



Faculty of Sciences
Department of Mathematics & Statistical Sciences

New Combinations of Generators for the Construction of Generalized Families and Classes of Distributions with Applications

by

Thatayaone Moakofi

Reg. No: 14000642

PhD (Statistics), BIUST, Botswana

A Dissertation/Thesis Submitted to the Faculty of Sciences in Partial Fulfilment
of the Requirements for the Award of the Degree of Doctor of Philosophy in
Statistics of BIUST

Supervisor:

Prof Broderick Oluyede
Department of Mathematics & Statistical Sciences,
Faculty of Sciences, BIUST
E-mail: oluyedeo@biust.ac.bw

DECLARATION REGARDING THE WORK AND COPYRIGHT

Candidate Thatayaone Moakofi

Student ID: 14000642

Thesis Titled: New Combinations of Generators for the Construction of Generalized Families and Classes of Distributions with Applications

I **Thatayaone Moakofi**, certify that the Thesis is all my own original work and that I have not obtained a degree in this University or elsewhere on the basis of any of this work.

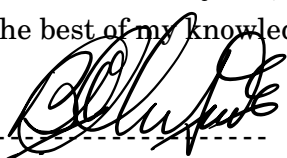
This dissertation/thesis is copyright material protected under the Berne Convention, the Copyright and Neighbouring Rights Act, Act. No. 8 of 2000 and other international and national enactments, in that behalf, on intellectual property. It must not be reproduced by any means, in full or in part, except for short extracts in fair dealing; for researcher private study, critical scholarly review or discourse with an acknowledgement, without the written permission of the office of the Postgraduate School, on behalf of both the author and the BIUST.

Signed: -----

Date: 08/01/2024-----

Primary Supervisor Prof Broderick Oluyede

I **Prof Broderick Oluyede**, hereby confirm that I have inspected the above titled thesis and, to the best of my knowledge, it is based on the original work of the candidate.

Signed: -----

Date: 08/01/2024-----

Abstract

Although, the baseline distributions such as Lindley, Weibull, logistic, gamma, exponential etc. have been widely used in statistical modeling, they however have shortcomings when it comes to modeling some new emerging data sets. Some of their shortcomings are:

- **Limited flexibility:** Classical distributions assume specific shapes for the density and hazard rate functions, limiting their flexibility in modeling various types of survival or failure times data. These means that they may fail to accurately describe or characterize data that exhibit different hazard rate patterns such as bathtub, upside-down bathtub and a combination of bathtub and upside-down bathtub.
- **Lack of tail variation and qualification:** Most of the standard distributions often assume either exponential decay or normal tail behavior. This may limit them to model data that may exhibit heavier or lighter tails.
- **Inadequate modeling of dependence:** Classical or standard distributions are typically used to model independent observations. However, in many real-world scenarios, data may have some form of dependence such as autocorrelation, spatial dependence, or temporal dependencies. Thus, standard distributions may fail to accurately model such forms of data sets.
- **Difficulty in handling missing or censored data:** Classical or standard distributions are typically used to model complete and continuous data. However, data may sometimes be missing or censored, bringing challenges for fitting standard distributions.
- **Limited support for skewed or asymmetric data:** In real world phenomena, data sets may be skewed or asymmetric. In that case, classical distribution such as the normal distribution that only assume symmetry around the mean may bring biased estimates or inappropriate inferences when characterizing such data sets.
- **Lack of interpretability:** Classical or standard distributions often have well-defined mathematical properties that enable straightforward interpretation and facilitate statistical inference. However, new generated classes and families of distributions may provide better fit to the data but lack the same level of interpretability. This brings challenges in drawing meaningful conclusions.

To address these limitations, different transformation methods are utilized to generate new distributions. These new distributions can accommodate various shapes of the hazard rate function commonly observed in real world data sets, hence providing more flexibility in capturing a wide range of data patterns and characteristics. This increased flexibility can help practitioners in selecting appropriate distributions for specific modeling scenarios, leading to better decision-making and more accurate predictions. To address the lack

of interpretability issue, careful model selection and comparison may be used to assess whether the goodness-of-fit justifies the loss in interpretability, this can be done by employing various model selection criteria, such as information criteria e.g., Akaike Information Criterion, Bayesian Information Criterion, Consistent Akaike Information Criterion or likelihood ratio tests. Another approach to address this issue is through the use of visualization techniques, such as density plots, or probability plots to visually compare the characteristics and behavior of new generated classes and families of distributions to equiparameter distributions as well as classical distributions. Researchers can also extend these new distributions via different transformation techniques or study them for different reasons, this may lead to improved modeling methodologies and techniques. Additionally, educators can incorporate these new distributions into their curriculum, equipping students with a broader understanding of statistical modeling techniques and providing them with practical tools for data analysis.

In this work, several new families of distributions are introduced. These new families of distributions were developed via the help of different methods of generating distributions. Chapters 2 and 3 involve extensions of Topp-Leone-G, Marshall-Olkin-G and Gompertz-G families of distributions using Type II Exponentiated Half Logistic generator. Chapter 4 uses Odd Burr III and Topp-Leone generators to obtain the new family of distributions. Chapter 5 uses compounding on the Type II Exponential Half Logistic-Topp-Leone-G to obtain a new generalized class of distributions. Chapter 6 are new heavy tailed families of distributions that are obtained via the extensions of the odd Power Generalized Weibull and Generalized Exponentiated Half Logistic generators. These new heavy-tailed families of distributions are particularly useful in finance, insurance, economic and related areas. Statistical properties of these new families of distributions, such as moments, moment generating function, incomplete moments, hazard function, quantile function, series representation of the probability density function, distribution of order statistics, Rényi entropy, probability weighted moments and moment of residual and reversed residual life are studied in detail. The unknown parameters of these new distributions are estimated using the maximum likelihood estimation technique. The consistency property of the maximum likelihood estimators is examined by considering a Monte Carlo simulation study. Some special distributions from these new family of distributions are also derived by specifying the baseline distribution. Applications to real-life data sets from different fields were considered to show the importance and adaptability of these new families of distributions. For the heavy-tailed families, actuarial and risk measures were derived/computed and numerically studied. Comparisons with several other heavy-tailed distributions was done to establish that they have heavier tails than those compared to or competing distributions, and then applied to real life data in finance, insurance and economics.

Dedications

This thesis is dedicated to the memory of my mother Botsalo Moakofi, who made me the person i am today by providing all forms of support.



Acknowledgements

I would like to express my sincere gratitude to Professor Broderick Oluyede for his support and guidance throughout my study and research at BIUST. Also, I would also like to send special thanks to my family (lost and alive) for their encouragement and support throughout my endeavor in education. Special thanks to Simons Foundation for the financial support.



List of Abbreviations and Publications

List of Abbreviations

Abbreviations	Meaning
cdf	cumulative distribution function
pdf	probability density function
hrf	hazard rate function
TL	Topp-Leone
MO	Marshall-Olkin
Kum	Kumaraswamy
RB	Ristić and Balakrishnan
Gom	Gompertz
BIII	Burr III
EHL	Exponentiated Half Logistic
THIEHL	Type II Exponentiated Half Logistic
OPGW	Odd Power Generalized Weibull
PS	Power Series
GEN-EHL	Generalized Exponentiated Half Logistic
TI-HT	Type I Heavy-Tailed
TII-TL-G	Type II Topp-Leone-G
Exp-G	exponentiated-G
GFPD	Generalized Family of Probability Distribution
PSCoD	Power Series Class of Distributions
CoD	Class of Distributions
PP	Probability Plots
Value-at-Risk	VaR
Tail-Value-at-Risk	TVaR
Tail Variance	TV
Tail Variance Premium	TVP
PWMs	Probability Weighted Moments
MLE	Maximum Likelihood Estimation
MLEs	Maximum Likelihood Estimates
GoF	Goodness-of-Fit
ABIAS	Average Bias
RMSE	Root Mean Squared Error
AIC	Akaike Information Criterion
CAIC	Consistent Akaike Information Criterion
BIC	Bayesian Information Criterion
W^*	Cramér-von Mises statistic
A^*	Anderson-Darling statistic
K-S	Kolmogorov-Smirnov statistic

Table 1: List of Abbreviations

List of Publications

Published Papers (in the thesis)

- Moakofi, T., Oluyede, B. and Chipepa, F., 2021. Type II exponentiated half-logistic Topp-Leone Marshall-Olkin-G family of distributions with applications. Heliyon, 7(12).
<https://www.sciencedirect.com/science/article/pii/S2405844021026931>
- Moakofi, T. and Oluyede, B., 2023. Type II Exponentiated Half-Logistic Gompertz-G Family of Distributions: Properties and Applications. Mathematica Slovaca, 73(3), 785-810.
<https://www.degruyter.com/document/doi/10.1515/ms-2023-0058/html>

- Moakofi, T., Oluyede, B. and Gabanakgosi, M., 2022. The Topp-Leone odd Burr III-G family of distributions: Model, properties and applications. *Statistics, Optimization Information Computing*, 10(1), 236-262.
<http://iapress.org/index.php/soic/article/view/1135>
- Moakofi, T., Oluyede, B. and Chipepa, F., 2021. Type II exponentiated half-logistic-Topp-Leone-G power series class of distributions with applications. *Pakistan Journal of Statistics and Operation Research*, pp.885-909.
<https://pjsor.com/pjsor/article/view/3775>
- Type I Heavy-Tailed Odd Power Generalized Weibull-G Family of Distributions with Applications: Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics.
<https://dergipark.org.tr/en/pub/cfsuasmas/issue/80844/1195058>
- Heavy-Tailed Generalized Exponentiated Half Logistic-G Family of Distributions with Applications: *International Journal of Mathematics in Operational Research*. <https://pjsor.com/pjsor/article/view/4458>

Other Papers Published During The Study

- Moakofi, T. and Oluyede, B., 2023. The Harris-Topp-Leone-G family of distributions: properties and applications. *International Journal of Mathematics in Operational Research*, 25(2), 208-241.
<https://www.inderscienceonline.com/doi/pdf/10.1504/IJMOR.2023.131390>
- Oluyede, B., Moakofi, T. and Chipepa, F., 2022. The odd power generalized Weibull-G power series class of distributions: properties and applications. *Statistics in Transition new series*, 23(1), pp.89-108.
<https://www.ceeol.com/search/article-detail?id=1037004>
- Chipepa, F., Moakofi, T. and Oluyede, B., 2022. The Marshall-Olkin-odd power generalized Weibull-G family of distributions with applications of COVID-19 data. *Journal of Probability and Statistical Science*, 20(1), 1-20.
<https://journals.uregina.ca/jpss/article/view/509>
- Oluyede, B. and Moakofi, T., 2022. Type II exponentiated half-logistic-Gompertz Topp-Leone-G family of distributions with applications. *Central European Journal of Economic Modelling and Econometrics*, 415-461.

<https://journals.pan.pl/Content/125437/mainFile.pdf>

- Oluyede, B. and Moakofi, T., 2023. The Gamma-Topp-Leone-Type II-Exponentiated Half Logistic-G Family of Distributions with Applications. *Stats*, 6(2), 706-733.
<https://www.mdpi.com/2571-905X/6/2/45>
- Warahena-Liyanage, G., Oluyede, B., Moakofi, T. and Sengweni, W., 2023. The New Exponentiated Half Logistic-Harris-G Family of Distributions with Actuarial Measures and Applications. *Stats*, 6(3), 773-801.
<https://www.mdpi.com/2571-905X/6/3/50>
- Chipepa, F., Oluyede, B., Wanduku, D. and Moakofi, T., 2022. The Exponentiated Half Logistic-Topp-Leone-G Power Series Class of Distributions: Model, Properties and Applications. *Methods of Mathematical Modelling and Computation for Complex Systems*, 341-374.
https://link.springer.com/chapter/10.1007/978-3-030-77169-0_14
- Gabanakgosi, M., Moakofi, T., Oluyede, B. and Makubate, B., 2021. The gamma odd power generalized Weibull-G family of distributions with applications. *Journal of Statistical Modelling: Theory and Applications*, 2(2), 79-101.
http://jsm.yazd.ac.ir/article_2708.html
- Pu, S., Moakofi, T. and Oluyede, B., 2023. The Ristić–Balakrishnan–Topp–Leone–Gompertz-G Family of Distributions with Applications. *Journal of Statistical Theory and Applications*, 22, 116-150.
<https://link.springer.com/article/10.1007/s44199-023-00053-9>

Contents

Contents	xii
List of Figures	xii
List of Tables	xvi
1 Introduction	1
1.1 Some Existing Distributions	1
1.2 Several Generators of New Distributions	4
1.2.1 Compounding Method	4
1.2.2 Transformed-Transformer (T-X) Method	4
1.2.3 The Marshall-Olkin-G (MO-G) Generator	5
1.2.4 Exponentiated-G Generator	5
1.2.5 Type II Transformation	5
1.2.6 Gompertz-G (Gom-G) Generator	6
1.2.7 Topp-Leone-G (TL-G) Generator	6
1.2.8 Type II Topp-Leone-G (TII-TL-G) Generator	6
1.2.9 Harris-G Generator	7
1.2.10 Heavy-Tailed Generator	7
1.2.11 Odd Burr III Generator	7
1.2.12 Exponentiated Half Logistic (EHL-G) Generator	8
1.2.13 Combination of Generators	8
1.3 Motivations	8
1.4 Thesis Outline	9
2 Type II Exponentiated Half-Logistic Topp-Leone-Marshall-Olkin-G (TIEHL-TL-MO-G) Generalized Family of Probability Distributions (GFPD)	10
2.1 TIEHL-TL-MO-G GFPD	10
2.1.1 The Model	10
2.1.2 Density Expansion	13
2.1.3 Order Statistics and Uncertainty Measure	13
2.1.4 Probability Weighted Moments, Moments and Incomplete Moments	15
2.1.5 Maximum Likelihood Estimation	17
2.1.6 Different Special Cases	18
2.1.7 Simulation Study	32
2.1.8 Applications	33
2.1.9 Concluding Remarks	44
3 Type II Exponentiated Half-Logistic-Gompertz-G (TIEHL-Gom-G) Gener-	

alized Family of Probability Distributions with Applications	45
3.1 Type II Exponentiated Half-Logistic-Gompertz-G (TIIHL-Gom-G) GFPD	45
3.1.1 The Model	45
3.1.2 Probability Weighted Moments and Moments	48
3.1.3 Order Statistics and Uncertainty Measure	49
3.1.4 Three Special Cases	51
3.1.5 Maximum Likelihood Estimation	60
3.1.6 Simulation Study	61
3.1.7 Applications	62
3.1.8 Concluding Remarks	70
4 Topp-Leone-Odd Burr III-G (TL-OBIII-G) Generalized Family of Probability Distributions	71
4.1 TL-OBIII-G GFPD	71
4.1.1 The New Model	71
4.1.2 Some Specialized Sub Models	74
4.1.3 Density Expansion	78
4.1.4 Moments and Generating Function	79
4.1.5 Incomplete Moments	80
4.1.6 Rényi Entropy	80
4.1.7 Order Statistics	81
4.1.8 Probability Weighted Moments (PWMs)	82
4.1.9 Maximum Likelihood Estimation	83
4.1.10 Simulation Study	83
4.1.11 Applications	84
4.1.12 Concluding Remarks	90
5 Type II Exponentiated Half Logistic-Topp-Leone-G Power Series (TIIHL-TL-GPS) Class of Distributions (CoD)	91
5.1 TIIHL-TL-GPS CoD	91
5.1.1 The Model	92
5.1.2 Order Statistics and Uncertainty Measure	96
5.1.3 Maximum Likelihood Estimation	97
5.1.4 Some Special Sub-Classes	98
5.1.5 Simulation Study	111
5.1.6 Applications	112
5.1.7 Conclusions	121
6 Type I Heavy-Tailed Odd Power Generalized Weibull-G (TI-HT-OPGW-G) Generalized Family of Probability Distributions and Heavy-Tailed Generalized Exponentiated Half Logistic-G (HT-GEN-EHL-G) Generalized Family of Probability Distributions	122
6.1 TI-HT-OPGW-G GFPD	122

6.1.1	The New Model	123
6.1.2	Reliability and Hazard Rate Functions	124
6.1.3	Sub-Families of TI-HT-OPGW-G GFPD	124
6.1.4	Linear Representation	125
6.1.5	Quantile Function	126
6.1.6	Statistical Properties	127
6.1.7	Moments and Generating Function	127
6.1.8	Order Statistics	127
6.1.9	Rényi Entropy	128
6.1.10	Moment of Residual and Reversed Residual Life	129
6.1.11	Special Cases	130
6.1.12	TI-HT-OPGW-Log-Logistic (TI-HT-OPGW-LLoG) Distribution	130
6.1.13	TI-HT-OPGW-Weibull (TI-HT-OPGW-W) Distribution	133
6.1.14	TI-HT-OPGW-Rayleigh (TI-HT-OPGW-R) Distribution	136
6.1.15	TI-HT-OPGW-Standard Half Logistic (TI-HT-OPGW-SHL) Distribution	139
6.1.16	Parameter Estimation	142
6.1.17	Monte Carlo Simulation Study	143
6.1.18	Actuarial Measures	145
6.1.19	Value-at-Risk (VaR)	145
6.1.20	Tail-Value-at-Risk (TVaR)	146
6.1.21	Tail Variance (TV)	146
6.1.22	Tail Variance Premium (TVP)	146
6.1.23	Numerical Study for the Risk Measures	146
6.1.24	Applications	147
6.1.25	Agriculture Data	148
6.1.26	Failure Times Data	152
6.1.27	Biomedical Sciences Data	156
6.1.28	Insurance Data	160
6.1.29	Concluding Remarks	163
6.2	HT-GEN-EHL-G GFPD	164
6.2.1	The Proposed Distribution	165
6.2.2	Quantile Function	165
6.2.3	Density Expansion	166
6.2.4	Statistical Properties	166
6.2.5	Moments, Generating Function and Incomplete Moments	166
6.2.6	Distribution of Order Statistics	167
6.2.7	Rényi Entropy	168
6.2.8	Maximum Likelihood Estimation	169
6.2.9	Simulations	169
6.2.10	Risk Measures	171
6.2.11	Value at Risk	171
6.2.12	Tail Value at Risk	171

6.2.13 Tail Variance	171
6.2.14 Tail Variance Premium	172
6.2.15 Numerical Study for the Risk Measures	172
6.2.16 Special Cases	173
6.2.17 HT-GEN-EHL-Weibull (HT-GEN-EHL-W) Distribution	173
6.2.18 HT-GEN-EHL-Log-Logistic (HT-GEN-EHL-LLoG) Distribution	176
6.2.19 HT-GEN-EHL-Standard Half Logistic (HT-GEN-EHL-SHL) Distribu- tion	179
6.2.20 Inference	183
6.2.21 Insurance Data	183
6.2.22 Remission Times Data	186
6.2.23 Silicon Nitride Data	189
6.2.24 Failure Times Data	192
6.2.25 Conclusions	195
7 Summary and Areas of Further Studies	197
Appendix	198
A Appendix	198

List of Figures

2.1	Visualization of the pdf and hrf for the TIIEHL-TL-MO-U distribution	19
2.2	3 D Plots of skewness and kurtosis for TIIEHL-TL-MO-U distribution	20
2.3	Visualization of the pdf and hrf for the TTIEHL-TL-MO-LLoG distribution .	21
2.4	3 D Plots of skewness and kurtosis for TIIEHL-TL-MO-LLoG distribution . .	23
2.5	Visualization of the pdf and hrf for the TIIEHL-TL-MO-W distribution	24
2.6	3 D Plots of skewness and kurtosis for TIIEHL-TL-MO-W distribution	26
2.7	Visualization of the pdf and hrf for the TIIEHL-TL-MO-N distribution	28
2.8	Visualization of the pdf and hrf for the TIIEHL-TL-MO-E distribution	30
2.9	3 D Plots of skewness and kurtosis for TIIEHL-TL-MO-E distribution	32
2.10	Profile log-likelihood function plots for parameters of TIIEHL-TL-MO-LLoG distribution on the toluene exposure concentrations data	35
2.11	Fitted Densities and Probability Plots of the Toluene Exposure Concentra- tions Data	37
2.12	Profile log-likelihood function plots for parameters of TIIEHL-TL-MO-LLoG distribution on the actual taxes revenue data	38
2.13	Fitted Densities and Probability Plots of the Actual Taxes Revenue Data . . .	40
2.14	Profile log-likelihood function plots for parameters of TIIEHL-TL-MO-LLoG distribution on the kevlar 49/epoxy data	41
2.15	Fitted Densities and Probability Plots of the Kevlar 49/epoxy Data	43
3.1	Visualization of Skewness and Kurtosis for the TIIEHL-Gom-P distribution .	53
3.2	Visualization of the pdf and hrf for the TIIEHL-Gom-P distribution	54
3.3	Visualization of Skewness and Kurtosis for the TIIEHL-Gom-W distribution	56
3.4	Visualization of the pdf and hrf for the TIIEHL-Gom-W distribution	57
3.5	Visualization of Skewness and Kurtosis for the TIIEHL-Gom-LLoG distribu- tion	59
3.6	Visualization of the pdf and hrf for the TIIEHL-Gom-LLoG distribution . . .	60
3.7	Profile log-likelihood function plots for parameters of TIIEHL-Gom-W distri- bution on the carbon fibres data	64
3.8	Fitted Densities and Probability Plots of the Carbon Fibres Data	66
3.9	Fitted Total Time on Test (TTT) and Kaplan-Meier (K-M) Plots for Carbon Fibres Data	66
3.10	Profile log-likelihood function plots for parameters of TIIEHL-Gom-W distri- bution on the breaking stress of carbon fibres data	67
3.11	Fitted Densities and Probability Plots of the Breaking Stress of Carbon Fibres	69
3.12	Fitted TTT and K-M Plots for Breaking Stress of Carbon Fibres	69
4.1	Density and hrf Plots for TL-OBIII-LLoG Distribution	75

4.2	Density and hrf Plots for TL-OBIII-Power Distribution	76
4.3	Density and hrf Plots for TL-OBIII-L Distribution	78
4.4	Profile log-likelihood function plots for parameters of TL-OBIII-LLoG distribution on the growth hormone data	86
4.5	Fitted Densities and Probability Plots for Dataset 1	87
4.6	Profile log-likelihood function plots for parameters of TL-OBIII-LLoG distribution on the airborne communication transceiver data	88
4.7	Fitted Densities and Probability Plots for Dataset 2	89
5.1	Visualization of the pdf and hrf for the TIIHL-TL-LLoGL distribution . . .	100
5.2	3 D Visualization of skewness and kurtosis for TIIHL-TL-LLoGL distribution	101
5.3	Visualization of the pdf and hrf plots for the TIIHL-TL-LLoGP distribution	102
5.4	3 D Visualization of skewness and kurtosis for TIIHL-TL-LLoGP distribution	104
5.5	Visualization of the pdf and hrf for the TIIHL-TL-WL distribution	106
5.6	3 D Visualization of skewness and kurtosis for TIIHL-TL-WL distribution .	108
5.7	Visualization of the pdf and hrf for the TIIHL-TL-WP distribution	109
5.8	3 D Visualization of skewness and kurtosis for TIIHL-TL-WL distribution .	111
5.9	Profile log-likelihood function plots for parameters of TIIHL-TL-WL distribution on the survival times data	114
5.10	Fitted Densities and Probability Plots for Survival Times Data	115
5.11	Profile log-likelihood function plots for parameters of TIIHL-TL-WL distribution on the kevlar 49/epoxy strands data	117
5.12	Fitted Densities and Probability Plots for kevlar 49/epoxy Data	118
5.13	Profile log-likelihood function plots for parameters of TIIHL-TL-WL distribution on the failure times of 50 components data	119
5.14	Fitted Densities and Probability Plots for Failure Times Data	120
6.1	Pdf and hrf Plots for TI-HT-OPGW-LLoG Distribution	131
6.2	3D Plots of skewness and kurtosis for the TI-HT-OPGW-LLoG distribution .	132
6.3	Pdf and hrf Plots for TI-HT-OPGW-W Distribution	134
6.4	3D Plots of skewness and kurtosis for the TI-HT-OPGW-W distribution	135
6.5	Pdf and hrf Plots for TI-HT-OPGW-R Distribution	137
6.6	3D Plots of skewness and kurtosis for the TI-HT-OPGW-R distribution	139
6.7	Pdf and hrf Plots for TI-HT-OPGW-SHL Distribution	140
6.8	3D Plots for the skewness and kurtosis for the TI-HT-OPGW-SHL distribution	142
6.9	Profile log-likelihood function plots for parameters of TI-HT-OPGW-W distribution on the agriculture data	149
6.10	Fitted Densities and Probability Plots for TFP Growth Data	150
6.11	Fitted TTT, K-M Survival, ECDF and hrf Plots for TFP Growth Data	151
6.12	Profile log-likelihood function plots for parameters of TI-HT-OPGW-W distribution on the failure times data	153

6.13 Fitted Densities and Probability Plots for Failure Times Data	154
6.14 Fitted TTT, K-M, ECDF and hrf Plots for Failure Times Data	155
6.15 Profile log-likelihood function plots for parameters of TI-HT-OPGW-W distribution on the biomedical sciences data	157
6.16 Fitted Densities and Probability Plots for Remission Times Data	158
6.17 Fitted TTT, KM, ECDF and hrf Plots for Remission Times Data	159
6.18 Profile log-likelihood function plots for parameters of TI-HT-OPGW-W distribution on the insurance data	161
6.19 Fitted Densities and Probability Plots for Insurance Data	162
6.20 Fitted TTT, K-M Survival, ECDF and hrf Plots for Insurance Data	163
6.21 Plots of the pdf and hrf for HT-GEN-EHL-W distribution	174
6.22 3D Plots of the skewness for HT-GEN-EHL-W distribution	175
6.23 3D Plots of the kurtosis for HT-GEN-EHL-W distribution	176
6.24 Plots of the pdf and hrf for HT-GEN-EHL-LLoG distribution	177
6.25 3D Plots of the skewness for HT-GEN-EHL-LLoG distribution	178
6.26 3D Plots of the kurtosis for HT-GEN-EHL-LLoG distribution	179
6.27 Plots of the pdf and hrf for HT-GEN-EHL-SHL distribution	180
6.28 3D Plots of the skewness for HT-GEN-EHL-SHL distribution	182
6.29 3D Plots of the kurtosis for HT-GEN-EHL-SHL distribution	182
6.30 Profile log-likelihood function plots for parameters of HT-GEN-EHL-LLoG distribution on the insurance data set	184
6.31 Fitted density superimposed on the histogram and observed probability vs expected Probability Plots for the insurance data	185
6.32 Fitted K-M survival curve, ECDF, the TTT scaled plot, and the fitted hrf for the insurance data	186
6.33 Profile log-likelihood function plots for parameters of HT-GEN-EHL-LLoG distribution on the remission times data	187
6.34 Fitted density superimposed on the histogram and observed probability vs expected Probability Plots for the remission times data	188
6.35 Fitted K-M survival curve, ECDF, the TTT scaled plot, and the fitted hrf for the remission times data	189
6.36 Profile log-likelihood function plots for parameters of HT-GEN-EHL-LLoG distribution on the silicon nitride data	190
6.37 Fitted density superimposed on the histogram and observed probability vs expected Probability Plots for the silicon nitride data	191
6.38 Fitted K-M survival curve, ECDF, the TTT scaled plot, and the fitted hrf for the silicon nitride data	192
6.39 Profile log-likelihood function plots for parameters of HT-GEN-EHL-LLoG distribution on the failure time data	193
6.40 Fitted density superimposed on the histogram and observed probability vs expected Probability Plots for the failure times data	194

6.41 Fitted K-M survival curve, ECDF, the TTT scaled plot, and the fitted hrf for the failure times data	195
---	-----



List of Tables

1	List of Abbreviations	vi
2.1	Table of Quantile for TIIHL-TL-MO-U Distribution	19
2.2	Moments of TIIHL-TL-MO-U Distribution for Selected Parameters	20
2.3	Table of Quantile for TIIHL-TL-MO-LLoG Distribution	22
2.4	Moments of TIIHL-TL-MO-LLoG Distribution for Selected Parameters	22
2.5	Table of Quantile for TIIHL-TL-MO-W Distribution	25
2.6	Moments of TIIHL-TL-MO-W Distribution for Selected Parameters	25
2.7	Table of Quantile for TIIHL-TL-MO-N Distribution	28
2.8	Moments of TIIHL-TL-MO-N Distribution for Selected Parameters	28
2.9	Table of Quantile for TIIHL-TL-MO-E Distribution	30
2.10	Moments of TIIHL-TL-MO-E Distribution for Selected Parameters	31
2.11	Monte Carlo Simulation Results	33
2.12	Monte Carlo Simulation Results	33
2.13	Estimates from the Toluene Exposure Concentrations Data	36
2.14	GoF Statistics for Toluene Exposure Concentrations Data	36
2.15	Estimates from the Actual Taxes Revenue Data	39
2.16	GoF Statistics for Actual Taxes Revenue Data	39
2.17	Estimates from the Time to Failure of Kevlar 49/epoxy Data	42
2.18	GoF Statistics for Time to Failure of Kevlar 49/epoxy Data	42
2.19	Likelihood Ratio Test Results	43
2.20	Likelihood Ratio Test Results	44
2.21	Likelihood Ratio Test Results	44
2.22	Likelihood Ratio Test Results	44
3.1	Table of Quantile for TIIHL-Gom-P Distribution	52
3.2	Moments of TIIHL-Gom-P Distribution for Selected Parameters	52
3.3	Table of Quantile for TIIHL-Gom-W Distribution	55
3.4	Moments of TIIHL-Gom-W Distribution for Selected Parameters	55
3.5	Table of Quantile for TIIHL-Gom-LLoG Distribution	58
3.6	Moments of TIIHL-Gom-LLoG Distribution for Selected Parameters	58
3.7	Monte Carlo Simulation Results	61
3.8	Monte Carlo Simulation Results	62
3.9	Estimates from the Carbon Fibres Data	65
3.10	GoF Statistics for Carbon Fibres Data	65
3.11	Estimates from the Breaking Stress of Carbon Fibres	68
3.12	GoF Statistics for Breaking Stress of Carbon Fibres	68
3.13	Likelihood Ratio Test Results	70

4.1	Table of Quantile for TL-OBIII-LLoG Distribution	75
4.2	Table of Quantile for TL-OBIII-P Distribution	77
4.3	Table of Quantile for TL-OBIII-L Distribution	78
4.4	Moments of TL-OBIII-LLoG Distribution for Selected Parameters	79
4.5	Moments of TL-OBIII-Lomax Distribution for Selected Parameters	80
4.6	Moments of TL-OBIII-Power Distribution for Selected Parameters	80
4.7	Monte Carlo Simulation Results: Mean, Average Bias and RMSE	84
4.8	Estimates of Models for Dataset 1	87
4.9	Estimates of Models for Dataset 2	89
5.1	Special Classes of the TIEHL-TL-GPS Distribution	93
5.2	Table of Quantile for TIEHL-TL-LLoGL Distribution	100
5.3	Moments of TIEHL-TL-LLoGL Distribution for Selected Parameters	100
5.4	Table of Quantile for TIEHL-TL-LLoGP Distribution	103
5.5	Moments of TIEHL-TL-LLoGP Distribution for Selected Parameters	103
5.6	Table of Quantile for TIEHL-TL-WL Distribution	107
5.7	Moments of TIEHL-TL-WL Distribution for Selected Parameters	107
5.8	Table of Quantile for TIEHL-TL-WP Distribution	110
5.9	Moments of TIEHL-TL-WP Distribution for Selected Parameters	110
5.10	Monte Carlo Simulation Results	112
5.11	MLEs and GoF Statistics	115
5.12	MLEs and GoF Statistics	118
5.13	MLEs and GoF Statistics	120
5.14	Likelihood Ratio Test Results	121
5.15	Likelihood Ratio Test Results	121
6.1	Table of Quantile for TI-HT-OPGW-LLoG Distribution	131
6.2	Moments of TI-HT-OPGW-LLoG Distribution for Selected Parameters	132
6.3	Table of Quantile for TI-HT-OPGW-W Distribution	134
6.4	Moments of TI-HT-OPGW-W Distribution for Selected Parameters	135
6.5	Table of Quantile for TI-HT-OPGW-R Distribution	137
6.6	Moments of TI-HT-OPGW-R Distribution for Selected Parameters	138
6.7	Table of Quantile for TI-HT-OPGW-SHL Distribution	141
6.8	Moments of TI-HT-OPGW-SHL Distribution for Selected Parameters	141
6.9	Monte Carlo Simulation Results	144
6.10	Monte Carlo Simulation Results	145
6.11	Simulation Results of VaR, TVaR, TV and TVP	147
6.12	Simulation Results of VaR, TVaR, TV and TVP	147
6.13	Estimates from the TFP Growth Data	150
6.14	GoF Statistics for TFP Growth Data	150
6.15	Estimates from the Failure Times Data	154
6.16	GoF Statistics for Failure Times Data	154
6.17	Estimates from the Remission Times Data	158

6.18 GoF Statistics for Remission Times Data	158
6.19 Estimates from the Insurance Data	162
6.20 GoF Statistics for Insurance Data	162
6.21 Monte Carlo Simulation Results	170
6.22 Monte Carlo Simulation Results	170
6.23 Simulation Results of VaR, TVaR, TV and TVP	173
6.24 Simulation Results of VaR, TVaR, TV and TVP	173
6.25 Table of Quantile for HT-GEN-EHL-W Distribution	175
6.26 Moments of HT-GEN-EHL-W Distribution for Selected Parameters	175
6.27 Table of Quantile for HT-GEN-EHL-LLoG Distribution	178
6.28 Moments of HT-GEN-EHL-LLoG Distribution for Selected Parameters	178
6.29 Table of Quantile for HT-GEN-EHL-SHL Distribution	181
6.30 Moments of HT-GEN-EHL-SHL Distribution for Selected Parameters	181
6.31 MLEs and GoF Statistics	185
6.32 MLEs and GoF Statistics	188
6.33 MLEs and GoF Statistics	191
6.34 MLEs and GoF Statistics	194



1. Introduction

This chapter reviews work on different techniques or methods of generating new probability distributions. The chapter is divided into four sections. Section 1.1 deals with available work in the literature related to methods of generating distributions considered in this thesis. Section 1.2 focuses on different generators or methods of developing distributions. Section 1.3 of this chapter provides motivations for developing the new distributions, while Section 1.4 presents the structure of the whole work.

1.1 Some Existing Distributions

Statistical distributions are widely studied and used in data analysis. These distributions have been utilized for describing data in several areas including engineering, finance, economics, environmental sciences, to name a few. Unfortunately, in most situations, classical distributions like Weibull, Lindley, Lomax, log-logistic, Pareto, Rayleigh distributions do not provide high precision in data analysis. For this reason, there has been an increased interest in constructing new families of continuous distributions by adding additional shape parameter(s) to well known distributions in order to improve their fitting power (flexibility). Some of the recently generated families available in the literature include the flexible Burr X-G distribution by [Al-Babtain et al. \(2021\)](#), the Gamma Kumaraswamy-G generalized family of probability distributions (GFPD) by [Arshad et al. \(2020\)](#), the odd Nadarajah-Haghighi GFPD by [Nascimento et al. \(2019\)](#), the odd Dagum GFPD by [Afify and Alizadeh \(2020\)](#), modified odd Weibull GFPD by [Chesneau and El Achi \(2020\)](#), the U-GFPD by [Jamal et al. \(2022\)](#), a new modified Lehmann type-II G CoD by [Balogun et al. \(2021\)](#), the Zeid-G GFPD by [Ampadu \(2020\)](#), the Burr III-Topp-Leone-G distribution by [Chipepa et al. \(2021\)](#), the Zografos-Balakrishnan-G GFPD by [Nadarajah et al. \(2015\)](#), new power generalized Weibull-G GFPD by [Oluyede et al. \(2021\)](#), the Gompertz-G GFPD by [Alizadeh et al. \(2017a\)](#), the Kumaraswamy Marshall-Olkin GFPD by [Alizadeh et al. \(2015a\)](#), cubic transmuted-G GFPD by [Aslam et al. \(2018\)](#), the Poisson-G GFPD by [Abouelmagd et al. \(2017\)](#), the modified beta transmuted GFPD by [Awodutire et al. \(2021\)](#), power Lindley-G GFPD by [Hassan and Nassr \(2019\)](#), the odd log-logistic Topp-Leone-G GFPD by [Alizadeh et al. \(2018a\)](#), a new Weibull-X distribution by [Ahmad et al. \(2018\)](#), generalized odd Weibull-G distribution by [Tanis and Karakaya \(2021\)](#), odd Lomax-G GFPD by [Cordeiro et al. \(2019\)](#), odd inverse Pareto-G CoD by [Aldahlan et al. \(2019\)](#), the Nadarajah Haghighi Topp Leone-G GFPD by [Reyad et al. \(2019a\)](#), the Weibull-G GFPD by [Bourguignon et al. \(2014\)](#), the odd generalized half-logistic Weibull-G GFPD by [Chipepa et al. \(2020a\)](#), beta-G family by [Eugene et al. \(2002\)](#), sine Topp-Leone-G family by [Al-Babtain et al. \(2020\)](#), the Zubair-G GFPD by [Ahmad \(2020\)](#), a new Kumaraswamy generalized GFPD by [Tahir et al. \(2020\)](#), beta Weibull-G GFPD by [Yousof et al. \(2017\)](#), the odd Burr III GFPD by [Jamal et al. \(2017\)](#), the shifted Gompertz-G GFPD by [Eghwerido and Agu \(2021\)](#), truncated Cauchy

power Weibull-G CoD by [Alotaibi et al. \(2022\)](#), modified alpha power GFPD by [Hussein et al. \(2022\)](#), truncated Cauchy power-inverted Topp-Leone distribution by [Mohamed et al. \(2023\)](#) and additive Chen distribution by [Méndez-González et al. \(2023\)](#).

Additionally, the Marshall-Olkin transformation has been utilized to give the Marshall-Olkin generalized-G GFPD by [Yousof et al. \(2018\)](#), the Marshall-Olkin odd exponential half logistic-G distribution by [Oluyede and Chipepa \(2022\)](#), alpha power Marshall-Olkin-G distribution by [Eghwerido et al. \(2021\)](#), Marshall-Olkin-Gompertz-G GFPD by [Chipepa and Oluyede \(2021\)](#), Marshall-Olkin exponentiated odd exponential half logistic-G family by [Rannona et al. \(2022\)](#), the Marshall-Olkin Odd Burr III-G GFPD by [Affy et al. \(2019\)](#), the beta Marshall-Olkin GFPD by [Alizadeh et al. \(2015b\)](#), the Marshall-Olkin-Topp-Leone-G GFPD by [Khaleel et al. \(2020\)](#), Marshall-Olkin alpha power-G distribution by [Nassar et al. \(2019\)](#) and the Burr III Marshall-Olkin-G GFPD by [Bhatti et al. \(2019\)](#), to mention a few.

In addition to the previously mentioned families, several other notable families have emerged in the literature including the exponentiated odd Lomax exponential distribution by [Dhungana and Kumar \(2022\)](#), bivariate exponentiated inverse flexible Weibull distribution by [El-Morshedy et al. \(2023\)](#), Burr X exponential-G GFPD by [Sanusi et al. \(2020\)](#), the exponentiated generalized CoD by [Cordeiro et al. \(2013\)](#), the exponentiated Gompertz GFPD by [Cordeiro et al. \(2016\)](#), generalized modified exponential-G GFPD by [Handique et al. \(2020\)](#), the odd exponentiated half-logistic-G GFPD by [Affy et al. \(2017\)](#), exponentiated Weibull-H GFPD by [Cordeiro et al. \(2017\)](#), the exponentiated generalized Topp-Leone-G GFPD by [Reyad et al. \(2019b\)](#), log-logistic tan generalized GFPD by [Zaidi et al. \(2021\)](#), the exponentiated half logistic odd Lindley-G GFPD by [Sengweni et al. \(2021\)](#), extended generalized inverted Kumaraswamy Weibull distribution by [Ramzan et al. \(2021\)](#), type-I extended-F GFPD by [Alfaer et al. \(2023\)](#), extended log-logistic distribution by [Alfaer et al. \(2021\)](#) and exponentiated odd log-logistic-G GFPD by [Alizadeh et al. \(2018b\)](#), the odd generalized gamma-G GFPD by [Nasir et al. \(2020\)](#), the odd power generalized Weibull-G GFPD by [Moakofi et al. \(2021c\)](#), the generalized transmuted-G GFPD proposed by [Nofal et al. \(2016\)](#), the generalized odd gamma-G GFPD by [Hosseini et al. \(2018\)](#), a novel generalized GFPD by [Salahuddin et al. \(2021\)](#), generalized truncated Fréchet generated family distributions by [ZeinEldin et al. \(2021\)](#); among others.

The type II transformations have been applied to several distributions, leading to generalized families including type II Kumaraswamy half-logistic-G GFPD by [El-Sherpieny and Elsehetry \(2019\)](#), type II exponentiated half-logistic-Gompertz Topp-Leone-G GFPD by [Oluyede and Moakofi \(2022\)](#), type II general exponential CoD by [Hamedani et al. \(2019\)](#), type II exponentiated half-logistic Topp-Leone-Marshall-Olkin-G GFPD by [Moakofi et al. \(2021a\)](#), type II Topp-Leone GFPD by [Elgarhy et al. \(2018\)](#), type II half logistic GFPD by [Soliman et al. \(2017\)](#), type II half-logistic Kumaraswamy distribution by [ZeinEldin et al. \(2020\)](#), type II exponentiated half-logistic GFPD by [Al-Mofleh et al. \(2020\)](#), type II power Topp-Leone GFPD by [Bantan et al. \(2020\)](#), type II general inverse exponential GFPD by [Jamal et al. \(2017\)](#), type II generalized Topp-Leone GFPD by [Hassan et al. \(2019\)](#), type

II Topp-Leone Inverse Rayleigh distribution introduced by [Mohammed and Yahia \(2019\)](#) and type II half-logistic odd Fréchet CoD by [Alyami et al. \(2022\)](#).

Several distributions suggested in the literature, encompassing the power series class of distributions (PSCoD), include the exponentiated generalized PSCoD by [Oluyede et al. \(2020c\)](#), a new generalized Lindley-Weibull PSCoD by [Makubate et al. \(2021\)](#), type II exponentiated half-logistic-Topp-Leone-G PSCoD by [Moakofi et al. \(2021b\)](#), the exponentiated power generalized Weibull PSCoD by [Aldahlan et al. \(2020\)](#), the exponentiated half logistic-Topp-Leone-G PSCoD by [Chipepa et al. \(2022\)](#), Weibull-PSCoD by [Morais and Barreto-Souza \(2011\)](#), complementary exponential PSCoD by [Flores et al. \(2013\)](#), complementary extended Weibull-PSCoD by [Cordeiro and Silva \(2014\)](#), Burr XII PSCoD by [Silva and Cordeiro \(2015\)](#), extended Weibull-PSCoD distribution by [Silva et al. \(2013\)](#), the T-R $\{Y\}$ PSCoD by [Osatohanmwon et al. \(2020\)](#), the Topp-Leone generalized exponential PSCoD by [Kunjiratanachot et al. \(2018\)](#), the odd Weibull-Topp-Leone-G PSCoD by [Oluyede et al. \(2021b\)](#) and the Burr-Weibull PSCoD by [Oluyede et al. \(2019\)](#).

The Topp-Leone transformation was applied to several distributions, for example; Topp-Leone odd Burr III-G distribution by [Moakofi et al. \(2022\)](#), Topp-Leone Inverse Weibull distribution by [Abbas et al. \(2017\)](#), [Afify et al. \(2021\)](#) developed the Topp-Leone odd log-logistic exponential distribution and applied it to the gauge lengths of 20 mm data and the time to failure (10^3 h) of turbocharger of one type of engine, Topp-Leone Weibull-Lomax distribution presented by [Jamal et al. \(2019\)](#), Topp-Leone Odd Fréchet GFRD by [Al-Marzouki et al. \(2020\)](#), Topp-Leone odd Lindley-G distribution by [Reyad et al. \(2018\)](#), Topp-Leone generated distribution by [Rezaei et al. \(2017\)](#) and [Aryal et al. \(2017\)](#) developed the Topp-Leone generated Weibull distribution.

Data sets in different fields, such as actuarial sciences, reliability, engineering, bio-medical sciences, economic, risk management are usually positive, right-skewed, unimodal with heavier tails. In this case, all above mentioned distributions are not quite suitable to characterise the data sets. These data sets need to be modelled by heavy-tailed distributions. Thus, there is need for development of heavy-tailed distributions. Some heavy-tailed distributions available in the literature include among others; the type-I heavy-tailed Weibull distribution by [Zhao et al. \(2020\)](#), heavy-tailed beta-power transformed Weibull distribution by [Zhao et al. \(2021\)](#), heavy-tailed log-logistic distribution by [Teamah et al. \(2021\)](#), heavy-tailed exponential distribution by [Afify et al. \(2020\)](#), the logit slash distribution by [Korkmaz \(2020\)](#), the LogPH distribution by [Ahn et al. \(2012\)](#), alpha power cosine-Weibull distribution by [Alghamdi and Abd El-Raouf \(2023\)](#), trapezoidal beta distribution by [Figueroa-Zúñiga et al. \(2022\)](#), new heavy-tailed Weibull distribution by [Arif et al. \(2021\)](#), type I heavy-tailed odd power generalized Weibull distribution by [Moakofi and Oluyede \(2023\)](#), alpha power transformed inverse Lomax distribution by [ZeinEldin et al. \(2020\)](#), alpha power Topp-Leone Weibull distribution by [Benkhelifa \(2022\)](#) and alpha power Weibull distribution by [Nassar et al. \(2017\)](#), to mention a few.

1.2 Several Generators of New Distributions

In this section, we discuss several methods of developing new distributions. We will let $G_B(x; \zeta)$ and $g_B(x; \zeta)$ represent the cumulative distribution function (cdf) and probability density function (pdf) of an arbitrary baseline distribution.

1.2.1 Compounding Method

Suppose N is a discrete random variable, truncated at zero, following a power series distribution, then its probability mass function (PMF) is obtained as

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, n = 1, 2, 3, \dots \quad (1.1)$$

In the above given context, the function $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, where $\theta > 0$ and $\{a_n\}_{n \geq 1}$ represents a sequence of positive real numbers.

Supposing $X_{(1)} = \min(X_1, X_2, \dots, X_N)$, then the marginal cdf and pdf of $X_{(1)}$ are defined by

$$F_{X_{(1)}}(x) = 1 - \frac{C(\theta S_B(x; \zeta))}{C(\theta)}, \quad (1.2)$$

and

$$f_{X_{(1)}}(x) = \frac{\theta g_B(x; \zeta) C'(\theta S_B(x; \zeta))}{C(\theta)}, \quad (1.3)$$

respectively, where $S_B(x; \zeta)$ is the survival function of the baseline distribution and ζ is a vector of parameters from the baseline pdf $g_B(x; \zeta)$. This method when properly utilized in concert with other generators allows for the construction of classes as well as families of generalized distributions that can characterize light, heavy, short, and long-tailed data.

The PSCoD include binomial, Poisson, geometric and logarithmic distributions ([Johnson et al. \(1994\)](#)).

1.2.2 Transformed-Transformer (T-X) Method

[Alzaatreh et al. \(2013\)](#) defined the T - X generator as follows

$$F(x; \zeta) = \int_a^{\omega(G_B(x; \zeta))} k(t) dt, \quad (1.4)$$

and

$$f(x; \zeta) = \left[\frac{d}{dx} \omega(G_B(x; \zeta)) \right] k[\omega(G_B(x; \zeta))], \quad (1.5)$$

where a is a real number, $k(t)$ is the generator pdf of a random variable T , and $\omega(G_B(x; \zeta))$ is the function of the cdf of the random variable X .

1.2.3 The Marshall-Olkin-G (MO-G) Generator

The MO-G (Marshall and Olkin (1997)) generator is given as

$$F_{MO-G}(x; \delta, \zeta) = 1 - \frac{\delta \bar{G}_B(x; \zeta)}{1 - \delta \bar{G}_B(x; \zeta)}, \quad (1.6)$$

for $\delta > 0$, $\bar{\delta} = 1 - \delta$ and $\bar{G}_B(x; \zeta) = 1 - G_B(x; \zeta)$, where $G_B(x; \zeta)$ is the baseline cdf. Note that when $\delta = 1$, we have $G_B(x; \zeta) = F(x; \zeta)$, the baseline distribution function. The corresponding pdf is given by

$$f_{MO-G}(x; \delta, \zeta) = \frac{\delta g_B(x; \zeta)}{(1 - \delta \bar{G}_B(x; \zeta))^2}. \quad (1.7)$$

Marshall and Olkin referred to the shape parameter δ as the tilt parameter due to the fact that for all $x > 0$, the hazard functions are ordered based on the values of δ , that is, $h_G(x; \delta, \zeta) \leq h_F(x)$ when $\delta \geq 1$, and $h_G(x; \delta, \zeta) \geq h_F(x; \zeta)$ when $0 < \delta \leq 1$.

1.2.4 Exponentiated-G Generator

The type I exponentiated-G distribution was studied by Lehmann (1953). The cdf and pdf are:

$$F_{E-G}(x; \alpha, \zeta) = (G_B(x; \zeta))^\alpha, \quad (1.8)$$

and

$$f_{E-G}(x; \alpha, \zeta) = \alpha g_B(x; \zeta) (G_B(x; \zeta))^{\alpha-1}, \quad (1.9)$$

respectively, for $\alpha > 0$ and parameter vector ζ .

Gupta et al. (1998) introduced the proportional hazard rate family or type II exponentiated-G technique having the cdf and pdf given as

$$F_{E-G}(x; b, \zeta) = 1 - [1 - G_B(x; \zeta)]^b, \quad (1.10)$$

and

$$f_{E-G}(x; b, \zeta) = b g_B(x; \zeta) [1 - G_B(x; \zeta)]^{b-1}, \quad (1.11)$$

respectively, for $b > 0$ and parameter vector ζ .

1.2.5 Type II Transformation

The Type II transformation method is obtained using the transformed-transformer $(T - X)$ method by Alzaatreh et al. (2013) and has the cdf and pdf given as

$$F(x; \zeta) = 1 - \int_0^{-\log(G_B(x; \zeta))} k(t) dt, \quad (1.12)$$

and

$$f(x; \zeta) = \frac{K'(-\log(G_B(x; \zeta)))g_B(x; \zeta)}{G_B(x; \zeta)}. \quad (1.13)$$

For example [Al-Mofleh et al. \(2020\)](#) introduced the type II exponentiated half logistic-G (TII-EHL-G) GFRD with the cdf given by

$$F_{TII-EHL-G}(x; a, \lambda, \zeta) = 1 - \left[\frac{1 - (G_B(x; \zeta))^\lambda}{1 + (G_B(x; \zeta))^\lambda} \right]^a, \quad (1.14)$$

for $a, \lambda > 0$ and baseline parameter vector ζ . In this work, the parameter λ is taken to be equal to 1.

1.2.6 Gompertz-G (Gom-G) Generator

[Alizadeh et al. \(2017a\)](#) constructed the Gompertz-G generator with the cdf and pdf given by

$$F_{GOM-G}(x; \theta, \gamma, \zeta) = 1 - \exp\left(\frac{\theta}{\gamma} (1 - [1 - G_B(x; \zeta)]^{-\gamma})\right), \quad (1.15)$$

and

$$f_{GOM-G}(x; \theta, \gamma, \zeta) = \theta [1 - G(x; \zeta)]^{-\gamma-1} \exp\left(\frac{\theta}{\gamma} (1 - [1 - G_B(x; \zeta)]^{-\gamma})\right) g_B(x; \zeta), \quad (1.16)$$

respectively, for $\theta, \gamma > 0$ and baseline parameter vector ζ . In this work, we let $\theta = 1$.

1.2.7 Topp-Leone-G (TL-G) Generator

[Al-Shomrani et al. \(2016\)](#) proposed the TL-G generator with the cdf and pdf given by

$$F_{TL-G}(x; b, \zeta) = \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b, \quad (1.17)$$

and

$$f_{TL-G}(x; b, \zeta) = 2b \left[1 - (\bar{G}_B(x; \zeta))^2\right]^{b-1} \bar{G}_B(x; \zeta) g_B(x; \zeta), \quad (1.18)$$

respectively, for $b > 0$ and parameter vector ζ .

1.2.8 Type II Topp-Leone-G (TII-TL-G) Generator

The TII-TL-G generator introduced by [Elgarhy et al. \(2018\)](#) has the cdf and pdf given by

$$F_{TII-TL-G}(x; b, \zeta) = 1 - \left[1 - (G_B(x; \zeta))^2\right]^b, \quad (1.19)$$

and

$$F_{TII-TL-G}(x; b, \zeta) = 2bG_B(x; \zeta)g_B(x; \zeta) \left[1 - (G_B(x; \zeta))^2 \right]^{b-1}, \quad (1.20)$$

respectively, for $b > 0$ and parameter vector ζ .

1.2.9 Harris-G Generator

[Pinho et al. \(2015\)](#) developed the Harris-G GFGPD with the cdf and pdf given by

$$F_{Harris-G}(x; v, \delta, \zeta) = 1 - \left[\frac{\delta (\bar{G}_B(x; \zeta))^v}{1 - \bar{\delta} (\bar{G}_B(x; \zeta))^v} \right]^{\frac{1}{v}}, \quad (1.21)$$

and

$$f_{Harris-G}(x; v, \delta, \zeta) = \frac{\delta^{\frac{1}{v}} g_B(x; \zeta)}{(1 - \bar{\delta} (\bar{G}_B(x; \zeta))^v)^{1 + \frac{1}{v}}}, \quad (1.22)$$

respectively, for $v, \delta > 0$, $\bar{\delta} = 1 - \delta$ and parameter vector ζ . If we take $v = 1$, then the Harris-G generator reduces to the MO-G generator by [Marshall and Olkin \(1997\)](#).

1.2.10 Heavy-Tailed Generator

The type-I heavy-tailed (TI-HT) GFGPD introduced by [Zhao et al. \(2020\)](#) have the cdf and pdf given by

$$F(x; \theta, \zeta) = 1 - \left(\frac{1 - G_B(x; \zeta)}{1 - (1 - \theta)G_B(x; \zeta)} \right)^{\theta}, \quad (1.23)$$

and

$$f(x; \theta, \zeta) = \frac{\theta^2 g_B(x; \zeta) (1 - G_B(x; \zeta))^{\theta-1}}{(1 - (1 - \theta)G_B(x; \zeta))^{\theta+1}}, \quad (1.24)$$

for $\theta > 0, x \in \mathbb{R}$.

1.2.11 Odd Burr III Generator

Let $B_G(x; \zeta) = \frac{G_B(x; \zeta)}{G_B(x; \zeta)}$. A random variable X has the odd Burr III-G distribution if its cdf and pdf are given by

$$F(x; \alpha, \beta, \zeta) = (1 + (B_G(x; \zeta))^{-\alpha})^{-\beta}, \quad (1.25)$$

and

$$f(x; \alpha, \beta, \zeta) = \alpha\beta (1 + (B_G(x; \zeta))^{-\alpha})^{-\beta-1} (B_G(x; \zeta))^{-\alpha-1} \frac{g_B(x; \zeta)}{(\bar{G}_B(x; \zeta))^2},$$

for $\alpha, \beta > 0$, and parameter vector ζ , respectively.

1.2.12 Exponentiated Half Logistic (EHL-G) Generator

The cdf and pdf of the EHL-G generator are given by

$$F_{EHL-G}(x; \alpha, \lambda, \zeta) = \left[\frac{1 - (\bar{G}_B(x; \zeta))^\lambda}{1 + (\bar{G}_B(x; \zeta))^\lambda} \right]^\alpha, \quad (1.26)$$

and

$$f_{EHL-G}(x; \alpha, \lambda, \zeta) = 2\alpha\lambda g_B(x; \zeta) (1 - G_B(x; \zeta))^{\lambda-1} \frac{[1 - (\bar{G}_B(x; \zeta))^\lambda]^{\alpha-1}}{[1 + (\bar{G}_B(x; \zeta))^\lambda]^{\alpha+1}}, \quad (1.27)$$

respectively, for $x > 0$, $\alpha, \lambda > 0$ and parameter vector ζ .

There are additional generators in the literature that include Beta-G (Eugene et al. (2002)), Kumaraswamy-G (Cordeiro and de Castro (2011)), half-logistic-G (Cordeiro et al. (2016)), Gamma-G (Zografos and Balakrishnan (2009)) and Gamma-G (Type II) by Ristic and Balakrishnan (2011).

1.2.13 Combination of Generators

As a new development, we consider putting together new generators, while taking control of issues of identifiability and estimability, via the use of appropriate combinations of generators to produce larger classes and families of distributions that can account for tail variation, control skewness, and kurtosis as well as model light, short, long and heavy-tailed data. In this thesis, we consider combining some of the generators considered in this section to generate new classes and families of distributions.

1.3 Motivations

The basic motivations for developing these new generalized families of distributions are as follows;

- to construct and generate distributions with symmetric, left-skewed, right-skewed, J and reversed-J shapes of the density function;
- to define special models that possesses various types of hazard rate functions including monotonic as well as non-monotonic shapes;
- to construct heavy-tailed distributions for modeling different real data sets;
- to make the skewness and kurtosis more flexible compared to that of the baseline distributon;

- to construct new statistical distributions with better properties than the baseline distribution and other competitive distributions;
- to construct new distributions that can be used to model light, long and short tailed distributed data sets.

1.4 Thesis Outline

This work is made up of 7 chapters inclusive of this one. Chapter 2 presents the type II exponentiated half-logistic-Topp-Leone-Marshall-Olkin-G (TIIEHL-TL-MO-G) GFPD. Chapter 3 is focused on a new family, namely, the type II exponentiated half-logistic-Gompertz-G GFPD. Chapter 4 introduces the Topp-Leone odd Burr III-G (TL-OBIII-G) GFPD. Chapter 5 deals with the type II exponentiated half-logistic-Topp-Leone-G power Series (TIIEHL-TL-GPS) class of distributions. Chapter 6 is more focused on heavy-tailed distributions and comprises of the type I heavy-tailed odd power generalized Weibull-G (TI-HT-OPGW-G) GFPD, and the heavy-tailed generalised exponentiated half logistic-G (HT-GEN-EHL-G) GFPD. Last but not least, Chapter 7 presents the summary of the work and areas of further studies.

2. Type II Exponentiated Half-Logistic Topp-Leone-Marshall-Olkin-G (TIIHL-TL-MO-G) Generalized Family of Probability Distributions (GFPD)

Moakofi, T., Oluyede, B. and Chipepa, F., 2021. Type II exponentiated half-logistic Topp-Leone Marshall-Olkin-G family of distributions with applications. Heliyon, 7(12).

<https://www.sciencedirect.com/science/article/pii/S2405844021026931>

This chapter involves extensions of Topp-Leone-G and Marshall-Olkin-G using Type II Exponentiated Half Logistic generator.

2.1 TIIHL-TL-MO-G GFPD

The aim of this section is to construct and study a new GFPD in order to gain flexibility. The new generalized distribution is named as TIIHL-TL-MO-G GFPD. Several mathematical properties of the new model have been obtained. These includes moments, generating function, distribution of the order statistics and Rényi entropy. The performance of the maximum likelihood estimates (MLEs) is evaluated via a simulation study. Lastly, we apply the generated family to three data sets.

2.1.1 The Model

Using the classes of distributions given by equations (1.14), (1.17) and (1.6), the cdf and pdf of the proposed TIIHL-TL-MO-G GFPD are given by

$$F(x; a, b, \delta, \zeta) = 1 - \left[\frac{1 - \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b}{1 + \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b} \right]^a \quad (2.1)$$

and

$$\begin{aligned} f(x; a, b, \delta, \zeta) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^{b-1} \left(1 - \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b\right)^{a-1} \\ &\times \left(1 + \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b\right)^{-(a+1)} (1 - \bar{\delta}\bar{G}_B(x; \zeta))^{-3} \\ &\times \bar{G}_B(x; \zeta)g_B(x; \zeta), \end{aligned} \quad (2.2)$$

respectively, for $a, b, \delta > 0$ being shape parameters and parameter vector ζ , where $\bar{F}_{MO-G}^2(x; \delta, \zeta) = \frac{\delta^2 (\bar{G}_B(x; \zeta))^2}{(1 - \delta \bar{G}_B(x; \zeta))^2}$. If a random variable X has the TIIEHL-TL-MO-G GFPD, we write $X \sim \text{TIIEHL-TL-MO-G}(a, b, \delta, \zeta)$.

Sub-Families of TIIEHL-TL-MO-G GFPD

In this subsection, some sub-families of the TIIEHL-TL-MO-G GFPD are presented.

- When $a = 1$, we obtain the type II half-logistic Topp-Leone-Marshall-Olkin-G (TIIHL-TL-MO-G) GFPD with the cdf

$$F(x; b, \delta, \zeta) = 1 - \left[\frac{1 - \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b}{1 + \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b} \right] \quad (2.3)$$

and for $b, \delta > 0$, and parameter vector ζ .

- If $b = 1$, we obtain the new GFPD with the cdf

$$F(x; a, \delta, \zeta) = 1 - \left[\frac{\bar{F}_{MO-G}^2(x; \delta, \zeta)}{1 + \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)} \right]^a \quad (2.4)$$

and for $a, \delta > 0$, and parameter vector ζ .

- If $\delta = 1$, we obtain the type II exponentiated half-logistic Topp-Leone-G (TIIEHL-TL-G) GFPD with the cdf

$$F(x; a, b, \zeta) = 1 - \left[\frac{1 - F_{TL-G}(x; b, \zeta)}{1 + F_{TL-G}(x; b, \zeta)} \right]^a \quad (2.5)$$

and for $a, b > 0$, and parameter vector ζ .

- If $a = b = 1$, we obtain the new GFPD with the cdf

$$F(x; \delta, \zeta) = 1 - \left[\frac{\bar{F}_{MO-G}^2(x; \delta, \zeta)}{1 + \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)} \right] \quad (2.6)$$

and for $\delta > 0$, and parameter vector ζ .

- When $a = \delta = 1$, we obtain the type II half-logistic Topp-Leone-G (TIIHL-TL-G) GFPD with the cdf

$$F(x; b, \zeta) = 1 - \left[\frac{1 - F_{TL-G}(x; b, \zeta)}{1 + F_{TL-G}(x; b, \zeta)} \right] \quad (2.7)$$

for $b > 0$, and parameter vector ζ .

- If $\delta = b = 1$, we obtain the new GFPD with the cdf

$$F(x; a, \zeta) = 1 - \left[\frac{(\bar{G}_B(x; \zeta))^2}{1 + (1 - (\bar{G}_B(x; \zeta))^2)} \right]^a \quad (2.8)$$

and for $a > 0$, and parameter vector ζ .

We also note that different baseline cdf $g_B(x; \zeta)$ can lead to several new TIIEHL-TL-MO-G distributions as sub-models.

Hazard Rate and Quantile Functions

In this section, we present the hazard rate function (hrf) and quantile function (qf) of the TIIEHL-TL-MO-G GFPD. The hrf of the TIIEHL-TL-MO-G GFPD is given by

$$\begin{aligned} h_F(x; \alpha, \beta, a, \zeta) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^{b-1} \left(1 - \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b\right)^{2a-1} \\ &\times \left(1 + \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b\right)^{-(2a+1)} (1 - \bar{\delta}\bar{G}_B(x; \zeta))^{-3} \\ &\times \bar{G}_B(x; \zeta)g_B(x; \zeta). \end{aligned}$$

The qf of the TIIEHL-TL-MO-G distribution is obtained by solving the non-linear equation:

$$F(x; a, b, \delta, \zeta) = 1 - \left[\frac{1 - \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b}{1 + \left(1 - \bar{F}_{MO-G}^2(x; \delta, \zeta)\right)^b} \right]^a = u, \quad (2.9)$$

for $0 \leq u \leq 1$, that is,

$$\bar{F}_{MO-G}^2(x; \delta, \zeta) = \left(1 - \left(\frac{1 - (1-u)^{\frac{1}{a}}}{1 + (1-u)^{\frac{1}{a}}}\right)^{\frac{1}{b}}\right)^{\frac{1}{b}},$$

so that

$$\bar{G}_B(x; \zeta) = \frac{\left(1 - \left(\frac{1 - (1-u)^{\frac{1}{a}}}{1 + (1-u)^{\frac{1}{a}}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}}}{\delta + \bar{\delta} \left(1 - \left(\frac{1 - (1-u)^{\frac{1}{a}}}{1 + (1-u)^{\frac{1}{a}}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}}}.$$

Consequently, the qf for the TIIEHL-TL-MO-G distribution is given by

$$Q_G(u; \alpha, \beta, a, \zeta) = X = G^{-1} \left[\frac{\left(1 - \left(\frac{1 - (1-u)^{\frac{1}{a}}}{1 + (1-u)^{\frac{1}{a}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{2}}}{\delta + \bar{\delta} \left(1 - \left(\frac{1 - (1-u)^{\frac{1}{a}}}{1 + (1-u)^{\frac{1}{a}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{2}}} \right]. \quad (2.10)$$

It follows therefore that random variates can be generated from the TIIEHL-TL-MO-G GFPD based on equation (2.10).

2.1.2 Density Expansion

In this section, we present the density expansion of the pdf of TIIEHL-TL-MO-G distribution. Using two generalized binomial series expansions given by

$$(1+z)^{-\beta} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta+k-1}{k} z^k, \quad (1-z)^{\beta} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} z^i$$

for $|z| < 1$, we have

$$f(x; a, b, \delta, \zeta) = \sum_{w=0}^{\infty} C_{w+1} g_{w+1}(x; \zeta), \quad (2.11)$$

where $g_{w+1}(x; \zeta) = (w+1)[G_B(x; \zeta)]^w g_B(x; \zeta)$ is the exponentiated-G (exp-G) pdf with the power parameter $(w+1)$ and parameter vector ζ , where

$$\begin{aligned} C_{w+1} &= \sum_{k,i,j,q=0}^{\infty} \delta^{2(j+1)} \bar{\delta}^q (-1)^{k+i+j+w} \binom{a+k}{k} \binom{a-1}{i} \binom{b(k+i+1)-1}{j} \\ &\times \binom{3+2j}{q} \binom{2j+q+1}{w} \left(\frac{4ab}{w+1} \right). \end{aligned} \quad (2.12)$$

(See the appendix for derivations).

Consequently, the statistical properties of the TIIEHL-TL-MO-G GFPD follow directly from those of the exponentiated-G distribution.

2.1.3 Order Statistics and Uncertainty Measure

Order statistics and uncertainty measures play vital roles in different fields of probability and statistics, such as reliability, lifetime data analysis and information theory. Here, we present the distribution of the r^{th} order statistic and Rényi entropy for the TIIEHL-TL-MO-G GFPD.

Order Statistics

In this subsection, the pdf of the r^{th} order statistic is presented. Let $X_1, X_2, \dots, X_n$ be independent and identically distributed TIIEHL-TL-MO-G random variables. Using the

binomial expansion

$$(1 - F(x))^{n-r} = \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j [F(x)]^j,$$

the pdf of the r^{th} order statistic from the TIIEHL-TL-MO-G pdf $f(x; \alpha, \beta, a, \zeta) = f(x)$ can be written as

$$\begin{aligned} f_{r:n}(x) &= \frac{n!f(x)}{(r-1)!(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} \\ &= \frac{n!f(x)}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} [F(x)]^{m+r-1}. \end{aligned} \quad (2.13)$$

Using equations (2.1) and (2.2), we have

$$f(x)[F(x)]^{m+r-1} = \sum_{w=0}^{\infty} \phi_{w+1} g_{w+1}(x; \zeta), \quad (2.14)$$

where $g_{w+1}(x; \zeta) = (w+1)[G_B(x; \zeta)]^w g_B(x; \zeta)$ is the exponentiated-G (exp-G) pdf with the power parameter $(w+1)$ and parameter vector ζ , and weights ϕ_{w+1} given by

$$\begin{aligned} \phi_{w+1} &= \sum_{k,i,l,j,q=0}^{\infty} \delta^{2(j+1)} \bar{\delta}^q (-1)^{k+i+l+j+w} \binom{m+r-1}{l} \binom{a(l+1)+k}{k} \\ &\times \binom{a(l+1)-1}{i} \binom{b(k+i+1)-1}{j} \binom{3+2j}{q} \binom{2j+q+1}{w} \left(\frac{4ab}{w+1} \right). \end{aligned}$$

(See the appendix for derivations).

Thus, by substituting (2.14) into (2.13), the pdf of the r^{th} order statistic for the TIIEHL-TL-MO-G GPDF can be written as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{w=0}^{\infty} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} \phi_{w+1} g_{w+1}(x; \zeta).$$

Rényi Entropy

Rényi entropy (Rényi (1961)) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^{\infty} [f(x; a, b, \delta, \zeta)]^v dx \right), v \neq 1, v > 0. \quad (2.15)$$

Rényi entropy reduces to Shannon entropy as $v \rightarrow 1$. Note that

$$\begin{aligned}
 \int_0^\infty f^v(x; a, b, \delta, \zeta) dx &= (4ab)^v \sum_{k,i,j,q,w=0}^\infty \delta^{2(v+j)} \bar{\delta}^q (-1)^{k+i+j+w} \binom{v(a+1)+k}{k} \\
 &\times \binom{v(a-1)}{i} \binom{b(k+i+v)-v}{j} \binom{3v+2j-1+q}{q} \\
 &\times \binom{2j+q+v}{w} \int_0^\infty (G_B(x; \zeta))^w (g_B(x; \zeta))^v dx.
 \end{aligned} \tag{2.16}$$

(See the appendix for derivations).

Consequently, Rényi entropy for the TIIEHL-TL-MO-G GFGPD is given by

$$\begin{aligned}
 I_R(v) &= \frac{1}{1-v} \log \left[(4ab)^v \sum_{k,i,j,q,w=0}^\infty \delta^{2(v+j)} \bar{\delta}^q (-1)^{k+i+j+w} \binom{v(a+1)+k}{k} \right. \\
 &\times \binom{v(a-1)}{i} \binom{b(k+i+v)-v}{j} \binom{3v+2j-1+q}{q} \\
 &\times \left. \binom{2j+q+v}{w} \frac{1}{[1+\frac{w}{v}]^v} \int_0^\infty \left(\left[1 + \frac{w}{v} \right] (G_B(x; \zeta))^{\frac{w}{v}} (g_B(x; \zeta))^v dx \right) \right] \\
 &= \frac{1}{1-v} \log \left[\sum_{w=0}^\infty \tau_w \exp((1-v)I_{REG}) \right],
 \end{aligned} \tag{2.17}$$

for $v > 0, v \neq 1$, where $I_{REG} = \frac{1}{1-v} \log \left[\int_0^\infty \left(\left[1 + \frac{w}{v} \right] (G_B(x; \zeta))^{\frac{w}{v}} (g_B(x; \zeta))^v dx \right) \right]$ is the Rényi entropy of exponentiated-G (E-G) distribution with power parameter $(1 + \frac{w}{v})$ and

$$\begin{aligned}
 \tau_w &= (4ab)^v \sum_{k,i,j,q=0}^\infty \delta^{2(v+j)} \bar{\delta}^q (-1)^{k+i+j+w} \binom{v(a+1)+k}{k} \binom{3v+2j-1+q}{q} \\
 &\times \binom{v(a-1)}{i} \binom{b(k+i+v)-v}{j} \binom{2j+q+v}{w} \frac{1}{[1+\frac{w}{v}]^v}.
 \end{aligned}$$

Therefore, Rényi entropy of the TIIEHL-TL-MO-G GFGPD can be obtained from those of the exponentiated-G (E-G) GFGPD.

2.1.4 Probability Weighted Moments, Moments and Incomplete Moments

In this section, probability weighted moments, moments, moment generating function and incomplete moments for the TIIEHL-TL-MO-G GFGPD are presented.

Probability Weighted Moments

The probability weighted moment of a random variable X is defined by

$$\eta_{m,r} = E(X^m F(x)^r) = \int_{-\infty}^\infty x^m F(x)^r f(x) dx.$$

We note that

$$F(x)^r f(x) = \sum_{w=0}^{\infty} C_{w+1}^* g_{w+1}(x; \zeta).$$

(See the appendix for derivations).

Thus, the probability weighted moment of TL-TII-EHL-GPS is given by

$$\eta_{m,r} = E(X^m F(X)^r) = \sum_{w=0}^{\infty} C_{w+1}^* \int_{-\infty}^{\infty} x^m g_{w+1}(x; \zeta) dx,$$

where $g_{w+1}(x; \zeta) = (w+1)[G_B(x; \zeta)]^w g_B(x; \zeta)$ is the exponentiated-G (exp-G) pdf with the power parameter $(w+1) > 0$ and parameter vector ζ , and

$$\begin{aligned} C_{w+1}^* &= \sum_{p,k,i,j,q=0}^{\infty} \delta^{2(j+1)} \bar{\delta}^q (-1)^{p+k+i+j+w} \binom{r}{p} \binom{a(p+1)+k}{k} \binom{a(p+1)-1}{i} \\ &\times \binom{b(k+i+1)-1}{j} \binom{3+2j}{q} \binom{2j+q+1}{w} \left(\frac{4ab}{w+1} \right). \end{aligned} \quad (2.18)$$

Moments and Generating Function

Let $Y_{w+1} \sim \text{Exponentiated} - G(w+1, \zeta)$, then the s^{th} raw moment, μ'_s of the TIIEHL-TL-MO-G GPDF is given by:

$$\mu'_s = E(X^s) = \int_{-\infty}^{\infty} x^s f(x) dx = \sum_{w=0}^{\infty} C_{w+1} E(Y_{w+1}^s),$$

where $E(Y_{w+1}^s)$ is the s^{th} moment of Y_{w+1} and C_{w+1} is given by equation (2.12). The moment generating function (MGF), for $|t| < 1$, is given by:

$$M_X(t) = \sum_{w=0}^{\infty} C_{w+1} M_{w+1}(t),$$

where $M_{w+1}(t)$ is the mgf of Y_{w+1} and C_{w+1} is given by equation (2.12).

Incomplete Moments

For lifetime models and measures of inequality such as Bonferonni and Lorenz curves, it is of particular interest to find the incomplete moments. The s^{th} incomplete moment of X denoted $I_s(t)$ is given by

$$I_s(t) = \int_{-\infty}^t x^s f(x; a, b, \delta, \zeta) dx. \quad (2.19)$$

Using equation (2.11), we obtain

$$I_s(t) = \sum_{w=0}^{\infty} C_{w+1} \int_{-\infty}^t x^s g_{w+1}(x; \zeta) dx. \quad (2.20)$$

By setting $s = 1$ in equation (2.20), we obtain the first incomplete moment of the TIIEHL-TL-MO-G GFRD.

2.1.5 Maximum Likelihood Estimation

Let X follow the TIIEHL-TL-MO-G distribution with the parameters a, b, δ , and ζ . Denoting the vector of model parameters as $\Delta = (a, b, \delta, \zeta)^T$. The log-likelihood function $\ell_n = \ell_n(\Delta)$ obtained from a random sample of size n from the TIIEHL-TL-MO-G GFRD is expressed as:

$$\begin{aligned} \ell_n(\Delta) = & n \ln(4ab\delta^2) + (b-1) \sum_{i=1}^n \ln \left(1 - \bar{F}_{MO-G}^2(x_i; \delta, \zeta) \right) \\ & + (a-1) \sum_{i=1}^n \ln \left(1 - \left(1 - \bar{F}_{MO-G}^2(x_i; \delta, \zeta) \right)^b \right) \\ & - (a+1) \sum_{i=1}^n \ln \left(1 + \left(1 - \bar{F}_{MO-G}^2(x_i; \delta, \zeta) \right)^b \right) \\ & - 3 \sum_{i=1}^n \ln \left(1 - \delta \bar{G}_B(x_i; \zeta) \right) + \sum_{i=1}^n \ln \bar{G}_B(x_i; \zeta) + \sum_{i=1}^n \ln (g_B(x_i; \zeta)), \end{aligned}$$

where $\bar{F}_{MO-G}^2(x_i; \delta, \zeta) = \frac{\delta^2 (\bar{G}_B(x_i; \zeta))^2}{(1 - \delta \bar{G}_B(x_i; \zeta))^2}$.

The maximum likelihood estimates (MLEs) of the parameters, denoted by $\hat{\Delta}$ can be obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \zeta_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure.

The estimated parameters values (standard error in parenthesis), along with -2log-likelihood statistic $(-2 \ln(L))$, Akaike Information Criterion ($AIC = 2p - 2 \ln(L)$), Bayesian Information Criterion ($BIC = p \ln(n) - 2 \ln(L)$) and Consistent Akaike Information Criterion ($AICC = AIC + 2 \frac{p(p+1)}{n-p-1}$), are provided. Here, $L = L(\hat{\Delta})$ represents the likelihood function value at the estimated parameters, n is the number of observations, and p is the count of estimated parameters are presented. In order to compare the models, we use the criteria stated above. We also obtain the goodness-of-fit statistics: Cramér-von Mises (W^*) and Anderson-Darling Statistics (A^*) described by [Chen and Balakrishnan \(1995\)](#), as well as Kolmogorov-Smirnov (KS) statistic and its P-value. It's noteworthy that, for the log-likelihood function at its maximum (ℓ_n), a larger value is preferable. Conversely, for AIC, AICC, BIC, and the goodness-of-fit statistics W^* , A^* and KS , smaller values are preferred.

