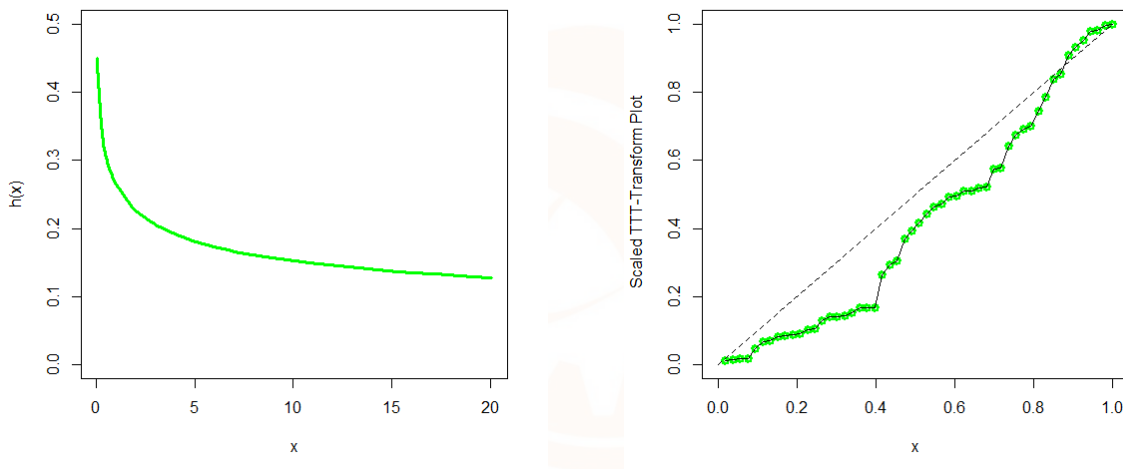


Figure 6.12: Fitted hrf and TTT plots for COVID-19 survival times data

Figures 6.10(a) and 6.10(b) shows the fitted density and probability plots of the active repair times data set using the WTL-LLoG-Poisson distribution. Figures 6.11(a) shows the survival function of the WTL-LLoG-Poisson model superimposed on the K-M plot and 6.11(b) shows the fitted cdf plotted on the empirical cdf. We can see that these plots are following the empirical plots closely. Figure 6.12(a) and Figure 6.12(b) shows the fitted hrf and the TTT-Transform plots, all these plots provide sufficient evidence that the WTL-LLoG-Poisson is capturing the characteristics of the COVID-19 survival times set well.

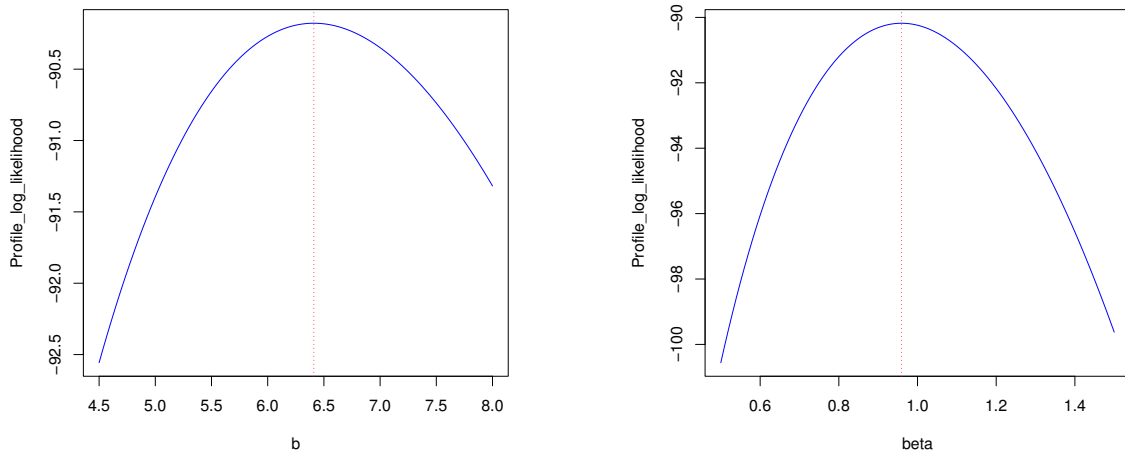


Figure 6.13: Profile plots for the covid-19 data

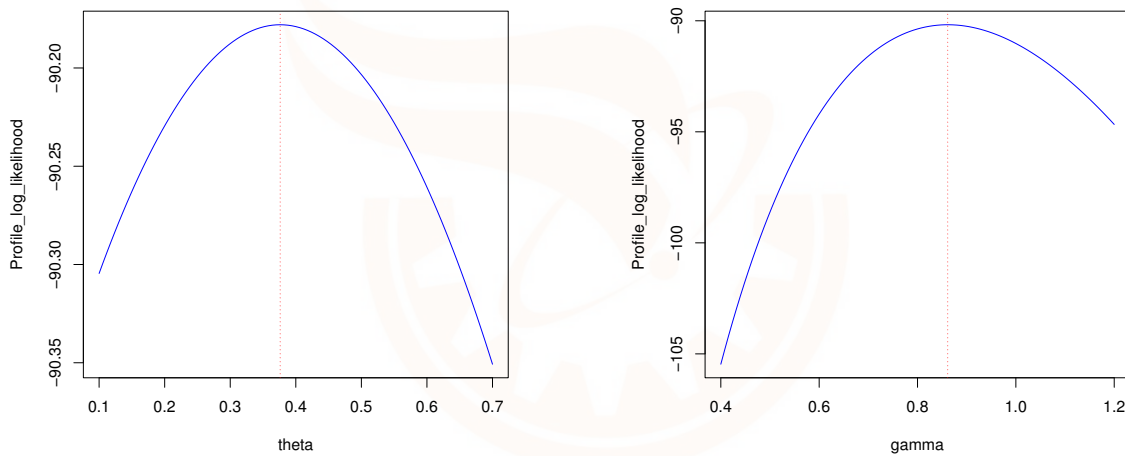


Figure 6.14: Profile plots for the covid-19 data

The profile plots in Figures 6.13 and 6.14 shows that the estimates of the WTL-LLoG-Poisson distribution are global maximums for the Covid-19 survival times data.

Active Repair Times Data

The second data set represents active repair times (hours) for an airborne communication transceiver, it was also analyzed by Jorgensen [68]. The data are: 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

The estimated parameters (SE of estimates in parenthesis) and the GoF statistics are presented in Table 6.5. The var-cov matrix for WTL-LLoG-Poisson distribution on the active repair times

(hours) data is given by

$$\begin{bmatrix} 3.5596 & -0.4210 & 0.2463 & 0.3056 \\ -0.4210 & 0.0890 & 0.0321 & -0.0712 \\ 0.2463 & 0.0321 & 0.2906 & -0.0515 \\ 0.3056 & -0.0712 & -0.0515 & 0.0716 \end{bmatrix}$$

and the 95% CIs for the model parameters are given by

$b \in [6.1180 \pm 3.6979]$, $\beta \in [0.9161 \pm 0.5848]$, $\theta \in [4.1603 \times 10^{-7} \pm 1.0566]$ and $\gamma \in [0.9262 \pm 0.5243]$.

Distribution	Estimates				statistics							
	b	β	θ	γ	-2 ln(L)	AIC	CAIC	BIC	W*	A*	K-S	P-value
WTL-LLoGP	6.1180 (1.8867)	0.9161 (0.2984)	4.1603×10^{-7} (0.5391)	0.9262 (0.2675)	180.3997	188.3997	189.5425	195.1552	0.0683	0.4433	0.1048	0.7717
WTL-LLoGP (β, θ, γ)	1	0.5427 (0.1029)	1.0438×10^{-9} (0.0045)	0.7941 (0.6421)	239.5521	245.5521	246.2187	250.6187	0.0832	0.5238	0.5169	1.0400×10^{-9}
WTL-LLoGP (b, θ, γ)	2.2207 (0.3518)	1	1.1266×10^{-8} (0.0142)	1.1963 (0.1009)	206.9595	212.9603	213.6270	218.0270	0.0667	0.4526	0.3461	0.0001
WTL-LLoGP (β, θ)	1	0.7679 (0.0749)	1.5160×10^{-8} (0.0160)	1	247.0630	251.0630	251.3873	254.4407	0.0673	0.4502	0.5897	1.6640×10^{-12}
WL	1808.300 (2.7994×10^{-5})	2.6055 (1.0821)	0.0221 (0.0135)	0.3119 (0.4861)	182.6578	190.6578	191.8006	197.4133	0.0751	0.5398	0.1186	0.6267
EWL	6.1176×10^{-5} (6.4071×10^{-5})	51.1130 (1.4660×10^{-6})	97.4950 (5.2250×10^{-7})	0.1758 (0.020261)	182.8809	190.8809	192.0238	197.6364	0.0786	0.5561	0.1302	0.5061
TL-WP	0.1826 (0.0774)	0.8197 (0.0138)	1.0540 (0.3064)	1.3388×10^{-8} (0.0135)	191.9508	199.9501	201.093	206.7056	0.1288	0.9335	0.1715	0.1901
LLOGWP	1.4155 (0.1717)	0.00001 (2.8894×10^{-9})	0.4743 (0.2206)	2.6307 (0.6642)	184.0882	192.0879	193.2307	198.8434	0.0829	0.5509	0.1099	0.7185
LBXHP	0.0308 (0.0751)	0.7010 (0.1393)	1.5834 (0.3939)	2.3302×10^{-8} (0.0302)	236.8118	244.8119	245.9547	251.5674	0.0676	0.4803	0.5433	1.112×10^{-10}

Table 6.5: Estimates and GoF Statistics for the Active Repair Times Data

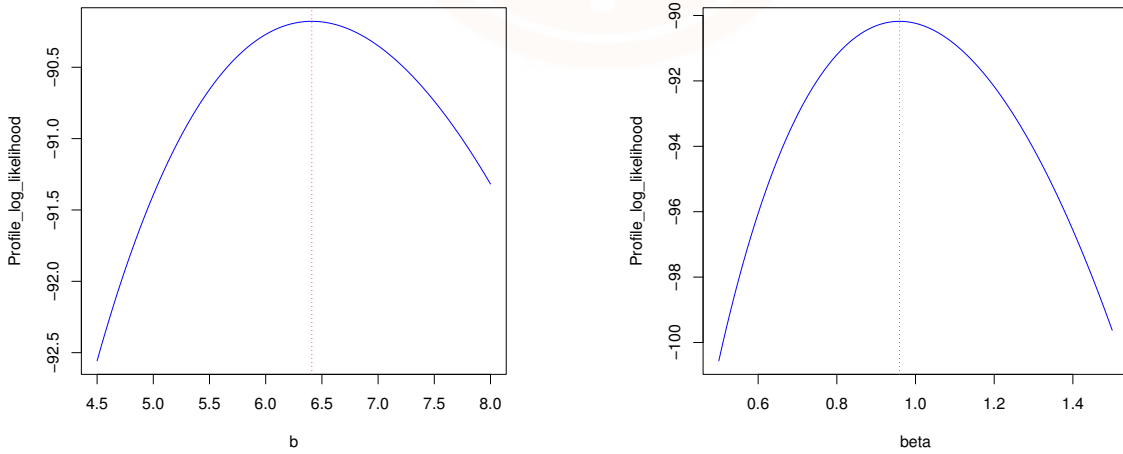


Figure 6.15: Profile plots for the active repair times data

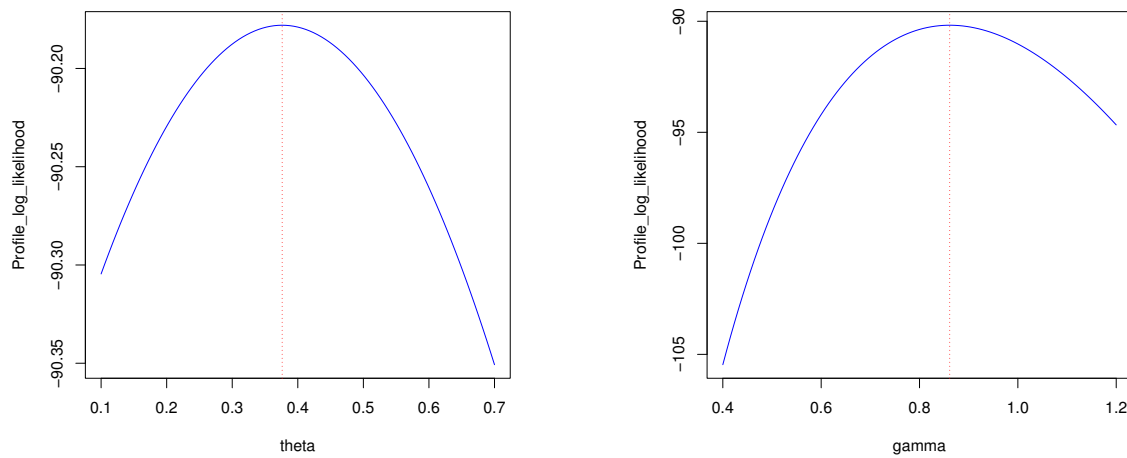


Figure 6.16: Profile plots for the active repair times data

The profile plots in Figures 6.20 and 6.21 shows that the estimates of the WTL-LLoG-Poisson distribution are global solutions for the active repair times data.

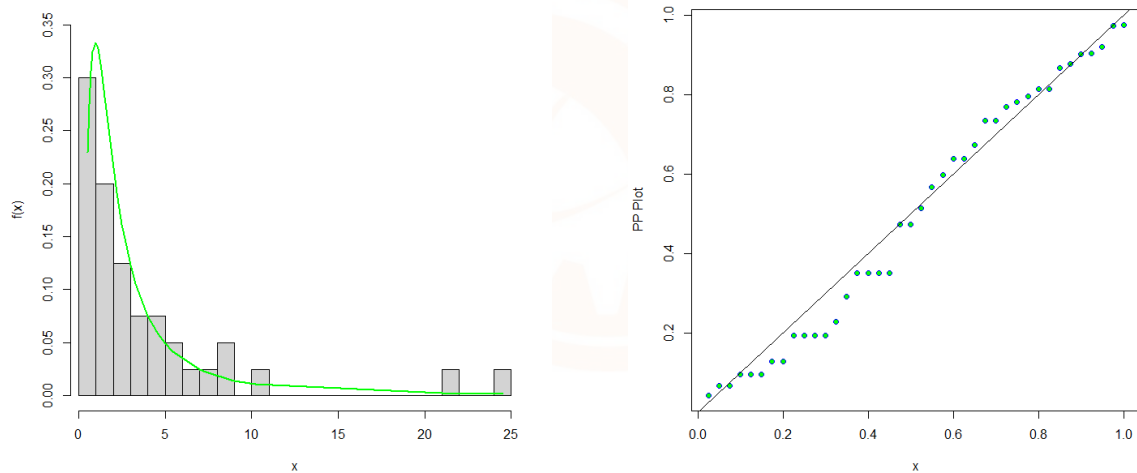


Figure 6.17: Fitted densities and probability plots for active repair times data

Table 6.5 shows the parameter estimates (SE in parenthesis) and some GoF statistics. The WTL-LLoG-Poisson model has the lowest values of the GoF statistics. Also, the WTL-LLoG-Poisson distribution has the largest p-value of the K-S statistic. We can conclude that the WTL-LLoG-Poisson model performs better than the nested and non-nested models considered.