We can use the likelihood ratio (LR) test to compare the fit of the TIIEHL-TL-MO-G distri-

bution with its sub-models for a given data set. For example, to test $b = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{a}, \hat{b}, \hat{\delta}, \hat{\zeta}_k)) - \ln(L(\tilde{a}, 1, \tilde{\delta}, \tilde{\zeta}_k))]$, where $\hat{a}$, $\hat{b}$, $\hat{\delta}$ and $\hat{\zeta}_k$ are the unrestricted estimates, and $\tilde{a}$, $\tilde{\delta}$ and $\tilde{\zeta}_k$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi_{\epsilon}^2$, where χ_{ϵ}^2 denote the upper 100 ϵ % point of the χ^2 distribution with 1 degree of freedom.

2.1.6 Different Special Cases

In this section, we consider several special cases of the TIIEHL-TL-MO-G GFPD, specifically when the distribution function $G_B(x; \zeta)$ is uniform, log-logistic, Weibull, Normal and exponential distributions, respectively.

TIIEHL-TL-MO-Uniform (TIIEHL-TL-MO-U) Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x; \theta) = \frac{x}{\theta}$ and $g(x; \theta) = \frac{1}{\theta}$ for $0 < x < \theta < \infty$. Then, the new TIIEHL-TL-MO-U distribution has cdf and pdf respectively given by

$$F(x; a, b, \delta, \theta) = 1 - \left[\frac{1 - \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^b}{1 + \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^b} \right]^a,$$

and

$$\begin{aligned} f(x; a, b, \delta, \theta) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^{b-1} \left(1 - \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^b\right)^{a-1} \\ &\times \left(1 + \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^b\right)^{-(a+1)} \left(1 - \delta \left(1 - \frac{x}{\theta}\right)\right)^{-3} \left(1 - \frac{x}{\theta}\right) \frac{1}{\theta}. \end{aligned}$$

The hrf is given by

$$\begin{aligned} h(x; a, b, \delta, \theta) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^{b-1} \left(1 - \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^b\right)^{a-1} \\ &\times \left(1 + \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^b\right)^{-(a+1)} \left(1 - \delta \left(1 - \frac{x}{\theta}\right)\right)^{-3} \\ &\times \left(1 - \frac{x}{\theta}\right) \frac{1}{\theta} \left[\frac{1 - \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^b}{1 + \left(1 - \bar{F}_{MO-U}^2(x; \delta, \theta)\right)^b} \right]^{-a} \end{aligned}$$

for $a, b, \delta > 0$ being shape parameters $\theta > 0$ is the scale parameter, $0 < x < \theta < \infty$, where $\bar{F}_{MO-U}^2(x; \delta, \theta) = \frac{\delta^2 \left(1 - \frac{x}{\theta}\right)^2}{(1 - \delta \left(1 - \frac{x}{\theta}\right))^2}$.

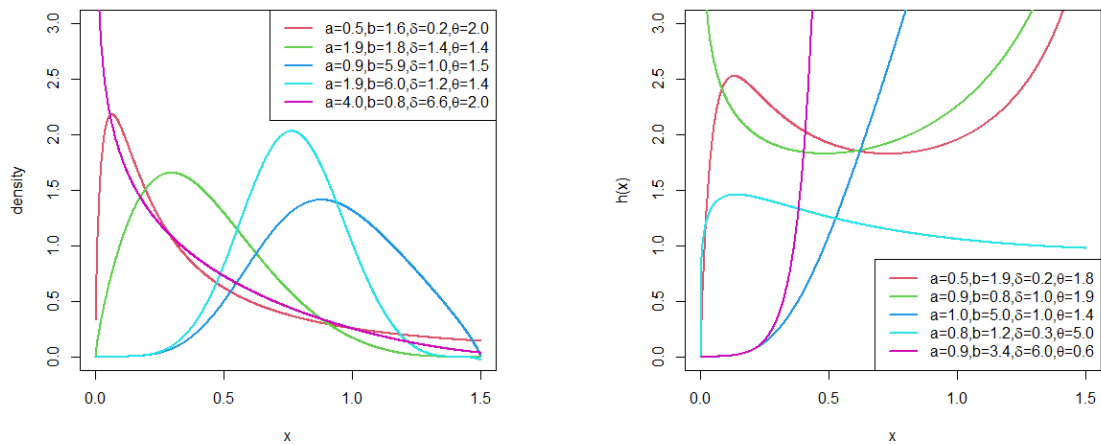


Figure 2.1: Visualization of the pdf and hrf for the TIIEHL-TL-MO-U distribution

Figure 2.1 show the plots of the pdf and hrf of the TIIEHL-TL-MO-U distribution. The pdf can take various shapes that include almost symmetric, decreasing, left, or right-skewed. Furthermore, the hrf for the TIIEHL-TL-MO-U distribution exhibit increasing, J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

	(a, b, δ, θ)				
u	(0.8, 1.3, 3.5, 3)	(1.4, 1.9, 2.5, 1.1)	(1, 0.5, 0.5, 2.2)	(3, 2.9, 0.4, 1.8)	(0.9, 0.5, 9.5, 8.6)
0.1	0.5755	0.2257	0.0058	0.1001	0.1361
0.2	0.9442	0.3202	0.0070	0.1428	0.5857
0.3	1.253	0.4001	0.0180	0.1731	1.3577
0.4	1.5284	0.4681	0.0363	0.2051	2.3915
0.5	1.7790	0.5355	0.0649	0.2372	3.5820
0.6	2.0174	0.5994	0.1096	0.2740	4.7914
0.7	2.2422	0.6716	0.1906	0.3141	5.9177
0.8	2.4637	0.7474	0.3226	0.3685	6.8994
0.9	2.6895	0.8437	0.5935	0.4560	7.7303

Table 2.1: Table of Quantile for TIIEHL-TL-MO-U Distribution

	(a, b, δ, θ)				
	(3.5,1.3,0.5,1.5)	(2.7,1.8,2.5,0.9)	(3.4,2.5,4.5,2.5)	(1,2.9,5.4,1.3)	(3.3,1.5,4.5,1)
$E(X)$	0.0923	0.3186	0.1327	0.3269	0.3811
$E(X^2)$	0.0154	0.1251	0.1064	0.2787	0.1806
$E(X^3)$	0.0037	0.0554	0.0888	0.2432	0.0968
$E(X^4)$	0.0012	0.0264	0.0763	0.2157	0.0564
$E(X^5)$	0.00050	0.0130	0.0668	0.1939	0.0349
$E(X^6)$	0.0002	0.0063	0.0595	0.1761	0.0226
SD	0.0829	0.1535	0.2979	0.4145	0.1881
CV	0.8985	0.4819	2.2450	1.2680	0.49367
CS	1.9338	0.1440	1.9344	0.5566	0.1563
CK	8.7532	1.9055	5.0055	1.429	2.2981

Table 2.2: Moments of TIIEHL-TL-MO-U Distribution for Selected Parameters

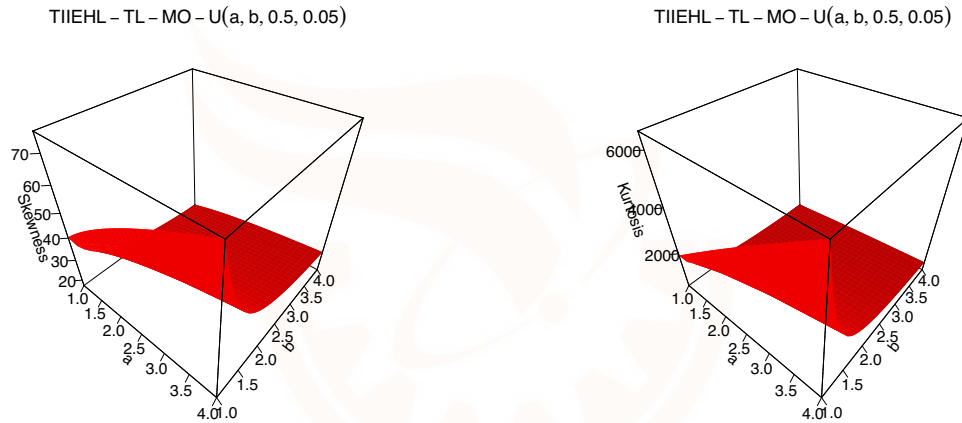


Figure 2.2: 3 D Plots of skewness and kurtosis for TIIEHL-TL-MO-U distribution

Figures 2.2 shows the 3D plots of skewness and kurtosis of the TIIEHL-TL-MO-U distribution. We observe that

- Fixing the parameters θ and λ , the skewness of the TIIEHL-TL-MO-U distribution is right skewed, while kurtosis is leptokurtic as a and b increase.

TIIEHL-TL-MO-Log-Logistic (TIIEHL-TL-MO-LLoG) Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$ for $c > 0$ and $x > 0$. The new TIIEHL-TL-MO-LLoG distribution has cdf and pdf respectively given by

$$F(x; a, b, \delta, c) = 1 - \left[\frac{1 - \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^b}{1 + \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^b} \right]^a$$

and

$$\begin{aligned} f(x; a, b, \delta, c) &= 4abc\delta^2 \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^{b-1} (1 - \delta(1+x^c)^{-1})^{-3} \\ &\times \left(1 - \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^b\right)^{a-1} (1+x^c)^{-1} x^{c-1} \\ &\times \left(1 + \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^b\right)^{-(a+1)} (1+x^c)^{-2}. \end{aligned}$$

The hrf is given by

$$\begin{aligned} h(x; a, b, \delta, c) &= 4abc\delta^2 \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^{b-1} (1 - \delta(1+x^c)^{-1})^{-3} \\ &\times \left(1 - \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^b\right)^{a-1} (1+x^c)^{-1} x^{c-1} (1+x^c)^{-2} \\ &\times \left(1 + \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^b\right)^{-(a+1)} \left[\frac{1 - \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^b}{1 + \left(1 - \bar{F}_{MO-LLoG}^2(x; \delta, c)\right)^b} \right]^{-a}, \end{aligned}$$

for $a, b, \delta, c > 0$ being shape parameters, and $x > 0$, where $\bar{F}_{MO-LLoG}^2(x; \delta, c) = \frac{\delta^2(1+x^c)^{-2}}{(1-\delta(1+x^c)^{-1})^2}$.

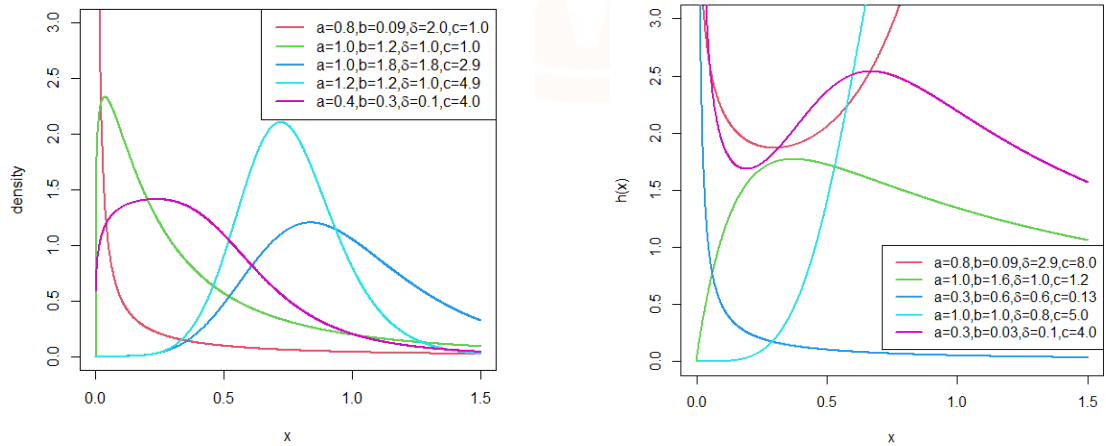


Figure 2.3: Visualization of the pdf and hrf for the TIIEHL-TL-MO-LLoG distribution

Figure 2.3 show the plots of the pdf and hrf of the TIIEHL-TL-MO-LLoG distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrf for the TIIEHL-TL-MO-LLoG distribution exhibit increasing, reverse-J, bathtub, upside bathtub and bathtub followed by upside bathtub shapes.

	(a, b, δ, c)				
u	(1.8, 0.6, 2, 0.9)	(0.4, 3.9, 0.4, 0.7)	(5, 3.5, 1.2, 0.5)	(6, 7.9, 1.8, 1.4)	(2.9, 1.5, 4.6, 4.5)
0.1	0.0050	0.1212	0.0441	0.9137	0.6729
0.2	0.0058	0.2258	0.0745	1.0421	0.7553
0.3	0.01512	0.3668	0.1108	1.1426	0.8181
0.4	0.02531	0.5780	0.1471	1.2298	0.8691
0.5	0.0495	0.9319	0.1954	1.3146	0.9192
0.6	0.0851	1.5785	0.2550	1.4068	0.9686
0.7	0.1400	2.9465	0.3376	1.5076	1.0184
0.8	0.2516	6.6974	0.4628	1.6372	1.0815
0.9	0.5190	25.0720	0.7057	1.8301	1.1757

Table 2.3: Table of Quantile for TIEHL-TL-MO-LLoG Distribution

	(a, b, δ, c)				
	(1.5, 0.3, 0.5, 2.0)	(3.7, 0.9, 1.8, 3.5)	(7.4, 3.5, 1.5, 0.5)	(2, 0.9, 2.3, 0.4)	(0.3, 2.5, 4.0, 9.5)
$E(X)$	0.1343	0.4694	0.2545	0.0772	0.0433
$E(X^2)$	0.0416	0.2512	0.1119	0.0336	0.0412
$E(X^3)$	0.0188	0.1479	0.0645	0.0211	0.0392
$E(X^4)$	0.0106	0.0939	0.0434	0.0153	0.0375
$E(X^5)$	0.0069	0.0635	0.0321	0.0119	0.0358
$E(X^6)$	0.0049	0.0452	0.0253	0.0098	0.0343
SD	0.1536	0.1755	0.2171	0.1664	0.1983
CV	1.1433	0.3740	0.8530	2.1549	4.5764
CS	1.9116	0.1889	1.1768	3.0819	4.3685
CK	7.2537	2.8438	3.9001	12.8611	20.1310

Table 2.4: Moments of TIEHL-TL-MO-LLoG Distribution for Selected Parameters

Figures 2.4 shows the 3D plots of skewness and kurtosis of the TIEHL-TL-MO-LLoG distribution. We observe that

- Fixing the parameters δ and c , the skewness and kurtosis of the TIEHL-TL-MO-LLoG distribution increases as a and b increase.
- Fixing the parameters a and b , the skewness and kurtosis of the TIEHL-TL-MO-LLoG distribution decreases and increases as δ and c increase.

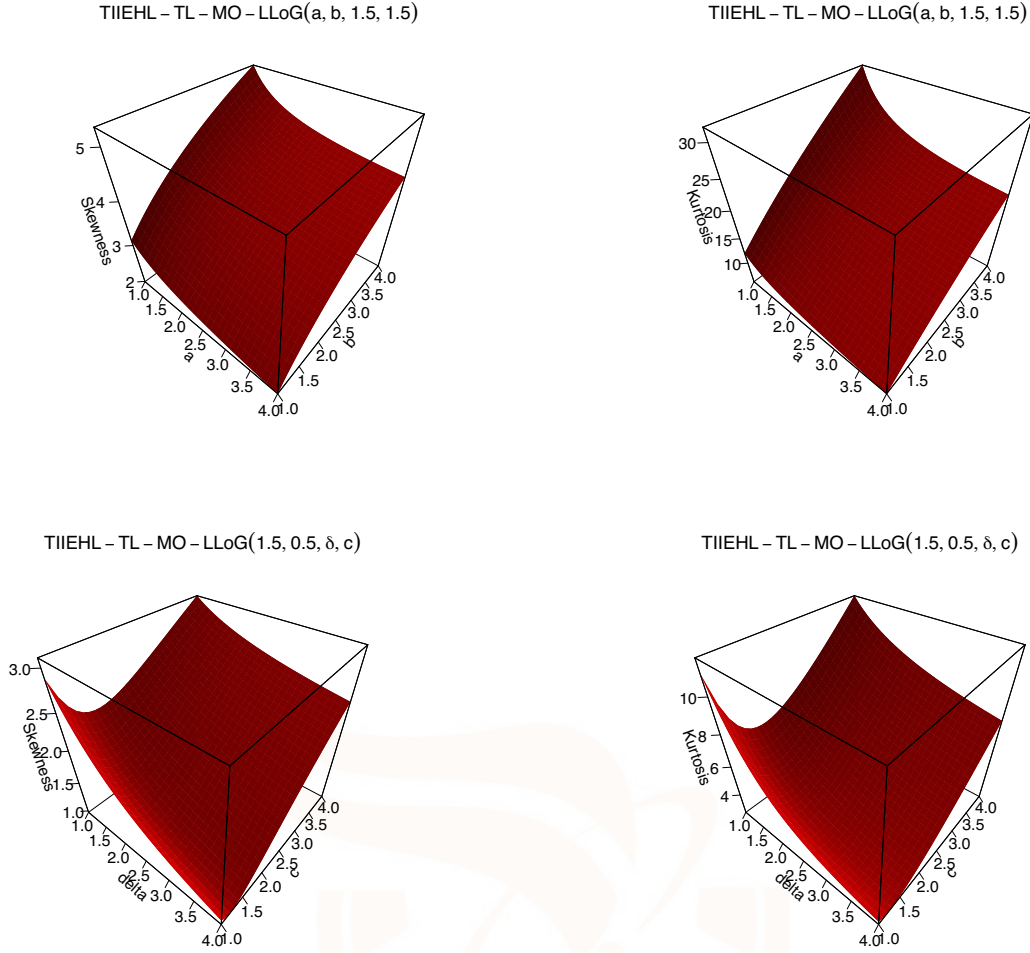


Figure 2.4: 3 D Plots of skewness and kurtosis for TIIEHL-TL-MO-LLoG distribution

TIIEHL-TL-MO-Weibull (TIIEHL-TL-MO-W) Distribution

Suppose the cdf and pdf of the Weibull distribution are given by $G(x; \lambda) = 1 - \exp(-x^\lambda)$, for $x \geq 0, \lambda > 0$ and $g(x; \lambda) = \lambda x^{\lambda-1} \exp(-x^\lambda)$, for $\lambda > 0$, and $x > 0$, then, the cdf and pdf of the TIIEHL-TL-MO-W distribution are respectively given by

$$F(x; a, b, \delta, \lambda) = 1 - \left[\frac{1 - \left(1 - \bar{F}_{MO-W}^2(x; \delta, \lambda)\right)^b}{1 + \left(1 - \bar{F}_{MO-W}^2(x; \delta, \lambda)\right)^b} \right]^a$$

and

$$\begin{aligned} f(x; a, b, \delta, \lambda) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-W}^2(x; \delta, \lambda)\right)^{b-1} \left(1 - \delta \exp(-x^\lambda)\right)^{-3} \\ &\times \left(1 - \left(1 - \bar{F}_{MO-W}^2(x; \delta, \lambda)\right)^b\right)^{a-1} \exp(-2x^\lambda) \\ &\times \left(1 + \left(1 - \bar{F}_{MO-W}^2(x; \delta, \lambda)\right)^b\right)^{-(a+1)} \lambda x^{\lambda-1} \exp(-x^\lambda). \end{aligned}$$

The hrf is given by

$$\begin{aligned}
 f(x; a, b, \delta, \lambda) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-W}^2(x; \delta, \lambda)\right)^{b-1} \left(1 - \bar{\delta} \exp(-x^\lambda)\right)^{-3} \\
 &\times \left(1 - \left(1 - \bar{F}_{MO-W}^2(x; \delta, \lambda)\right)^b\right)^{a-1} \exp(-2x^\lambda) \\
 &\times \left(1 + \left(1 - \bar{F}_{MO-W}^2(x; \delta, \lambda)\right)^b\right)^{-(a+1)} \lambda x^{\lambda-1} \left(\exp(-x^\lambda)\right) \vartheta,
 \end{aligned}$$

for $a, b, \delta, \lambda > 0$ being shape parameters, and $x > 0$, where $\vartheta = \left[\frac{1 - \left(1 - \bar{F}_{MO-W}^2(x; \delta, \lambda)\right)^b}{1 + \left(1 - \frac{\delta^2 \exp(-2x^\lambda)}{(1 - \bar{\delta} \exp(-x^\lambda))^2}\right)^b} \right]^{-a}$,

$$\bar{F}_{MO-W}^2(x; \delta, \lambda) = \frac{\delta^2 \exp(-2x^\lambda)}{(1 - \bar{\delta} \exp(-x^\lambda))^2}.$$

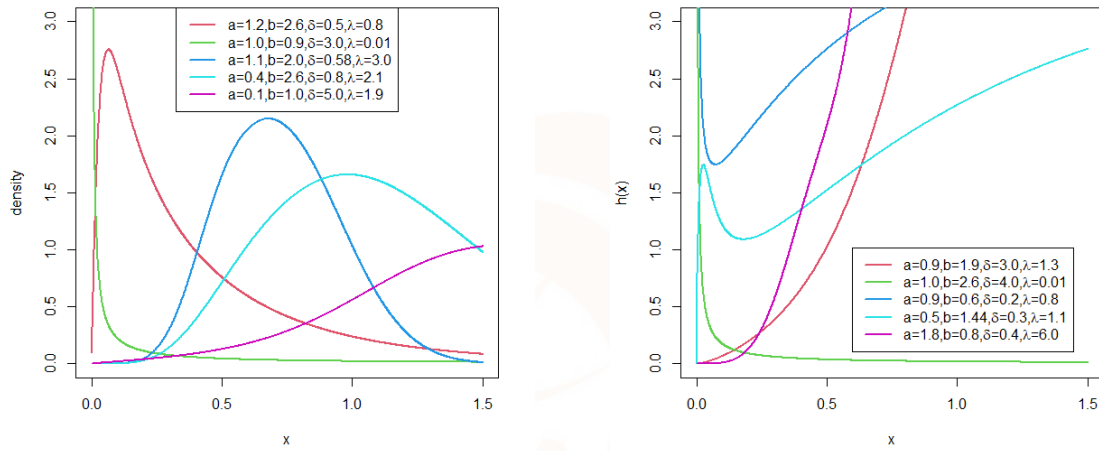


Figure 2.5: Visualization of the pdf and hrf for the TIIEHL-TL-MO-W distribution

Figure 2.5 show the plots of the pdf and hrf of the TIIEHL-TL-MO-W distribution. The pdf can take various shapes that include almost symmetric, increasing, reverse-J, left, or right-skewed. Furthermore, the hrf for the TIIEHL-TL-MO-W distribution exhibit increasing, reverse-J, bathtub and upside bathtub followed by bathtub shapes.

	(a, b, δ, λ)				
u	(1, 0.5, 0.3, 1.9)	(0.6, 0.9, 1.4, 1.7)	(1.3, 0.8, 0.7, 0.9)	(0.2, 1.6, 0.8, 3.4)	(0.9, 1.1, 1.6, 2.5)
0.1	0.0171	0.1672	0.0061	0.6479	0.3301
0.2	0.0349	0.2767	0.0088	0.7873	0.4367
0.3	0.0596	0.3870	0.0214	0.9040	0.5246
0.4	0.0852	0.4956	0.03301	1.0112	0.6023
0.5	0.1200	0.6157	0.0527	1.1170	0.6826
0.6	0.1620	0.7510	0.0813	1.2236	0.7652
0.7	0.2180	0.9099	0.1265	1.3381	0.8582
0.8	0.2962	1.1139	0.2025	1.4664	0.9701
0.9	0.4311	1.4168	0.3553	1.6440	1.1226

Table 2.5: Table of Quantile for TIIEHL-TL-MO-W Distribution

	(a, b, δ, λ)				
	(0.5, 3.6, 3.5, 4.9)	(0.7, 2.9, 5.8, 0.5)	(1.4, 6.5, 3.5, 2.5)	(0.5, 0.9, 0.2, 3.4)	(0.3, 3.5, 0.9, 0.7)
$E(X)$	0.1290	0.0729	0.0556	0.1853	0.2766
$E(X^2)$	0.1223	0.0533	0.0523	0.1327	0.1848
$E(X^3)$	0.1163	0.0420	0.0493	0.1045	0.1376
$E(X^4)$	0.1108	0.0347	0.0467	0.0865	0.1092
$E(X^5)$	0.1059	0.0295	0.0443	0.0738	0.0903
$E(X^6)$	0.1013	0.0257	0.0421	0.0644	0.0770
SD	0.3251	0.2190	0.2219	0.3136	0.3291
CV	2.5198	3.0041	3.9849	1.6918	1.1900
CS	2.1319	2.9642	3.7495	1.4079	0.7434
CK	5.5688	10.4562	15.1181	3.3953	2.0664

Table 2.6: Moments of TIIEHL-TL-MO-W Distribution for Selected Parameters

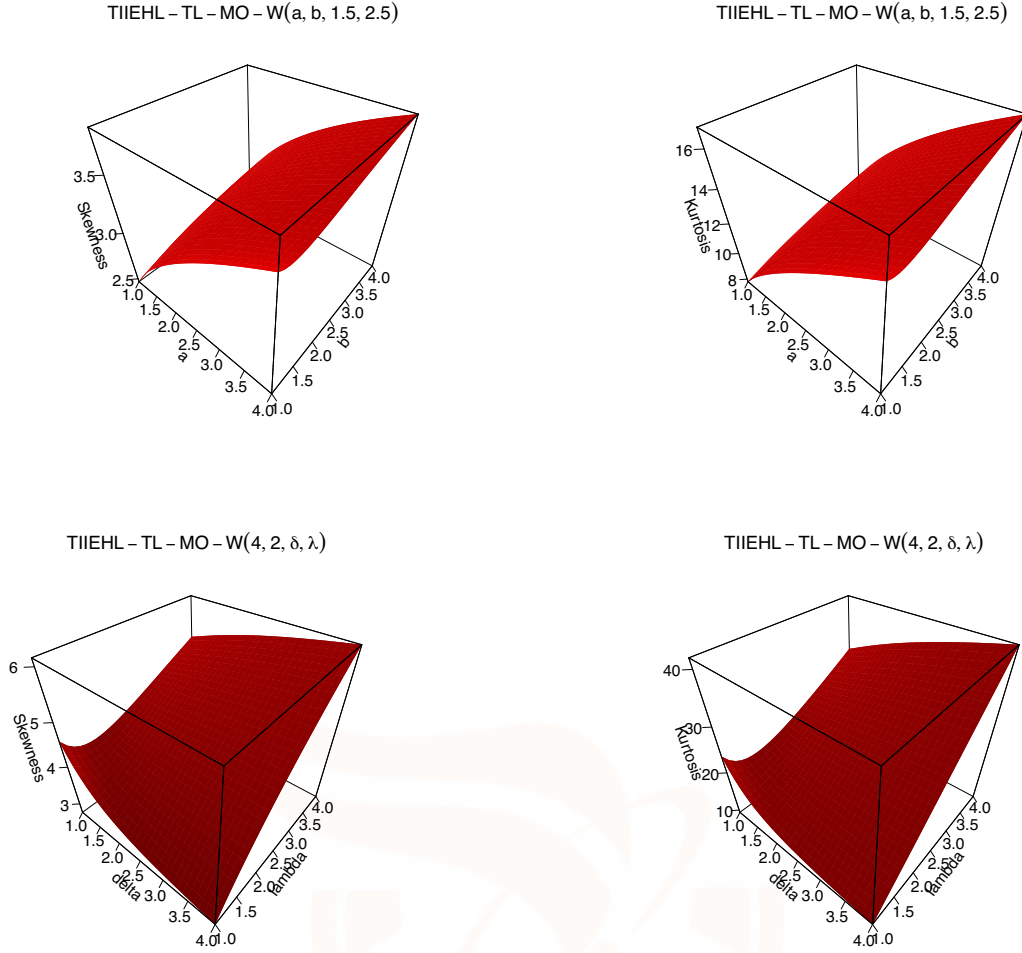


Figure 2.6: 3 D Plots of skewness and kurtosis for TIIEHL-TL-MO-W distribution

Figures 2.6 shows the 3D plots of skewness and kurtosis of the TIIEHL-TL-MO-W distribution. We observe that

- Fixing the parameters δ and λ , the skewness and kurtosis of the TIIEHL-TL-MO-W distribution increases as a and b increase.
- Fixing the parameters a and b , the skewness and kurtosis of the TIIEHL-TL-MO-W distribution increases as δ and λ increase.

TIIEHL-TL-MO-Normal (TIIEHL-TL-MO-N) Distribution

Considering the normal distribution with pdf $g(x; \mu, \sigma) = \sigma^{-1} \phi\left(\frac{x-\mu}{\sigma}\right)$ and cdf $G(x; \mu, \sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right)$, for $\mu \in \mathbb{R}$ and $\sigma > 0$, as the baseline distribution, we obtain the TIIEHL-TL-MO-N cdf and pdf given by

$$F(x; a, b, \delta, \mu, \sigma) = 1 - \left[\frac{1 - \left(1 - \bar{F}_{MO-N}^2(x; \delta, \mu, \sigma)\right)^b}{1 + \left(1 - \bar{F}_{MO-N}^2(x; \delta, \mu, \sigma)\right)^b} \right]^a$$

and

$$\begin{aligned} f(x; a, b, \delta, \mu, \sigma) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-N}^2(x; \delta, \mu, \sigma)\right)^{b-1} \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)^2 \\ &\times \left(1 + \left(1 - \frac{\delta^2 \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)^2}{\left(1 - \bar{\delta} \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)\right)^2}\right)^b\right)^{-(a+1)} \\ &\times \left(1 - \left(1 - \bar{F}_{MO-N}^2(x; \delta, \mu, \sigma)\right)^b\right)^{a-1} \sigma^{-1} \phi\left(\frac{x-\mu}{\sigma}\right) \\ &\times \left(1 - \bar{\delta} \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)\right)^{-3}. \end{aligned}$$

The hrf is given by

$$\begin{aligned} h(x; a, b, \delta, \mu, \sigma) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-N}^2(x; \delta, \mu, \sigma)\right)^{b-1} \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)^2 \\ &\times \left(1 + \left(1 - \frac{\delta^2 \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)^2}{\left(1 - \bar{\delta} \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)\right)^2}\right)^b\right)^{-(a+1)} \\ &\times \left(1 - \left(1 - \bar{F}_{MO-N}^2(x; \delta, \mu, \sigma)\right)^b\right)^{a-1} \sigma^{-1} \phi\left(\frac{x-\mu}{\sigma}\right) \\ &\times \left(1 - \bar{\delta} \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)\right)^{-3} \left[\frac{1 - \left(1 - \bar{F}_{MO-N}^2(x; \delta, \mu, \sigma)\right)^b}{1 + \left(1 - \bar{F}_{MO-N}^2(x; \delta, \mu, \sigma)\right)^b} \right]^{-a}, \end{aligned}$$

for $a, b, \delta > 0$ being shape parameters $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter, $x \in (-\infty, \infty)$, where $\bar{F}_{MO-N}^2(x; \delta, \mu, \sigma) = \frac{\delta^2 \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)^2}{\left(1 - \bar{\delta} \left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right)\right)^2}$.

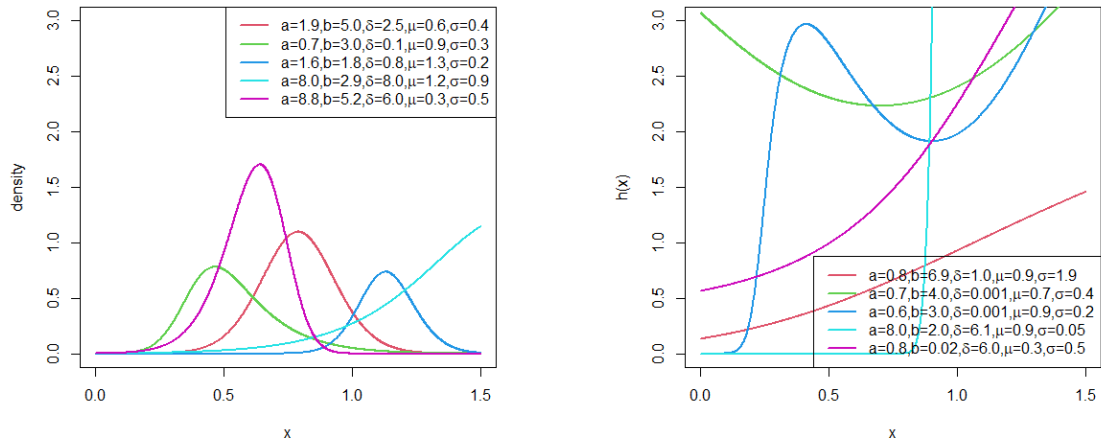


Figure 2.7: Visualization of the pdf and hrf for the TIIEHL-TL-MO-N distribution

Figure 2.7 show the plots of the pdf and hrf of the TIIEHL-TL-MO-N distribution. The pdf can take various shapes that include almost symmetric, increasing, left, or right-skewed. Furthermore, the hrf for the TIIEHL-TL-MO-N distribution exhibit increasing, J, bathtub, and upside bathtub followed by bathtub shapes.

	$(a, b, \delta, \mu, \sigma)$				
u	(0.9, 2.8, 1.3, 0.5, 1.3)	(0.7, 4.6, 2.4, 0.9, 0.5)	(4.9, 8.3, 3.7, 2.8, 3.2)	(0.4, 3.5, 0.8, 0.6, 2.3)	(0.5, 1.9, 0.6, 0.1, 1.0)
0.1	1.20119	2.3011	4.4809	0.5292	0.52815
0.2	1.2653	2.4059	4.5349	0.6546	0.5572
0.3	1.3145	2.4882	4.5770	0.7526	0.5762
0.4	1.3574	2.5635	4.6060	0.8412	0.5936
0.5	1.3996	2.6408	4.6380	0.9410	0.6148
0.6	1.4449	2.7296	4.6662	1.0554	0.6323
0.7	1.4984	2.8416	4.7013	1.2153	0.6539
0.8	1.5698	3.0000	4.7456	1.4291	0.6845
0.9	1.6839	3.2659	4.8105	1.4459	0.7036

Table 2.7: Table of Quantile for TIIEHL-TL-MO-N Distribution

	$(a, b, \delta, \mu, \sigma)$				
	(0.5, 0.9, 0.8, 0.4, 1.2)	(0.7, 0.8, 0.9, 0.5, 0.7)	(0.4, 0.5, 0.5, 0.5, 2.0)	(0.5, 0.3, 0.09, 0.4, 1.2)	(0.8, 0.9, 0.6, 0.8, 1.0)
$E(X)$	0.2003	0.2260	0.2243	0.0517	0.4162
$E(X^2)$	0.0971	0.1101	0.1372	0.0325	0.2540
$E(X^3)$	0.0597	0.0679	0.0993	0.0239	0.1782
$E(X^4)$	0.0413	0.0471	0.0780	0.0189	0.1347
$E(X^5)$	0.0306	0.0352	0.0640	0.0157	0.1068
$E(X^6)$	0.0238	0.0276	0.0547	0.0134	0.0874
SD	0.2387	0.2429	0.2948	0.1728	0.2841
CV	1.1917	1.0744	1.3139	3.3403	0.6827
CS	1.2845	1.1411	1.1543	3.7089	0.2296
CK	3.6833	3.3551	3.0060	16.2650	1.83987151

Table 2.8: Moments of TIIEHL-TL-MO-N Distribution for Selected Parameters

TIIEHL-TL-MO-Exponential (TIIEHL-TL-MO-E) Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x; \lambda) = 1 - e^{-\lambda x}$, $x \geq 0$ and $g(x; \lambda) = \lambda e^{-\lambda x}$, $x > 0, \lambda > 0$. Then, the new TIIEHL-TL-MO-E distribution has cdf and pdf respectively given by

$$F(x; a, b, \delta, \lambda) = 1 - \left[\frac{1 - \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^b}{1 + \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^b} \right]^a$$

and

$$\begin{aligned} f(x; a, b, \delta, \lambda) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^{b-1} \left(1 - \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^b\right)^{a-1} \\ &\times \left(1 + \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^b\right)^{-(a+1)} \left(1 - \delta e^{-\lambda x}\right)^{-3} \\ &\times e^{-\lambda x} \lambda e^{-\lambda x}. \end{aligned}$$

The hrf is given by

$$\begin{aligned} h(x; a, b, \delta, \lambda) &= 4ab\delta^2 \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^{b-1} \left(1 - \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^b\right)^{a-1} \\ &\times \left(1 + \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^b\right)^{-(a+1)} \left(1 - \delta e^{-\lambda x}\right)^{-3} \\ &\times e^{-\lambda x} \lambda e^{-\lambda x} \left[\frac{1 - \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^b}{1 + \left(1 - \bar{F}_{MO-E}^2(x; \delta, \lambda)\right)^b} \right]^{-a}, \end{aligned}$$

for $a, b, \delta, \lambda > 0$, being shape parameters, where $\bar{F}_{MO-E}^2(x; \delta, \lambda) = \frac{\delta^2 e^{-2\lambda x}}{(1 - \delta e^{-\lambda x})^2}$.

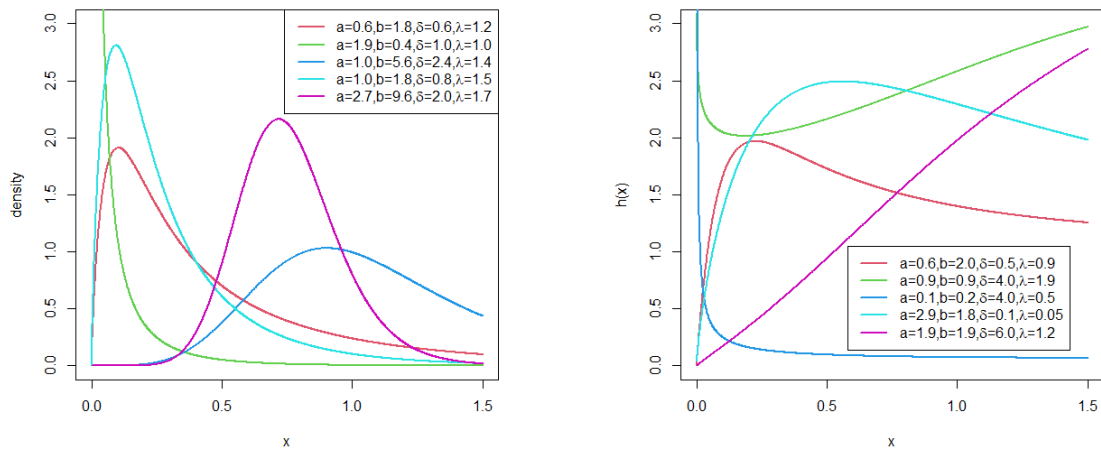


Figure 2.8: Visualization of the pdf and hrf for the TIIEHL-TL-MO-E distribution

Figure 2.8 show the plots of the pdf and hrf of the TIIEHL-TL-MO-E distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrf for the TIIEHL-TL-MO-E distribution exhibit increasing, reverse-J, bathtub, and upside bathtub shapes.

	(a, b, δ, λ)				
u	(5, 1.5, 2.3, 0.9)	(3.6, 1.9, 2.4, 0.7)	(4.3, 3.8, 4.7, 1.9)	(0.5, 0.6, 1.8, 0.4)	(2.9, 3.1, 2.6, 0.5)
0.1	0.0611	0.1866	0.3569	0.0542	0.7439
0.2	0.1004	0.2822	0.4333	0.18470	0.9688
0.3	0.1428	0.3666	0.4900	0.3991	1.1467
0.4	0.1808	0.4479	0.5408	0.7096	1.3100
0.5	0.2235	0.5320	0.5870	1.1411	1.4740
0.6	0.2718	0.6248	0.6349	1.7206	1.6448
0.7	0.3290	0.7328	0.6878	2.5117	1.8370
0.8	0.4017	0.8684	0.7486	3.6423	2.0813
0.9	0.5214	1.0816	0.8376	5.5314	2.4533

Table 2.9: Table of Quantile for TIIEHL-TL-MO-E Distribution

	(a, b, δ, λ)				
	(0.5,0.8,1.5,0.9)	(2.7,5.9,0.8,1.5)	(0.4,3.5,1.5,5.5)	(6.5,0.9,1.2,0.4)	(0.8,0.2,0.9,0.7)
$E(X)$	0.2300	0.3632	0.3049	0.0957	0.0882
$E(X^2)$	0.1336	0.1532	0.1342	0.0212	0.0417
$E(X^3)$	0.0932	0.0737	0.0746	0.0075	0.0267
$E(X^4)$	0.0713	0.0398	0.0483	0.0036	0.0195
$E(X^5)$	0.0577	0.0237	0.0345	0.0021	0.0153
$E(X^6)$	0.0483	0.0154	0.0264	0.0013	0.0126
SD	0.2841	0.1458	0.2030	0.1101	0.1843
CV	1.2353	0.4015	0.6659	1.1504	2.0887
CS	1.1050	0.8285	1.0211	2.4010	2.7329
CK	3.0045	3.9972	3.6565	11.1723	10.3139

Table 2.10: Moments of TIIEHL-TL-MO-E Distribution for Selected Parameters

Figures 2.9 shows the 3D plots of skewness and kurtosis of the TIIEHL-TL-MO-E distribution. We observe that

- Fixing the parameters δ and λ , the skewness of the TIIEHL-TL-MO-E distribution decreases, while kurtosis of the TIIEHL-TL-MO-E distribution becomes leptokurtic as a and b increase.
- Fixing the parameters a and b , the skewness of the TIIEHL-TL-MO-E distribution decreases, while kurtosis of the TIIEHL-TL-MO-E distribution becomes leptokurtic as δ and λ increase.

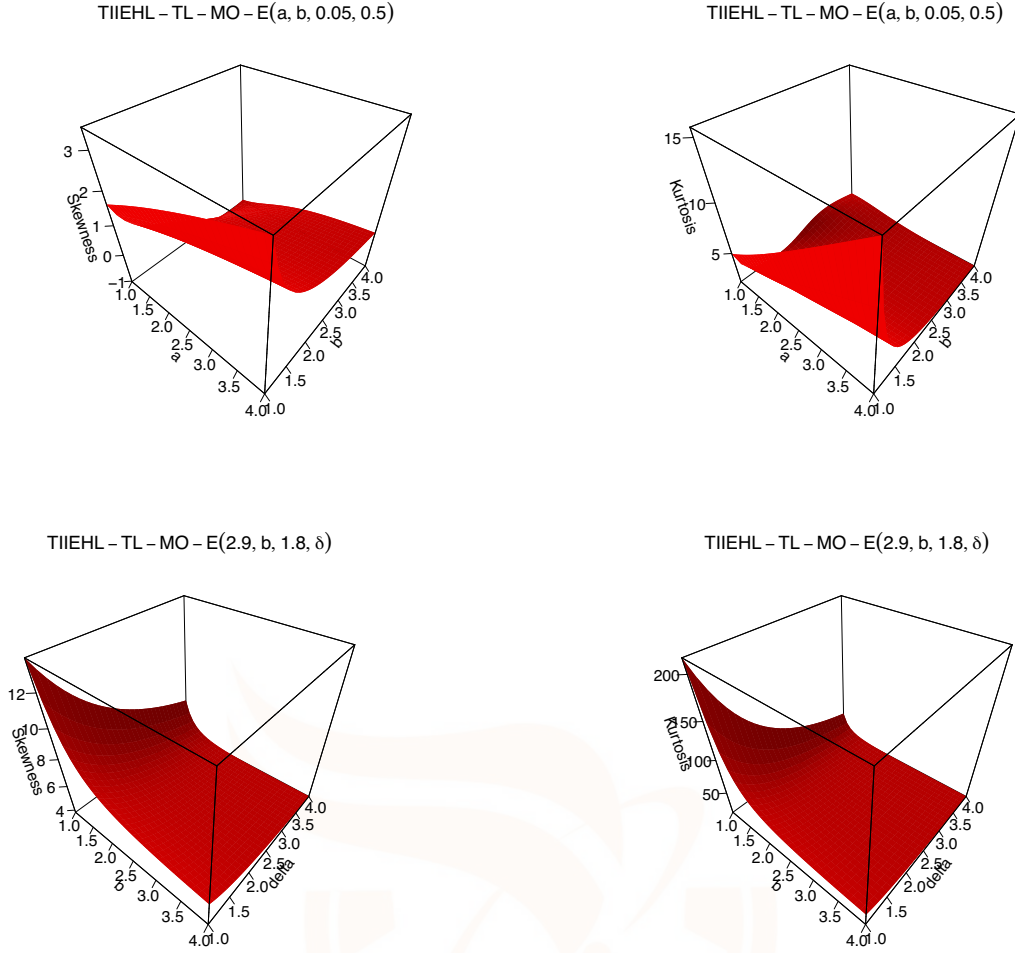


Figure 2.9: 3 D Plots of skewness and kurtosis for TIIEHL-TL-MO-E distribution

2.1.7 Simulation Study

The performance of the TIIEHL-TL-MO-LLoG distribution with the true parameter values b set to be equal to 1 is examined by conducting various simulations for different sizes ($n=50, 100, 200, 400, 800, 1600$) via the R package rootSolve (R Software (2011)). We simulate $N = 3000$ samples for the true parameters values given in Tables 2.11 and 2.12 when b and δ are set to be equal to 1, respectively. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. From the results, it can be seen that as the sample size n increases, the estimates tend to be closer to the true parameter values. Also as the sample size increases the RMSEs decreases in all cases.

Parameter	Sample Size	(1.5, 1.5, 1.5)			(0.9, 0.9, 0.9)		
		Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
a	50	1.0090	1.2137	-0.5909	1.4620	2.5028	0.5620
	100	1.0414	1.0523	-0.5586	1.3009	1.7143	0.4009
	200	1.1382	0.9821	-0.4617	1.1550	1.4973	0.2550
	400	1.2585	0.8653	-0.3414	1.0844	1.1783	0.1844
	800	1.4499	0.5240	-0.1501	0.9962	0.8569	0.0962
	1600	1.5831	0.3753	-0.0169	0.9086	0.6417	0.0086
δ	50	1.0425	1.1188	-0.5574	0.9179	0.1265	0.0179
	100	1.2193	0.9864	-0.3806	0.9177	0.0967	0.0177
	200	1.2640	0.8794	-0.3359	0.9309	0.0827	0.0309
	400	1.3504	0.7426	-0.2495	0.9212	0.0657	0.0212
	800	1.4384	0.5807	-0.1615	0.9191	0.0517	0.0191
	1600	1.5044	0.3212	-0.0955	0.9164	0.0385	0.0164
c	50	1.0384	1.1169	-0.5615	0.9759	0.1255	0.0759
	100	1.2086	0.9899	-0.3913	0.9631	0.0953	0.0630
	200	1.2373	0.8730	-0.3626	0.9314	0.0823	0.0314
	400	1.3441	0.7483	-0.2558	0.9239	0.0663	0.0239
	800	1.4368	0.5861	-0.1631	0.9207	0.0528	0.0207
	1600	1.5993	0.3155	-0.0006	0.9179	0.0400	0.0179

Table 2.11: Monte Carlo Simulation Results

Parameter	Sample Size	(0.9, 2.0, 0.2)			(1.3, 0.9, 1.3)		
		Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
a	50	1.3908	2.3557	0.4908	1.4180	0.3148	0.1180
	100	1.3096	1.7793	0.4096	1.4023	0.2503	0.1023
	200	1.1742	1.5373	0.2742	1.3346	0.1199	0.0346
	400	1.0379	1.1324	0.1379	1.3227	0.0926	0.0227
	800	0.9997	0.9063	0.0997	1.3161	0.0707	0.0161
	1600	0.9116	0.6729	0.0116	1.3103	0.0513	0.0103
δ	50	1.2691	2.0031	-0.7308	1.7233	3.8328	0.8233
	100	1.6803	1.9549	-0.3196	1.4035	1.9091	0.5035
	200	1.8626	1.8799	-0.1373	1.2501	1.6082	0.3501
	400	2.1891	2.2101	0.1291	1.1868	1.2772	0.2868
	800	2.1246	1.6598	0.1246	1.1302	1.0011	0.2302
	1600	2.0608	0.9272	0.0608	1.0680	0.7817	0.1680
c	50	0.5492	0.6878	0.3492	1.4303	0.2440	0.1303
	100	0.4992	0.6757	0.2992	1.3615	0.1899	0.0615
	200	0.4686	0.6067	0.2686	1.3318	0.1186	0.0318
	400	0.3792	0.4378	0.1792	1.3221	0.0925	0.0221
	800	0.2738	0.2612	0.0738	1.3166	0.0713	0.0166
	1600	0.2481	0.1835	0.04812	1.3099	0.0507	0.0099

Table 2.12: Monte Carlo Simulation Results

2.1.8 Applications

In this section, the TIIEHL-TL-MO-LLoG and TIIEHL-TL-MO-E distributions are fitted to three data sets to demonstrate their usefulness when it comes to data modelling. The TIIEHL-TL-MO-LLoG and TIIEHL-TL-MO-E distributions are compared to its nested models and to the following non-nested models: type II exponentiated half logistic Weibull (TIIEHLW) distribution by [Al-Mofleh et al. \(2020\)](#), type II general inverse exponential Burr III (TIIGIE-BIII) distribution by [Jamal et al. \(2017\)](#), type II general inverse exponen-

tial Lomax (TIIGIE-Lx) distribution by [Hamedani et al. \(2019\)](#), exponentiated modified Weibull by [Elbatal \(2011\)](#), and generalized Weibull log-logistic (GWLLoG) distribution by [Cordeiro et al. \(2015\)](#).

Plots of the fitted densities, the histogram of the data and probability plots (PP) ([Chambers et al. \(1983\)](#)) are given in Figure 2.11, Figure 2.13 and Figure 2.15. For the probability plot, $F(x_{(i)}; \hat{\alpha}, \hat{\beta}, \hat{a}, \hat{c})$ was plotted against $\frac{i - 0.375}{n + 0.25}$, $i = 1, 2, \dots, n$, where $x_{(i)}$ are the ordered values of the observed data. The assessment of closeness is quantified by the sum of squares (SS) given by

$$SS = \sum_{i=1}^n \left[F(x_{(i)}) - \left(\frac{i - 0.375}{n + 0.25} \right) \right]^2.$$

The goodness-of-fit (GoF) statistics W^* and A^* , described by Chen and Balakrishnan [Chen and Balakrishnan \(1995\)](#) are also presented in the tables. These statistical measures can aid in assessing the suitability of various distributions to the data. Typically, a superior fit is indicated by lower values of W^* and A^* .

Toluene Exposure Concentrations Data

[Kumagai et al. \(1989\)](#) presented the following time series data for toluene exposure concentrations (8 hr TWAs) for a worker doing stain removing. **(See the data in the appendix).**

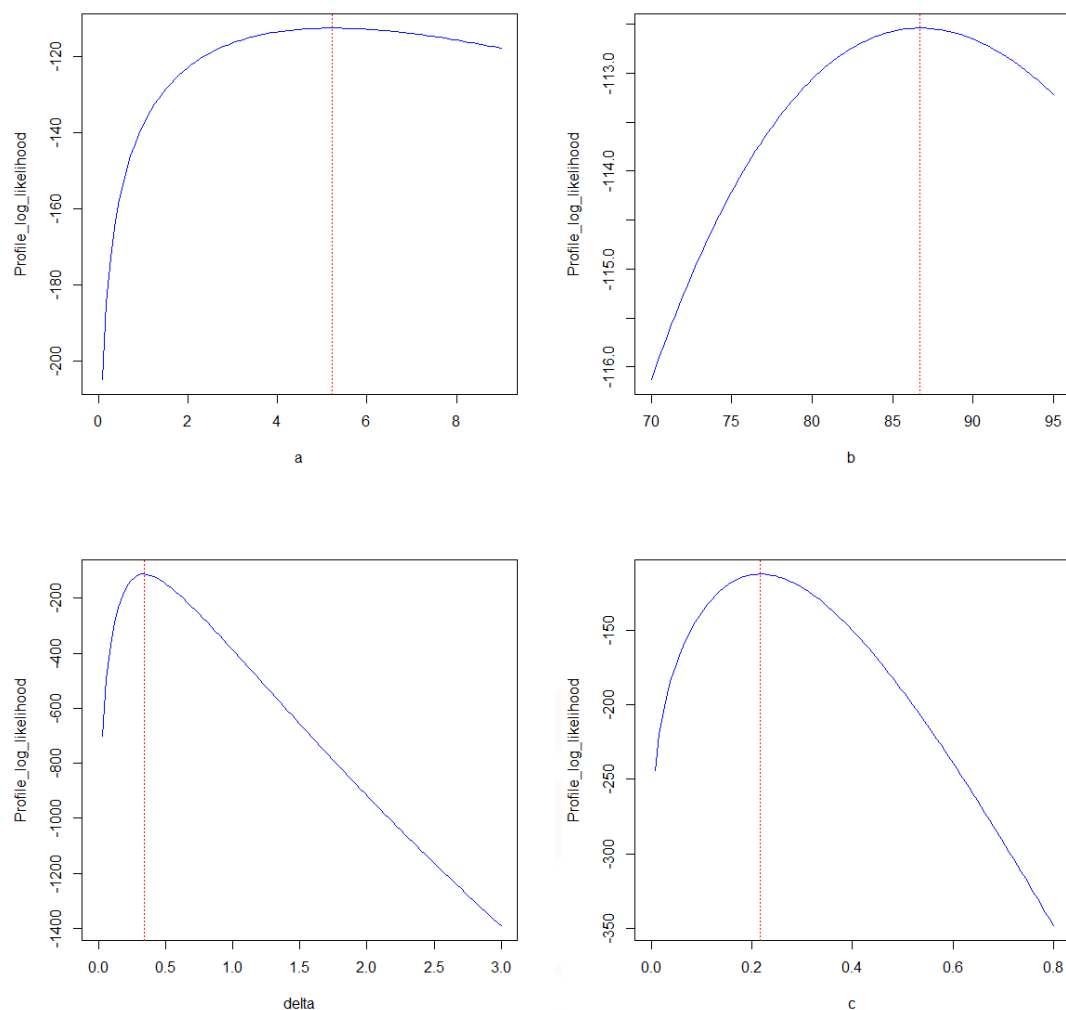


Figure 2.10: Profile log-likelihood function plots for parameters of TIIEHL-TL-MO-LLoG distribution on the toluene exposure concentrations data

Figure 2.10 shows the profile log-likelihood plots for parameters of the TIIEHL-TL-MO-LLoG distribution on the toluene exposure concentrations data. From the profile log-likelihood plots, we can see that the MLEs for the parameters of the TIIEHL-TL-MO-LLoG distribution are unique.

Model	Estimates			
	a	b	δ	c
TIEHL-TL-MO-LLoG	5.2239 (7.9433)	86.7008 (0.0669)	0.3397 (0.0468)	0.2175 (0.1168)
TIEHL-TL-MO-LLoG(1, b , δ , c)	1 -	1.0223 (0.1863)	0.6333 (0.0881)	0.3201 (0.0536)
TIEHL-TL-MO-LLoG(a , 1, δ , c)	0.0771 (0.0564)	1 -	2.0168 (1.5085)	3.1183 (1.9105)
TIEHL-TL-MO-LLoG(a , b , 1, c)	0.0738 (0.0505)	1.9484 (1.7591)	1 -	3.0954 (1.9212)
TIEHL-TL-MO-LLoG(a , b , δ , 1)	2.7353×10^{-01} (5.5024×10^{-02})	4.0111×10^{05} (3.3090×10^{-11})	3.2258×10^{-03} (5.3816×10^{-04})	1 -
TIEHL-TL-MO-LLoG(1, 1, δ , c)	1 -	1 -	1.0000 (0.1455)	0.3572 (0.0581)
TIEHL-TL-MO-E	1.4168×10^{-01} (5.1931×10^{-02})	1.0000×10^{02} (1.1228×10^{-05})	2.9055×10^{-02} (1.5948×10^{-02})	1.3873×10^{-01} (7.9153×10^{-02})
TIEHL-TL-MO-E(1, b , δ , λ)	1 -	0.7557 (0.2009)	2.1094 (2.4046)	0.0272 (0.0165)
TIEHL-TL-MO-E(a , 1, δ , λ)	0.2500 (0.2446)	1 -	0.8540 (0.5221)	0.0938 (0.1128)
TIEHL-TL-MO-E(a , b , 1, λ)	3.2356 (0.0033)	0.9604 (0.1593)	1 -	0.0049 (0.0018)
TIIGIE-BIII	347.83331 (0.1581)	29.6836 (64.3819)	0.1103 (0.0991)	2.8376 (2.8956)
TIEHLW	4.4543×10^{02} (1.2021×10^{-05})	1.5685×10^{03} (3.4351×10^{-05})	5.1012 (5.4118×10^{-02})	2.4218×10^{-02} (3.1984×10^{-03})
TIIGIE-Lx	1.2893 (9.9255×10^{-01})	5.6814×10^{01} (1.1208×10^{-06})	1.0434×10^{-02} (5.3240×10^{-03})	8.8122 (1.7888×10^{-01})
EMW	9.0070×10^{-01} (2.1531×10^{-01})	5.5644×10^{-02} (1.3927×10^{-02})	1.7001×10^{01} (1.3496×10^{-18})	1.0000×10^{-04} (2.2940×10^{-13})
GWLLoG	0.0012 (0.0023)	6.9653 (0.0062)	0.0115 (0.0154)	0.3558 (0.0869)

Table 2.13: Estimates from the Toluene Exposure Concentrations Data

Model	Criteria for Model Selection							
	$-2\log L$	AIC	AICC	BIC	W^*	A^*	K-S	p-value
TIEHL-TL-MO-LLoG	225.0787	233.0787	234.6787	238.6835	0.0261	0.2597	0.0961	0.9445
TIEHL-TL-MO-LLoG(1, b , δ , c)	382.636	388.636	389.5591	392.8396	0.0260	0.2587	0.9131	2.2×10^{-16}
TIEHL-TL-MO-LLoG(a , 1, δ , c)	238.7153	244.7154	245.6384	248.9189	0.0579	0.4543	0.2425	0.0586
TIEHL-TL-MO-LLoG(a , b , 1, c)	240.0699	246.0700	246.9931	250.2736	0.0646	0.5065	0.2514	0.0451
TIEHL-TL-MO-LLoG(a , b , δ , 1)	235.7401	241.7401	242.6632	245.9437	0.1474	1.0349	0.2212	0.1061
TIEHL-TL-MO-LLoG(1, 1, δ , c)	349.1219	353.1219	353.5663	355.9243	0.0265	0.2620	0.8509	2.2×10^{-16}
TIEHL-TL-MO-E	223.8926	231.8925	233.4925	237.4973	0.0489	0.3899	0.1014	0.9171
TIEHL-TL-MO-E(1, b , δ , λ)	229.4803	235.4803	236.4034	239.6839	0.1152	0.8339	0.1178	0.7993
TIEHL-TL-MO-E(a , 1, δ , λ)	228.4432	234.4432	235.3662	238.6467	0.1080	0.7860	0.1134	0.8345
TIEHL-TL-MO-E(a , b , 1, λ)	227.8431	233.8431	234.7662	238.0467	0.0927	0.6977	0.1174	0.8025
TIIGIE-BIII	225.3964	233.3964	234.9964	239.0012	10.2462	60.3286	0.9870	2.2×10^{-16}
TIEHLW	226.4908	234.4908	236.0908	240.0956	0.0697	0.5494	0.1080	0.8752
TIIGIE-Lx	227.2764	235.2764	236.8764	240.8812	0.0725	0.5687	0.1007	0.9211
EMW	228.9093	236.9093	238.5093	242.514	0.1212	0.8744	0.1446	0.5570
GWLLoG	226.4904	234.4904	236.0904	240.0952	0.0695	0.5479	0.1049	0.8957

Table 2.14: GoF Statistics for Toluene Exposure Concentrations Data

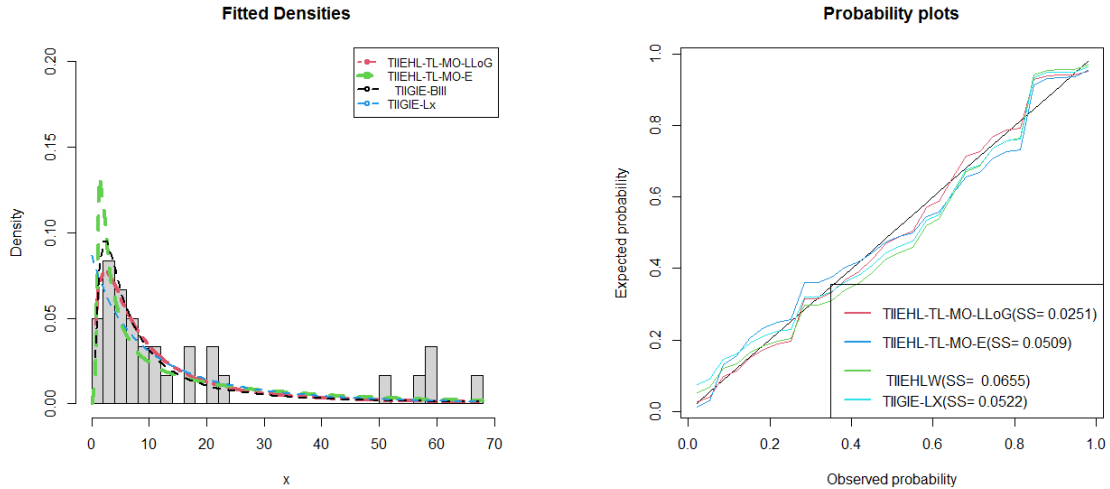


Figure 2.11: Fitted Densities and Probability Plots of the Toluene Exposure Concentrations Data

Estimates of the parameters of TIIEHL-TL-MO-LLoG distribution (standard errors in parenthesis), AIC, CAIC, BIC are given in Table 2.14 for the toluene exposure concentrations data. The SS and the goodness-of-fit statistics W^* and A^* are also given. Plots of histogram and fitted densities, and observed probability versus predicted probability for the toluene exposure concentrations data are given in Figure 2.11.

The values of the goodness-of-fit statistics: W^* , A^* , K-S and its p-value indicate that the TIIEHL-TL-MO-LLoG distribution provide more adequate fits than the other competitive models. The value of SS from the probability plots in Figure 2.11 are smaller for TIIEHL-TL-MO-LLoG distribution. Also, the goodness-of-fit statistics K-S, W^* and A^* are smaller for TIIEHL-TL-MO-LLoG distribution as compared to nested and non-nested distributions for the toluene exposure concentrations data. So there is a clear need of the TIIEHL-TL-MO-LLoG distribution when modelling real life data.

Also, from the results in Table 2.14, the smallest goodness-of-fit statistics $-2 \log(L)$, AIC, AICC, BIC, K-S, W^* and A^* and the highest p-value are acquired for the TIIEHL-TL-MO-E distribution compared to to nested and non-nested distributions. Therefore, the TIIEHL-TL-MO-E distribution gives significantly better fits to the toluene exposure concentrations data.

Actual Taxes Revenue Data

The data consists of the monthly actual taxes revenue in Egypt from January 2006 to November 2010. The data was used by Mead (2016). (See the data in the appendix).

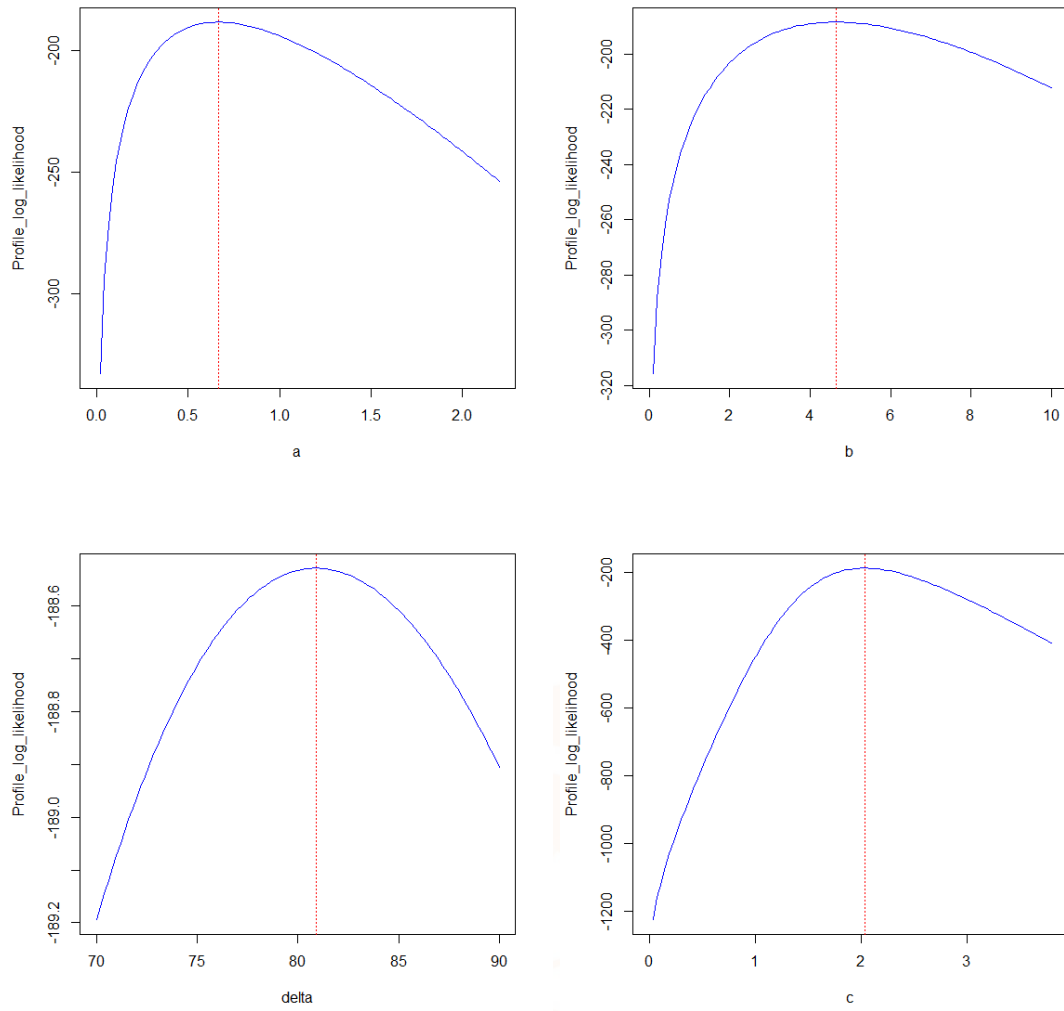


Figure 2.12: Profile log-likelihood function plots for parameters of TIIEHL-TL-MO-LLoG distribution on the actual taxes revenue data

The identifiability of model parameters can be assessed by plotting the log-likelihood function. From the profile log-likelihood plots in Figure 2.12, we can see that the MLEs for the TIIEHL-TL-MO-LLoG distribution are identifiable.

Model	Estimates			
	a	b	δ	c
TIEHL-TL-MO-LLoG	0.6666 (0.2703)	4.6432 (1.7266)	80.9241 (0.0330)	2.0368 (0.2706)
TIEHL-TL-MO-LLoG(1, b , δ , c)	1	59.3484 (10.7883)	0.6003 (0.0801)	0.5671 (0.0558)
TIEHL-TL-MO-LLoG(a , 1, δ , c)	0.0453 (0.0071)	1	3.5505 (0.2881)	4.8233 (0.4091)
TIEHL-TL-MO-LLoG(a , b , 1, c)	0.0544 (0.0212)	0.6407 (0.4039)	1	3.4983 (1.3054)
TIEHL-TL-MO-LLoG(a , b , δ , 1)	2.5867×10^{-01} (3.6414×10^{-02})	2.1243×10^{07} (7.6114×10^{-12})	1.5689×10^{-03} (1.7712×10^{-04})	1 -
TIEHL-TL-MO-LLoG(1, 1, δ , c)	1 -	1 -	4.5479×10^{03} (5.9718×10^{-06})	2.8148 (7.4657×10^{-21})
TIEHL-TL-MO-E	a 0.1595 (0.0476)	b 3.8249 (1.7200)	δ 20.2438 (7.2341)	λ 0.4617 (0.0865)
TIEHL-TL-MO-E(1, b , δ , λ)	1 -	1.4350 (0.2840)	9.7541 (4.7294)	0.1271 (0.0210)
TIEHL-TL-MO-E(a , 1, δ , λ)	0.1792 (0.0798)	1 -	45.3755 (41.1994)	0.4263 (0.1529)
TIEHL-TL-MO-E(a , b , 1, λ)	0.0921 (0.0202)	0.1042 (0.0497)	1 -	0.2890 (0.0626)
TIIGIE-BIII	λ 7.6219×10^{-04} (6.1103×10^{-06})	θ 4.5353×10^{-05} (6.0072×10^{-06})	c 4.4469 (3.5220×10^{-07})	k 2.0411×10^{-03} (1.1326×10^{-03})
TIEHLW	a 2.3148×10^{03} (1.1991×10^{-06})	λ 2.4003×10^{03} (2.2597×10^{-05})	δ 5.0733 (5.4696×10^{-02})	γ 3.9979×10^{-02} (3.6625×10^{-03})
TIIGIE-Lx	λ 2.6446×10^{-04} (2.2147×10^{-04})	α 2.9428×10^{02} (1.2416×10^{-08})	a 6.2474×10^{-03} (5.9137×10^{-04})	b 1.7495×10^{-01} (4.5779×10^{-07})
EMW	γ 5.5304 (1.4344)	δ 1.7867×10^{-01} (2.3287×10^{-02})	λ 1.7001×10^{01} (7.6147×10^{-17})	θ 1.0000×10^{-04} (1.2948×10^{-11})
GWLLoG	α 0.0002 (0.0004)	β 9.3926 (0.0098)	a 0.1799 (0.1068)	θ 0.5356 (0.1064)

Table 2.15: Estimates from the Actual Taxes Revenue Data

Criteria for Model Selection								
Model	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIEHL-TL-MO-LLoG	377.0577	385.0577	385.7984	393.3678	0.0395	0.2497	0.0731	0.9102
TIEHL-TL-MO-LLoG(1, b , δ , c)	414.5659	420.5658	421.0022	426.7985	0.0480	0.2828	0.1993	0.0183
TIEHL-TL-MO-LLoG(a , 1, δ , c)	504.9258	510.9258	511.3622	517.1585	0.0528	0.3089	0.4503	8.064×10^{-11}
TIEHL-TL-MO-LLoG(a , b , 1, c)	522.0963	528.0964	528.5327	534.329	0.0586	0.3396	0.4782	3.793×10^{-12}
TIEHL-TL-MO-LLoG(a , b , δ , 1)	432.0772	438.0776	438.514	444.3102	0.0404	0.3158	0.3862	4.514×10^{-08}
TIEHL-TL-MO-LLoG(1, 1, δ , c)	384.0819	388.0819	388.2962	392.237	0.1611	0.9444	0.0987	0.6125
TIEHL-TL-MO-E	377.697	385.6971	386.4378	394.0072	0.0655	0.3951	0.0997	0.5994
TIEHL-TL-MO-E(1, b , δ , λ)	395.6143	401.6143	402.0507	407.847	0.2986	1.8804	0.1239	0.3255
TIEHL-TL-MO-E(a , 1, δ , λ)	387.8638	393.8639	394.3002	400.0965	0.2060	1.2320	0.1275	0.2922
TIEHL-TL-MO-E(a , b , 1, λ)	462.3705	468.3705	468.8068	474.6031	0.2653	1.6643	0.4011	1.134×10^{-08}
TIIGIE-BIII	λ 567.3549	θ 575.3546	c 576.0954	k 583.6648	20.0518	118.3187	0.74007	2.2×10^{-16}
TIEHLW	a 388.9579	λ 396.9579	δ 397.6986	γ 405.268	0.2275	1.4295	0.1355	0.2287
TIIGIE-Lx	λ 395.0752	α 403.0752	a 403.8159	b 411.3853	0.2947	1.8805	0.1398	0.1987
EMW	382.4471	390.4471	391.1878	398.7572	0.1442	0.8709	0.1148	0.4179
GWLLoG	389.7551	397.7551	398.4959	406.0653	0.2344	1.4784	0.1353	0.2297

Table 2.16: GoF Statistics for Actual Taxes Revenue Data

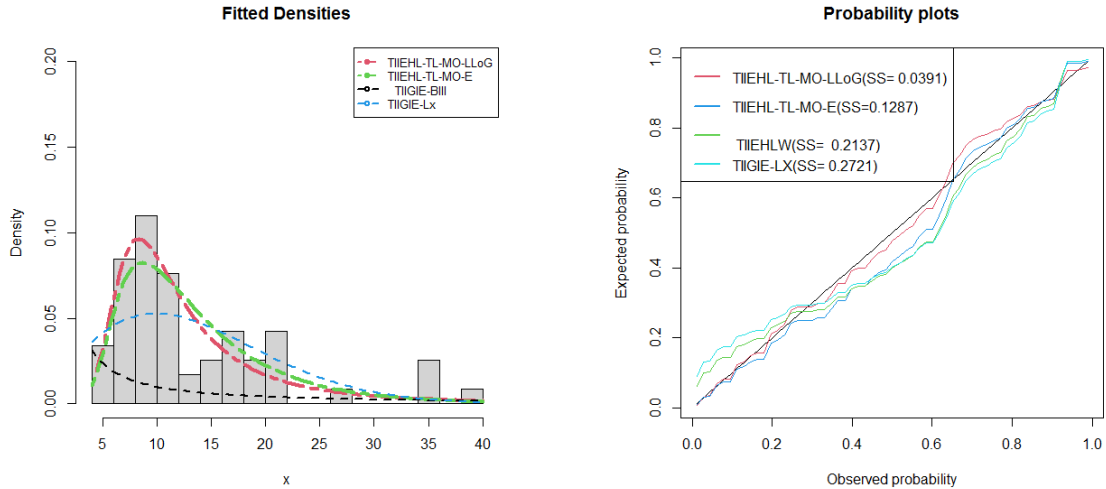


Figure 2.13: Fitted Densities and Probability Plots of the Actual Taxes Revenue Data

The TIIEHL-TL-MO-LLoG distribution give the smallest AIC , $AICC$, BIC , $K-S$, W^* , A^* , and the largest p-value in table 2.16. These statistics indicates that the TIIEHL-TL-MO-LLoG distribution fits the data better than other competitive distributions for the actual taxes revenue data.

Also, the values of all statistics in Table 2.16 indicate that the TIIEHL-TL-MO-E distribution is the best model to the actual taxes revenue data compared to all the nested and competitive distributions considered.

Time to Failure of Kevlar 49/epoxy Data

The third data is concerned with the examination of the lifetimes of kevlar 49/epoxy spherical pressure vessels. These vessels undergo continous sustained pressure until vessel failure, a phenomenon commonly known as static fatigue or stress-rupture. The data set consists of 101 observations of stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all have failed, so that the complete data set with the exact times of failure is recorded. **(See the data in the appendix).**

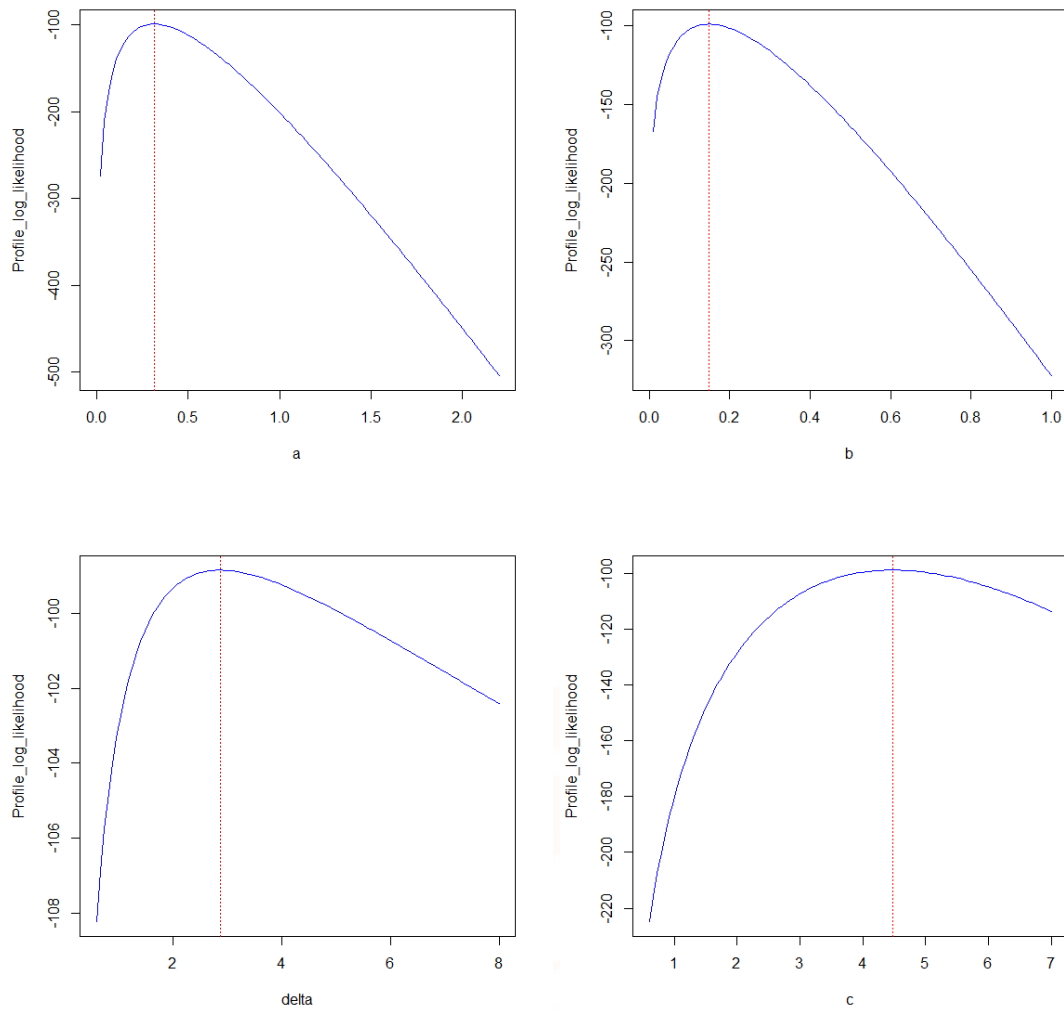


Figure 2.14: Profile log-likelihood function plots for parameters of TIIEHL-TL-MO-LLoG distribution on the kevlar 49/epoxy data

The identifiability of model parameters can be assessed by plotting the log-likelihood function. From the profile log-likelihood plots in Figure 2.14, we can see that the MLEs for the TIIEHL-TL-MO-LLoG distribution are achieved at single and different points, hence identifiable.

Model	Estimates			
	a	b	δ	c
TIEHL-TL-MO-LLoG	0.3160 (0.0905)	0.1492 (0.0513)	2.87993 (2.3074)	4.47287 (0.9793)
TIEHL-TL-MO-LLoG(1, b , δ , c)	1 -	0.3361 (0.0916)	18.2993 (14.1954)	2.6166 (0.5757)
TIEHL-TL-MO-LLoG(a , 1, δ , c)	2.3103×10^{04} (6.2632×10^{-07})	1 -	9.1549×10^{04} (1.5804×10^{-07})	9.2591×10^{-01} (6.9253×10^{-02})
TIEHL-TL-MO-LLoG(a , b , 1, c)	1.0995×10^{06} (1.1365×10^{-06})	5.0645×10^{01} (3.5889×10^{-01})	1 -	5.3607×10^{-02} (4.1187×10^{-03})
TIEHL-TL-MO-LLoG(a , b , δ , 1)	6.4554×10^{02} (2.7519×10^{-06})	9.2630×10^{-01} (1.3411×10^{-02})	4.5168×10^{03} (3.6409×10^{-07})	1 -
TIEHL-TL-MO-LLoG(1, 1, δ , c)	1 -	1 -	2.5428 (0.3798)	1.1861 (0.0979)
TIEHL-TL-MO-E	a	b	δ	λ
	0.2974 (0.1666)	0.8168 (0.1834)	2.2406 (1.3285)	1.7804 (0.8803)
TIEHL-TL-MO-E(1, b , δ , λ)	1 -	1.1093 (0.1440)	0.0006 (0.0074)	0.0002 (0.0033)
TIEHL-TL-MO-E(a , 1, δ , λ)	9.8952×10^{02} (5.9375×10^{-08})	1 -	1.0424×10^{02} (5.6306×10^{-07})	2.5000×10^{-02} (2.4184×10^{-03})
TIEHL-TL-MO-E(a , b , 1, λ)	1.5893 (1.5457)	1.0132 (0.1116)	1 -	0.1934 (0.2279)
TIIGIE-BIII	λ	θ	c	k
	5.1829 (3.4987)	3.3191×10^{03} (3.2371×10^{-04})	5.9244×10^{-02} (9.7386×10^{-03})	1.0274×10^{01} (3.3749×10^{-01})
TIEHLW	a	λ	δ	γ
	0.3523 (0.4424)	0.7289 (0.2131)	1.8270 (3.1301)	1.2123 (0.2386)
TIIGIE-Lx	λ	α	a	b
	1.7010×10^{02} (7.5375×10^{-07})	1.0186×10^{-03} (2.1156×10^{-04})	9.0000 (1.0787×10^{-04})	9.1000×10^{-01} (2.5260×10^{-01})
EMW	γ	δ	λ	θ
	8.6634×10^{-01} (1.0983×10^{-01})	8.8828×10^{-01} (1.2011×10^{-01})	1.7894×10^{01} (2.0310×10^{-19})	1.0000×10^{-04} (3.6356×10^{-14})
GWLLoG	α	β	a	θ
	3.3576 (0.8568)	0.2257 (0.1307)	3.2678 (0.9299)	4.3802 (2.4759)

Table 2.17: Estimates from the Time to Failure of Kevlar 49/epoxy Data

Model	Criteria for Model Selection							
	$-2 \log(L)$	AIC	AICC	BIC	W^*	A^*	K-S	p-value
TIEHL-TL-MO-LLoG	197.7035	205.7035	206.1202	216.164	0.0390	0.3242	0.0630	0.8174
TIEHL-TL-MO-LLoG(1, b , δ , c)	204.0467	210.0467	210.2942	217.8921	0.1514	0.8744	0.0865	0.4364
TIEHL-TL-MO-LLoG(a , 1, δ , c)	205.9536	212.0748	212.3223	219.9202	0.1986	1.1112	0.1041	0.2234
TIEHL-TL-MO-LLoG(a , b , 1, c)	207.0532	213.0532	213.3006	220.8985	0.2487	1.3520	0.1007	0.2571
TIEHL-TL-MO-LLoG(a , b , δ , 1)	205.9536	211.9536	212.201	219.799	0.1988	1.1119	0.0907	0.3768
TIEHL-TL-MO-LLoG(1, 1, δ , c)	216.259	220.259	220.3814	225.4892	0.4268	2.3053	0.1113	0.1637
TIEHL-TL-MO-E	203.6599	211.6599	212.0766	222.1204	0.1122	0.7107	0.0714	0.6807
TIEHL-TL-MO-E(1, b , δ , λ)	219.6099	225.6099	225.8573	233.4552	0.4794	2.5839	0.1261	0.0801
TIEHL-TL-MO-E(a , 1, δ , λ)	207.5157	213.5157	213.7631	221.361	0.1710	0.9870	0.0916	0.3643
TIEHL-TL-MO-E(a , b , 1, λ)	208.2926	214.2926	214.5400	222.1380	0.2696	1.4651	0.1037	0.2267
TIIGIE-BIII	233.0062	241.0063	241.4229	251.4667	32.0684	197.6112	0.9981	2.2×10^{-16}
TIEHLW	206.6925	214.6925	215.1092	225.153	0.1673	0.9811	0.0891	0.3978
TIIGIE-Lx	221.7345	229.7345	230.1511	240.1949	0.4843	2.6166	0.1706	0.0056
EMW	205.6399	213.6399	214.0566	224.1004	0.1786	1.0183	0.0887	0.4045
GWLLoG	206.3484	214.3485	214.7652	224.809	0.2208	1.2124	0.0930	0.3460

Table 2.18: GoF Statistics for Time to Failure of Kevlar 49/epoxy Data

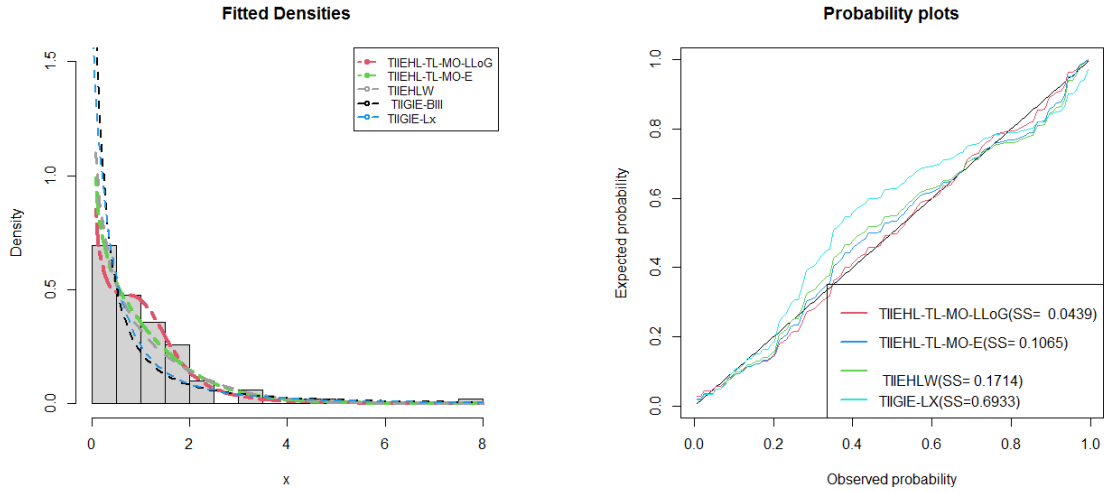


Figure 2.15: Fitted Densities and Probability Plots of the Kevlar 49/epoxy Data

In Table 2.18, we note that the TIIEHL-TL-MO-LLoG model gives the smallest figures of AIC , $AICC$, BIC , $K-S$, W^* , A^* , and the largest p-value. So, the TIIEHL-TL-MO-LLoG model can be chosen as the “best” model for the time to failure of kevlar 49/epoxy data set.

We also note that, the TIIEHL-TL-MO-E model gives the lowest values of AIC , $AICC$, BIC , $K-S$, W^* , A^* , and the largest p-value among all fitted models. Hence, the TIIEHL-TL-MO-E model could be chosen as a good alternative to explain the kevlar 49/epoxy data set.

Likelihood Ratio Test

The likelihood ratio test results for comparing the full and nested models are given in this section.

Model	df	Toluene Exposure Concentrations Data	Actual Taxes Revenue Data
		$\chi^2(p - value)$	$\chi^2(p - value)$
TIIEHL-TL-MO-LLoG(1, b , δ , c)	1	157.5573(<0.00001)	37.5082(<0.00001)
TIIEHL-TL-MO-LLoG(a , 1, δ , c)	1	13.6366(0.00022)	127.8681(<0.00001)
TIIEHL-TL-MO-LLoG(a , b , 1, c)	1	14.9912(0.00011)	145.0386(<0.00001)
TIIEHL-TL-MO-LLoG(a , b , δ , 1)	1	10.6614(0.00109)	55.0195(<0.00001)
TIIEHL-TL-MO-LLoG(1, 1, δ , c)	2	124.0432(<0.00001)	7.0242(0.02983)

Table 2.19: Likelihood Ratio Test Results

Time to Failure of Kevlar 49/epoxy Data		
Model	df	$\chi^2(p - value)$
TIIEHL-TL-MO-LLoG(1, b , δ , c)	1	6.3432(0.01178)
TIIEHL-TL-MO-LLoG(a , 1, δ , c)	1	8.2501(0.00407)
TIIEHL-TL-MO-LLoG(a , b , 1, c)	1	9.3497(0.00223)
TIIEHL-TL-MO-LLoG(a , b , δ , 1)	1	8.2501(0.00407)
TIIEHL-TL-MO-LLoG(1, 1, δ , c)	2	18.5555(0.00009)

Table 2.20: Likelihood Ratio Test Results

The likelihood ratio test results in Table 2.19 and 2.20 indicates that the TIIEHL-TL-MO-LLoG performs better than its nested models at 5% level of significance, since all p-values are less than 0.05 among all the data sets considered.

Toluene Exposure Concentrations Data		Actual Taxes Revenue Data	
Model	df	$\chi^2(p - value)$	$\chi^2(p - value)$
TIIEHL-TL-MO-E(1, b , δ , λ)	1	5.5877(0.0181)	17.9173(0.00002)
TIIEHL-TL-MO-E(a , 1, δ , λ)	1	4.5506(0.0329)	10.1668(0.0014)
TIIEHL-TL-MO-E(a , b , 1, λ)	1	3.9505(0.0468)	84.6735(<0.00001)

Table 2.21: Likelihood Ratio Test Results

Time to Failure of Kevlar 49/epoxy Data		
Model	df	$\chi^2(p - value)$
TIIEHL-TL-MO-E(1, b , δ , λ)	1	15.95(0.00006)
TIIEHL-TL-MO-E(a , 1, δ , λ)	1	3.8558(0.0495)
TIIEHL-TL-MO-E(a , b , 1, λ)	1	4.6327(0.0313)

Table 2.22: Likelihood Ratio Test Results

The likelihood ratio test results in Table 2.21 and 2.22 indicates that the TIIEHL-TL-MO-E performs better than its nested models at 5% level of significance, since all p-values are less than 0.05 among all the data sets considered.

2.1.9 Concluding Remarks

This paper provides a new type II exponentiated half-logistic Topp-Leone-Marshall-Olkin-G (TIIEHL-TL-MO-G) GFPD. The general mathematical properties are obtained for the TIIEHL-TL-MO-G GFPD. The unknown parameters of the new distribution are estimated using the maximum likelihood estimation method and evaluated via a simulation study. The importance of the TIIEHL-TL-MO-G distribution are assessed empirically using three real-life data sets. Based on the results, it is concluded that the suggested distribution is a more efficient model as compared to other distributions in the literature that were considered in this paper.

3. Type II Exponentiated Half-Logistic-Gompertz-G (TIIHL-Gom-G) Generalized Family of Probability Distributions with Applications

Moakofi, T. and Oluyede, B., 2023. Type II Exponentiated Half-Logistic Gompertz-G Family of Distributions: Properties and Applications. *Mathematica Slovaca*, 73(3), 785-810.

<https://www.degruyter.com/document/doi/10.1515/ms-2023-0058/html>

This chapter utilizes the Type II Exponentiated Half Logistic generator to obtain the new family of distributions.

3.1 Type II Exponentiated Half-Logistic-Gompertz-G (TIIHL-Gom-G) GFPD

In this section, a new family of distributions referred to as type II exponentiated half-logistic-Gompertz-G (TIIHL-Gom-G) GFPD is introduced and studied. Some of the main statistical properties of these GFPD are derived. The model parameters are estimated using the MLE technique and consistency of MLEs is evaluated by performing a simulation study. The importance and versatility of the TIIHL-Gom-G GFPD is demonstrated in an application to two real data sets from different fields.

3.1.1 The Model

By utilizing equations (1.14) and (1.15), the cdf and pdf of the proposed TIIHL-Gom-G GFPD are given by

$$F(x; \alpha, \gamma, \zeta) = 1 - \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)}{1 + \left[1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)\right]} \right]^\alpha \quad (3.1)$$

and

$$\begin{aligned}
 f(x; \alpha, \gamma, \zeta) &= 2\alpha \exp\left(\frac{\alpha}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right) [1 - G_B(x; \zeta)]^{-\gamma-1} \\
 &\times \left(1 + \left[1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)\right]\right)^{-(\alpha+1)} \\
 &\times g_B(x; \zeta),
 \end{aligned} \tag{3.2}$$

respectively, for $\alpha, \gamma > 0$ being shape parameters and parameter vector ζ .

Sub-Families

In this sub-section, some sub-families of the TIEHL-Gom-G GFRD are presented.

- When $\alpha = 1$, we obtain the type II half-logistic-Gom-G (TIIHL-Gom-G) GFRD with the cdf

$$F(x; \gamma, \zeta) = 1 - \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)}{1 + \left[1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)\right]} \right], \tag{3.3}$$

for $\gamma > 0$, and parameter vector ζ .

- If $\gamma = 1$, we obtain the new GFRD with the cdf

$$F(x; \alpha, \zeta) = 1 - \left[\frac{\exp\left(\{1 - [1 - G_B(x; \zeta)]^{-1}\}\right)}{1 + \left[1 - \exp\left(\{1 - [1 - G_B(x; \zeta)]^{-1}\}\right)\right]} \right]^\alpha, \tag{3.4}$$

for $\alpha > 0$, and parameter vector ζ .

- If $\alpha = \gamma = 1$, we obtain the new GFRD with the cdf

$$F(x; \zeta) = 1 - \left[\frac{\exp\left(\{1 - [1 - G_B(x; \zeta)]^{-1}\}\right)}{1 + \left[1 - \exp\left(\{1 - [1 - G_B(x; \zeta)]^{-1}\}\right)\right]} \right], \tag{3.5}$$

for parameter vector ζ .

Hazard Rate and Quantile Functions

The hrf of the TIEHL-Gom-G GFRD is given by

$$\begin{aligned}
 h_F(x; \alpha, \gamma, \zeta) &= 2\alpha [1 - G_B(x; \zeta)]^{-\gamma-1} g_B(x; \zeta) \\
 &\times \left(1 + \left[1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)\right]\right)^{-1}.
 \end{aligned} \tag{3.6}$$

The qf of the TIIEHL-Gom-G GFPD is obtained by solving the non-linear equation:

$$F(x; \alpha, \gamma, \zeta) = 1 - \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)}{1 + \left[1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)\right]} \right]^\alpha = u, \quad (3.7)$$

for $0 \leq u \leq 1$, that is,

$$\exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right) = \frac{2(1-u)^{\frac{1}{\alpha}}}{1 + (1-u)^{\frac{1}{\alpha}}},$$

so that

$$G_B(x; \zeta) = 1 - \left(1 - \gamma \log \left[\frac{2(1-u)^{\frac{1}{\alpha}}}{1 + (1-u)^{\frac{1}{\alpha}}} \right] \right)^{\frac{-1}{\gamma}}.$$

Consequently, the qf for the TIIEHL-Gom-G GFPD is given by

$$Q_G(u; \alpha, \gamma, \zeta) = G^{-1} \left[1 - \left(1 - \gamma \log \left[\frac{2(1-u)^{\frac{1}{\alpha}}}{1 + (1-u)^{\frac{1}{\alpha}}} \right] \right)^{\frac{-1}{\gamma}} \right]. \quad (3.8)$$

It follows therefore that random numbers can be generated from the TIIEHL-Gom-G GFPD based on equation (3.8).

Density Expansion

In this section, we present the density expansion of the TIIEHL-Gom-G distribution. Using the generalized binomial and Taylor series expansions given by

$$(1+z)^{-\beta} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta+k-1}{k} z^k, \quad \text{for } |z| < 1, \quad \text{and} \quad e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!},$$

we have

$$f(x; \alpha, \gamma, \zeta) = \sum_{q=0}^{\infty} C_{q+1} g_{q+1}(x; \zeta), \quad (3.9)$$

where $g_{q+1}(x; \zeta) = (q+1)[G_B(x; \zeta)]^q g_B(x; \zeta)$ is the exponentiated-G (Exp-G) pdf with the power parameter $q+1 > 0$ and parameter vector ζ , where

$$C_{q+1} = \sum_{k,m,l,i=0}^{\infty} \binom{\alpha+k}{k} \binom{k}{m} \binom{l}{i} \binom{\gamma(i+1)+q}{q} (-1)^{k+m+i+q} \frac{\left(\frac{k+\alpha}{\gamma}\right)^l}{l!} \left(\frac{2\alpha}{q+1}\right). \quad (3.10)$$

(See the Appendix for derivations).

Consequently, the properties of the TIIEHL-Gom-G GFPD follow directly from those of the Exp-G distribution.

3.1.2 Probability Weighted Moments and Moments

In this section, probability weighted moments, moments and moment generating function for the TIIEHL-Gom-G GFPD are presented.

Probability Weighted Moments (PWMs)

The PWMs of a random variable X is defined by

$$\phi_{w,p} = E(X^w (F(X))^p) = \int_{-\infty}^{\infty} x^w F(x)^p f(x) dx.$$

We note that

$$\begin{aligned} F(x)^p f(x) &= \left(1 - \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)}{1 + [1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)]} \right]^\alpha \right)^p \\ &\times 2\alpha \exp\left(\frac{\alpha}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right) [1 - G_B(x; \zeta)]^{-\gamma-1} \\ &\times \left(1 + \left[1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right) \right] \right)^{-(\alpha+1)} g_B(x; \zeta) \\ &= \sum_{j=0}^{\infty} \binom{p}{j} (-1)^j 2\alpha \exp\left(\frac{\alpha(j+1)}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right) [1 - G_B(x; \zeta)]^{-\gamma-1} \\ &\times \left(1 + \left[1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right) \right] \right)^{-(\alpha(j+1)+1)} g_B(x; \zeta). \end{aligned}$$

Now following the same steps leading to equation (3.9), we get that

$$F(x)^p f(x) = \sum_{q=0}^{\infty} C_{q+1}^* g_{q+1}(x; \zeta).$$

Thus, the probability weighted moment of TIIEHL-Gom-G GFPD is given by

$$\phi_{w,p} = E(X^w (F(X))^p) = \sum_{q=0}^{\infty} C_{q+1}^* \int_{-\infty}^{\infty} x^w g_{q+1}(x; \zeta) dx,$$

where $g_{q+1}(x; \zeta) = (q+1)[G_B(x; \zeta)]^q g_B(x; \zeta)$ is the Exp-G pdf with the power parameter $q+1 > 0$ and parameter vector ζ , and

$$\begin{aligned} C_{q+1}^* &= \sum_{j,k,m,l,i=0}^{\infty} \binom{p}{j} \binom{\alpha(j+1)+k}{k} \binom{k}{m} \binom{l}{i} \binom{\gamma(i+1)+q}{q} (-1)^{j+k+m+i+q} \\ &\times \frac{\left(\frac{k+\alpha(j+1)}{\gamma}\right)^l}{l!} \left(\frac{2\alpha}{q+1}\right). \end{aligned}$$

Moments and Generating Function

Let $Y_{q+1} \sim \text{Exp} - G(q+1, \zeta)$, then the s^{th} raw moment, μ'_s of the TIIHL-Gom-G GFRD is given by

$$\mu'_s = E(X^s) = \int_{-\infty}^{\infty} x^s f(x) dx = \sum_{q=0}^{\infty} C_{q+1} E(Y_{q+1}^s),$$

where $E(Y_{q+1}^s)$ is the s^{th} moment of Y_{q+1} and C_{q+1} is given by equation (3.10). The moment generating function (MGF), for $|t| < 1$, is given by:

$$M_X(t) = \sum_{q=0}^{\infty} C_{q+1} M_{q+1}(t),$$

where $M_{q+1}(t)$ is the mgf of Y_{q+1} and C_{q+1} is given by equation (3.10).

3.1.3 Order Statistics and Uncertainty Measure

In this section, we derive the distribution of the r^{th} order statistic and uncertainty measure for the TIIHL-Gom-G GFRD.

Order Statistics

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed TIIHL-Gom-G random variables. Using the binomial expansion

$$(1 - F(x))^{n-r} = \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j [F(x)]^j,$$

the pdf of the r^{th} order statistic from the TIIHL-Gom-G pdf $f(x; \alpha, \gamma, \zeta) = f(x)$ can be written as

$$f_{r:n}(x) = \frac{n! f(x)}{(r-1)!(n-r)!} \sum_{z=0}^{n-r} (-1)^z \binom{n-r}{z} [F(x)]^{z+r-1}. \quad (3.11)$$

Using equations (3.1) and (3.2) we have

$$\begin{aligned} f(x)[F(x)]^{z+r-1} &= \sum_{p=0}^{\infty} \binom{z+r-1}{p} (-1)^p 2\alpha \exp\left(\frac{\alpha(p+1)}{\gamma} \{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right) \\ &\times \left(1 + \left[1 - \exp\left(\frac{1}{\gamma} \{1 - [1 - G_B(x; \zeta)]^{-\gamma}\}\right)\right]\right)^{-(\alpha(p+1)+1)} g_B(x; \zeta) \\ &\times [1 - G_B(x; \zeta)]^{-\gamma-1}. \end{aligned}$$

Now following the same steps leading to equation (3.9), we obtain

$$f(x)[F(x)]^{z+r-1} = \sum_{q=0}^{\infty} b_{q+1} g_{q+1}(x; \zeta), \quad (3.12)$$

where $g_{q+1}(x; \zeta) = (q+1)[G_B(x; \zeta)]^q g_B(x; \zeta)$ is the Exp-G pdf with the power parameter $(q+1)$ and parameter vector ζ , where

$$b_{q+1} = \sum_{p,k,m,l,i=0}^{\infty} \binom{z+r-1}{p} \binom{\alpha(p+1)+k}{k} \binom{k}{m} \binom{l}{i} \binom{\gamma(i+1)+q}{q} (-1)^{k+m+i+q} \\ \times \frac{\left(\frac{k+\alpha(p+1)}{\gamma}\right)^l}{l!} \left(\frac{2\alpha}{q+1}\right).$$

Thus, by substituting (3.12) into (3.11), the pdf of the r^{th} order statistic for the TIIHL-Gom-G GFPD can be written as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{u=0}^{\infty} \sum_{z=0}^{n-r} (-1)^z \binom{n-r}{z} b_{q+1} g_{q+1}(x; \zeta).$$

Uncertainty Measure

Rényi entropy (Rényi (1961)) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^{\infty} [f(x; \alpha, \gamma, \zeta)]^v dx \right), v \neq 1, v > 0. \quad (3.13)$$

Rényi entropy for the TIIHL-Gom-G GFPD is given by

$$I_R(v) = \frac{1}{1-v} \log \left[\sum_{k,m,l,i,q=0}^{\infty} \binom{v(\alpha+1)+k-1}{k} \binom{l}{i} \binom{\gamma(v+i)+v+q-1}{q} \right. \\ \times (-1)^{k+m+i+q} \frac{\left(\frac{v\alpha+k}{\gamma}\right)^l}{l!} \frac{(2\alpha)^v}{\left[1+\frac{q}{v}\right]^v} \int_0^{\infty} \left(\left[1+\frac{q}{v}\right] (G_B(x; \zeta))^{\frac{q}{v}} (g_B(x; \zeta)) \right)^v dx \Big] \\ = \frac{1}{1-v} \log \left[\sum_{q=0}^{\infty} \tau_q \exp((1-v)I_{REG}) \right], \quad (3.14)$$

for $v > 0, v \neq 1$, where $I_{REG} = \frac{1}{1-v} \log \left[\int_0^{\infty} \left(\left[1+\frac{q}{v}\right] (G_B(x; \zeta))^{\frac{q}{v}} (g_B(x; \zeta)) \right)^v dx \right]$ is the Rényi entropy of Exp-G distribution with power parameter $(\frac{q}{v} + 1)$ and

$$\tau_q = \sum_{k,m,l,i=0}^{\infty} \binom{v(\alpha+1)+k-1}{k} \binom{l}{i} \binom{\gamma(v+i)+v+q-1}{q} \\ \times (-1)^{k+m+i+q} \frac{\left(\frac{v\alpha+k}{\gamma}\right)^l}{l!} \frac{(2\alpha)^v}{\left[1+\frac{q}{v}\right]^v}.$$

Therefore, Rényi entropy of the TIIHL-Gom-G GFPD can be obtained from those of the Exp-G GFPD. (See the Appendix for derivations).

3.1.4 Three Special Cases

In this section, we consider some special cases of the TIIEHL-Gom-TL-G GPPD, specifically when the baseline distribution function $G_B(x; \zeta)$ is Power, Weibull and Log-logistic distributions, respectively.

TIIEHL-Gom-Power (TIIEHL-Gom-P) Distribution

Suppose the cdf and pdf of the Power distribution are given by $G(x; \theta, k) = (\theta x)^k$, for $x \geq 0, \theta, k > 0$ and $g(x; \theta, k) = k\theta^k x^{k-1}$, for $\theta, k > 0$, and $x > 0$, then, the cdf and pdf of the TIIEHL-Gom-P distribution are respectively given by

$$F(x; \alpha, \gamma, \theta, k) = 1 - \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [1 - (\theta x)^k]^{-\gamma}\}\right)}{1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - (\theta x)^k]^{-\gamma}\}\right)\right)} \right]^\alpha$$

and

$$\begin{aligned} f(x; \alpha, \gamma, \theta, k) &= 2\alpha \exp\left(\frac{\alpha}{\gamma}\{1 - [1 - (\theta x)^k]^{-\gamma}\}\right) [1 - (\theta x)^k]^{-\gamma-1} \\ &\times \left(1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - (\theta x)^k]^{-\gamma}\}\right)\right)\right)^{-(\alpha+1)} k\theta^k x^{k-1}, \end{aligned}$$

respectively, for $\alpha, \gamma, k > 0$, being shape parameters, $\theta > 0$ being the scale parameter and $x > 0$. The hrf is given by

$$\begin{aligned} h(x) &= 2\alpha \exp\left(\frac{\alpha}{\gamma}\{1 - [1 - (\theta x)^k]^{-\gamma}\}\right) [1 - (\theta x)^k]^{-\gamma-1} \\ &\times \left(1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - (\theta x)^k]^{-\gamma}\}\right)\right)\right)^{-(\alpha+1)} k\theta^k x^{k-1} \\ &\times \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [1 - (\theta x)^k]^{-\gamma}\}\right)}{1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [1 - (\theta x)^k]^{-\gamma}\}\right)\right)} \right]^{-\alpha}. \end{aligned}$$

	$(\alpha, \gamma, \theta, k)$				
u	(0.5, 3.8, 0.3, 2.9)	(0.90.6, 0.4, , 0.7)	(2.80.3, 0.7, , 1.9)	(0.6, 5.5, 0.8, 1.4)	(0.1, 4.9, 0.6, 0.5)
0.1	1.4452	0.0423	0.1763	0.1905	0.1072
0.2	1.7739	0.1201	0.2588	0.2871	0.2199
0.3	1.9820	0.2245	0.3332	0.3604	0.3050
0.4	2.1399	0.3576	0.4016	0.4186	0.3691
0.5	2.2629	0.5185	0.4703	0.4701	0.4257
0.6	2.3721	0.7126	0.5439	0.5183	0.4768
0.7	2.4659	0.9439	0.6197	0.5622	0.5252
0.8	2.5584	1.2236	0.7168	0.6098	0.5752
0.9	2.6546	1.5764	0.8446	0.6612	0.6351

Table 3.1: Table of Quantile for TIIHL-Gom-P Distribution

	$(\alpha, \gamma, \theta, k)$				
	(0.5,0.5,0.8,0.3)	(0.7,0.8,0.9,0.5)	(0.4,0.5,0.5,0.5)	(0.5,0.2,0.9,0.4)	(0.8,0.9,0.6,0.4)
E(X)	0.1713	0.2260	0.2243	0.1713	0.2333
E(X ²)	0.0913	0.1101	0.1372	0.0954	0.1135
E(X ³)	0.0618	0.0679	0.0993	0.0659	0.0678
E(X ⁴)	0.0465	0.0471	0.0780	0.0503	0.0450
E(X ⁵)	0.0372	0.0352	0.0643	0.0406	0.0318
E(X ⁶)	0.0310	0.0276	0.0547	0.0341	0.0236
SD	0.2490	0.2429	0.2948	0.2571	0.2430
CV	1.4534	1.0744	1.3139	1.5003	1.0418
CS	1.6124	1.1411	1.1543	1.5856	0.9601
CK	4.600	3.3551	3.0060	4.4315	2.8309

Table 3.2: Moments of TIIHL-Gom-P Distribution for Selected Parameters

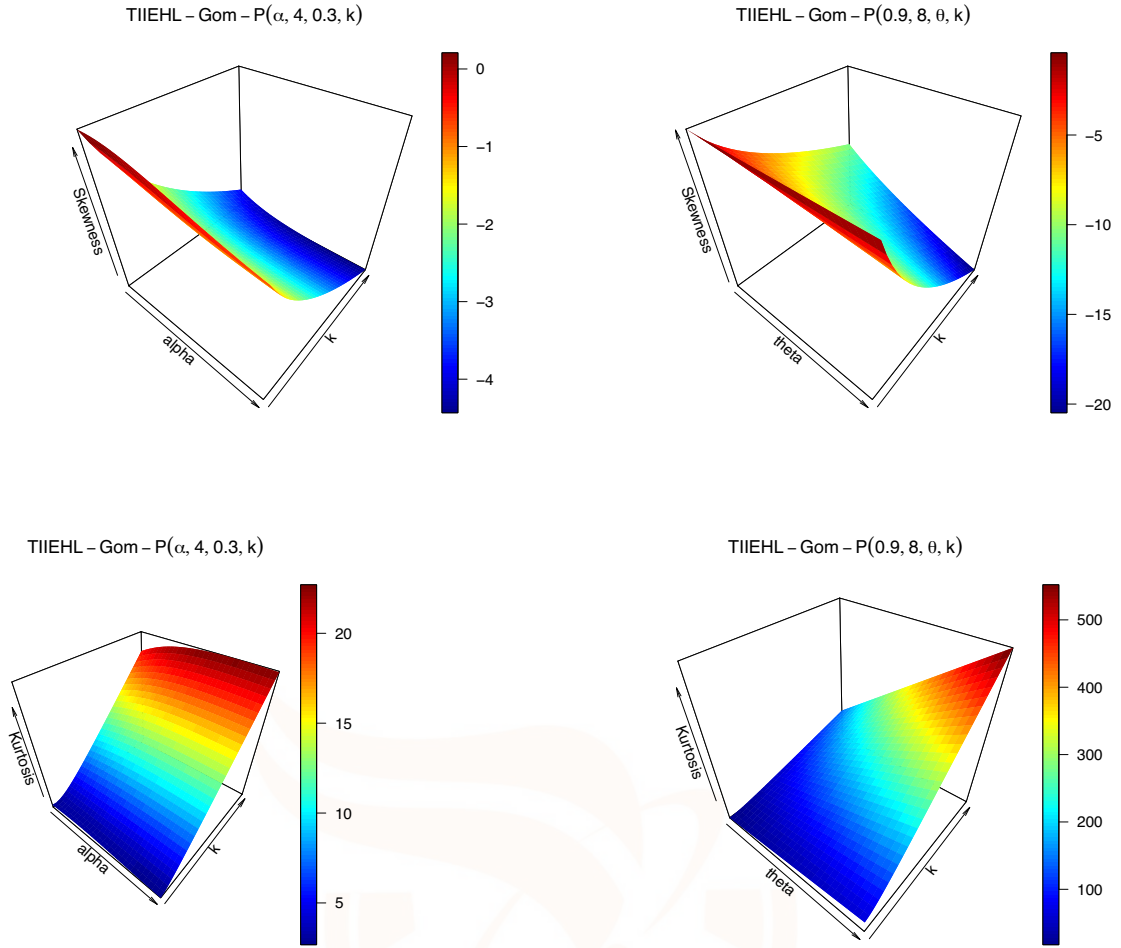


Figure 3.1: Visualization of Skewness and Kurtosis for the TIIEHL-Gom-P distribution

Figure 3.1 shows the plots of skewness and kurtosis for the TIIEHL-Gom-P distribution. It is clear that the skewness and kurtosis can be right skewed or left skewed with increasing values of α and k and also with increasing values θ and k . Thus, we conclude that the parameters α , θ and k affect the shape of the TIIEHL-Gom-P distribution.

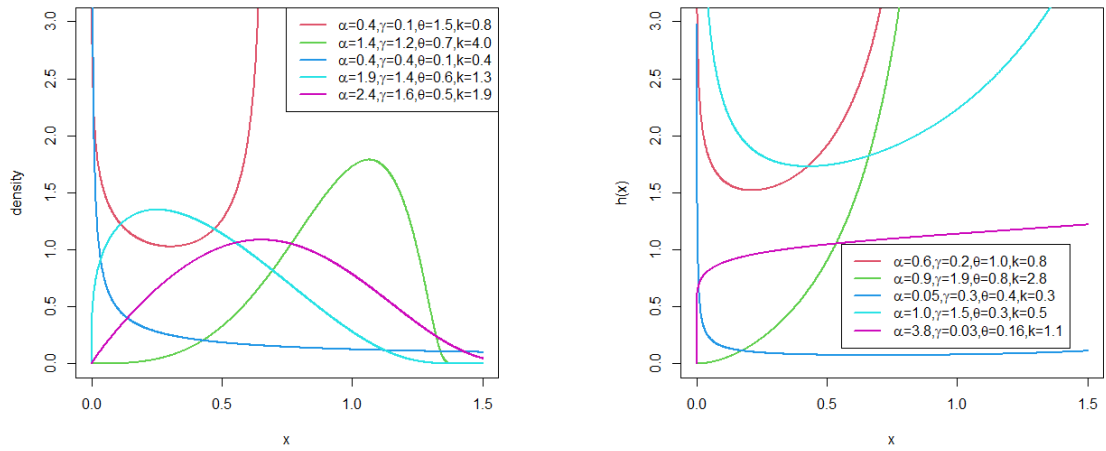


Figure 3.2: Visualization of the pdf and hrf for the TIIHL-Gom-P distribution

Figure 3.2 shows the plots of the pdf and hrf of the TIIHL-Gom-P distribution. The pdf can take various shapes that include almost symmetric, inverse-J, U-shape, left, or right-skewed. Furthermore, the hrfs for the TIIHL-Gom-P distribution exhibits increasing, inverse-J, bathtub and upside bathtub shapes.

TIIHL-Gom-Weibull (TIIHL-Gom-W) Distribution

Suppose the cdf and pdf of the Weibull distribution are given by $G(x; \delta, \lambda) = 1 - \exp(-\delta x^\lambda)$, for $x \geq 0, \delta, \lambda > 0$ and $g(x; \delta, \lambda) = \delta \lambda x^{\lambda-1} \exp(-\delta x^\lambda)$, for $\delta, \lambda > 0$, and $x > 0$, then, the cdf and pdf of the TIIHL-Gom-W distribution are respectively given by

$$F(x; \alpha, \gamma, \delta, \lambda) = 1 - \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [\exp(-\delta x^\lambda)]^{-\gamma}\}\right)}{1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [\exp(-\delta x^\lambda)]^{-\gamma}\}\right)\right)} \right]^\alpha$$

and

$$\begin{aligned} f(x; \alpha, \gamma, \delta, \lambda) &= 2\alpha \exp\left(\frac{\alpha}{\gamma}\{1 - [\exp(-\delta x^\lambda)]^{-\gamma}\}\right) [\exp(-\delta x^\lambda)]^{-\gamma-1} \\ &\times \left(1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [\exp(-\delta x^\lambda)]^{-\gamma}\}\right)\right)\right)^{-(\alpha+1)} \\ &\times \delta \lambda x^{\lambda-1} \exp(-\delta x^\lambda), \end{aligned}$$

respectively, for $\alpha, \gamma, \lambda > 0$ being shape parameters, $\delta > 0$ being a scale parameter and $x > 0$. The hrf is given by

$$h(x) = 2\alpha \exp\left(\frac{\alpha}{\gamma}\{1 - [\exp(-\delta x^\lambda)]^{-\gamma}\}\right) [\exp(-\delta x^\lambda)]^{-\gamma-1} \\ \times \left(1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [\exp(-\delta x^\lambda)]^{-\gamma}\}\right)\right)\right)^{-(\alpha+1)} \delta \lambda x^{\lambda-1} \\ \times \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [\exp(-\delta x^\lambda)]^{-\gamma}\}\right)}{1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [\exp(-\delta x^\lambda)]^{-\gamma}\}\right)\right)}\right]^{-\alpha} \exp(-\delta x^\lambda).$$

	$(\alpha, \gamma, \delta, \lambda)$				
u	(1.3, 2.8, 0.6, 0.9)	(0.2, 0.4, 4.9, 3.7)	(1.7, 2.3, 0.8, 0.9)	(1.8, 1.5, 0.9, 0.8)	(3.6, 0.9, 7.1, 2.5)
0.1	0.0482	0.4641	0.0271	0.0157	0.0822
0.2	0.1065	0.5707	0.0572	0.0346	0.1140
0.3	0.1714	0.6499	0.0973	0.0610	0.1382
0.4	0.2430	0.7149	0.1420	0.0930	0.1607
0.5	0.3263	0.7713	0.1930	0.1349	0.1776
0.6	0.4211	0.8221	0.2533	0.1908	0.2012
0.7	0.5292	0.8631	0.3264	0.2598	0.2225
0.8	0.6734	0.9064	0.4240	0.3585	0.2508
0.9	0.8781	0.9569	0.5717	0.5276	0.2896

Table 3.3: Table of Quantile for TIIHL-Gom-W Distribution

	$(\alpha, \gamma, \delta, \lambda)$				
	(5.5,0.5,0.8,1.3)	(2.8,4.7,0.9,1.5)	(0.5,0.4,3.5,0.5)	(0.2,0.5,0.9,8.4)	(0.9,1.8,1.6,0.4)
E(X)	0.1774	0.2457	0.1337	0.2487	0.0658
E(X ²)	0.0511	0.0784	0.0554	0.2224	0.0157
E(X ³)	0.0198	0.0291	0.0321	0.2010	0.0059
E(X ⁴)	0.0094	0.0119	0.0218	0.1833	0.0028
E(X ⁵)	0.0052	0.0053	0.0162	0.1684	0.0016
E(X ⁶)	0.0032	0.0025	0.0128	0.1557	0.0010
SD	0.1401	0.1345	0.1937	0.4007	0.1066
CV	0.7899	0.5475	1.4480	1.6109	1.6206
CS	1.3717	0.3929	2.0193	1.0230	2.7867
CK	5.3566	2.5602	6.8557	2.1087	12.8420

Table 3.4: Moments of TIIHL-Gom-W Distribution for Selected Parameters

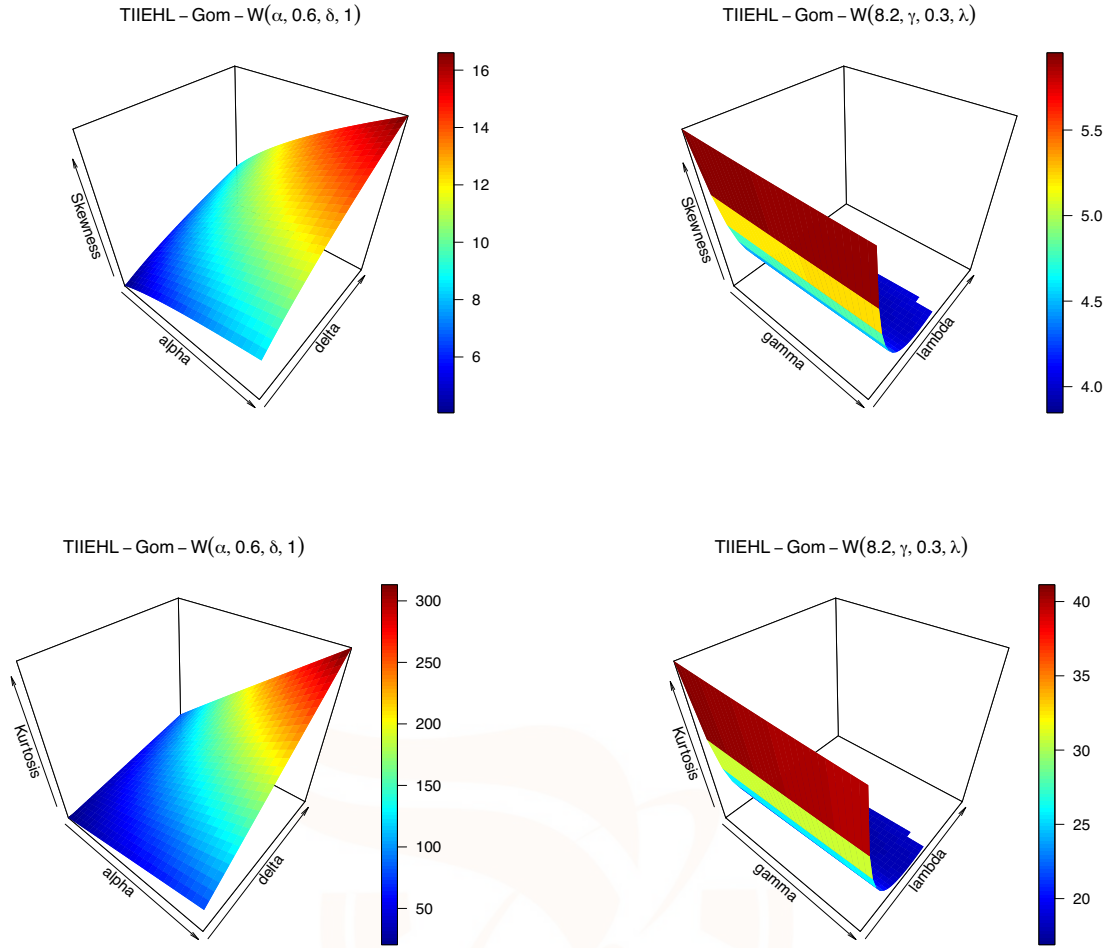


Figure 3.3: Visualization of Skewness and Kurtosis for the TIIEHL-Gom-W distribution

Figure 3.3 shows the plots of skewness and kurtosis for the TIIEHL-Gom-W distribution. It is clear that the skewness and kurtosis becomes left skewed with increasing values of α and δ and right skewed with increasing values of γ and λ . Thus, we conclude that the parameters α , δ , γ and λ affect the shape of the TIIEHL-Gom-W distribution.

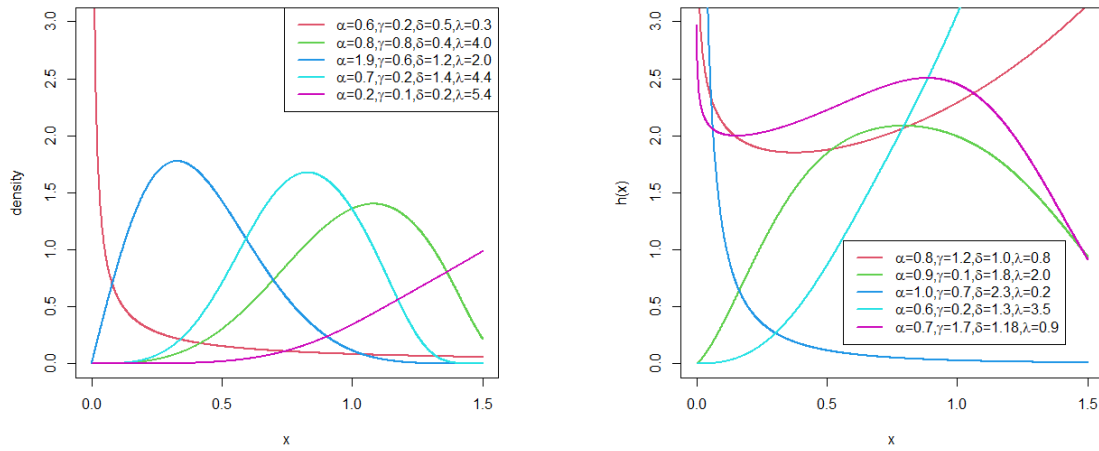


Figure 3.4: Visualization of the pdf and hrf for the TIEHL-Gom-W distribution

Figure 3.4 shows the plots of the pdf and hrf of the TIEHL-Gom-W distribution. The pdf can take various shapes that include almost symmetric, inverse-J, increasing, left, or right-skewed. Furthermore, the hrfs for the TIEHL-Gom-W distribution exhibits increasing, inverse-J, bathtub, upside bathtub and bathtub followed by upside bathtub shapes.

TIEHL-Gom-Log-Logistic (TIEHL-Gom-LLoG) Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$ for $c > 0$ and $x > 0$. The new TIEHL-Gom-LLoG distribution has cdf and pdf respectively given by

$$F(x; \alpha, \gamma, c) = 1 - \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [(1 + x^c)^{-1}]^{-\gamma}\}\right)}{1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [(1 + x^c)^{-1}]^{-\gamma}\}\right)\right)} \right]^\alpha$$

and

$$\begin{aligned} f(x; \alpha, \gamma, c) &= 2\alpha cx^{c-1} \exp\left(\frac{\alpha}{\gamma}\{1 - [(1 + x^c)^{-1}]^{-\gamma}\}\right) [(1 + x^c)^{-1}]^{-\gamma-1} \\ &\times \left(1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [(1 + x^c)^{-1}]^{-\gamma}\}\right)\right)\right)^{-(\alpha+1)} (1 + x^c)^{-2}, \end{aligned}$$

respectively, for $\alpha, \gamma, c > 0$ being shape parameters and $x > 0$. The hazard rate function (hrf) is given by

$$h(x) = 2\alpha c x^{c-1} \exp\left(\frac{\alpha}{\gamma}\{1 - [(1+x^c)^{-1}]^{-\gamma}\}\right) [(1+x^c)^{-1}]^{-\gamma-1} \\ \times \left(1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [(1+x^c)^{-1}]^{-\gamma}\}\right)\right)\right)^{-(\alpha+1)} (1+x^c)^{-2} \\ \times \left[\frac{\exp\left(\frac{1}{\gamma}\{1 - [(1+x^c)^{-1}]^{-\gamma}\}\right)}{1 + \left(1 - \exp\left(\frac{1}{\gamma}\{1 - [(1+x^c)^{-1}]^{-\gamma}\}\right)\right)}\right]^{-\alpha}.$$

	(α, γ, c)				
u	(0.3, 1.3, 1.6)	(2.4, 0.5, 3.9)	(5.3, 1.7, 0.8)	(0.5, 0.8, 0.9)	(0.9, 0.6, 4.1)
0.1	0.3496	0.3768	0.0050	0.0879	0.5064
0.2	0.5781	0.4590	0.0097	0.2167	0.6126
0.3	0.7991	0.5203	0.0141	0.3979	0.6964
0.4	1.0261	0.5756	0.0248	0.6429	0.7713
0.5	1.2693	0.6256	0.0312	0.9886	0.8437
0.6	1.5292	0.6792	0.0453	1.4782	0.9183
0.7	1.8283	0.7344	0.0697	2.2187	1.0058
0.8	2.1904	0.8062	0.0950	3.4314	1.1111
0.9	2.7043	0.9073	0.1552	5.8477	1.2646

Table 3.5: Table of Quantile for TIEHL-Gom-LLoG Distribution

	(α, γ, c)				
	(0.2,0.5,0.8)	(1.7,0.8,1.9)	(3.4,1.5,0.5)	(2.5,3.2,1.9)	(1.9,0.9,0.6)
E(X)	0.0806	0.4095	0.0472	0.3552	0.1383
E(X ²)	0.0476	0.2357	0.0119	0.1581	0.0598
E(X ³)	0.0336	0.1565	0.0055	0.0813	0.0364
E(X ⁴)	0.0259	0.1138	0.0034	0.0463	0.0258
E(X ⁵)	0.0210	0.0880	0.0024	0.0286	0.0199
E(X ⁶)	0.0177	0.0712	0.0018	0.0188	0.0162
SD	0.2028	0.2608	0.0986	0.1788	0.2016
CV	2.5142	0.6369	2.0868	0.5035	1.4580
CS	2.7713	0.2385	4.2563	0.4238	2.0599
CK	9.9267	2.2253	26.3970	2.7278	6.9313

Table 3.6: Moments of TIEHL-Gom-LLoG Distribution for Selected Parameters

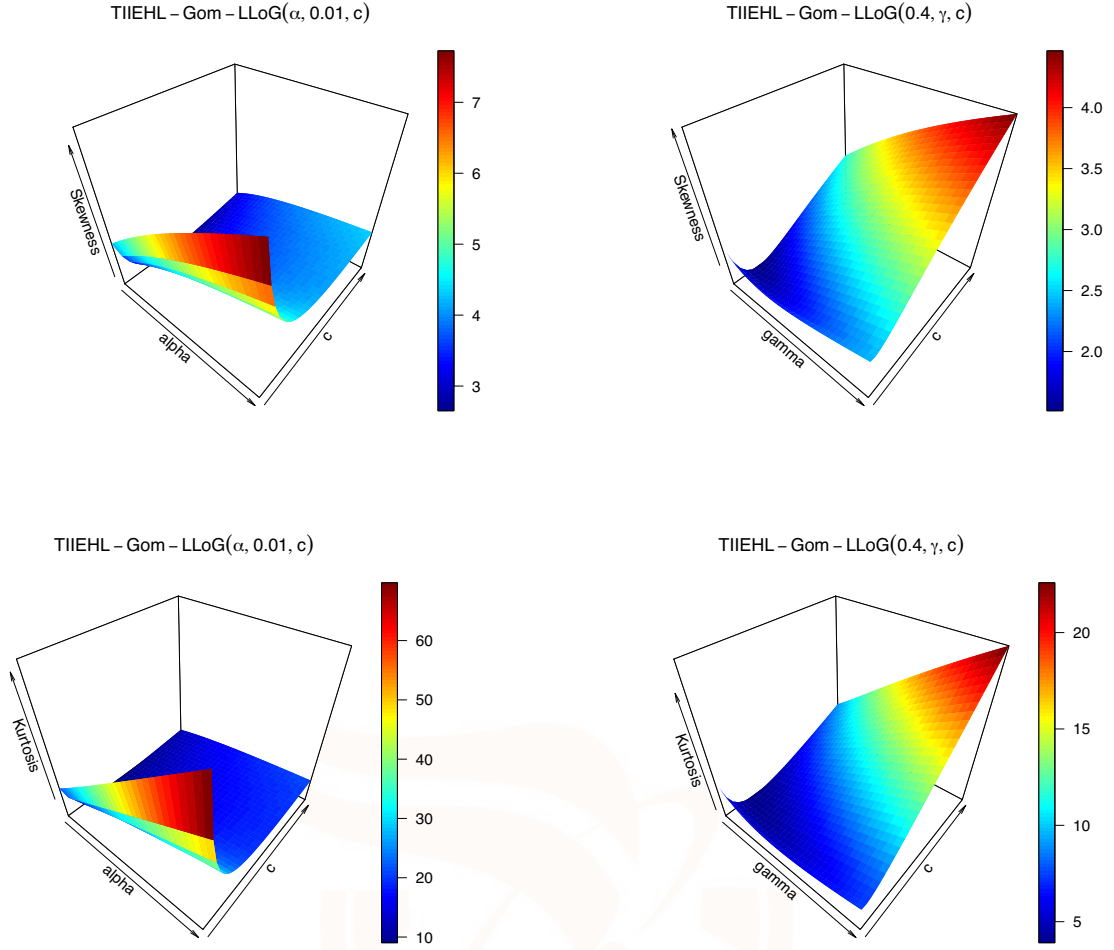


Figure 3.5: Visualization of Skewness and Kurtosis for the TIEHL-Gom-LLoG distribution

Figure 3.5 shows the plots of skewness and kurtosis for the TIEHL-Gom-LLoG distribution. We can see the skewness and kurtosis becomes right skewed with increasing values of α and c and left skewed with increasing values γ and c . Thus, we conclude that the parameters α , c and γ affect the shape of the TIEHL-Gom-LLoG distribution.

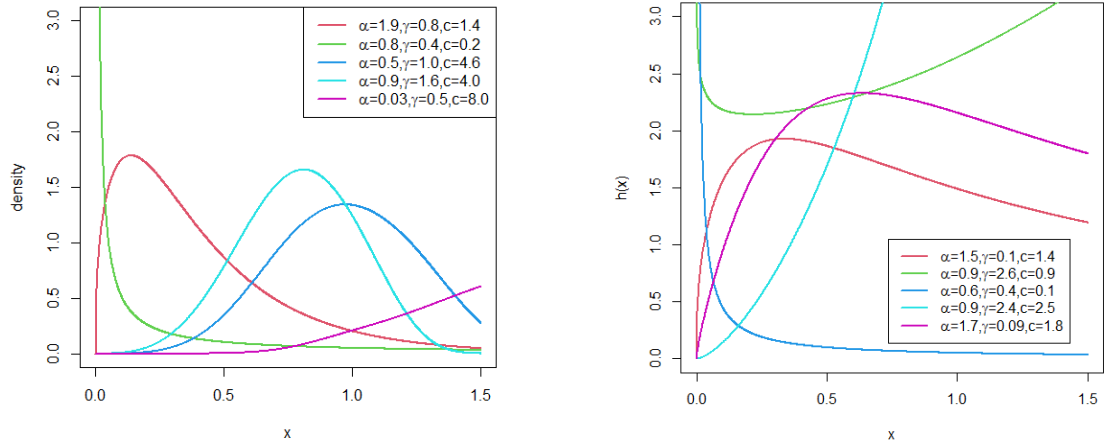


Figure 3.6: Visualization of the pdf and hrf for the TIEHL-Gom-LLoG distribution

Figure 3.6 shows the plots of the pdf and hrf of the TIEHL-Gom-LLoG distribution. The pdf can take various shapes that include almost symmetric, inverse-J, increasing, left, or right-skewed. Furthermore, the hrfs for the TIEHL-Gom-LLoG distribution exhibits increasing, inverse-J, bathtub and upside bathtub shapes.

3.1.5 Maximum Likelihood Estimation

Let $X \sim \text{TIEHL-Gom-G}(\alpha, \gamma, \zeta)$ and $\Delta = (\alpha, \gamma, \zeta)^T$ be the vector of model parameters. The log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the TIEHL-Gom-G GPDF is given by

$$\begin{aligned} \ell_n(\Delta) &= n \ln(2\alpha) + \sum_{i=1}^n \left(\frac{\alpha}{\gamma} \{1 - [1 - G_B(x_i; \zeta)]^{-\gamma}\} \right) + (-\gamma - 1) \sum_{i=1}^n \ln[1 - G_B(x_i; \zeta)] \\ &\quad - (\alpha + 1) \sum_{i=1}^n \ln \left(1 + \left[1 - \exp \left(\frac{1}{\gamma} \{1 - [1 - G_B(x_i; \zeta)]^{-\gamma}\} \right) \right] \right) + \sum_{i=1}^n \ln(g_B(x_i; \zeta)). \end{aligned}$$

Elements of score vector are given in the appendix. The MLEs of the parameters, denoted by $\hat{\Delta}$ can be obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \gamma}, \frac{\partial \ell_n}{\partial \zeta_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by

$$J(\Delta) = \begin{pmatrix} J_{\alpha, \alpha}(\Delta) & J_{\alpha, \gamma}(\Delta) & J_{\alpha, \zeta}(\Delta) \\ J_{\gamma, \alpha}(\Delta) & J_{\gamma, \gamma}(\Delta) & J_{\gamma, \zeta}(\Delta) \\ J_{\zeta, \alpha}(\Delta) & J_{\zeta, \gamma}(\Delta) & J_{\zeta, \zeta}(\Delta) \end{pmatrix}, \quad (3.15)$$

where $J_{i,j} = -\frac{\partial^2 \ell_n(\Delta)}{\partial_i \partial_j}$ for $i, j = \alpha, \gamma, \zeta$.

The multivariate normal distribution $N_{q+2}(\underline{0}, J(\hat{\Delta})^{-1})$, is employed for constructing confidence intervals and regions for individual model parameters as well as for the survival and hazard rate functions.. Here, the mean vector is denoted as $\underline{0} = (0, 0, \underline{0})^T$ and $J(\hat{\Delta})^{-1}$ represents the observed Fisher information matrix evaluated at $\hat{\Delta}$.

3.1.6 Simulation Study

The performance of the MLEs of the TIIEHL-Gom-W distribution is assessed by performing a simulation study for different sample sizes ($n = 25, 50, 100, 200, 400, 800, 1600$) via the R package. We generate $N = 3000$ samples based on the true parameter values outlined in 3.7 and 3.8. The tables present the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). All simulations were done using R software. The ABias and RMSE for the estimated parameter, say, $\hat{\theta}$, are computed as follows:

$$AverageBias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

Parameter	Sample Size	(1.3, 1.3, 1.3, 0.3)			(2.0, 0.3, 1.3, 0.8)		
		Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
α	25	1.5846	0.5875	0.2846	3.7422	3.8869	1.7422
	50	1.4603	0.4238	0.1603	3.4929	3.6547	1.4929
	100	1.3912	0.3501	0.0912	3.4185	3.4698	1.4185
	200	1.3400	0.2521	0.0400	3.2524	3.2394	1.2524
	400	1.3281	0.1975	0.0281	2.7982	2.4837	0.7982
	800	1.3024	0.1492	0.0024	2.5559	1.0059	0.5559
	1600	1.3018	0.1142	0.0018	2.3264	0.5142	0.3264
γ	25	1.4512	0.4213	0.1512	0.4558	0.6731	0.1558
	50	1.3874	0.3441	0.0874	0.4332	0.6125	0.1332
	100	1.3389	0.2527	0.0389	0.3845	0.4993	0.0845
	200	1.3259	0.1948	0.0259	0.3417	0.2885	0.0417
	400	1.3085	0.1459	0.0085	0.3273	0.1880	0.0273
	800	1.3066	0.1148	0.0066	0.3092	0.1336	0.0092
	1600	1.3058	0.0926	0.0058	0.3023	0.0956	0.0023
δ	25	2.2219	2.1319	0.9219	1.4909	0.4908	0.1909
	50	2.1947	2.1808	0.8947	1.4000	0.3989	0.1001
	100	2.1219	2.0268	0.8219	1.3595	0.3314	0.0595
	200	1.9965	1.9209	0.6965	1.3340	0.2456	0.0340
	400	1.8131	1.5710	0.5131	1.3128	0.1931	0.0128
	800	1.6662	1.1288	0.3662	1.3072	0.1118	0.0072
	1600	1.3625	0.9074	0.0625	1.3054	0.0906	0.0054
λ	25	0.4586	0.8699	0.1586	1.3549	1.2095	0.5549
	50	0.4271	0.6379	0.1271	1.3493	1.3646	0.5493
	100	0.3859	0.5135	0.0859	1.3425	1.2832	0.5425
	200	0.3348	0.2403	0.0348	1.1648	1.0865	0.3648
	400	0.3252	0.1694	0.0252	1.0567	0.8400	0.2567
	800	0.3125	0.1338	0.0125	1.0141	0.6775	0.2141
	1600	0.3055	0.0963	0.0055	0.9493	0.5324	0.1493

Table 3.7: Monte Carlo Simulation Results

Parameter	Sample Size	(0.3, 2.0, 2.0, 0.3)			(0.8, 2.0, 0.8, 0.3)		
		Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
α	25	0.4441	0.7291	0.1441	0.8941	0.2616	0.0941
	50	0.3953	0.5367	0.0953	0.8441	0.2057	0.0441
	100	0.3373	0.3816	0.0373	0.8343	0.1448	0.0343
	200	0.3313	0.2218	0.0313	0.8219	0.1133	0.0219
	400	0.3254	0.1716	0.0254	0.8115	0.0848	0.0115
	800	0.3081	0.1234	0.0081	0.8102	0.0635	0.0102
	1600	0.3032	0.0958	0.0032	0.8097	0.0560	0.0097
	25	2.4400	0.8902	0.4400	0.9793	0.3616	0.1793
	50	2.2669	0.6634	0.2669	0.9030	0.2589	0.1030
	100	2.1443	0.5216	0.1443	0.8592	0.2073	0.0592
	200	2.0731	0.3889	0.0731	0.8371	0.1500	0.0371
	400	2.0397	0.2981	0.0397	0.8311	0.1169	0.0311
	800	2.0142	0.1757	0.0142	0.8186	0.0875	0.0186
	1600	2.0135	0.1397	0.0135	0.8181	0.0575	0.0181
	25	3.6715	3.6596	1.6715	0.4277	0.6104	0.1277
	50	3.4872	3.5812	1.4872	0.3922	0.5877	0.0922
	100	3.4018	3.4639	1.4018	0.3555	0.4149	0.0555
	200	3.2031	3.1751	1.2031	0.3256	0.2163	0.0256
	400	2.8250	2.5276	0.8250	0.3254	0.1815	0.0254
	800	2.2247	1.2276	0.2247	0.3115	0.1338	0.0115
	1600	2.1573	0.4978	0.1573	0.3057	0.0949	0.0057
	25	0.4679	0.8842	0.1679	2.2849	0.7448	0.2849
	50	0.4225	0.5391	0.1225	2.2063	0.6232	0.2063
	100	0.3775	0.4472	0.0775	2.0945	0.4990	0.0945
	200	0.3430	0.2857	0.0430	2.0629	0.3661	0.0629
	400	0.3254	0.1800	0.0254	2.0365	0.2903	0.0365
	800	0.3115	0.1339	0.0115	2.0216	0.1715	0.0216
	1600	0.3011	0.0941	0.0011	2.0100	0.1403	0.0100

Table 3.8: Monte Carlo Simulation Results

The figures in Table 3.7 and 3.8 indicate that the estimates are quite stable as they get close to the true values as the sample size increase. Furthermore, as the sample size increases, the MSEs decreases as expected

3.1.7 Applications

In this section, we employ two real data sets to compare the fits of TIEHL-Gom-W distribution to its nested models and to the following non-nested models: type II general inverse exponential Burr III (TIIGIE-BIII) distribution by [Jamal et al. \(2017\)](#), type II exponentiated half-logistic Topp-Leone-Weibull logarithmic (TIEHL-TL-WL) distribution by [Moakofi et al. \(2021b\)](#), type II general inverse exponential Lomax (TIIGIE-Lx) distribution by [Hamedani et al. \(2019\)](#), exponentiated half logistic-power generalized Weibull-log-logistic (EHL-PGW-LLoG) distribution by [Oluyede et al. \(2020a\)](#) and odd exponentiated half logistic- Burr XII (OEHL-BXII) distribution by [Aldahlan and Afify \(2018\)](#), generalized Weibull Gompertz (GWG) distribution by [Nadarajah and Kotz \(2005\)](#), Harris extended Gompertz (HEG) distribution by [Batsidis and Lemonte \(2015\)](#), Kumaraswamy inverse Gompertz (KuIG) distribution by [El-Morshedy et al. \(2020\)](#) and generalized Gompertz (GG) distribution by [El-Gohary et al. \(2013\)](#). The pdf's of these non-nested models are given in the appendix.

The MLEs of the parameters of TIEHL-Gom-W distribution (standard errors in parenthesis) are given in Table 3.9 and 3.11, AIC, CAIC, BIC are given in Table 3.10 and 3.12 for Dataset 1 and Dataset 2, respectively.

The tables also include the GoF statistics W^* and A^* , as introduced by Chen and Balakr-

ishnan [Chen and Balakrishnan \(1995\)](#). These metrics serve to verify which distribution fits better to the data. In general, the smaller the values of AIC, CAIC, BIC, W^* and A^* , the better the fit.

Dataset 1

The data set consists of 66 observations on breaking stress of carbon fibres (Gba). The data set was reported by [Nichols and Padgett \(2006\)](#), and also used by [Al-Babtain et al. \(2020\)](#). **(The observations are given in the appendix)**. The estimated variance-covariance matrix for TIIHL-Gom-W model on carbon fibres is given by

$$\begin{bmatrix} 8.2199 \times 10^{-07} & -3.0389 \times 10^{-10} & 3.4907 \times 10^{-08} & -2.9117 \times 10^{-05} \\ -3.0389 \times 10^{-10} & 1.1234 \times 10^{-13} & -1.2905 \times 10^{-11} & 1.0764 \times 10^{-08} \\ 3.4907 \times 10^{-08} & -1.2905 \times 10^{-11} & 1.6431 \times 10^{-09} & -1.2365 \times 10^{-06} \\ -2.9117 \times 10^{-05} & 1.0764 \times 10^{-08} & -1.2365 \times 10^{-06} & 1.0314 \times 10^{-03} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by

$\alpha \in [1.4311 \pm 1.7770 \times 10^{-03}]$, $\gamma \in [5.3947 \times 10^{03} \pm 6.5696 \times 10^{-07}]$, $\delta \in [9.0989 \times 10^{-04} \pm 7.9449 \times 10^{-05}]$ and $\lambda \in [4.1176 \times 10^{-01} \pm 6.2947 \times 10^{-02}]$, respectively.

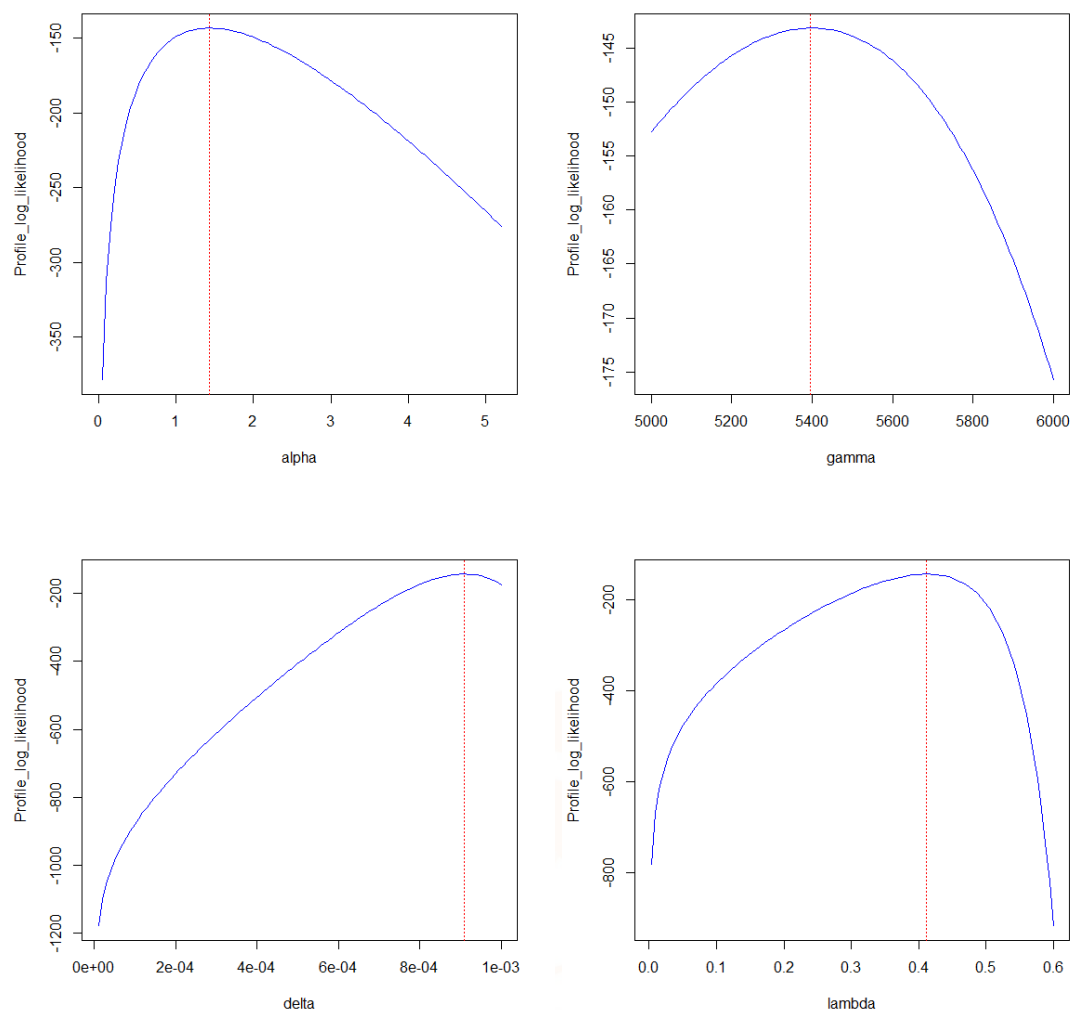


Figure 3.7: Profile log-likelihood function plots for parameters of TIIHL-Gom-W distribution on the carbon fibres data

The MLE of the TIIHL-Gom-W parameters on carbon fibres data can be uniquely identified, as shown in Figure 3.7.

Model	Estimates			
	α	γ	δ	λ
TIIEHL-Gom-W	1.4311 (9.0664 $\times 10^{-04}$)	5.3947 $\times 10^{03}$ (3.3519 $\times 10^{-07}$)	9.0989 $\times 10^{-04}$ (4.0535 $\times 10^{-05}$)	4.1176 $\times 10^{-01}$ (3.2116 $\times 10^{-02}$)
TIIEHL-Gom-W(1, γ , δ , λ)	1	6.0257 $\times 10^{05}$ (5.4597 $\times 10^{-18}$)	1.9423 $\times 10^{-05}$ (1.8467 $\times 10^{-07}$)	9.0935 $\times 10^{-02}$ (3.6625 $\times 10^{-12}$)
TIIEHL-Gom-W(α , 1, δ , λ)	7.9142 $\times 10^{03}$ (1.3194 $\times 10^{-14}$)	1	3.2366 $\times 10^{-05}$ (3.2366 $\times 10^{-06}$)	7.0761 $\times 10^{-01}$ (1.0425 $\times 10^{-10}$)
TIIEHL-Gom-W(α , γ , δ , 1)	2.1834 (0.71132)	64.4268 (0.0210)	0.0145 (0.0014)	1
TIIEHL-Gom-W(1, γ , 1, λ)	1	2.5576 $\times 10^{-09}$ (1.2493 $\times 10^{-02}$)	1	6.8029 $\times 10^{-01}$ (5.0466 $\times 10^{-02}$)
TIIGIE-BIII	7.7081 (5.54640)	8.4126 $\times 10^{04}$ (2.2992 $\times 10^{-05}$)	1.1417 $\times 10^{-01}$ (1.9800 $\times 10^{-02}$)	1.6152 $\times 10^{01}$ (5.9742 $\times 10^{-01}$)
TII-EHL-TL-WL	1.0950 (4.0099 $\times 10^{-01}$)	9.0645 (2.3093)	1.1350 $\times 10^{-08}$ (1.1409 $\times 10^{-02}$)	2.7996 $\times 10^{-01}$ (4.6888 $\times 10^{-02}$)
TIIGIE-Lx	2.9529 $\times 10^{02}$ (5.5597 $\times 10^{-12}$)	3.6376 $\times 10^{04}$ (4.5155 $\times 10^{-14}$)	1.2838 $\times 10^{-04}$ (1.2813 $\times 10^{-05}$)	3.6203 $\times 10^{03}$ (4.5352 $\times 10^{-13}$)
EHL-PGW-LLoG	1.2499 (63.3087)	0.6264 (0.2445)	2.6141 (0.9515)	1.4037 (71.1000)
OEHL-BXII	0.3078 (0.0616)	0.0019 (0.0024)	11.9671 (0.0016)	0.4005 (0.0666)
GWG	109.0200 (6.5990 $\times 10^{-06}$)	2.1192 $\times 10^{-07}$ (0.0766)	1.4717 $\times 10^{-03}$ (2.9757 $\times 10^{-04}$)	1.7907 (0.0780)
HEG	9.9090 $\times 10^{03}$ (2.6853 $\times 10^{-03}$)	0.2965 (0.0437)	1.1760 (0.0726)	29.2790 (6.1996)
KuIG	4.6994 (0.4404)	6.1088 $\times 10^{-09}$ (9.6699 $\times 10^{-03}$)	4.6677 (0.82435)	
GG	1.0132 (0.1110)	2.1021e-08 (0.0284)	7.7882 (1.6027)	

Table 3.9: Estimates from the Carbon Fibres Data

Model	Criteria for Model Selection							
	$-2 \log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-Gom-W	286.3741	294.374	294.7951	304.7947	0.0667	0.5418	0.06946	0.7201
TIIEHL-Gom-W(1, γ , δ , λ)	394.6079	400.6082	400.8582	408.4237	0.1495	0.7650	0.30693	1.314 $\times 10^{-08}$
TIIEHL-Gom-W(α , 1, δ , λ)	453.6041	459.6051	459.8551	467.4206	0.1796	0.9270	0.3778	7.983 $\times 10^{-13}$
TIIEHL-Gom-W(α , γ , δ , 1)	294.7265	300.7265	300.9765	308.5420	0.1295	1.0563	0.0884	0.4139
TIIEHL-Gom-W(1, γ , 1, λ)	614.0529	618.0529	618.1766	623.2632	0.1820	0.9424	0.7488	2.2 $\times 10^{-16}$
TIIGIE-BIII	305.2645	313.2647	313.6858	323.6854	31.27947	196.2417	1.0000	2.2 $\times 10^{-16}$
TII-EHL-TL-WL	471.9434	479.9446	480.3656	490.3653	0.3030	1.6324	0.4448	2.2 $\times 10^{-16}$
TIIGIE-Lx	392.5156	400.5156	400.9366	410.9362	0.1488	0.7618	0.3203	2.435 $\times 10^{-09}$
EHL-PGW-LLoG	286.7524	294.7524	295.1734	305.1730	0.1568	0.7964	0.1003	0.2664
OEHL-BXII	318.6305	326.6305	327.0515	337.0512	0.2041	1.4189	0.1301	0.0679
GWG	309.2974	317.2975	317.7186	327.7182	0.0903	0.8189	0.1726	0.0051
HEG	318.8496	326.8409	327.2619	337.2615	0.3931	2.2153	0.2163	0.0001
KuIG	311.0403	317.0405	317.2905	324.856	0.3858	2.1263	0.1280	0.0751
GG	292.3646	298.3646	298.6146	306.1801	0.2267	1.1859	0.1077	0.1959

Table 3.10: GoF Statistics for Carbon Fibres Data

These results show that the TIIEHL-Gom-W distribution provide a significantly better fit than its nested and non-nested models considered for comparison. This can be confirmed by the smallest values of goodness-of-fit statistics K-S, AIC , $CAIC$, BIC , W^* and A^* and largest p-value for the TIIEHL-Gom-W distribution as compared to nested and non-nested distributions.

Plots of histogram and fitted densities, and observed probability versus predicted probability for the carbon fibres data given in Figure 3.8 support the results shown in Table 3.10.