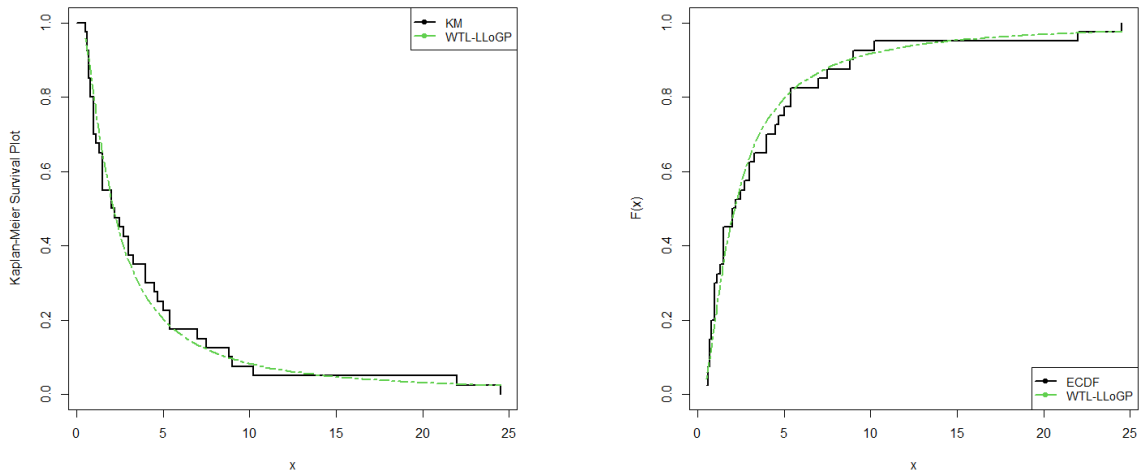


Figure 6.18: Fitted K-M Survival and CDF plots for active repair times data

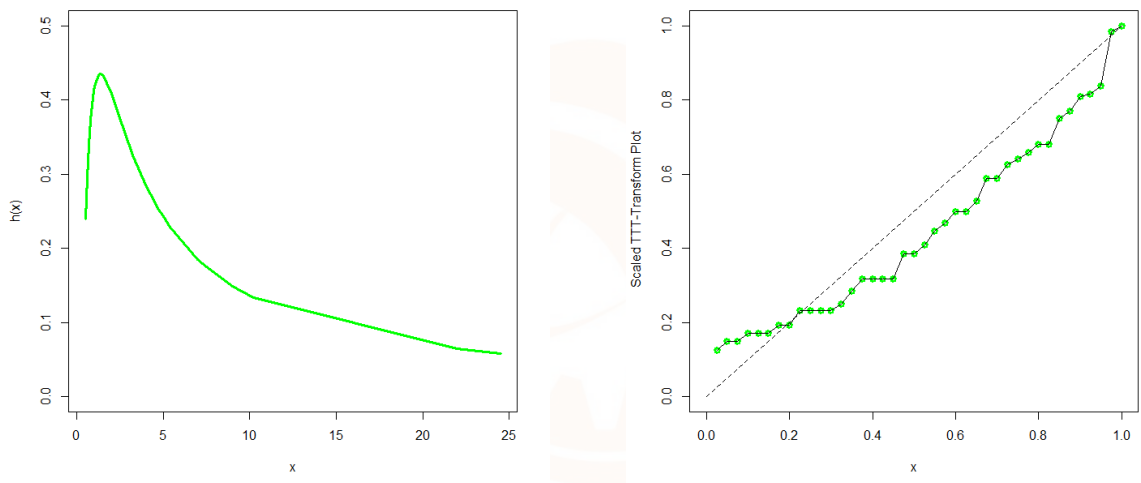


Figure 6.19: Fitted hrf and TTT plots for active repair times data

Figures 6.17(a) and 6.17(b) shows the fitted density and probability plots of the active repair times data set using the WTL-LLoG-Poisson distribution. Figure 6.18(a) shows the survival function of the WTL-LLoG-Poisson model superimposed on the K-M plot and Figure 6.18(b) shows the fitted cdf plotted on the empirical cdf. We can see that these plots are following the empirical plots closely. Figure 6.19(a) and Figure 6.19(b) shows the fitted hrf and the TTT-Transform plots, all these plots provide sufficient evidence that the WTL-LLoG-Poisson is capturing the characteristics of the active repair times data set well.

Bladder Cancer Patients Data

The third data consists of the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [75]. The data are: 0.08, 4.98, 25.74, 3.70, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.50, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85,

3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.40, 11.98, 6.54, 6.93, 6.94, 0.40, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26, 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.50, 2.02, 0.20, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26, 4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.90, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07.

The estimated parameters (SE of estimates in parenthesis) and the GoF statistics are presented in Table 6.6. The var-cov matrix for WTL-LLoG-Poisson model on the bladder cancer patients data is given by

$$\begin{bmatrix} 0.9839 & -1.0145 & -0.0072 & 0.0874 \\ -1.0145 & 1.1225 & 0.0327 & -0.0931 \\ -0.0072 & 0.0327 & 0.0411 & -0.0024 \\ 0.0874 & -0.0931 & -0.0024 & 0.0080 \end{bmatrix}$$

and the 95% CIs for the model parameters are given by

$b \in [3.8022 \pm 1.9442]$, $\beta \in [3.0470 \pm 2.0766]$, $\theta \in [9.6432 \times 10^{-6} \pm 0.3972]$ and $\gamma \in [0.3098 \pm 0.1753]$.

Distribution	Estimates				statistics							
	b	β	θ	γ	-2ln(L)	AIC	CAIC	BIC	W^*	A^*	K-S	P-value
WTL-LLoGP	3.8022 (0.9919)	3.0470 (1.0595)	9.6432×10^{-6} (0.2027)	0.3098 (0.0895)	822.1322	830.1322	830.4574	841.5403	0.0522	0.3429	0.0504	0.901
WTL-LLoGP (β, θ, γ)	1	5.1335×10^{-9} (0.0615)	9.2852×10^{-9} (0.0074)	1.2015 (0.2426)	1170.496	1176.496	1176.689	1185.052	0.5004	3.1335	0.6824	2.2×10^{-16}
WTL-LLoGP (b, θ, γ)	2.3449 (0.2095)	1	2.4126×10^{-8} (0.0103)	0.4888 (0.0336)	1010.229	1016.229	1016.423	1024.785	0.1718	1.1431	0.5327	2.2×10^{-16}
WTL-LLoGP (β, θ)	1	0.5476 (0.0316)	1.2913×10^{-8} (0.0087)	1	1167.249	1171.249	1171.345	1176.953	0.3907	2.4850	0.6771	2.2×10^{-16}
WL	94.9827 (0.0058)	6.1701 (2.3846)	0.0544 (0.0101)	0.0068 (0.0203)	823.4175	831.4175	831.7427	842.8256	0.0692	0.4445	0.0569	0.8025
EWL	α 0.0452 (0.0057)	v 0.0776 (0.0167)	a 0.0259 (0.0111)	b 2.8300 (0.5557)	972.8692	980.8692	981.1944	992.2773	0.7088	4.3423	0.32617	2.97×10^{-12}
TL-WP	α 0.1176 (0.0301)	β 0.7209 (0.0704)	b 1.0207 (0.1402)	θ 1.1352×10^{-8} (0.0074)	854.1563	862.1563	873.5613	873.5613	0.0662	0.4102	0.2064	3.672×10^{-5}
LLOGWP	c 0.0052 (0.0273)	α 0.1503 (0.0304)	θ 1.0075 (0.0601)	θ 4.7724 (0.6144)	867.7323	875.7323	876.0575	887.1404	0.5304	3.2348	0.1366	0.0169
LBXIIP	λ 0.0001 (0.0120)	c 0.5813 (0.3455)	k 1.5207 (1.1232)	θ 1.4878×10^{-9} (0.0032)	1101.164	1109.172	1109.497	1120.58	0.1489	0.9984	0.6431	2.2×10^{-16}

Table 6.6: Estimates and GoF Statistics for Bladder Cancer Patients Data

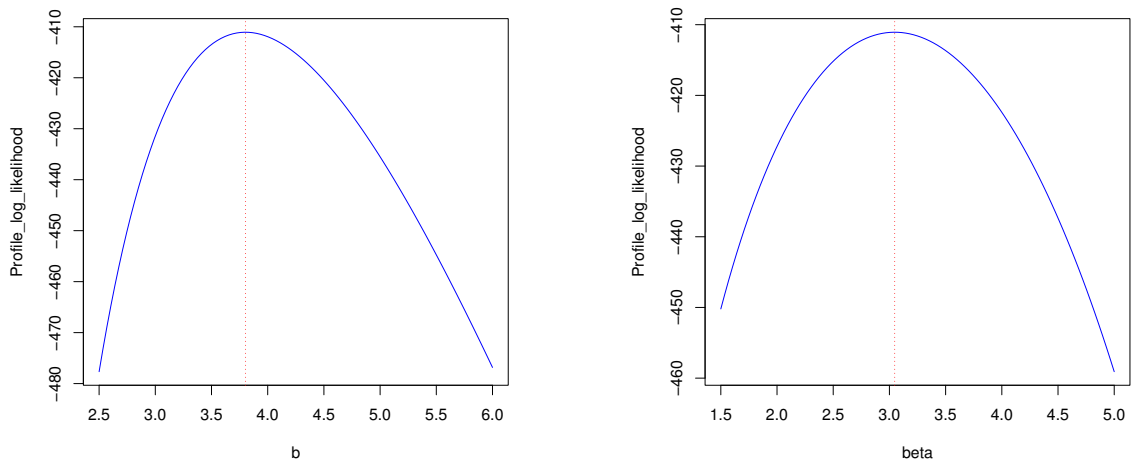


Figure 6.20: Profile plots for the bladder cancer patients remission times data

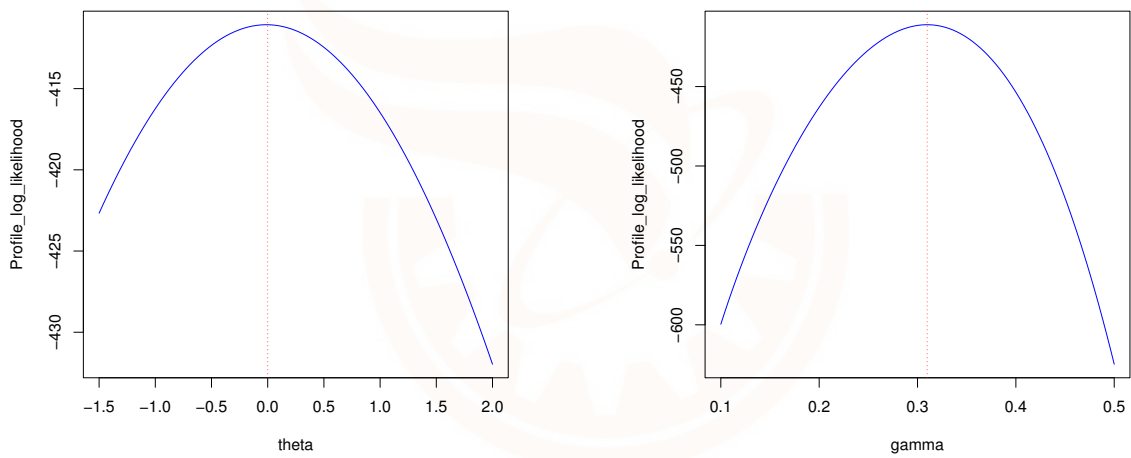


Figure 6.21: Profile plots for the bladder cancer patients remission times data

The profile plots in Figure 6.20 and Figure 6.21 shows that the estimates of the WTL-LLoG-Poisson distribution are global maximums for the bladder cancer patients remission times data.

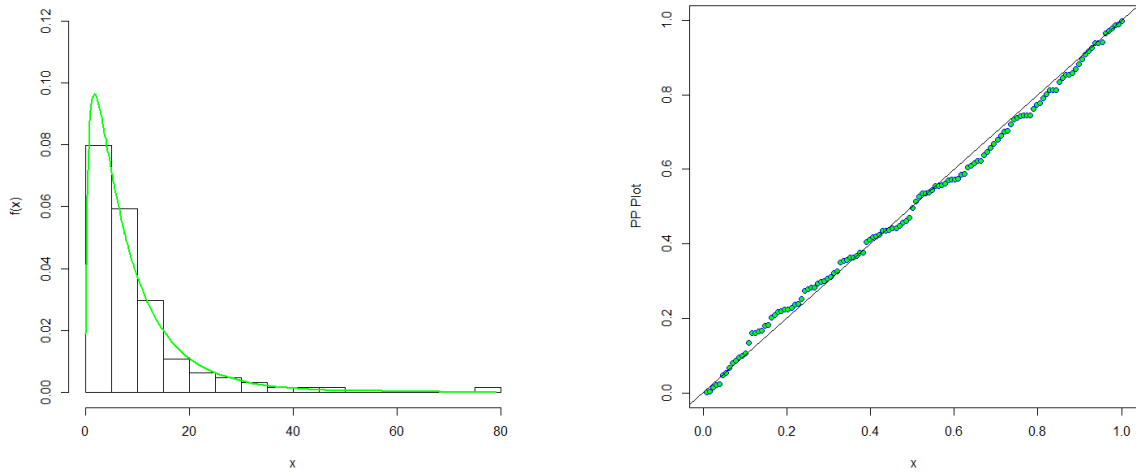


Figure 6.22: Fitted densities and probability plots for bladder cancer patients data

Presented in Table 6.6 are some GoF statistics. From these statistics, we can infer that the WTL-LLoG-Poisson model performs better than the nested and non-nested models presented since it has the lowest values for all the GoF statistics and the largest value of the p-value of the K-S statistic.

Figures 6.22(a) and 6.22(b) shows the fitted density and probability plots of the WTL-LLoG-Poisson distribution for the bladder cancer patients data set. Figure 6.23(a) shows the survival function of the WTL-LLoG-Poisson model superimposed on the K-M plot and Figure 6.23(b) shows the fitted cdf plotted on the empirical cdf where we can see that the WTL-LLoG-Poisson model is doing well in modelling and capturing the characteristics of the bladder cancer patients. Figure 6.24(a) and Figure 6.24(b) shows the fitted hrf and the TTT-Transform plots. The fitted hrf exhibits an inverted bathtub shape, all pointing to the flexibility of the WTL-LLoG-Poisson model in data modelling.

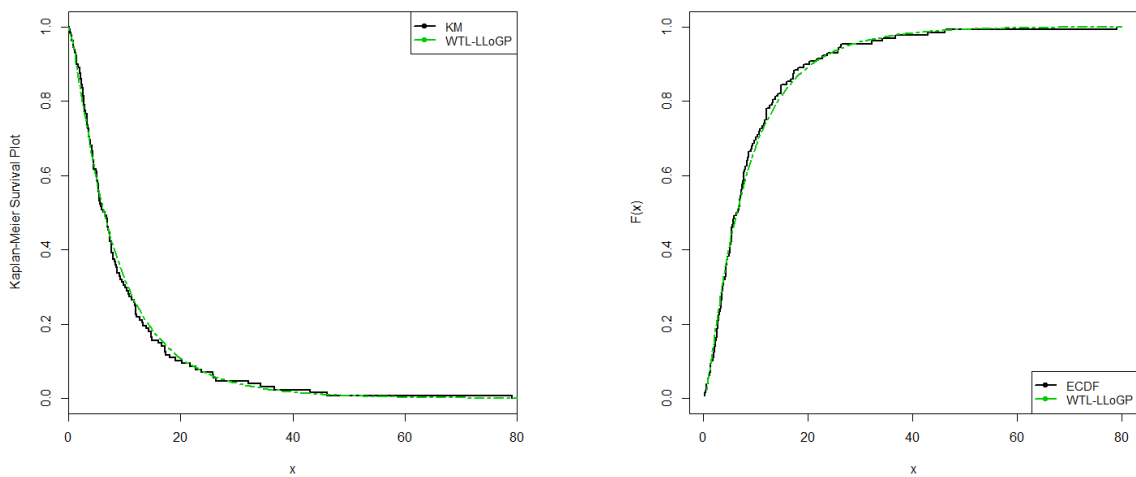


Figure 6.23: Fitted K-M Survival and CDF plots for bladder cancer patients data

Model	df	Covid-19 survival times		Active repair times data set		Bladder cancer patients data set	
		Chi-Square	p-value	Chi-Square	p-value	Chi-Square	p-value
WTL-LLoGP (β, θ, γ)	1	57.8217	< 0.00001	59.1524	< 0.00001	348.3638	< 0.00001
WTL-LLoGP (b, θ, γ)	1	8.2025	0.00418	26.5598	< 0.00001	188.0968	< 0.00001
WTL-LLoGP (β, θ)	2	59.9691	< 0.00001	66.6633	< 0.00001	345.1168	< 0.00001

Table 6.7: Likelihood Ratio Test Results

The likelihood ratio (LR) test results presented in Table 6.7 indicates that there is a significant difference between the WTL-LLoGP and its nested models (WTL-LLoGP (β, θ, γ) , WTL-LLoGP (b, θ, γ) and WTL-LLoGP (β, θ)) for the three data sets data sets. We can conclude that the WTL-LLoGP distribution provides the best fit to the Covid-19 survival times data, active repair times data and the bladder cancer patients data compared to the models presented in this paper.