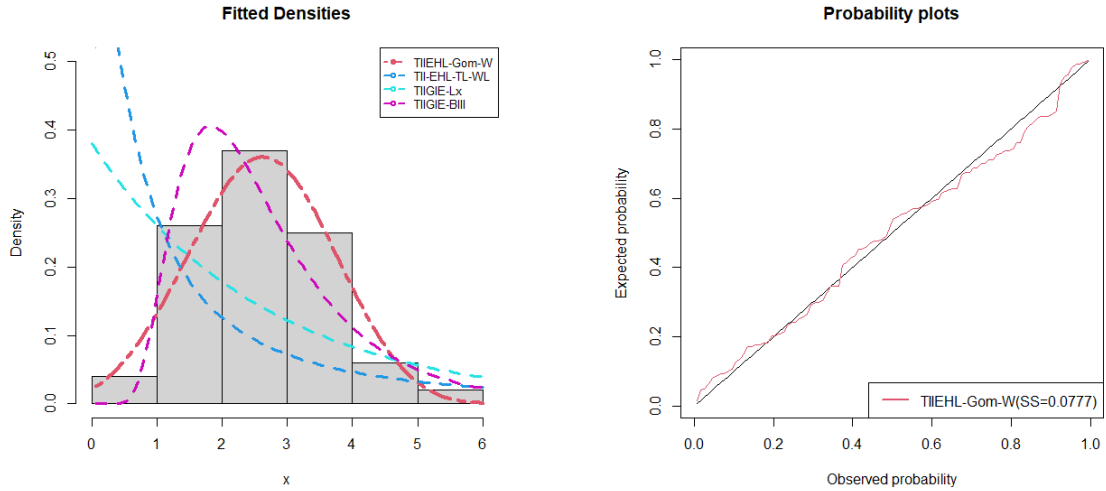


Figure 3.8: Fitted Densities and Probability Plots of the Carbon Fibres Data

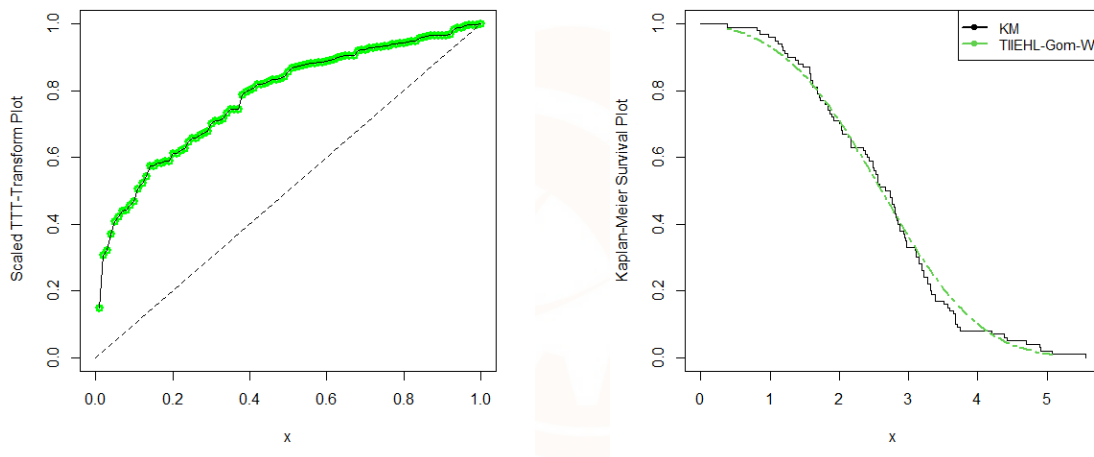


Figure 3.9: Fitted Total Time on Test (TTT) and Kaplan-Meier (K-M) Plots for Carbon Fibres Data

Figure 3.9 shows the TTT scaled plots, observed and the fitted K-M curves. We conclude that our model performs better since the observed and fitted K-M curves are close to each other.

Dataset 2

The data set is on the breaking stress of carbon fibres of 50 mm length (GPa). The data has been previously used by [Cordeiro and Lemonte \(2011\)](#) and [Al-Aqtash et al. \(2014\)](#). (**The observations are given in the appendix**).

The estimated variance-covariance matrix for TIIEHL-Gom-W model on carbon fibres is given by

$$\begin{bmatrix} 3.2330 \times 10^{-05} & -1.2304 \times 10^{-07} & 2.5058 \times 10^{-06} & -4.1914 \times 10^{-04} \\ -1.2304 \times 10^{-07} & 4.6828 \times 10^{-10} & -9.5366 \times 10^{-09} & 1.5951 \times 10^{-06} \\ 2.5058 \times 10^{-06} & -9.5366 \times 10^{-09} & 2.0564 \times 10^{-07} & -3.2486 \times 10^{-05} \\ -4.1914 \times 10^{-04} & 1.5951 \times 10^{-06} & -3.2486 \times 10^{-05} & 5.4339 \times 10^{-03} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\alpha \in [1.4376 \pm 1.1144 \times 10^{-02}]$, $\gamma \in [5.0130 \times 10^{02} \pm 4.2414 \times 10^{-05}]$, $\delta \in [4.4951 \times 10^{-03} \pm 8.8881 \times 10^{-04}]$ and $\lambda \in [7.5286 \times 10^{-01} \pm 1.4448 \times 10^{-01}]$, respectively.

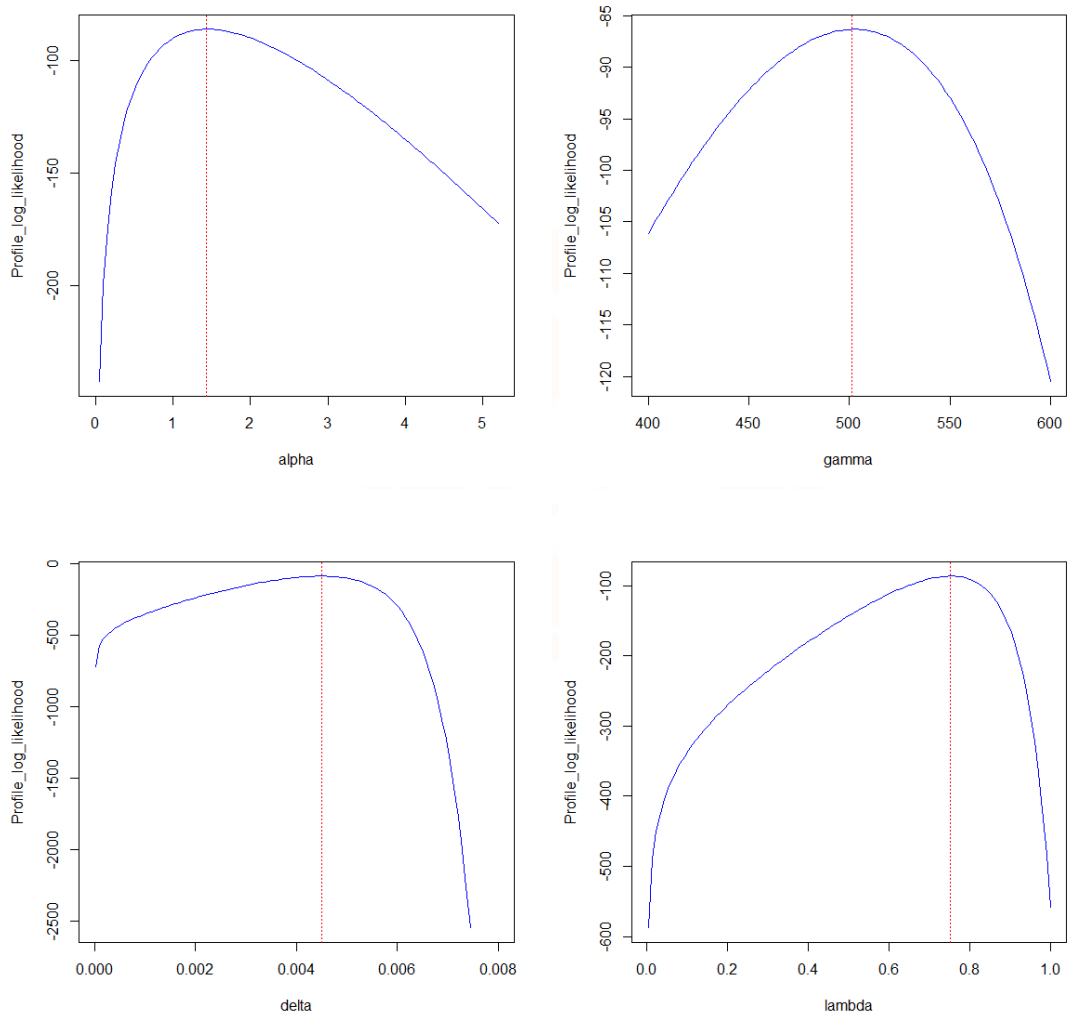


Figure 3.10: Profile log-likelihood function plots for parameters of TIIEHL-Gom-W distribution on the breaking stress of carbon fibres data

Figure 3.10 shows that the MLEs of the TIIEHL-Gom-W parameters on breaking stress of carbon fibres data can be uniquely identified.

Model	Estimates			
	α	γ	δ	λ
TIEHL-Gom-W	1.4376 (5.6860 $\times 10^{-03}$)	5.0130 $\times 10^{02}$ (2.1640 $\times 10^{-05}$)	4.4951 $\times 10^{-03}$ (4.5348 $\times 10^{-04}$)	7.5286 $\times 10^{-01}$ (7.3715 $\times 10^{-02}$)
TIEHL-Gom-W(1, γ , δ , λ)	1	6.4835 $\times 10^{05}$ (5.3721 $\times 10^{-18}$)	1.8106 $\times 10^{-05}$ (2.0903 $\times 10^{-07}$)	9.1171 $\times 10^{-02}$ (4.0075 $\times 10^{-12}$)
TIEHL-Gom-W(α , 1, δ , λ)	2.4099 $\times 10^{03}$ (2.9740 $\times 10^{-13}$)	1	7.6352 $\times 10^{-05}$ (9.3983 $\times 10^{-06}$)	9.8446 $\times 10^{-01}$ (7.6663 $\times 10^{-10}$)
TIEHL-Gom-W(α , γ , δ , 1)	0.0273 (0.0311)	0.9319 (1.1783)	1.1933 (1.4619)	1
TIEHL-Gom-W(1, γ , 1, λ)	1	1.0711 $\times 10^{-08}$ (2.3879 $\times 10^{-02}$)	1	5.0147 $\times 10^{-01}$ (5.2539 $\times 10^{-02}$)
TIIGIE-BIII	4.0810 $\times 10^{01}$ (7.9437 $\times 10^{-06}$)	2.5313 $\times 10^{-01}$ (1.9957 $\times 10^{-02}$)	8.3944 (5.8002 $\times 10^{-04}$)	1.9747 $\times 10^{-04}$ (9.5760 $\times 10^{-01}$)
TII-EHL-TL-WL	1.2011 $\times 10^{-01}$ (1.9236 $\times 10^{-02}$)	7.6931 $\times 10^{-02}$ (2.9969 $\times 10^{-02}$)	4.3739 $\times 10^{-09}$ (1.2388 $\times 10^{-02}$)	1.0050 (1.1282 $\times 10^{-01}$)
TIIGIE-Lx	3.1829 $\times 10^{02}$ (6.0958 $\times 10^{-12}$)	3.3438 $\times 10^{04}$ (5.8043 $\times 10^{-14}$)	1.2576 $\times 10^{-04}$ (1.5454 $\times 10^{-05}$)	3.6985 $\times 10^{03}$ (5.2456 $\times 10^{-13}$)
EHL-PGW-LLoG	1.8986 (0.1152)	0.6360 (0.1994)	3.2914 (1.0464)	0.9252 (0.2365)
OEHL-BXII	0.2813 (0.0810)	0.0001 (0.0003)	7.9388 (0.0169)	0.8500 (0.1947)
GWG	1.1443 $\times 10^{-04}$ (3.0457 $\times 10^{-05}$)	1.5797 $\times 10^{-07}$ (2.3349 $\times 10^{-03}$)	7.1155 (3.5485 $\times 10^{-04}$)	0.2264 (0.0215)
HEG	1.0497 (3.5673)	0.0558 (0.1819)	2.5501 (0.3068)	0.0003 (0.3448)
KuIG	5.9700 (0.5953)	2.6144 $\times 10^{-08}$ (0.0257)	5.9055 (1.1880)	
GG	1.0075 (0.2732)	4.2552 $\times 10^{-07}$ (0.1126)	9.1992 (3.1828)	

Table 3.11: Estimates from the Breaking Stress of Carbon Fibres

Model	Criteria for Model Selection							
	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIEHL-Gom-W	172.6052	180.6053	181.261	189.3639	0.0723	0.5093	0.0984	0.5446
TIEHL-Gom-W(1, γ , δ , λ)	266.2306	272.2292	272.6163	278.7982	0.2445	1.3219	0.3510	1.718 $\times 10^{-07}$
TIEHL-Gom-W(α , 1, δ , λ)	267.8032	273.8027	274.1898	280.3717	0.2481	1.3442	0.3608	6.88 $\times 10^{-08}$
TIEHL-Gom-W(α , γ , δ , 1)	177.7984	183.7984	184.1855	190.3674	0.1240	0.8971	0.1166	0.3305
TIEHL-Gom-W(1, γ , 1, λ)	432.6558	436.6559	436.8463	441.0352	0.3213	1.7661	0.7648	2.2 $\times 10^{-16}$
TIIGIE-BIII	218.6713	226.6713	227.327	235.4299	18.37841	125.4600	0.9999	2.2 $\times 10^{-16}$
TII-EHL-TL-WL	324.8422	332.842	333.4977	341.6006	0.1565	0.8295	0.4631	1.005 $\times 10^{-12}$
TIIGIE-Lx	265.8472	273.8472	274.503	282.6058	0.2457	1.3308	0.3579	9.075 $\times 10^{-08}$
EHL-PGW-LLoG	182.8946	190.8946	191.5503	199.6532	0.2567	1.3839	0.1433	0.1326
OEHL-BXII	194.4549	202.4549	203.1107	211.2136	0.1948	1.2537	0.1612	0.0646
GWG	195.2118	203.2109	203.8667	211.9696	0.1068	0.5794	0.2184	0.0036
HEG	177.5911	185.5911	186.2468	194.3497	0.0919	0.6492	0.1121	0.3775
KuIG	206.9517	212.9517	213.3388	219.5207	0.5184	2.9179	0.1850	0.0217
GG	190.7447	196.7447	197.1318	203.3137	0.3472	1.9082	0.1549	0.0841

Table 3.12: GoF Statistics for Breaking Stress of Carbon Fibres

The TIEHL-Gom-W distribution gives the smallest AIC , $AICC$, BIC , K-S, W^* , A^* , and the largest p-value in Table 3.12. These statistics indicate that the TIEHL-Gom-W distribution fits the data better than other competitive distributions for breaking stress of carbon fibres.

Fitted density plots, histogram of the data and probability plots for the breaking stress of carbon fibres are given in Figure 3.11. These plots also support the conclusion reached from Table 3.12.

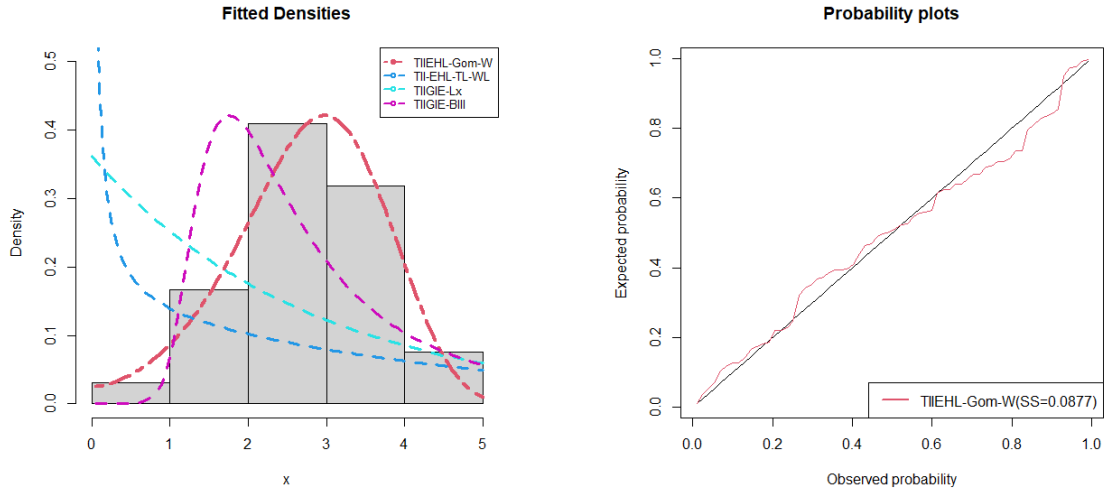


Figure 3.11: Fitted Densities and Probability Plots of the Breaking Stress of Carbon Fibres

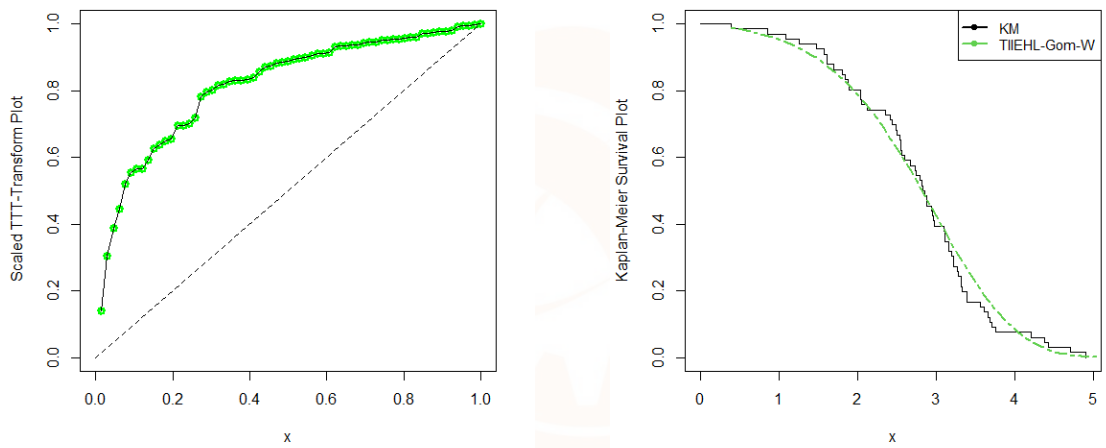


Figure 3.12: Fitted TTT and K-M Plots for Breaking Stress of Carbon Fibres

Figure 3.12 shows the TTT scaled plots, observed and the fitted K–M curves. We conclude that the TIIEHL-Gom-W distribution performs better since the observed and fitted K–M curves are close to each other.

Likelihood Ratio Test

The likelihood ratio test results for comparing the full and nested models are given in this section.

Model	df	Dataset 1	Dataset 2
		$\chi^2(p - value)$	$\chi^2(p - value)$
TIIEHL-Gom-W($1, \gamma, \delta, \lambda$)	1	108.2338(<0.00001)	93.6254(<0.00001)
TIIEHL-Gom-W($\alpha, 1, \delta, \lambda$)	1	167.2300(<0.00001)	95.1980(<0.00001)
TIIEHL-Gom-W($\alpha, \gamma, \delta, 1$)	1	8.3524(.0038)	5.1932(0.0226)
TIIEHL-Gom-W($1, \gamma, 1, \lambda$)	2	327.6788(<0.00001)	260.0506(<0.00001)

Table 3.13: Likelihood Ratio Test Results

The likelihood ratio test results in Table 3.13 indicates that the TIIEHL-Gom-TL-W performs better than its nested models at 5% level of significance, since all p-values are less than 0.05 among all the data sets considered.

3.1.8 Concluding Remarks

In this paper, we derived a new GFPD based on the type II transformation. Some properties of the TIIEHL-Gom-G GFD such as quantile function, linear representation, moments, moment generating function, order statistics and Rényi entropy are derived. The unknown parameters of the new distribution are estimated using the maximum likelihood estimation method and evaluated via a simulation study. Finally, two data sets have been analyzed to show the importance of the new GFD.

4. Topp-Leone-Odd Burr III-G (TL-OBIII-G) Generalized Family of Probability Distributions

Moakofi, T., Oluyede, B. and Gabanakgosi, M., 2022. The Topp-Leone odd Burr III-G Family of Distributions: Model, Properties and Applications. *Statistics, Optimization & Information Computing*, 10(1), 236-262.

<http://iapress.org/index.php/soic/article/view/1135>

This chapter utilizes the Odd Burr III and Topp-Leone generators to obtain the new family of distributions.

4.1 TL-OBIII-G GFPD

In this section, we propose and investigate a new generalized family of distributions called the Topp-Leone Odd Burr III-G (TL-OBIII-G) GFPD. We present the sub-families of this new GFPD. Properties of the new distribution including sub-models, quantile function, moments, incomplete and probability weighted moments, distribution of the order statistics, and Renyi entropy are derived. The MLE technique is used to estimate the model parameters, and a Monte Carlo simulation study is employed to examine the performance of the model. Two real-life data sets are used to show the importance of the TL-OBIII-G GFPD.

A random variable X has the odd Burr III-G GFPD if its cdf and pdf are given by

$$F(x; \alpha, \beta, \zeta) = (1 + (O_G(x; \zeta))^{-\alpha})^{-\beta},$$

and

$$f(x; \alpha, \beta, \zeta) = \alpha\beta (1 + (O_G(x; \zeta))^{-\alpha})^{-\beta-1} (O_G(x; \zeta))^{-\alpha-1} \frac{g_B(x; \zeta)}{(\bar{G}_B(x; \zeta))^2},$$

respectively, for $\alpha, \beta > 0$ being shape parameters and parameter vector ζ , where $O_G(x; \zeta) = \frac{G_B(x; \zeta)}{\bar{G}_B(x; \zeta)}$, and $\bar{G}_B(x; \zeta) = 1 - \bar{G}_B(x; \zeta)$.

4.1.1 The New Model

Inserting the cdf of the OBIII-G GFPD into equation (1.17), we obtain the new GFPD herein referred to as Topp-Leone Odd Burr III-G GFPD. The cdf and pdf of the proposed

Topp-Leone Odd Burr III-G (TL-OBIII-G) GFRD are given by

$$F(x; \alpha, \beta, b, \zeta) = \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta} \right)^2 \right]^b \quad (4.1)$$

and

$$\begin{aligned} f(x; \alpha, \beta, b, \zeta) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\ &\times \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta} \right) \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta-1} \\ &\times (O_G(x; \zeta))^{-\alpha-1} \frac{g_B(x; \zeta)}{(\bar{G}_B(x; \zeta))^2}, \end{aligned} \quad (4.2)$$

respectively, for $\alpha, \beta, b > 0$ being shape parameters and parameter vector ζ .

The hrf and qf of the TL-OBIII-G GFRD are respectively given below. The hrf is given by

$$\begin{aligned} h_F(x; \alpha, \beta, b, \zeta) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\ &\times \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta} \right) \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta-1} \\ &\times (O_G(x; \zeta))^{-\alpha-1} \frac{g_B(x; \zeta)}{(\bar{G}_B(x; \zeta))^2} \\ &\times \left(1 - \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta} \right)^2 \right]^{-b} \right)^{-1}, \end{aligned} \quad (4.3)$$

for $\alpha, \beta, \delta > 0$ and parameter vector ζ .

The qf of the TL-OBIII-G GFRD is obtained by solving the non-linear equation:

$$F(x; \alpha, \beta, b, \zeta) = \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta} \right)^2 \right]^b = u, \quad (4.4)$$

for $0 \leq u \leq 1$, that is,

$$O_G(x; \zeta) = \left(\left(1 - \left(1 - u^{\frac{1}{b}} \right)^{\frac{1}{2}} \right)^{\frac{-1}{\beta}} - 1 \right)^{\frac{-1}{\alpha}},$$

so that

$$G_B(x; \zeta) = \left(\left(\left(1 - \left(1 - u^{\frac{1}{b}} \right)^{\frac{1}{2}} \right)^{\frac{-1}{\beta}} - 1 \right)^{\frac{-1}{\alpha}} + 1 \right)^{-1}. \quad (4.5)$$

Consequently, the qf for the TL-OBIII-G GFRD is given by

$$Q_G(u; \alpha, \beta, b, \zeta) = X = G^{-1} \left[\left(\left(\left(1 - \left(1 - u^{\frac{1}{b}} \right)^{\frac{1}{2}} \right)^{\frac{-1}{\beta}} - 1 \right)^{\frac{-1}{\alpha}} + 1 \right)^{-1} \right]. \quad (4.6)$$

It follows therefore that random numbers can be generated from the TL-OBIII-G GPDF based on equation (4.6).

Sub-Families of TL-OBIII-G GPDF

In this subsection, some sub-families of the TL-OBIII-G GPDF are presented.

- When $\alpha = 1$, the Topp-Leone Odd Burr III-G (TL-OBIII-G) GPDF reduces to a new distribution with the cdf

$$F(x; \beta, b, \zeta) = \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-1} \right)^{-\beta} \right)^2 \right]^b,$$

for $\beta, b > 0$, and parameter vector ζ .

- If $\beta = 1$, we obtain the new distribution with the cdf

$$F(x; \alpha, b, \zeta) = \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-1} \right)^2 \right]^b,$$

for $\alpha, b > 0$, and parameter vector ζ .

- If $b = 1$ we obtain the new distribution with the cdf

$$F(x; \alpha, \beta, \zeta) = \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-\beta} \right)^2 \right],$$

for $\alpha, \beta > 0$ and parameter vector ζ .

- If $\beta = \alpha = 1$ we obtain the new distribution with the cdf

$$F(x; b, \zeta) = \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-1} \right)^{-1} \right)^2 \right]^b,$$

for $b > 0$ and parameter vector ζ .

- If $\beta = b = 1$ we obtain the new distribution with the cdf

$$F(x; \alpha, \zeta) = \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-\alpha} \right)^{-1} \right)^2 \right],$$

for $\alpha > 0$ and parameter vector ζ .

- If $\beta = \alpha = b = 1$ we obtain the new distribution with the cdf

$$F(x; \zeta) = \left[1 - \left(1 - \left(1 + (O_G(x; \zeta))^{-1} \right)^{-1} \right)^2 \right],$$

for parameter vector ζ .

4.1.2 Some Specialized Sub Models

In this subsection, we present sub-models of TL-OBIII-G GFRD by specifying $G_B(x; \alpha, b, \zeta)$ and $g_B(x; \alpha, b, \zeta)$ in equations (4.1) and (4.2).

TL-OBIII-LLoG Distribution

If the baseline cdf and pdf are given by $G(x; \lambda) = 1 - (1 + x^\lambda)^{-1}$ and $g(x; \lambda) = \lambda x^{\lambda-1} (1 + x^\lambda)^{-2}$, for $\lambda > 0$, and $x > 0$, the cdf and pdf of Topp-Leone Odd Burr III-log-logistic (TL-OBIII-LLoG) distribution are given by

$$F(x; \alpha, \beta, b, \lambda) = \left[1 - \left(1 - \left(1 + (O_{LLoG}(x; \lambda))^{-\alpha} \right)^{-\beta} \right)^2 \right]^b$$

and

$$\begin{aligned} f(x; \alpha, \beta, b, \lambda) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + (O_{LLoG}(x; \lambda))^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\ &\times \left(1 - \left(1 + (O_{LLoG}(x; \lambda))^{-\alpha} \right)^{-\beta} \right) \frac{\lambda x^{\lambda-1} (1 + x^\lambda)^{-2}}{\left((1 + x^\lambda)^{-1} \right)^2} \\ &\times \left(1 + (O_{LLoG}(x; \lambda))^{-\alpha} \right)^{-\beta-1} (O_{LLoG}(x; \lambda))^{-\alpha-1}, \end{aligned}$$

respectively, for $\alpha, \beta, b, \lambda > 0$ being shape parameters and $x > 0$. The hrf is given by

$$\begin{aligned} h_F(x; \alpha, \beta, b, \lambda) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + (O_{LLoG}(x; \lambda))^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\ &\times \left(1 - \left(1 + (O_{LLoG}(x; \lambda))^{-\alpha} \right)^{-\beta} \right) \frac{\lambda x^{\lambda-1} (1 + x^\lambda)^{-2}}{\left((1 + x^\lambda)^{-1} \right)^2} \\ &\times \left(1 + (O_{LLoG}(x; \lambda))^{-\alpha} \right)^{-\beta-1} (O_{LLoG}(x; \lambda))^{-\alpha-1} \\ &\times \left(1 - \left[1 - \left(1 - \left(1 + (O_{LLoG}(x; \lambda))^{-\alpha} \right)^{-\beta} \right)^2 \right]^b \right)^{-1}, \end{aligned}$$

where $O_{LLoG}(x; \lambda) = \frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}}$.

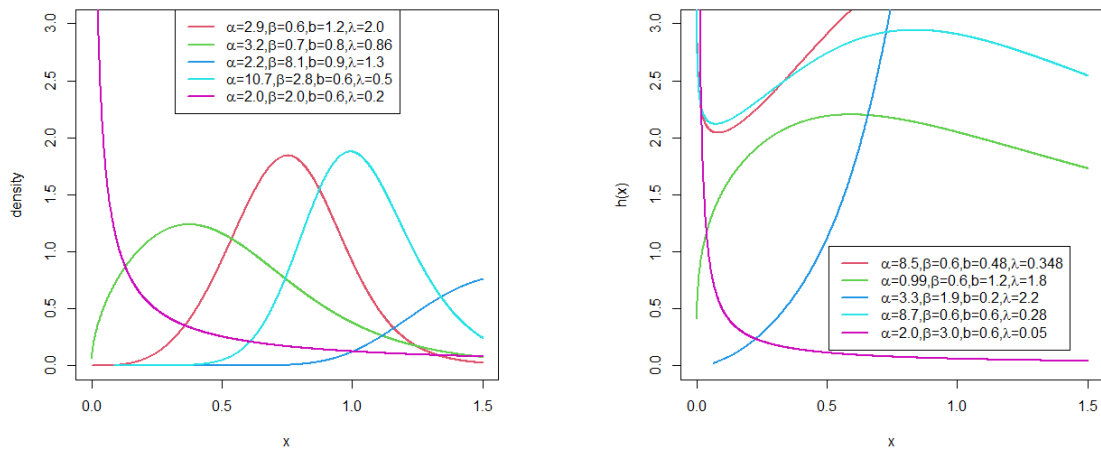


Figure 4.1: Density and hrf Plots for TL-OBIII-LLoG Distribution

Figure 4.1 shows the plots of pdf and hrf of the TL-OBIII-LLoG distribution, respectively. The pdf can take several shapes including increasing, right-skewed, left-skewed, unimodal and reverse-J shapes. The TL-OBIII-LLoG hrf displays increasing, decreasing, bathtub, bathtub followed by upside-down bathtub and upside-down bathtub shapes. These non-monotonic shapes are likely to be encountered in real life situations.

	$(\alpha, \beta, b, \lambda)$				
u	(1.4, 1.0, 0.4, 1.8)	(0.6, 1.3, 2.2, 1.0)	(0.8, 1.0, 2.6, 1.7)	(2.0, 1.0, 1.0, 2.0)	(1.0, 1.8, 0.6, 2.5)
0.1	0.0773	0.2157	0.4168	0.4799	0.3786
0.2	0.1555	0.4281	0.5756	0.5885	0.5050
0.3	0.2330	0.7007	0.7224	0.6643	0.6133
0.4	0.3154	1.0671	0.8773	0.7354	0.7139
0.5	0.4046	1.5836	1.0497	0.8022	0.8174
0.6	0.5027	2.3724	1.2586	0.8735	0.9319
0.7	0.6194	3.6896	1.5376	0.9544	1.0676
0.8	0.7768	6.3088	1.9540	1.0557	1.2502
0.9	1.0327	13.9007	2.7804	1.2141	1.5600

Table 4.1: Table of Quantile for TL-OBIII-LLoG Distribution

TL-OBIII-Power Distribution

The cdf and pdf of power distribution are given by $G(x; \theta, k) = (\theta x)^k$ and $g(x; \theta, k) = k\theta^k x^{k-1}$, for $\theta, k > 0$, and $x > 0$. The cdf and pdf of Topp-Leone Odd Burr III-Power (TL-OBIII-P) distribution has cdf and pdf given by

$$F(x; \alpha, \beta, b, \theta, k) = \left[1 - \left(1 - \left(1 + (O_P(x; \theta, k))^{-\alpha} \right)^{-\beta} \right)^2 \right]^b$$

and

$$\begin{aligned}
 f(x; \alpha, \beta, b, \theta, k) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + (O_P(x; \theta, k))^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\
 &\times \left(1 - \left(1 + (O_P(x; \theta, k))^{-\alpha} \right)^{-\beta} \right) \left(1 + (O_P(x; \theta, k))^{-\alpha} \right)^{-\beta-1} \\
 &\times (O_P(x; \theta, k))^{-\alpha-1} \frac{k\theta^k x^{k-1}}{\left(1 - (\theta x)^k \right)^2},
 \end{aligned}$$

respectively, for $\alpha, \beta, b, k > 0$ being shape parameters, $\theta > 0$ being the scale parameter and $x > 0$. The hrf is given by

$$\begin{aligned}
 h_F(x; \alpha, \beta, b, \theta, k) &= 2\alpha\beta b \left[1 - \left(1 - \left(1 + (O_P(x; \theta, k))^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\
 &\times \left(1 - \left(1 + (O_P(x; \theta, k))^{-\alpha} \right)^{-\beta} \right) \left(1 + (O_P(x; \theta, k))^{-\alpha} \right)^{-\beta-1} \\
 &\times (O_P(x; \theta, k))^{-\alpha-1} \frac{k\theta^k x^{k-1}}{\left(1 - (\theta x)^k \right)^2} \\
 &\times \left(1 - \left[1 - \left(1 - \left(1 + (O_P(x; \theta, k))^{-\alpha} \right)^{-\beta} \right)^2 \right]^b \right)^{-1},
 \end{aligned}$$

where $O_P(x; \theta, k) = \frac{(\theta x)^k}{1 - (\theta x)^k}$.

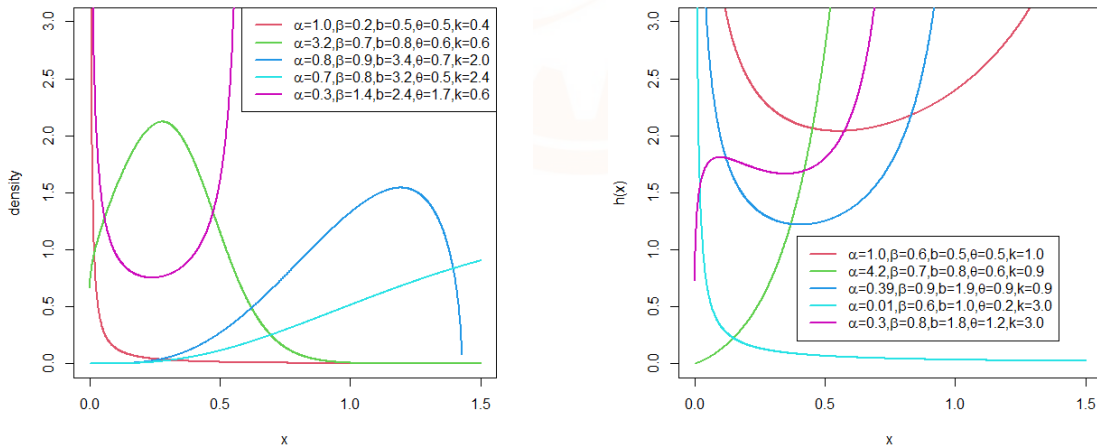


Figure 4.2: Density and hrf Plots for TL-OBIII-Power Distribution

Figure 4.2 shows the plots of pdf and hrf of the TL-OBIII-P distribution, respectively. The pdf can take several shapes including increasing, right-skewed, U-shape and reverse-J shapes. The TL-OBIII-P hrf displays increasing, reverse-J, bathtub and upside-down bathtub followed by bathtub shapes.

	$(\alpha, \beta, b, \theta, k)$				
u	(1.4, 1, 0.4, 1.8, 0.3)	(0.6, 1.3, 2.2, 1, 1.2)	(0.8, 1, 2.6, 1.7, 0.5)	(2, 1.0, 1.0, 2, 1.0)	(1, 1.8, 0.6, 2.5, 0.5)
0.1	0.7849	0.7447	0.1180	0.2946	0.7221
0.2	1.2038	1.0361	0.1813	0.4059	0.9034
0.3	1.5896	1.2875	0.2397	0.4897	1.0497
0.4	1.9807	1.5323	0.2989	0.5605	1.1866
0.5	2.4011	1.7882	0.3628	0.6240	1.3261
0.6	2.8779	2.0726	0.4353	0.6837	1.4782
0.7	3.4540	2.4114	0.5231	0.7420	1.6568
0.8	4.2202	2.8583	0.6399	0.8019	1.8902
0.9	5.4586	3.5807	0.8293	0.8694	2.2648

Table 4.2: Table of Quantile for TL-OBIII-P Distribution

TL-OBIII-Lomax Distribution

Consider Lomax distribution as the baseline distribution with cdf and pdf are given by $G(x; a, \theta) = 1 - (1 + \theta x)^{-a}$ and $g(x; a, \theta) = a\theta(1 + \theta x)^{-a-1}$, for $a, \theta > 0$, and $x > 0$. The TL-OBIII-Lomax (TL-OBIII-L) distribution has cdf and pdf given by

$$F(x; \alpha, \beta, b, a, \theta) = \left[1 - \left(1 - (1 + (O_L(x; \theta, a))^{-\alpha})^{-\beta} \right)^2 \right]^b$$

and

$$\begin{aligned} f(x; \alpha, \beta, b, a, \theta) &= 2\alpha\beta b \left[1 - \left(1 - (1 + (O_L(x; \theta, a))^{-\alpha})^{-\beta} \right)^2 \right]^{b-1} \\ &\times \left(1 - (1 + (O_L(x; \theta, a))^{-\alpha})^{-\beta} \right) \frac{a\theta(1 + \theta x)^{-a-1}}{((1 + \theta x)^{-a})^2} \\ &\times (1 + (O_L(x; \theta, a))^{-\alpha})^{-\beta-1} (O_L(x; \theta, a))^{-\alpha-1}, \end{aligned}$$

respectively, for α, β, b, a being shape parameters, $\theta > 0$ being the scale parameter and $x > 0$. The hrf is given by

$$\begin{aligned} h_F(x; \alpha, \beta, b, a, \theta) &= 2\alpha\beta b \left[1 - \left(1 - (1 + (O_L(x; \theta, a))^{-\alpha})^{-\beta} \right)^2 \right]^{b-1} \\ &\times \left(1 - (1 + (O_L(x; \theta, a))^{-\alpha})^{-\beta} \right) \frac{a\theta(1 + \theta x)^{-a-1}}{((1 + \theta x)^{-a})^2} \\ &\times (1 + (O_L(x; \theta, a))^{-\alpha})^{-\beta-1} (O_L(x; \theta, a))^{-\alpha-1} \\ &\times \left(1 - \left[1 - \left(1 - (1 + (O_L(x; \theta, a))^{-\alpha})^{-\beta} \right)^2 \right]^b \right)^{-1}, \end{aligned}$$

where $O_L(x; \theta, a) = \frac{1 - (1 + \theta x)^{-a}}{(1 + \theta x)^{-a}}$.

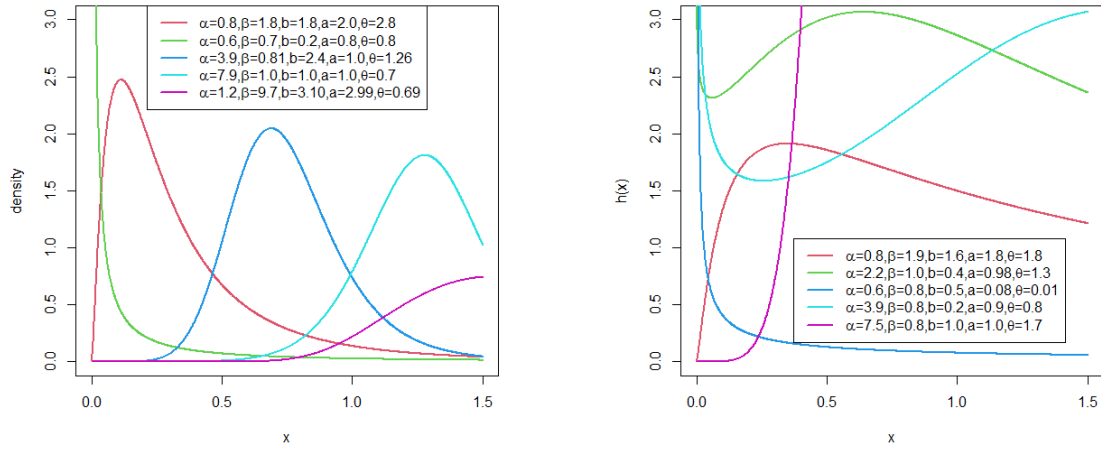


Figure 4.3: Density and hrf Plots for TL-OBIII-L Distribution

Figure 4.3 shows the plots of pdf and hrf functions of TL-OBIII-L distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, almost symmetric, and reverse-J shapes. The TL-OBIII-L hrf displays increasing, decreasing, bathtub, bathtub followed by upside-down bathtub and upside-down bathtub shapes.

	$(\alpha, \beta, b, a, \theta)$				
u	(1.8, 0.9, 0.8, 0.8, 1.0)	(2.0, 0.8, 0.5, 0.4, 2.0)	(2.5, 0.6, 0.7, 1.5, 1.8)	(1.4, 1.3, 1.9, 2.0, 2.0)	(0.8, 1.7, 0.8, 0.4, 1.9)
0.1	0.1414	0.0448	0.0249	0.1008	0.1205
0.2	0.2545	0.1167	0.0501	0.1340	0.2808
0.3	0.3619	0.2034	0.0749	0.1657	0.5182
0.4	0.4836	0.3169	0.0995	0.1934	0.8791
0.5	0.6176	0.4601	0.1250	0.2282	1.4738
0.6	0.7764	0.6440	0.1597	0.2629	2.5450
0.7	0.9849	0.9101	0.1918	0.3094	4.8028
0.8	1.2809	1.3367	0.2352	0.3733	10.9256
0.9	1.8431	2.2512	0.3060	0.4878	40.3184

Table 4.3: Table of Quantile for TL-OBIII-L Distribution

4.1.3 Density Expansion

In this section, we write the pdf of TL-OBIII-G GFGPD as a linear combination of exponentiated-G (Exp-G) distribution, which will be used for further computations. Applying the following expansions

$$(1 - z)^k = \sum_{j=0}^{\infty} (-1)^j \binom{k}{j} z^j \quad \text{and} \quad (1 + z)^{-\beta-1} = \sum_{l=0}^{\infty} \frac{\Gamma((\beta+1)+l)}{\Gamma(\beta+1)l!} (-1)^l z^l,$$

for $|z| < 1$, the pdf of TL-OBIII-G GFGPD can be written as

$$f(x; \alpha, \beta, b, \zeta) = \sum_{v=0}^{\infty} C_{v+1} g_{v+1}(x; \zeta), \quad (4.7)$$

where $g_{v+1}(x; \zeta) = (v+1)[G_B(x; \zeta)]^v g_B(x; \zeta)$ is the Exp-G pdf with the power parameter $v+1 > 0$ and parameter vector ζ ,

$$C_{v+1} = 2\alpha\beta b \sum_{j,p,l,k=0}^{\infty} \binom{b-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\ \times (-1)^{j+p+l+k} \binom{k+\alpha(l+1)-1}{v} \frac{1}{v+1}. \quad (4.8)$$

(See the appendix for derivations).

Therefore, the pdf of TL-OBIII-G GFPD reduces to an infinite linear combination of Exp-G densities. The structural properties of the TL-OBIII-G GFPD follows from those of Exp-G distribution.

4.1.4 Moments and Generating Function

From now onwards, let $Y_{v+1} \sim \text{Exp} - G(v+1)$. The r^{th} raw moment, μ'_r of the TL-OBIII-G GFPD can be derived from equation (4.7) as

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx = \sum_{v=0}^{\infty} C_{v+1} E(Y_{v+1}^r).$$

The moment generating function (MGF) $M_X(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{v=0}^{\infty} C_{v+1} M_{v+1}(t),$$

where $M_{v+1}(t)$ is the mgf of Y_{v+1} and C_{v+1} is given by equation (4.8).

Tables 6.30, 4.5 and 4.6 presents the first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for different parameter values.

	$(\alpha, \beta, b, \lambda)$				
	(0.4, 0.5, 0.2, 0.5)	(0.9, 1.2, 1.0, 0.6)	(0.3, 2.2, 1.0, 0.2)	(3.0, 2.0, 0.5, 0.6)	(0.8, 2.0, 1.2, 0.5)
$E(X)$	0.0053	0.1599	0.0220	0.3490	0.1208
$E(X^2)$	0.0024	0.0830	0.0111	0.2261	0.0689
$E(X^3)$	0.0015	0.0552	0.0074	0.1637	0.0478
$E(X^4)$	0.0011	0.0412	0.0055	0.1270	0.0365
$E(X^5)$	0.0009	0.0328	0.0044	0.1032	0.0295
$E(X^6)$	0.0007	0.0272	0.0037	0.0866	0.0247
SD	0.0006	0.0232	0.0032	0.0745	0.0213
CV	0.0005	0.0203	0.0028	0.0653	0.0187
CS	0.0005	0.0180	0.0024	0.0581	0.0166
CK	0.0004	0.0162	0.0022	0.0522	0.0150

Table 4.4: Moments of TL-OBIII-LLoG Distribution for Selected Parameters

	$(\alpha, \beta, b, a, \theta)$				
	(0.4, 0.5, 0.2, 0.5)	(0.9, 1.2, 1.0, 0.6)	(0.3, 2.2, 1.0, 0.2)	(3.0, 2.0, 0.5, 0.6)	(0.8, 2.0, 1.2, 0.5)
$E(X)$	0.2291	0.0822	0.07376	0.0170	0.1920
$E(X^2)$	0.1075	0.0396	0.0344	0.0036	0.0480
$E(X^3)$	0.0662	0.0257	0.0222	0.0015	0.0155
$E(X^4)$	0.0469	0.0189	0.0163	0.0009	0.0064
$E(X^5)$	0.0360	0.0150	0.0129	0.0006	0.0032
$E(X^6)$	0.0291	0.0124	0.0107	0.0004	0.0019
SD	0.0244	0.0105	0.0091	0.0003	0.0013
CV	0.0210	0.0092	0.0079	0.0003	0.0010
CS	0.0184	0.0081	0.0070	0.0002	0.0007
CK	0.0164	0.0073	0.0063	0.0002	0.0006

Table 4.5: Moments of TL-OBIII-Lomax Distribution for Selected Parameters

	$(\alpha, \beta, b, \theta, k)$				
	(0.4, 0.5, 0.2, 0.5, 0.2)	(0.9, 1.2, 1.0, 0.6, 2.0)	(0.3, 2.2, 1.0, 0.2, 3.0)	(0.5, 2.0, 0.5, 0.6, 0.8)	(0.5, 2.0, 1.2, 0.5, 0.2)
$E(X)$	0.0046	0.3398	0.0323	0.1908	0.1570
$E(X^2)$	0.0025	0.2507	0.0233	0.1119	0.0931
$E(X^3)$	0.0017	0.1979	0.0182	0.0799	0.0667
$E(X^4)$	0.0013	0.1632	0.0149	0.0624	0.0522
$E(X^5)$	0.0011	0.1387	0.0127	0.0513	0.0429
$E(X^6)$	0.0009	0.1205	0.0109	0.0436	0.0364
SD	0.0008	0.1065	0.0096	0.0379	0.0316
CV	0.0007	0.0954	0.0087	0.0335	0.0279
CS	0.0006	0.0863	0.0078	0.0301	0.0251
CK	0.0005	0.0788	0.0072	0.0273	0.0228

Table 4.6: Moments of TL-OBIII-Power Distribution for Selected Parameters

4.1.5 Incomplete Moments

When it comes to computing Bonferroni and Lorenz curves, incomplete moments are useful. The s^{th} incomplete moment of X denoted by $\eta_s(t)$ is given by

$$\eta_s(t) = \int_{-\infty}^t x^s f(x) dx.$$

Using equation (4.7), we obtain

$$\eta_s(t) = \sum_{v=0}^{\infty} C_{v+1} \int_{-\infty}^t x^s g_{v+1}(x; \zeta) dx. \quad (4.9)$$

By setting $s = 1$ in equation (4.9), we obtain the first incomplete moment of the TL-OBIII-G GFDP.

4.1.6 Rényi Entropy

Rényi entropy (Rényi (1961)) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^{\infty} [f(x; \alpha, \beta, b, \zeta)]^v dx \right), v \neq 1, v > 0. \quad (4.10)$$

Rényi entropy tends to Shannon entropy as $v \rightarrow 1$. Note that

$$\begin{aligned} [f(x; \alpha, \beta, b, \zeta)]^v &= (2\alpha\beta b)^v \sum_{j,p,l,k,q=0}^{\infty} \binom{v(b-1)}{j} \binom{2j+v}{p} \binom{k+\alpha(v+l)-v}{q} \\ &\times (-1)^{j+p+l} \frac{\Gamma(\alpha(v+l)+v+k)}{\Gamma(\alpha(v+l)+v)k!} \frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!} (G_B(x; \zeta))^q g_B(x; \zeta)^v. \end{aligned}$$

(See the appendix for derivations).

Consequently, Rényi entropy for the TL-OBIII-G GFPD is given by

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left[(2\alpha\beta b)^v \sum_{j,p,l,k,q=0}^{\infty} \binom{v(b-1)}{j} \binom{2j+v}{p} \binom{k+\alpha(v+l)-v}{q} \right. \\ &\times (-1)^{j+p+l} \frac{\Gamma(\alpha(v+l)+v+k)}{\Gamma(\alpha(v+l)+v)k!} \frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!} \frac{1}{[\frac{q}{v}+1]^v} \\ &\times \left. \int_0^{\infty} \left([\frac{q}{v}+1] (G_B(x; \zeta))^{\frac{q}{v}} g_B(x; \zeta) \right)^v dx \right] \\ &= \frac{1}{1-v} \log \left[\sum_{j,p,l,k,q=0}^{\infty} w_q^* \exp((1-v)I_{REG}) \right], \end{aligned} \quad (4.11)$$

for $v > 0, v \neq 1$, where

$I_{REG} = \frac{1}{1-v} \log \left[\int_0^{\infty} \left([\frac{q}{v}+1] (G_B(x; \zeta))^{\frac{q}{v}} g_B(x; \zeta) \right)^v dx \right]$ is the Rényi entropy of Exp-G distribution with power parameter $(\frac{q}{v}+1)$, and

$$\begin{aligned} w_q^* &= (2\alpha\beta b)^v \sum_{j,p,l,k,q=0}^{\infty} \binom{v(b-1)}{j} \binom{2j+v}{p} \binom{k+\alpha(v+l)-v}{q} \\ &\times (-1)^{j+p+l} \frac{\Gamma(\alpha(v+l)+v+k)}{\Gamma(\alpha(v+l)+v)k!} \frac{\Gamma(\beta(v+p)+v+l)}{\Gamma(\beta(v+p)+v)l!} \frac{1}{[\frac{q}{v}+1]^v}. \end{aligned}$$

4.1.7 Order Statistics

Let $X_1, X_2, \dots, X_n$ be a simple random sample from the TL-OBIII-G GFPD. Using the binomial expansion $(1 - F(x))^{n-i} = \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F(x)]^m$,

the pdf of the i^{th} order statistic can be expressed as

$$f_{i:n}(x) = \frac{n!f(x)}{(i-1)!(n-i)!} \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} [F(x)]^{m+i-1}. \quad (4.12)$$

Based on equations (4.1) and (4.2), we can write

$$f(x)F(x)^{m+i-1} = \sum_{v=0}^{\infty} b_{v+1}^* g_{v+1}(x; \zeta), \quad (4.13)$$

where

$$\begin{aligned} b_{v+1}^* &= 2\alpha\beta b \sum_{j,p,l,k,v=0}^{\infty} \binom{b(m+i)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\ &\times (-1)^{j+p+l+k} \binom{k+\alpha(l+1)-1}{v} \frac{1}{v+1}. \end{aligned}$$

(See the appendix for derivations).

Substituting equation (4.13) into equation (4.12), we obtain

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{v=0}^{\infty} \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} b_{v+1}^* g_{v+1}(x; \zeta), \quad (4.14)$$

where $g_{v+1}(x; \zeta) = (v+1)[G_B(x; \zeta)]^v g_B(x; \zeta)$ is the exponentiated-G (Exp-G) pdf with the power parameter $(v+1)$ and parameter vector ζ . Thus, the density function of the TL-OBIII-G order statistics is a linear combination of Exp-G densities.

4.1.8 Probability Weighted Moments (PWMs)

The $(p, r)^{th}$ PWMs of X with TL-OBIII-G distribution denoted $M_{p,r}$ is given by

$$M_{p,r} = E(X^p (F(X))^r) = \int_{-\infty}^{\infty} x^p (F(x))^r f(x) dx.$$

Utilizing equations (4.1) and (4.2), we can write

$$f(x)(F(x))^r = \sum_{v=0}^{\infty} C_{v+1}^* g_{v+1}(x; \zeta), \quad (4.15)$$

where $g_{v+1}(x; \zeta) = (v+1)[G_B(x; \zeta)]^v g_B(x; \zeta)$ is the Exp-G pdf with the power parameter $v+1 > 0$ and parameter vector ζ , and

$$\begin{aligned} C_{v+1}^* &= 2\alpha\beta b \sum_{j,p,l,k=0}^{\infty} \binom{b(r+1)-1}{j} \binom{2j+1}{p} \frac{\Gamma(\beta(p+1)+l+1)}{\Gamma(\beta(p+1)+1)l!} \frac{\Gamma(\alpha(l+1)+1+k)}{\Gamma(\alpha(l+1)+1)k!} \\ &\times (-1)^{j+p+l+k} \binom{k+\alpha(l+1)-1}{v} \frac{1}{v+1}. \end{aligned}$$

(See the appendix for derivations).

Consequently, the PWMs of the TL-OBIII-G GFRD can be written as

$$M_{p,r} = \int_{-\infty}^{\infty} x^p \sum_{v=0}^{\infty} C_{v+1}^* g_{v+1}(x; \zeta) dx = \sum_{v=0}^{\infty} C_{v+1}^* \int_{-\infty}^{\infty} x^p g_{v+1}(x; \zeta) dx$$

which shows that the $(p, r)^{th}$ PWMs of TL-OBIII-G GFPD can be obtained from the moments of the Exp-G distribution.

4.1.9 Maximum Likelihood Estimation

The MLE technique is the commonly used method of parameter estimation among others in the literature. Here, we employ the MLE to estimate model parameters of TL-OBIII-G GFPD. Let $x_1, x_2, \dots, x_n$ be a random sample from TL-OBIII-G GFPD with the parameter vector $\Delta = (\alpha, \beta, b, \zeta)^T$, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ is given by

$$\begin{aligned} \ell_n(\Delta) &= n \ln(2\alpha\beta b) + (b-1) \sum_{i=1}^n \ln \left[1 - \left(1 - (1 + (O_G(x_i; \zeta))^{-\alpha})^{-\beta} \right)^2 \right] \\ &+ \sum_{i=1}^n \ln \left(1 - (1 + (O_G(x_i; \zeta))^{-\alpha})^{-\beta} \right) + (-\alpha - 1) \sum_{i=1}^n \ln (O_G(x_i; \zeta)) \\ &+ (-\beta - 1) \sum_{i=1}^n \ln (1 + (O_G(x_i; \zeta))^{-\alpha}) + \sum_{i=1}^n \ln g_B(x_i; \zeta) - 2 \sum_{i=1}^n \ln (\bar{G}_B(x_i; \zeta)). \end{aligned}$$

Elements of the score vector $U(\Delta) = \left(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \zeta_k} \right)$ are given in the appendix.

The MLEs, say $\hat{\Delta} = (\hat{\alpha}, \hat{\beta}, \hat{b}, \hat{\zeta})^T$ can be obtained by equating the system of non linear equations $\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \zeta_k}$ to zero and solving simultaneously. These equations cannot be solved analytically as they are not in closed form. Thus, numerical methods such as Newton-Raphson procedure can be used to solve them numerically. We maximize the likelihood function using the function `nlm` in R ([R Software \(2011\)](#)). The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic ($-2 \ln(L)$), AIC, BIC, and AICC are presented. Also, the goodness-of-fit statistics: (W^*) and (A^*) as well as K-S statistic and its p-value are presented for model comparison. Note that for AIC, AICC, BIC, and the goodness-of-fit statistics W^* , A^* and KS , smaller values are preferred.

4.1.10 Simulation Study

The performance assessment of the TL-OBIII-LLoG distribution when $\beta = 1$ is conducted through various simulations for different sizes ($n=25, 50, 100, 200, 400$) via the R package. Simulation results for other parameter values are available on request from the authors. We simulate $N = 1000$ samples for the true parameters values given in Table 4.7. The table provides the mean MLEs of the model parameters, accompanied by the corresponding bias and root mean squared errors (RMSEs).

The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

Parameter	n	I (1.2, 1.0, 2.1.4)			II (1.0, 2.0, 2.4)			III (1.0, 3.0, 2.0)		
		Mean	Average Bias	RMSE	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
α	25	1.2151	0.0151	0.1254	1.0577	0.0577	0.1700	1.0383	0.0383	0.1646
	50	1.2105	0.0105	0.0930	1.0347	0.0347	0.1065	1.0377	0.0377	0.1095
	100	1.2056	0.0056	0.0649	1.0216	0.0216	0.0745	1.01595	0.0159	0.0690
	200	1.1973	-0.0027	0.0469	1.0096	0.0096	0.0506	1.0121	0.0121	0.0470
	400	1.2017	0.0017	0.0357	1.0037	0.0037	0.0339	1.0085	0.0085	0.0380
b	25	1.0465	0.0465	0.1831	2.0189	0.0189	0.3054	3.1034	0.1034	0.7515
	50	1.0232	0.0232	0.1410	2.0202	0.0202	0.2077	3.0286	0.0286	0.4635
	100	1.0163	0.0163	0.1008	2.0030	0.0030	0.1401	2.9788	-0.0212	0.2998
	200	1.0151	0.0152	0.0668	1.9979	-0.0021	0.0978	3.0087	0.0087	0.2200
	400	1.0035	0.0035	0.0489	2.0033	0.0033	0.0757	3.0031	0.0031	0.1555
λ	25	1.4122	0.0122	0.1120	2.4087	0.0086	0.0617	2.0148	0.0148	0.0920
	50	1.4079	0.0079	0.0818	2.4052	0.0052	0.0421	2.0190	0.0190	0.0739
	100	1.4035	0.0035	0.0576	2.4023	0.0024	0.0322	1.9979	-0.0020	0.0533
	200	1.3969	-0.0031	0.0405	2.3994	-0.0006	0.0234	1.9853	-0.0147	0.0423
	400	1.4009	0.0009	0.0306	2.3991	-0.0009	0.0159	1.9891	-0.0109	0.0359

Table 4.7: Monte Carlo Simulation Results: Mean, Average Bias and RMSE

From the results, we can clearly verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

4.1.11 Applications

The TL-OBIII-LLoG distribution is fitted to two data sets and compared to fits of several non-nested four parameter distributions. The TL-OBIII-LLoG distribution is compared to the new modified Weibull (NMW) distribution introduced by [Doostmoradi et al. \(2014\)](#), beta generalized exponential (BGE) distribution introduced by [Barreto-Souza et al. \(2010\)](#), beta generalized Lindley (BGL) distribution by [Oluyede and Yang \(2015\)](#), exponentiated modified Weibull distribution by [Elbatal \(2011\)](#), Weibull Lomax (WL) distribution by [Tahir et al. \(2015\)](#), generalized Weibull log-logistic (GWLLoG) distribution by [Cordeiro et al. \(2015\)](#) and Burr III Marshall Olkin-Lindley (BIIIMO-L) distribution by [Bhatti et al. \(2019\)](#). The pdf's of the four parameter NMW, BGE, BGL, EMW, WL, GWLLoG, BIIIMO-L distributions are given in the appendix.

Fitted density plots, histogram of the data and probability plots ([Chambers et al. \(1983\)](#)) are given in Figure 4.5 and Figure 4.7 for the two data sets considered in this section. For the probability plot, we plotted

$F_{TL-OBIII-LLoG}(x_{(i)}; \hat{\alpha}, \hat{\beta}, \hat{b}, \hat{\lambda})$ against $\frac{i - 0.375}{n + 0.25}$, $i = 1, 2, \dots, n$, where $x_{(i)}$ are the ordered values of the observed data. The measure of closeness is given by the sum of squares (SS) as

$$SS = \sum_{i=1}^n \left[F_{TL-OBIII-LLoG}(x_{(i)}; \hat{\alpha}, \hat{\beta}, \hat{b}, \hat{\lambda}) - \left(\frac{i - 0.375}{n + 0.25} \right) \right]^2.$$

The goodness-of-fit statistics W^* and A^* , described by [Chen and Balakrishnan \(1995\)](#) as well as K-S statistic, its P-value and SS are also presented in the tables. These statistics help determine the best-fitting distribution for the data. Usually, a superior fit is associated with smaller values of W^* , A^* and K-S statistics.

Dataset 1

The first dataset involves the duration of growth hormone medication until children reach the target age, as examined by [Alizadeh et al. \(2017b\)](#). The analysis highlights the superiority of the Exponentiated Power Lindley Power Series (EPLPS) class of distributions. **(See the appendix for the detailed data).**

The estimated variance-covariance matrix for TL-OBIII-LLoG distribution on growth hormone data is given by

$$\begin{bmatrix} 7.1817 \times 10^{-05} & 7.4727 \times 10^{-04} & 1.2539 \times 10^{-06} & 2.6815 \times 10^{-07} \\ 7.4727 \times 10^{-04} & 9.8899 \times 10^{-03} & 1.6593 \times 10^{-05} & 2.7902 \times 10^{-06} \\ 1.2539 \times 10^{-06} & 1.6593 \times 10^{-05} & 2.7841 \times 10^{-08} & 4.6818 \times 10^{-09} \\ 2.6815 \times 10^{-07} & 2.7902 \times 10^{-06} & 4.6818 \times 10^{-09} & 1.0012 \times 10^{-09} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\alpha \in [0.0710 \pm 1.6610 \times 10^{-02}]$, $\beta \in [0.5180 \pm 0.1949]$, $b \in [168.7000 \pm 3.2704 \times 10^{-04}]$ and $\lambda \in [19.0390 \pm 6.2018 \times 10^{-05}]$, respectively.

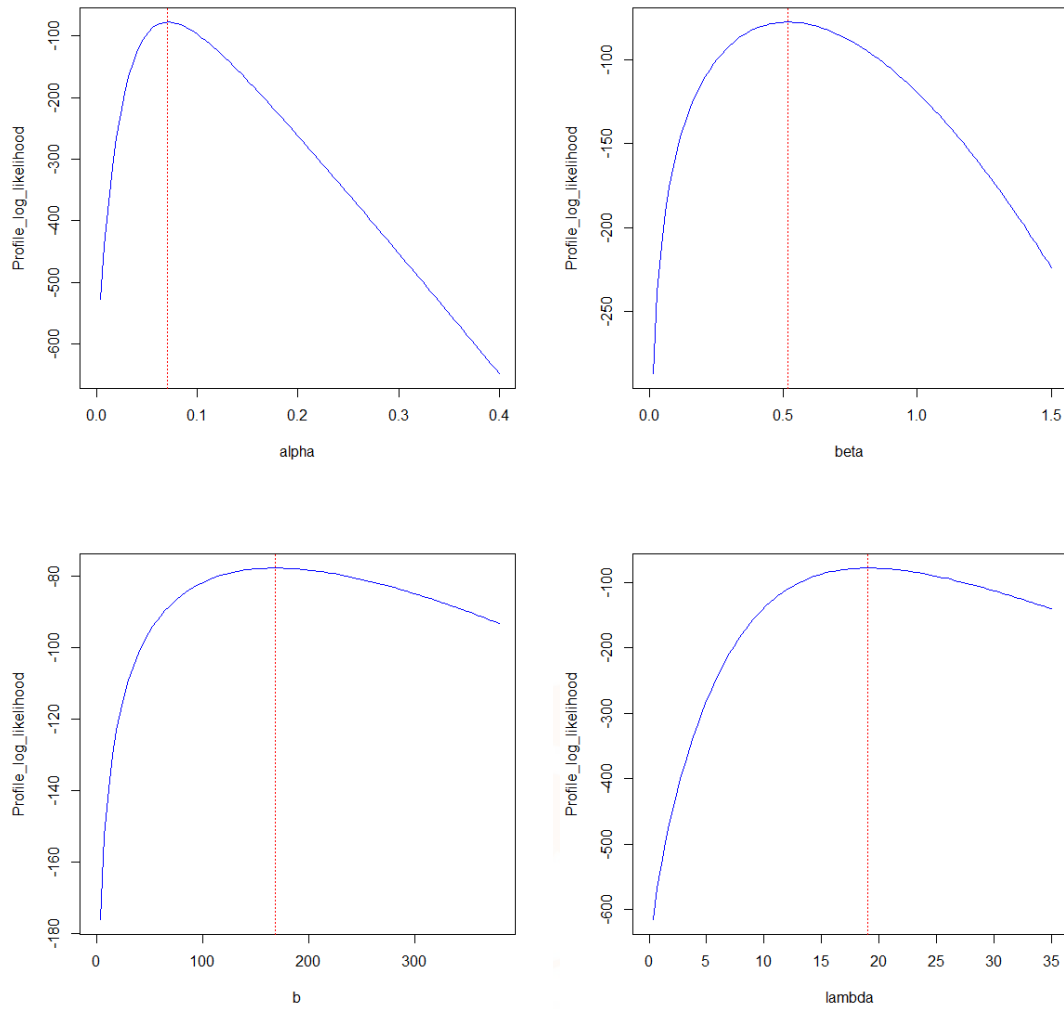


Figure 4.4: Profile log-likelihood function plots for parameters of TL-OBIII-LLoG distribution on the growth hormone data

The MLE of TL-OBIII-LLoG parameters on growth hormone data can be uniquely identified, as shown in Figure 4.4.

Table 4.8 presents the MLEs of the TL-OBIII-LLoG distribution, along with those of non-nested models, including standard errors. Additional metrics like AIC, BIC, and goodness-of-fit statistics such as W^* , A^* , K-S and its p-value, as well as SS, are included. Figure 4.5 complements this with plots of fitted densities against the histogram and observed probability versus predicted probability.

Model	Estimates				Statistics								
	α	β	b	λ	$-2\log(L)$	AIC'	$AICC'$	BIC'	W^*	A^*	KS	P-value	SS
TL-OBIII-LLoG	0.0710	0.5180	168.7000	19.0390	155.4418	163.4418	164.7751	169.6632	0.0389	0.2685	0.0873	0.9522	0.0389
	(8.4745×10^{-03})	(9.9448×10^{-02})	(1.6686×10^{-04})	(3.1642×10^{-05})									
BGE	α	λ	a	b									
	0.7487	0.0323	6.3824	19.9198	159.7534	167.7534	169.0867	173.9748	0.0944	0.6166	0.1184	0.7097	0.0907
	(0.5541)	(0.0111)	(7.1232)	(0.5568)									
NMW	α	γ	λ	β									
	0.0006	2.8548	4.0927	0.0018	168.0796	176.0796	177.413	182.301	0.0551	0.3873	0.1618	0.3187	0.1439
	(0.0014)	(0.8086)	(0.7964)	(0.0021)									
BGL	α	θ	a	b									
	0.1179	1.1290×10^{-07}	3.0140×10^{-01}	1.0100×10^{01}	342.7928	350.8082	352.1415	357.0296	0.0685	0.4632	0.4833	1.579×10^{-07}	2.6551
	(9.1449×10^{-03})	(2.4083×10^{-06})	(8.1743×10^{-04})	(2.6267×10^{-05})									
EMW	γ	δ	λ	θ									
	6.5130	0.4820	17.0010	1.0000×10^{-04}	158.1951	166.1951	167.5284	172.4165	0.0703	0.4747	0.1012	0.8656	0.0754
	(2.2693)	(0.07981)	(1.7949×10^{-16})	(3.0495×10^{-11})									
WL	a	b	α	β									
	0.0002	4.3897	0.3987	0.0380	164.004	172.0040	173.3373	178.2254	0.1508	0.9496	0.1504	0.4069	0.1533
	(0.0003)	(0.0111)	(0.0653)	(0.0507)									
GWLLoG	α	β	a	θ									
	0.2962	2.7285	2.3479	1.4831	159.4803	167.4803	168.8137	173.7017	0.0863	0.5686	0.1158	0.7352	0.0829
	(0.5048)	(1.3142)	(1.6100)	(0.7834)									
BIIMO-L	α	β	λ	b									
	0.1177	0.1109	0.1097	5.8707×10^{-06}	556.1171	564.1172	565.4505	570.3386	0.0723	0.4860	0.8985	2.2000×10^{-16}	8.8178
	(0.0205)	(0.0168)	(0.0185)	(3.3596×10^{-05})									

Table 4.8: Estimates of Models for for Dataset 1

The values in the table above showed that the TL-OBIII-LLoG distribution has the smallest values of AIC, AICC, BIC, W^* , A^* , K-S and the largest p-value compared to the corresponding values for the fitted non-nested models. So, we conclude that the TL-OBIII-LLoG distribution could be chosen as the best fitting model for for dataset 1.

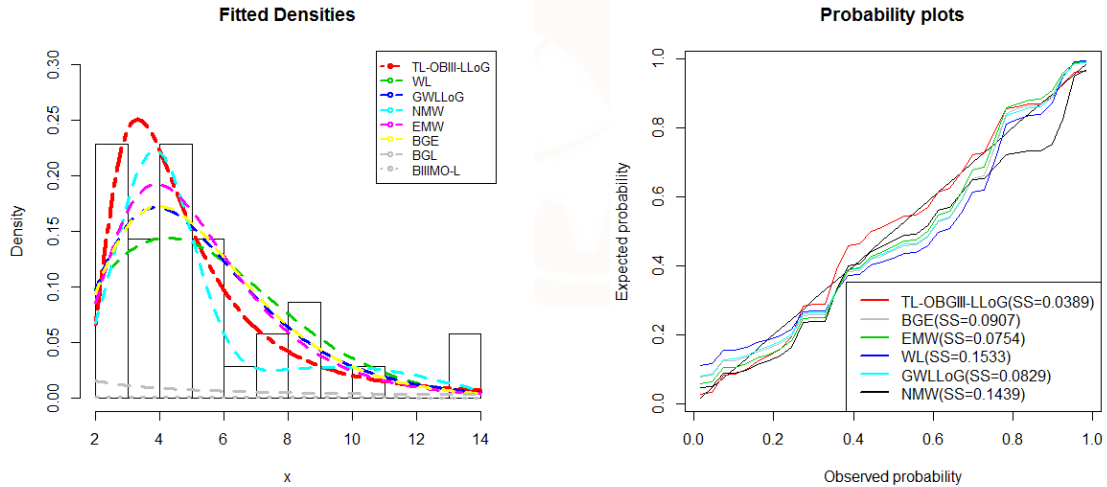


Figure 4.5: Fitted Densities and Probability Plots for Dataset 1

Dataset 2

The second dataset pertains to maintenance on active repair times (in hours) for an airborne communication transceiver, with a sample size of $n=46$, sourced from [Leiva et al. \(2009\)](#) and [Chhikara and Folks \(1977\)](#). (The data is given in the appendix).

The estimated variance-covariance matrix for TL-OBIII-LLoG distribution on airborne communication transceiver data is given by

$$\begin{bmatrix} 2.2434 \times 10^{-03} & -5.1834 \times 10^{-05} & 9.6028 \times 10^{-08} & 6.7024 \times 10^{-04} \\ -5.1834 \times 10^{-05} & 1.1809 \times 10^{-04} & 5.7151 \times 10^{-08} & -1.5485 \times 10^{-05} \\ 9.6028 \times 10^{-08} & 5.7151 \times 10^{-08} & 3.4262 \times 10^{-11} & 2.8688 \times 10^{-08} \\ 6.7024 \times 10^{-04} & -1.5485 \times 10^{-05} & 2.8688 \times 10^{-08} & 2.0023 \times 10^{-04} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\alpha \in [0.4601 \pm 0.0928]$, $\beta \in [0.1392 \pm 0.0212]$, $b \in [144.0183 \pm 1.1472 \times 10^{-05}]$ and $\lambda \in [1.5403 \pm 0.0277]$, respectively.

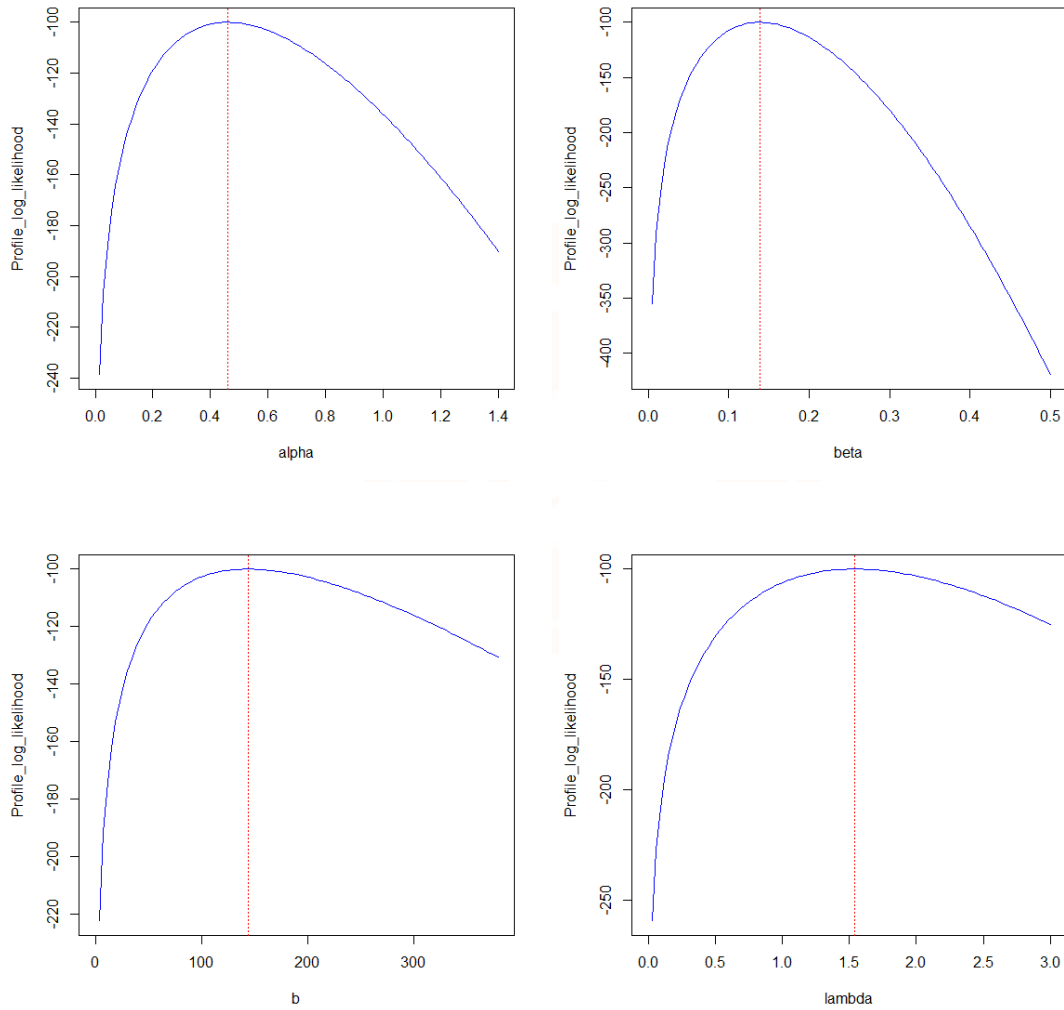


Figure 4.6: Profile log-likelihood function plots for parameters of TL-OBIII-LLoG distribution on the airborne communication transceiver data

The profile plots presented in Figure 4.6 give evidence that the MLE of the TL-OBIII-LLoG model parameters can be uniquely identified.

MLEs of TL-OBIII-LLoG distribution and non-nested models (standard error in parenthe-

ses), AIC, BIC, W^* , A^* , K-S and its p-value as well as SS are given in Table 4.9. Fitted density plots, the histogram and observed probability vs predicted probability are given in Figure 4.7.

Model	Estimates				Statistics								
	α	β	b	λ	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	KS	P-value	SS
TL-OBIII-LLoG	0.4601 (0.0473)	0.1392 (0.0108)	144.0183 (5.8534×10^{-06})	1.5403 (0.0141)	199.9626	207.9626	208.9382	215.2772	0.0515	0.3127	0.0762	0.9520	0.0508
BGE	10.9759 (20.9289)	1.2799 (0.7340)	0.1848 (0.3193)	0.1855 (0.0620)	201.8082	209.8083	210.7839	217.1229	0.0563	0.4558	0.1109	0.6228	0.0890
NMW	0.0377 (0.0333)	1.0873 (0.2863)	1.3624 (0.2228)	0.2518 (0.0803)	218.5555	226.5555	227.5311	233.8701	0.0681	0.4514	0.1604	0.1870	0.3884
BGL	0.1179 (8.1272×10^{-03})	1.3719×10^{-07} (2.4697×10^{-06})	0.3014 (7.7978×10^{-04})	10.1000 (2.2593×10^{-05})	382.0483	390.1551	391.1307	397.4696	0.0837	0.5519	0.4630	5.4390×10^{-09}	3.2424
EMW	0.9582 (0.1897)	0.2693 (0.0543)	17.0010 (3.4209×10^{-18})	1.0000×10^{-04} (5.8159×10^{-13})	209.9658	217.9658	218.9414	225.2804	0.1441	1.0004	0.1519	0.2385	0.1889
WL	1.6956×10^3 (7.6319×10^{-05})	2.5654 (1.3635)	0.0198 (0.0158)	0.2078 (0.4281)	200.603	208.603	209.5786	215.9176	0.0563	0.3634	0.0937	0.8132	0.0509
GWLLoG	8.2995 (1.8675)	0.1697 (0.1866)	35.4344 (1.2987)	5.3096 (5.7670)	208.9809	216.9809	217.9565	224.2954	0.1295	0.8973	0.1199	0.5218	0.1147
BIIMO-L	0.1184 (0.0187)	0.1121 (0.0175)	0.1107 (0.0163)	1.1385×10^{-05} (1.8930×10^{-05})	659.9823	667.9820	668.9577	675.2966	0.0902	0.6006	0.8711	2.2000×10^{-16}	11.4417

Table 4.9: Estimates of Models for for Dataset 2

In Table 4.9 the values of AIC, AICC, BIC, W^* , A^* , K-S for TL-OBIII-LLoG distribution are smaller than those for fitted non-nested distributions, which indicates that the TL-OBIII-LLoG distribution provides a better fit for dataset 2. Also, we can conclude that the TL-OBIII-LLoG distribution is the better fit from the p-value of the KS statistic as it is large for TL-OBIII-LLoG distribution as compared to those from the other non-nested distributions for for dataset 2.

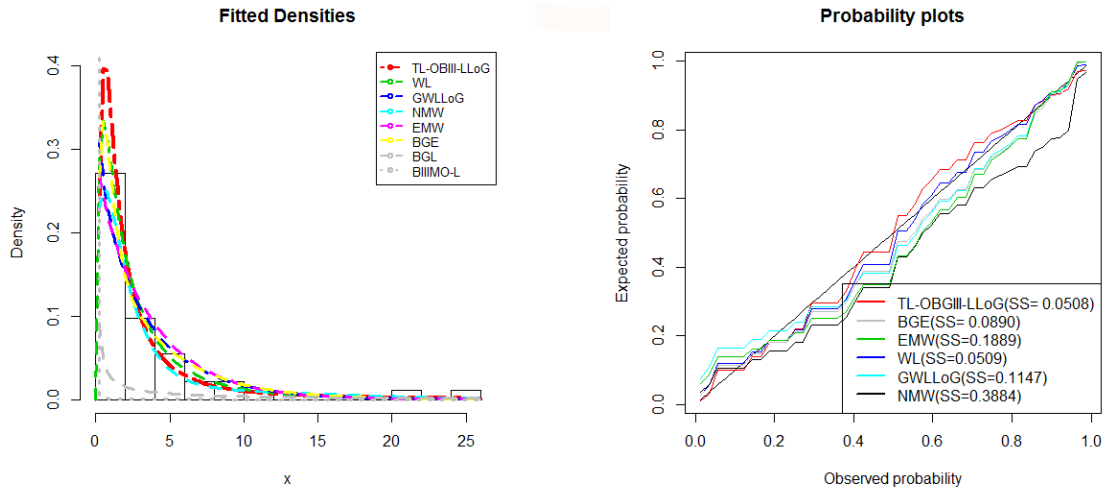


Figure 4.7: Fitted Densities and Probability Plots for Dataset 2

4.1.12 Concluding Remarks

We have developed a new GFPD called the Topp-Leone Odd Burr III-G (TL-OBIII-G) GFD. Properties of this new distribution such as sub-models, quantile function, moments, incomplete moments, probability weighted moments, order statistics and Rényi entropy were studied. The MLEs have been computed via the maximum likelihood estimation method. Monte Carlo simulation study was employed to check the performance of the MLEs. Lastly, we illustrated that the TL-OBIII-G GFD is useful for lifetime applications by fitting its special case of TL-OBIII-LLoG distribution to two real data sets.



5. Type II Exponentiated Half Logistic-Topp-Leone-G Power Series (TIIHL-TL-GPS) Class of Distributions (CoD)

Moakofi, T., Oluyede, B. and Chipepa, F., 2021. Type II Exponentiated Half-Logistic-Topp-Leone-G Power Series Class of Distributions with Applications. *Pakistan Journal of Statistics and Operation Research*, 885-909.

<https://pjsor.com/pjsor/article/view/3775>

This chapter uses compounding on the Type II Exponential Half Logistic-Topp-Leone-G families of distributions to obtain the new generalized CoD.

5.1 TIIHL-TL-GPS CoD

In this section, we propose a new class of distributions TIIHL-TL-GPS CoD. An attractive feature about the model is that the extra parameter introduced has the capability to control both the weights at the tails of the density function. Also, the new CoD can model different types of failure rate functions that are available in different areas like reliability, engineering and biological studies. The hazard rate function from the special cases exhibits increasing, decreasing, bathtub and upside bathtub shapes. Of note is the upside bathtub followed by bathtub shape of the hazard rate exhibited in some cases. The method of estimation utilized was also used by [Karakaya and Tanis \(2020a\)](#), [Karakaya and Tanis \(2020b\)](#), [Tanis and Karakaya \(2021\)](#) and [Tanis \(2021\)](#).

The cdf and pdf of the type II exponentiated half-logistic Topp-Leone-G (TIIHL-TL-G) GFPD are given by

$$F(x; a, b, \zeta) = 1 - \left[\frac{1 - \left[1 - (\bar{G}_B(x; \zeta))^2 \right]^b}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2 \right]^b} \right]^a \quad (5.1)$$

and

$$f(x; a, b, \zeta) = \frac{4abg_B(x; \zeta) \left[1 - (\bar{G}_B(x; \zeta))^2\right]^{b-1} \bar{G}_B(x; \zeta) \left(1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b\right)^{a-1}}{\left(1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b\right)^{a+1}}, \quad (5.2)$$

respectively, for $a, b > 0$ being shape parameters and parameter vector ζ .

5.1.1 The Model

Let $X_1, X_2, \dots, X_N$ be N identically and independently distributed (iid) random variables following the TIIHL-TL-G distribution. Let $X_{(1)} = \min(X_1, X_2, \dots, X_N)$, then the cdf of $X_{(1)}|N = n$ is given by

$$F_{X_{(1)}|N=n}(x; a, b, \theta, \zeta) = 1 - \left(\frac{\left[1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b\right]^a}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right)^n, \quad (5.3)$$

for $a, b, \theta > 0$, $n \geq 1$ and parameter vector ζ . The type II exponentiated half-logistic Topp-Leone power series (TIIHL-TL-GPS) CoD denoted by TIIHL-TL-GPS (a, b, θ, ζ) is defined by the marginal distribution of $X_{(1)}$, that is,

$$F_{X_{(1)}}(x) = 1 - \frac{C \left(\theta \frac{\left[1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b\right]^a}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right)}{C(\theta)}, \quad (5.4)$$

for $a, b > 0$ being shape parameters, $\theta > 0$ being a scale parameter, $C(\theta)$ is a normalization function and parameter vector ζ . The corresponding pdf is given by

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{4ab\theta g_B(x; \zeta) \left[1 - (\bar{G}_B(x; \zeta))^2\right]^{b-1} \bar{G}_B(x; \zeta) \left(1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b\right)^{a-1}}{\left(1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b\right)^{a+1}} \\ &\times \frac{C' \left(\theta \frac{\left[1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b\right]^a}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right)}{C(\theta)}. \end{aligned} \quad (5.5)$$

The hrf is given by

$$h_F(x) = \frac{4ab\theta g_B(x; \zeta) \left[1 - (\bar{G}_B(x; \zeta))^2\right]^{b-1} \bar{G}_B(x; \zeta) \left(1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b\right)^{a-1}}{\left(1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b\right)^{a+1}} \times \frac{C' \left(\theta \left[\frac{1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right]^a \right)}{C \left(\theta \left[\frac{1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right]^a \right)}. \quad (5.6)$$

Table 5.1 below presents the special classes of TIEHL-TL-GPS distribution when $C(\theta)$ is specified in equation (5.4) and $\{a_n\}_{n \geq 1}$ represents a sequence of positive real numbers.

Distribution	$C(\theta)$	a_n	cdf
TIEHL-TL-G Poisson	$e^\theta - 1$	$(n!)^{-1}$	$1 - \frac{\exp \left(\theta \left[\frac{1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right]^a \right)}{\exp(\theta) - 1} - 1$
TIEHL-TL-G Geometric	$\theta(1 - \theta)^{-1}$	1	$1 - \frac{(1 - \theta) \left(\left[\frac{1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right]^a \right)}{\left(1 - \theta \left(\left[\frac{1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right]^a \right) \right)}$
TIEHL-TL-G Logarithmic	$-\log(1 - \theta)$	n^{-1}	$1 - \frac{\log \left(1 - \theta \left(\left[\frac{1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right]^a \right) \right)}{\log(1 - \theta)}$
TIEHL-TL-G Binomial	$(1 + \theta)^m - 1$	$\binom{m}{n}$	$1 - \frac{\left(1 + \theta \left(\left[\frac{1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right]^a \right) \right)^m}{(1 + \theta)^m - 1} - 1$

Table 5.1: Special Classes of the TIEHL-TL-GPS Distribution

Sub-Classes of TIEHL-TL-GPS Class of Distributions

- When $a = 1$, we obtain the type II half-logistic Topp-Leone-G power series (TIEHL-TL-GPS) CoD with the cdf

$$F(x; b, \theta, \zeta) = 1 - \frac{C \left(\theta \left[\frac{1 - \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b}{1 + \left[1 - (\bar{G}_B(x; \zeta))^2\right]^b} \right] \right)}{C(\theta)},$$

for $b, \theta > 0$ and parameter vector ζ . This is a new CoD.

- When $b = 1$, we obtain the new CoD with the cdf

$$F(x; a, \theta, \zeta) = 1 - \frac{C \left(\theta \left[\frac{(\bar{G}_B(x; \zeta))^2}{1 + [1 - (\bar{G}_B(x; \zeta))^2]} \right]^a \right)}{C(\theta)},$$

for $a, \theta > 0$ and parameter vector ζ .

- When $a = b = 1$, we obtain the new CoD with the cdf

$$F(x; \theta, \zeta) = 1 - \frac{C \left(\theta \left[\frac{(\bar{G}_B(x; \zeta))^2}{1 + [1 - (\bar{G}_B(x; \zeta))^2]} \right] \right)}{C(\theta)},$$

for $\theta > 0$ and parameter vector ζ .

- When $\theta \rightarrow 0^+$, we obtain the type II exponentiated half-logistic Topp-Leone-G (TIEHL-TL-G) CoD with the cdf

$$F(x; a, b, \zeta) = 1 - \left[\frac{1 - [1 - (\bar{G}_B(x; \zeta))^2]^b}{1 + [1 - (\bar{G}_B(x; \zeta))^2]^b} \right]^a,$$

for $a, b > 0$ and parameter vector ζ . This is a new CoD.

- When $a = 1$ and $\theta \rightarrow 0^+$, we obtain the type II half-logistic Topp-Leone-G (TIIHL-TL-G) CoD with the cdf

$$F(x; b, \zeta) = 1 - \left[\frac{1 - [1 - (\bar{G}_B(x; \zeta))^2]^b}{1 + [1 - (\bar{G}_B(x; \zeta))^2]^b} \right],$$

for $b > 0$ and parameter vector ζ . This is a new CoD.

- When $b = 1$ and $\theta \rightarrow 0^+$, we obtain the new CoD with the cdf

$$F(x; a, \zeta) = 1 - \left[\frac{(\bar{G}_B(x; \zeta))^2}{1 + [1 - (\bar{G}_B(x; \zeta))^2]} \right]^a,$$

for $a > 0$ and parameter vector ζ .

- When $a = b = 1$ and $\theta \rightarrow 0^+$, we obtain the new CoD with the cdf

$$F(x; \zeta) = 1 - \left[\frac{(\bar{G}_B(x; \zeta))^2}{1 + [1 - (\bar{G}_B(x; \zeta))^2]} \right],$$

for parameter vector ζ .

Quantile Function

Let X be a random variable with cdf defined by equation (5.4). The qf $Q_{X(1)}(u)$ is defined by $F_{X(1)}(Q_{X(1)}(u)) = u, 0 \leq u \leq 1$. Note that

$$1 - \frac{C \left(\theta \left[\frac{1 - [1 - (\bar{G}_B(x; \zeta))^2]^b}{1 + [1 - (\bar{G}_B(x; \zeta))^2]^b} \right]^a \right)}{C(\theta)} = u,$$

so that

$$C \left(\theta \left[\frac{1 - [1 - (\bar{G}_B(x; \zeta))^2]^b}{1 + [1 - (\bar{G}_B(x; \zeta))^2]^b} \right]^a \right) = C(\theta) (1 - u).$$

This is equivalent to

$$\left[\frac{1 - [1 - (\bar{G}_B(x; \zeta))^2]^b}{1 + [1 - (\bar{G}_B(x; \zeta))^2]^b} \right]^a = \frac{C^{-1}(C(\theta)(1 - u))}{\theta},$$

that is,

$$\bar{G}_B(x; \zeta) = \left(1 - \left(\frac{1 - \left(\frac{C^{-1}(C(\theta)(1 - u))}{\theta} \right)^{\frac{1}{a}}}{1 + \left(\frac{C^{-1}(C(\theta)(1 - u))}{\theta} \right)^{\frac{1}{a}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{2}}.$$

The expression further simplifies to

$$G_B(x; \zeta) = 1 - \left(1 - \left(\frac{1 - \left(\frac{C^{-1}(C(\theta)(1 - u))}{\theta} \right)^{\frac{1}{a}}}{1 + \left(\frac{C^{-1}(C(\theta)(1 - u))}{\theta} \right)^{\frac{1}{a}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{2}}.$$

Therefore, the qf of the TIIEHL-TL-GPS CoD is given by,

$$Q_{X(1)}(u) = G^{-1} \left[1 - \left(1 - \left(\frac{1 - \left(\frac{C^{-1}(C(\theta)(1 - u))}{\theta} \right)^{\frac{1}{a}}}{1 + \left(\frac{C^{-1}(C(\theta)(1 - u))}{\theta} \right)^{\frac{1}{a}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{2}} \right]. \quad (5.7)$$

It follows therefore that equation (5.7) can be used to produce random numbers from the TIIEHL-TL-GPS distribution with the aid of statistical software such as R, MATLAB and SAS.

Expansion of Density

In this subsection, we present the expanded form of the density function for the TIIHL-TL-GPS CoD. The TIIHL-TL-GPS CoD can be expressed as an infinite linear combination of exponentiated-G (Exp-G) densities as

$$f_{X(1)}(x) = \sum_{m=0}^{\infty} \tau_{m+1} g_{m+1}(x; \zeta), \quad (5.8)$$

where $g_{m+1}(x; \zeta) = (m+1) (G_B(x; \zeta))^m g_B(x; \zeta)$ is the exponentiated-G (Exp-G) distribution with power parameter $(m+1)$ and

$$\begin{aligned} \tau_{m+1} &= \sum_{n=1}^{\infty} \sum_{j,k,l=0}^{\infty} \frac{4ab\theta n a_n \theta^n}{C(\theta)} \binom{an-1}{j} \binom{an+k}{k} \binom{b(j+k+1)-1}{l} \\ &\times \binom{2l+1}{m} \frac{(-1)^{k+j+l+m}}{m+1}. \end{aligned} \quad (5.9)$$

(See Appendix section for details of the derivation).

Calculating Moments and Generating Function

If X follows the TIIHL-TL-GPS distribution and $Y \sim \text{Exp-G}(m+1)$. Then using equation (5.8), the p^{th} raw moment, μ'_p of the TIIHL-TL-GPS CoD is obtained as

$$\mu'_p = E(X^p) = \int_{-\infty}^{\infty} x^p f(x) dx = \sum_{m=0}^{\infty} \tau_{m+1} E(Y^p),$$

where τ_{m+1} is given by equation (5.9). The moment generating function (MGF) $M(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{m=0}^{\infty} \tau_{m+1} M_Y(t),$$

where $M_Y(t)$ is the MGF of Y and τ_{m+1} is given by equation (5.9).

5.1.2 Order Statistics and Uncertainty Measure

In this section, we present the distribution of the i^{th} order statistic and the uncertainty measure.

Distribution of Order Statistics

Order statistics are fundamental in many areas of statistical theory and practice. Let $X_1, X_2, \dots, X_n$ be a random sample from TIIHL-TL-GPS CoD. Then, the pdf the k^{th} order statistic from TIIHL-TL-GPS CoD is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{m=0}^{\infty} \sum_{p=0}^{n-i} \binom{n-i}{p} (-1)^p h_{m+1} g_{m+1}(x; \zeta), \quad (5.10)$$

where $g_{m+1}(x; \zeta) = (m+1)g_B(x; \zeta) (G_B(x; \zeta))^m$ is an Exp-G with power parameter $(m+1)$ and the linear component

$$h_{m+1} = \sum_{n,z=1}^{\infty} \sum_{j,k,l=0}^{\infty} \frac{4ab\theta n a_n d_{q,z} \theta^{n+z} (-1)^{q+j+l+m}}{C^{q+1}(\theta) (m+1)} \binom{i+p-1}{q} \\ \times \binom{a(n+z)-1}{j} \binom{a(n+z)+k}{k} \binom{b(j+k+1)-1}{l} \binom{2l+1}{m}.$$

(See Appendix section for details of the derivation).

Uncertainty Measure

Here, Rényi entropy for TIIEHL-TL-GPS CoD is derived. An entropy is a measure of uncertainty of a random variable. Rényi entropy [Rényi \(1961\)](#) is a generalization of Shannon entropy [Shannon \(1951\)](#). Rényi entropy of the TIIEHL-TL-GPS distribution is defined by

$$I_R(v) = \frac{1}{1-v} \log \left(\sum_{m=0}^{\infty} w_m^* e^{(1-v)I_{REG}} \right), \quad (5.11)$$

where $I_{REG} = \int_0^{\infty} [(1+m/v)g_B(x; \zeta) (G_B(x; \zeta))^{m/v}]^v dx$ is Rényi entropy for an Exp-G distribution with power parameter $(m/\nu + 1)$ and

$$w_m^* = \sum_{n=1}^{\infty} \sum_{j,k,l=0}^{\infty} (-1)^{k+j+l+m} \frac{n d_{v,n} \theta^{v+n-1}}{(C(\theta))^v} (4ab)^v \binom{b(v+j+k)-v}{l} \\ \times \binom{a(v+n-1)-v}{j} \binom{a(v+n-1)+v-1+k}{k} \binom{2l+v}{m} \frac{1}{(1+m/\nu)^v}. \quad (5.12)$$

Consequently, Rényi entropy for TIIEHL-TL-GPS CoD can be obtained from Rényi entropy of the Exp-G distribution. (See Appendix section for details of the derivation).

5.1.3 Maximum Likelihood Estimation

Here we use the MLE technique to find the MLEs of the parameters of the TIIEHL-TL-GPS CoD. Let $X_i \sim TIIEHL-TL-GPS(x; a, b, \theta, \zeta)$ and $\Delta = (a, b, \theta, \zeta)^T$ be the vector of unknown parameters. The total log-likelihood $\ell = \ell(\Delta)$ function is given by

$$\ell(\Delta) = n \ln(4ab\theta) + (b-1) \sum_{i=1}^n \ln \left[1 - (\bar{G}_B(x_i; \zeta))^2 \right] - n \ln(C(\theta)) \\ + (a-1) \sum_{i=1}^n \left(1 - \left[1 - (\bar{G}_B(x_i; \zeta))^2 \right]^b \right) - (a+1) \sum_{i=1}^n \ln \left(1 + \left[1 - (\bar{G}_B(x_i; \zeta))^2 \right]^b \right) \\ + \sum_{i=1}^n \ln \left(C' \left(\theta \left[\frac{1 - \left[1 - (\bar{G}_B(x_i; \zeta))^2 \right]^b}{1 + \left[1 - (\bar{G}_B(x_i; \zeta))^2 \right]^b} \right]^a \right) \right) + \sum_{i=1}^n \ln(g_B(x_i; \zeta)) + \sum_{i=1}^n \ln(\bar{G}_B(x_i; \zeta)).$$

The MLEs of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation

$(\frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \zeta_k})^T = \mathbf{0}$, using numerical methods. The multivariate normal distribution $N_{q+3}(\underline{0}, J(\hat{\Delta})^{-1})$, can be utilized to construct confidence intervals and confidence regions for the individual model parameters, where $\underline{0} = (0, 0, 0, 0)^T$ is the mean vector and $J(\hat{\Delta})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$. **(Elements of score vector are given in the appendix).**

5.1.4 Some Special Sub-Classes

In this section, special classes of TIEHL-TL-GPS CoD are presented by specifying the baseline cdf $G_B(x; \zeta)$ and pdf $g_B(x; \zeta)$ in equations (5.4) and (5.5).

Type II Exponentiated Half-Logistic Topp-Leone-Log-Logistic Power Series (TIEHL-TL-LLoGPS) CoD

If the baseline cdf and pdf are given by $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1} (1 + x^c)^{-2}$, for $c > 0$, and $x > 0$, then the cdf and pdf of the TIEHL-TL-LLoGPS CoD are given by

$$F_{X(1)}(x) = 1 - \frac{C \left(\theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b} \right]^a \right)}{C(\theta)}, \quad (5.13)$$

and

$$\begin{aligned} f_{X(1)}(x) &= \frac{4ab\theta cx^{c-1} \bar{G}^2(x; c) [1 - \bar{G}^2(x; c)]^{b-1} (1 + x^c)^{-1}}{(1 + [1 - \bar{G}^2(x; c)]^b)^{a+1}} \\ &\times (1 - [1 - \bar{G}^2(x; c)]^b)^{a-1} \frac{C' \left(\theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b} \right]^a \right)}{C(\theta)}, \end{aligned} \quad (5.14)$$

respectively. The hrf is given by

$$\begin{aligned} h_F(x) &= \frac{4ab\theta cx^{c-1} \bar{G}^2(x; c) [1 - \bar{G}^2(x; c)]^{b-1} (1 + x^c)^{-1}}{(1 + [1 - \bar{G}^2(x; c)]^b)^{a+1}} \\ &\times (1 - [1 - \bar{G}^2(x; c)]^b)^{a-1} \frac{C' \left(\theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b} \right]^a \right)}{C \left(\theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b} \right]^a \right)}, \end{aligned} \quad (5.15)$$

for $a, b, c > 0$ being shape parameters, $\theta > 0$ being a scale parameter and $x > 0$, where $\bar{G}^2(x; c) = (1 + x^c)^{-2}$.

Type II Exponentiated Half-Logistic Topp-Leone-Log-Logistic Logarithmic (TIIHL-TL-LLoGL) Distribution

The cdf and pdf of TIIHL-TL-LLoGL distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{-\log \left(1 - \theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b} \right]^a \right)}{-\log(1 - \theta)},$$

and

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{4ab\theta c x^{c-1} \bar{G}^2(x; c) [1 - \bar{G}^2(x; c)]^{b-1} (1 + x^c)^{-1}}{(1 + [1 - \bar{G}^2(x; c)]^b)^{a+1}} \\ &\times \left(1 - [1 - \bar{G}^2(x; c)]^b \right)^{a-1} \frac{\left(1 - \theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b} \right]^a \right)^{-1}}{-\log(1 - \theta)}, \end{aligned}$$

respectively. The hrf is given by

$$\begin{aligned} h_F(x) &= \frac{4ab\theta c x^{c-1} \bar{G}^2(x; c) [1 - \bar{G}^2(x; c)]^{b-1} (1 + x^c)^{-1}}{(1 + [1 - \bar{G}^2(x; c)]^b)^{a+1}} \\ &\times \left(1 - [1 - \bar{G}^2(x; c)]^b \right)^{a-1} \frac{\left(1 - \theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b} \right]^a \right)^{-1}}{-\log \left(1 - \theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b} \right]^a \right)}, \end{aligned}$$

for $a, b, c > 0$ being shape parameters, $\theta > 0$ being a scale parameter and $x > 0$, where $\bar{G}^2(x; c) = (1 + x^c)^{-2}$.

Figure 5.1 shows the pdfs of the TIIHL-TL-LLoGL distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrf for the TIIHL-TL-LLoGL distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

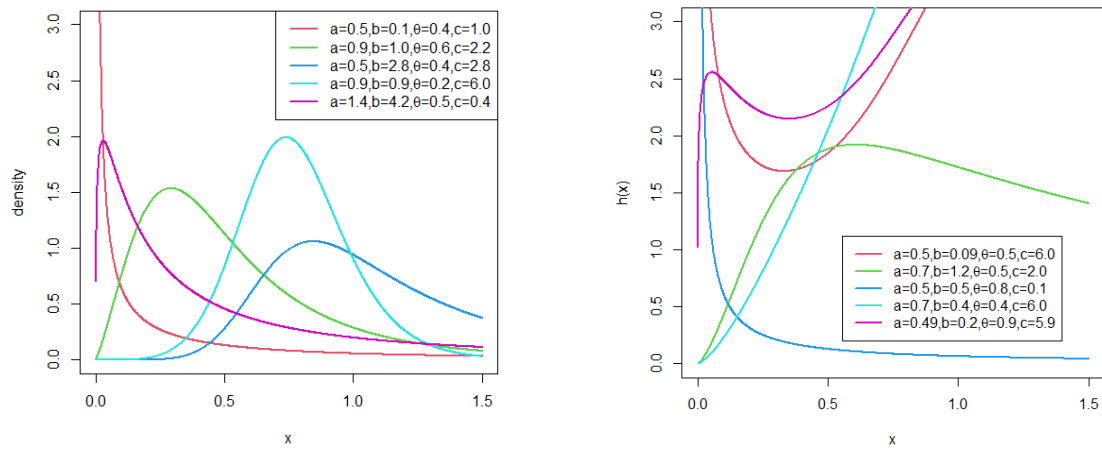


Figure 5.1: Visualization of the pdf and hrf for the TIEHL-TL-LLoGL distribution

(a, b, θ, c)					
u	(0.8, 1.6, 0.3, 0.9)	(1.4, 4.9, 0.2, 3)	(2.5, 0.6, 0.7, 1.7)	(1.6, 5.9, 0.8, 0.8)	(0.2, 3.1, 0.6, 2)
0.1	0.0723	0.7463	0.0086	0.3114	0.7081
0.2	0.1335	0.8194	0.0173	0.4196	0.9257
0.3	0.2080	0.8783	0.0280	0.5235	1.1514
0.4	0.2995	0.9299	0.04483	0.6353	1.4336
0.5	0.4189	0.9836	0.0625	0.7637	1.8146
0.6	0.5946	1.0417	0.0834	0.9247	2.3857
0.7	0.8740	1.1100	0.1182	1.1508	3.3584
0.8	1.3938	1.2034	0.1656	1.5102	5.4078
0.9	2.7587	1.3548	0.2558	2.2785	12.3311

Table 5.2: Table of Quantile for TIEHL-TL-LLoGL Distribution

(a, b, θ, c)					
	(2.5, 1.8, 0.5, 1.1)	(0.7, 0.9, 0.8, 0.5)	(2.4, 0.5, 0.9, 1)	(1.5, 0.9, 0.2, 5.4)	(0.8, 0.6, 0.9, 0.4)
$E(X)$	0.3092	0.1758	0.3301	0.3372	0.1386
$E(X^2)$	0.1284	0.0936	0.2077	0.2940	0.0742
$E(X^3)$	0.0615	0.0631	0.1500	0.2609	0.0503
$E(X^4)$	0.0322	0.0474	0.1168	0.2346	0.0380
$E(X^5)$	0.0180	0.0379	0.0954	0.2132	0.0305
$E(X^6)$	0.0105	0.0316	0.0805	0.1955	0.0255
SD	0.1811	0.2504	0.3143	0.4246	0.2345
CV	0.5857	1.4237	0.9519	1.2591	1.6915
CS	0.2633	1.5667	0.5224	0.5237	1.9245
CK	2.2184	4.4657	1.9430	1.3694	5.8083

Table 5.3: Moments of TIEHL-TL-LLoGL Distribution for Selected Parameters

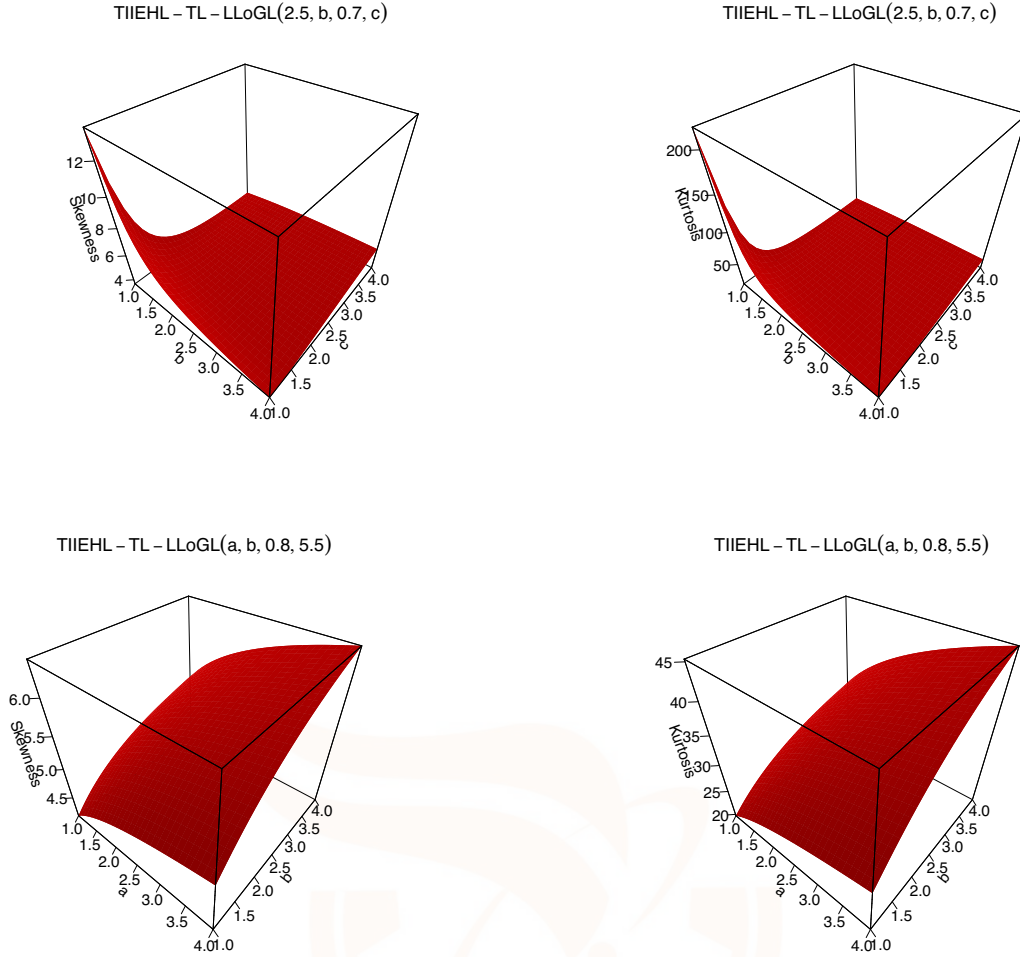


Figure 5.2: 3 D Visualization of skewness and kurtosis for TIIHL-TL-LLoGL distribution

We present in Figure 5.2, 3D plots of skewness and kurtosis of the TIIHL-TL-LLoGL distribution. We observe that

- Fixing the parameters a and θ , the skewness of the TIIHL-TL-LLoGL distribution becomes right skewed, while kurtosis is leptokurtic as b and c increase.
- Fixing the parameters θ and c , the skewness and kurtosis of the TIIHL-TL-LLoGL distribution increases as a and b increase.

Type II Exponentiated Half-Logistic Topp-Leone-Log-Logistic Poisson (TIIHL-TL-LLoGP) Distribution

The cdf and pdf of TIIHL-TL-LLoGLP distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{\exp\left(\theta \left[\frac{1 - [1 - \bar{G}^2(x;c)]^b}{1 + [1 - \bar{G}^2(x;c)]^b} \right]^a - 1\right)}{\exp(\theta - 1)},$$

and

$$f_{X_{(1)}}(x) = \frac{4ab\theta c x^{c-1} \bar{G}^2(x; c) [1 - \bar{G}^2(x; c)]^{b-1} (1 + x^c)^{-1}}{\left(1 + [1 - \bar{G}^2(x; c)]^b\right)^{a+1}} \times \left(1 - [1 - \bar{G}^2(x; c)]^b\right)^{a-1} \frac{\exp\left(\theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b}\right]^a\right)}{\exp(\theta - 1)},$$

respectively. The hrf is given by

$$h_F(x) = \frac{4ab\theta c x^{c-1} \bar{G}^2(x; c) [1 - \bar{G}^2(x; c)]^{b-1} (1 + x^c)^{-1}}{\left(1 + [1 - \bar{G}^2(x; c)]^b\right)^{a+1}} \times \left(1 - [1 - \bar{G}^2(x; c)]^b\right)^{a-1} \frac{\exp\left(\theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b}\right]^a\right)}{\exp\left(\theta \left[\frac{1 - [1 - \bar{G}^2(x; c)]^b}{1 + [1 - \bar{G}^2(x; c)]^b}\right]^a - 1\right)},$$

for $a, b, c > 0$ being shape parameters, $\theta > 0$ being a scale parameter and $x > 0$, where $\bar{G}^2(x; c) = (1 + x^c)^{-2}$.

Figure 5.3 shows the pdfs of the TIIHL-TL-LLoGP distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrf for the TIIHL-TL-LLoGP distribution exhibit increasing, reverse-J, bathtub, upside bathtub and bathtub followed by upside bathtub shapes.

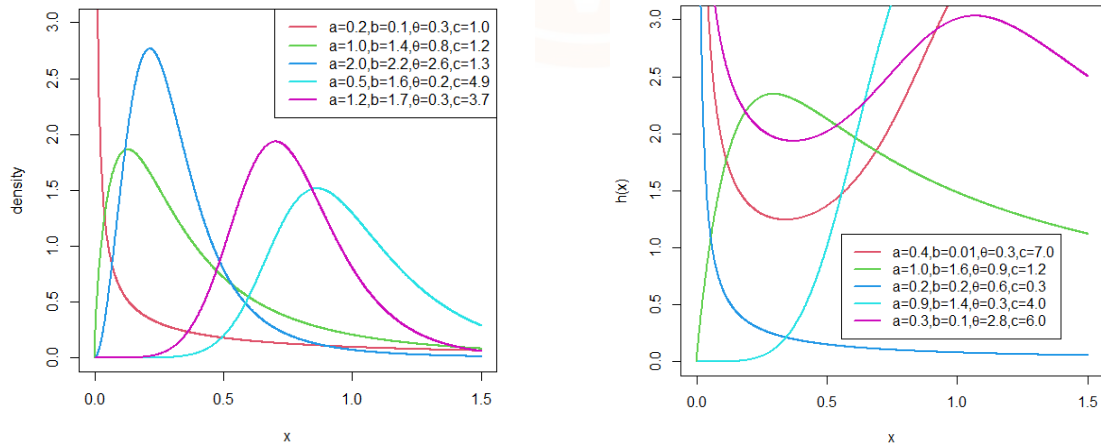


Figure 5.3: Visualization of the pdf and hrf plots for the TIIHL-TL-LLoGP distribution

	(a, b, θ, c)				
u	(0.9, 1.3, 1, 0.8)	(2.4, 3.2, 0.9, 5)	(0.5, 3.7, 1.6, 0.7)	(0.6, 0.8, 0.9, 0.8)	(0.8, 0.3, 2.1, 6)
0.1	0.0088	0.2884	0.0177	0.0091	0.7373
0.2	0.0155	0.3411	0.0367	0.0292	0.7958
0.3	0.0332	0.3778	0.0639	0.0593	0.8426
0.4	0.0524	0.4123	0.0950	0.1056	0.8864
0.5	0.0870	0.4461	0.1373	0.1825	0.9268
0.6	0.1401	0.4786	0.2082	0.3121	0.9676
0.7	0.2295	0.5131	0.3182	0.5610	1.0225
0.8	0.4109	0.5588	0.5451	1.1223	1.0887
0.9	0.9423	0.6209	1.2676	3.0442	1.2038

Table 5.4: Table of Quantile for TIIHL-TL-LLoGP Distribution

	(a, b, θ, c)				
	(0.5, 3.5, 1.9, 1.7)	(0.2, 0.9, 0.4, 0.5)	(1.4, 3.9, 0.6, 1.8)	(3.5, 6.2, 1.9, 3)	(2.8, 1.9, 0.6, 4.4)
$E(X)$	0.4128	0.0649	0.0923	0.4025	0.3157
$E(X^2)$	0.2413	0.0332	0.0212	0.1719	0.1216
$E(X^3)$	0.1595	0.0222	0.0085	0.0774	0.0546
$E(X^4)$	0.1150	0.0167	0.0047	0.0367	0.0277
$E(X^5)$	0.0882	0.0133	0.0032	0.0183	0.0156
$E(X^6)$	0.0709	0.0111	0.0023	0.0096	0.0096
SD	0.2662	0.1703	0.1125	0.0993	0.1482
CV	0.6449	2.6244	1.2195	0.2468	0.4693
CS	0.0710	3.3027	2.9874	0.3207	0.7211
CK	2.2389	13.9159	15.5031	3.8181	3.5907

Table 5.5: Moments of TIIHL-TL-LLoGP Distribution for Selected Parameters

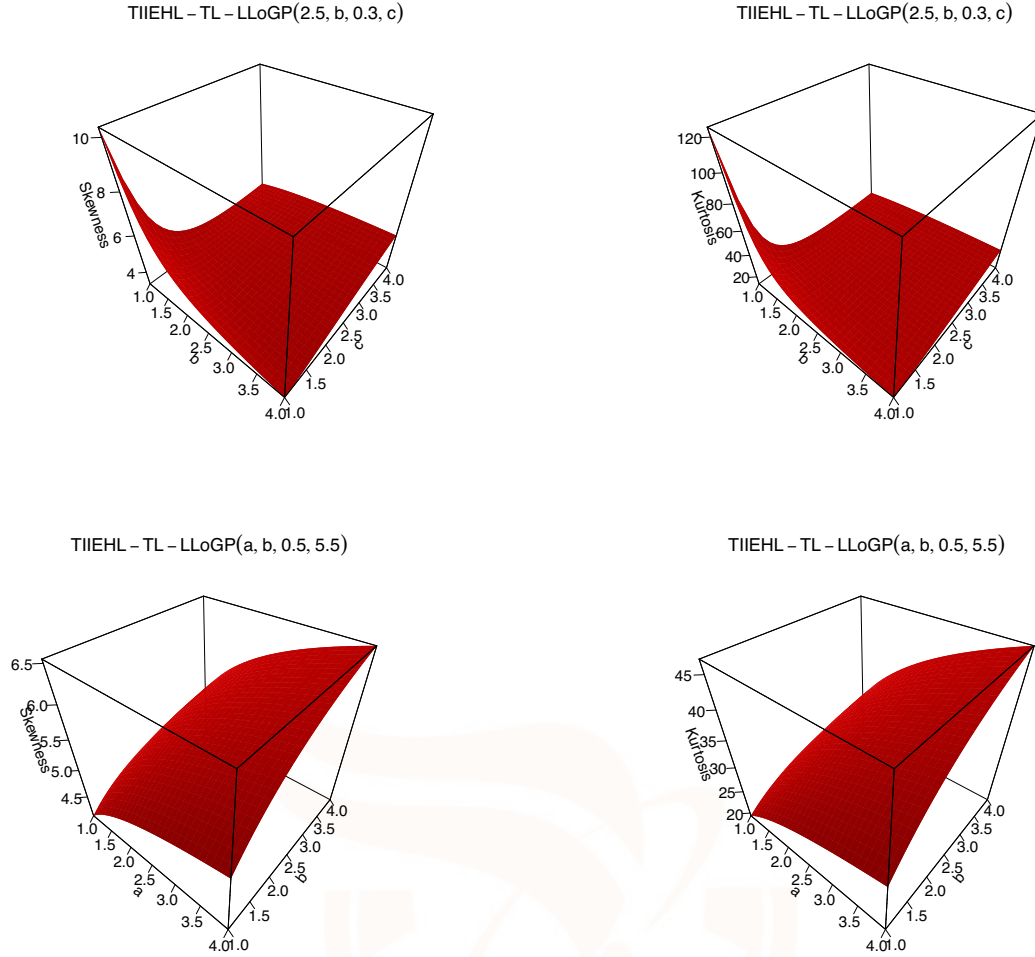


Figure 5.4: 3 D Visualization of skewness and kurtosis for TIIHL-TL-LLoGP distribution

We present in Figure 5.4, 3D plots of skewness and kurtosis of the TIIHL-TL-LLoGP distribution. We observe that

- Fixing the parameters a and θ , the skewness of the TIIHL-TL-LLoGP distribution becomes right skewed, while kurtosis is leptokurtic as b and c increase.
- Fixing the parameters θ and c , the skewness and kurtosis of the TIIHL-TL-LLoGP distribution increases as a and b increase.

Type II Exponentiated Half-Logistic Topp-Leone-Weibull Power Series (TIIHL-TL-WPS) CoD

Suppose the cdf and pdf of the Weibull distribution are given by $G(x; \lambda) = 1 - \exp(-x^\lambda)$, for $x \geq 0, \lambda > 0$ and $g(x; \lambda) = \lambda x^{\lambda-1} \exp(-x^\lambda)$, for $\lambda > 0$, and $x > 0$, then, the cdf and pdf of

the TIEHL-TL-WGPS CoD are given by

$$F_{X_{(1)}}(x) = 1 - \frac{C \left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b} \right]^a \right)}{C(\theta)},$$

and

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{4ab\theta\lambda x^{\lambda-1} \exp(-x^\lambda) [1 - \exp(-2x^\lambda)]^{b-1} \exp(-x^\lambda)}{(1 + [1 - \exp(-2x^\lambda)]^b)^{a+1}} \\ &\times \left(1 - [1 - \exp(-2x^\lambda)]^b \right)^{a-1} \frac{C' \left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b} \right]^a \right)}{C(\theta)}, \end{aligned}$$

respectively. The hrf is given by

$$\begin{aligned} h_F(x) &= \frac{4ab\theta\lambda x^{\lambda-1} \exp(-x^\lambda) [1 - \exp(-2x^\lambda)]^{b-1} \exp(-x^\lambda)}{(1 + [1 - \exp(-2x^\lambda)]^b)^{a+1}} \\ &\times \left(1 - [1 - \exp(-2x^\lambda)]^b \right)^{a-1} \frac{C' \left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b} \right]^a \right)}{C \left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b} \right]^a \right)}, \end{aligned}$$

for $a, b, \lambda > 0$ being shape parameters, $\theta > 0$ being a scale parameter and $x > 0$.

Type II Exponentiated Half-Logistic Topp-Leone-Weibull Logarithmic (TIEHL-TL-WL) Distribution

The cdf and pdf of TIEHL-TL-WL distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{-\log \left(1 - \theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b} \right]^a \right)}{-\log(1 - \theta)},$$

and

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{4ab\theta\lambda x^{\lambda-1} \exp(-x^\lambda) [1 - \exp(-2x^\lambda)]^{b-1} \exp(-x^\lambda)}{(1 + [1 - \exp(-2x^\lambda)]^b)^{a+1}} \\ &\times \left(1 - [1 - \exp(-2x^\lambda)]^b \right)^{a-1} \frac{\left(1 - \theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b} \right]^a \right)^{-1}}{-\log(1 - \theta)}, \end{aligned}$$

respectively. The hrf is given by

$$h_F(x) = \frac{4ab\theta\lambda x^{\lambda-1} \exp(-x^\lambda) [1 - \exp(-2x^\lambda)]^{b-1} \exp(-x^\lambda)}{\left(1 + [1 - \exp(-2x^\lambda)]^b\right)^{a+1}} \\ \times \left(1 - [1 - \exp(-2x^\lambda)]^b\right)^{a-1} \frac{\left(1 - \theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b}\right]^a\right)^{-1}}{-\log\left(1 - \theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b}\right]^a\right)},$$

for $a, b, \lambda > 0$ being shape parameters, $\theta > 0$ being a scale parameter and $x > 0$.

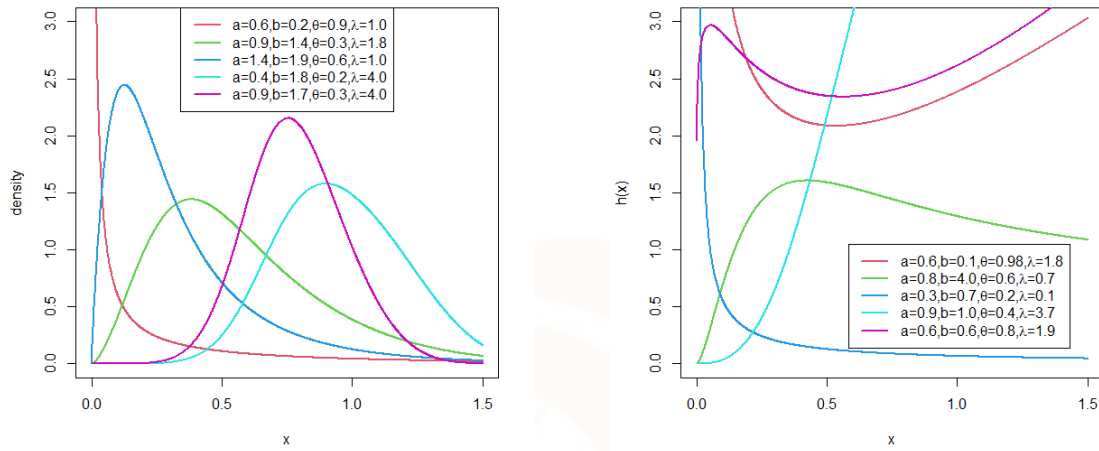


Figure 5.5: Visualization of the pdf and hrf for the TIIHL-TL-WL distribution

Figure 5.5 shows the pdfs of the TIIHL-TL-WL distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrf for the TIIHL-TL-WL distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

	(a, b, θ, λ)				
u	(1.9, 1.4, 0.3, 2.8)	(2.4, 0.9, 0.6, 3)	(8.5, 0.6, 0.7, 1.7)	(0.6, 0.9, 0.8, 6.8)	(1.8, 2.5, 0.9, 4)
0.1	0.3016	0.1606	0.0050	0.5275	0.5280
0.2	0.3719	0.2165	0.0058	0.6038	0.5805
0.3	0.4212	0.2609	0.0085	0.6602	0.6203
0.4	0.4693	0.3048	0.0091	0.7098	0.6522
0.5	0.5154	0.3454	0.0206	0.7558	0.6859
0.6	0.5646	0.3874	0.0268	0.8077	0.7192
0.7	0.6167	0.4394	0.0357	0.8612	0.7571
0.8	0.6776	0.4997	0.0500	0.9219	0.8049
0.9	0.7701	0.5924	0.0726	1.0059	0.8706

Table 5.6: Table of Quantile for TIIHL-TL-WL Distribution

	(a, b, θ, λ)				
	(4.5, 0.9, 0.5, 1.3)	(2.2, 1.4, 0.9, 0.6)	(0.4, 3.6, 0.8, 0.8)	(0.5, 0.9, 0.2, 0.9)	(3.8, 0.6, 0.7, 5.4)
E(X)	0.0765	0.0482	0.2935	0.2217	0.3738
E(X ²)	0.0116	0.0118	0.1774	0.1195	0.1602
E(X ³)	0.0026	0.0055	0.1231	0.0804	0.0762
E(X ⁴)	0.0008	0.0033	0.0930	0.0602	0.0393
E(X ⁵)	0.0003	0.0023	0.0744	0.0481	0.0217
E(X ⁶)	0.0002	0.0018	0.0618	0.0399	0.0126
SD	0.0757	0.0978	0.3022	0.2654	0.1432
CV	0.9889	2.0249	1.0296	1.1971	0.3832
CS	2.1070	4.2935	0.6326	1.2148	0.3267
CK	9.7808	26.8613	2.1534	3.4191	2.7883

Table 5.7: Moments of TIIHL-TL-WL Distribution for Selected Parameters

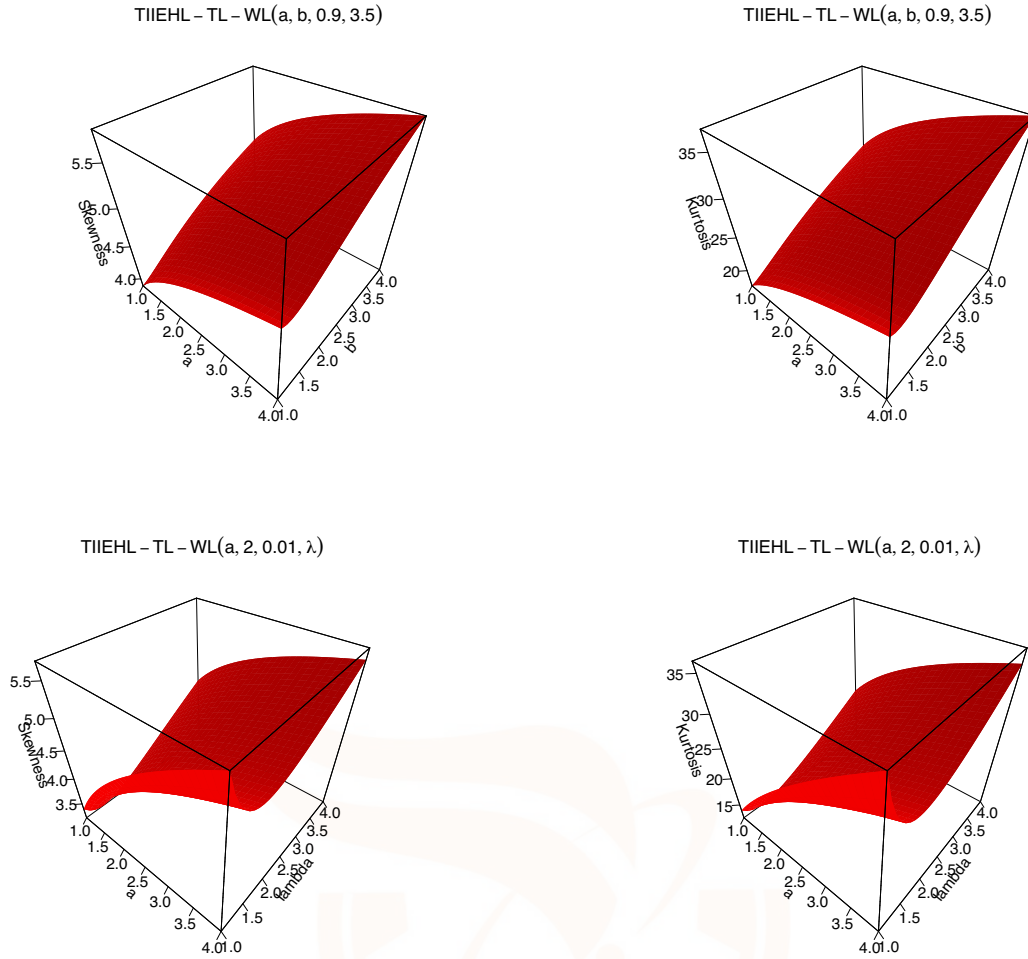


Figure 5.6: 3 D Visualization of skewness and kurtosis for TIEHL-TL-WL distribution

We present in Figure 5.6, 3D plots of skewness and kurtosis of the TIEHL-TL-WL distribution. We observe that

- Fixing the parameters θ and λ , the skewness and kurtosis of the TIEHL-TL-WL distribution increases as a and b increase.
- Fixing the parameters b and θ , the skewness and kurtosis of the TIEHL-TL-WL distribution decreases and increases as a and λ increase.

Type II Exponentiated Half-Logistic Topp-Leone-Weibull Poisson (TIEHL-TL-WP) Distribution

The cdf and pdf of TIEHL-TL-WP distribution are given by

$$F_{X(1)}(x) = 1 - \frac{\exp\left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b} \right]^a - 1\right)}{\exp(\theta - 1)},$$

and

$$f_{X_{(1)}}(x) = \frac{4ab\theta\lambda x^{\lambda-1} \exp(-x^\lambda) [1 - \exp(-2x^\lambda)]^{b-1} \exp(-x^\lambda)}{(1 + [1 - \exp(-2x^\lambda)]^b)^{a+1}} \\ \times \left(1 - [1 - \exp(-2x^\lambda)]^b\right)^{a-1} \frac{\exp\left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b}\right]^a\right)}{\exp(\theta - 1)},$$

respectively. The hrf is given by

$$h_F(x) = \frac{4ab\theta\lambda x^{\lambda-1} \exp(-x^\lambda) [1 - \exp(-2x^\lambda)]^{b-1} \exp(-x^\lambda)}{(1 + [1 - \exp(-2x^\lambda)]^b)^{a+1}} \\ \times \left(1 - [1 - \exp(-2x^\lambda)]^b\right)^{a-1} \frac{\exp\left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b}\right]^a\right)}{\exp\left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b}\right]^a - 1\right)},$$

for $a, b, \lambda > 0$ being shape parameters, $\theta > 0$ being a scale parameter and $x > 0$.

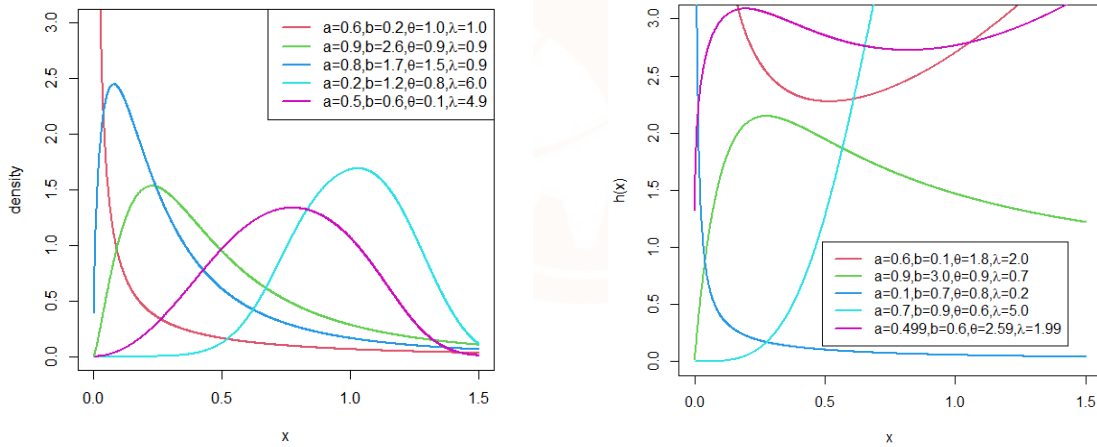


Figure 5.7: Visualization of the pdf and hrf for the TIEHL-TL-WP distribution

Figure 5.7 shows the pdfs of the TIEHL-TL-WP distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrf for the TIEHL-TL-WP distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

	(a, b, θ, λ)				
u	(0.9, 1.3, 2.4, 0.8)	(1.4, 0.7, 1.9, 2)	(3.5, 2.7, 3.6, 0.7)	(0.9, 0.8, 0.3, 4.8)	(0.8, 1.2, 0.5, 1.3)
0.1	0.0840	0.2848	0.0536	0.0960	0.0076
0.2	0.1365	0.3569	0.0779	0.1592	0.0090
0.3	0.1852	0.4152	0.0982	0.2225	0.0255
0.4	0.2446	0.4660	0.1225	0.2876	0.0477
0.5	0.3125	0.5142	0.1444	0.3572	0.0796
0.6	0.3955	0.5730	0.1707	0.4362	0.1294
0.7	0.5166	0.6435	0.2045	0.5216	0.2038
0.8	0.7038	0.7255	0.2470	0.6330	0.3306
0.9	1.0820	0.8555	0.3303	0.7812	0.5925

Table 5.8: Table of Quantile for TIIHL-TL-WP Distribution

	(a, b, θ, λ)				
	(1.5, 0.5, 0.9, 0.7)	(0.2, 0.9, 0.4, 2.8)	(1.4, 1.8, 1.6, 0.9)	(1.6, 0.7, 0.9, 3.9)	(0.8, 3.2, 8.6, 0.4)
$E(X)$	0.0893	0.3054	0.1832	0.5129	0.3295
$E(X^2)$	0.0305	0.2012	0.0659	0.2972	0.1921
$E(X^3)$	0.0165	0.1496	0.0343	0.1878	0.1301
$E(X^4)$	0.0109	0.1190	0.0219	0.1271	0.0966
$E(X^5)$	0.0081	0.0987	0.0158	0.0909	0.0762
$E(X^6)$	0.0064	0.0843	0.0122	0.0681	0.0627
SD	0.1500	0.3285	0.1797	0.1848	0.2890
CV	1.6794	1.0754	0.9806	0.3604	0.8771
CS	2.8957	0.6279	1.7926	0.0551	0.4852
CK	12.5288	1.9469	6.4264	2.8310	2.1406

Table 5.9: Moments of TIIHL-TL-WP Distribution for Selected Parameters

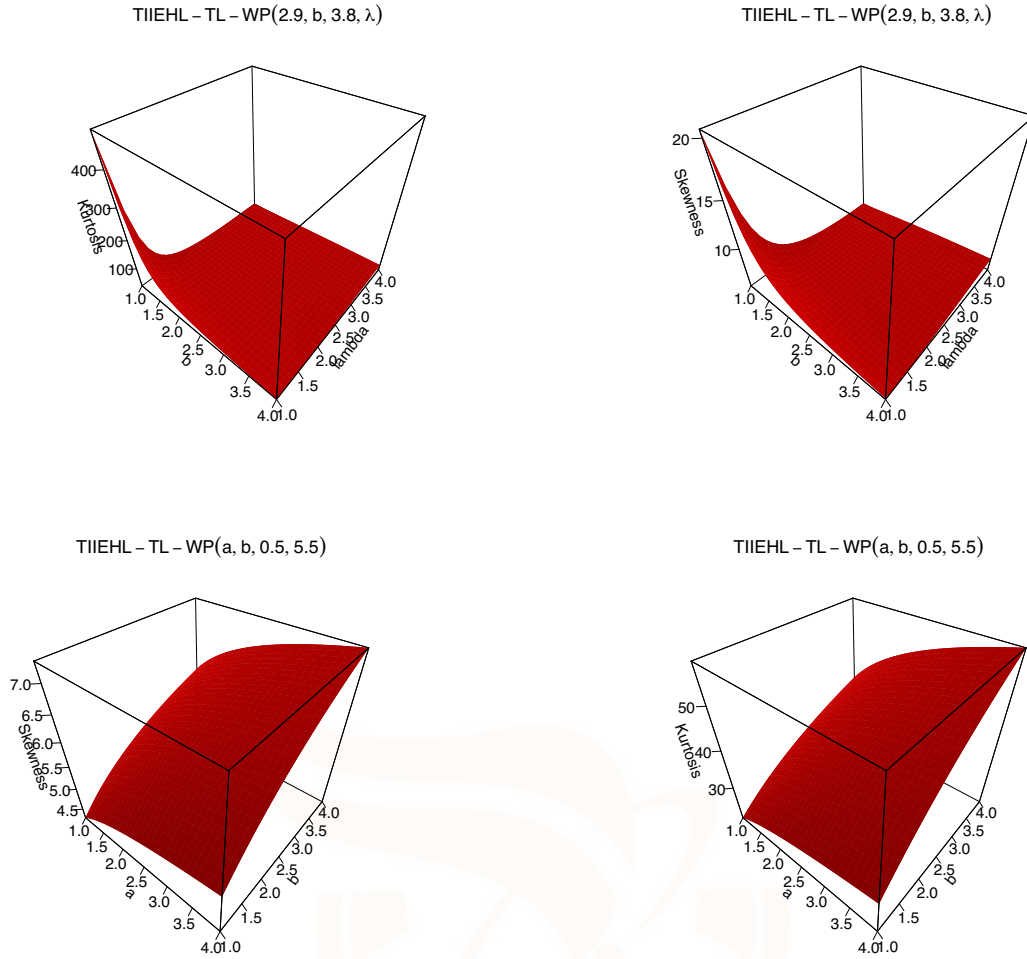


Figure 5.8: 3 D Visualization of skewness and kurtosis for TIIEHL-TL-WL distribution

We present in Figure 5.8, 3D plots of skewness and kurtosis of the TIIEHL-TL-WP distribution. We observe that

- Fixing the parameters a and θ , the skewness of the TIIEHL-TL-WP distribution becomes right skewed, while kurtosis is leptokurtic as b and λ increase.
- Fixing the parameters θ and λ , the skewness and kurtosis of the TIIEHL-TL-WP distribution increases as a and b increase.

5.1.5 Simulation Study

In this section, the performance of the TIIEHL-TL-WL distribution is examined by conducting various simulations for different sizes ($n=25, 50, 100, 200, 400$ and 800). We simulate $N = 3000$ samples for the true parameters values given in Table 5.10. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The precision of the MLEs is discussed by means of the following measures: mean, mean square error (MSE) and average bias.

The ABias and RMSE for the MLEs, say, $\hat{\theta}$, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

parameter	Sample Size	(1.5, 1.0, 0.9, 0.6)			(1.0, 0.5, 0.5, 1.0)			(0.7, 0.4, 0.4, 1.0)		
		Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
a	25	1.9519	1.8074	0.4519	1.5811	0.9439	0.5811	0.7142	0.5022	0.0142
	50	1.9491	1.4855	0.4491	1.3136	0.6695	0.3136	0.8110	1.0306	0.1110
	100	1.7883	1.2965	0.2883	1.2610	0.5273	0.2610	0.7431	0.2660	0.0431
	200	1.6299	0.8137	0.1299	1.1773	0.3629	0.1773	0.7219	0.1962	0.0219
	400	1.5915	0.6830	0.0915	1.1275	0.2876	0.1275	0.7045	0.1395	0.0045
	800	1.5061	0.3518	0.0061	1.0643	0.1726	0.0643	0.7030	0.0990	0.0030
b	25	1.3639	1.0016	0.3639	0.4056	0.2439	-0.0943	0.3518	0.3813	-0.0481
	50	1.2581	0.7241	0.2581	0.4474	0.2104	-0.0525	0.3186	0.1624	-0.0813
	100	1.1998	0.7133	0.1998	0.4516	0.1688	-0.0483	0.3188	0.1349	-0.0811
	200	1.0563	0.4182	0.0563	0.4577	0.1595	-0.0422	0.3220	0.1230	-0.0779
	400	1.0169	0.3898	0.0169	0.4723	0.1321	-0.0276	0.3305	0.1077	-0.0694
	800	0.9833	0.2126	-0.0166	0.5074	0.1090	0.0074	0.3527	0.0827	-0.0472
θ	25	0.7043	0.3329	-0.1956	0.7530	0.3573	0.2530	0.7075	0.4034	0.3075
	50	0.7716	0.2709	-0.1283	0.7483	0.3547	0.2483	0.7115	0.4024	0.3115
	100	0.7890	0.2659	-0.1109	0.7229	0.3299	0.2229	0.6827	0.3751	0.2827
	200	0.8380	0.2141	-0.0619	0.6908	0.3039	0.1908	0.6611	0.3553	0.2611
	400	0.8810	0.1298	-0.0189	0.6606	0.2835	0.1606	0.6409	0.3310	0.2409
	800	0.8972	0.1220	-0.0027	0.5940	0.2620	0.0940	0.5683	0.2905	0.1683
λ	25	0.8853	0.8499	0.2853	1.5311	0.9999	0.5311	1.2960	0.5125	0.2960
	50	0.7385	0.4875	0.1385	1.3954	0.7584	0.3954	1.2479	0.4104	0.2479
	100	0.7017	0.3609	0.1017	1.3538	0.5857	0.3538	1.1968	0.3186	0.1968
	200	0.6890	0.2650	0.0890	1.2874	0.4607	0.2874	1.1580	0.2651	0.1580
	400	0.6571	0.1780	0.0571	1.2136	0.3445	0.2136	1.1145	0.1936	0.1145
	800	0.6535	0.1411	0.0535	1.0962	0.1973	0.0962	1.0716	0.1362	0.0716

Table 5.10: Monte Carlo Simulation Results

The results show that the estimation method used is appropriate for estimating the TIIEHL-TL-WL model parameters as the means of the parameters tend to be closer to the true parameter values when n increases.

5.1.6 Applications

In this section, we present two real data applications to demonstrate the importance and applicability of the TIIEHL-TL-WL distribution. The R software was used for data fitting and model diagnostics. The following goodness-of-fit statistics $-2\log(L)$, AIC , $AICC$, BIC , K-S statistic (and its p-value), W^* , (A^*) , and sum of squares (SS) are used to assess the performance of the model. The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is regarded as the best model.

The TIIEHL-TL-WL distribution was compared to its nested models and to the following non-nested models: type II exponentiated half logistic Weibull (TIIEHLW) distribution by [Al-Mofleh et al. \(2020\)](#), type II general inverse exponential Burr III (TIIGIE-BIII) distribution by [Jamal et al. \(2017\)](#), and type II general inverse exponential Lomax (TIIGIE-Lx) distribution by [Hamedani et al. \(2019\)](#). The pdf's of these competing distributions are given in the appendix.

Application results are shown in Tables [5.11](#), [5.12](#) and [5.13](#). Histogram of data, fitted densities and probability plots are shown in Figures [5.10](#), [5.12](#) and [5.14](#).

Dataset 1

The first real data set is a subset of data reported by [Bekker et al. \(2000\)](#) which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. **(The data is given in the appendix).**

The estimated variance-covariance matrix for TIIEHL-TL-WL distribution on survival times is given by

$$\begin{bmatrix} 1.3252 \times 10^{-02} & 7.2034 \times 10^{-02} & -2.5728 \times 10^{-05} & -2.7290 \times 10^{-02} \\ 7.2034 \times 10^{-02} & 5.0952 \times 10^{-01} & -4.4650 \times 10^{-05} & -1.7131 \times 10^{-01} \\ -2.5728 \times 10^{-05} & -4.4650 \times 10^{-05} & 1.9335 \times 10^{-04} & 3.4986 \times 10^{-05} \\ -2.7290 \times 10^{-02} & -1.7131 \times 10^{-01} & 3.4986 \times 10^{-05} & 7.1600 \times 10^{-02} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $a \in [3.2823 \times 10^{-01} \pm 0.2256]$, $b \in [1.4055 \pm 1.3990]$, $\theta \in [1.0674 \times 10^{-06} \pm 0.0272]$ and $\lambda \in [1.0753 \pm 0.5244]$, respectively.

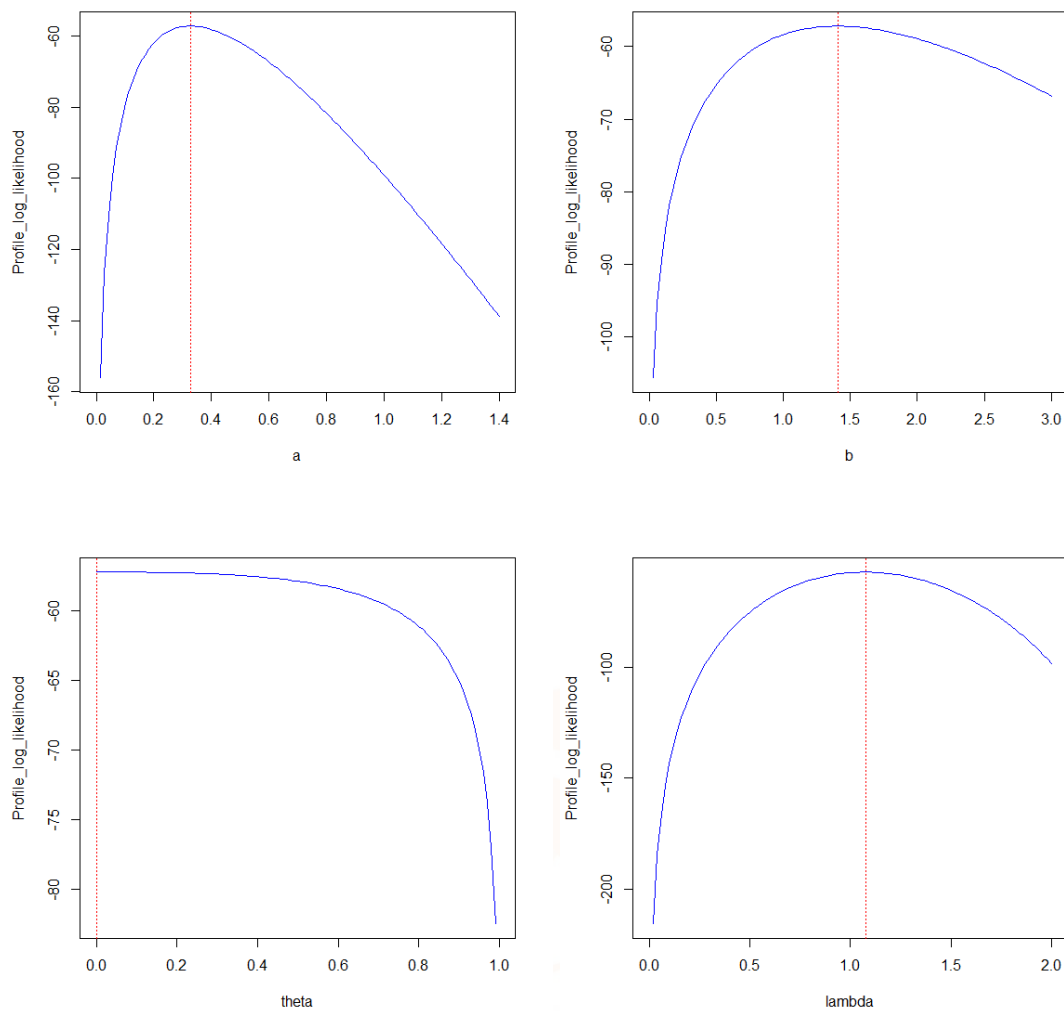


Figure 5.9: Profile log-likelihood function plots for parameters of TIIEHL-TL-WL distribution on the survival times data

Figure 5.9 shows the profile log-likelihood plots for parameters of the TIIEHL-TL-WL distribution on the survival times data set. It can be seen that the MLEs obtained for the TIIEHL-TL-WL distribution are unique. This shows that the parameters of the TIIEHL-TL-WL distribution are identifiable.

Model	Estimates				Statistics							
	a	b	θ	λ	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIIEHL-TL-WL	3.2823×10^{-01} (1.1512×10^{-01})	1.4055 (7.1381×10^{-10})	1.0674×10^{-06} (1.3905×10^{-01})	1.0753 (2.6758×10^{-01})	114.412	122.412	123.412	129.6387	0.0556	0.3916	0.0960	0.7647
TIIIEHL-TL-WL(1, b, θ , λ)	1 -	1.1021 (1.5849×10^{-01})	1.3610×10^{-10} (1.8371×10^{-03})	6.8916×10^{-01} (7.1734×10^{-02})	196.9675	202.9643	203.5497	208.3843	0.0613	0.4242	0.5308	2.321×10^{-12}
TIIIEHL-TL-WL(a, 1, θ , λ)	9.7192×10^{-01} (1.4490×10^{-01})	1 -	1.9653×10^{-10} (2.2104×10^{-03})	2.4731×10^{-01} (3.4972×10^{-02})	268.0722	274.073	274.6584	279.493	0.0571	0.4111	0.7854	2.2×10^{-16}
TIIIEHL-TL-WL(a, b, θ , 1)	9.9680×10^{-01} (2.2545×10^{-01})	1.9004 (3.8360×10^{-10})	2.7632×10^{-09} (8.1565×10^{-03})	1 -	174.8767	180.8763	181.4617	186.2963	0.0596	0.4164	0.3571	1.21×10^{-05}
TIIIEHL-TL-WL(1, 1, θ , λ)	1 -	1 -	16.3267×10^{-10} (4.1664×10^{-03})	7.0003×10^{-02} (1.0384×10^{-02})	385.1095	389.1099	389.3956	392.7232	0.0598	0.4362	0.8895	2.2×10^{-16}
TIIIEHL-TL-WL(1, b, θ , 1)	1 -	2.8752 (3.7626×10^{-01})	1.4472×10^{-08} (2.1368×10^{-02})	1 -	166.6634	170.6634	170.9491	174.2767	0.0565	0.4008	0.3138	0.0002
TIIIEHL-TL-WL(a, 1, θ , 1)	9.9617×10^{-01} (1.4850×10^{-01})	1 -	4.3253×10^{-10} (3.4541×10^{-03})	1 -	211.5891	215.5888	215.8745	219.2021	0.0785	0.5277	0.4819	3.87×10^{-10}
TIIIE-BIII	λ	θ	c	k	119.7969	127.7969	128.7969	135.0236	15.8450	90.2978	0.9901	2.2×10^{-16}
	17.0303 (0.0038)	1.1525 (0.5246)	1.3717 (0.6776)	0.0720 (0.0967)								
TIIIEHLW	a	λ	δ	γ	116.2558	124.2558	125.2558	131.4824	0.0811	0.5427	0.1094	0.6152
	3.5837×10^{02} (1.6288×10^{-03})	1.1514 (1.7314)	12.4872×10^{-03} (2.2467×10^{-02})	9.1589×10^{-01} (1.3684)								
TIIIE-Lx	λ	α	a	b	115.9789	123.9789	124.9789	131.2055	0.0882	0.5867	0.1164	0.5365
	6.8295 (1.1302×10^{01})	3.7824×10^{04} (1.9479×10^{-07})	9.1447×10^{-02} (1.3345×10^{-01})	3.5879×10^{04} (4.0338×10^{-07})								

Table 5.11: MLEs and GoF Statistics

From the results in Table 5.11, we conclude that the TIIIEHL-TL-WL distribution is the “best” model compared to the selected models since it has the lowest values for the goodness-of-fit statistics $-2\log(L)$, AIC , $AICC$, BIC , A^* , W^* and K-S (and the largest p-value for the K-S statistic). Also, the plots in figure 5.10 show that the TIIIEHL-TL-WL fitted the data better than the other models of comparison and also has the smallest SS value, hence termed the better model.

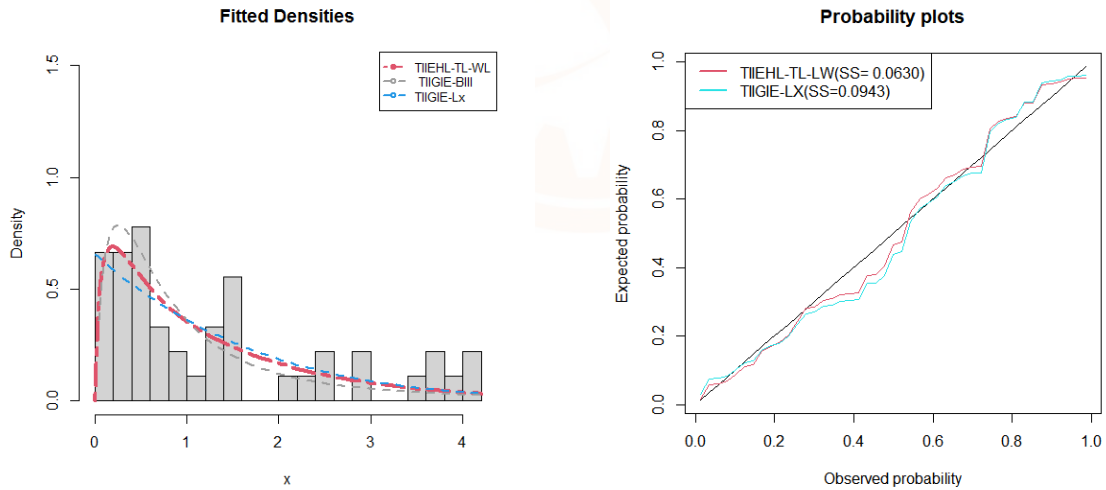


Figure 5.10: Fitted Densities and Probability Plots for Survival Times Data

Dataset 2

The second data is concerned with the study of the lifetimes of kevlar 49/epoxy spherical pressure vessels that are subjected to a constant sustained pressure until vessel failure, commonly known as static fatigue or stress-rupture. The data set consists of 101 observations of stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all have failed, so that the complete data

set with the exact times of failure is recorded. These failure times in hours, are originally given by [Barlow et al. \(1984\)](#). **(The data is given in the appendix).**

The estimated variance-covariance matrix for TIIHL-TL-WL distribution on kevlar 49/e-poxy data is given by

$$\begin{bmatrix} 0.0231 & 0.0212 & -0.1657 & -0.0348 \\ 0.0212 & 0.0437 & -0.1105 & -0.0419 \\ -0.1657 & -0.1105 & 1.3521 & 0.2350 \\ -0.0348 & -0.0419 & 0.2350 & 0.0633 \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $a \in [0.3180 \pm 0.2983]$, $b \in [0.7098 \pm 0.4100]$, $\theta \in [0.0494 \pm 2.2791]$ and $\lambda \in [1.2281 \pm 0.4931]$, respectively.



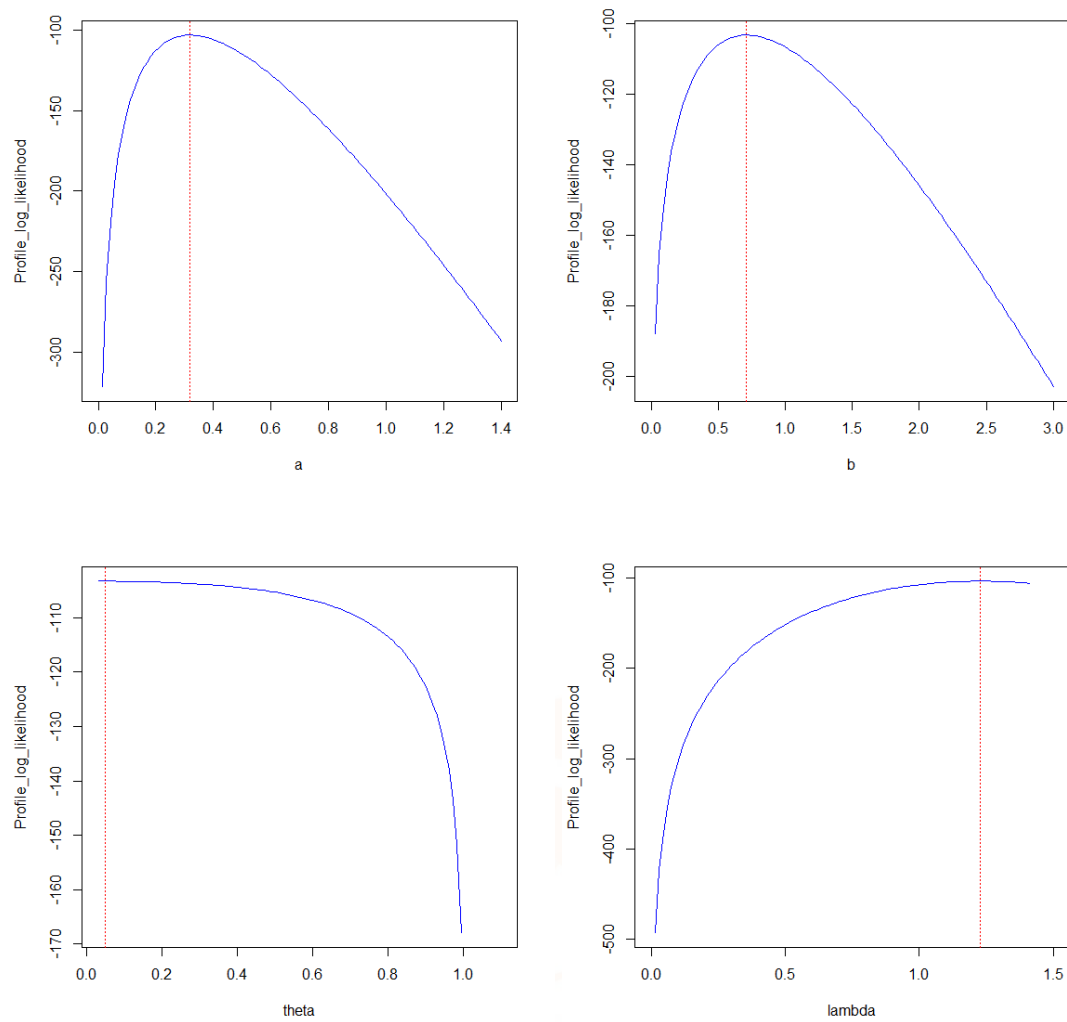


Figure 5.11: Profile log-likelihood function plots for parameters of TIIHL-TL-WL distribution on the kevlar 49/epoxy strands data

Figure 5.11 shows the profile log-likelihood plots for parameters of the TIIHL-TL-WL distribution on the kevlar 49/epoxy strands data set. From the plots we can see that the MLEs for the TIIHL-TL-WL distribution are achieved at single and different points.