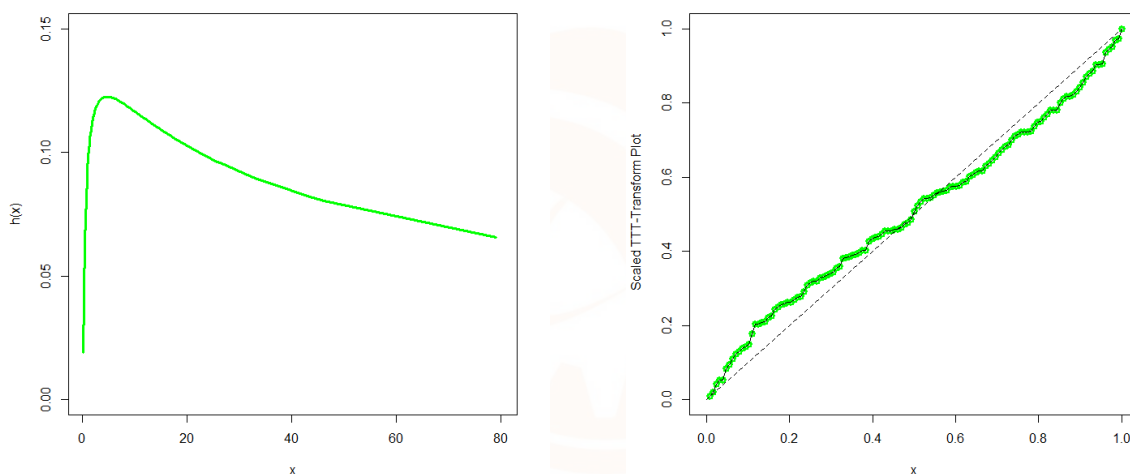


Figure 6.24: Fitted hrf and TTT plots for bladder cancer patients data

6.1.8 Concluding Remarks

We developed a new generalized FoD called the Weibull Topp-Leone-G Power Series (WTL-GPS) CoD. The WTL-GPS CoD possesses hrf that is quite flexible and captures both monotone and non-monotone shapes. Expressions for the moments, distribution of order statistics, incomplete moments and Rényi entropy were also obtained. The method of maximum likelihood estimation (MLE) was used to estimate the model parameters. The performance of the special case of the WTL-LLoG-Poisson distribution was evaluated by conducting various simulations for different sample sizes. The special case of the WTL-LLoG-Poisson distribution was fitted to real data sets to demonstrate the applicability and potentiality of the proposed CoD.

6.2 The Burr III-Exponentiated Half Logistic-G Family of Distributions with Applications

Oluyede, B. and Sengweni, W., The Burr III Exponentiated Half-Logistic-G Family of Distributions. (*Submitted: Pakistan Journal of Statistics and Operations Research*)

6.2.1 Introduction

In this chapter, we develop and study a new generalized FoD called Burr III- Exponentiated Half Logistic-G (BIII-EHL-G) FoD. The Burr III distribution was introduced by Burr [32] to model lifetime data or survival data. The Burr III distribution has wide applications in the scientific fields ranging from income, wealth and wage distributions, forestry, meteorology, survival analysis, environmental studies and reliability theory (Burr and Cislak [31], MielKe [87], Mokhlis [88] and Gove et al. [62]). Kanak and Vinod [80] introduced a new unit Burr III distribution. The Burr III distribution has a pdf and cdf are respectively given by

$$g(x; a, b) = abx^{-b-1}(1 + x^{-b})^{-a-1} \quad (6.26)$$

and

$$G(x; a, b) = (1 + x^{-b})^{-a}, \quad (6.27)$$

for $a > 0, b > 0$ and $x > 0$.

By generalizing the Burr III distribution and the exponentiated half logistic distribution we obtained a very flexible FoD called the Burr III- Exponentiated Half Logistic-G (BIII-EHL-G) distribution which has a density function that has diverse shapes: left-skewed, almost symmetric, reversed-J shape and right-skewed. The hrf for the BIII-EHL-G FoD it can capture both non-monotonic and monotonic shapes such as decreasing, increasing, bathtub, inverted bathtub, and bathtub followed by inverted bathtub shapes. The flexibility we gained by the generalization of these distribution has motivated us to come up with this new FoD.

This Section is arranged as follows. In subsection 6.2.2 we provide basic results, the BIII-EHL-G distribution and its sub-families, hrf, and the qf are presented. The moments and mgf are given in subsection 6.2.3. Subsection 6.2.4 results on the distribution of order statistics and Rényi entropy. In subsection 6.2.5, results on the estimation of the parameters of the BIII-EHL-G FoD via the method of maximum likelihood are presented. A Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators in subsection 6.2.6. Some examples on the applicability of the model are given in Section 6.2.7, finally some concluding remarks are provided in subsection 6.2.8.

6.2.2 The Generalized Model and Its Mathematical Properties

In this subsection, some mathematical and statistical properties of the BIII-EHL-G FoD are presented, including hrf and qf, and expansions of the density function.

The New Generalized Family of Distributions

The cdf and pdf of the proposed model are respectively given by

$$\begin{aligned} F(x; \alpha, \beta, \delta, \zeta) &= \int_0^{-\log \left(1 - \left[W_G(x; \zeta) \right]^\delta \right)} \alpha \beta t^{-\beta-1} (1+t^{-\beta})^{-\alpha-1} dt \\ &= \left[1 + \left(-\log \left(1 - \left[W_G(x; \zeta) \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha} \end{aligned} \quad (6.28)$$

and

$$\begin{aligned} f(x; \alpha, \beta, \delta, \zeta) &= \frac{2\alpha\delta\beta g(x; \zeta) G^{\delta-1}(x; \zeta) \left[1 + \left(-\log \left(1 - \left[W_G(x; \zeta) \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha-1}}{(1 + \overline{G}(x; \zeta))^{\delta+1} \left[1 - \left[W_G(x; \zeta) \right]^\delta \right]} \\ &\quad \times \left(-\log \left(1 - \left[W_G(x; \zeta) \right]^\delta \right) \right)^{-\beta-1}, \end{aligned} \quad (6.29)$$

where $W_G(x; \zeta) = \frac{G(x; \zeta)}{1 + \overline{G}(x; \zeta)}$, $\alpha, \beta, \delta > 0$ and parameter vector ζ .

Hazard Rate and Quantile Functions

In this subsection, we present expressions for the hrf and the qf. We set $f(x; \alpha, \beta, \delta, \zeta) = f(x)$ for convenience. The hrf of the BIIE-EHL-G FoD is given by

$$\begin{aligned} h(x) &= \frac{2\alpha\delta\beta g(x; \zeta) G^{\delta-1}(x; \zeta) \left[1 + \left(-\log \left(1 - \left[W_G(x; \zeta) \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha-1}}{(1 + \overline{G}(x; \zeta))^{\delta+1} \left[1 - \left[W_G(x; \zeta) \right]^\delta \right]} \\ &\quad \times \left(1 - \left[1 + \left(-\log \left(1 - \left[W_G(x; \zeta) \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha} \right)^{-1} \\ &\quad \times \left(-\log \left(1 - \left[W_G(x; \zeta) \right]^\delta \right) \right)^{-\beta-1}. \end{aligned} \quad (6.30)$$

The qf $Q(u)$ is obtained by solving the non-linear equation

$$F_{BIIE-EHL-G}(Q(u)) = \left[1 + \left(-\log \left(1 - \left[\frac{G(Q(u))}{1 + \overline{G}(Q(u))} \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha} = u,$$

for $0 < u < 1$.

Thus, the qf of the BIIE-EHL-G FoD is given by

$$Q(u) = G^{-1} \left[\frac{2(1 - \exp(-(u^{\frac{-1}{\alpha}} - 1)^{\frac{-1}{\beta}}))}{2 - \exp(-(u^{\frac{-1}{\alpha}} - 1)^{\frac{-1}{\beta}}))} \right],$$

for $0 < u < 1$. This function is extremely vital in many statistical applications and Monte-Carlo methods. It is used to generate random numbers from a distribution.

Density Expansion

In this subsection, we present the pdf of the BIII-EHL-G FoD as an infinite linear combination of exponentiated-G densities. The BIII-EHL-G pdf is

$$\begin{aligned}
 f(x) &= 2\alpha\delta\beta g(x; \zeta) G^{\delta-1}(x; \zeta) \left[1 + \left(-\log \left(1 - \left[W_G(x; \zeta) \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha-1} \\
 &\times (1 + \overline{G}(x; \zeta))^{-(\delta+1)} \left[1 - \left[W_G(x; \zeta) \right]^\delta \right]^{-1} \\
 &\times \left(-\log \left(1 - \left[W_G(x; \zeta) \right]^\delta \right) \right)^{-\beta-1}.
 \end{aligned} \tag{6.31}$$

Using the series expansion $(1 - z)^{-n} = \sum_{j=0}^{\infty} \frac{\Gamma(n+j)}{\Gamma(n)j!} z^j$, equation (6.31) can be written as

$$\begin{aligned}
 f(x) &= 2\alpha\delta\beta g(x; \zeta) G^{\delta-1}(x; \zeta) \sum_{k,p=0}^{\infty} \frac{\Gamma(k+1)}{k!} (-1)^k \binom{-\alpha-1}{p} \\
 &\times (1 + \overline{G}(x; \zeta))^{-(\delta+1)} \left[\frac{G(x; \zeta)}{1 + \overline{G}(x; \zeta)} \right]^{\delta k} \\
 &\times \left(-\log \left(1 - \left[\frac{G(x; \zeta)}{1 + \overline{G}(x; \zeta)} \right]^\delta \right) \right)^{-\beta p - \beta - 1}.
 \end{aligned}$$

Let $y = \left[W_G(x; \zeta) \right]^\delta$, $0 < y < 1$. Using the series representation

$$-\log(1 - y) = \sum_{i=1}^{\infty} \frac{y^{i+1}}{i+1}, \tag{6.32}$$

we get

$$[-\log(1 - y)]^{-\beta(p+1)-1} = y^{-\beta(p+1)-1} \sum_{q=0}^{\infty} \binom{-\beta(p+1)-1}{q} y^q \sum_{s=0}^{\infty} \left(\frac{y^s}{s+2} \right)^q.$$

Applying the result on a power series raised to a positive integer with $a_s = (s+2)^{-1}$ (see, Gradshteyn and Ryzhik [63]), that is,

$$\sum_{s=0}^{\infty} a_s y^s = \sum_{s=0}^{\infty} b_{s,q} y^s,$$

where

$$b_{s,q} = (sa_0)^{-1} \sum_{i=1}^s [q(l+i) - s] a_l b_{s-l,q} \quad \text{and} \quad b_{0,q} = a_0^m. \tag{6.33}$$

Now, the pdf can be written as

$$\begin{aligned}
 f(x) &= 2\alpha\delta\beta g(x; \zeta) \sum_{k,p,q,s=0}^{\infty} \frac{\Gamma(k+1)}{k!} (-1)^k \binom{-\alpha-1}{p} \binom{-\beta(p+1)-1}{q} \\
 &\times (1 + \overline{G}(x; \zeta))^{-\delta(k+1)+\beta(p+1)-q-s} G(x; \zeta)^{\delta(k+1)-\beta(p+1)+q+s-2} \\
 &= 2\alpha\delta\beta g(x; \zeta) \sum_{k,p,q,s,m=0}^{\infty} \frac{\Gamma(k+1)}{k!} (-1)^k \binom{-\alpha-1}{p} \binom{-\beta(p+1)-1}{q} \\
 &\times \binom{-\delta(k+1)+\beta(p+1)-q-s}{m} \overline{G}^m(x; \zeta) (1 - \overline{G}(x; \zeta))^{\delta(k+1)-\beta(p+1)+q+s-2} \\
 &= 2\alpha\delta\beta g(x; \zeta) \sum_{k,p,q,s,m,d,l=0}^{\infty} \frac{\Gamma(k+1)}{k!} (-1)^{k+d+l} \binom{-\alpha-1}{p} \binom{-\beta(p+1)-1}{q} \\
 &\times \binom{-\delta(k+1)+\beta(p+1)-q-s}{m} \binom{\delta(k+1)-\beta(p+1)+q+s-2}{d} \\
 &\times \binom{m+d}{l} G^l(x; \zeta) \\
 &= \sum_{l=0}^{\infty} \vartheta_{l+1}^* h_{l+1}(x; \zeta), \tag{6.34}
 \end{aligned}$$

where

$$\begin{aligned}
 \vartheta_{l+1}^* &= 2\alpha\delta\beta \sum_{k,p,q,s,m,d=0}^{\infty} \frac{\Gamma(k+1)(-1)^{k+d+l}}{k!(l+1)} \binom{-\alpha-1}{p} \binom{-\beta(p+1)-1}{q} \\
 &\times \binom{-\delta(k+1)+\beta(p+1)-q-s}{m} \binom{m+d}{l}, \tag{6.35}
 \end{aligned}$$

$h_{l+1}(x; \zeta) = (l+1)g(x; \zeta)G(x; \zeta)^l$ is the Exp-G density with power parameter $(l+1)$. Having expressed the pdf as a linear combination of Exp-G density, other mathematical properties of the BIIE-EHL-G FoD follows from those of the Exp-G distribution.

6.2.3 Moments and Generating Function

This subsection presents ordinary moments, incomplete moments, central moments and the mgf of the BIIE-EHL-G FoD are presented. Let Y_{l+1} be an Exp-G distribution with power parameter $l+1$, then the r^{th} ordinary moment of the BIIE-EHL-G FoD is given by

$$\mu_r' = E(X^r) = \sum_{l=0}^{\infty} \vartheta_{l+1}^* E(Y_{l+1}^r), \tag{6.36}$$

where ϑ_{l+1}^* is as given in equation (6.35) and $E(Y_{l+1}^r)$ is the r^{th} moment of the Exp-G distribution. The r^{th} incomplete moment of the BIIE-EHL-G FoD is given by

$$\delta_r(y) = \int_{-\infty}^y x^r f(x; \alpha, \beta, \delta, \zeta) dx = \sum_{l=0}^{\infty} \vartheta_{l+1}^* \int_{-\infty}^y x^r h_{l+1}(x; \zeta) dx,$$

where $\int_{-\infty}^y x^r h_{l+1}(x; \zeta) dx$ is the r^{th} incomplete moment of the Exp-G distribution. Incomplete moments are used in the estimation of Bonferroni and Lorenz curves which are useful in the field of medicine, reliability, insurance and demography. Other measures such as the mean deviations and conditional moments can be readily obtained from the ordinary and incomplete moments. The q^{th} central moment of the BIII-EHL-G FoD is given

$$\mu_q = \sum_{r=0}^q \binom{q}{r} (-\mu'_1)^{q-r} E(X^r) = \sum_{r=0}^q \sum_{l=0}^{\infty} \vartheta_{l+1}^* \binom{q}{r} (-\mu'_1)^{q-r} E(Y_{l+1}^r).$$

The mgf $M_X(t) = E(e^{tX})$ of the BIII-EHL-G FoD can be derived from equation (6.29) and is given by

$$M_X(t) = \sum_{l=0}^{\infty} \vartheta_{l+1}^* M_{l+1}(t),$$

where $M_{l+1}(t)$ is the mgf of Y_{l+1} . Hence, $M_X(t)$ can be determined from the Exp-G generating function.

Plots of coefficient of skewness (CS) and kurtosis (CK) for the BIII-EHL-W distribution is presented in Figure 6.25 and Figure 6.26. We plotted CS and CK versus α and θ , it can be seen from the plots that by varying the values of these two parameters we capture different levels of skewness and kurtosis.

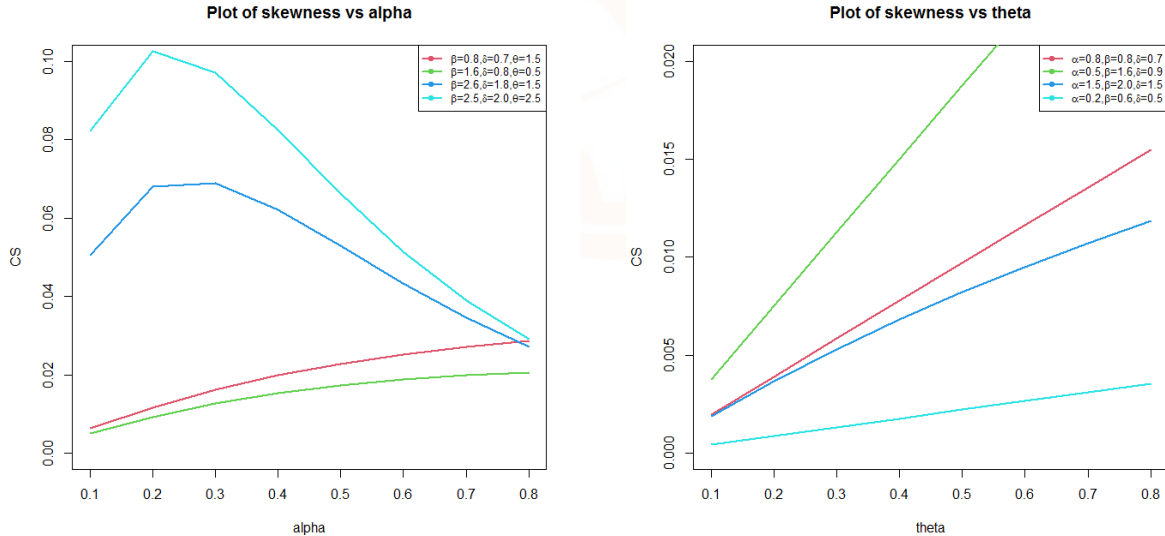


Figure 6.25: Plots of the skewness of the BIII-EHL-W distribution

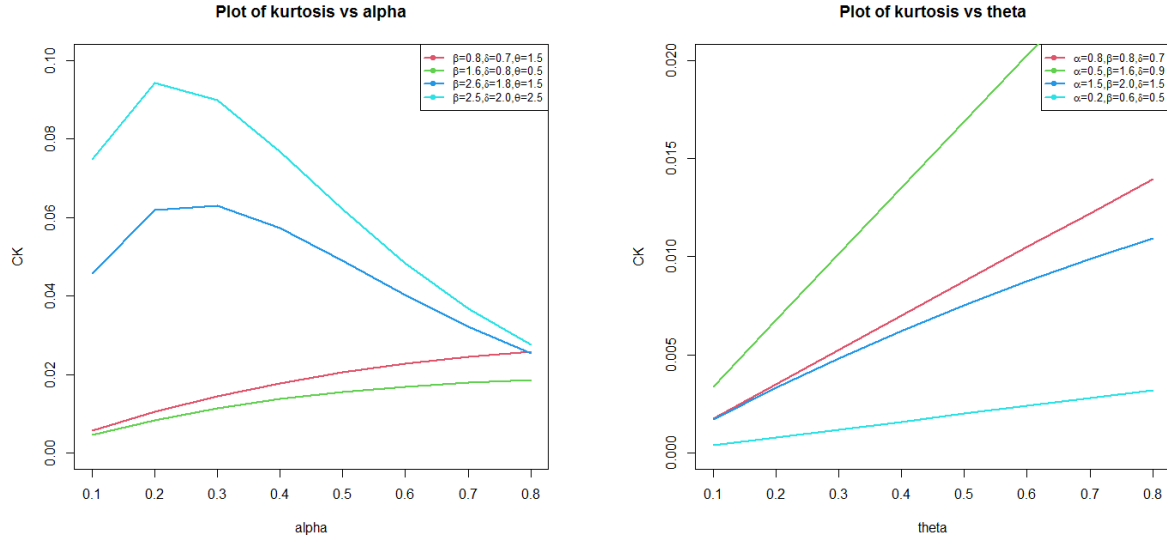


Figure 6.26: Plots of the kurtosis of the BIII-EHL-W distribution

Some Special Cases

Special cases of the BIII-EHL-G FoD are provided in this subsection. We considered the Weibull, uniform and log-logistic distributions as baseline distributions.

BIII-EHL-Weibull Distribution

If $G(x; \theta) = 1 - e^{-x^\theta}$ and $g(x; \theta) = \theta x^{\theta-1} e^{-x^\theta}$ is the baseline distribution for $\theta > 0$ and $x > 0$, then, the BIII-EHL-Weibull (BIII-EHL-W) distribution has cdf and pdf given by

$$F(x; \alpha, \beta, \delta, \theta) = \left[1 + \left(-\log \left(1 - \left[\frac{1 - e^{-x^\theta}}{1 + e^{-x^\theta}} \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha}$$

and

$$\begin{aligned} f(x; \alpha, \beta, \delta, \theta) &= \frac{2\alpha\delta\beta\theta x^{\theta-1} e^{-x^\theta} \left[1 + \left(-\log \left(1 - \left[\frac{1 - e^{-x^\theta}}{1 + e^{-x^\theta}} \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha-1}}{(1 + e^{-x^\theta})^{\delta+1} \left[1 - \left[\frac{1 - e^{-x^\theta}}{1 + e^{-x^\theta}} \right]^\delta \right]} \\ &\times \left(-\log \left(1 - \left[\frac{1 - e^{-x^\theta}}{1 + e^{-x^\theta}} \right]^\delta \right) \right)^{-\beta-1} (1 - e^{-x^\theta})^{\delta-1}, \end{aligned}$$

respectively, for $\alpha > 0, \beta > 0, \delta > 0, \theta > 0$. We plotted the pdf and the hrf of the BIII-EHL-W distribution. Figure 6.27 reveals that the pdf has a variety of shapes including left-skewed, right-skewed, reverse-J and almost symmetric. Graphs of the hrf exhibits decreasing, increasing, inverted bathtub, and bathtub followed by inverted bathtub shapes.

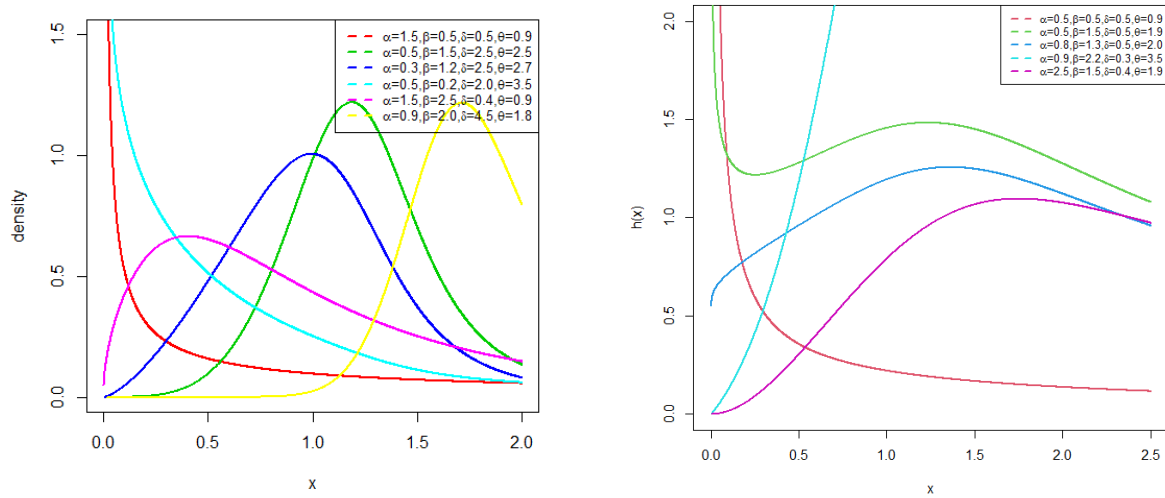


Figure 6.27: Plots of the pdf and hrf of the BIII-EHL-W distribution

BIII-EHL-Uniform Distribution

If we take the uniform distribution with cdf, $G(x; \gamma) = \frac{x}{\gamma}$ and $g(x; \gamma) = \frac{1}{\gamma}$, for $0 < x < \gamma$, as the baseline distribution, we obtain the BIII-EHL-Uniform (BIII-EHL-U) distribution with cdf and pdf given by

$$F(x; \alpha, \beta, \delta, \gamma) = \left[1 + \left(-\log \left(1 - \left[\frac{\frac{x}{\gamma}}{1 + (1 - \frac{x}{\gamma})} \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha}$$

and

$$\begin{aligned} f(x; \alpha, \beta, \delta, \gamma) &= \frac{2\alpha\delta\beta\frac{1}{\gamma}(\frac{x}{\gamma})^{\delta-1} \left[1 + \left(-\log \left(1 - \left[\frac{\frac{x}{\gamma}}{1 + (1 - \frac{x}{\gamma})} \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha-1}}{(1 + (1 - \frac{x}{\gamma}))^{\delta+1} \left[1 - \left[\frac{\frac{x}{\gamma}}{1 + (1 - \frac{x}{\gamma})} \right]^\delta \right]} \\ &\times \left(-\log \left(1 - \left[\frac{\frac{x}{\gamma}}{1 + (1 - \frac{x}{\gamma})} \right]^\delta \right) \right)^{-\beta-1}, \end{aligned}$$

respectively, respectively, for $\alpha > 0, \beta > 0, \delta > 0, \gamma > 0$. Graphs of the pdf and hrf of the BIII-EHL-U distribution are presented in Figure 6.28. The pdf can be left-skewed, right-skewed, reverse-J and almost symmetric. The hrf shows decreasing, increasing and bathtub shapes.

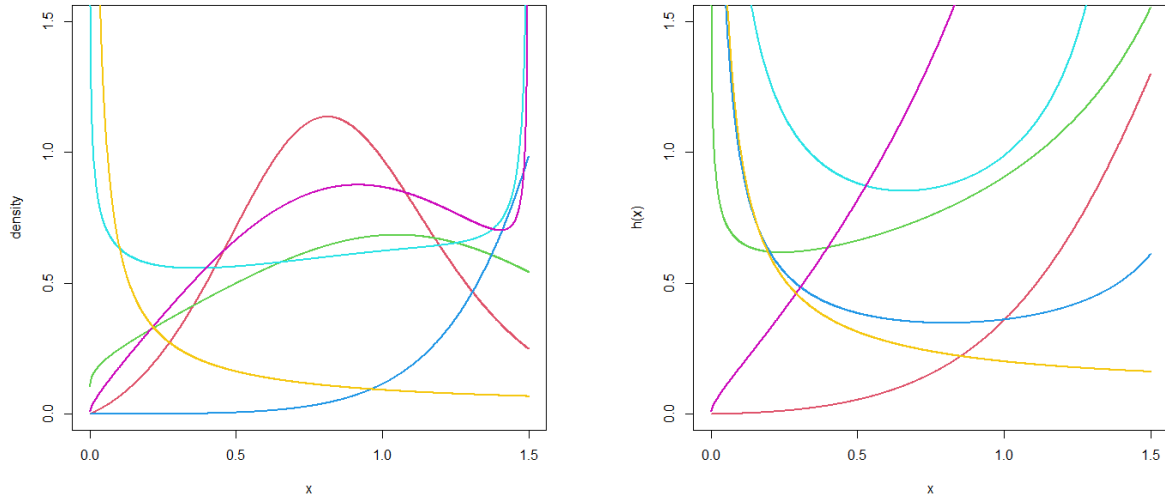


Figure 6.28: Plots of the pdf and hrf of the BIII-EHL-U distribution

BIII-EHL-Log-Logistic Distribution

Consider the log-logistic distribution as the baseline distribution. The log-logistic distribution has pdf and cdf given by $g(x) = cx^{c-1}(1+x^c)^{-2}$ and $G(x) = 1 - (1+x^c)^{-1}$, for $c > 0$, respectively. Therefore, the Burr III-Exponentiated Half Logistic-Log-Logistic (BIII-EHL-LLoG) distribution have the cdf and pdf given by

$$F(x; \alpha, \beta, \delta, c) = \left[1 + \left(-\log \left(1 - \left[\frac{1 - (1+x^c)^{-1}}{1 + (1+x^c)^{-1}} \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha}$$

and

$$\begin{aligned} f(x; \alpha, \beta, \delta, c) &= \frac{2\alpha\delta\beta cx^{c-1}(1+x^c)^{-2} \left[1 + \left(-\log \left(1 - \left[\frac{1 - (1+x^c)^{-1}}{1 + (1+x^c)^{-1}} \right]^\delta \right) \right)^{-\beta} \right]^{-\alpha-1}}{(1 + (1+x^c)^{-1})^{\delta+1} \left[1 - \left[\frac{1 - (1+x^c)^{-1}}{1 + (1+x^c)^{-1}} \right]^\delta \right]} \\ &\times \left(-\log \left(1 - \left[\frac{1 - (1+x^c)^{-1}}{1 + (1+x^c)^{-1}} \right]^\delta \right) \right)^{-\beta-1} (1 - (1+x^c)^{-1})^{\delta-1}, \end{aligned}$$

respectively, for $\alpha > 0, \beta > 0, \delta > 0, c > 0$. Plots of the pdf and hrf of the BIII-EHL-LLoG distribution are presented in Figure 6.29. The pdf assumes shapes such as right-skewed, reverse-J, almost symmetric and left-skewed. The hrf shows decreasing, increasing, and inverted bathtub shapes.

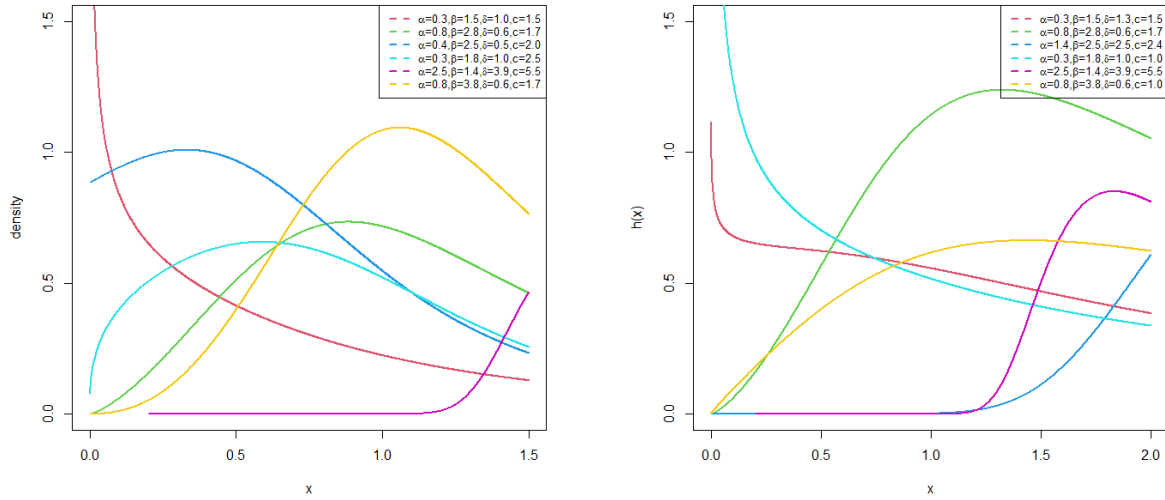


Figure 6.29: Plots of the pdf and hrf of the BIII-EHL-LLoG distribution

6.2.4 Order Statistics, Probability Weighted Moments (PWMs) and Rényi Entropy

This subsection provides expressions for the distribution of order statistics, PWMs and RE are presented.

Order Statistics

Suppose $X_1, X_2, X_3, \dots, X_n$ are i.i.d. BIII-EHL-G rvs. Then, the pdf of the i^{th} order statistics ($X_{i:n}$) can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{j+i-1}, \quad (6.37)$$

where $B(.,.)$ is the beta function. Note that

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= 2\alpha\delta\beta g(x; \zeta) G^{\delta-1}(x; \zeta) \left(-\log \left(1 - [W_G(x; \zeta)]^\delta \right) \right)^{-\beta-1} \\ &\times (1 + \overline{G}(x; \zeta))^{-(\delta+1)} \left[1 - [W_G(x; \zeta)]^\delta \right]^{-1} \\ &\times \left[1 + \left(-\log \left(1 - [W_G(x; \zeta)]^\delta \right) \right)^{-\beta} \right]^{-\alpha(j+i)-1}. \end{aligned}$$

Using the equations (6.32) and (6.33), we have

$$\begin{aligned} f(x)[F(x)]^{j+i-1} &= 2\alpha\delta\beta g(x; \zeta) \sum_{k,p,q,s,d,m,l=0}^{\infty} \frac{\Gamma(k+1)}{k!} (-1)^{k+d+l} \binom{-\alpha(j+i)-1}{p} \\ &\times \binom{-\beta(p+1)-1}{q} \binom{-\delta(k+1)+\beta(p+1)-q-s}{m} \\ &\times \binom{\delta(k+1)-\beta(p+1)+q+s-2}{d} \binom{m+d}{l} b_{s,q} G^l(x; \zeta). \quad (6.38) \end{aligned}$$

Substituting equation (6.38) into equation (6.37), we can write the pdf of $X_{i:n}$ as

$$f_{i:n}(x) = \sum_{l=0}^{\infty} \Upsilon_{l+1}^* h_{l+1}(x; \zeta), \quad (6.39)$$

where

$$\begin{aligned} \Upsilon_{l+1}^* &= 2\alpha\delta\beta \sum_{k,p,q,s,d,m=0}^{\infty} \sum_{j=0}^{n-i} \frac{\Gamma(k+1)(-1)^{k+d+l+j}}{k!B(i, n-i+1)(l+1)} \binom{-\alpha(j+i)-1}{p} \\ &\times \binom{-\beta(p+1)-1}{q} \binom{-\delta(k+1)+\beta(p+1)-q-s}{m} \\ &\times \binom{\delta(k+1)-\beta(p+1)+q+s-2}{d} \binom{m+d}{l} b_{s,q}, \end{aligned}$$

$h_{l+1}(x; \zeta) = (l+1)g(x; \zeta)G(x; \zeta)^l$ is the Exp-G density with power parameter $(l+1)$.