Model	Estimates				Statistics							
	a	b	θ	λ	$-2 \log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-TL-WL	0.3180 (0.1522)	0.7098 (0.2092)	0.0494 (1.1628)	1.2281 (0.2516)	206.5335	214.5335	214.9502	224.994	0.1581	0.9407	0.0850	0.4584
TIIEHL-TL-WL(1, b, θ , λ)	1	4.3039 (5.6589×10^{-01})	3.3319×10^{-05} (1.9637×10^{-02})	4.3119×10^{-01} (4.8213×10^{-02})	224.1607	230.1607	230.4081	238.006	0.5829	3.1213	0.1752	0.0040
TIIEHL-TL-WL(a, 1, θ , λ)	9.9909×10^{-01} (9.9890×10^{-02})	1	1.8028×10^{-09} (4.4638×10^{-03})	9.7888×10^{-02} (9.6435×10^{-03})	700.859	706.8599	707.1073	714.7052	0.5696	3.0477	0.8371	2.2×10^{-16}
TIIEHL-TL-WL(a, b, θ , 1)	9.9787×10^{-01} (1.6895×10^{-01})	9.0097×10^{-01} (1.3937×10^{-01})	1.1956×10^{-08} (1.4043×10^{-02})	1	341.6717	347.6722	347.9196	355.5176	0.1802	1.0344	0.4630	2.2×10^{-16}
TIIEHL-TL-WL(1, 1, θ , λ)	1	1	1.4105×10^{-09} (3.9364×10^{-03})	7.0003×10^{-02} (6.9093×10^{-03})	771.8714	775.8723	775.9948	781.1026	0.5885	3.1505	0.8669	2.2×10^{-16}
TIIEHL-TL-WL(1, b, θ , 1)	1	9.0000×10^{-01} (8.1773×10^{-02})	1.3713×10^{-08} (1.4051×10^{-02})	1	342.503	346.5031	346.6255	351.7333	0.1801	1.0333	0.4638	2.2×10^{-16}
TIIEHL-TL-WL(a, 1, θ , 1)	4.9999×10^{-01} (4.9759×10^{-02})	1	3.4527×10^{-09} (6.9473×10^{-03})	1	214.2777	218.2774	218.3998	223.5076	0.2330	1.2794	0.2026	0.0005
TIIGIE-BIII	λ	θ	c	k								
	5.1829 (3.4987)	3.3191×10^{03} (3.2371×10^{-04})	5.9244×10^{-02} (9.7386×10^{-03})	1.0274×10^{01} (3.3749×10^{-01})	233.0062	241.0063	241.4229	251.4667	32.0684	197.6112	0.9981	2.2×10^{-16}
TIIEHLW	a	λ	δ	γ								
	0.3523 (0.4424)	0.7289 (0.2131)	1.8270 (3.1301)	1.2123 (0.2386)	206.6925	214.6925	215.1092	225.153	0.1673	0.9811	0.0891	0.3978
TIIGIE-Lx	λ	α	a	b								
	1.7010×10^{02} (7.5375×10^{-07})	1.0186×10^{-03} (2.1156×10^{-04})	9.0000 (1.0787×10^{-04})	9.1000×10^{-01} (2.5260×10^{-01})	221.7345	229.7345	230.1511	240.1949	0.4843	2.6166	0.1706	0.0056

Table 5.12: MLEs and GoF Statistics

from the results shown in Table 5.12, we conclude that the TIIEHL-TL-WL distribution is indeed the “best” model compared to several selected models since it is associated with the lowest values for all the goodness-of-fit statistics (and the largest p-value for the K-S statistic). Also, the plots in Figure 5.12 show that the TIIEHL-TL-WL fitted the data better than the other models of comparison and also has the smallest SS value.

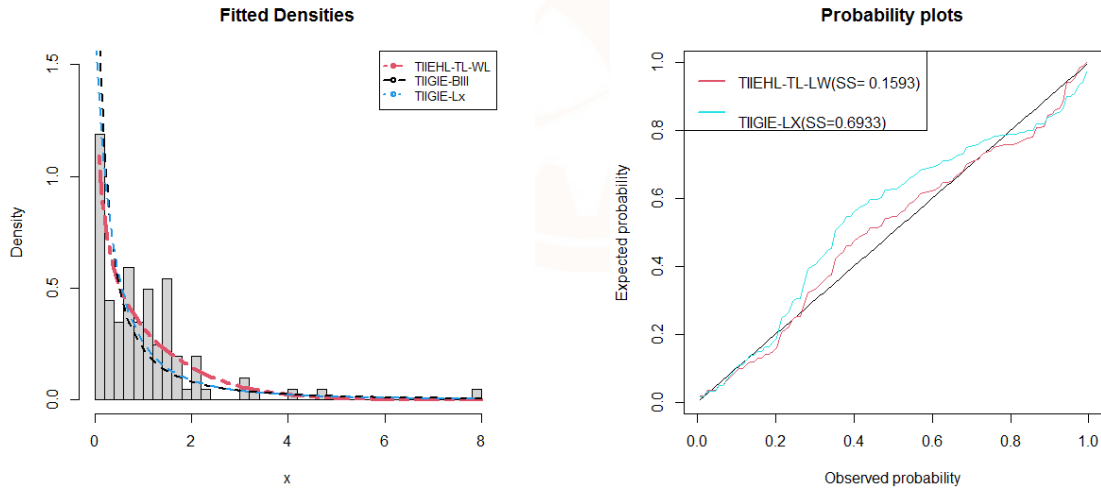


Figure 5.12: Fitted Densities and Probability Plots for kevlar 49/epoxy Data

Dataset 3

The third data set taken from Murthy et al. (2004) represents the failure times of 50 components (per 1000 hours). (The data is given in the appendix).

The estimated variance-covariance matrix for TIIEHL-TL-WL distribution on kevlar 49/epoxy data is given by

$$\begin{bmatrix} 0.0133 & 0.0641 & -0.0537 & -0.0164 \\ 0.0641 & 0.5204 & -0.1024 & -0.0983 \\ -0.0537 & -0.1024 & 0.5187 & 0.0492 \\ -0.0164 & -0.0983 & 0.0492 & 0.0253 \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $a \in [0.2564 \pm 0.2264]$, $b \in [1.5151 \pm 1.4139]$, $\theta \in [3.2136 \times 10^{-03} \pm 1.4116]$ and $\lambda \in [0.6676 \pm 0.3119]$, respectively.

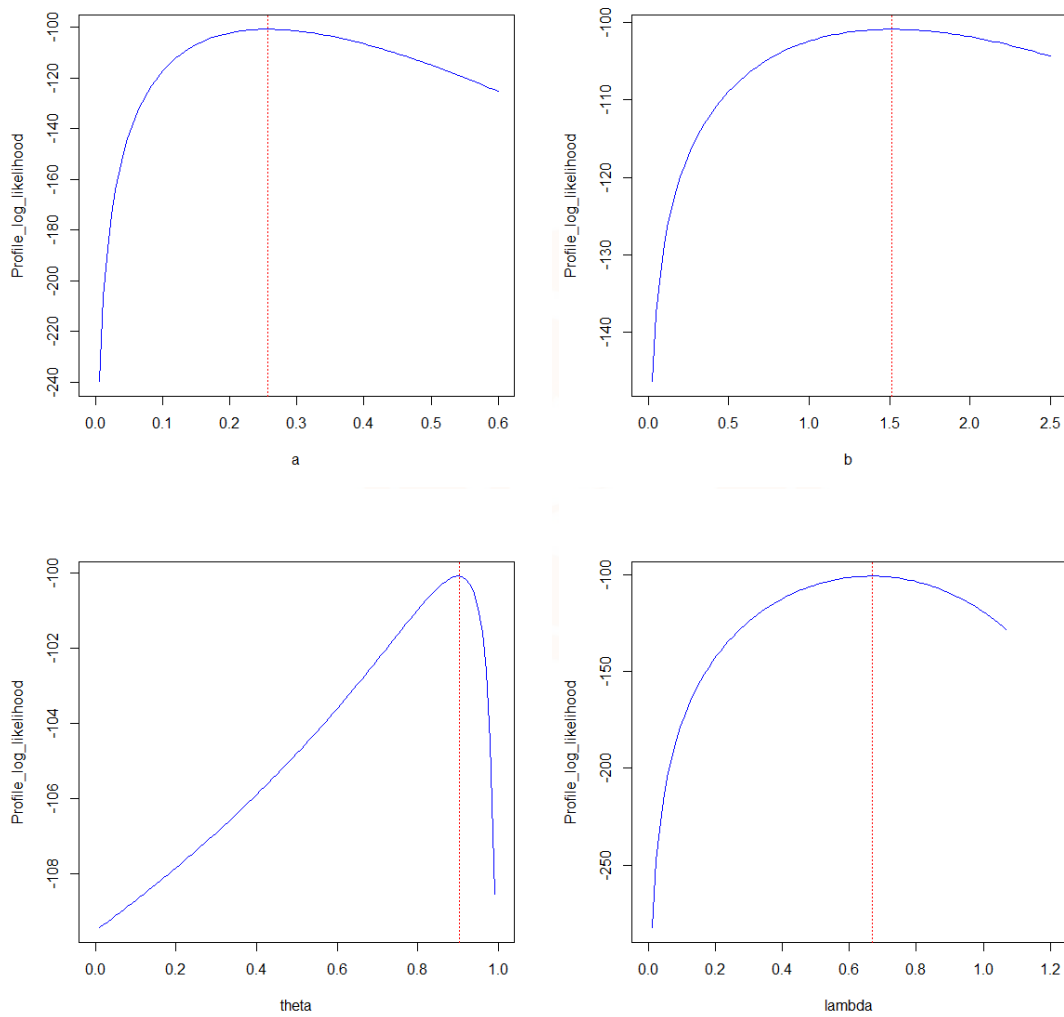


Figure 5.13: Profile log-likelihood function plots for parameters of TIIEHL-TL-WL distribution on the failure times of 50 components data

The MLE of TIIEHL-TL-WL parameters on failure times of 50 components data can be uniquely identified, as shown in Figure 5.13.

Model	Estimates				Statistics							
	a	b	θ	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-TL-WL	0.2564 (0.1155)	1.5151 (0.7213)	3.2136×10^{-03} (0.7202)	0.6676 (0.1591)	200.1913	208.1913	209.0802	215.8394	0.1179	0.7430	0.1157	0.479
TIIEHL-TL-WL(1, b, θ , λ)	1	1.1021 (1.5167×10^{-01})	3.7550×10^{-10} (2.8821×10^{-03})	3.9163×10^{-01} (3.6788×10^{-02})	323.7181	329.7176	330.2394	335.4537	0.1505	0.9358	0.5971	8.882×10^{-16}
TIIEHL-TL-WL(a, 1, θ , λ)	9.4878×10^{-01} (1.4217×10^{-01})	1	3.8253×10^{-09} (9.7940×10^{-03})	2.4777×10^{-01} (3.1658×10^{-02})	340.7781	346.778	347.2997	352.514	0.1606	0.9952	0.7332	8.882×10^{-16}
TIIEHL-TL-WL(a, b, θ , 1)	8.5440×10^{-01} (1.9039×10^{-01})	9.2572×10^{-01} (2.0649×10^{-01})	3.0344×10^{-09} (9.2514×10^{-03})	1 -	546.6781	552.6787	553.2005	558.4148	0.3004	2.3604	0.4657	1.849×10^{-10}
TIIEHL-TL-WL(1, 1, θ , λ)	1 -	1 -	1.2491×10^{-09} (7.2458×10^{-03})	7.0003×10^{-02} (9.7717×10^{-03})	459.5549	463.5551	463.8104	467.3792	0.1758	1.0886	0.8858	8.882×10^{-16}
TIIEHL-TL-WL(1, b, θ , 1)	1 -	2.0371 (0.2601)	0.0001 (0.1831)	1 -	606.0939	610.0939	610.3492	613.9179	0.1176	0.8470	0.4614	2.909×10^{-10}
TIIEHL-TL-WL(a, 1, θ , 1)	4.7096×10^{-01} (6.6607×10^{-02})	1 -	4.9115×10^{-09} (1.0310×10^{-02})	1 -	326.3046	330.307	330.5623	334.131	0.1504	0.9845	0.3885	2.628×10^{-07}
TIIGIE-BIII	λ	θ	c	k								
	1.4492 (2.4554)	1.1742×10^{03} (1.9884×10^{-03})	7.4801×10^{-02} (2.3451×10^{-02})	9.9836 (1.3601)	204.9782	212.9781	213.8679	220.6262	17.9579	99.9618	0.9813	8.882×10^{-16}
TIIEHLW	a	λ	δ	γ								
	1.0704 (7.5006×10^{-04})	1.5713×10^{-01} (1.8898×10^{-02})	8.9786×10^{-05} (4.5116×10^{-05})	3.5666 (8.8732×10^{-04})	201.4852	209.4852	210.3741	217.1333	0.1193	0.8021	0.1360	0.2865
TIIGIE-Lx	λ	α	a	b								
	1.4663×10^{-03} (8.1400×10^{-03})	1.6231×10^{01} (1.2061×10^{-05})	4.0485×10^{-02} (4.8355×10^{-03})	1.2245×10^{-04} (9.8404×10^{-04})	204.6059	212.6059	213.4948	220.254	0.1517	0.9504	0.1252	0.3812

Table 5.13: MLEs and GoF Statistics

Furthermore, from the results shown in Table 5.13, we conclude that the TIIEHL-TL-WL distribution is indeed the “best” model compared to several selected models since it is associated with the lowest values for all the goodness-of-fit statistics (and the largest p-value for the K-S statistic). Also, the plots in Figure 5.14 show that the TIIEHL-TL-WL fitted the data better than the other models of comparison and also has the smallest SS value.

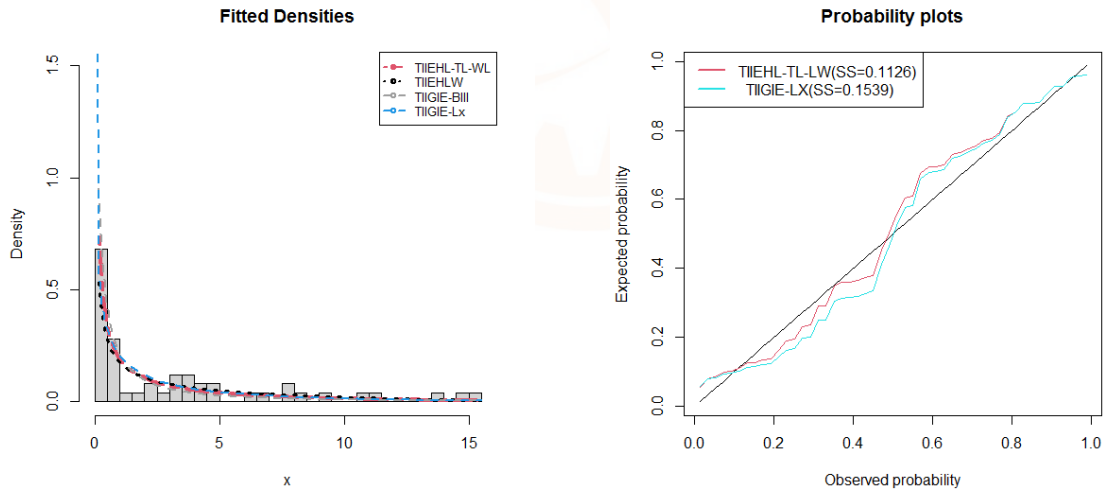


Figure 5.14: Fitted Densities and Probability Plots for Failure Times Data

Likelihood Ratio Test

Model	df	Survival Times (in years)	Time to failure of kevlar 49/epoxy
		$\chi^2(p - value)$	$\chi^2(p - value)$
TIIHL-TL-WL(1, b, θ , λ)	1	82.5525(<0.00001)	17.6272(0.00003)
TIIHL-TL-WL(a, 1, θ , λ)	1	153.6602(<0.00001)	494.3255(<0.00001)
TIIHL-TL-WL(a, b, θ , 1)	1	60.4647(<0.00001)	135.1382(<0.00001)
TIIHL-TL-WL(1, 1, θ , λ)	2	270.6975(<0.00001)	565.3379(<0.00001)
TIIHL-TL-WL(1, b, θ , 1)	2	52.2514(<0.00001)	135.9695(<0.00001)
TIIHL-TL-WL(a, 1, θ , 1)	2	97.1771(<0.00001)	7.7442(0.0208)

Table 5.14: Likelihood Ratio Test Results

Model	df	Failure Times of 50 Components Data Set
		$\chi^2(p - value)$
TIIHL-TL-WL(1, b, θ , λ)	1	123.5628(<0.00001)
TIIHL-TL-WL(a, 1, θ , λ)	1	140.5868(<0.00001)
TIIHL-TL-WL(a, b, θ , 1)	1	346.4868(<0.00001)
TIIHL-TL-WL(1, 1, θ , λ)	2	259.3636(<0.00001)
TIIHL-TL-WL(1, b, θ , 1)	2	405.9026(<0.00001)
TIIHL-TL-WL(a, 1, θ , 1)	2	126.1133(<0.00001)

Table 5.15: Likelihood Ratio Test Results

The likelihood ratio test results in Table 5.14 and 5.15 indicates that the TIIHL-TL-WL performs better than its nested models at 5% level of significance, since all p-values are less than 0.05 among all the data sets considered.

5.1.7 Conclusions

A new CoD called the type II exponentiated half-logistic-Topp-Leone-G power series (TIIHL-TL-GPS) CoD is introduced. Some mathematical properties including moments and moment generating function, order statistics, entropy and quantiles are provided. Model parameters are estimated using the maximum likelihood method and the performance of the estimates is assessed by means of a simulation study. The potentiality of the new model is demonstrated by means of three real data sets.

6. Type I Heavy-Tailed Odd Power Generalized Weibull-G (TI-HT-OPGW-G) Generalized Family of Probability Distributions and Heavy-Tailed Generalized Exponentiated Half Logistic-G (HT-GEN-EHL-G) Generalized Family of Probability Distributions

Moakofi, T. and Oluyede, B., 2023. Type I Heavy-Tailed Odd Power Generalized Weibull-G Family of Distributions with Applications. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 72(4), 921-958.

<https://dergipark.org.tr/en/pub/cfsuasmas/issue/80844/1195058>

This chapter introduces new heavy tailed families of distributions that are obtained via the extensions of the odd Power Generalized Weibull and new Generalized Exponentiated Half Logistic generators.

6.1 TI-HT-OPGW-G GFPD

In this section, we propose a new heavy-tailed distribution, namely, the type I heavy-tailed odd power generalized Weibull-G GFPD. Several statistical properties including hazard function, quantile function, moments, distribution of the order statistics and Rényi entropy are presented. Actuarial measures such as value at risk and tail value at risk are also derived. To obtain the estimates of the parameters of the new GFPD, we adopt the maximum likelihood estimation method and assess the consistency property via a Monte Carlo simulation. Finally, we illustrate the usefulness of the new GFPD by analyzing four real life data sets from different fields such as insurance, engineering, bio-medical and environmental sciences.

6.1.1 The New Model

Moakofi et al. (2021c) introduced the odd power generalized Weibull-G GFPD with the cdf and pdf given as

$$F(x; \alpha, \beta, \zeta) = 1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right), \quad (6.1)$$

and

$$\begin{aligned} f(x; \alpha, \beta, \zeta) &= \alpha \beta \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^{\beta-1} \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^{\alpha-1} \\ &\times \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right) \frac{g_B(x; \zeta)}{(1 - G_B(x; \zeta))^2}, \end{aligned} \quad (6.2)$$

respectively, for $\alpha, \beta > 0$ being shape parameters and parameter vector ζ .

The aim of this section is to develop a new family of heavy-tailed distributions namely, type I heavy-tailed odd power generalized Weibull-G (TI-HT-OPGW-G) family of distributions by combining equations (1.23), (1.24), (6.1) and (6.2).

The cdf and pdf of the TI-HT-OPGW-G GFPD are

$$F(x; \theta, \alpha, \beta, \zeta) = 1 - \left(\frac{\exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right)}{1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right) \right]} \right)^\theta, \quad (6.3)$$

and

$$\begin{aligned} f(x; \theta, \alpha, \beta, \zeta) &= \theta^2 \alpha \beta \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^{\beta-1} \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^{\alpha-1} \\ &\times \exp \left(\theta \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right) \right) \frac{g_B(x; \zeta)}{(1 - G_B(x; \zeta))^2} \\ &\times \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right) \right] \right)^{-(\theta+1)}, \end{aligned} \quad (6.4)$$

respectively, for $\theta, \alpha, \beta > 0$ being shape and parameter vector ζ . Note that $\bar{\theta} = 1 - \theta$.

6.1.2 Reliability and Hazard Rate Functions

The survival function, and hrf of the TI-HT-OPGW-G GFPD are given respectively by

$$S(x; \theta, \alpha, \beta, \zeta) = \left(\frac{\exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right)}{1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right) \right]} \right)^\theta, \quad (6.5)$$

and

$$\begin{aligned} h(x; \theta, \alpha, \beta, \zeta) &= \theta^2 \alpha \beta \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^{\beta-1} \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^{\alpha-1} \\ &\times \exp \left(\theta \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right) \right) \frac{g_B(x; \zeta)}{(1 - G_B(x; \zeta))^2} \\ &\times \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right) \right] \right)^{-(\theta+1)} \\ &\times \left[\left(\frac{\exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right)}{1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right) \right]} \right)^\theta \right]^{-1}, \end{aligned} \quad (6.6)$$

for $\alpha, \beta, \theta > 0$, $\bar{\theta} = 1 - \theta$, and parameter vector ζ .

6.1.3 Sub-Families of TI-HT-OPGW-G GFPD

- When $\theta = 1$, we obtain the odd power generalized Weibull-G (OPGW-G) GFPD ([Moakofi et al. \(2021c\)](#)) with the cdf

$$F(x; \alpha, \beta, \zeta) = 1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right]^\beta \right),$$

for $\alpha, \beta > 0$ and parameter vector ζ .

- When $\beta = 1$, we obtain the new type I heavy-tailed Weibull-G (TI-HT-W-G) GFPD with the cdf

$$F(x; \theta, \alpha, \zeta) = 1 - \left(\frac{\exp \left(- \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right)}{1 - \bar{\theta} \left[1 - \exp \left(- \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right) \right]} \right)^\theta,$$

for $\theta, \alpha > 0$ and parameter vector ζ . This is a new GFPD.

- When $\alpha = 1$, we obtain the new type I heavy-tailed odd Nadarajah Haghghi-G (TI-

HT-ONH-G) GFPD with the cdf

$$F(x; \theta, \beta, \zeta) = 1 - \left(\frac{\exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\beta \right] \right)}{1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\beta \right] \right) \right]} \right)^\theta,$$

for $\theta, \beta > 0$, $\bar{\theta} = (1 - \theta)$, and parameter vector ζ . This is a new GFD.

- When $\theta = \beta = 1$, we obtain the Weibull-G (W-G) GFD (Bourguignon et al. (2014)) with the cdf

$$F(x; \alpha, \zeta) = 1 - \exp \left(- \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\alpha \right),$$

for $\alpha > 0$, and parameter vector ζ .

- If $\theta = \alpha = 1$, we obtain the odd Nadarajah Haghighi-G (ONH-G) (Nascimento et al. (2019)) GFD with the cdf

$$F(x; \beta, \zeta) = 1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^\beta \right] \right),$$

for $\beta > 0$, and parameter vector ζ .

- If $\theta = \alpha = \beta = 1$, we obtain the odd exponential-G (OE-G) GFD with the cdf

$$F(x; \zeta) = 1 - \exp \left(- \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right) \right),$$

for parameter vector ζ . This is a new GFD.

- If $\theta = \beta = 1, \alpha = 2$ we obtain the odd Rayleigh-G (OR-G) GFD with the cdf

$$F(x; \zeta) = 1 - \exp \left(- \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^2 \right),$$

for parameter vector ζ . This is a new GFD.

6.1.4 Linear Representation

Here, in this sub-section, we express the pdf of TI-HT-OPGW-G GFD as an infinite linear combination of exponentiated-G (Exp-G) GFD. Using the following generalized binomial and Taylor series expansions

$$(1 + z)^{-a} = \sum_{k=0}^{\infty} (-1)^k \binom{a + k - 1}{k} z^k \quad \text{for } |z| < 1, \quad \text{and} \quad e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!},$$

respectively, we can write

$$f(x; \theta, \alpha, \beta, \zeta) = \sum_{p=0}^{\infty} \varphi_{p+1} g_{p+1}(x; \zeta), \quad (6.7)$$

where $g_{p+1}(x; \zeta) = (p+1)[G_B(x; \zeta)]^p g_B(x; \zeta)$ is the Exp-G pdf with the power parameter $(p+1)$ and parameter vector ζ , and

$$\begin{aligned} \varphi_{p+1} &= \theta^2 \alpha \beta \sum_{h,i,j,k,l,m=0}^{\infty} \bar{\theta}^h \binom{\theta+h}{h} \binom{h}{i} \binom{j}{k} \binom{\beta(k+1)-1}{l} \binom{\alpha(l+1)-1}{m} \\ &\times \binom{\alpha(l+1)-m+p}{p} \frac{(\theta+i)^j (-1)^{i+k+m}}{j! (p+1)}. \end{aligned} \quad (6.8)$$

Consequently, the mathematical and statistical properties of the TI-HT-OPGW-G GFPD follow directly from those of the Exp-G GFPD. (See the Appendix for derivations).

6.1.5 Quantile Function

The quantile function(qf) is used in Monte Carlo simulations to generate random numbers for a specified probability distribution. It is obtained by inverting the cdf of a distribution. If the random variable X is from the TI-HT-OPGW-G GFPD, then the qf of X can be obtained as follows:

$$F(x; \theta, \alpha, \beta, \zeta) = 1 - \left(\frac{\exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^{\alpha} \right]^{\beta} \right)}{1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^{\alpha} \right]^{\beta} \right) \right]} \right)^{\theta} = u, \quad (6.9)$$

for $0 \leq u \leq 1$, that is,

$$G_B(x; \zeta) = \left(\left[\left(1 - \log \left(\theta \left[(1-u)^{\frac{-1}{\theta}} - \bar{\theta} \right]^{-1} \right) \right)^{\frac{1}{\beta}} - 1 \right]^{\frac{-1}{\alpha}} + 1 \right)^{-1}. \quad (6.10)$$

Therefore, the qf of the TI-HT-OPGW-G GFPD is given by

$$Q_X(u) = G^{-1} \left[\left(\left[\left(1 - \log \left(\theta \left[(1-u)^{\frac{-1}{\theta}} - \bar{\theta} \right]^{-1} \right) \right)^{\frac{1}{\beta}} - 1 \right]^{\frac{-1}{\alpha}} + 1 \right)^{-1} \right]. \quad (6.11)$$

Consequently, variates of the TI-HT-OPGW-G GFPD can be obtained using equation (6.11) for specified baseline cdf G .

6.1.6 Statistical Properties

In this section, we derived some statistical features of the TI-HT-OPGW-G GFPD, specifically the moments, moment generating function, distribution of order statistics and Rényi entropy. Let the pdf of the TI-HT-OPGW-G GFPD be denoted by $f(x)$.

6.1.7 Moments and Generating Function

Let $Y_{p+1} \sim \text{Exponentiated} - G(p+1, \zeta)$, then the n^{th} raw moment, μ'_n of the TI-HT-OPGW-G GFPD is given by

$$\mu'_n = E(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx = \sum_{p=0}^{\infty} \varphi_{p+1} E(Y_{p+1}^n),$$

where $E(Y_{p+1}^n)$ is the n^{th} moment of Y_{p+1} and φ_{p+1} is given by equation (6.8). The moment generating function (MGF), for $|t| < 1$, is given by:

$$M_X(t) = \sum_{p=0}^{\infty} \varphi_{p+1} M_{p+1}(t),$$

where $M_{p+1}(t)$ is the mgf of Y_{p+1} and φ_{p+1} is given by equation (6.8).

6.1.8 Order Statistics

Order statistics have many applications in survival, reliability, failure analysis, and it is a natural way to perform a reliability analysis of a system. Suppose $X_1, X_2, \dots, X_n$ are independent and identically distributed (i.i.d) random variables from the TI-HT-OPGW-G distribution. The pdf of the r^{th} order statistic from the TI-HT-OPGW-G pdf $f(x)$ can be written as

$$f_{r:n}(x) = \frac{n!f(x)}{(r-1)!(n-r)!} \sum_{z=0}^{n-r} (-1)^z \binom{n-r}{z} [F(x)]^{z+r-1}. \quad (6.12)$$

Using equations (6.3) and (6.4), we have

$$\begin{aligned} f(x)[F(x)]^{z+r-1} &= \theta^2 \alpha \beta \sum_{q=0}^{\infty} \binom{z+r-1}{q} (-1)^q \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^{\alpha} \right]^{\beta-1} \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^{\alpha-1} \\ &\times \exp \left(\theta(q+1) \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^{\alpha} \right]^{\beta} \right) \right) \frac{g_B(x; \zeta)}{(1 - G_B(x; \zeta))^2} \\ &\times \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x; \zeta)}{1 - G_B(x; \zeta)} \right)^{\alpha} \right]^{\beta} \right) \right] \right)^{-(\theta(q+1)+1)}. \end{aligned}$$

Now following the same steps leading to equation (6.7), we obtain

$$f(x)[F(x)]^{z+r-1} = \sum_{p=0}^{\infty} \rho_{p+1} g_{p+1}(x; \zeta), \quad (6.13)$$

where $g_{p+1}(x; \zeta) = (p+1)[G_B(x; \zeta)]^p g_B(x; \zeta)$ is the Exp-G pdf with the power parameter $(p+1)$ and parameter vector ζ , and

$$\begin{aligned} \rho_{p+1} &= \theta^2 \alpha \beta \sum_{h,i,j,k,l,m,q=0}^{\infty} \bar{\theta}^h \binom{z+r-1}{q} \binom{\theta(q+1)+h}{h} \binom{h}{i} \binom{j}{k} \binom{\beta(k+1)-1}{l} \\ &\times \binom{\alpha(l+1)-1}{m} \binom{\alpha(l+1)-m+p}{p} \frac{(\theta(q+1)+i)^j (-1)^{q+i+k+m}}{j! (p+1)}. \end{aligned}$$

Thus, by substituting (6.13) into (6.12), the pdf of the r^{th} order statistic from the TI-HT-OPGW-G GFPD can be written as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{p=0}^{\infty} \sum_{z=0}^{n-r} (-1)^z \binom{n-r}{z} \rho_{p+1} g_{p+1}(x; \zeta).$$

6.1.9 Rényi Entropy

Rényi entropy is a very important tool in information theory, as a measure of randomness or uncertainty in the system. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^{\infty} [f(x; \theta, \alpha, \beta, \zeta)]^v dx \right), v \neq 1, v > 0. \quad (6.14)$$

Rényi entropy for the TI-HT-OPGW-G GFPD is given by

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left[\sum_{h,i,j,k,l,m,p=0}^{\infty} \bar{\theta}^h \binom{v(\theta+1)+h-1}{h} \binom{h}{i} \frac{(v\theta+i)^j (-1)^{i+k+m}}{j!} \binom{j}{k} \right. \\ &\times \binom{\beta(k+v)-v}{l} \binom{\alpha(l+v)-v}{m} \binom{\alpha(l+v)+v-m+p-1}{p} \frac{(\theta^2 \alpha \beta)^v}{[1+\frac{p}{v}]^v} \\ &\times \left. \int_0^{\infty} \left(\left[1 + \frac{p}{v} \right] (G_B(x; \zeta))^{\frac{p}{v}} (g_B(x; \zeta)) \right)^v dx \right] \\ &= \frac{1}{1-v} \log \left[\sum_{p=0}^{\infty} \tau_p \exp((1-v)I_{REG}) \right], \end{aligned} \quad (6.15)$$

for $v > 0, v \neq 1$, where $I_{REG} = \frac{1}{1-v} \log \left[\int_0^{\infty} \left(\left[1 + \frac{p}{v} \right] (G_B(x; \zeta))^{\frac{p}{v}} (g_B(x; \zeta)) \right)^v dx \right]$ is the Rényi entropy of Exp-G distribution with power parameter $(\frac{p}{v} + 1)$ and

$$\begin{aligned} \tau_p &= \sum_{h,i,j,k,l,m=0}^{\infty} \bar{\theta}^h \binom{v(\theta+1)+h-1}{h} \binom{h}{i} \frac{(v\theta+i)^j (-1)^{i+k+m}}{j!} \binom{j}{k} \\ &\times \binom{\beta(k+v)-v}{l} \binom{\alpha(l+v)-v}{m} \binom{\alpha(l+v)+v-m+p-1}{p} \frac{(\theta^2 \alpha \beta)^v}{[1 + \frac{p}{v}]^v}. \end{aligned}$$

Therefore, Rényi entropy of the TI-HT-OPGW-G GFPD can be obtained from those of the Exp-G GFPD. (**See the Appendix for derivations**).

6.1.10 Moment of Residual and Reversed Residual Life

Moments of the residual life distribution are used to obtain the mean, variance and coefficient of variation of residual life which are extensively used in reliability analysis.

The s^{th} moment of the residual life, say $\kappa_s(t)$ of a random variable X is

$$\kappa_s(t) = E[(X-t)^s | X > t] = \frac{1}{\bar{F}(t)} \int_t^{\infty} (x-t)^s f(x) dx.$$

Consequently, $\kappa_s(t)$ for the TI-HT-OPGW-G GFPD is given as follows:

$$\kappa_s(t) = \frac{1}{\bar{F}(t)} \sum_{p,u=0}^{\infty} \binom{s}{u} (-t)^{s-u} \varphi_{p+1} \int_t^{\infty} x^u g_{p+1}(x; \zeta) dx, \quad (6.16)$$

where $\bar{F}(t)$ is the survival function of the TI-HT-OPGW-G GFPD, φ_{p+1} is as defined in equation (6.8) and $g_{p+1}(x; \zeta)$ denotes the Exp-G distribution with power parameter $(p+1)$. The mean excess function of the TI-HT-OPGW-G GFPD is obtained from the above formula with $s = 1$. The s^{th} moment of the reversed residual life, say $\vartheta_s(t)$ of a random variable X is

$$\vartheta_s(t) = E[(t-X)^s | X \leq t] = \frac{1}{F(t)} \int_0^t (t-x)^s f(x) dx.$$

Subsequently, $\vartheta_s(t)$ for the TI-HT-OPGW-G GFPD is given as follows:

$$\vartheta_s(t) = \frac{1}{F(t)} \sum_{p,u=0}^{\infty} \binom{s}{u} (-t)^{s-u} \varphi_{p+1} \int_0^t x^u g_{p+1}(x; \zeta) dx,$$

where $F(t)$ is the cdf of the TI-HT-OPGW-G GFPD, φ_{p+1} is as defined in equation (6.8) and $g_{p+1}(x; \zeta)$ denotes the Exp-G distribution with power parameter $(p+1)$. The mean inactivity

time of the TI-HT-OPGW-G GFPD is obtained from the above formula with $s = 1$.

6.1.11 Special Cases

In this section, we present special cases of the TI-HT-OPGW-G GFPD, when the baseline distributions are log-logistic, Weibull, Rayleigh and standard half logistic distributions.

6.1.12 TI-HT-OPGW-Log-Logistic (TI-HT-OPGW-LLoG) Distribution

Given the cdf and pdf of the log-logistic distribution as $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$ for $c > 0$ and $x > 0$, we define the cdf and pdf of the TI-HT-OPGW-LLoG distribution as follows

$$F(x; \theta, \alpha, \beta, c) = 1 - \left(\frac{\exp \left(1 - \left[1 + \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right)^\alpha \right]^\beta \right)}{1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right)^\alpha \right]^\beta \right) \right]} \right)^\theta,$$

and

$$\begin{aligned} f(x; \theta, \alpha, \beta, c) &= \theta^2 \alpha \beta \left[1 + \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right)^\alpha \right]^{\beta-1} \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right)^{\alpha-1} \\ &\times \exp \left(\theta \left(1 - \left[1 + \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right)^\alpha \right]^\beta \right) \right) \frac{cx^{c-1}(1 + x^c)^{-2}}{((1 + x^c)^{-1})^2} \\ &\times \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right)^\alpha \right]^\beta \right) \right] \right)^{-(\theta+1)}, \end{aligned}$$

respectively, for $\theta, \alpha, \beta, c > 0$, being shape parameters and $x > 0$, where $\bar{\theta} = 1 - \theta$. The hrf for the TI-HT-OPGW-LLoG distribution is given by

$$\begin{aligned} h(x; \theta, \alpha, \beta, c) &= \theta^2 \alpha \beta \left[1 + \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right)^\alpha \right]^{\beta-1} (1 - (1 + x^c)^{-1})^{\alpha-1} \\ &\times \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right)^\alpha \right]^\beta \right) \right] \right)^{-1} \\ &\times \frac{cx^{c-1}(1 + x^c)^{-2}}{((1 + x^c)^{-1})^{\alpha+1}}, \end{aligned}$$

for $\theta, \alpha, \beta, c > 0$, and $\bar{\theta} = 1 - \theta$.

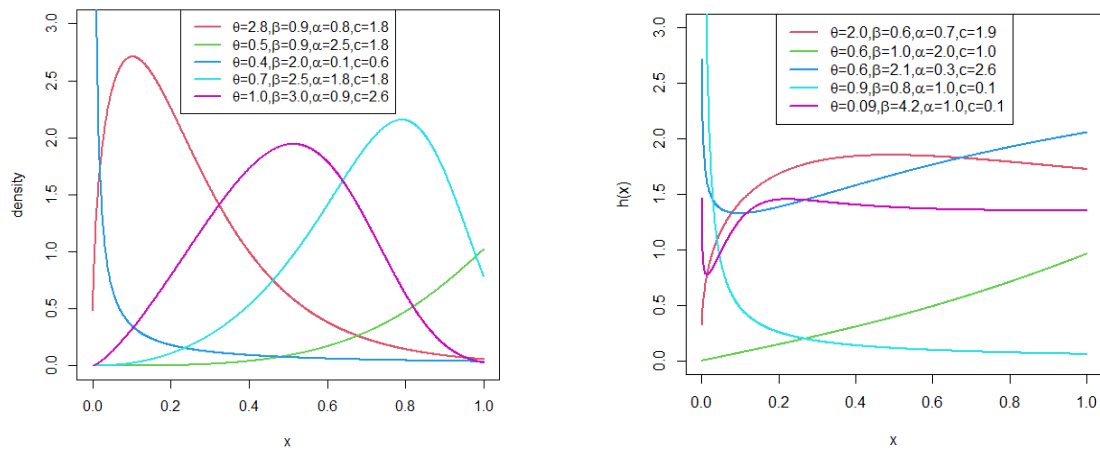


Figure 6.1: Pdf and hrf Plots for TI-HT-OPGW-LLoG Distribution

Figure 6.1 shows the plots of pdf and hazard functions of TI-HT-OPGW-LLoG distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, uni-modal, J and reverse-J shapes. The TI-HT-OPGW-LLoG hrf displays increasing, decreasing, bathtub, upside-down bathtub and bathtub followed by upside-down bathtub shapes.

	$(\theta, \beta, \alpha, c)$				
u	(4.3, 0.9, 3.4, 0.8)	(0.1, 0.5, 6.4, 1)	(13.7, 1.5, 4.6, 0.9)	(0.7, 0.3, 6.3, 4.8)	(12.2, 5.8, 9.5, 0.3)
0.1	0.1538	1.5243	0.1483	1.0000	0.0403
0.2	0.2063	1.6876	0.1777	1.0286	0.0569
0.3	0.2485	1.8180	0.1976	1.0528	0.0628
0.4	0.2838	1.9408	0.2195	1.0708	0.0739
0.5	0.3183	2.0658	0.2353	1.0875	0.0809
0.6	0.3575	2.1976	0.2515	1.1099	0.0920
0.7	0.3964	2.3450	0.2726	1.1282	0.1027
0.8	0.4489	2.5239	0.2917	1.1502	0.1113
0.9	0.5197	2.7808	0.3175	1.1827	0.1272

Table 6.1: Table of Quantile for TI-HT-OPGW-LLoG Distribution

	$(\theta, \beta, \alpha, c)$				
	(0.5,1.5,0.3,1.7)	(0.7,0.5,0.3,0.8)	(1.8,0.3,0.6,0.9)	(1.7,0.1,2.9,1)	(2.0,0.8,3.0,0.1)
$E(X)$	0.0481	0.0344	0.1587	0.1357	0.1367
$E(X^2)$	0.0240	0.0186	0.0889	0.1033	0.0676
$E(X^3)$	0.0155	0.0127	0.0614	0.0830	0.0446
$E(X^4)$	0.0113	0.0097	0.0468	0.0692	0.0333
$E(X^5)$	0.0088	0.0078	0.0378	0.0592	0.0265
$E(X^6)$	0.0072	0.0065	0.0317	0.0516	0.0220
SD	0.1473	0.1321	0.2524	0.2914	0.2212
CV	3.0585	3.8384	1.5897	2.1478	1.6176
CS	3.8356	4.7373	1.6847	1.8559	2.0379
CK	18.3858	26.5074	4.7783	4.7821	6.4438

Table 6.2: Moments of TI-HT-OPGW-LLoG Distribution for Selected Parameters

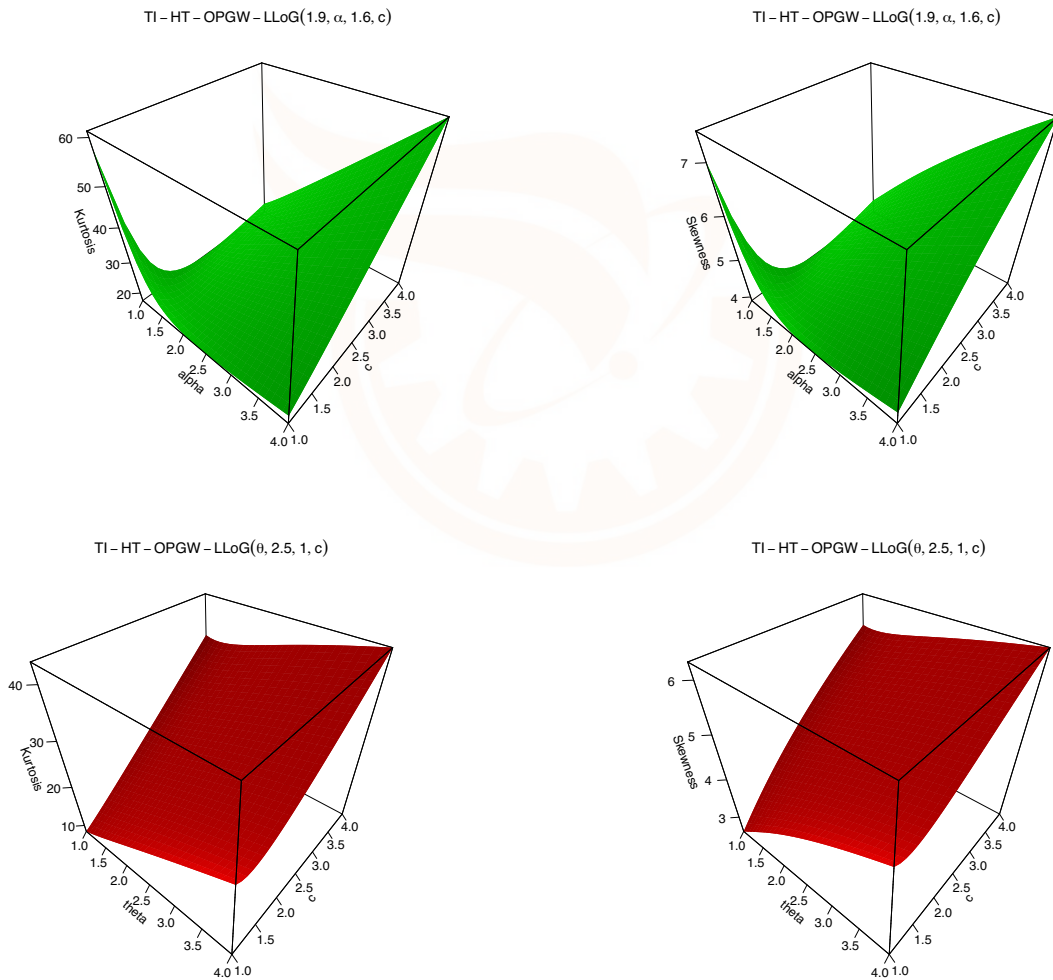


Figure 6.2: 3D Plots of skewness and kurtosis for the TI-HT-OPGW-LLoG distribution

Figure 6.2 shows the 3D plots of skewness and kurtosis of the TI-HT-OPGW-LLoG distribution. We observe that

- Fixing the parameters θ and β , the skewness and kurtosis of the TI-HT-OPGW-LLoG distribution changes from decreasing to increasing as α and c increases.
- For fixed values of the parameters α and β , the skewness and kurtosis of the TI-HT-OPGW-LLoG distribution increases as θ and c increases.

6.1.13 TI-HT-OPGW-Weibull (TI-HT-OPGW-W) Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x; \lambda) = 1 - e^{-x^\lambda}$, and $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$, $x > 0, \lambda > 0$, then, the cdf and pdf of the TI-HT-OPGW-W distribution are given by

$$F(x; \theta, \alpha, \beta, \lambda) = 1 - \left(\frac{\exp \left(1 - \left[1 + \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^\alpha \right]^\beta \right)}{1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^\alpha \right]^\beta \right) \right]} \right)^\theta,$$

and

$$\begin{aligned} f(x; \theta, \alpha, \beta, \lambda) &= \theta^2 \alpha \beta \left[1 + \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^\alpha \right]^{\beta-1} \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^{\alpha-1} \\ &\times \exp \left(\theta \left(1 - \left[1 + \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^\alpha \right]^\beta \right) \right) \frac{\lambda x^{\lambda-1} e^{-x^\lambda}}{(e^{-x^\lambda})^2} \\ &\times \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^\alpha \right]^\beta \right) \right] \right)^{-(\theta+1)}, \end{aligned}$$

respectively, for $\theta, \alpha, \beta, \lambda > 0$, being shape parameters and $x > 0$, where $\bar{\theta} = 1 - \theta$. The hrf for TI-HT-OPGW-W distribution is given by

$$\begin{aligned} h(x; \theta, \alpha, \beta, \lambda) &= \theta^2 \alpha \beta \left[1 + \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^\alpha \right]^{\beta-1} (1 - e^{-x^\lambda})^{\alpha-1} \\ &\times \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^\alpha \right]^\beta \right) \right] \right)^{-1} \frac{\lambda x^{\lambda-1} e^{-x^\lambda}}{(e^{-x^\lambda})^{\alpha+1}}, \end{aligned}$$

for $\theta, \alpha, \beta, \lambda > 0$, and $\bar{\theta} = 1 - \theta$.

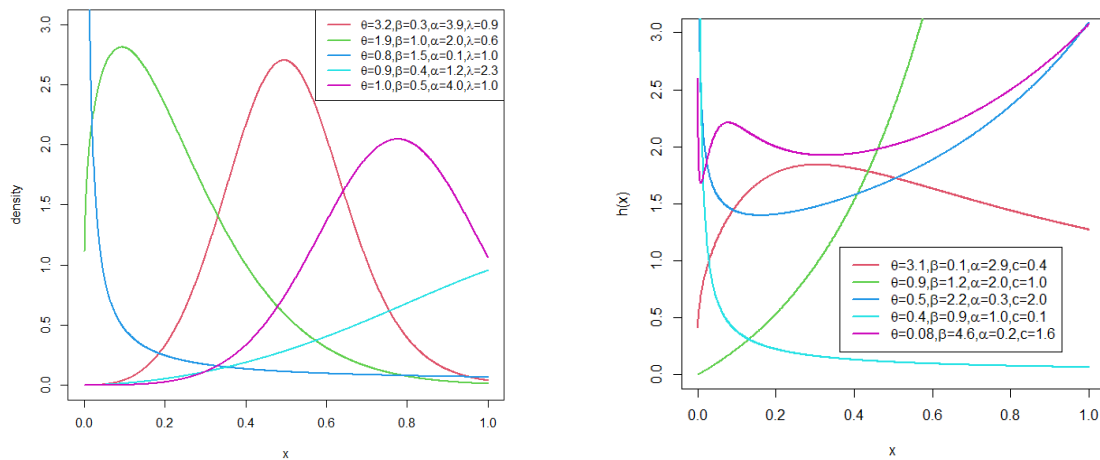


Figure 6.3: Pdf and hrf Plots for TI-HT-OPGW-W Distribution

Figure 6.3 shows the plots of pdf and hrf of TI-HT-OPGW-W distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, unimodal, J and reverse-J shapes. The TI-HT-OPGW-W hrf displays increasing, decreasing, bathtub, upside-down bathtub and bathtub followed by upside-down bathtub shapes.

	$(\theta, \beta, \alpha, \lambda)$				
u	(9.3, 0.09, 13.4, 0.9)	(10.1, 7.5, 0.8, 10)	(3.7, 18.5, 5.6, 0.3)	(0.7, 6.3, 0.3, 14.8)	(12.2, 0.8, 4.5, 2.3)
0.1	0.5084	0.3278	0.0050	0.4569	0.4887
0.2	0.5351	0.3612	0.0084	0.5264	0.5190
0.3	0.5527	0.3837	0.0106	0.5709	0.5404
0.4	0.5672	0.4000	0.0141	0.6092	0.5590
0.5	0.5767	0.4156	0.0206	0.6360	0.5730
0.6	0.5880	0.4344	0.0237	0.6630	0.5885
0.7	0.6001	0.4497	0.0269	0.6894	0.6014
0.8	0.6086	0.4673	0.0331	0.7129	0.6193
0.9	0.6283	0.4921	0.0357	0.7491	0.6381

Table 6.3: Table of Quantile for TI-HT-OPGW-W Distribution

	$(\theta, \beta, \alpha, \lambda)$				
	(8.5,1.5,1.3,2.7)	(6.2,0.04,7.3,0.5)	(11.8,0.01,16.6,0.9)	(0.1,9.7,0.9,2)	(2.0,0.09,2.0,3.1)
$E(X)$	0.2397	0.4427	0.6530	0.2669	0.3297
$E(X^2)$	0.0635	0.2178	0.4295	0.0974	0.2837
$E(X^3)$	0.0182	0.1171	0.2844	0.0438	0.2481
$E(X^4)$	0.0055	0.0684	0.1897	0.0230	0.2199
$E(X^5)$	0.0018	0.0431	0.1274	0.0136	0.1970
$E(X^6)$	0.0006	0.0291	0.0861	0.0088	0.1781
SD	0.0781	0.1478	0.0549	0.1617	0.4183
CV	0.3259	0.3339	0.0840	0.6058	1.2688
CS	0.1351	0.4255	0.0170	0.9182	0.5351
CK	2.8414	3.9806	6.4853	3.8659	1.3792

Table 6.4: Moments of TI-HT-OPGW-W Distribution for Selected Parameters

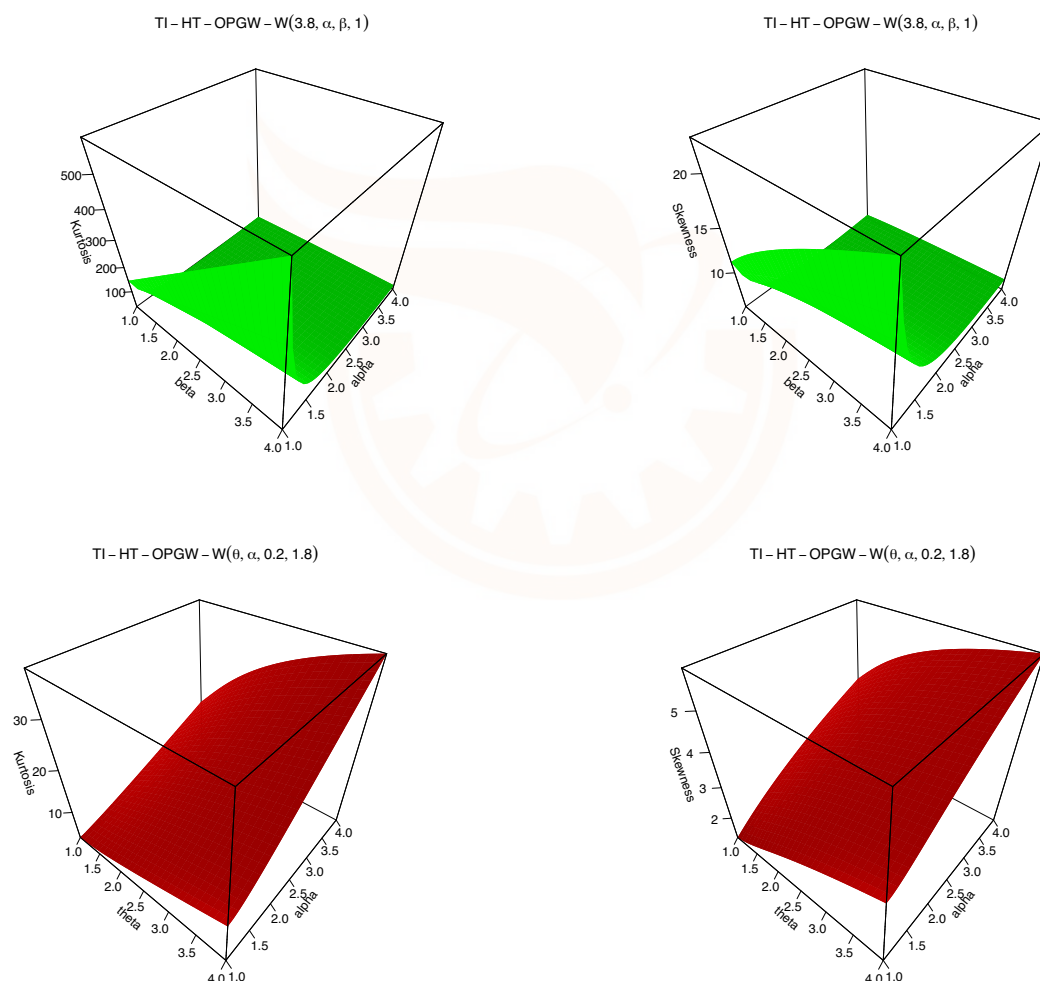


Figure 6.4: 3D Plots of skewness and kurtosis for the TI-HT-OPGW-W distribution

Figure 6.4 shows the 3D plots of skewness and kurtosis of the TI-HT-OPGW-W distribution. We observe that

- For fixed values of the parameters θ and λ , the skewness and kurtosis of the TI-HT-OPGW-W distribution decrease as α and β increases.
- For fixed values of the parameters β and λ , the skewness and kurtosis of the TI-HT-OPGW-W distribution increases as θ and α increases.

6.1.14 TI-HT-OPGW-Rayleigh (TI-HT-OPGW-R) Distribution

If the baseline cdf and pdf are given by $G(x; \lambda) = 1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)$ and $g(x; \lambda) = \frac{x}{\lambda^2} \exp\left(-\frac{x^2}{2\lambda^2}\right)$ for $x > 0$, and $\lambda > 0$, then, the cdf and pdf of the TI-HT-OPGW-R distribution are given by

$$F(x; \theta, \alpha, \beta, \lambda) = 1 - \left(\frac{\exp\left(1 - \left[1 + \left(\frac{1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^\alpha\right]^\beta\right)}{1 - \bar{\theta} \left[1 - \exp\left(1 - \left[1 + \left(\frac{1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^\alpha\right]^\beta\right)\right]} \right)^\theta,$$

and

$$\begin{aligned} f(x; \theta, \alpha, \beta, \lambda) &= \theta^2 \alpha \beta \left[1 + \left(\frac{1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^\alpha\right]^{\beta-1} \left(\frac{1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^{\alpha-1} \\ &\times \exp\left(\theta \left(1 - \left[1 + \left(\frac{1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^\alpha\right]^\beta\right)\right) \frac{\frac{x}{\lambda^2} \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\left(\exp\left(-\frac{x^2}{2\lambda^2}\right)\right)^2} \\ &\times \left(1 - \bar{\theta} \left[1 - \exp\left(1 - \left[1 + \left(\frac{1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^\alpha\right]^\beta\right)\right]\right)^{-(\theta+1)}, \end{aligned}$$

respectively, for $\theta, \alpha, \beta > 0$ being shape parameters, $\lambda > 0$ being a scale parameters and $x > 0$, where $\bar{\theta} = 1 - \theta$. The hrf for TI-HT-OPGW-R distribution is given by

$$\begin{aligned} h(x; \theta, \alpha, \beta, \lambda) &= \theta^2 \alpha \beta \left[1 + \left(\frac{1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^\alpha\right]^{\beta-1} \left(1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)\right)^{\alpha-1} \\ &\times \left(1 - \bar{\theta} \left[1 - \exp\left(1 - \left[1 + \left(\frac{1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^\alpha\right]^\beta\right)\right]\right)^{-1} \\ &\times \frac{\frac{x}{\lambda^2} \exp\left(-\frac{x^2}{2\lambda^2}\right)}{\left(\exp\left(-\frac{x^2}{2\lambda^2}\right)\right)^{\alpha+1}}, \end{aligned}$$

for $\theta, \alpha, \beta, \lambda > 0$, and $\bar{\theta} = 1 - \theta$.

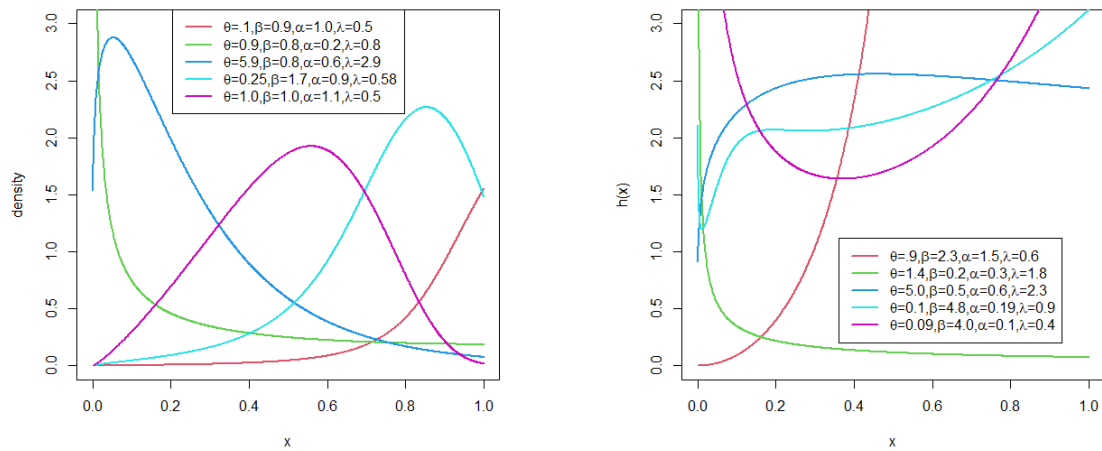


Figure 6.5: Pdf and hrf Plots for TI-HT-OPGW-R Distribution

Figure 6.5 shows the plots of pdf and hrf of TI-HT-OPGW-R distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, unimodal, J and reverse-J shapes. The TI-HT-OPGW-R hrf displays increasing, decreasing, bathtub and upside-down bathtub shapes.

u	$(\theta, \beta, \alpha, \lambda)$				
	(0.03, 0.9, 1.4, 4.9)	(1.1, 0.05, 9.8, 16)	(0.07, 10.5, 15.6, 2.3)	(11.7, 3.3, 6.3, 4.8)	(2.2, 5.8, 41.5, 0.3)
0.1	9.2014	19.8672	2.5932	3.2932	0.3361
0.2	9.9412	21.2309	2.6070	3.4724	0.3401
0.3	10.4954	22.5673	2.6143	3.5860	0.3369
0.4	10.9524	23.9820	2.6213	3.6795	0.3400
0.5	11.3572	25.5128	2.6296	3.7602	0.3437
0.6	11.7344	27.2064	2.6351	3.8316	0.3450
0.7	12.1073	29.1392	2.6386	3.9063	0.3450
0.8	12.5064	31.4741	2.6448	3.9898	0.3440
0.9	12.9940	34.6819	2.6524	4.0933	0.3490

Table 6.5: Table of Quantile for TI-HT-OPGW-R Distribution

	$(\theta, \beta, \alpha, \lambda)$				
	(10.5,0.05,0.3,12.7)	(1.2,0.4,0.03,5.5)	(20.8,1.1,1.6,6.9)	(0.7,0.01,0.9,0.09)	(0.9,3,1,1.1)
$E(X)$	0.1658	0.0130	0.2067	0.2271	0.4414
$E(X^2)$	0.0922	0.0065	0.1641	0.1739	0.2547
$E(X^3)$	0.0632	0.0044	0.1357	0.1410	0.1675
$E(X^4)$	0.0479	0.0033	0.1156	0.1187	0.1202
$E(X^5)$	0.0386	0.0026	0.1005	0.1025	0.0916
$E(X^6)$	0.0322	0.0022	0.0889	0.0902	0.0731
SD	0.2543	0.0801	0.3483	0.3497	0.2445
CV	1.5341	6.1559	1.6851	1.5398	0.5538
CS	1.6095	8.0842	1.2216	1.0760	0.1561
CK	4.5327	75.0272	2.7121	2.4351	2.3114

Table 6.6: Moments of TI-HT-OPGW-R Distribution for Selected Parameters

Figure 6.6 shows the 3D plots of skewness and kurtosis of the TI-HT-OPGW-R distribution. We observe that

- For fixed values of the parameters θ and β , the skewness and kurtosis of the TI-HT-OPGW-R distribution decreases as α and λ increases.
- For fixed values of the parameters α and λ , the skewness and kurtosis of the TI-HT-OPGW-R distribution decreases and increases as θ and β increases.

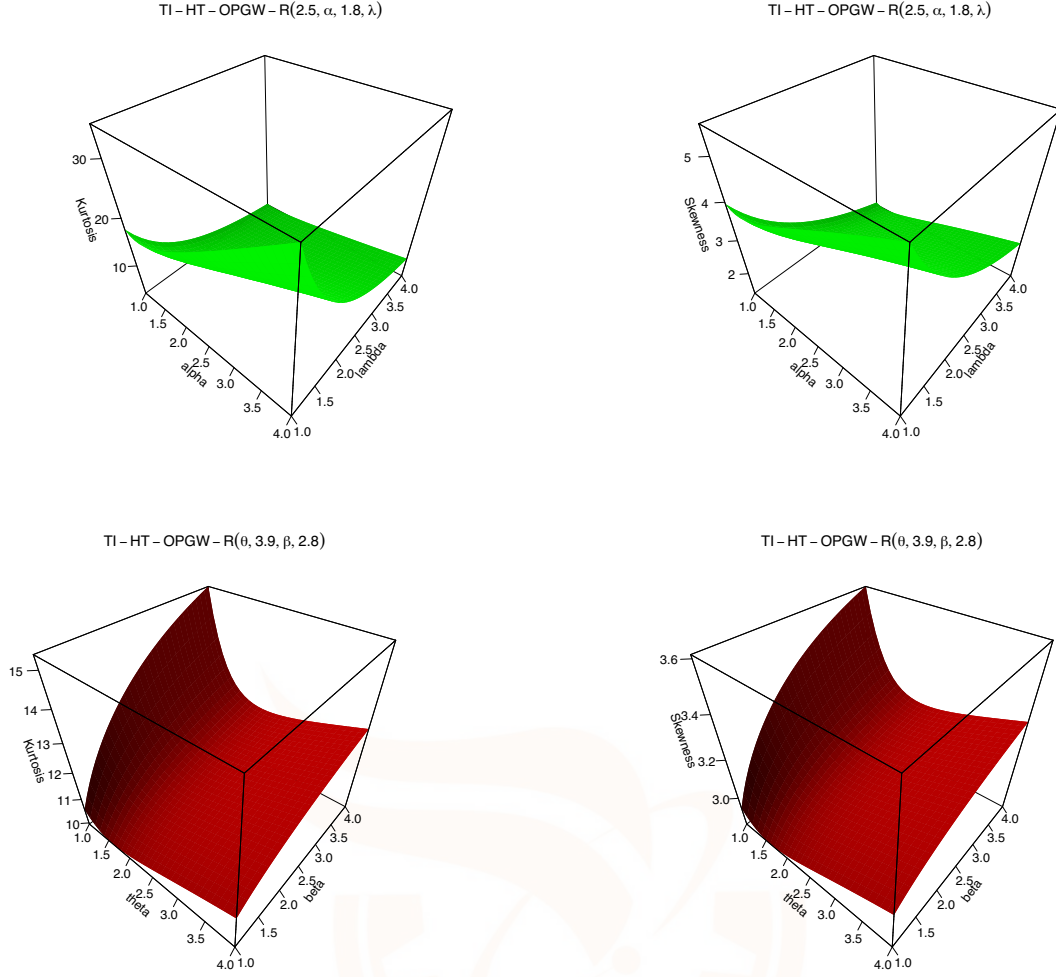


Figure 6.6: 3D Plots of skewness and kurtosis for the TI-HT-OPGW-R distribution

6.1.15 TI-HT-OPGW-Standard Half Logistic (TI-HT-OPGW-SHL) Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x) = \frac{1-e^{-x}}{1+e^{-x}}$, and $g(x) = \frac{2e^{-x}}{(1+e^{-x})^2}$, $x > 0$, then, the cdf and pdf of the TI-HT-OPGW-SHL distribution are given by

$$F(x; \theta, \alpha, \beta) = 1 - \left(\frac{\exp \left(1 - \left[1 + \left(\frac{1-e^{-x}}{1+e^{-x}} \right)^\alpha \right]^\beta \right)}{1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{1-e^{-x}}{1+e^{-x}} \right)^\alpha \right]^\beta \right) \right]} \right)^\theta,$$

and

$$\begin{aligned}
 f(x; \theta, \alpha, \beta) &= \theta^2 \alpha \beta \left[1 + \left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1 - \frac{1-e^{-x}}{1+e^{-x}}} \right)^\alpha \right]^{\beta-1} \left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1 - \frac{1-e^{-x}}{1+e^{-x}}} \right)^{\alpha-1} \\
 &\times \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1 - \frac{1-e^{-x}}{1+e^{-x}}} \right)^\alpha \right]^\beta \right) \right] \right)^{-(\theta+1)} \\
 &\times \exp \left(\theta \left(1 - \left[1 + \left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1 - \frac{1-e^{-x}}{1+e^{-x}}} \right)^\alpha \right]^\beta \right) \right) \frac{\frac{2e^{-x}}{(1+e^{-x})^2}}{\left(1 - \frac{1-e^{-x}}{1+e^{-x}} \right)^2},
 \end{aligned}$$

respectively, for $\theta, \alpha, \beta > 0$ being shape parameters and $x > 0$, where $\bar{\theta} = 1 - \theta$. The hrf for TI-HT-OPGW-SHL distribution is given by

$$\begin{aligned}
 h(x; \theta, \alpha, \beta) &= \theta^2 \alpha \beta \left[1 + \left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1 - \frac{1-e^{-x}}{1+e^{-x}}} \right)^\alpha \right]^{\beta-1} \left(\frac{1-e^{-x}}{1+e^{-x}} \right)^{\alpha-1} \\
 &\times \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1 - \frac{1-e^{-x}}{1+e^{-x}}} \right)^\alpha \right]^\beta \right) \right] \right)^{-1} \frac{\frac{2e^{-x}}{(1+e^{-x})^2}}{\left(1 - \frac{1-e^{-x}}{1+e^{-x}} \right)^{\alpha+1}},
 \end{aligned}$$

for $\theta, \alpha, \beta > 0$, and $\bar{\theta} = 1 - \theta$.

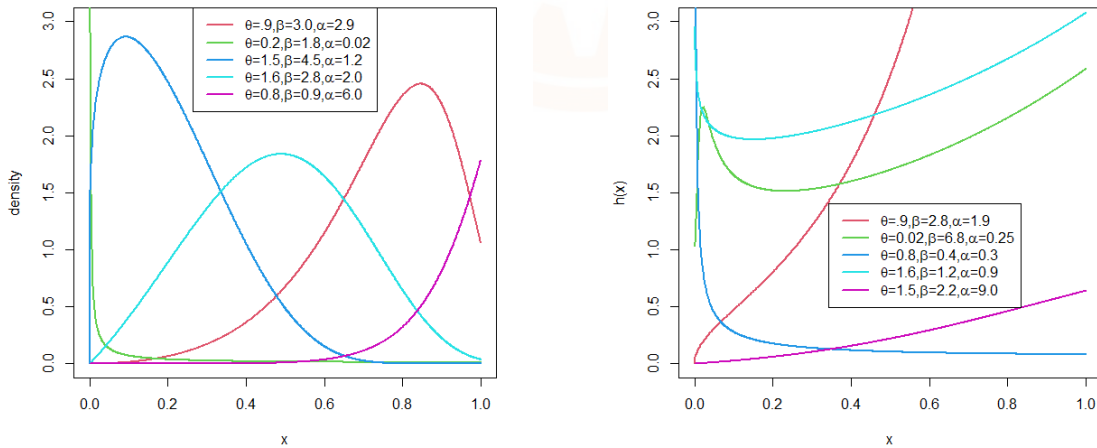


Figure 6.7: Pdf and hrf Plots for TI-HT-OPGW-SHL Distribution

Figure 6.7 shows the plots of pdf and hrf of TI-HT-OPGW-SHL distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, unimodal, J and reverse-J shapes. The TI-HT-OPGW-SHL hrf displays increasing, decreasing, bathtub, and upside-down bathtub followed by bathtub shapes.

(θ, β, α)					
u	(0.3, 0.1, 0.3)	(1.5, 0.2, 0.4)	(0.7, 2.8, 1.3)	(6.8, 9.9, 3.5)	(0.06, 0.08, 9.9)
0.1	21.6679	0.0676	0.2302	0.1667	2.8785
0.2	31.8039	0.4775	0.3563	0.2050	3.2792
0.3	37.4256	1.3335	0.4589	0.2334	3.5994
0.4	37.4281	2.4227	0.5423	0.2535	3.8851
0.5	37.4271	3.6040	0.6176	0.2745	4.1585
0.6	37.4232	4.8751	0.6902	0.2950	4.4286
0.7	37.4337	6.3142	0.7631	0.3186	4.7077
0.8	37.4331	8.0891	0.8448	0.3413	5.0208
0.9	37.4309	10.6831	0.9412	0.3783	5.4214

Table 6.7: Table of Quantile for TI-HT-OPGW-SHL Distribution

(θ, β, α)					
	(0.05,0.8,0.2)	(0.08,0.9,4.0)	(1.5,0.5,0.4)	(3.2,0.9,0.5)	(2.2,1.6,0.09)
$E(X)$	0.0677	0.1266	0.1330	0.3301	0.1121
$E(X^2)$	0.0391	0.0405	0.0744	0.1995	0.0812
$E(X^3)$	0.0272	0.0201	0.0514	0.1407	0.0637
$E(X^4)$	0.0208	0.0126	0.0392	0.1080	0.0524
$E(X^5)$	0.0169	0.0089	0.0317	0.0873	0.0446
$E(X^6)$	0.0141	0.0069	0.0266	0.0731	0.0388
SD	0.1857	0.1566	0.2381	0.3008	0.2620
CV	2.7430	1.2372	1.7896	0.9112	2.3375
CS	3.1088	2.3010	1.9570	0.5551	2.1811
CK	12.1710	9.2094	5.8553	2.0673	6.2671

Table 6.8: Moments of TI-HT-OPGW-SHL Distribution for Selected Parameters

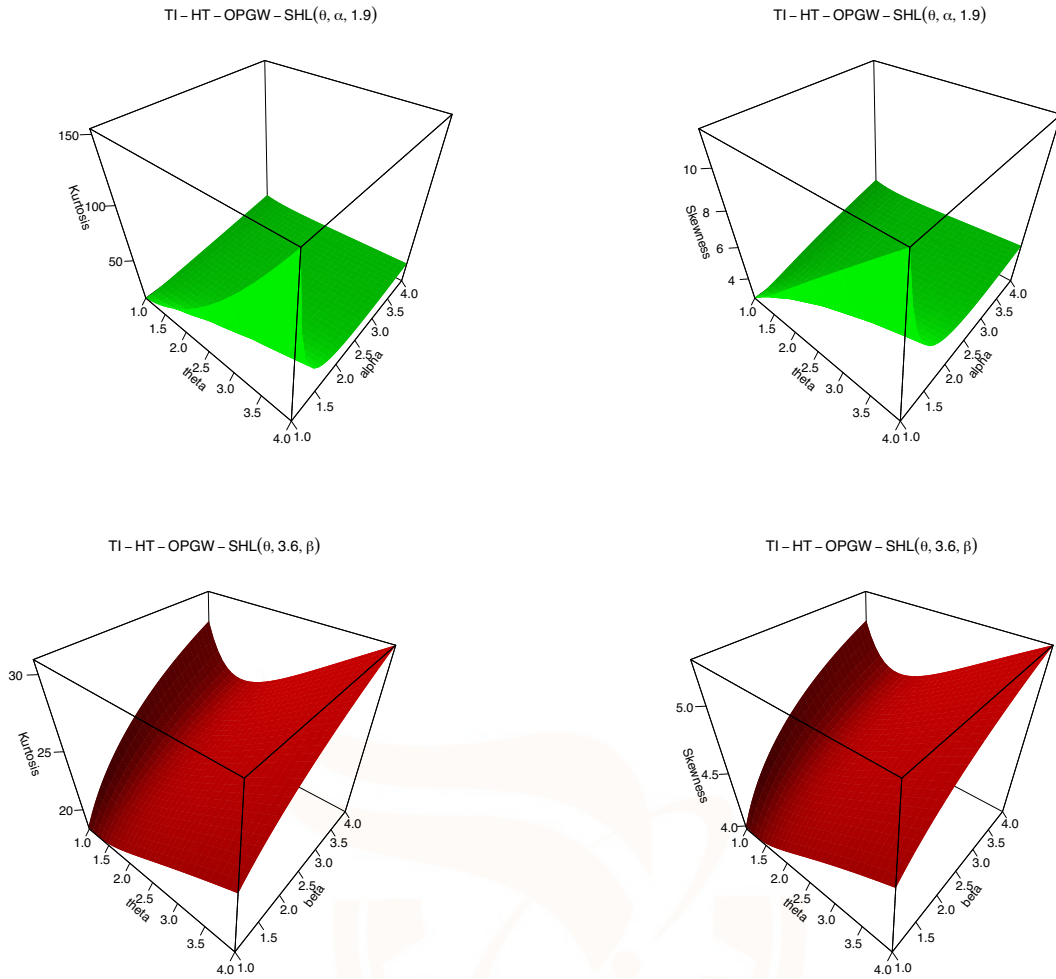


Figure 6.8: 3D Plots for the skewness and kurtosis for the TI-HT-OPGW-SHL distribution

Figure 6.8 shows the 3D plots of skewness and kurtosis of the TI-HT-OPGW-SHL distribution. We observe that

- For fixed values of the parameters β , the skewness and kurtosis of the TI-HT-OPGW-SHL distribution decreases as θ and α increases.
- For fixed values of the parameters α , the skewness and kurtosis of the TI-HT-OPGW-SHL distribution increases as θ and β increases.