Probability Weighted Moments (PWMs)

The $(i, j)^{th}$ PWMs of a rv $X \sim BIII - EHL - G$, say $\wp_{i,j}$ is given by

$$\wp_{i,j} = E(X^i F(X)^j) = \int_{-\infty}^{\infty} x^i f(x) [F(x)]^j dx. \quad (6.40)$$

From equation (6.38), we can write $f(x)[F(x)]^j$ as

$$\begin{aligned} f(x)[F(x)]^j &= 2\alpha\delta\beta g(x; \zeta) \sum_{k,p,q,s,d,m,l=0}^{\infty} \frac{\Gamma(k+1)(-1)^{k+d+l}}{k!} \binom{-\alpha(j+1)-1}{p} \\ &\times \binom{-\delta(k+1)+\beta(p+1)-q-s}{m} \binom{-\beta(p+1)-1}{q} \\ &\times \binom{\delta(k+1)-\beta(p+1)+q+s-2}{d} \binom{m+d}{l} b_{s,q} G^l(x; \zeta) \\ &= \sum_{l=0}^{\infty} \omega_{l+1}^* h_{l+1}(x; \zeta), \end{aligned}$$

where

$$\begin{aligned} \omega_{l+1}^* &= 2\alpha\delta\beta \sum_{k,p,q,s,d,m=0}^{\infty} \frac{\Gamma(k+1)(-1)^{k+l+d}}{k!} \binom{-\alpha(j+1)-1}{p} \binom{m}{l} \\ &\times \binom{-\delta(k+1)+\beta(p+1)-q-s}{m} \binom{-\beta(p+1)-1}{q} \\ &\times \binom{\delta(k+1)-\beta(p+1)+q+s-2}{d} \binom{m+d}{l} b_{s,q}. \end{aligned}$$

Consequently, the PWM of X can be written as

$$\wp_{i,j} = \sum_{l=0}^{\infty} \omega_{l+1}^* \int_{-\infty}^{\infty} x^i h_{l+1}(x; \zeta) dx = \sum_{l=0}^{\infty} \omega_{l+1}^* E(Y_{l+1}^i). \quad (6.41)$$

Rényi Entropy (RE)

The RE is defined to be

$$I_R(\nu) = \frac{1}{1-\nu} \log \left(\int_0^\infty [f(x)]^\nu dx \right), \nu \neq 1, \nu > 0. \quad (6.42)$$

From the BIII-EHL-G pdf, $f^\nu(x)$ can be expressed as

$$\begin{aligned} f^\nu(x) &= (2\alpha\delta\beta)^\nu g^\nu(x; \zeta) G^{\nu(\delta-1)}(x; \zeta) \left(-\log \left(1 - [W_G(x; \zeta)]^\delta \right) \right)^{-\nu(\beta+1)} \\ &\times (1 + \overline{G}(x; \zeta))^{-\nu(\delta+1)} [1 - [W_G(x; \zeta)]^\delta]^{-\nu} \\ &\times \left[1 + \left(-\log \left(1 - [W_G(x; \zeta)]^\delta \right) \right)^{-\beta} \right]^{-\nu(\alpha+1)}. \end{aligned} \quad (6.43)$$

Using the binomial series expansion, equation (6.43) can be written as

$$\begin{aligned} f^\nu(x) &= (2\alpha\delta\beta)^\nu g^\nu(x; \zeta) G^{\nu(\delta-1)}(x; \zeta) \sum_{k,p=0}^{\infty} (-1)^k \binom{-\nu}{k} \binom{-\nu(\alpha+1)}{p} \\ &\times (1 + \overline{G}(x; \zeta))^{-\nu(\delta+1)} [W_G(x; \zeta)]^{\delta k} \\ &\times \left(-\log \left(1 - [W_G(x; \zeta)]^\delta \right) \right)^{-\beta p - \beta \nu - \nu}. \end{aligned}$$

Let $y = [W_G(x; \zeta)]^\delta$, $0 < y < 1$. Using the series representation

$$-\log(1-y) = \sum_{i=1}^{\infty} \frac{y^{i+1}}{i+1}, \quad (6.44)$$

we get

$$[-\log(1-y)]^{-\beta(p+\nu)-\nu} = y^{-\beta(p+\nu)-\nu} \sum_{q=0}^{\infty} \binom{-\beta(p+\nu)-\nu}{q} y^q \left(\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^q.$$

Applying the result on a power series raised to a positive integer with $a_s = (s+2)^{-1}$ (see, Gradshteyn and Ryzhik [63]), that is

$$\left(\sum_{s=0}^{\infty} a_s y^s \right)^q = \sum_{s=0}^{\infty} b_{s,q} y^s,$$

where

$$b_{s,q} = (s a_0)^{-1} \sum_{l=1}^s [q(l+1) - s] a_l b_{s-l,q} \quad \text{and} \quad b_{0,q} = a_0^m. \quad (6.45)$$

Now,

$$\begin{aligned}
f^\nu(x) &= (2\alpha\delta\beta)^\nu g^\nu(x; \zeta) \sum_{k,p,q,s=0}^{\infty} (-1)^k \binom{-\nu}{k} \binom{-\nu(\alpha+1)}{p} \binom{-\beta(p+\nu)-\nu}{q} b_{s,q} \\
&\times (1 + \overline{G}(x; \zeta))^{-\delta(k-\nu)+\beta(p+\nu)-q-s} (G(x; \zeta))^{\delta(k+\nu)-\beta(p+\nu)+q+s-2\nu} \\
&= (2\alpha\delta\beta)^\nu g^\nu(x; \zeta) \sum_{k,p,q,s=0}^{\infty} (-1)^k \binom{-\nu}{k} \binom{-\nu(\alpha+1)}{p} \binom{-\beta(p+\nu)-\nu}{q} b_{s,q} \\
&\times \binom{-\delta(k-\nu)+\beta(p+\nu)-q-s}{m} \overline{G}^m(x; \zeta) (1 - \overline{G}(x; \zeta))^{\delta(k+\nu)-\beta(p+\nu)+q+s-2\nu} \\
&= (2\alpha\delta\beta)^\nu g^\nu(x; \zeta) \sum_{k,p,q,m,s,d,l=0}^{\infty} (-1)^{k+d+l} \binom{-\nu}{k} \binom{-\nu(\alpha+1)}{p} \binom{-\beta(p+\nu)-\nu}{q} b_{s,q} \\
&\times \binom{-\delta(k-\nu)+\beta(p+\nu)-q-s}{m} \binom{\delta(k+\nu)-\beta(p+\nu)+q+s-2\nu}{d} \\
&\times \binom{m+d}{l} G(x; \zeta)^l.
\end{aligned}$$

Consequently, the RE of the BIII-EHL-G FoD can be written as

$$I_R(v) = \frac{1}{1-v} \log \left[\sum_{l=0}^{\infty} \Psi_{l+1}^* e^{(1-v)I_{REG}} \right], \quad (6.46)$$

where

$$\begin{aligned}
\Psi_{l+1}^* &= (2\alpha\delta\beta)^\nu \sum_{k,p,q,m,s,d=0}^{\infty} (-1)^{k+d+l} \binom{-\nu}{k} \binom{-\nu(\alpha+1)}{p} \binom{-\beta(p+\nu)-\nu}{q} b_{s,q} \\
&\times \binom{-\delta(k-\nu)+\beta(p+\nu)-q-s}{m} \binom{\delta(k+\nu)-\beta(p+\nu)+q+s-2\nu}{d} \\
&\times \binom{m+d}{l} \left(\frac{l}{\nu} + 1 \right)
\end{aligned}$$

and $I_{REG} = \int_0^\infty [(\frac{l}{\nu} + 1)g(x; \varphi)G(x; \varphi)^{\frac{l}{\nu}}]^\nu dx$ is the RE of the Exp-G distribution with power parameter $\frac{l}{\nu}$. This implies that the RE of the BIII-EHL-G FoD can be calculated directly from that of the Exp-G distribution.

6.2.5 Maximum Likelihood Estimation

In this subsection, we provide the maximum likelihood technique for estimating the parameters of the BIII-EHL-G FoD. Suppose $X \sim BIII - EHL - G(\alpha, \beta, \delta, \zeta)$ and $\Delta = (\alpha, \beta, \delta, \zeta)^T$ be the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random

sample of size n from the BIII-EHL-G FoD is given by

$$\begin{aligned}
\ell_n(\mathbf{\Delta}) &= n \ln(2\alpha) + n \ln(\delta) + n \ln(\beta) + \sum_{i=1}^n \ln(g(x_i; \zeta)) + (\delta - 1) \sum_{i=1}^n \ln(G(x_i; \zeta)) \\
&- (\alpha + 1) \sum_{i=1}^n \ln \left[1 + \left(-\ln \left(1 - \left[\frac{G(x_i; \zeta)}{1 + \bar{G}(x_i; \zeta)} \right]^\delta \right) \right)^{-\beta} \right] \\
&- (\delta + 1) \sum_{i=1}^n \ln(1 + \bar{G}(x_i; \zeta)) - \sum_{i=1}^n \ln \left[1 - \left[W_G(x_i; \zeta) \right]^\delta \right] \\
&- (\beta + 1) \sum_{i=1}^n \ln \left(-\ln \left(1 - \left[W_G(x_i; \zeta) \right]^\delta \right) \right). \tag{6.47}
\end{aligned}$$

Differentiating the $\ell_n = \ell_n(\mathbf{\Delta})$ w.r.t each component of the parameter vector $\mathbf{\Delta} = (\alpha, \beta, \delta, \zeta)^T$, gives the elements of the score vector $U(\mathbf{\Delta})$. The MLEs of the parameters, denoted by $\hat{\mathbf{\Delta}}$ are obtained by numerically solving the nonlinear equation $(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \zeta_k})^T = \mathbf{0}$, using a numerical approximations such as Newton-Raphson technique. Under the standard regularity conditions, the MLEs asymptotically follows the multivariate normal distribution $N_{p+3}(\mathbf{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\mathbf{0} = (0, 0, 0, \mathbf{0})^T$ and $J(\hat{\mathbf{\Delta}})^{-1}$ is the observed FIM evaluated at $\hat{\mathbf{\Delta}}$, can be used to build CIs and confidence regions for the individual model parameters and for the survival functions and hrf.

6.2.6 Simulation Results

Table 6.8 and Table 6.9 provides results from the simulations carried out to assess the performance of the maximum likelihood estimators. We generated $N = 1000$ samples for the true parameters. The ABias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

(0.8, 2.5, 1.5, 1.3)					(0.8, 2.5, 0.8, 1.3)		
parameter	Sample Size	Mean	RMSE	ABias	Mean	RMSE	ABias
α	100	2.7871	18.8171	1.9871	7.4179	23.5575	6.6179
	200	0.8669	0.4395	0.0669	2.1906	10.0177	1.3906
	400	0.8692	0.2913	0.0692	0.9522	2.4029	0.1522
	800	0.8542	0.2262	0.0542	0.7523	0.1577	-0.0477
β	100	7.5421	7.9378	5.0421	8.4843	10.9829	5.9843
	200	7.7327	7.8298	5.2327	8.5412	10.2800	6.0412
	400	6.1877	6.0902	3.6877	7.5655	8.1603	5.0655
	800	4.7488	4.4489	2.2488	6.2661	5.9375	3.7661
δ	100	1.9923	2.6427	0.4923	1.0011	1.3377	0.2011
	200	1.7379	2.0952	0.2379	0.9057	0.8255	0.1057
	400	1.5588	1.5213	0.0588	0.7790	0.3757	-0.0210
	800	1.4680	0.8972	-0.0320	0.7568	0.1778	-0.0432
θ	100	0.9074	0.6719	-0.3926	0.9489	0.7079	-0.3511
	200	0.8489	0.6806	-0.4511	0.8585	0.7322	-0.4415
	400	0.9328	0.5860	-0.3672	0.7930	0.6944	-0.5070
	800	1.0453	0.4726	-0.2547	0.8366	0.6405	-0.4634

Table 6.8: Simulation Results

parameter	Sample Size	(2.1, 2.1, 2.1, 1.8)			(2.2, 2.1, 2.8, 1.8)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
α	100	7.9908	29.4510	5.8908	7.3693	17.5408	5.1693
	200	6.7058	25.4756	4.6058	5.0913	8.2486	2.8913
	400	3.2620	3.4367	1.1620	4.3750	6.3853	2.1750
	800	2.9814	2.1382	0.8814	3.2484	2.5121	1.0484
β	100	6.2780	6.5665	4.1780	6.2992	7.2857	4.1992
	200	5.4462	5.5120	3.3462	5.4294	6.0380	3.3294
	400	3.9339	3.6003	1.8339	3.5926	3.2854	1.4926
	800	2.8701	1.6736	0.7701	2.6075	1.0825	0.5075
δ	100	3.0989	5.1519	0.9989	4.1824	7.1151	1.3824
	200	2.2821	3.4772	0.1821	3.0691	5.2235	0.2691
	400	1.7332	1.0539	-0.3668	2.3088	1.5460	-0.4912
	800	1.7033	0.7875	-0.3967	2.2553	1.0855	-0.5447
θ	100	1.2238	0.8654	-0.5762	1.2943	0.8171	-0.5057
	200	1.2635	0.7823	-0.5365	1.3239	0.7412	-0.4761
	400	1.4296	0.5922	-0.3704	1.5090	0.5034	-0.2910
	800	1.5635	0.3650	-0.2365	1.6335	0.2571	-0.1665

Table 6.9: Simulation Results

Table 6.8 and Table 6.9 shows the simulation results for some randomly selected parameter values of the BIII-EHL-W distribution. We noted that the mean estimates of the parameters are approaching the true parameter values as we increase the sample size n . Moreover, the RMSEs and the bias are decaying toward zero with increasing sample size, which suggest strong consistency of the MLE of the model parameters.

6.2.7 Applications

For the purpose of illustrating the applicability of the BIII-EHL-W distribution for data modeling, we present some real data examples in this subsection. A number of GoF statistics were used to evaluate the performance of the BIII-EHL-W distribution, these includes: $-2 \ln(L)$, AIC, CAIC, BIC, W^* and A^* , (see Chen and Balakrishnan [36] for details), SS, K-S statistic and its p-value. The fits of the BIII-EHL-W are compared to the fits of the EHL distribution, the Half Logistic distribution (Johnson et al. [69]), the Type II Topp-Leone Power Lomax distribution by Al-Marzouki et al. [16], the Type II Topp-Leone Dagum (TIITLD) distribution (Sakthivel and Dhivakar [115]), Exponentiated Weibull Lindley (EWL) distribution (Codeiro et al. [49]), Weibull Lomax (WL) distribution (Tahir et al. [123]), and the Beta Generalized Lindley (BGL) distribution (Oluyede and Yang [99]). The pdfs of these distributions are given in the appendix.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [34]) are given in Figure 6.32 and Figure 6.36. Figure 6.33 and Figure 6.37 shows the estimated cumulative density function, K-M survival plots and the estimated hazard rate plot for the BIII-EHL-W distribution.

The MLEs of the parameters with SEs in parenthesis and the values of the statistics ($-2\ln(L)$, AIC, CAIC, BIC, W^* , A^* , K-S and its p-Value) are given in Table 6.10 and Table 6.11.

Silicon Nitride Data

The BIII-EHL-W distribution was fitted to the silicon nitride data set which was taken from R base package in the Indometh object [111]. The data are as follows: 5.50,5.00,4.90,6.40,5.10,5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10,4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

Distribution	Estimates				statistics								
	α	β	δ	θ	-2 ln(L)	AIC	CAIC	BIC	W*	A*	K-S	P-value	SS
BIII-EHL-W	0.2805 (0.0783)	33.4952 (0.0037)	1.2292 (0.1556)	0.3180 (0.0398)	335.7144	343.7144	344.0653	354.8309	0.0515	0.3129	0.0536	0.8842	0.0500
BIII-EHL-W(β, δ, θ)	1 (2.3037 $\times 10^{-4}$)	33.2520 (0.0217)	0.8449 (0.0217)	0.1938 (0.0121)	356.1379	362.1379	362.3466	370.4753	0.3466	2.1826	0.0798	0.4352	0.1765
BIII-EHL-W(α, δ, θ)	0.2114 (0.0408)	1 (12.3014)	77.4315 (0.0275)	3.2500 (0.0275)	390.0846	396.0846	396.2933	404.4220	0.7786	4.6134	0.1578	0.0054	0.8518
BIII-EHL-W(α, θ)	1 (3.2425)	1 (3.2425)	22.6753 (0.0355)	1.0254 (0.0355)	439.0101	443.0101	443.1136	448.5684	1.4782	8.2918	0.2601	2.037 $\times 10^{-7}$	1.8073
EHL	a 16.7180 (3.61945)	λ 0.9248 (0.0603)			372.8243	376.8243	376.9277	382.3825	0.6515	3.8525	0.1515	0.0085	0.6638
HL	λ 0.3523 (0.0251)				544.4416	546.4416	546.4757	549.2207	0.3292	2.0076	0.3826	1.443 $\times 10^{-15}$	6.2259
W	a 4.9649 (0.3563)	λ 4.7134 (0.0914)			337.4137	341.4137	341.5171	346.9719	0.0919	0.5639	0.0720	0.5679	0.0821
TIITLPL	θ 34722 (0.0007)	α 12.531 (3.7263)	β 2.4895 (0.1799)	λ 110410 (0.0004)	337.4273	345.4273	345.7782	356.5438	0.0922	0.5660	0.0723	0.5632	0.0825
TIITLD	τ 19456 (7.3729 $\times 10^{-7}$)	δ 11376 (2.2933 $\times 10^{-6}$)	θ 2.6481 (0.1045)	β 0.9423 (0.0319)	337.4394	345.4394	345.7903	356.5559	0.0925	0.5679	0.0724	0.5605	0.0828
BGL	α 0.1165 (0.0049)	θ 9.8667 $\times 10^{-8}$ (1.3062 $\times 10^{-6}$)	a 0.3013 (0.0004)	b 10.100 (1.3859 $\times 10^{-5}$)	1143.6190	1151.6910	1152.0420	1162.8070	0.4850	2.9093	0.5003	2.2 $\times 10^{-16}$	9.5129
WL	a 0.0145 (0.0499)	b 3.7304 (1.3818)	α 2.0221 (1.3345)	β 4.6738 (0.3329)	337.1172	345.1172	345.4681	356.2337	0.0797	0.4852	0.0679	0.6422	0.0751
EWL	α 0.9397 (1.2996)	v 2.8860 (3.2416)	a 0.1489 (0.1352)	b 0.7955 (1.3964)	337.0783	345.0783	345.4292	356.1948	0.0705	0.4373	0.0643	0.7090	0.0665

Table 6.10: MLEs and GoF statistics for silicon nitride data

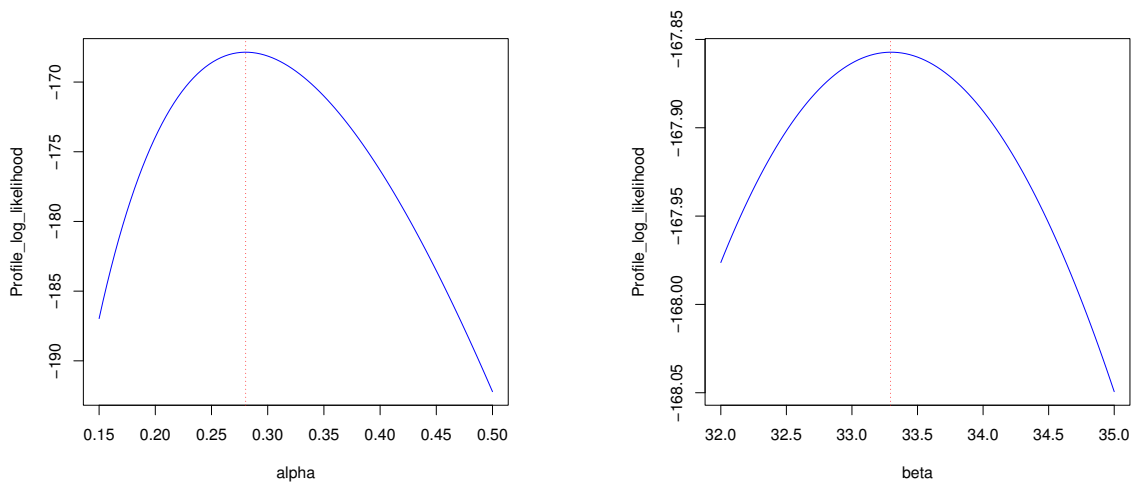


Figure 6.30: Profile plots for the silicon nitride data

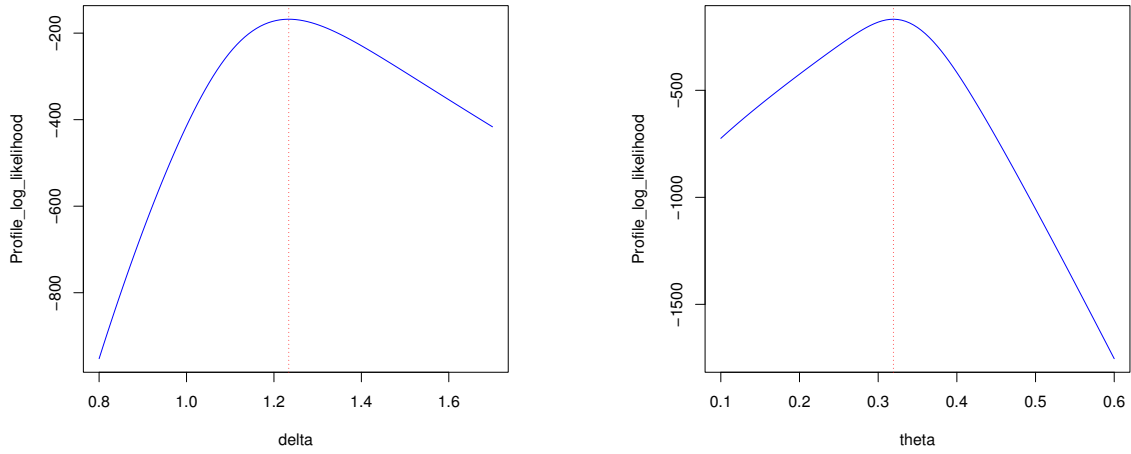


Figure 6.31: Profile plots for the silicon nitride data

The profile plots in Figures 6.30 and 6.31 shows that the estimates of the BIII-EHL-W distribution are global solutions for the silicon nitride data. The estimated var-cov matrix for the strengths of glass fibres data set is given by,

$$\begin{bmatrix} 0.0061 & -0.0003 & -0.0112 & -0.0028 \\ -0.0003 & 1.4059 \times 10^{-05} & 0.0006 & 0.0001 \\ -0.0112 & 0.0006 & 0.0242 & 0.0062 \\ -0.0028 & 0.0001 & 0.0062 & 0.0016 \end{bmatrix}$$

and the 95% CIs for the parameters α, β, δ and θ are: 0.2805 ± 0.1535 , 33.4952 ± 0.0073 , 1.2292 ± 0.3050 and 0.3180 ± 0.0780 , respectively.

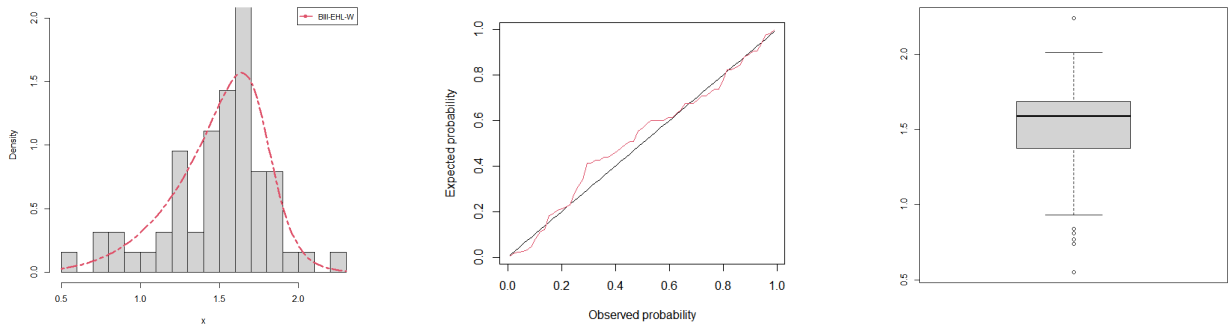


Figure 6.32: Fitted densities, probability plots and box-plot for silicon nitride data

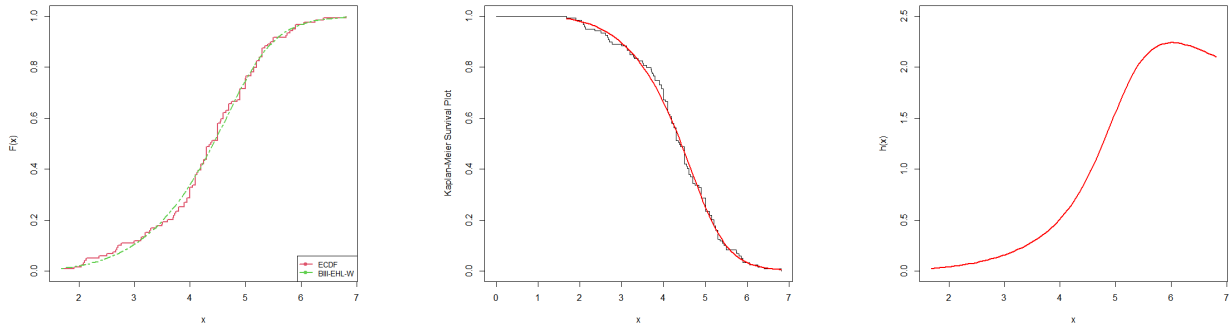


Figure 6.33: Empirical cdf, Kaplan Meier and Fitted hrf for silicon nitride data

Figure 6.32 provides the histogram, fitted densities and probability plots for the strength of glass fibre data. According to the plots and the GoF statistics, the BIII-EHL-W distribution is out performing the other models presented since it has the smallest GoF statistics, the largest K-S p-value and the smallest SS value, thus, we can infer that the BIII-EHL-W distribution provides a better fit compared to the other models presented in Table 6.10.

Strength of Glass Fibres Data

The BIII-EHL-W was fitted to the strength of glass fibre data set, the data consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. Smith and Naylor [121] also analysed this data set. The data is as follows 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74 ,1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11 ,1.28 ,1.42 ,1.50 ,1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55 ,1.61, 1.62, 1.66 ,1.70 ,1.77, 1.84, 0.84, 1.24, 1.30 ,1.48, 1.51, 1.55 ,1.61 ,1.63, 1.67, 1.70, 1.78, 1.89.

Distribution	Estimates				$-2 \ln(L)$	AIC	CAIC	BIC	W^*	A^*	K-S	p-value	SS
	α	β	δ	θ									
BIII-EHL-W	0.2281 (0.0687)	32.4975 (0.0021)	0.8936 (0.0817)	0.5569 (0.0961)	23.6597	31.6597	32.3493	40.2322	0.1093	0.6049	0.1270	0.2614	0.1167
BIII-EHL-W(β, δ, θ)	1	10.2792 (9.7631)	0.8604 (0.2701)	0.7021 (0.5277)	44.1803	50.1803	50.5871	56.6097	0.4753	2.6295	0.1532	0.1039	0.2906
BIII-EHL-W(α, δ, θ)	3.2773 (1.1073)	1	0.5133 (0.3211)	4.0152 (0.6787)	68.2618	308.8538	309.2606	315.2833	0.6574	3.5995	0.7695	2.2×10^{-16}	11.0957
BIII-EHL-W(α, θ)	1	1	3.5390 (0.5289)	2.4699 (0.1900)	72.6366	76.6366	76.8366	80.9228	0.8933	4.9028	0.2150	0.0059	0.7853
EHL	a 17.1769 (4.9708)	λ 2.6559 (0.2297)			61.2453	65.2453	65.4453	69.5316	0.7577	4.1418	0.2237	0.0036	0.7530
HL	λ 1.0127 (0.0989)				155.0062	157.0062	157.0718	159.1493	0.5209	2.8596	0.4137	8.59×10^{-10}	3.6496
W	a 5.7807 (0.5761)	λ 1.6281 (0.0371)			30.4137	34.4137	34.6137	38.7000	0.2372	1.3037	0.1522	0.1078	0.2111
TIITLPL	θ 22003 (0.0070)	α 42.396 (7.2957)	β 2.9003 (0.2893)	λ 25756 (0.0120)	30.4442	38.4442	39.1339	47.0168	0.2381	1.3081	0.1524	0.1073	0.2115
TIITLD	τ 16255 (6.7509×10^{-6})	δ 10815 (1.3584×10^{-5})	θ 4.2938 (0.3219)	β 0.6737 (0.0185)	30.4194	38.4194	39.1090	46.9919	0.2374	1.3047	0.1523	0.1076	0.2113
BGL	α 0.1211 (0.0070)	θ 2.2903×10^{-6} (4.7317×10^{-6})	a 0.3017 (0.0006)	b 10.100 (2.2207×10^{-5})	492.2649	500.2704	500.9601	508.8429	0.6376	3.4917	0.5212	2.776×10^{-15}	5.1508
WL	α 0.0125 (0.0060)	b 2.8402 (0.2702)	α 101.5900 (0.0046)	β 95.3490 (0.0048)	28.7920	36.7920	37.4816	45.3645	0.1732	0.9745	0.1362	0.1930	0.1673
EWL	α 4.1949 (5.9402)	v 1.8082 (0.9809)	a 0.0197 (0.0283)	b 0.7171 (1.1459)	28.2194	36.2194	36.90904	44.7919	0.1596	0.8990	0.1310	0.2296	0.1495