6.1.16 Parameter Estimation

In this section, we adopt the MLE technique for estimating the parameters of the TI-HT-OPGW-G GFDP. Let $x_1, x_2, \dots, x_n$ be the realization from the TI-HT-OPGW-G GFDP with the vector of model parameters $\Delta = (\theta, \alpha, \beta, \zeta)^T$. Then, the corresponding log-likelihood function $\ell_n = \ell_n(\Delta)$ of this sample has the form

$$\begin{aligned}
\ell_n(\Delta) = & n \ln(\theta^2 \alpha \beta) + (\beta - 1) \sum_{i=1}^n \log \left[1 + \left(\frac{G_B(x_i; \zeta)}{1 - G_B(x_i; \zeta)} \right)^\alpha \right] \\
& + (\alpha - 1) \sum_{i=1}^n \log \left(\frac{G_B(x_i; \zeta)}{1 - G_B(x_i; \zeta)} \right) + \sum_{i=1}^n \log(g_B(x_i; \zeta)) \\
& + \sum_{i=1}^n \left(\theta \left(1 - \left[1 + \left(\frac{G_B(x_i; \zeta)}{1 - G_B(x_i; \zeta)} \right)^\alpha \right]^\beta \right) \right) - 2 \sum_{i=1}^n \log(1 - G_B(x_i; \zeta)) \\
& - (\theta + 1) \sum_{i=1}^n \log \left(1 - \bar{\theta} \left[1 - \exp \left(1 - \left[1 + \left(\frac{G_B(x_i; \zeta)}{1 - G_B(x_i; \zeta)} \right)^\alpha \right]^\beta \right) \right] \right).
\end{aligned} \tag{6.17}$$

To obtain the MLEs of the unknown parameters, denoted by $\hat{\Delta} = (\hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\zeta})^T$, we set the nonlinear system of equations $(\frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \zeta})^T = \mathbf{0}$, and solve them simultaneously. However, since these equations are not in closed form, the MLEs can be found by maximizing $\ell_n(\Delta)$ numerically with respect to the parameters, using a numerical method such as Newton-Raphson procedure. We maximized the likelihood function using the function `nlm` in R. The partial derivatives of the log-likelihood function with respect to each component of the parameter vector are given in the appendix. The observed Fisher information matrix is given by

$$J(\hat{\Delta}) = \begin{pmatrix} J_{\theta, \theta}(\hat{\Delta}) & J_{\theta, \alpha}(\hat{\Delta}) & J_{\theta, \beta}(\hat{\Delta}) & J_{\theta, \zeta}(\hat{\Delta}) \\ J_{\alpha, \theta}(\hat{\Delta}) & J_{\alpha, \alpha}(\hat{\Delta}) & J_{\alpha, \beta}(\hat{\Delta}) & J_{\alpha, \zeta}(\hat{\Delta}) \\ J_{\beta, \theta}(\hat{\Delta}) & J_{\beta, \alpha}(\hat{\Delta}) & J_{\beta, \beta}(\hat{\Delta}) & J_{\beta, \zeta}(\hat{\Delta}) \\ J_{\zeta, \theta}(\hat{\Delta}) & J_{\zeta, \alpha}(\hat{\Delta}) & J_{\zeta, \beta}(\hat{\Delta}) & J_{\zeta, \zeta}(\hat{\Delta}) \end{pmatrix}, \tag{6.18}$$

where $J_{i,j} = -\frac{\partial^2 \ell_n(\hat{\Delta})}{\partial_i \partial_j}$ for $i, j = \theta, \alpha, \beta, \zeta$.

6.1.17 Monte Carlo Simulation Study

In this section, we evaluate the efficiency and consistency property of the MLEs of TI-HT-OPGW-W distribution via a Monte Carlo simulation based on the following: $N=3000$ samples of size $n=50, 100, 200, 400, 800, 1600$ generated from the TI-HT-OPGW-W distribution for different parameter values. The process is carried out as follows:

- 3000 random samples of size $n=50, 100, 200, 400, 800, 1600$ was generated from the TI-HT-OPGW-W distribution.
- Different combinations for the true parameters are selected such as for Table 1: $\theta = 0.3, \alpha = 1.8, \beta = 0.3, \lambda = 1.5$; $\theta = 0.3, \alpha = 1.8, \beta = 0.3, \lambda = 0.6$, and for $\theta = 1.5, \alpha = 0.6, \beta = 0.3, \lambda = 0.3$. For Table 2: $\theta = 1.8, \alpha = 0.8, \beta = 1.0, \lambda = 1.0$; $\theta = 0.5, \alpha = 0.8, \beta = 1.0, \lambda = 1.0$, and $\theta = 1.0, \alpha = 0.8, \beta = 1.0, \lambda = 2.5$. These values are unknown, hence chosen arbitrary and then estimated using the TI-HT-OPGW-W distribution.

- Two statistical quantities (ABIAS and RMSE) are calculated to evaluate the consistency of the MLEs.

The simulation results from the TI-HT-OPGW-W distribution are presented in Tables 1 and 2. These tables report the average estimates (Mean), average bias (ABIAS) and root mean squared errors (RMSEs). The ABIAS and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$ABIAS(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

parameter	Sample Size	(0.3, 1.8, 0.3, 1.5)			(0.3, 1.8, 0.3, 0.6)			(1.5, 0.6, 0.3, 0.3)		
		Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
θ	50	0.4131	0.3320	0.1131	0.4025	0.3280	0.1025	1.9393	1.2849	0.4393
	100	0.3709	0.2615	0.0709	0.3627	0.2634	0.0627	1.8984	1.2093	0.3984
	200	0.3329	0.1659	0.0329	0.3323	0.1811	0.0323	1.8523	1.1403	0.3523
	400	0.3138	0.1020	0.0138	0.3130	0.1066	0.0130	1.7255	0.8857	0.2255
	800	0.3054	0.0655	0.0054	0.3038	0.0654	0.0038	1.7153	0.6723	0.2153
	1600	0.3033	0.0413	0.0033	0.3010	0.0421	0.0010	1.5500	0.4883	0.0500
α	50	2.8345	4.2902	1.0345	2.7212	3.9147	0.9212	0.8534	0.5010	0.2534
	100	2.1531	1.6306	0.3531	2.1017	1.6751	0.3017	0.7606	0.3709	0.1606
	200	1.9319	0.8064	0.1319	1.9023	0.8054	0.1023	0.6981	0.2622	0.0981
	400	1.8671	0.3683	0.0671	1.8506	0.3632	0.0506	0.6492	0.1579	0.0492
	800	1.8460	0.2244	0.0460	1.8195	0.2293	0.0195	0.6227	0.0930	0.0227
	1600	1.8287	0.1493	0.0287	1.8035	0.1513	0.0035	0.6227	0.0930	0.0227
β	50	0.2388	0.2563	-0.0611	0.3370	0.3062	0.0370	0.2218	0.1673	-0.0781
	100	0.2858	0.2929	-0.0141	0.3244	0.2383	0.0244	0.2387	0.1480	-0.0612
	200	0.3140	0.2744	0.0140	0.3152	0.1753	0.0152	0.2433	0.1306	-0.0566
	400	0.3065	0.2098	0.0065	0.3064	0.1087	0.0064	0.2606	0.1106	-0.0393
	800	0.3043	0.1543	0.0043	0.3034	0.0677	0.0034	0.2637	0.1009	-0.0362
	1600	0.2963	0.0612	-0.0036	0.3006	0.0408	0.0006	0.2858	0.0782	-0.0141
λ	50	2.2032	1.2778	0.7032	0.7148	0.3051	0.1148	0.4404	0.3475	0.1404
	100	1.9400	0.9330	0.4400	0.6682	0.2302	0.0682	0.3811	0.2420	0.0811
	200	1.7680	0.6684	0.2680	0.6349	0.1569	0.0349	0.3441	0.1859	0.0441
	400	1.6370	0.3791	0.1370	0.6156	0.0933	0.0156	0.3261	0.1673	0.0261
	800	1.5583	0.2246	0.0583	0.6034	0.0581	0.0034	0.3166	0.1470	0.0166
	1600	1.5143	0.0984	0.0143	0.6004	0.0386	0.0004	0.3035	0.1027	0.0035

Table 6.9: Monte Carlo Simulation Results

parameter	Sample Size	(1.0, 0.8, 1.0, 1.0)			(0.5, 0.8, 1.0, 1.0)			(1.0, 0.8, 1.0, 2.5)		
		Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
θ	50	1.4765	1.2119	0.4765	0.9008	0.8341	0.4008	1.4641	1.1880	0.4641
	100	1.3266	1.0090	0.3266	0.7723	0.6818	0.2723	1.2951	0.9814	0.2951
	200	1.1939	0.6730	0.1939	0.6529	0.4772	0.1529	1.1603	0.7228	0.1603
	400	1.0873	0.3151	0.0873	0.5586	0.2767	0.0586	1.0603	0.3698	0.0603
	800	1.0480	0.2065	0.0480	0.5255	0.1497	0.0255	1.0305	0.1995	0.0305
	1600	1.0343	0.1418	0.0343	0.5160	0.1074	0.0160	1.0176	0.1385	0.0176
α	50	0.6694	0.4028	-0.1305	0.7109	0.2963	-0.0890	0.6270	0.4801	-0.1729
	100	0.7054	0.2882	-0.0945	0.7314	0.2234	-0.0685	0.6734	0.3882	-0.1265
	200	0.7451	0.2180	-0.0548	0.7490	0.1776	-0.0509	0.7296	0.3072	-0.0703
	400	0.7697	0.1535	-0.0302	0.7791	0.1314	-0.0208	0.7621	0.2162	-0.0378
	800	0.7810	0.1067	-0.0189	0.7929	0.0878	-0.0070	0.7853	0.1568	-0.0146
	1600	0.7890	0.0757	-0.0109	0.7955	0.0575	-0.0044	0.7915	0.1107	-0.0084
β	50	1.3852	0.9164	0.3852	0.7714	0.6878	-0.2285	1.3469	0.8878	0.3469
	100	1.2787	0.7790	0.2787	0.8657	0.6477	-0.1342	1.2998	0.8349	0.2998
	200	1.2316	0.5893	0.2316	0.9453	0.5986	-0.0546	1.2538	0.6952	0.2538
	400	1.1744	0.4825	0.1744	1.0284	0.4209	0.0284	1.1921	0.5701	0.1921
	800	1.1190	0.3480	0.1190	1.0134	0.3097	0.0134	1.1502	0.4588	0.1502
	1600	1.0720	0.2444	0.0720	0.9976	0.1883	-0.0023	1.0938	0.3273	0.0938
λ	50	1.5395	0.9209	0.5395	1.4667	0.7605	0.4667	3.7769	2.2398	1.2769
	100	1.3303	0.6322	0.3303	1.3161	0.5817	0.3161	3.2118	1.5261	0.7118
	200	1.1754	0.4109	0.1754	1.1890	0.4010	0.1890	2.8540	0.9718	0.3540
	400	1.0872	0.2718	0.0872	1.0970	0.2440	0.0970	2.6753	0.6612	0.1753
	800	1.0468	0.1834	0.0468	1.0400	0.1368	0.0400	2.5916	0.4537	0.0916
	1600	1.0246	0.1229	0.0246	1.0157	0.0936	0.0157	2.5420	0.3094	0.0420

Table 6.10: Monte Carlo Simulation Results

From Tables 6.1 and 6.2, it can be verified that the mean tend to the true parameters, the RMSEs and average bias decreases when n increases, thus showing that the estimates are consistent.

6.1.18 Actuarial Measures

In actuarial sciences, actuaries need to evaluate the exposure of market risk in a portfolio of instruments. This section introduces some important risk measures for the TI-HT-OPGW-G GFPD such as value-at-risk (VaR), tail-value-at-risk (TVaR), tail variance (TV) and tail variance premium (TVP).

6.1.19 Value-at-Risk (VaR)

Value-at-risk (VaR) is an important and well known risk measure. It is also known as the quantile risk measure. The VaR of a random variable X is the q^{th} quantile of its cdf. If X is a random variable from TI-HT-OPGW-G GFPD, then

$$VaR_q = x_q = G^{-1} \left[\left(\left[\left(1 - \log \left(\theta \left[(1-q)^{\frac{-1}{\theta}} - \bar{\theta} \right]^{-1} \right) \right)^{\frac{1}{\beta}} - 1 \right]^{\frac{-1}{\alpha}} + 1 \right)^{-1} \right], \quad (6.19)$$

where $q \in (0, 1)$ is a specified level of significance.

6.1.20 Tail-Value-at-Risk (TVaR)

TVaR is used to measure the expected loss given that an event outside a given probability level has occurred. Let X follows from TI-HT-OPGW-W-G GFPD, then TVaR of X is defined as

$$TVaR_q = \frac{1}{1-q} \int_{VaR_q}^{\infty} x f(x) dx = \frac{1}{1-q} \sum_{p=0}^{\infty} \int_{VaR_q}^{\infty} x \varphi_{p+1} g_{p+1}(x; \zeta) dx, \quad (6.20)$$

where $g_{p+1}(x; \zeta) = (p+1)[G_B(x; \zeta)]^p g_B(x; \zeta)$ is the Exp-G pdf with the power parameter $(p+1)$ and parameter vector ζ , and φ_{p+1} is given by equation (6.8). Thus, TVaR of TI-HT-OPGW-W-G GFPD can be obtained from those of Exp-G distribution.

6.1.21 Tail Variance (TV)

The tail variance is one of the most important actuarial measures which looks at the variance beyond the VaR. The TV of the TI-HT-OPGW-W-G GFPD can be defined as

$$\begin{aligned} TV_q &= E(X^2 | X > x_q) - (TVaR_q)^2 \\ &= \frac{1}{1-q} \int_{VaR_q}^{\infty} x^2 f(x) dx - (TVaR_q)^2 \\ &= \frac{1}{1-q} \sum_{p=0}^{\infty} \int_{VaR_q}^{\infty} x^2 \varphi_{p+1} g_{p+1}(x; \zeta) dx - (TVaR_q)^2, \end{aligned} \quad (6.21)$$

where $g_{p+1}(x; \zeta) = (p+1)[G_B(x; \zeta)]^p g_B(x; \zeta)$ is the Exp-G pdf with the power parameter $p+1$ and parameter vector ζ , and φ_{p+1} is given by equation (6.8). Thus, TV of TI-HT-OPGW-W-G GFPD can be obtained from those of Exp-G distribution.

6.1.22 Tail Variance Premium (TVP)

The TVP is an important actuarial measure that plays an essential role in insurance sciences. The TVP of the TI-HT-OPGW-W-G GFPD takes the form

$$TVP_q = TVaR_q + \delta (TV_q), \quad (6.22)$$

where $0 < \delta < 1$. The TVP of the TI-HT-OPGW-W-G GFPD can be obtained by substituting the equations (6.20) and (6.21) into equation (6.22).

6.1.23 Numerical Study for the Risk Measures

This sub-section deals with the the numerical study of VaR, TVaR, TV and TVP for the TI-HT-OPGW-W distribution for different sets of parameters. The VaR, TVaR, TV and TVP of the TI-HT-OPGW-W distribution are compared with the type-I heavy-tailed Weibull (TI-

HT-W) distribution, the half logistic generalized Weibull (HLGW) and Weibull distribution. The process of obtaining the results is described as follows:

1. Random samples of size $n = 100$ are generated from each one of used distributions and parameters have been estimated via maximum likelihood method.
2. 1000 repetitions are made to calculate the VaR, TVaR, TV and TVP for these distributions.

Tables 6.11 and 6.12 present the simulated results of VaR, TVaR, TV and TVP of the compared distributions. A model with higher values of VaR, TVaR, TV and TVP is said to have a heavier tail. The simulated results provided in Tables 6.11 and 6.12 shows that the proposed TI-HT-OPGW-W distribution has higher values of the risk measures than the TI-HT-W, HLGW, and Weibull distributions.

Significance level	0.7	0.75	0.8	0.85	0.9	0.95	0.99
TI-HT-OPGW-W($\theta = 0.9, \alpha = 1.0, \beta = 1.0, \lambda = 0.5$)	VaR	1.9926	2.6118	3.4848	4.8008	7.0294	11.8902
	TVaR	7.6930	8.7744	10.2127	12.2525	15.4771	21.8847
	TV	73.5835	80.9009	90.1785	102.5321	120.3521	150.9723
	TVP	59.2015	69.4501	82.3555	99.4047	123.7940	165.3084
TI-HT-W($\theta = 0.9, \alpha = 1.0, \beta = 1.0$)	VaR	1.3277	1.6403	2.0526	2.6306	3.5303	5.2882
	TVaR	3.5823	4.0032	4.5450	5.2855	6.4073	8.5284
	TV	8.1989	8.7637	9.4695	10.4026	11.7593	14.1829
	TVP	9.3215	10.5760	12.1206	14.1278	16.9907	22.0022
HLGW($w = 0.9, \lambda = 1.0, \gamma = 1.0$)	VaR	1.8490	2.0965	2.3944	2.7727	3.2999	4.1968
	TVaR	3.1698	3.4099	3.7023	4.0781	4.6077	5.5185
	TV	1.8390	1.8495	1.8675	1.8971	1.9480	2.0518
	TVP	4.4571	4.7971	5.1964	5.6907	6.3609	7.4678
W($\lambda = 0.5$)	VaR	1.4484	1.9202	2.5892	3.6011	5.3150	9.0315
	TVaR	5.9073	6.7544	7.8846	9.4938	12.0546	17.2314
	TV	51.1712	56.9057	64.4260	74.9502	91.3972	124.0315
	TVP	41.7272	49.4338	59.4254	73.2015	94.3121	135.0614

Table 6.11: Simulation Results of VaR, TVaR, TV and TVP

Significance level	0.7	0.75	0.8	0.85	0.9	0.95	0.99
TI-HT-OPGW-W($\theta = 1.1, \alpha = 1.0, \beta = 0.7, \lambda = 0.85$)	VaR	1.9364	2.3451	2.8822	3.6337	4.8049	7.1085
	TVaR	4.8986	5.4519	6.1648	7.1416	8.6279	11.4624
	TV	16.0559	17.3282	18.9573	21.1721	24.5035	30.7003
	TVP	16.1378	18.4481	21.3306	25.1379	30.6811	40.6277
TI-HT-W($\theta = 1.1, \alpha = 1.0, \beta = 0.7$)	VaR	1.3891	1.7162	2.1476	2.7523	3.6937	5.5329
	TVaR	3.7480	4.1884	4.7553	5.5301	6.7037	8.9230
	TV	8.9912	9.6105	10.3845	11.4077	12.8957	15.5534
	TVP	10.0419	11.3964	13.0630	15.2267	18.3099	23.6987
HLGW($w = 1.0, \lambda = 1.0, \gamma = 1.7$)	VaR	1.0243	1.1469	1.2926	1.4750	1.7249	2.1408
	TVaR	1.6523	1.7659	1.9030	2.0774	2.3201	2.7303
	TV	0.3841	0.3812	0.3790	0.3781	0.3792	0.3856
	TVP	1.9212	2.0518	2.2062	2.3988	2.6614	3.0967
W($\lambda = 0.85$)	VaR	1.2421	1.4647	1.7443	2.1149	2.6543	3.6157
	TVaR	2.5691	2.8131	3.1167	3.5156	4.0910	5.1058
	TV	2.0797	2.1306	2.1905	2.2641	2.3622	2.5175
	TVP	4.0249	4.4111	4.8691	5.4401	6.2170	7.4974

Table 6.12: Simulation Results of VaR, TVaR, TV and TVP

6.1.24 Applications

In this section, we illustrate the fitting power of the new TI-HT-OPGW-W distribution by analyzing four real life data sets from different fields. The choice of this special case is motivated by the applicability of the Weibull distribution in different fields as compared to other baseline distributions considered in defining other presented special cases. The goodness-of-fit of the TI-HT-OPGW-W distribution was compared to those of other well-known heavy-tailed distributions and some generalizations of the Weibull distribu-

tion. These distributions are: alpha power Topp-Leone Weibull (APTLW) distribution by Benkhelifa (2022), the type-I heavy-tailed Weibull (TI-HT-W) distribution by Zhao et al. (2020), the heavy-tailed beta-power transformed Weibull (HTBPT-W) distribution by Zhao et al. (2021), the half logistic generalized Weibull (HLGW) distribution by Anwar and Bibi (2018), odd generalized half-logistic Weibull-Weibull (OGHLW-W) by Chipepa et al. (2020b), the Kumaraswamy-Weibull (KW) distribution by Cordeiro et al. (2010), and type II exponentiated half logistic Weibull (TIIHLW) distribution by Al-Mofleh et al. (2020). The pdf's of these competing models are given in the appendix.

For comparison purposes, we used well-known goodness-of-fit statistics such as -2log-likelihood statistic ($-2 \ln(L)$), Akaike Information Criterion ($AIC = 2p - 2 \ln(L)$) by Akaike (1974@), Consistent Akaike Information Criterion ($CAIC = AIC + 2 \frac{p(p+1)}{n-p-1}$) by Bozdogan (1987), Bayesian Information Criterion ($BIC = p \ln(n) - 2 \ln(L)$) by Schwarz (1978), (n is the number of observations, and p is the number of estimated parameters), Cramér-von Mises (W^*) statistic, and Anderson-Darling statistic (A^*) described by Chen and Balakrishnan (1995), Kolmogorov-Smirnov (K-S) statistic by Chakravarti et al. (1967), and its p -value. It is known that the smaller the values of all the goodness-of-fit statistics, except for the p -value of K-S statistic, the better the model for fitting the data set.

For the probability plot, we plotted $F(x_{(i)}) = F(x_{(i)}; \hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda})$ against $\frac{i - 0.375}{n + 0.25}$, $i = 1, 2, \dots, n$, where $x_{(i)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{i=1}^n \left[F(x_{(i)}) - \left(\frac{i - 0.375}{n + 0.25} \right) \right]^2.$$

6.1.25 Agriculture Data

This dataset represents the total factor productivity (TFP) growth agricultural production for thirty-seven African countries from 2001-2010, see <https://dataverse.harvard.edu/dataset.xhtml?persistentId=doi:10.7910/DVN/9IOAKR>, accessed on 30 June 2022. (See the data in the Appendix).

The estimated variance-covariance matrix for TI-HT-OPGW-W model on agriculture data is given by

$$\begin{bmatrix} 1.834502 \times 10^{-22} & 6.582383 \times 10^{-21} & 1.316762 \times 10^{-16} & 7.006136 \times 10^{-21} \\ 6.582383 \times 10^{-21} & 2.361826 \times 10^{-19} & 4.724676 \times 10^{-15} & 2.513874 \times 10^{-19} \\ 1.316762 \times 10^{-16} & 4.724676 \times 10^{-15} & 9.451397 \times 10^{-11} & 5.028836 \times 10^{-15} \\ 7.006136 \times 10^{-21} & 2.513874 \times 10^{-19} & 5.028836 \times 10^{-15} & 2.675709 \times 10^{-19} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\alpha \in [73.8740 \pm 2.6546 \times 10^{-11}]$, $\gamma \in [0.9815 \pm 9.5253 \times 10^{-10}]$, $\delta \in [7.6304 \times 10^{-05} \pm 1.9054 \times 10^{-05}]$ and $\lambda \in [1.4386 \pm 1.0138 \times 10^{-09}]$, respectively.

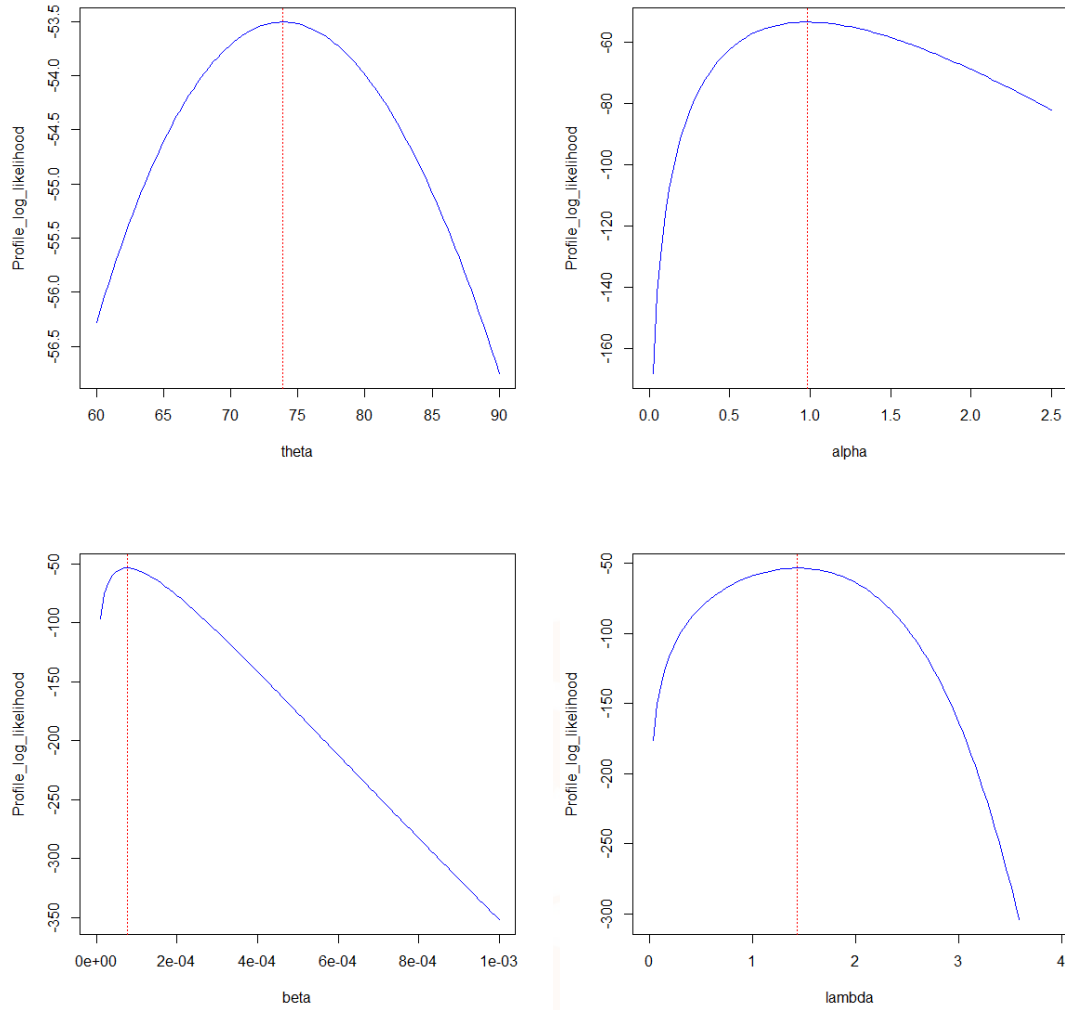


Figure 6.9: Profile log-likelihood function plots for parameters of TI-HT-OPGW-W distribution on the agriculture data

The identifiability of model parameters can be assessed by plotting the log-likelihood function. From the profile log-likelihood plots in Figure 6.9, we can see that the MLEs for the TI-HT-OPGW-W distribution are identifiable.

Model	Estimates			
	θ	α	β	λ
TI-HT-OPGW-W	73.8740 (1.3544e $\times 10^{-11}$)	0.9815 (4.8599 $\times 10^{-10}$)	7.6304 $\times 10^{-05}$ (9.7218 $\times 10^{-06}$)	1.4386 (5.1727 $\times 10^{-10}$)
APTLW	3.1266 (5.8242)	12.1322 (29.4512)	0.6406 (0.5086)	1.0002 (1.0759)
TI-HT-W	0.8309 (0.1212)	0.2991 (0.1187)	2.7494 (1.1363)	
HTBPT-W	0.6588 (0.1140)	1.1740 (0.2605)	0.0286 (0.0330)	
HLGW	797.4800 (1.1333 $\times 10^{-07}$)	0.5000 (7.3374 $\times 10^{-02}$)	9.0808 $\times 10^{-04}$ (9.6790 $\times 10^{-05}$)	
KW	0.1293 (0.0446)	1.8134 (1.0046)	7.4198 (2.5729)	0.2015 (0.0385)
TIIEHLW	1.6908 (1.5011)	3.9314 (4.8099)	0.9271 (0.8245)	0.6168 (0.4828)
OGHLWW	1.0774 $\times 10^{-04}$ (1.2797 $\times 10^{-04}$)	0.6420 (8.7916 $\times 10^{-02}$)	13.6300 (3.5390 $\times 10^{-03}$)	0.1275 (2.3757 $\times 10^{-02}$)

Table 6.13: Estimates from the TFP Growth Data

Model	Statistics							
	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	$K-S$	P-value
TI-HT-OPGW-W	107.0129	115.0129	116.2629	121.4566	0.0293	0.1812	0.0798	0.9722
APTLW	107.9218	115.9218	117.1718	122.3655	0.0368	0.2264	0.1049	0.8098
TI-HT-W	111.3917	117.3918	118.1191	122.2246	0.0368	0.2240	0.1197	0.6635
HTBPT-W	115.7192	121.7192	122.4464	126.5519	0.0311	0.1928	0.1807	0.1781
HLGW	117.7719	123.7719	124.4991	128.6046	0.0325	0.2050	0.2142	0.0670
KW	109.6151	117.6158	118.8658	124.0595	0.0745	0.4569	0.1043	0.8151
TIIEHLW	109.7821	117.7821	119.0321	124.2258	0.0643	0.4020	0.1255	0.6043
OGHLWW	108.3969	116.3967	117.6467	122.8403	0.0428	0.2752	0.0889	0.9316

Table 6.14: GoF Statistics for TFP Growth Data

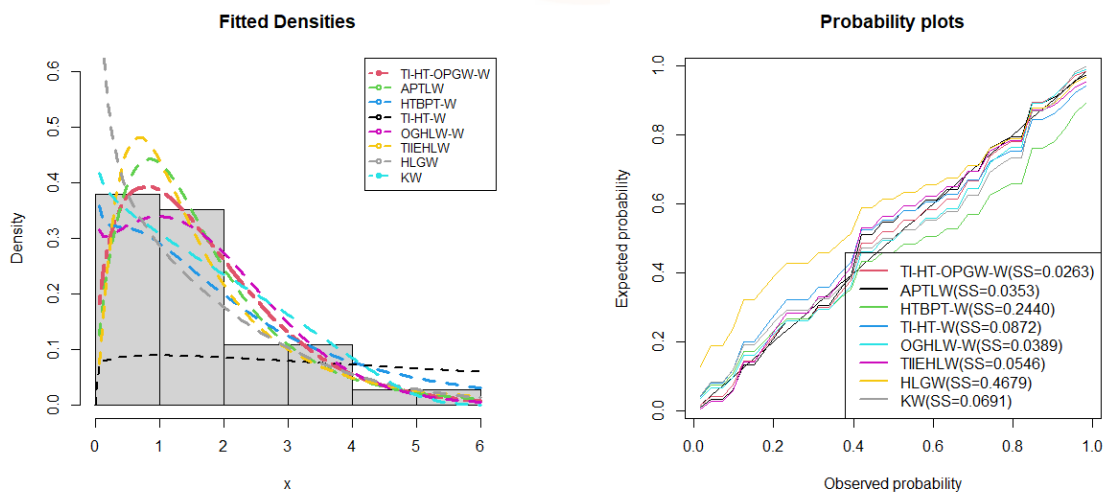


Figure 6.10: Fitted Densities and Probability Plots for TFP Growth Data

The parameter estimates, standard error (in parentheses) are given in Table 6.13. The

goodness-of-fit statistics: AIC , BIC , $CAIC$, W^* , A^* , K-S statistic, and its p-value are given in Table 6.14. The values in Table 6.14 shows that the TI-HT-OPGW-W distribution gives the smallest values for the goodness-of-fit statistics and the largest p-value of K-S statistic. Thus, the TI-HT-OPGW-W distribution provides better fit than the rest of the distributions for the TFP growth data. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 6.10.

Figure 6.11 presents the TTT scaled plots, empirical and theoretical K-M survival plots, cumulative frequency curve of the observed data with the fitted cdf of the TI-HT-OPGW-W distribution and also, the hrf plot. It is visible that, the fitted empirical and theoretical plots are close to each other, hence we conclude that our model provide the better fit for the data. The TTT scaled plot clearly demonstrates that the data fit the increasing hazard rate structure. Furthermore, the estimated hrf for TFP growth data is an increasing shape.

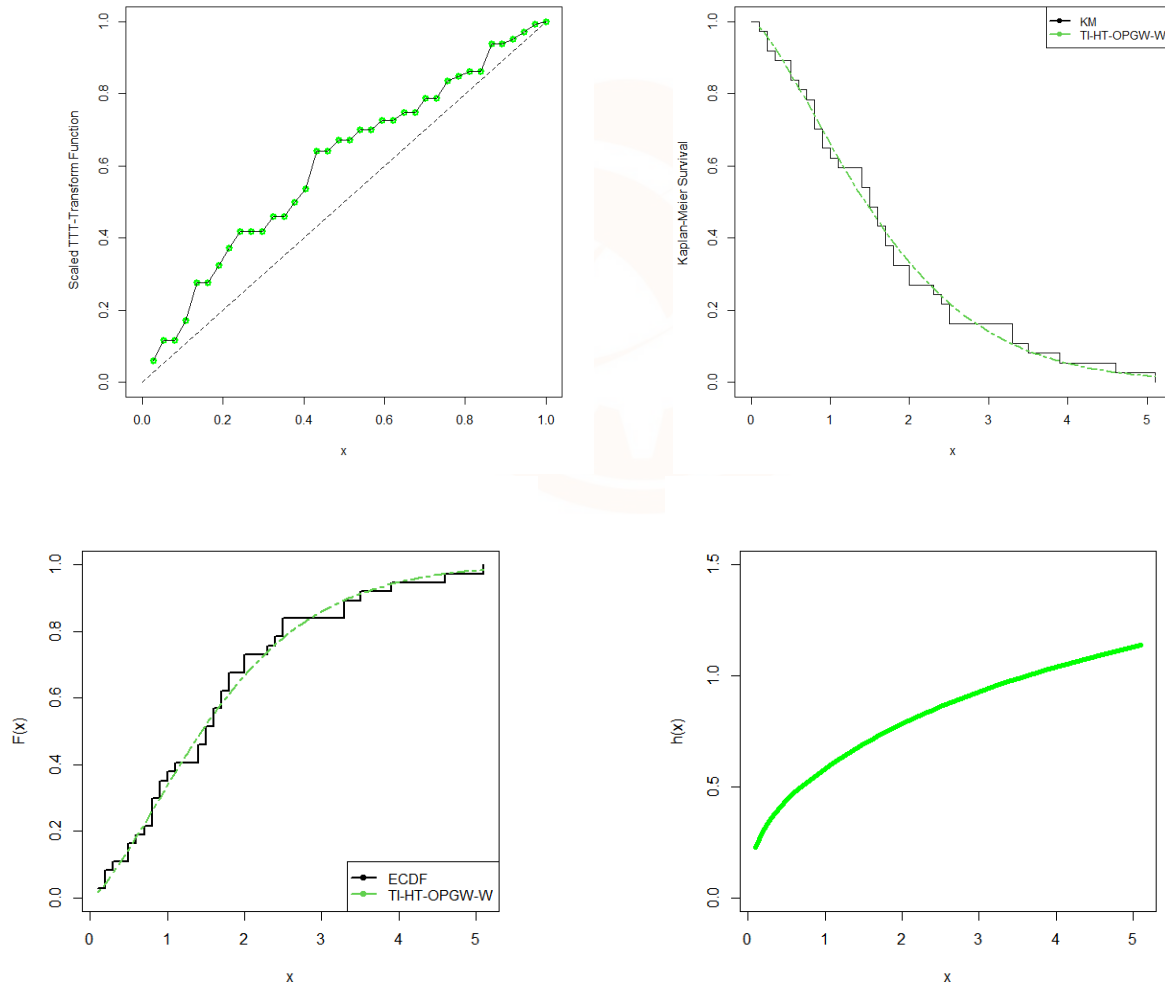


Figure 6.11: Fitted TTT, K-M Survival, ECDF and hrf Plots for TFP Growth Data

6.1.26 Failure Times Data

These data represent the failure time of a machine from Babel tyres factory (Iraq) in hours. The data was obtained from [AL-Kazrajy \(2001\)](#). (See the data in the Appendix).

The estimated variance-covariance matrix for TI-HT-OPGW-W model on failure times data is given by

$$\begin{bmatrix} 1.2819 \times 10^{-02} & -1.2822 \times 10^{-06} & -1.5772 \times 10^{-03} & 4.3067 \times 10^{-03} \\ -1.2822 \times 10^{-06} & 1.6146 \times 10^{-10} & 1.9861 \times 10^{-07} & -6.4167 \times 10^{-07} \\ -1.5772 \times 10^{-03} & 1.9861 \times 10^{-07} & 2.4431 \times 10^{-04} & -7.8929 \times 10^{-04} \\ 4.3067 \times 10^{-03} & -6.4167 \times 10^{-07} & -7.8929 \times 10^{-04} & 2.8590 \times 10^{-03} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\alpha \in [0.1548 \pm 0.2219]$, $\gamma \in [24.2130 \pm 0.00002]$, $\delta \in [0.0461 \pm 0.0306]$ and $\lambda \in [0.1683 \pm 0.1048]$, respectively.



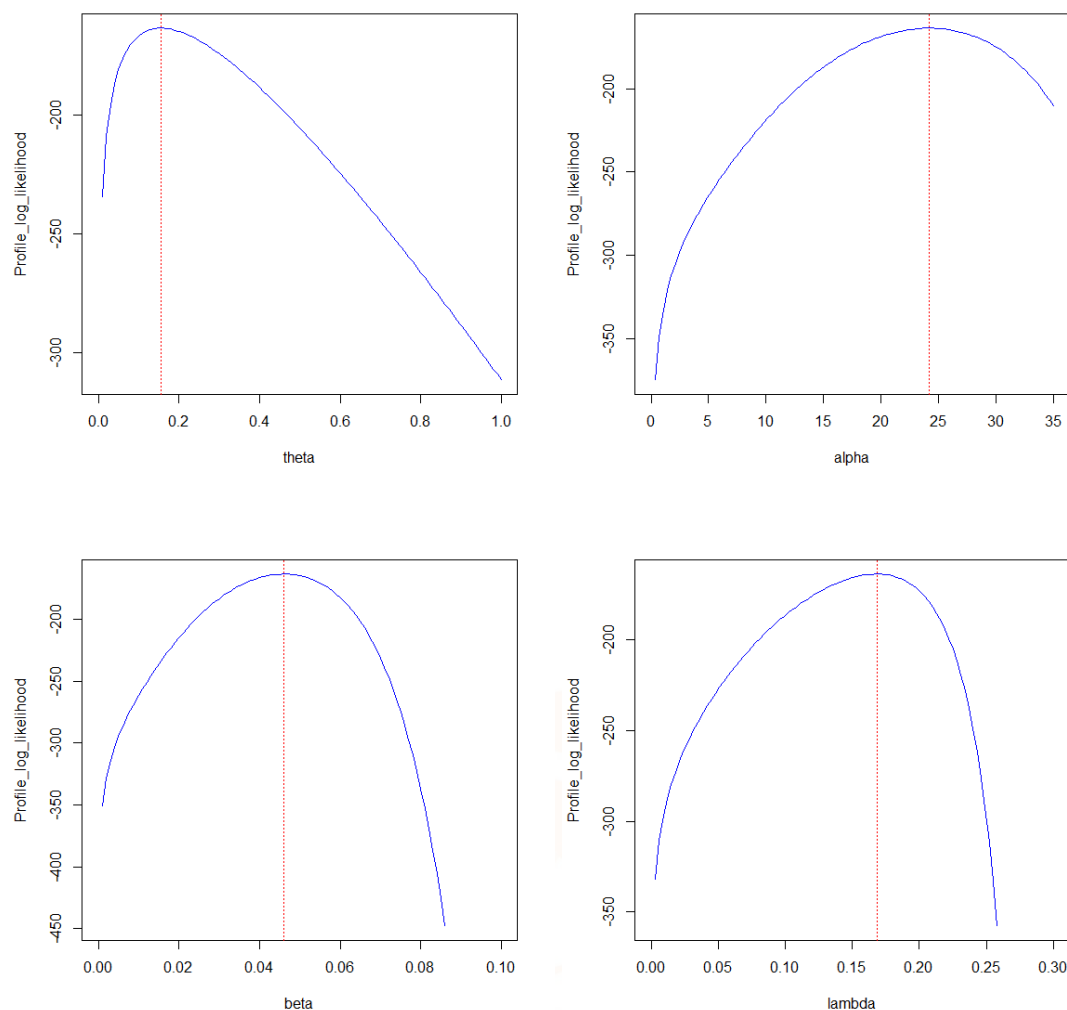


Figure 6.12: Profile log-likelihood function plots for parameters of TI-HT-OPGW-W distribution on the failure times data

The MLEs of TI-HT-OPGW-W parameters on failure times data can be uniquely identified, as shown in Figure 6.12.

Model	Estimates			
	θ	α	β	λ
TI-HT-OPGW-W	0.1548 (0.1132)	24.2130 (1.2707×10^{-05})	0.0461 (0.0156)	0.1683 (0.0534)
APTLW	0.5500 (0.0748)	0.0173 (0.0675)	1.3216 (0.0378)	0.0001 (0.0001)
TI-HT-W	0.6980 (0.0756)	0.1196 (0.0486)	0.4071 (0.1391)	
HTBPT-W	0.8987 (0.0783)	0.0125 (0.0052)	1.0002 (0.4193)	
HLGW	0.2370 (0.0500)	0.9780 (0.1847)	0.5827 (0.4439)	
KW	15.4546 (4.7782)	62.0537 (0.1384)	0.0990 (0.0165)	0.4869 (1.1236)
TIIEHLW	27.4140 (3.2565×10^{-04})	109.6900 (7.7848×10^{-04})	2.6706 (8.9301×10^{-02})	0.0500 (6.7886×10^{-03})
OGHLWW	2.1083 (8.5735×10^{-06})	0.4143 (6.8140×10^{-04})	21.2990 (1.3243×10^{-05})	0.0510 (5.8370×10^{-03})

Table 6.15: Estimates from the Failure Times Data

Model	Statistics							
	$-2 \log(L)$	AIC	AICC	BIC	W^*	A^*	$K-S$	P-value
TI-HT-OPGW-W	327.3327	335.3327	336.9994	340.8019	0.0660	0.3691	0.1344	0.6705
APTLW	328.3628	336.3628	338.0295	341.8320	0.0845	0.4480	0.1495	0.5353
TI-HT-W	332.8800	338.8801	339.8401	342.9819	0.0687	0.3736	0.2413	0.0682
HTBPT-W	333.4933	339.4933	340.4533	343.5952	0.1034	0.5659	0.2861	0.0173
HLGW	333.0898	339.0899	340.0499	343.1918	0.0598	0.3901	0.1786	0.3132
KW	327.7001	335.7001	337.3668	341.1693	0.0696	0.3815	0.1365	0.6518
TIIEHLW	327.5437	335.8219	337.4885	341.2911	0.0731	0.3973	0.1377	0.6409
OGHLWW	331.0865	339.0866	340.7532	344.5557	0.1181	0.6422	0.1616	0.4348

Table 6.16: GoF Statistics for Failure Times Data

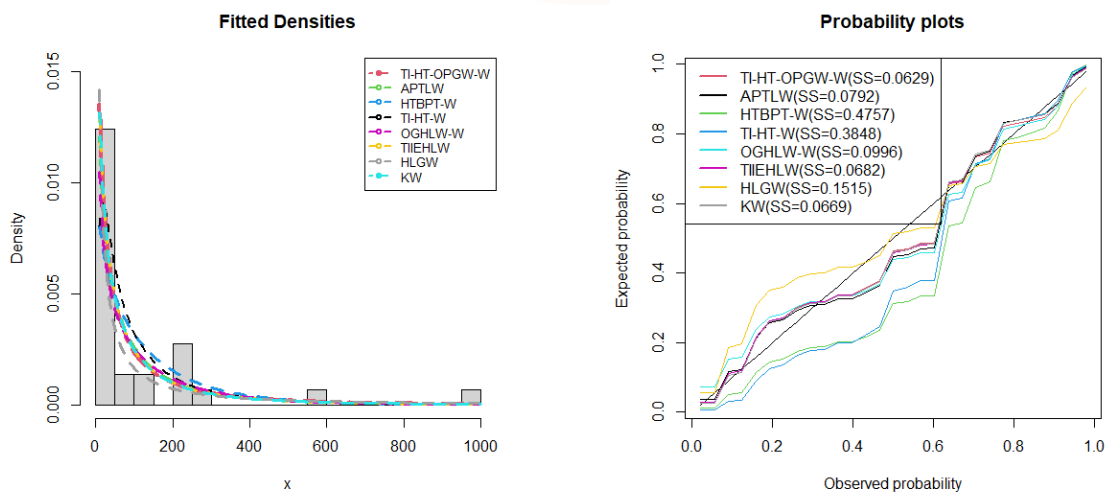


Figure 6.13: Fitted Densities and Probability Plots for Failure Times Data

Table 6.15 and 6.16 presents numerical values of the MLEs with standard errors (in paren-

thesis), $-2\ln(L)$, AIC , $CAIC$, BIC , W^* , A^* , K-S statistic, and its p – value for the failure times data. It can be seen that the TI-HT-OPGW-W distribution has the lowest value for all goodness-of-fit statistics and a larger p – value between all fitted distributions which gives it the superiority for fitting the failure times data. To support the best fitting power of the TI-HT-OPGW-W distribution, plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 6.13.

Figure 6.14 depicts the TTT scaled plots, observed and the fitted K-M survival curves, theoretical and ECDF and hrf plots. The green line in the K-M and ECDF plots illustrates the good fit to the data. The TTT scaled plot demonstrates that the data follow a decreasing hazard rate shape. Furthermore, the hazard rate function exhibit a decreasing shape for the failure times data.

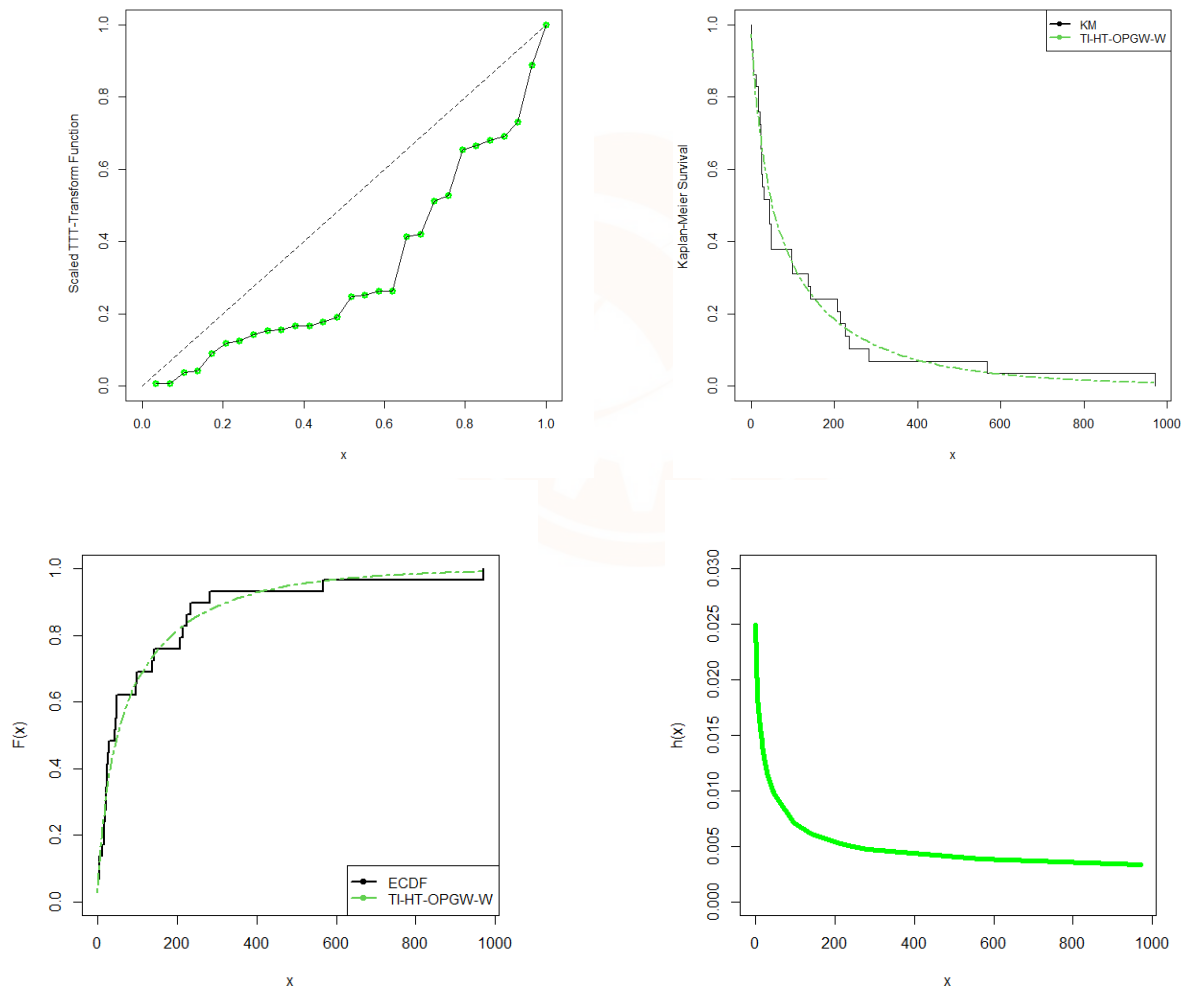


Figure 6.14: Fitted TTT, K-M, ECDF and hrf Plots for Failure Times Data

6.1.27 Biomedical Sciences Data

The data set is on remission times (months) of 128 bladder cancer patients by [Lee and Wang \(2003\)](#). **(See the data in the Appendix).**

The estimated variance-covariance matrix for TI-HT-OPGW-W distribution on remission times data is given by

$$\begin{bmatrix} 0.0040 & 0.0075 & -0.0056 & 0.0017 \\ 0.0075 & 7.0579 & -0.4969 & -0.0208 \\ -0.0056 & -0.4969 & 0.0422 & -0.0014 \\ 0.0017 & -0.0208 & -0.0014 & 0.0013 \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\theta \in [0.1575 \pm 0.1241]$, $\alpha \in [4.6733 \pm 5.2071]$, $\beta \in [0.3432 \pm 0.4030]$ and $\lambda \in [0.2149 \pm 0.0723]$, respectively.



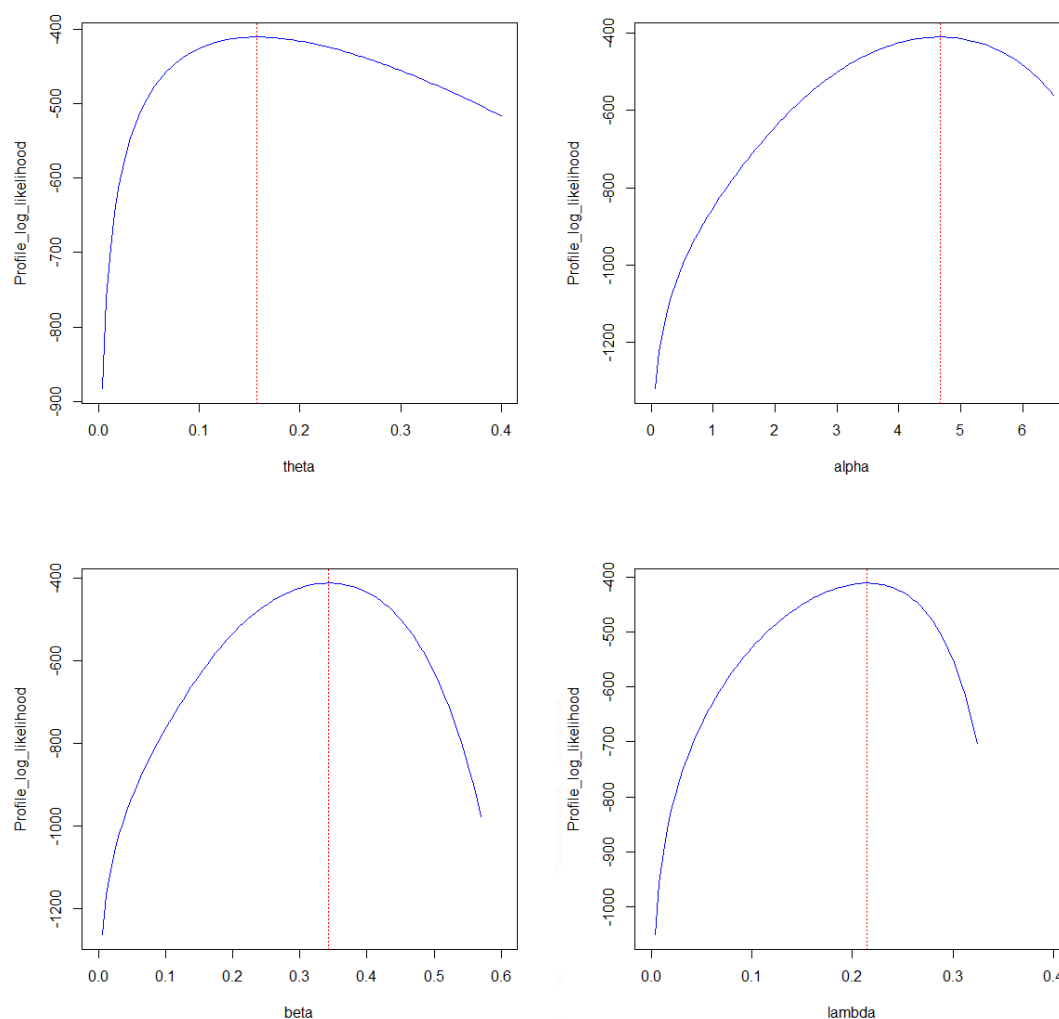


Figure 6.15: Profile log-likelihood function plots for parameters of TI-HT-OPGW-W distribution on the biomedical sciences data

The profile log-likelihood plots for parameters of the TI-HT-OPGW-W distribution can be used to check for identifiability of parameters. From the plots in Figure 6.15, we can see that the MLEs for the TI-HT-OPGW-W distribution are unique, hence we conclude that the parameters are identifiable.

Model	Estimates			
	θ	α	β	λ
TI-HT-OPGW-W	0.1575 (0.0633)	4.6733 (2.6563)	0.3432 (0.2056)	0.2149 (0.0368)
APTLW	θ 0.3632 (0.1239)	α 2.6272×10^{02} (4.8767×10^{-05})	β 0.8610 (0.1241)	λ 0.1120 (0.0540)
TI-HT-W	α 1.1030 (0.6049)	θ 1.2551 (3.5043)	β 0.0564 (0.3501)	
HTBPT-W	α 1.0478 (0.0675)	γ 0.0938 (0.0190)	β 0.9999 (0.7280)	
HLGW	w 6.2967 (9.9402)	λ 0.5000 (0.0776)	γ 0.0516 (0.0804)	
KW	a 8.8278 (4.6441)	b 206.7178 (0.0352)	c 0.1835 (0.0580)	λ 0.0301 (0.0679)
TIIEHLW	a 1.0297×10^{03} (1.8761×10^{-06})	λ 1.0759×10^{02} (1.9066×10^{-04})	δ 2.4004 (2.1528×10^{-02})	γ 0.0500 (3.2111×10^{-03})
OGHLWW	α 2.1269×10^{-05} (3.4793×10^{-06})	β 0.6471 (4.2738×10^{-04})	λ 14.4550 (1.9132×10^{-05})	γ 0.0774 (4.3136×10^{-03})

Table 6.17: Estimates from the Remission Times Data

Model	Statistics							
	$-2 \log(L)$	AIC	AICC	BIC	W^*	A^*	$K-S$	P-value
TI-HT-OPGW-W	822.0841	830.0841	830.4093	841.4922	0.0507	0.3255	0.0488	0.9196
APTLW	827.4507	835.4507	835.7759	846.8588	0.1275	0.7620	0.0727	0.5065
TI-HT-W	826.9924	832.9924	833.1859	841.5485	0.1150	0.6940	0.0684	0.5863
HTBPT-W	828.1738	834.1738	834.3673	842.7298	0.1313	0.7864	0.0700	0.5570
HLGW	851.9690	857.9690	858.1626	866.5251	0.2699	1.5921	0.1647	0.0019
KW	823.5203	831.5204	831.8456	842.9285	0.0776	0.4738	0.0567	0.8036
TIIEHLW	823.0934	831.0934	831.4186	842.5015	0.0710	0.4392	0.0549	0.8351
OGHLWW	838.0349	846.0347	846.3599	857.4428	0.2477	1.4603	0.0952	0.1962

Table 6.18: GoF Statistics for Remission Times Data

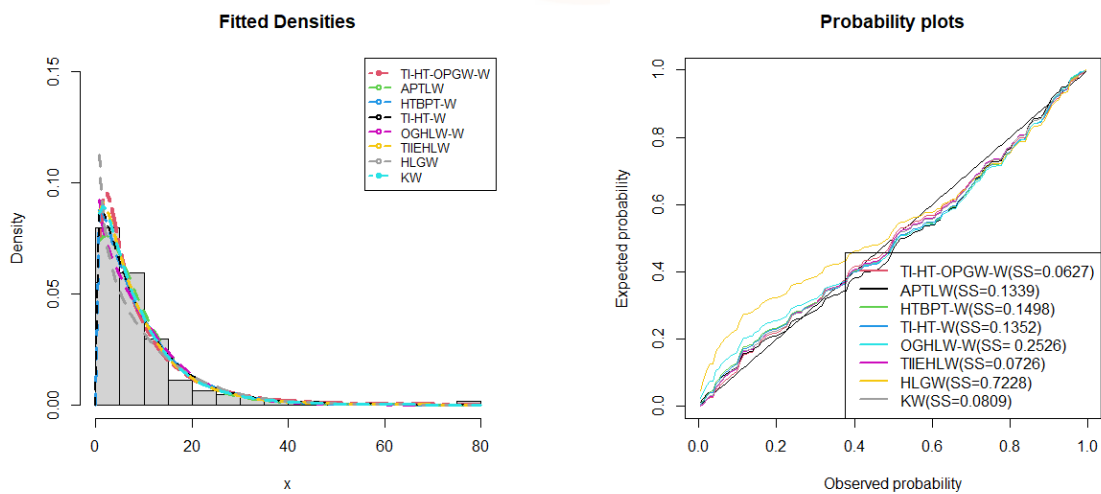


Figure 6.16: Fitted Densities and Probability Plots for Remission Times Data

Table 6.17 gives the MLEs of the fitted distributions together with the standard errors (in

parenthesis). Table 6.18, gives the values of all considered goodness-of-fit statistics. It is evident that the TI-HT-OPGW-W distribution provides the best fit among the competitors since it has the lowest value of $-2 \ln(L)$, AIC , $CAIC$, BIC , W^* , A^* , K-S statistic, and larger p – value. From the plots in Figure 6.16, we can see that the TI-HT-OPGW-W distribution follows the fitted histogram closely and has the smallest sum of squares (SS) value from the probability plots. This supports the conclusion made from Table 6.18.

The total test time (TTT) scaled plots, observed and the fitted Kaplan-Meier (K-M) survival curves, theoretical and empirical cumulative distribution function (ECDF) and hazard rate function (hrf) plots of the TI-HT-OPGW-W distribution are shown in Figure 6.17. From the K-M and ECDF plots, it is clear that the TI-HT-OPGW-W distribution is a good candidate for modeling the biomedical data. The TTT scaled plot demonstrates that the data follow an upside-down bathtub hazard rate shape. Furthermore, the hazard rate function exhibit a non-monotonic shape for the remission times data.

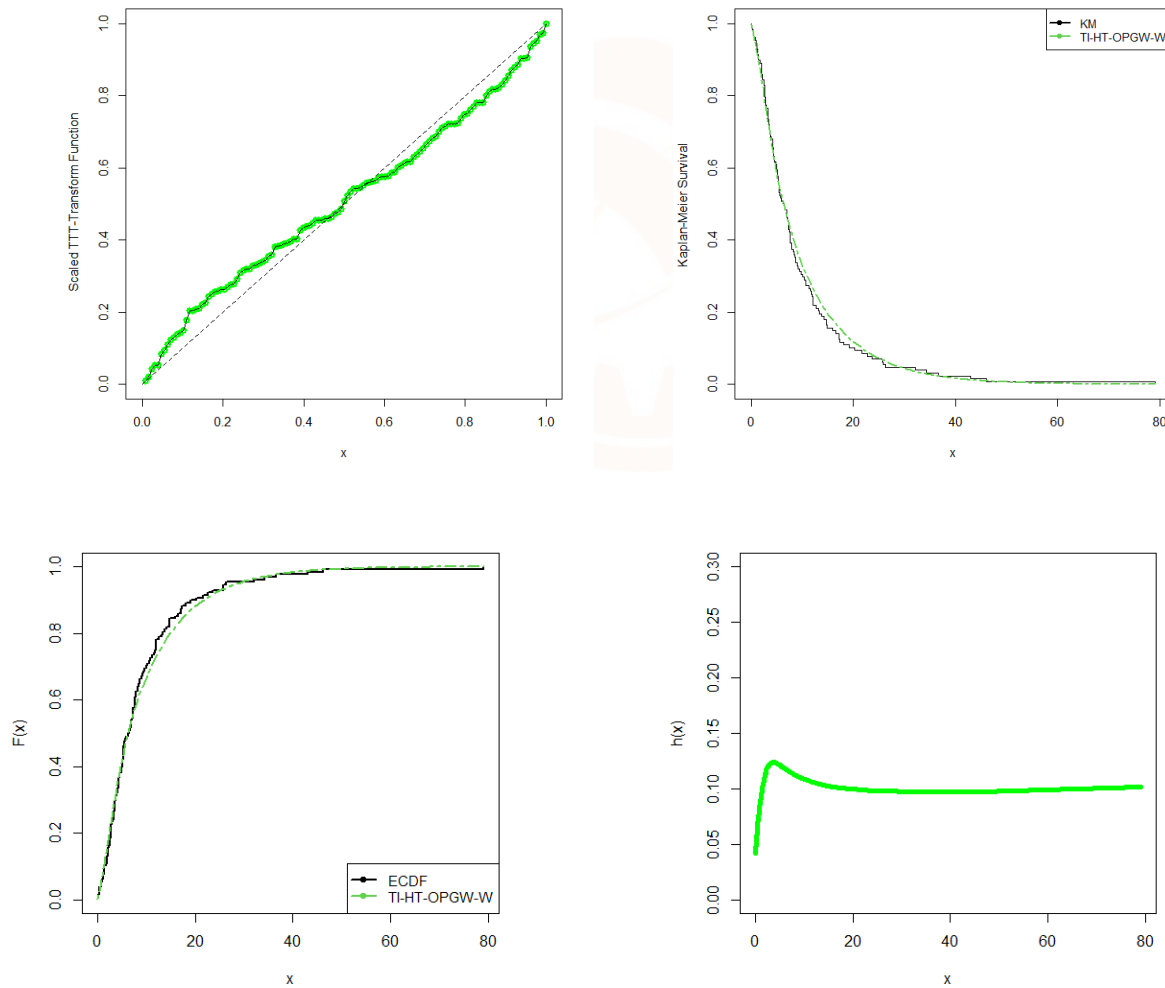


Figure 6.17: Fitted TTT, KM, ECDF and hrf Plots for Remission Times Data

6.1.28 Insurance Data

This data set from the insurance field represents monthly metrics on unemployment insurance from July 2008 to April 2013 from the department of labor, licensing and regulation. It consists of 58 observations and 21 variables, we studied the variable number 6. It is available at: <https://catalog.data.gov/dataset/unemployment-insurance-data-july-2008-to-april-2013>. **(See the data in the Appendix).**

The estimated variance-covariance matrix for TI-HT-OPGW-W distribution on insurance data is given by

$$\begin{bmatrix} 3.1522 \times 10^{-02} & -5.9576 \times 10^{-06} & -1.7866 \times 10^{-03} & 4.4816 \times 10^{-02} \\ -5.9576 \times 10^{-06} & 1.3425 \times 10^{-09} & 4.0137 \times 10^{-07} & -1.0400 \times 10^{-05} \\ -1.7866 \times 10^{-03} & 4.0137 \times 10^{-07} & 1.200 \times 10^{-04} & -3.1083 \times 10^{-03} \\ 4.4816 \times 10^{-02} & -1.0400 \times 10^{-05} & -3.1083 \times 10^{-03} & 8.1063 \times 10^{-02} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\theta \in [0.3099 \pm 0.3479]$, $\alpha \in [2.0529 \pm 7.1816 \times 10^{-05}]$, $\beta \in [0.0078 \pm 0.0214]$ and $\lambda \in [1.1668 \pm 0.5580]$, respectively.

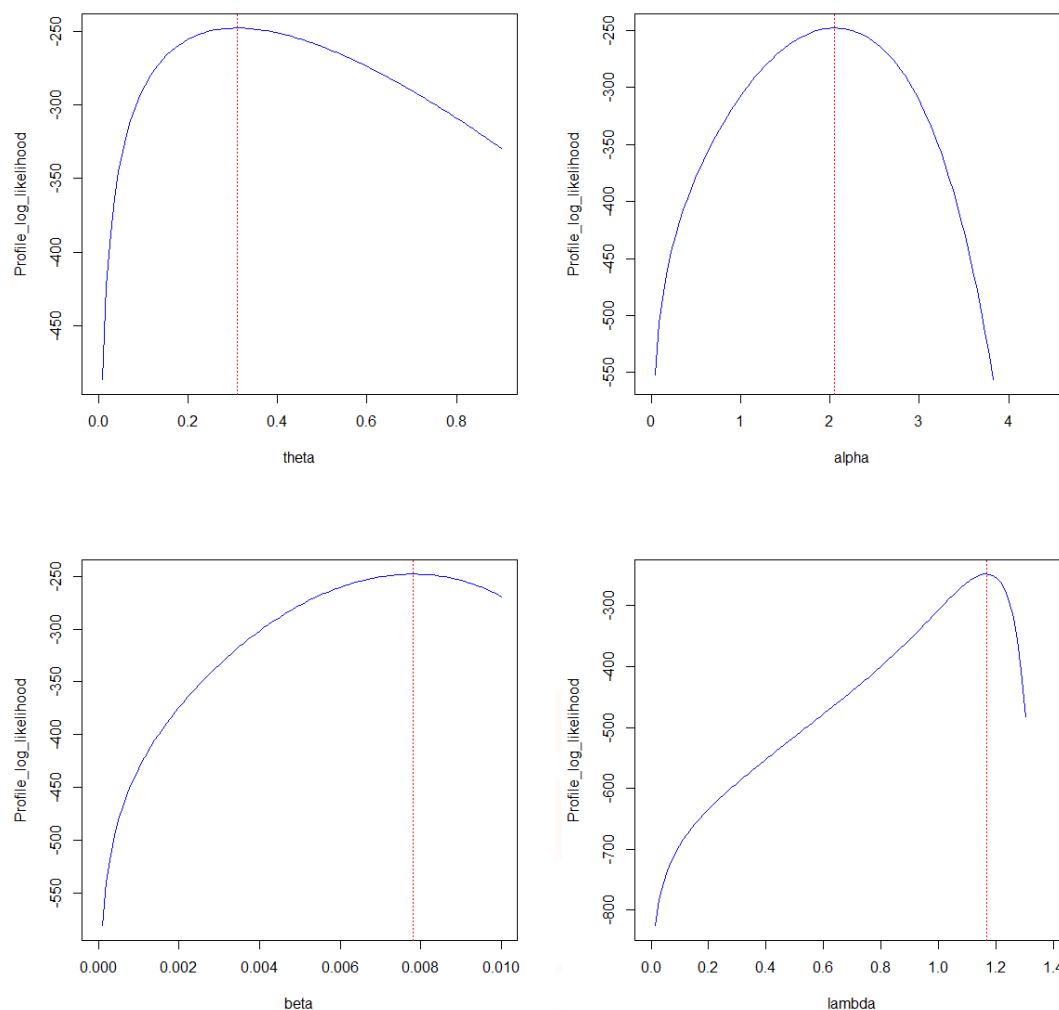


Figure 6.18: Profile log-likelihood function plots for parameters of TI-HT-OPGW-W distribution on the insurance data

Figure 6.18 shows the profile likelihood plots for parameters of the TI-HT-OPGW-W distribution on the insurance data set. From the plots we can see that the MLEs for the TI-HT-OPGW-W distribution are achieved at single and different points.

Model	Estimates			
	θ	α	β	λ
TI-HT-OPGW-W	0.3099 (0.1775)	2.0529 (3.6641 $\times 10^{-05}$)	0.0078 (0.0109)	1.1668 (0.2846)
APTLW	θ 1.3112 (2.3870 $\times 10^{-10}$)	α 2.3022 (5.1581 $\times 10^{-11}$)	β 2.3222 (2.4955 $\times 10^{-09}$)	λ 7.6494 $\times 10^{-05}$ (8.0428 $\times 10^{-06}$)
TI-HT-W	α 1.1963 (0.1374)	θ 0.1221 (0.0537)	β 0.0916 (0.0567)	
HTBPT-W	α 1.2034 (0.1125)	γ 0.0262 (0.01207)	β 7.1487 $\times 10^{-05}$ (6.8034 $\times 10^{-05}$)	
HLGW	w 545.6400 (1.2046 $\times 10^{-13}$)	λ 1.0121 (2.7123 $\times 10^{-10}$)	γ 3.2890 $\times 10^{-05}$ (2.0129 $\times 10^{-06}$)	
KW	a 1.2341 $\times 10^{02}$ (2.0464 $\times 10^{-07}$)	b 9.4224 $\times 10^{03}$ (3.0326 $\times 10^{-10}$)	c 0.1119 (6.1342 $\times 10^{-04}$)	λ 1.1220 $\times 10^{02}$ (6.9732 $\times 10^{-08}$)
TIIEHLW	a 1.4145 $\times 10^{04}$ (9.5420 $\times 10^{-08}$)	λ 2.3843 $\times 10^{03}$ (3.7577 $\times 10^{-05}$)	δ 4.4752 (9.1240 $\times 10^{-02}$)	γ 0.0500 (5.0324 $\times 10^{-03}$)
OGHLWW	α 0.0008 (0.0004)	β 1.2387 (0.2632)	λ 1.7631 (0.1435)	γ 0.3265 (0.0490)

Table 6.19: Estimates from the Insurance Data

Model	Statistics							
	$-2 \log(L)$	AIC	AICC	BIC	W^*	A^*	$K-S$	P-value
TI-HT-OPGW-W	495.8290	503.8290	504.5838	512.0708	0.0578	0.3621	0.0719	0.9249
APTLW	496.5398	504.5398	505.2945	512.7816	0.1102	0.5583	0.1054	0.5394
TI-HT-W	525.6656	531.6656	532.1100	537.8469	0.3263	1.7175	0.2036	0.0162
HTBPT-W	505.3288	511.3291	511.7735	517.5104	0.2356	1.2313	0.1188	0.3853
HLGW	512.5666	518.5643	519.0087	524.7456	0.0841	0.4385	0.2242	0.0058
KW	498.6628	506.6629	507.4176	514.9047	0.1545	0.7836	0.1241	0.3328
TIIEHLW	498.9921	506.9921	507.7468	515.2338	0.1601	0.8126	0.1261	0.3145
OGHLWW	496.0044	504.0044	504.7591	512.2462	0.0599	0.3634	0.0741	0.9069

Table 6.20: GoF Statistics for Insurance Data

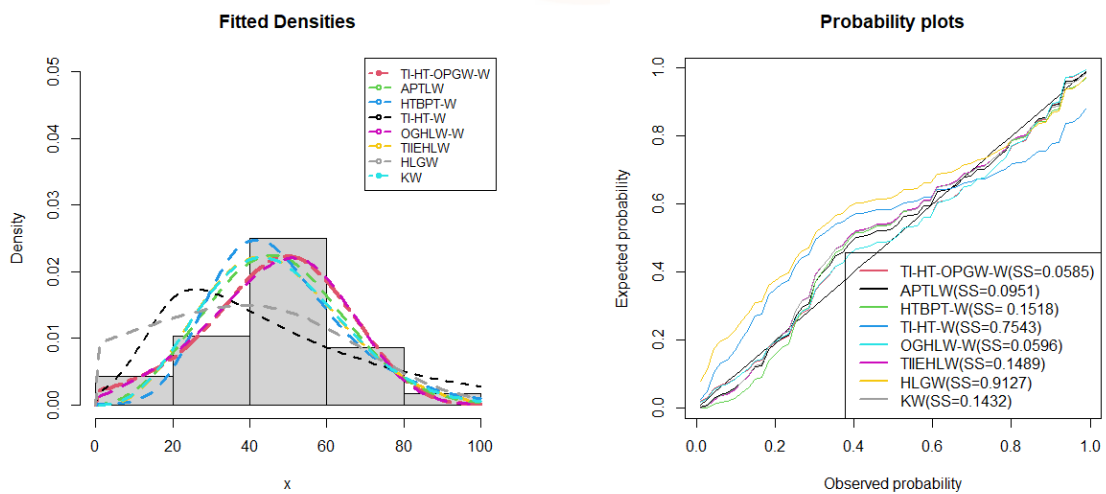


Figure 6.19: Fitted Densities and Probability Plots for Insurance Data

For these data, the MLEs, standard errors (in parenthesis) are given in Table 6.19. From

the goodness-of-fit statistics given in Table 6.20, we observe that the proposed TI-HT-OPGW-W distribution is the best choice to implement for fitting the insurance data since it has the smallest values of goodness-of-fit statistics and larger p-value compared to other fitted distributions. To support the results in Table 6.20, a visual illustrations is provided in Figure 6.19.

Figure 6.20 shows the TTT scaled plots, observed and the fitted K-M survival curves, ECDF and hrf plots. We can see that the TI-HT-OPGW-W distribution follows the empirical cdf, and KM survival curves very closely. The TTT scaled plot shows an increasing hrf, allowing us to fit the heavy-tailed insurance data using TI-HT-OPGW-W distribution. Furthermore, the estimated hazard rate function for insurance data is an increasing shape.

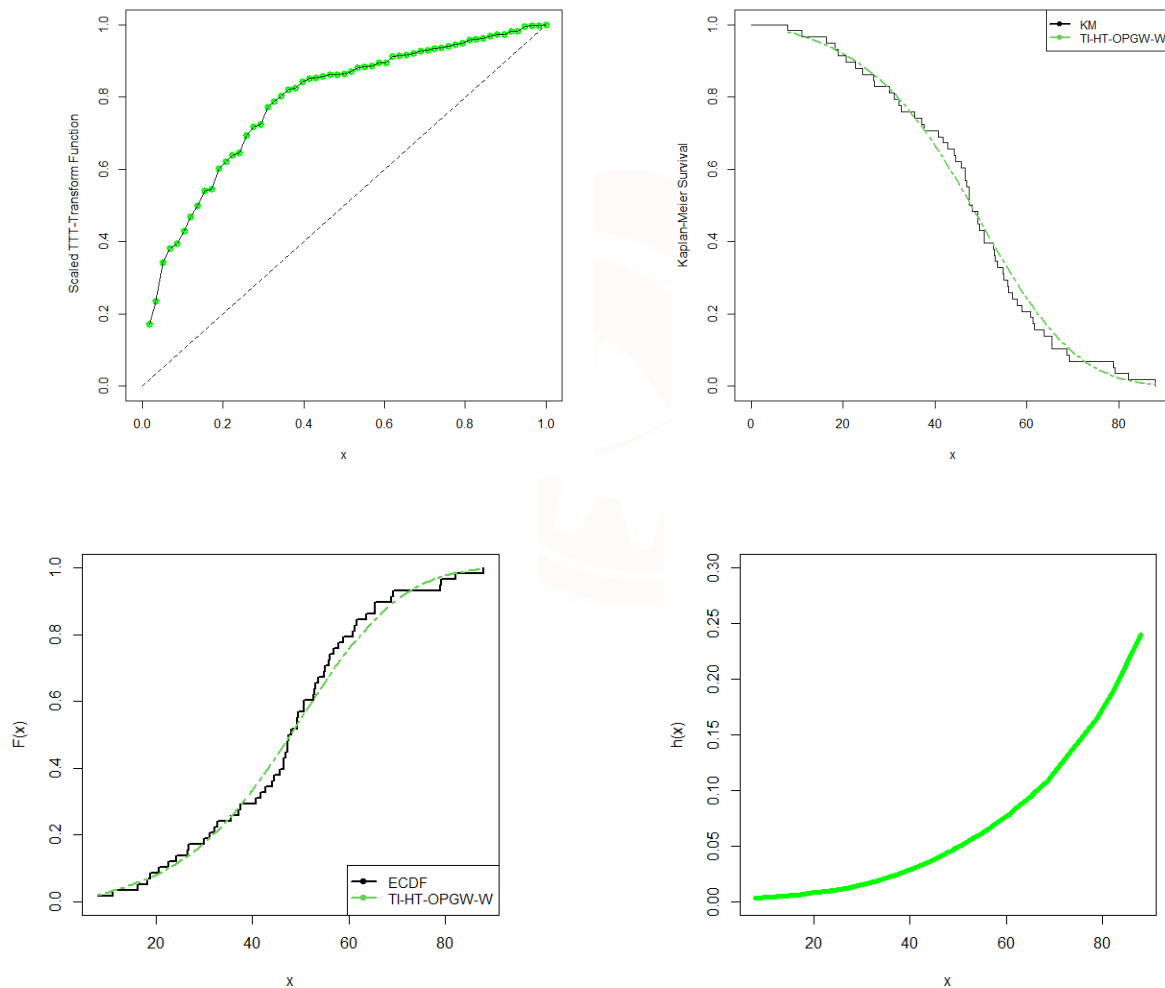


Figure 6.20: Fitted TTT, K-M Survival, ECDF and hrf Plots for Insurance Data

6.1.29 Concluding Remarks

We propose and study a new heavy-tailed GFGPD called type I heavy-tailed odd power generalized Weibull-G (TI-HT-OPGW-G) distribution. Many of its statistical properties such

as quantile function, linear representation, moments, moment generating function, distribution of order statistics and Rényi entropy were derived. The maximum likelihood estimation method was derived and evaluated via a simulation study. Actuarial measures for the proposed distribution were also derived. Numerical comparisons of the actuarial measures with other distributions was conducted. Finally, the superiority and importance of the TI-HT-OPGW-G family of distributions was illustrated by using four real data sets from different fields. The TI-HT-OPGW-G-W model as a special case to this new GFRD was applied to four datasets and from the results it is evident that the new proposed model performs better than several heavy tailed distributions.