Table 6.11: MLEs and GoF Statistics for Strength of Glass Fibre Data

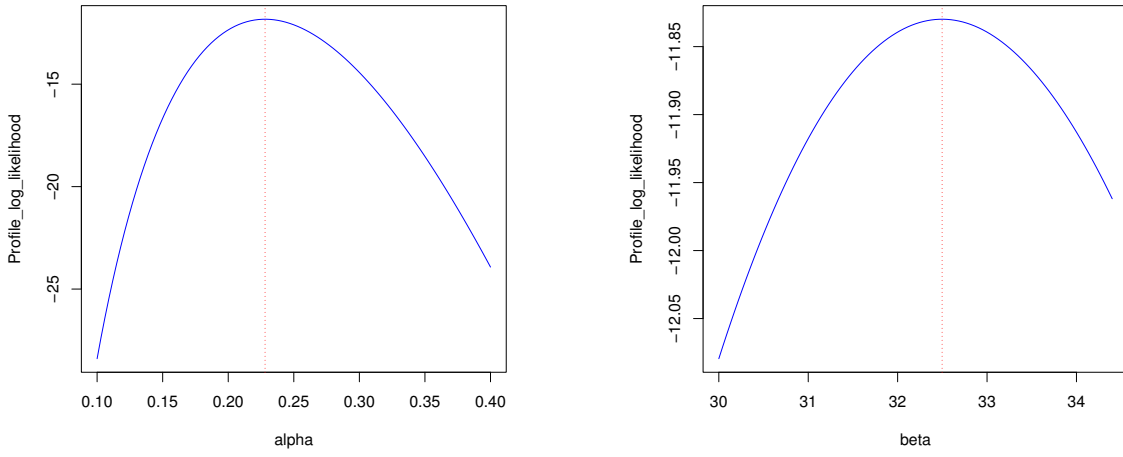


Figure 6.34: Profile plots for the strengths of glass fibres data

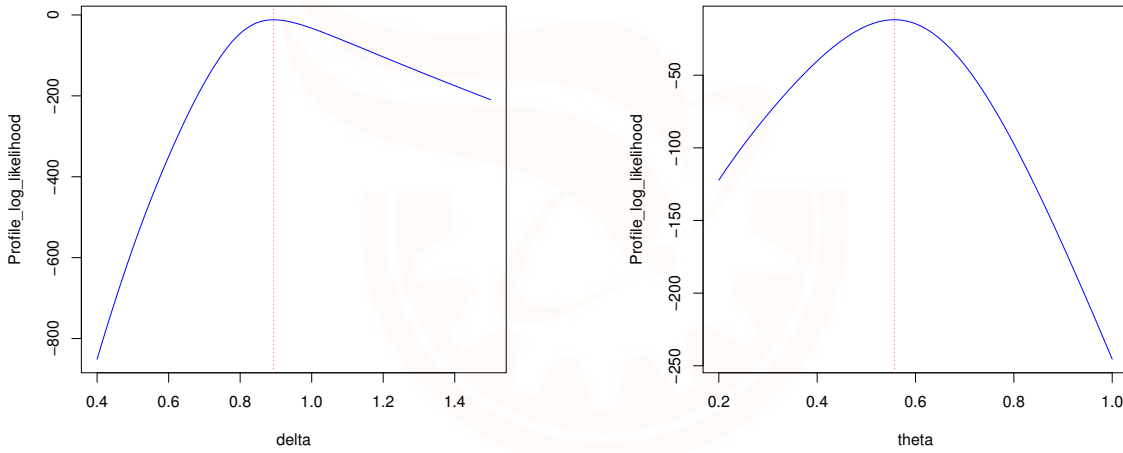


Figure 6.35: Profile plots for the strengths of glass fibres data

The profile plots in Figures 6.34 and 6.35 shows that the estimates of the BIII-EHL-W distribution are global maximums for the strengths of glass fibres data. The estimated var-cov matrix for the strengths of glass fibres data is given by,

$$\begin{bmatrix} 0.0047 & -0.0001 & -0.0050 & -0.0056 \\ -0.0001 & 4.3077 \times 10^{-6} & 0.0002 & 0.0002 \\ -0.0050 & 0.0002 & 0.0067 & 0.0077 \\ -0.0056 & 0.0002 & 0.0077 & 0.0092 \end{bmatrix}$$

and the 95% CIs for the parameters α, β, δ and θ are: 0.2281 ± 0.1347 , 32.4975 ± 0.0041 , 0.8936 ± 0.1601 , and 0.5569 ± 0.1884 , respectively.

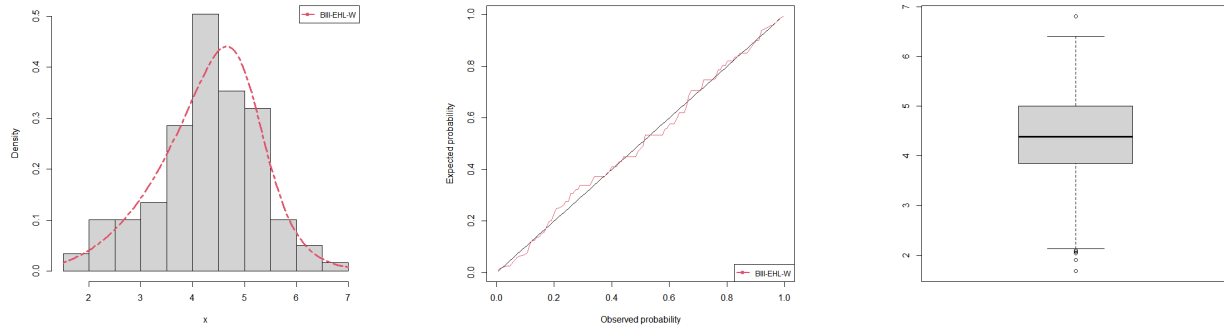


Figure 6.36: Fitted densities, probability plots and box-plot for strength of glass fibre data

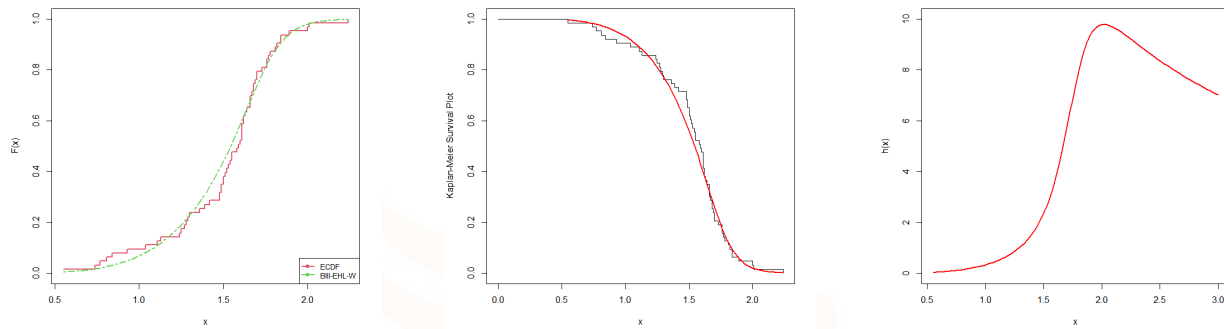


Figure 6.37: Empirical cdf, K-M and fitted hazard rate for strength of glass fibre data

From the GoF statistics results presented in Table 6.11, we can see that the BIII-EHL-W distribution has the least values of the GoF statistics, the biggest K-S p-value and the least SS value, hence we can infer that the BIII-EHL-W distribution provide a better fit to the glass fibres data set compared to models given in Table 6.11.

Model	df	Silicon Nitride Data		Strength of Glass Fibre Data	
		Chi-Square	p-value	Chi-Square	p-value
BIII-EHL-W($1, \beta, \delta, \theta$)	1	20.4235	<0.00001	20.5206	<0.00001
BIII-EHL-W($\alpha, 1, \delta, \theta$)	1	54.3702	<0.00001	44.6021	<0.00001
BIII-EHL-W($1, 1, \delta, \theta$)	2	103.2856	<0.00001	48.9769	<0.00001

Table 6.12: Likelihood ratio test results

The likelihood ratio (LR) test are presented in Table 6.12. The results show that there is a significant difference between the BIII-EHL-W distribution and its nested models: BIII-EHL-W($1, \beta, \delta, \theta$), BIII-EHL-W($\alpha, 1, \delta, \theta$) and BIII-EHL-W($1, 1, \delta, \theta$) for both data sets

6.2.8 Concluding Remarks

We developed a new generalized FoD called the Burr III-Exponentiated Half Logistic-G (BIII-EHL-G) distribution. The shapes of the hrf of the BIII-EHL-G FoD can be monotone as well as non-monotone. We also derived expressions for the moments, incomplete moments, PWMs,

distribution of order statistics and RE. We used the maximum likelihood estimation to estimate the parameters of our model. The effectiveness of the BIII-EHL-G FoD was evaluated by carrying various MCS for different sizes. Finally, the special case of the BIII-EHL-G FoD, namely BIII-EHL-W distribution was fitted to real data sets to demonstrate the importance of the proposed FoD.



7. Conclusions and Areas for Further Studies

7.1 Conclusions

In this thesis, we proposed, developed and presented nine new generalized families of distributions with their applications to real data sets. Some mathematical and statistical properties of these distributions including the series expansion of pdfs, moments, conditional moments, hrfs, qfs and sub-models, PWMs, RE and distribution of order statistics are derived and presented. We noted that the newly developed models can be instrumental in explaining the tail variations better than the classical model. We performed Monte-Carlo simulations to study the behaviour of the MLEs and we found out that the RMSEs and the ABias decreases with an increase in sample size which indicates that the MLEs are asymptotically unbiased and consistent. We applied the models to real data sets to illustrate their applicability and potentiality, and we noted that these new families of distributions are flexible and performs better than the classical models which we intended to improve. We observed that the developed models are quite flexible and produce better GoF statistics compared to other existing models, hence they can be used in many applied areas such as insurance, finance, engineering, survival analysis, probability and applied statistics, change detection, climate modelling.

7.2 Suggestions for Further Studies

Further studies could include employing other parameter estimation techniques such as weighted least squares, Anderson-Darling, Cramér-von Mises distance measures, Bayesian techniques. Another area could be applying the proposed models to regression analysis (that is, the incorporation of covariates into the models) as well as estimation of the model parameters under various censoring schemes. Bivariate and multivariate extensions can also be considered for the developed FoD.

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7.3 Appendix

$$\begin{aligned}
 f_{BOL-U}(x; a, b, \lambda, \theta) &= \frac{1}{B(a, b)} \left[1 - \frac{\lambda + (1 - x/\theta)}{(1 + \lambda)(1 - x/\theta)} \exp \left\{ -\lambda \frac{x}{(\theta - x)} \right\} \right]^{a-1} \\
 &\times \left[\frac{\lambda + (1 - x/\theta)}{(1 + \lambda)(1 - x/\theta)} \exp \left\{ -\lambda \frac{x}{(\theta - x)} \right\} \right]^{b-1} \\
 &\times \frac{\lambda^2}{(1 + \lambda)} \frac{\theta^2}{(\theta - x)^3} \exp \left\{ -\lambda \frac{x}{(\theta - x)} \right\},
 \end{aligned}$$

for $a, b, \lambda, \theta > 0$,

$$\begin{aligned}
 f_{BOL-E}(x; a, b, \lambda, \theta) &= \frac{1}{B(a, b)} \left[1 - \frac{\lambda + e^{-\theta x}}{(1 + \lambda)e^{-\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\} \right]^{a-1} \\
 &\times \left[\frac{\lambda + e^{-\theta x}}{(1 + \lambda)e^{-\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\} \right]^{b-1} \\
 &\times \frac{\lambda^2}{(1 + \lambda)} \frac{(\theta e^{-\theta x})}{e^{-3\theta x}} \exp \left\{ -\lambda \frac{1 - e^{-\theta x}}{e^{-\theta x}} \right\},
 \end{aligned}$$

for $a, b, \lambda, \theta > 0$,

$$\begin{aligned}
 f_{OEHLBXII}(x; \alpha, \lambda, a, b) &= \frac{2\alpha\lambda abx^{a-1} \exp\{\lambda[1 - (1 + x^a)^b]\}}{(1 + x^a)^{-b+1}(1 + \exp\{\lambda[1 - (1 + x^a)^b]\})^{\alpha+1}} \\
 &\times (1 - \exp\{\lambda[1 - (1 + x^a)^b]\})^{\alpha-1},
 \end{aligned}$$

for $\alpha, \lambda, a, b > 0$,

$$\begin{aligned}
 f_{WBXII}(x; \alpha, \beta, a, b) &= \alpha\beta abx^{\alpha-1}(1 + x^\alpha)^{\beta b-1}[1 - (1 + x^\alpha)^{-\beta}]^{b-1} \\
 &\times \exp \left\{ -a[(1 + x^\alpha)^\beta - 1]^b \right\},
 \end{aligned}$$

for $\alpha, \beta, a, b > 0$,

$$f_{EHL}(x) = \frac{2a\lambda e^{-\lambda x}}{1 - e^{-2\lambda x}} \left[\frac{1 - e^{-\lambda x}}{1 + e^{-\lambda x}} \right]^{a-1},$$

for $a, \lambda, x > 0$,

$$f_{HL}(x) = \frac{2\lambda e^{-\lambda x}}{(1 + e^{-2\lambda x})^2},$$

for $\lambda, x > 0$,

$$f_W(x) = \frac{a}{\lambda} \left(\frac{x}{\lambda} \right)^{a-1} e^{\left(\frac{x}{\lambda} \right)^a},$$

for $a, \lambda, x > 0$,

$$f_{TIITLPL}(x) = \frac{2\theta\alpha\beta}{\lambda} x^{\beta-1} \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha-1} \left[1 - \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha}\right] \\ \times \left\{1 - \left[1 - \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha}\right]^2\right\}^{\theta-1},$$

for $\theta, \alpha, \beta, \lambda, x > 0$,

$$f_{TIITLD}(x) = 2a\tau\delta\theta\beta x^{-\theta-1} (1 + \delta x^{-\theta})^{-2\beta-1} [1 - (1 + \delta x^{-\theta})^{-2\beta}]^{\tau-1},$$

for $\tau, \delta, \theta, \beta, x > 0$,

$$f_{OLLE-W}(x) = \frac{\theta\beta\gamma x^{\beta-1} e^{-(\frac{x}{\alpha})^\beta} [1 - e^{-(\frac{x}{\alpha})^\beta}]^{\gamma\theta-1} \{1 - [1 - e^{-(\frac{x}{\alpha})^\beta}]^\alpha\}^{\theta-1}}{\alpha^\beta \{[1 - e^{-(\frac{x}{\alpha})^\beta}]^{\gamma\theta} + \{1 - [1 - e^{-(\frac{x}{\alpha})^\beta}]^\gamma\}^\theta\}^2}, \quad (7.1)$$

for $\alpha, \beta, \theta, \gamma > 0$ and $x > 0$,

$$f_{Kum-W}(x) = ab\beta\alpha^\beta x^{\beta-1} e^{-(\alpha x)^\beta} (1 - e^{-(\alpha x)^\beta})^{a-1} \{1 - [1 - e^{-(\alpha x)^\beta}]^a\}^{b-1}, \quad (7.2)$$

for $\alpha, \beta, a, b > 0$ and $x > 0$,

$$f_{WL}(x) = \frac{ab\alpha}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{b\alpha-1} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1} \\ \times \exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^\alpha - 1\right\}\right\}, \quad (7.3)$$

for $\alpha, \beta, a, b > 0$ and $x > 0$,

$$f_{BGL}(x) = \frac{\alpha\theta^2(1+x)e^{-\theta x}}{B(a,b)(1+\theta)} \left[1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}\right]^{\alpha\alpha-1} \\ \times \left[1 - \left(1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}\right)^a\right]^{b-1}, \quad (7.4)$$

for $\alpha, \theta, a, b > 0$ and $x > 0$,

$$f_{GWLL}(x) = \frac{\alpha\beta\gamma x^{\gamma-1}}{a^\theta} \left(1 + \left(\frac{x}{a}\right)^\gamma\right)^{-1} [\log(1 + \left(\frac{x}{a}\right)^\gamma)]^{\beta-1} \\ \times \exp\{-\alpha[\log(1 + \left(\frac{x}{a}\right)^\gamma)]^\beta\}, \quad (7.5)$$

for $\alpha, \beta, a, \gamma > 0$ and $x > 0$,

$$f_{EHLOL-W}(x) = 2\alpha\lambda^2 ab^{-a} x^{a-1} e^{-(x/b)^a} \times \frac{\exp\left[\frac{-\lambda(1-e^{-(x/b)^a})}{e^{-(x/b)^a}}\right] \left\{1 - \frac{\lambda+e^{-(x/b)^a}}{(1+\lambda)e^{-(x/b)^a}} \exp\left[\frac{-\lambda(1-e^{-(x/b)^a})}{e^{-(x/b)^a}}\right]\right\}^{\alpha-1}}{(1+\lambda)(e^{-(x/b)^a})^3 \left\{1 + \frac{\lambda+e^{-(x/b)^a}}{(1+\lambda)e^{-(x/b)^a}} \exp\left[\frac{-\lambda(1-e^{-(x/b)^a})}{e^{-(x/b)^a}}\right]\right\}^{\alpha+1}},$$

for $x > 0, \gamma, \alpha, a, b > 0$,

$$f_{MO-Gom-W}(x; \delta, \theta, \lambda, \gamma) = \frac{\delta\theta\lambda\gamma e^{-x^\lambda} [e^{-x^\lambda}]^{-\gamma-1} e^{\frac{\theta}{\gamma}(1-[e^{-x^\lambda}]^{-\gamma})}}{\left(1 - \bar{\delta} \left(e^{\frac{\theta}{\gamma}(1-[e^{-x^\lambda}]^{-\gamma})}\right)\right)^2},$$

for $x > 0, \delta, \theta, \lambda, \gamma > 0$,

$$f_{MOLLW}(x; c, \alpha, \beta, \delta) = \frac{\delta(1+x^c)^{-1} e^{-\alpha x^\beta} \{\alpha\beta x^{\beta-1} + cx^{c-1}(1+x^c)^{-1}\}}{[1 - \bar{\delta}(1+x^c)^{-1} e^{-\alpha x^\beta}]^2},$$

for $x > 0, c, \alpha, \beta, \delta > 0$,

$$f_{MOEW}(x; \alpha, \beta, \lambda, k) = \frac{\alpha(\lambda + \beta k x^{k-1}) e^{-\lambda x - \beta x^k}}{(1 - \bar{\alpha} e^{-(\lambda x + \beta x^k)})^2},$$

for $x > 0, \alpha, \beta, \lambda, k > 0$,

$$\begin{aligned} f_{EWL}(x) &= \frac{abv(1+\alpha)\alpha^2(1+x)\exp(\alpha x)}{(1+\alpha+\alpha x)^2} \exp\left\{-a\left[\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right]^b\right\} \\ &\times \left(1 - \exp\left\{-a\left[\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right]^b\right\}\right)^{v-1} \\ &\times \left\{\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right\}^{b-1}, \end{aligned} \tag{7.6}$$

for $\alpha, v, a, b > 0$ and $x > 0$,

$$f_{TL-WP}(x; \alpha, \beta, b, \theta) = \frac{2b\theta\alpha\beta x^{\beta-1} e^{-2\alpha x^\beta} (1 - e^{-2\alpha x^\beta})^{b-1} e^{\theta(1-(1-e^{-2\alpha x^\beta})^b)}}{e^\theta - 1},$$

for $x > 0, \alpha, \beta, b, \theta > 0$,

$$f_{LLoGWP}(x; c, \alpha, \beta, \theta) = \theta \frac{e^{\theta(1-(1+x^c)^{-1}e^{-\alpha x^\beta})}}{e^\theta - 1} (1+x^c)^{-1} e^{-\alpha x^\beta} \\ \times [(1+x^c)^{-1} c x^{c-1} + \alpha \beta x^{\beta-1}],$$

for $x > 0, c, \alpha, \beta, \theta > 0$, and

$$f_{LBXHP}(x; \lambda, c, k, \theta) = \theta e^{\theta((1+\lambda+\lambda x)(1+\lambda)^{-1}e^{-\lambda x}(1+x^c)^{-k})} \\ \times \frac{(1+x^c)^{-k}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x^c) + \frac{k c x^{c-1}(1+\lambda+\lambda x)}{1+x^c} \right] \\ \times (e^\theta - 1)^{-1},$$

for $x > 0, \lambda, c, k, \theta > 0$.



7.4 Appendix

```

##PDF PLOTS
f1=function(x,alpha,b,theta, gamma)
{
y= 4*b*alpha*theta*gamma*x^(theta-1)*(1+x^theta)^(-gamma-1)
*(1-(1+x^theta)^(-gamma))*(1-(1-(1+x^theta)^(-gamma))^2)^(b-1)
*(1-(1-(1-(1+x^theta)^(-gamma))^2)^b)^(alpha-1)
*(1+(1-(1-(1+x^theta)^(-gamma))^2)^b)^(-alpha-1)

#insert the pdf of the model
}
x=seq(0,1.0,by=0.001)
y1=f1(x,0.7,0.5,0.5,0.9)
plot(x,y1,ylim=c(0,2.5),col=2,'l',lwd=2,xlab="x",ylab="density")
y2=f1(x,0.7,1.5,3.5,2.5)
lines(x,y2,col=3,lwd=2,'l')
legend("topright",c(
expression(paste(alpha,'=0.7',',b','=0.5',',theta','=0.5',',gamma','=0.9')),
expression(paste(alpha,'=0.7',',b','=1.5',',theta','=3.5',',gamma','=2.5')),
col=c(2,3),lty="dashed",lwd=c(2,2),cex = 0.8)
#-----
##HRF PLOTS
f1=function(x,alpha,b,theta, gamma)
{
y= #insert the hrf of the model
}
x=seq(0,1.0,by=0.001)
y1=f1(x,0.7,0.5,0.5,0.9)
plot(x,y1,ylim=c(0,2.5),col=2,'l',lwd=2,xlab="x",ylab="h(x)")
y2=f1(x,0.7,1.5,3.5,2.5)
lines(x,y2,col=3,lwd=2,'l')
legend("topright",c(
expression(paste(alpha,'=0.7',',b','=0.5',',theta','=0.5',',gamma','=0.9')),
expression(paste(alpha,'=0.7',',b','=1.5',',theta','=3.5',',gamma','=2.5')),
col=c(2,3),lty="dashed",lwd=c(2,2),cex = 0.8)
#-----
##SKEWNESS AND KURTOSIS 3D PLOTS
library(plot3Drgl)
F=function(x,par){
alpha=par[1];b=par[2];beta=par[3];c=par[4]
C= #Insert the cdf here

```

```
  return(C)}
#PDF function:
f = function(x,par){
  alpha=par[1];b =par[2];beta = par[3]; c=par[4]
  D = #Insert the pdf here
  return(D)}
#Hazard function:
h = function(x,par) f(x,par)/(1-F(x,par))
#The program is developed in R to obtain the value of skewness:
library(moments)
beta = seq(1,4,.1)
c = seq(1,4,.1)
G_skew <- function(beta, c) {
  x <- seq(0, 10, le = 2*10^4)
  U=f(x,c(2.5,1.5, beta, c))
  Skewness = skewness(U) # or kurt =kurtosis(U)
  return(Skewness)}
#Skew
G_skew <-Vectorize(G_skew, c('beta', 'c'))
Skewness<-outer(beta,c,G_skew)
persp3D(beta,c,Skewness,main=expression(paste(EHL-WTL-LLoG(2.5,
1.5,beta,c))), xlab ="beta", ylab="c", zlab="Skewness", phi=20)
#
#APPLICATION
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library('bbmle')
x<- c(#Inser your data here )
hist(x)
#EHL-WTL-LLoG
fEHLWTLLLoG<- function(alpha,b, beta,c) {
  -sum(log(#Insert your pdf here))
}
EHLWTLLLoG.result <-mle2(fEHLWTLLLoG
, hessian = NULL, start=list(alpha=1.5,b=0.3
,beta= 0.3,c=1.5),optimizer="nllminb",lower=0)
summary(EHLWTLLLoG.result)
vcov(EHLWTLLLoG.result)
output1<-vcov(EHLWTLLLoG.result)
```

```

1.96*sqrt(diag(output1))
solve(output1)
#-----
#FITTED DENSITIES
par(mfrow=c(0,4))
x<-c(#Insert data here)
##EHLWTLLLoGDistribution
EHLWTLLLoG<-function(alpha,b, beta,c,x)
{
#Insert PDF here
}
#You can add models here
hist(x, prob=TRUE, br=15,ylim=c(0,2.0), main='')
curve(EHLWTLLLoG(x, alpha=3.1117e+02,
b=1.3488e+00, beta=1.6658e+01,c=1.0795e-02),
lty=6, col=2, add=TRUE, lwd=3)
par(new=TRUE)
#-----
#PROBABILITY PLOTS
data=c(#Insert data set)
data<-sort(data)
n=length(data)
#write the estimated parameter values
alpha1=3.1117e+02
b1=1.3488e+00
beta1=1.6658e+01
c1=1.0795e-02
##
require(gsl)
require(stats)

F_observed<-c(rep(0,n))
m=n+0.25
for (i in 1:n)
{
F_observed[i]<-(i-0.375)/m
}

EHLWTLLLoG.cdf <- function(data,alpha,b, beta,c,n) {
F<-c(rep(0,n))
for (i in 1:n)
{

```

```
x<-data[i]

F[i]<- #Insert cdf here
}
return(F)
}
F1<-EHLWTLLLoG.cdf(data,alpha1,b1,beta1,c1,n)
F1
#-----
SS<-function(F,F_line,n){
  SS<-0
  for (i in 1:n)
  {
    SS<-SS+(F[i]-F_line[i])^2
  }
  return(SS)
}

#-----
SS1<-SS(F1,F_observed,n)
SS1
op <- par(mfrow=c(1,1))
plot(F_observed,F_observed,type='l',
xlab="Observed probability",ylab="Expected probability")
lines(F_observed,F1,col=2)
legend("bottomright",pch=1,cex=0.8,
c("EHLWTLLLoG",""),col=c(2,3),lwd=c(2,2))
#-----
# Codes for plotting the empirical cdf of the model
x=c(#Insert data here)
m=length(x)
# Write down the parameters of the model
and the corresponding estimated values
alpha=3.1117e+02
b=1.3488e+00
beta=1.6658e+01
c=1.0795e-02
x<-sort(x)
F1<-ecdf(x) ##### ecdf stands for the estimated cdf
ecdf<-F1(c(x))
EHLWTLLLoGcdf<-function(x,alpha,b,beta,c){
```

```

#Insert CDF here}
plot(x ,ecdf , lty=1, lwd=2.5, type="s", xlab="x",
ylab="F(x)", ylim=c(0,1),xlim=c( 0.05,2.72), col="2")
par(new=TRUE)
curve(EHLWTLLLoGcdf(x, alpha=3.1117e+02,b=1.3488e+00,
  beta=1.6658e+01,c=1.0795e-02),lty=6, col="3", add=TRUE, lwd=2)
legend("bottomright",pch=1,cex=0.8,c("ECDF","EHLWTLLLoG"),
col=c(2,3),lwd=c(2,2))
#
# Codes for plotting Kaplan–Meier survival plot of the model
x=c(#Insert data here)
library(survival)
delta=rep(1,length(x))
x<-sort(x)
km = survfit(Surv(x,delta)~1)
plot(km, conf.int=FALSE, ylab="Kaplan–Meier Survival Plot", xlab="x")
## Write down the parameters of the model
and the corresponding estimated values
alpha=3.1117e+02
b=1.3488e+00
beta=1.6658e+01
c=1.0795e-02
ss <-function(x)
{
  #Insert the survival function here
}
lines(seq(0.05,2.72, length.out=100),ss(seq(0.05,
2.72, length.out=100)),col="green", lwd=2)

#
#MONTE-CARLO SIMULATION STUDY
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library(rootSolve)

#Define PDF
EHLWTLLLoG.pdf<-function(alpha,b, beta,c,x)
{
#Insert the EHLWTLLLoG PDF here

```

```

}
#Define cdf EHLWTLLLoG
EHLWTLLLoG.cdf=function(alpha,b, beta,c,x){
y= #Insert the EHLWTLLLoG cdf here
}
EHLWTLLLoG.neglogl=function(alpha,b, beta,c,x){
-sum(log(EHLWTLLLoG.pdf(alpha,b, beta,c,x)))}
EHLWTLLLoG.hazard=function(alpha,b, beta,c,x){
  f=EHLWTLLLoG.pdf(alpha,b, beta,c,x)
  F=EHLWTLLLoG.cdf(alpha,b, beta,c,x)
  y=f/(1-F)
  return(y)}

#-----
## define EHLWTLLLoG quantile
EHLWTLLLoG.quantile =function(parameter,u){
alpha=parameter[1]
b=parameter[2]
beta=parameter[3]
c=parameter[4]
f=function(x){
EHLWTLLLoG.cdf(alpha,b, beta,c,x)-u
}
x=min(uniroot.all(f,lower=0,upper=10,tol=1000))
return(x)
}
EHLWTLLLoG.QuantileTable=function(parameter_matrix){
u=seq(0.1,0.9,0.1)
size=dim(parameter_matrix)[1]
Table.Quantile=matrix(NA,nrow=length(u),ncol=size)
row.names(Table.Quantile)=u
colnames(Table.Quantile)= apply(parameter_matrix,
1,function(x){paste0('(',paste0(x,collapse=','),')')})
Table.Quantile
for(iter in 1:size){
parameter=parameter_matrix[iter,]
for(i in 1:length(u)){
Table.Quantile[i,iter]=EHLWTLLLoG.quantile(parameter,u[i])
}
}
return(Table.Quantile)
}
#-----

```

```

## table of quantile
parameter_matrix=as.matrix(rbind(
  par1=c(1.8,0.8,0.7,1.5),
  par2=c(0.5,1.6,0.8,0.5),
  par3=c(3.5,2.0,1.5,1.5),
  par4=c(2.9,2.0,1.8,0.5),
  par5=c(0.8,4.5,1.6,2.4)
))
EHLWTLLLoG_QuantileTable(parameter_matrix)
print(parameter_matrix)
print(EHLWTLLLoG_QuantileTable(parameter_matrix))
#End of Quantile
#-----
#define pdf EHLWTLLLoG
EHLWTLLLoG_pdf=function(alpha,b,beta,c,x){
  y=#Insert PDF here
}
EHLWTLLLoG_moments=function(alpha,b,beta,c,n){
  f=function(alpha,b,beta,c,n,x){
    (x^n)*(EHLWTLLLoG_pdf(alpha,b,beta,c,x))
  }
  y=integrate(f,lower=0,upper=1,subdivisions=10, alpha=alpha,
    b=b,beta=beta,c=c,n=n)
  return(y$value)
}
#table of moments
parameter_matrix_Moments=matrix(c(0.8,4.5,1.6,0.4),ncol=4,byrow=T)
parameter_matrix_Moments
list_Moments=c('EX','EX2','EX3','EX4','EX5',
  'EX6','SD','CV','CS','CK')
Table_Moments=matrix(0,nrow=length(list_Moments),
  ncol=dim(parameter_matrix_Moments)[1])
row.names(Table_Moments)=list_Moments
Table_Moments

for(iter in 1:dim(parameter_matrix_Moments)[1]){
  parameter=parameter_matrix_Moments[iter,]
  for(i in 1:10){
    Table_Moments[i,iter]=EHLWTLLLoG_moments(parameter[1],
      parameter[2],parameter[3],parameter[4],i)
  }
}

```

```

print(Table_Moments)
####moments end

##-----

EHLWTLLLoG_simulation=function(size=c(25,50,100,200,400,800),
samp, par){
  set.seed(2000)
  K=length(par)
  Mean=vector()
  RMSE=vector()
  Bias=vector()
  samplesize=as.vector(t(mapply(rep, size, K)))
  for(iter_size in 1:length(size)){
    coef1=matrix(NA, samp, K)
    colnames(coef1)=c('alpha', 'b', 'beta', 'c')
    for(nsamp in 1:samp){
      tryCatch(
        {
          x1_EHLWTLLLoG=NULL
          q=runif(size[iter_size], 0, 1)
          x1=apply(q, EHLWTLLLoG_quantile, parameter=par)
          # MLE
          x1_EHLWTLLLoG=mle2(EHLWTLLLoG_neglogl, start=list(alpha=par[1],
            b=par[2], beta=par[3], c=par[4]), method="L-BFGS-B",
            data=list(x=x1), lower=c(alpha=0, b=0, beta=0, c=0),
            , upper=c(alpha=Inf, b=Inf, beta=Inf, c=Inf),
            use.ginv=TRUE) coef1[nsamp,]=coef(x1_EHLWTLLLoG)
        }, error=function(e){})
    }
    Mean[length(size)*(0:(K-1))+iter_size]=apply(coef1, 2, mean, na.rm=TRUE)
    RMSE[length(size)*(0:(K-1))+iter_size]=
      apply((coef1-matrix(rep(par, nsamp), ncol=K, byrow=T))^2,
        2, function(x){sqrt(mean(x, na.rm=TRUE))})
    }
    Bias=as.vector(apply(1:K, function(x){Bias[(length(size)*(x-1)+1)
      :(length(size)*x)]=Mean[(length(size)*(x-1)+1)
      :(length(size)*x)]-par[x]}))
    return(cbind(samplesize, Mean, RMSE, Bias))
  }
}

```

```
#
par=c(0.2,0.5,0.7,0.3)
par_k=c(0.4)
for(i in 1:length(par)){
  for(j in 1:length(par)){
    for(l in 1:length(par)){
      for(p in 1:length(par_k)){
        print(c(par[i],par[j],par[l],par_k[p]))
        print(EHLWTLLLoG_simulation(samp=1000,par=c(par[i],par[j],
par[l],par_k[p])))
      }
    }
  }
}
#
```