6.2 HT-GEN-EHL-G GFRD

Moakofi, T, Puoetsile, A. and Oluyede, B., 2023. Heavy-Tailed Generalized Exponentiated Half Logistic-G Family of Distributions with Applications. Pakistan Journal of Statistics and Operation Research, 20(2), 233-260.

<https://pjsor.com/pjsor/article/view/4458>

The heavy-tailed generalized exponentiated half logistic-G (HT-GEN-EHL-G) GFRD is introduced and investigated in this section. This work derives and studies some of the statistical characteristics of the HT-GEN-EHL-G GFRD. The maximum likelihood estimation technique was used to estimate the model parameters, and a simulation study is used to assess the consistency of the estimators. Additionally, actuarial metrics (risk measures) for this distribution were developed. Applications to four real life data sets from various fields shows the importance and adaptability of the new HT-GEN-EHL-G GFRD.

The cdf and pdf of generalised exponentiated half logistic-G (GEN-EHL-G) GFRD are given by

$$F_{GEN-EHL-G}(x; \alpha, \beta, \zeta) = 1 - (\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta, \quad (6.23)$$

and

$$\begin{aligned} f_{GEN-EHL-G}(x; \alpha, \beta, \zeta) &= 2\alpha\beta g_B(x; \zeta) (G_B(x; \zeta))^{\alpha-1} (1 + \bar{G}_B(x; \zeta))^{-(\alpha+1)} \\ &\times (\bar{F}_{EHL-G}(x; \alpha, \zeta))^{\beta-1}, \end{aligned} \quad (6.24)$$

respectively, for $\alpha, \beta > 0$, being shape parameters and baseline vector of parameters ζ , where

$$\bar{F}_{EHL-G}(x; \alpha, \zeta) = 1 - \left[\frac{G_B(x; \zeta)}{1 + \bar{G}_B(x; \zeta)} \right]^\alpha.$$

6.2.1 The Proposed Distribution

In this section, we present the new HT-GEN-EHL-G GFPD by specifying the cdf, pdf, survival function (sf) and hazard rate function (hrf). By substituting equation (6.23) into equation (1.23), both equations (6.23) and (6.24) into equation (1.24), respectively, we obtain the cdf and pdf of the HT-GEN-EHL-G GFPD given by

$$F_{HT-GEN-EHL-G}(x; \theta, \alpha, \beta, \zeta) = 1 - \left[\frac{(\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta}{1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta\right)} \right]^\theta \quad (6.25)$$

and

$$\begin{aligned} f_{HT-GEN-EHL-G}(x; \theta, \alpha, \beta, \zeta) &= 2\alpha\beta\theta^2 g_B(x; \zeta) (G_B(x; \zeta))^{\alpha-1} (1 + \bar{G}_B(x; \zeta))^{-(\alpha+1)} \\ &\times (\bar{F}_{EHL-G}(x; \alpha, \zeta))^{\beta\theta-1} \\ &\times \left[1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta\right) \right]^{-(\theta+1)}, \end{aligned} \quad (6.26)$$

respectively, for $\theta, \alpha, \beta > 0$ being shape parameters and baseline vector of parameters ζ . The sf and hrf are given by

$$\begin{aligned} S(x; \theta, \alpha, \beta, \zeta) &= 1 - F_{HT-GEN-EHL-G}(x; \theta, \alpha, \beta, \zeta) \\ &= \left[\frac{(\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta}{1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta\right)} \right]^\theta, \end{aligned}$$

and

$$\begin{aligned} h_F(x; \theta, \alpha, \beta, \zeta) &= 2\alpha\beta\theta^2 g_B(x; \zeta) (G_B(x; \zeta))^{\alpha-1} (1 + \bar{G}_B(x; \zeta))^{-(\alpha+1)} \\ &\times (\bar{F}_{EHL-G}(x; \alpha, \zeta))^{-1} \\ &\times \left[1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta\right) \right]^{-1}, \end{aligned}$$

respectively, for $\theta, \alpha, \beta > 0$ and baseline vector of parameters ζ .

6.2.2 Quantile Function

The qf is one the important statistical quantities. It is used to generate random numbers from a probability distribution. Also, the quantity can be used to obtain other statistical measures such as median, kurtosis and skewness. The qf of the HT-GEN-EHL-G GFPD is obtained by solving the non-linear equation:

$$1 - \left[\frac{(\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta}{1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta\right)} \right]^\theta = u,$$

for $0 \leq u \leq 1$, that is,

$$Q(u) = G^{-1} \left[2 \left(\left[1 - \left(\theta \left[(\bar{u})^{\frac{-1}{\theta}} - (1 - \theta) \right]^{-1} \right)^{\frac{1}{\beta}} \right]^{\frac{-1}{\alpha}} + 1 \right)^{-1} \right], \quad (6.27)$$

where G is the baseline cdf and $\bar{u} = (1 - u)$. Thus, (6.27) can be used to generate random numbers from the HT-GEN-EHL-G GFPD.

6.2.3 Density Expansion

This section is devoted to the series expansion and representation of the pdf of the HT-GEN-EHL-G GFPD as an infinite linear combination of exponentiated-G (Exp-G) densities. Using the generalized binomial series expansions (**see in the appendix**), we can express the pdf of HT-GEN-EHL-G GFPD as:

$$f_{HT-GEN-EHL-G}(x; \theta, \alpha, \beta, \zeta) = f(x) = \sum_{k=0}^{\infty} b_{k+1}^* g_{k+1}^*(x; \zeta), \quad (6.28)$$

where

$$\begin{aligned} b_{k+1}^* &= \sum_{q,s,j,t,p=0}^{\infty} \frac{(1-\theta)^q (-1)^{q+s+j+t+k}}{(k+1)} \binom{-(\theta+1)}{q} \binom{q}{s} \binom{\beta s + \beta\theta - 1}{j} \\ &\times \binom{\alpha(j+1) - 1}{t} \binom{-(\alpha(j+1) + 1)}{p} \binom{p+t}{k}, \end{aligned} \quad (6.29)$$

and $g_{k+1}^*(x; \zeta) = (k+1)g_B(x; \zeta) (G_B(x; \zeta))^k$ is the pdf of Exp-G distribution with power parameter $(k+1)$. Hence, the mathematical and statistical properties of the HT-GEN-EHL-G GFPD can be obtained directly from those of the Exp-G distribution.

6.2.4 Statistical Properties

In this section, we present some of the useful mathematical and statistical properties for the HT-GEN-EHL-G GFPD, namely, moments and related measures, distribution of order statistics, Rényi entropy and estimation of model parameters.

6.2.5 Moments, Generating Function and Incomplete Moments

Let Y_{k+1} denote the Exp-G random variable with power parameter $(k+1)$. Then, the i^{th} raw moment of X , say μ'_i can be obtained directly from equation (6.28) as

$$\mu'_i = E(X^i) = \sum_{k=0}^{\infty} b_{k+1}^* E(Y_{k+1}^i), \quad (6.30)$$

where $E(Y_{k+1}^i)$ is the i^{th} raw moment of Exp-G distribution. The moment generating function (mgf) $M_X(t) = E(e^{tX})$ of the HT-GEN-EHL-G GFPD can be derived from equation

(6.28) as

$$M_X(t) = \sum_{k=0}^{\infty} b_{k+1}^* M_{k+1}(t),$$

where b_{k+1}^* is as given in equation (6.29) and $M_{k+1}(t)$ is the mgf of Y_{k+1} .

The i^{th} incomplete moment of the HT-GEN-EHL-G GFPD is given by

$$m_i(t) = \int_{-\infty}^t x^i f(x; \theta, \alpha, \beta, \zeta) dx = \sum_{k=0}^{\infty} b_{k+1}^* \int_{-\infty}^t x^i g_{k+1}^*(x; \zeta) dx, \quad (6.31)$$

where b_{k+1}^* is as given in equation (6.29) and $g_{k+1}^*(x; \zeta) = (k+1)g_B(x; \zeta) (G_B(x; \zeta))^k$ is the pdf of Exp-G distribution with power parameter $(k+1)$. A general equation for $m_1(t)$ can be obtained from equation (6.31) as

$$m_1(t) = \sum_{k=0}^{\infty} b_{k+1}^* B_{k+1}(t),$$

where $B_{k+1}(t) = \int_{-\infty}^t x g_{k+1}^*(x; \zeta) dx$ is the first incomplete moment of the Exp-G distribution.

6.2.6 Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be i.i.d random variables distributed according to 6.26 and suppose $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ are the order statistics. Let the pdf and cdf of HT-GEN-EHL-G distribution be denoted by $f(x)$ and $F(x)$, respectively. Then, the pdf of the i^{th} order statistic is given by

$$\begin{aligned} f_{i:n}(x; \theta, \alpha, \beta, \zeta) &= f(x) \sum_{l=0}^{n-i} \frac{n!(-1)^l}{(n-i)!(i-1)!} \binom{n-i}{l} [F(x)]^{l+i-1} \\ &= \sum_{l=0}^{n-i} \sum_{e=0}^{\infty} \frac{n!(-1)^{l+e}}{(n-i)!(i-1)!} \binom{n-i}{l} \binom{i+l-1}{e} \\ &\quad \times 2\alpha\beta\theta^2 g_B(x; \zeta) (G_B(x; \zeta))^{\alpha-1} (1 + \bar{G}_B(x; \zeta))^{-(\alpha+1)} (\bar{F}_{EHL-G}(x; \alpha, \zeta))^{\beta\theta(e+1)-1} \\ &\quad \times \left[1 - (1-\theta) \left(1 - (\bar{F}_{EHL-G}(x; \alpha, \zeta))^{\beta} \right) \right]^{-(\theta(e+1)+1)}. \end{aligned} \quad (6.32)$$

Using the binomial series expansions (see in the appendix).

Thus, we can write

$$f_{i:n}(x; \theta, \alpha, \beta, \zeta) = \sum_{k=0}^{\infty} c_{k+1}^* g_{k+1}^*(x; \zeta), \quad (6.33)$$

where

$$\begin{aligned}
 c_{k+1}^* &= 2\alpha\beta\theta^2 \sum_{l=0}^{n-i} \sum_{e,q,s,j,t,p=0}^{\infty} \frac{n!(1-\theta)^q(-1)^{q+s+j+t+k+l+e}}{(n-i)!(i-1)!(k+1)} \binom{n-i}{l} \binom{i+l-1}{e} \\
 &\times \binom{-(\theta(e+1)+1)}{q} \binom{q}{s} \binom{\beta s + \beta\theta(e+1)-1}{j} \binom{\alpha(j+1)-1}{t} \\
 &\times \binom{-(\alpha(j+1)+1)}{p} \binom{p+t}{k}, \tag{6.34}
 \end{aligned}$$

and $g_{k+1}^*(x; \zeta) = (k+1)g_B(x; \zeta) (G_B(x; \zeta))^k$ is the pdf of Exp-G distribution with power parameter $(k+1)$.

6.2.7 Rényi Entropy

Rényi entropy is defined to be the measure of variation or uncertainty for a random variable X . Rényi entropy is defined as

$$I_R(v) = (1-v)^{-1} \log \left[\int_{-\infty}^{\infty} f^v(x) dx \right],$$

where $v > 0$ and $v \neq 1$. Note that Rényi entropy of the HT-GEN-EHL-G GfPD is obtained as

$$\begin{aligned}
 I_R(v) &= (1-v)^{-1} \log \left[\int_{-\infty}^{\infty} (2\alpha\beta\theta^2)^v (g_B(x; \zeta))^v (G_B(x; \zeta))^{v(\alpha-1)} (1 + \bar{G}_B(x; \zeta))^{-v(\alpha+1)} \right. \\
 &\times \left. (\bar{F}_{EHL-G}(x; \alpha, \zeta))^{v(\beta\theta-1)} \left[1 - (1-\theta) \left(1 - (\bar{F}_{EHL-G}(x; \alpha, \zeta))^\beta \right) \right]^{-v(\theta+1)} dx \right].
 \end{aligned}$$

Applying the series expansions (see in the appendix).

We have

$$\begin{aligned}
 I_R(v) &= (1-v)^{-1} \log \left[(2\alpha\beta\theta^2)^v \sum_{q,s,j,t,p,k=0}^{\infty} \frac{(1-\theta)^q(-1)^{q+s+j+t+k}}{(k+1)} \binom{-v(\theta+1)}{q} \binom{q}{s} \right. \\
 &\times \binom{\beta s + v(\beta\theta-1)}{j} \binom{\alpha(j+v)-v}{t} \binom{-(\alpha(j+v)+v)}{p} \binom{p+t}{k} \frac{1}{\left[1 + \frac{k}{v}\right]^v} \\
 &\times \int_{-\infty}^{\infty} \left(\left[1 + \frac{k}{v}\right] (G_B(x; \zeta))^{\frac{k}{v}} g_B(x; \zeta) \right)^v \Bigg] \\
 &= \frac{1}{1-v} \log \left[\sum_{k=0}^{\infty} d_k^* e^{(1-v)I_{REG}} \right], \tag{6.35}
 \end{aligned}$$

where

$$\begin{aligned}
 d_k^* &= (2\alpha\beta\theta^2)^v \sum_{q,s,j,t,p=0}^{\infty} \frac{(1-\theta)^q (-1)^{q+s+j+t+k}}{k+1} \binom{-v(\theta+1)}{q} \binom{q}{s} \binom{\beta s + v(\beta\theta - 1)}{j} \\
 &\times \binom{-(\alpha(j+v) + v)}{t} \binom{-(\alpha(j+v) + v)}{p} \binom{p+t}{k} \frac{1}{[1 + \frac{k}{v}]^v}, \quad (6.36)
 \end{aligned}$$

for $v > 0$, $v \neq 1$, and $I_{REG} = (1-v)^{-1} \log \left[\int_{-\infty}^{\infty} \left([1 + \frac{k}{v}] (G_B(x; \zeta))^{\frac{k}{v}} g_B(x; \zeta) \right)^v dx \right]$ is the Rényi entropy of the Exp-G distribution with power parameter $(1 + \frac{k}{v})$.

6.2.8 Maximum Likelihood Estimation

In this sub-section, we use the MLE technique to estimate the parameters of the HT-GEN-EHL-G GFPD. Let $X \sim HT-GEN-EHL-G(\theta, \alpha, \beta, \zeta)$ and $\Delta = (\theta, \alpha, \beta, \zeta)^T$ be the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the HT-GEN-EHL-G GFPD is given by

$$\begin{aligned}
 \ell_n(\Delta) &= (n) \ln(2\alpha\beta\theta^2) + \sum_{i=1}^n \ln(g_B(x_i; \zeta)) + (\alpha - 1) \sum_{i=1}^n \ln(G_B(x_i; \zeta)) \\
 &- (\alpha - 1) \sum_{i=1}^n \ln(1 + \bar{G}_B(x_i; \zeta)) + (\beta\theta - 1) \sum_{i=1}^n \ln(\bar{F}_{EHL-G}(x; \alpha, \zeta)) \\
 &- (\theta + 1) \sum_{i=1}^n \ln \left[1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-G}(x; \alpha, \zeta))^{\beta} \right) \right].
 \end{aligned}$$

In order to obtain the estimates of the unknown parameters from the HT-GEN-EHL-G GFPD, we solve $U = (\frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \zeta})^T = \mathbf{0}$. However, these equations are not in closed form, hence cannot be solved explicitly. To obtain the results we use numerical methods such as Newton-Raphson procedure in R. The elements of the score vector U are given in the appendix.

6.2.9 Simulations

In this section, we assess the performance of MLEs. A simulation study was carried out by generating $N = 3000$ random samples from the heavy-tailed generalized exponentiated half logistic-log-logistic (HT-GEN-EHL-LLoG) distribution for various sample sizes of $n=25, 50, 100, 200, 400$, and 800 .

The performance of the estimates was judged using the mean estimate (Mean), average bias (ABias) and root mean square error (RMSE). The ABias and RMSE for the estimated

parameter, say, $\hat{\psi}$, are given by:

$$ABias(\hat{\psi}) = \frac{1}{N} \sum_{i=1}^N (\hat{\psi}_i - \psi), \quad \text{and} \quad RMSE(\hat{\psi}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\psi}_i - \psi)^2}{N}},$$

respectively.

(1.3, 0.8, 1.1, 0.7)					(1.1, 1.1, 0.7, 1.1)		
Parameter	Sample Size	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
θ	25	2.2900	2.2846	0.9900	1.5317	0.9401	0.4317
	50	1.7571	1.3008	0.4571	1.4817	0.8883	0.3817
	100	1.4562	0.7131	0.1562	1.4300	0.7992	0.3300
	200	1.3360	0.5192	0.0360	1.3330	0.6279	0.2330
	400	1.3185	0.2847	0.0185	1.2267	0.4107	0.1267
	800	1.3138	0.2321	0.0138	1.1838	0.2578	0.0838
α	25	0.8667	0.3455	0.0667	1.6136	1.0195	0.5136
	50	0.8604	0.2796	0.0604	1.4547	0.7577	0.3547
	100	0.8387	0.2211	0.0387	1.3591	0.5887	0.2591
	200	0.8292	0.1767	0.0292	1.2855	0.4560	0.1855
	400	0.8096	0.1394	0.0096	1.2147	0.3252	0.1147
	800	0.8056	0.1106	0.0056	1.1929	0.2330	0.0929
b	25	1.4398	1.5736	0.3398	0.6351	0.3163	-0.0648
	50	1.2899	1.0615	0.1899	0.6410	0.2419	-0.0589
	100	1.2071	0.6805	0.1071	0.6498	0.1914	-0.0501
	200	1.1537	0.2794	0.0537	0.6591	0.1481	-0.0408
	400	1.1364	0.1642	0.0364	0.6712	0.1096	-0.0287
	800	1.1193	0.1106	0.0193	0.6818	0.0804	-0.0181
c	25	0.7986	0.6778	0.0986	3.0771	8.5920	1.9771
	50	0.7298	0.4365	0.0298	1.6230	2.8247	0.5230
	100	0.7161	0.2969	0.0161	1.2179	0.9114	0.1179
	200	0.7073	0.1869	0.0073	1.1254	0.3668	0.0254
	400	0.7068	0.1282	0.0068	1.1151	0.2331	0.0151
	800	0.7019	0.1080	0.0019	1.1150	0.1543	0.0150

Table 6.21: Monte Carlo Simulation Results

(1.3, 0.8, 1.3, 1.3)					(0.5, 0.7, 1.1, 1.1)		
Parameter	Sample Size	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
θ	25	1.5933	0.9817	0.2933	0.9223	2.8341	0.4223
	50	1.5987	0.9974	0.2987	0.6122	1.6025	0.1122
	100	1.5614	0.9320	0.2614	0.4634	0.1933	-0.0365
	200	1.4791	0.6800	0.1791	0.4686	0.1508	-0.0313
	400	1.4382	0.5177	0.1382	0.4714	0.1170	-0.0285
	800	1.3728	0.3429	0.0728	0.4834	0.0821	-0.0165
α	25	0.7537	0.3849	-0.0462	0.5570	0.3991	-0.1429
	50	0.7547	0.2903	-0.0452	0.5617	0.3349	-0.138
	100	0.7605	0.2372	-0.0394	0.6005	0.2759	-0.0994
	200	0.7650	0.1890	-0.0349	0.6255	0.2065	-0.0744
	400	0.7729	0.1472	-0.0270	0.6579	0.1450	-0.0420
	800	0.7742	0.1154	-0.0257	0.6807	0.1029	-0.0192
b	25	1.6600	0.8605	0.3600	1.5450	0.9446	0.4450
	50	1.6343	0.8443	0.3343	1.4631	0.8472	0.3631
	100	1.6262	0.7910	0.3262	1.3802	0.7165	0.2802
	200	1.5519	0.6602	0.2519	1.2458	0.4407	0.1458
	400	1.4666	0.5111	0.1666	1.1580	0.2102	0.0580
	800	1.4120	0.3714	0.1120	1.1290	0.1098	0.0290
c	25	1.8153	1.1167	0.5153	2.3603	1.9729	1.2603
	50	1.6420	0.8214	0.3420	2.0786	1.6155	0.9786
	100	1.5581	0.6660	0.2581	1.7554	1.2459	0.6554
	200	1.4899	0.5252	0.1899	1.4887	0.8399	0.3887
	400	1.4282	0.4088	0.1282	1.2864	0.5024	0.1864
	800	1.4003	0.3247	0.1003	1.1433	0.2673	0.0433

Table 6.22: Monte Carlo Simulation Results

The results presented in Tables 6.21 and 6.22 indicate the the mean values approximate

the true parameter values, RMSE and ABias decreases with increasing sample size for all the parameter values. Thus, we conclude that MLEs of the HT-GEN-EHL-LLoG distribution performs quite well.

6.2.10 Risk Measures

In this section, we discuss risk metrics including: value at risk (VaR), tail value at risk (TVaR), tail variance (TV), and tail variance premium (TVP) which are routinely employed by financial and actuarial professionals to assess the exposure to market risk in a portfolio of instruments.

6.2.11 Value at Risk

VaR is an actuarial measure that is frequently used to assess risk in the financial markets. It is often referred to as the quantile risk measure or the quantile premium principle, and it is always provided with a stated degree of confidence, such as (90%, 95%, or 99%). The VaR of the HT-GEN-EHL-G GFPD is given by

$$VaR_q = G^{-1} \left[2 \left(\left[1 - \left(\theta \left[(\bar{q})^{\frac{-1}{\theta}} - (1 - \theta) \right]^{-1} \right)^{\frac{1}{\beta}} \right]^{\frac{-1}{\alpha}} + 1 \right)^{-1} \right], \quad (6.37)$$

where $q \in (0, 1)$ is a specified level of significance and $\bar{q} = (1 - q)$.

6.2.12 Tail Value at Risk

Another significant and extensively used actuarial measure is the TVaR, which is used to express the expected value of loss in the case that an event beyond the predetermined probability threshold has actually occurred. The conditional tail expectation (CTE) or tail conditional expectation (TCE) are some of its other names. The TVaR of the HT-GEN-EHL-G GFPD is defined as

$$\begin{aligned} TVaR_q &= E(X | X > x_q) = \frac{1}{1 - q} \int_{VaR_q}^{\infty} x f(x) dx \\ &= \frac{1}{1 - q} \sum_{k=0}^{\infty} \int_{VaR_q}^{\infty} x b_{k+1}^* g_{k+1}^*(x; \zeta) dx, \end{aligned} \quad (6.38)$$

where b_{k+1}^* is given by equation (6.29) and $g_{k+1}^*(x; \zeta) = (k + 1)[G_B(x; \zeta)]^k g_B(x; \zeta)$ is the pdf of Exp-G distribution with the power parameter $(k + 1)$.

6.2.13 Tail Variance

One of the most significant actuarial measures that examines variation outside of the VaR is the tail variance. The TV of the HT-GEN-EHL-G GFPD is given by

$$\begin{aligned}
TV_q &= E(X^2 | X > x_q) - (TVaR_q)^2 \\
&= \frac{1}{1-q} \int_{VaR_q}^{\infty} x^2 f(x) dx - (TVaR_q)^2 \\
&= \frac{1}{1-q} \sum_{k=0}^{\infty} b_{k+1}^* \int_{VaR_q}^{\infty} x^2 g_{k+1}^*(x; \zeta) dx - (TVaR_q)^2,
\end{aligned} \tag{6.39}$$

where $g_{k+1}^*(x; \zeta) = (k+1)[G_B(x; \zeta)]^k g_B(x; \zeta)$ is the pdf of Exp-G distribution with the power parameter $(k+1)$ and parameter vector ζ , and b_{k+1}^* is given by equation (6.29). Thus, TV of HT-GEN-EHL-G GFPD can be obtained from those of Exp-G distribution.

6.2.14 Tail Variance Premium

The TVP is a significant actuarial measure that is crucial to the study of insurance. The TVP of the HT-GEN-EHL-G family distributions has the following shape:

$$TVP_q = TVaR_q + \delta(TV_q), \tag{6.40}$$

where $0 < \delta < 1$. The TVP of the HT-GEN-EHL-G GFPD can be obtained by substituting the equations (6.38) and (6.39) into equation (6.40).

6.2.15 Numerical Study for the Risk Measures

This section is devoted to the numerical study of the risk measures; VaR, TVaR, TV and TVP via simulation results. The VaR, TVaR, TV and TVP of the heavy-tailed-generalised exponentiated half logistic-log-logistic (HT-GEN-EHL-LLoG) distribution are compared with those of the generalised exponentiated half logistic-log-logistic (GEN-EHL-LLoG) distribution, exponentiated half logistic-log-logistic (EHL-LLoG) distribution, log-logistic (LLoG) distribution, alpha power exponentiated log-logistic (APExLLD) distribution by Teamah et al. (2021), type-I heavy-tailed Weibull (TI-HT-W) distribution by Zhao et al. (2021), and alpha power Weibull (APW) distribution by Nassar et al. (2017).

Simulation results are obtained as follows:

1. Random samples of size $n = 100$ are generated from each one of the distributions and parameters have been estimated via maximum likelihood method.
2. 1000 repetitions are made to calculate the VaR, TVaR, TV and TVP for these distributions.

Tables 6.23 and 6.24 show the numerical findings of VaR, TVaR, TV and TVP for the seven compared distributions. A model with higher values of VaR, TVaR, TV and TVP is said to have a heavier tail. From the figures in Tables 6.23 and 6.24, we conclude that the HT-GEN-EHL-LLoG distribution has heavier tail than the GEN-EHL-LLoG, EHL-LLoG, LLoG, APExLLD, TI-HT-W and APW distributions, hence it is suitable for modelling heavy-tailed data.

Significance level		0.7	0.75	0.8	0.85	0.9	0.95
HT-GEN-EHL-LLoG($\theta = 1.0, \alpha = 0.9, b = 1.3, c = 1.2$)	VaR	2.3195	2.7796	3.4041	4.3257	5.8954	9.5552
	TVaR	3.7158	3.9512	4.1704	5.2835	7.9110	12.4236
	TV	98.8350	118.2512	147.5152	196.4162	293.0749	549.0890
	TVP	72.9004	92.6396	122.1826	171.2373	267.6784	522.0581
GEN-EHL-LLoG($\alpha = 0.9, b = 1.8, c = 1.2$)	VaR	1.5093	1.7843	2.1476	2.6658	3.5080	5.3306
	TVaR	2.9948	3.2656	3.5928	3.9940	4.4676	4.6651
	TV	27.3131	32.3323	39.8748	52.5097	78.0143	154.8006
	TVP	22.1140	27.5148	35.4927	48.6272	74.6805	151.7257
EHL-LLoG($b = 1.8, c = 1.2$)	VaR	1.7160	2.0227	2.4291	3.0111	3.9623	6.0400
	TVaR	3.2330	3.5071	3.8300	4.2065	4.5894	4.3498
	TV	36.0763	42.8387	53.0220	70.1086	104.5927	207.2717
	TVP	28.4865	35.6361	46.2476	63.7989	98.7229	201.2579
LLoG($c = 1.2$)	VaR	1.6965	1.8496	2.0420	2.3021	2.6999	3.4938
	TVaR	2.6635	2.8421	3.0672	3.3681	3.8100	4.5821
	TV	4.0172	4.6265	5.5261	7.0004	9.9036	18.5599
	TVP	5.4756	6.3120	7.4881	9.3184	12.7233	22.2140
APExLLD($\alpha = 0.5, a = 0.9, b = 1.3, c = 0.2$)	VaR	0.1093	0.1907	0.3395	0.6327	1.3061	3.5984
	TVaR	2.6487	3.1492	3.8721	5.0067	7.0513	11.9344
	TV	56.9289	66.5748	80.1787	100.8745	136.4837	216.0148
	TVP	42.4990	53.0803	68.0151	90.7500	129.8867	217.1484
TIHT-W($\theta = 1.3, \alpha = 1.0, \beta = 0.9$)	VaR	0.8135	1.0051	1.2578	1.6119	2.1633	3.2405
	TVaR	2.1951	2.4530	2.7850	3.2388	3.9262	5.2259
	TV	3.0846	3.2971	3.5627	3.9137	4.4241	5.3359
	TVP	4.3544	4.9259	5.6352	6.5655	7.9079	10.2951
APW($\alpha = 1.5, \beta = 0.9, \lambda = 1.3$)	VaR	0.8689	1.0201	1.2090	1.4580	1.8179	2.4525
	TVaR	1.6567	1.7872	1.9415	2.1328	2.3918	2.8233
	TV	0.7782	0.8367	0.9294	1.0803	1.3371	1.7921
	TVP	2.2015	2.4148	2.6851	3.0511	3.5952	4.5258

Table 6.23: Simulation Results of VaR, TVaR, TV and TVP

Significance level		0.7	0.75	0.8	0.85	0.9	0.95
HT-GEN-EHL-LLoG($\theta = 1.5, \alpha = 1.9, b = 0.8, c = 0.9$)	VaR	6.1149	7.5904	9.7399	13.2029	19.8450	38.6519
	TVaR	17.7573	19.9465	22.7884	26.4085	31.8615	36.5175
	TV	284.7419	312.5032	349.5170	407.2236	524.5157	903.4977
	TVP	217.0766	254.3239	302.6391	372.7486	503.9257	894.8404
GEN-EHL-LLoG($\alpha = 1.9, b = 1.0, c = 2.9$)	VaR	2.1943	2.3744	2.6014	2.9089	3.3793	4.3154
	TVaR	3.2568	3.4519	3.6941	4.0100	4.4529	5.1275
	TV	6.3073	7.3365	8.8721	11.4227	16.5306	32.0426
	TVP	7.6719	8.9543	10.7918	13.7193	19.3305	35.5681
EHL-LLoG($b = 1.8, c = 1.9$)	VaR	1.4059	1.5591	1.7494	2.0023	2.3791	3.0998
	TVaR	2.3439	2.5166	2.7332	3.0213	3.444	4.1990
	TV	2.6044	2.8225	3.2885	4.0449	5.164	9.8382
	TVP	4.0970	4.6336	5.3640	6.4595	8.4089	13.5453
LLoG($c = 2.9$)	VaR	1.7045	1.8597	2.0560	2.3192	2.7238	3.5329
	TVaR	2.6835	2.8643	3.0921	3.3964	3.8424	4.6166
	TV	2.6835	2.8643	3.0921	3.3964	3.8424	4.6166
	TVP	5.6498	6.5290	7.7678	9.7007	13.3082	23.4066
APExLLD($\alpha = 0.8, a = 1.9, b = 0.3, c = 2.2$)	VaR	0.7674	0.7973	0.9173	1.0916	1.3843	2.0588
	TVaR	1.6623	1.8446	2.0923	2.4570	3.0740	4.4862
	TV	6.7109	7.8345	9.4544	12.0116	16.7216	28.7950
	TVP	6.3599	7.7206	9.6558	12.6669	18.1235	31.8415
TIHT-W($\theta = 0.3, \alpha = 0.8, \beta = 1.9$)	VaR	3.4337	4.2421	5.3084	6.8032	9.1301	13.6763
	TVaR	9.2644	10.3531	11.7542	13.6693	16.5705	22.05603
	TV	54.9557	58.7411	63.4720	69.7257	78.8151	95.0706
	TVP	47.7334	54.4090	62.5319	72.9362	87.5041	112.3731
APW($\alpha = 1.5, \beta = 0.9, \lambda = 1.3$)	VaR	2.1550	2.3150	2.4971	2.7144	2.9950	3.4241
	TVaR	2.0266	2.1310	2.2775	2.4988	2.8806	3.7666
	TV	2.3028	2.4629	2.6156	2.7181	2.8845	0.8814
	TVP	3.6386	3.9783	4.3700	4.8092	5.2067	4.6040

Table 6.24: Simulation Results of VaR, TVaR, TV and TVP

6.2.16 Special Cases

In this section, we define special cases of the HT-GEN-EHL-G GPDF by specifying the baseline distribution. Plots of the pdfs, hrf and 3D plots of skewness and kurtosis are presented.

6.2.17 HT-GEN-EHL-Weibull (HT-GEN-EHL-W) Distribution

Let the one parameter Weibull distribution be the baseline distribution with pdf and cdf given by $g(x; a) = ax^{a-1} \exp(-x^a)$ and $G(x; a) = 1 - \exp(-x^a)$, for $a > 0$, respectively. Then, we define the cdf and pdf of HT-GEN-EHL-W distribution as

$$F_{HT-GEN-EHL-W}(x; \theta, \alpha, \beta, a) = 1 - \left[\frac{(\bar{F}_{EHL-W}(x; \alpha, a))^\beta}{1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-W}(x; \alpha, a))^\beta\right)} \right]^\theta$$

and

$$\begin{aligned}
 f_{HT-GEN-EHL-W}(x; \theta, \alpha, \beta, a) &= 2\alpha\beta\theta^2 a x^{a-1} \exp(-x^a) (1 - \exp(-x^a))^{\alpha-1} \\
 &\times (1 + \exp(-x^a))^{-(\alpha+1)} (\bar{F}_{EHL-W}(x; \alpha, a))^{\beta\theta-1} \\
 &\times \left[1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-W}(x; \alpha, a))^\beta \right) \right]^{-(\theta+1)},
 \end{aligned}$$

respectively, for $\theta, \alpha, \beta, a > 0$ being shape parameters and $x > 0$. The hrf is given by

$$\begin{aligned}
 h_F(x; \theta, \alpha, \beta, a) &= 2\alpha\beta\theta^2 a x^{a-1} \exp(-x^a) (1 - \exp(-x^a))^{\alpha-1} \\
 &\times (1 + \exp(-x^a))^{-(\alpha+1)} (\bar{F}_{EHL-W}(x; \alpha, a))^{-1} \\
 &\times \left[1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-W}(x; \alpha, a))^\beta \right) \right]^{-1},
 \end{aligned}$$

for $\theta, \alpha, \beta, a > 0$, where

$$\bar{F}_{EHL-W}(x; \alpha, a) = 1 - \left[\frac{1 - \exp(-x^a)}{1 + \exp(-x^a)} \right]^\alpha.$$

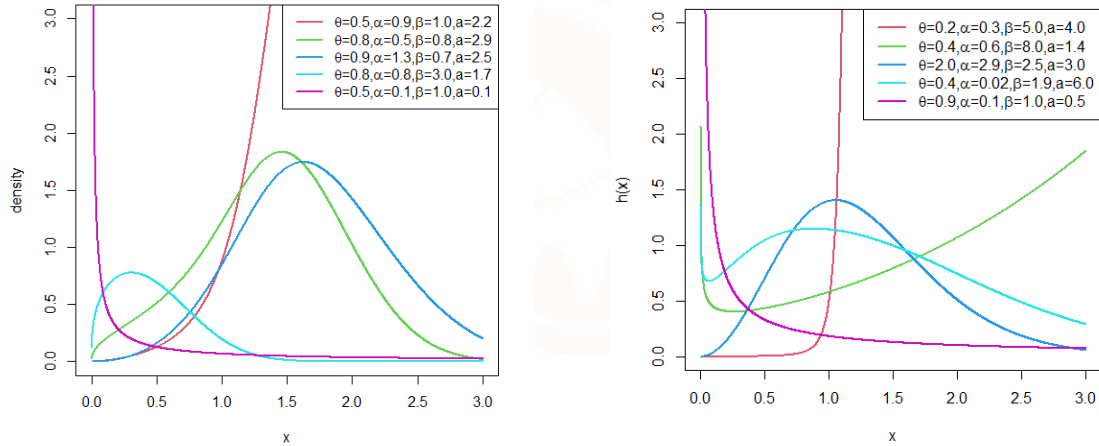


Figure 6.21: Plots of the pdf and hrf for HT-GEN-EHL-W distribution

Plots of the pdf and hrf of the HT-GEN-EHL-W distribution are presented in Figure 6.21. The pdf can be J-shaped, reverse-J, unimodal, right-skewed and left-skewed. On the other hand, the hrf of the HT-GEN-EHL-W distribution can be decreasing, increasing, bathtub, upside-down bathtub, and bathtub followed by upside-down bathtub shaped.

	$(\theta, \alpha, \beta, a)$				
u	(0.3, 3.9, 5.4, 1.9)	(0.1, 0.5, 0.8, 1)	(0.7, 1.5, 15, 10.3)	(1.7, 13.3, 0.3, 14.8)	(3.2, 1.8, 9.5, 3.3)
0.1	1.2097	3.7118	0.8115	1.0639	0.3917
0.2	1.3362	5.5373	0.8472	1.0762	0.4448
0.3	1.4247	7.3041	0.8751	1.0810	0.4835
0.4	1.4939	9.2567	0.8928	1.0880	0.5168
0.5	1.5651	11.5414	0.9064	1.0984	0.5435
0.6	1.6375	14.3318	0.9213	1.1036	0.5720
0.7	1.7164	17.9255	0.9376	1.1106	0.6035
0.8	1.8130	22.9950	0.9533	1.1180	0.6363
0.9	1.9592	31.6614	0.9752	1.1369	0.6818

Table 6.25: Table of Quantile for HT-GEN-EHL-W Distribution

	$(\theta, \alpha, \beta, a)$				
	(2.5,0.9,0.3,0.7)	(0.2,1.4,8.3,0.5)	(0.08,0.1,7.6,0.9)	(0.07,0.6,2.3,0.3)	(0.9,0.5,1.5,0.1)
$E(X)$	0.2354	0.1331	0.2754	0.0219	0.0192
$E(X^2)$	0.1353	0.0869	0.1618	0.0139	0.0095
$E(X^3)$	0.0941	0.0641	0.1151	0.0102	0.0062
$E(X^4)$	0.0719	0.0506	0.0895	0.00815	0.0047
$E(X^5)$	0.0581	0.0417	0.0732	0.0067	0.00375
$E(X^6)$	0.0487	0.0355	0.0620	0.0057	0.0031
SD	0.2827	0.2631	0.2932	0.1161	0.0955
CV	1.2010	1.9753	1.0647	5.3032	4.9529
CS	1.0894	1.8739	0.9224	5.9938	6.6021
CK	2.9856	5.1733	2.5743	40.0434	50.8681

Table 6.26: Moments of HT-GEN-EHL-W Distribution for Selected Parameters

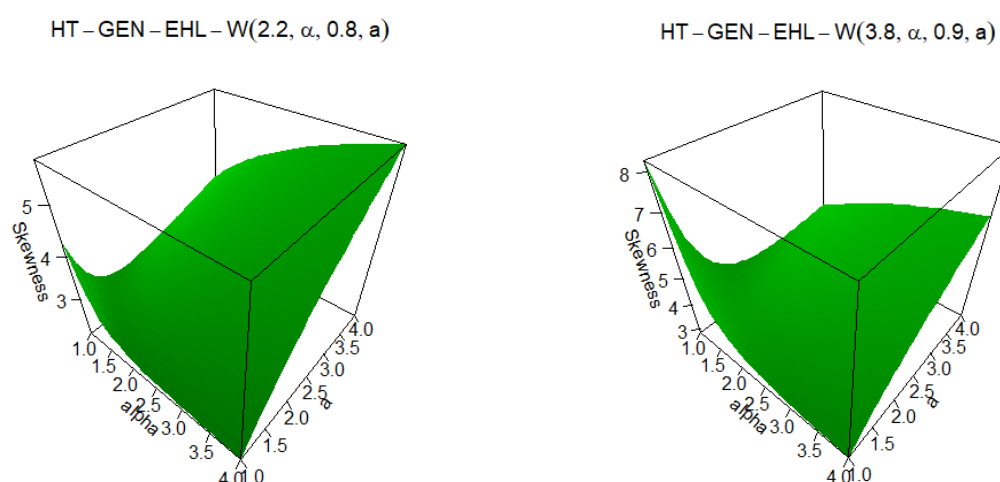


Figure 6.22: 3D Plots of the skewness for HT-GEN-EHL-W distribution

Figure 6.22 shows plots of skewness for the HT-GEN-EHL-W distribution. We can see that

for higher values of α and a , there is an increase in skewness, low values of skewness are observed for high α and low values of a , when $\theta = 2.2$ and $\beta = 0.8$. Furthermore, high values of the HT-GEN-EHL-W skewness are noticed for lower values of α and a , when $\theta = 3.8$ and $\beta = 0.9$.

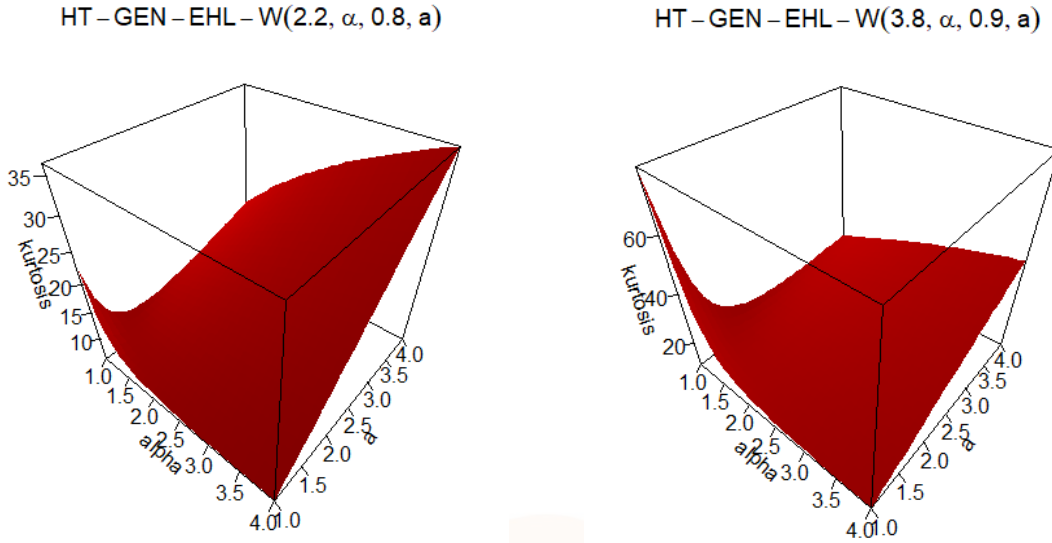


Figure 6.23: 3D Plots of the kurtosis for HT-GEN-EHL-W distribution

The HT-GEN-EHL-W has its highest kurtosis in Figure 6.23 for large values of α and a , low values of kurtosis are observed for high α and low values of a , when $\theta = 2.2$ and $\beta = 0.8$. On another note, high values of kurtosis are noticed for lower values of α and a , when $\theta = 3.8$ and $\beta = 0.9$.

6.2.18 HT-GEN-EHL-Log-Logistic (HT-GEN-EHL-LLoG) Distribution

Consider the log-logistic distribution as the baseline distribution with parameter $c > 0$ having cdf and pdf $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$, respectively. Then, the cdf and pdf of HT-GEN-EHL-LLoG distribution are given by

$$F_{HT-GEN-EHL-LLoG}(x; \theta, \alpha, \beta, c) = 1 - \left[\frac{(\bar{F}_{EHL-LLoG}(x; \alpha, c))^\beta}{1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-LLoG}(x; \alpha, c))^\beta\right)} \right]^\theta$$

and

$$\begin{aligned} f_{HT-GEN-EHL-LLoG}(x; \theta, \alpha, \beta, c) &= 2\alpha\beta\theta^2 cx^{c-1}(1 + x^c)^{-2}(1 - (1 + x^c)^{-1})^{\alpha-1} \\ &\times (1 + (1 + x^c)^{-1})^{-(\alpha+1)} (\bar{F}_{EHL-LLoG}(x; \alpha, c))^{\beta\theta-1} \\ &\times \left[1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-LLoG}(x; \alpha, c))^\beta\right) \right]^{-(\theta+1)}, \end{aligned}$$

respectively, for $\theta, \alpha, \beta, c > 0$ being shape parameters and $x > 0$. The hrf of the HT-GEN-EHL-LLoG distribution is given by

$$h_F(x; \theta, \alpha, \beta, c) = 2\alpha\beta\theta^2 cx^{c-1}(1+x^c)^{-2}(1-(1+x^c)^{-1})^{\alpha-1}(1+(1+x^c)^{-1})^{-(\alpha+1)} \\ \times (\bar{F}_{EHL-LLoG}(x; \alpha, c))^{-1} \left[1 - (1-\theta) \left(1 - (\bar{F}_{EHL-LLoG}(x; \alpha, c))^\beta \right) \right]^{-1},$$

for $\theta, \alpha, \beta, c > 0$, where

$$\bar{F}_{EHL-LLoG}(x; \alpha, c) = 1 - \left[\frac{1 - (1+x^c)^{-1}}{1 + (1+x^c)^{-1}} \right]^\alpha.$$

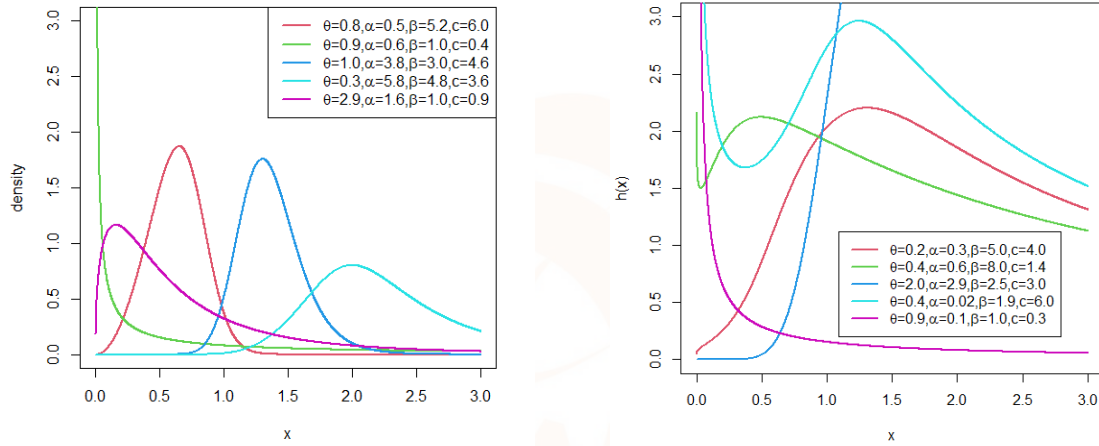


Figure 6.24: Plots of the pdf and hrf for HT-GEN-EHL-LLoG distribution

Figure 6.24 shows the plots of the pdf and the hrf of the HT-GEN-EHL-LLoG distribution for different parameter values. The pdf of the HT-GEN-EHL-LLoG distribution can be J-shaped, unimodal, right-skewed and left-skewed. The shapes of the hrf of the HT-GEN-EHL-LLoG distribution include decreasing, increasing, upside-down bathtub, and bathtub followed by upside-down bathtub.

u	$(\theta, \alpha, \beta, c)$				
	(2.3, 8.9, 4.4, 2.9)	(2.7, 3.7, 1.8, 0.8)	(0.7, 0.5, 1.3, 0.7)	(1.2, 1.9, 0.4, 1.6)	(13.2, 0.8, 3.5, 7.3)
0.1	1.3515	0.6962	0.0111	1.1705	0.2482
0.2	1.4491	1.0050	0.0815	1.7632	0.2831
0.3	1.5204	1.2913	0.2562	2.4136	0.3071
0.4	1.583	1.5893	0.6082	3.2149	0.3264
0.5	1.6442	1.9259	1.2879	4.3082	0.3458
0.6	1.7071	2.3283	2.6178	5.9524	0.3618
0.7	1.7750	2.8517	5.4899	8.7906	0.3827
0.8	1.8638	3.6246	13.1470	14.8958	0.4000
0.9	1.9950	5.1118	47.3990	36.1252	0.4276

Table 6.27: Table of Quantile for HT-GEN-EHL-LLoG Distribution

	$(\theta, \alpha, \beta, c)$				
	(1.5,1.9,0.9,0.2)	(2.2,0.4,1.3,0.5)	(0.8,3.1,4.6,3.9)	(0.7,1.6,0.3,0.6)	(0.7,1.6,0.3,0.6)
$E(X)$	0.0453	0.0192	0.0861	0.0118	0.0195
$E(X^2)$	0.0245	0.0071	0.0781	0.0074	0.0136
$E(X^3)$	0.0168	0.0043	0.0713	0.0053	0.0104
$E(X^4)$	0.0127	0.0030	0.0656	0.0042	0.0084
$E(X^5)$	0.0102	0.0023	0.0607	0.0034	0.0071
$E(X^6)$	0.0086	0.0019	0.0564	0.0029	0.0061
SD	0.1500	0.0825	0.2658	0.0852	0.1153
CV	3.3063	4.2948	3.0854	7.2068	5.8944
CS	4.0409	6.9260	2.7907	8.2527	6.3177
CK	19.7399	58.8597	8.8760	74.9976	43.4547

Table 6.28: Moments of HT-GEN-EHL-LLoG Distribution for Selected Parameters

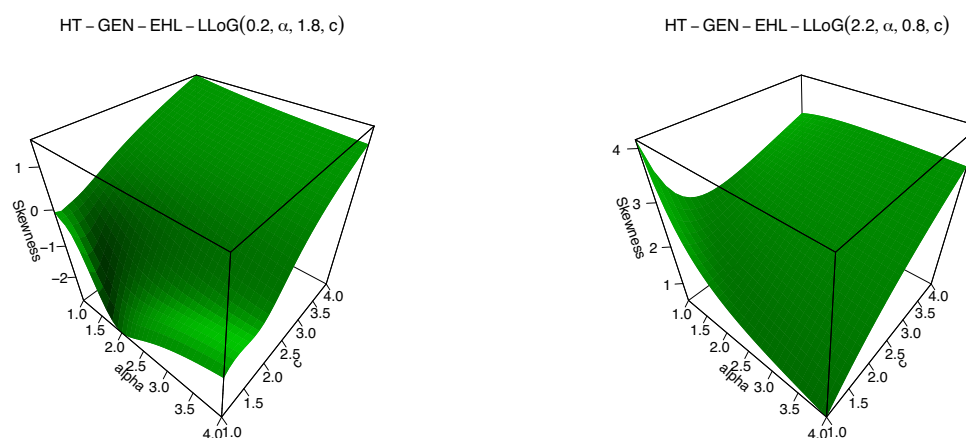


Figure 6.25: 3D Plots of the skewness for HT-GEN-EHL-LLoG distribution

As shown by Figure 6.25, for $\theta = 0.2$ and $\beta = 1.8$, the HT-GEN-EHL-LLoG exhibits higher

levels of skewness for larger values of the parameter c across lower values of α . We can also see high values of skewness for small values of α and larger values of c , when $\theta = 2.2$ and $\beta = 0.8$.

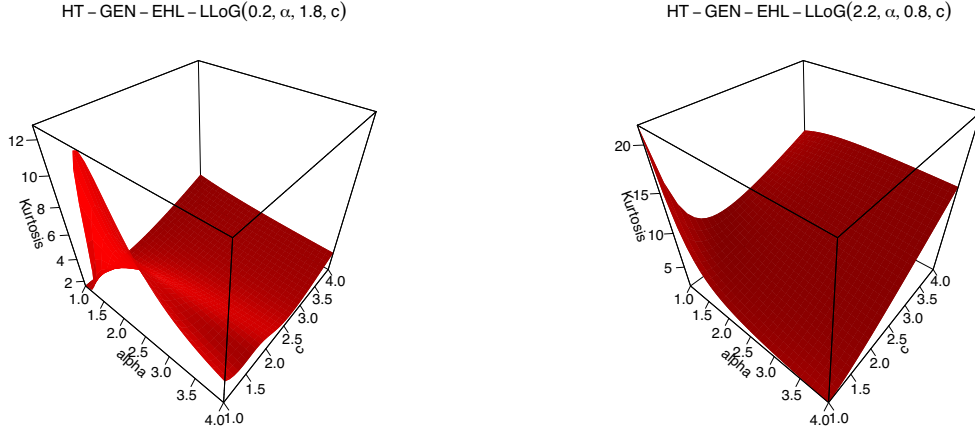


Figure 6.26: 3D Plots of the kurtosis for HT-GEN-EHL-LLoG distribution

Figure 6.26 shows that the HT-GEN-EHL-LLoG distribution has the highest values of kurtosis for smaller values of the parameter c and the mid-range values of α when $\theta = 0.2$ and $\beta = 1.8$. We can see high values of kurtosis for lower values of both α and c , the kurtosis also reduces for higher values of c and smaller values of α , when $\theta = 2.2$ and $\beta = 0.8$.

6.2.19 HT-GEN-EHL-Standard Half Logistic (HT-GEN-EHL-SHL) Distribution

Let the baseline distribution be standard half logistic distribution with pdf and cdf given by $g(x) = \frac{2\exp(-x)}{(1+\exp(-x))^2}$ and $G(x) = \frac{1-\exp(-x)}{1+\exp(-x)}$, for $x > 0$, respectively. Then, the cdf and pdf of HT-GEN-EHL-SHL distribution are given by

$$F_{HT-GEN-EHL-SHL}(x; \theta, \alpha, \beta) = 1 - \left[\frac{(\bar{F}_{EHL-SHL}(x; \alpha))^\beta}{1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-SHL}(x; \alpha))^\beta\right)} \right]^\theta$$

and

$$\begin{aligned} f_{HT-GEN-EHL-SHL}(x; \theta, \alpha, \beta) &= 2\alpha\beta\theta^2 \frac{2\exp(-x)}{(1+\exp(-x))^2} ((1+\exp(-x)))^{\alpha-1} \\ &\times \left(2 - \frac{1-\exp(-x)}{1+\exp(-x)}\right)^{-(\alpha+1)} (\bar{F}_{EHL-SHL}(x; \alpha))^{\beta\theta-1} \\ &\times \left[1 - (1 - \theta) \left(1 - (\bar{F}_{EHL-SHL}(x; \alpha))^\beta\right)\right]^{-(\theta+1)}, \end{aligned}$$

respectively, for $\theta, \alpha, \beta > 0$ being shape parameters and $x > 0$. The corresponding hrf is given by

$$\begin{aligned} h_F(x; \theta, \alpha, \beta) &= 2\alpha\beta\theta^2 \frac{2\exp(-x)}{(1+\exp(-x))^2} \left(\frac{1-\exp(-x)}{1+\exp(-x)}\right)^{\alpha-1} \\ &\times \left(2 - \frac{1-\exp(-x)}{1+\exp(-x)}\right)^{-(\alpha+1)} (\bar{F}_{EHL-SHL}(x; \alpha))^{-1} \\ &\times \left[1 - (1-\theta) \left(1 - (\bar{F}_{EHL-SHL}(x; \alpha))^\beta\right)\right]^{-1}, \end{aligned}$$

for $\theta, \alpha, \beta > 0$, where

$$\bar{F}_{EHL-SHL}(x; \alpha) = 1 - \left[\frac{\frac{1-\exp(-x)}{1+\exp(-x)}}{2 - \frac{1-\exp(-x)}{1+\exp(-x)}} \right]^\alpha.$$

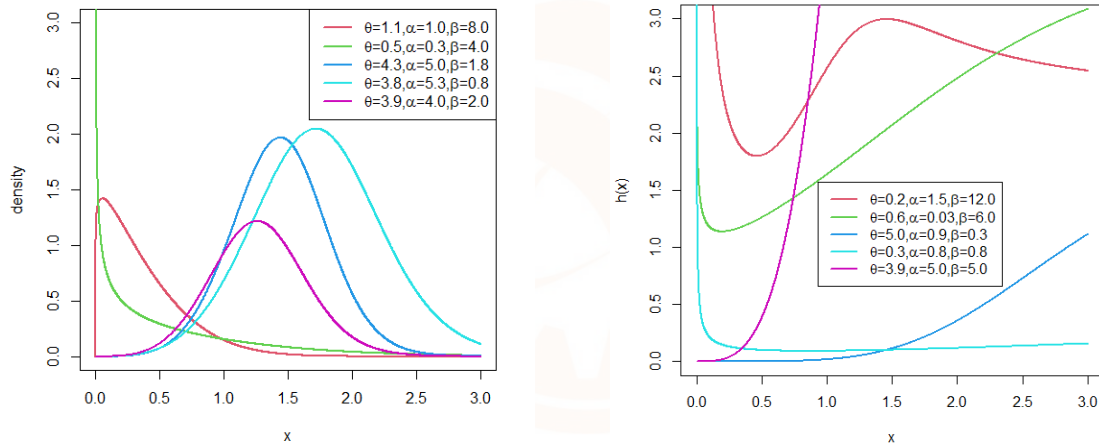


Figure 6.27: Plots of the pdf and hrf for HT-GEN-EHL-SHL distribution

Figure 6.27 shows that the pdf of the HT-GEN-EHL-SHL distribution can be reversed-J shaped, unimodal, right-skewed or left-skewed. Possible shapes of the hrf of the HT-GEN-EHL-SHL distribution include among others; decreasing, increasing, bathtub and bathtub followed by upside-down bathtub.

(θ, α, β)					
u	(0.2, 8.3, 2.4)	(0.5, 0.4, 0.9)	(10.7, 1.3, 6.8)	(6.8, 9.5, 2.9)	(0.8, 3.9, 0.1)
0.1	3.7442	0.2667	0.0050	1.5194	4.2460
0.2	4.2809	0.8255	0.0095	1.6456	6.1095
0.3	4.7013	1.3914	0.0120	1.7292	8.0657
0.4	5.1064	1.9408	0.01590	1.8003	10.2475
0.5	5.5377	2.5003	0.0206	1.8653	12.7646
0.6	6.0370	3.1085	0.0242	1.9301	15.7749
0.7	6.6573	3.8263	0.0309	1.9945	19.5741
0.8	7.5154	4.7789	0.0348	2.0703	24.8255
0.9	8.9679	6.3470	0.0462	2.1758	33.6652

Table 6.29: Table of Quantile for HT-GEN-EHL-SHL Distribution

(θ, α, β)					
	(0.5,2.2,1.8)	(4.8,1,0.9)	(0.5,0.4,2.5)	(1.2,0.09,5.9)	(2,0.03,1.2)
$E(X)$	0.3736	0.2590	0.2526	0.1114	0.0331
$E(X^2)$	0.1980	0.1074	0.1805	0.0956	0.0229
$E(X^3)$	0.1244	0.0560	0.1389	0.0836	0.0175
$E(X^4)$	0.0871	0.0336	0.1122	0.0742	0.0142
$E(X^5)$	0.0656	0.0222	0.0938	0.0666	0.0119
$E(X^6)$	0.0520	0.0157	0.0805	0.0604	0.0103
SD	0.2417	0.2008	0.3416	0.2884	0.1478
CV	0.6470	0.7754	1.3522	2.5875	4.4554
CS	0.4768	0.8964	0.8612	2.2666	4.7549
CK	2.5212	3.2819	2.1133	6.2960	25.2576

Table 6.30: Moments of HT-GEN-EHL-SHL Distribution for Selected Parameters

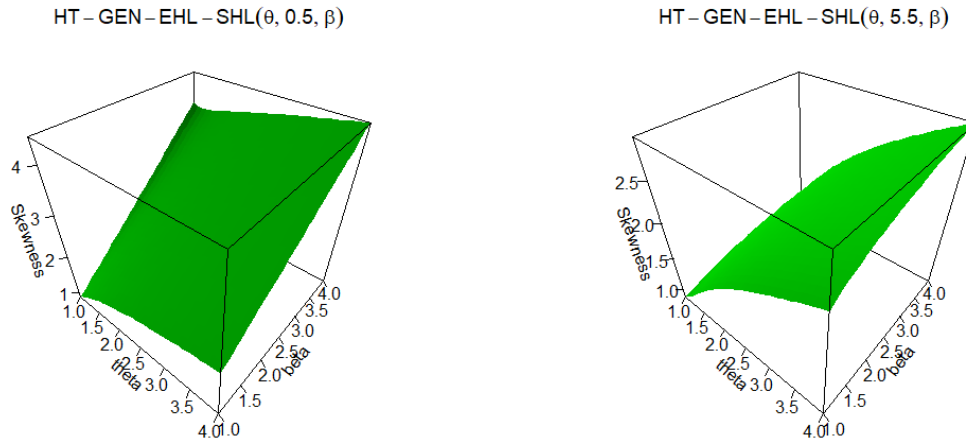


Figure 6.28: 3D Plots of the skewness for HT-GEN-EHL-SHL distribution

Figure 6.28 above shows a uniform increase in skewness when θ and β values are increasing, low values of skewness are observed for high θ and low values of β . For $\alpha = 5.5$, we observe highest values of the HT-GEN-EHL-SHL skewness for higher values of θ across all values of β .

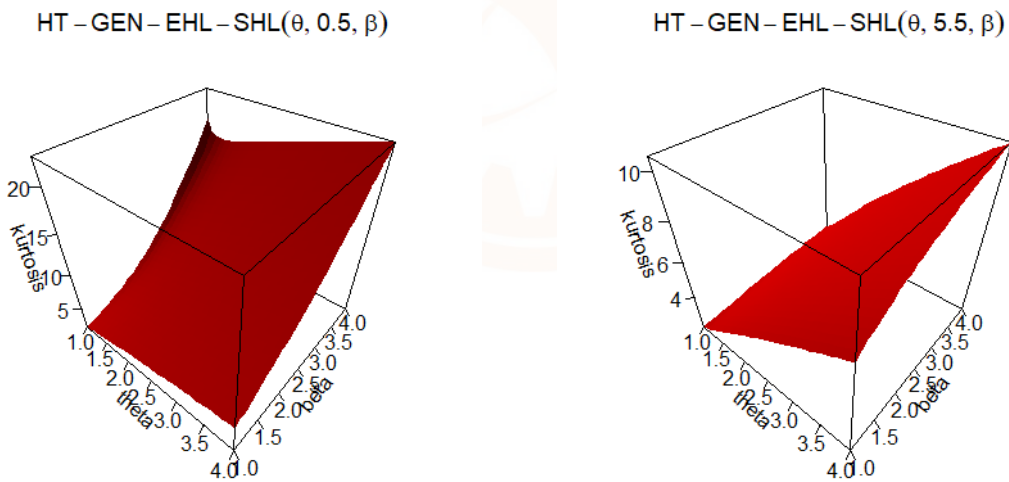


Figure 6.29: 3D Plots of the kurtosis for HT-GEN-EHL-SHL distribution

Figure 6.29 shows plots of kurtosis for the HT-GEN-EHL-SHL distribution. When $\alpha = 0.5$, we see lower values of kurtosis for low β values, across the range of θ values, higher values of kurtosis are observed for high θ and β . The highest values of the HT-GEN-EHL-SHL kurtosis can be observed for higher values of θ across all values of β , when $\alpha = 5.5$.

6.2.20 Inference

In this section, the special case of the HT-GEN-EHL-G GFPD, namely, the HT-GEN-EHL-LLoG distribution is fitted to four real data sets. The fit of the HT-GEN-EHL-LLoG distribution is compared to that of the new gamma exponentiated Lindley-log-logistic (GELLoG) distribution by Moakofi et al. (2020), alpha power exponentiated log-logistic distribution by Teamah et al. (2021), type-I heavy-tailed Weibull (TI-HT-W) distribution by Zhao et al. (2020), heavy-tailed beta-power transformed Weibull (HTBPT-W) distribution by Zhao et al. (2021), and alpha power Weibull (APW) distribution by Nassar et al. (2017). The pdf of these competing models are given in the appendix.

In order to determine which distribution the sample data might come from, the following well-known goodness-of-fit statistics: $-2\log$ -likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Consistent Akaike Information Criterion ($CAIC = AIC + 2\frac{p(p+1)}{n-p-1}$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), (where L is the likelihood function, n is the number of observations, and p is the number of estimated parameters), Cramér-von Mises (W^*) statistic, Anderson-Darling statistic (A^*), Kolmogorov-Smirnov (K-S) statistic and its p -value were considered. The data is likely to be obtained from the distribution with the smaller values of all the goodness-of-fit statistics, except for the p -value of K-S statistic.

6.2.21 Insurance Data

This data set is from the insurance and represents monthly metrics on unemployment insurance from July 2008 to April 2013 from the department of labor, licensing and regulation. It consists of 58 observations and 21 variables, we studied the variable number 6. It is available at: <https://catalog.data.gov/dataset/unemployment-insurance-data-july-2008-to-april-2013>. (See the data in the Appendix).

The estimated variance-covariance matrix for HT-GEN-EHL-LLoG distribution on insurance data is given by

$$\begin{bmatrix} 5.5540 \times 10^{-07} & -0.0012 & 4.2971 \times 10^{-05} & -1.3001 \times 10^{-05} \\ -1.1927 \times 10^{-03} & 2.561 & -9.2280 \times 10^{-02} & 2.7921 \times 10^{-02} \\ 4.2971 \times 10^{-05} & -0.0922 & 3.3246 \times 10^{-03} & -1.0059 \times 10^{-03} \\ -1.3001 \times 10^{-05} & 0.0279 & -1.0059 \times 10^{-03} & 3.0956 \times 10^{-04} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\theta \in [3.4296 \times 10^{03} \pm 0.0014]$, $\alpha \in [3.0125 \times 10^{01} \pm 3.1368]$, $\beta \in [2.2193 \times 10^{01} \pm 0.1130]$ and $c \in [2.0119 \times 10^{-01} \pm 0.0344]$, respectively.

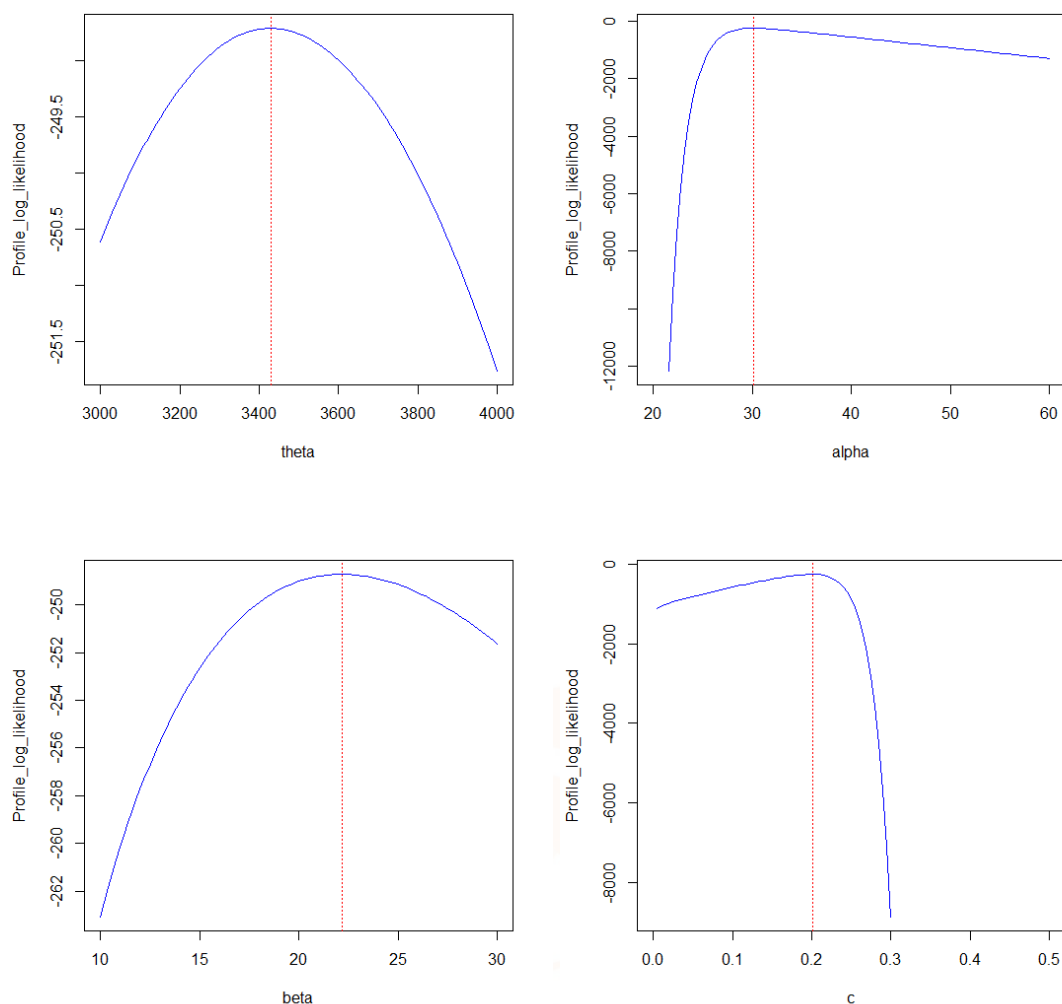


Figure 6.30: Profile log-likelihood function plots for parameters of HT-GEN-EHL-LLoG distribution on the insurance data set

Figure 6.30 shows the profile log-likelihood plots for parameters of the HT-GEN-EHL-LLoG distribution on the insurance data. It can be seen that the MLEs obtained are unique. This indicates that the parameters of the HT-GEN-EHL-LLoG distribution are indeed the correct maximums.

Model	Estimates				Statistics							
	θ	α	β	c	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
HT-GEN-EHL-LLoG	3.4296×10^{03} (7.4525×10^{-04})	3.0125×10^{01} (1.6004)	2.2193×10^{01} (5.7660×10^{-02})	2.0119×10^{-01} (1.7594×10^{-02})	497.425	505.425	506.1798	513.6668	0.1302	0.6582	0.1161	0.4143
GELLoG	λ 0.2317 (0.0761)	c 5.2945×10^{-07} (9.5966×10^{-03})	α 3.9148×10^{-03} (0.0234)	δ 14.8790 (9.1622)	498.5636	506.5636	507.3184	514.8054	0.1360	0.6981	0.1230	0.3441
APExLLD	a 179.8383 (255.0455)	a 2.1907 (0.1874)	b 1.1440 (0.3332)	c 399.9338 (1.7515)	529.4586	537.4586	538.2133	545.7003	0.6264	3.3509	0.1737	0.0602
TIHT-W	α 1.1963 (0.1374)	θ 0.1221 (0.0537)	β 0.0916 (0.0567)		525.6656	531.6656	532.1100	537.8469	0.3263	1.7175	0.2036	0.0162
HTBPT-W	α 1.2034 (0.1125)	γ 0.0262 (0.01207)	β 7.1487×10^{-05} (6.8034×10^{-06})		505.3288	511.3291	511.7735	517.5104	0.2356	1.2313	0.1188	0.3853
APW-W	α 6.8279×10^{05} (5.7285×10^{-09})	β 0.9198 (0.0980)	λ 0.0907 (0.0348)		508.4963	514.4963	514.9407	520.6776	0.3186	1.6798	0.1514	0.1398

Table 6.31: MLEs and GoF Statistics

Table 6.31 offers the estimates of fitted model parameters, standard errors (in parenthesis) and goodness-of-fit statistics. We conclude that our suggested HT-GEN-EHL-LLoG distribution delivers a superior fit than the other fitted models since it is associated with smaller values of $-2\ln(L)$, AIC , $CAIC$, BIC , W^* , A^* , K-S and larger p-value of the K-S statistic. Furthermore, the fitted density plots and probability-probability (PP) plots of the fitted distributions are presented in Figure 6.31.

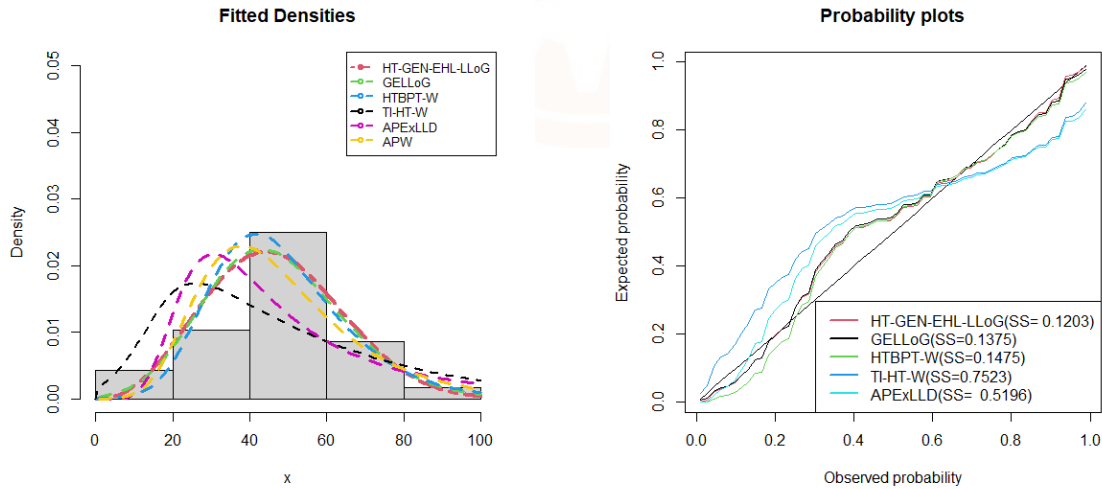


Figure 6.31: Fitted density superimposed on the histogram and observed probability vs expected Probability Plots for the insurance data

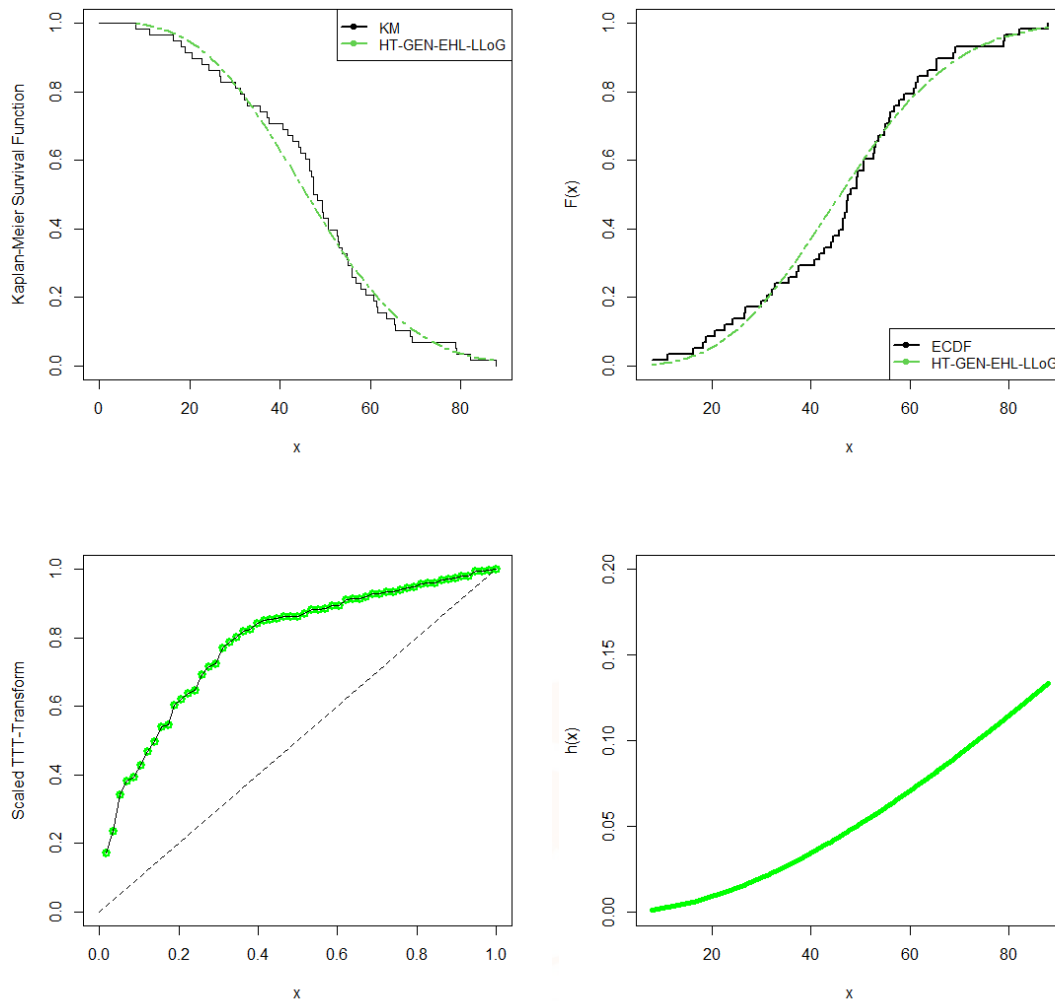


Figure 6.32: Fitted K-M survival curve, ECDF, the TTT scaled plot, and the fitted hrf for the insurance data

The observed and the fitted K-M survival curves, theoretical and ECDF, TTT scaled plot, and estimated hrf plots are shown in Figure 6.32. From the K-M survival and ECDF plots, it is clear that the HT-GEN-EHL-LLoG distribution is very close to the empirical plots, this indicate that the HT-GEN-EHL-LLoG distribution capture information of the insurance data well. The TTT scaled plot demonstrates that the insurance data possess increasing hazard rate structure. The estimated hrf of our proposed model also shows an increasing hazard rate shape.

6.2.22 Remission Times Data

The data set is on remission times (months) of 128 bladder cancer patients by [Lee and Wang \(2003\)](#). (See the data in the Appendix).

The estimated variance-covariance matrix for HT-GEN-EHL-LLoG distribution on remis-

sion times data is given by

$$\begin{bmatrix} 0.0101 & 0.0408 & -1.0008 \times 10^{-04} & -1.5893 \times 10^{-03} \\ 0.0408 & 0.1933 & -4.7380 \times 10^{-04} & -4.0805 \times 10^{-03} \\ -0.0001 & -0.0005 & 1.1613 \times 10^{-06} & 1.0024 \times 10^{-05} \\ -0.0016 & -0.0041 & 1.0024 \times 10^{-05} & 5.5861 \times 10^{-04} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\theta \in [2.4359 \times 10^{-01} \times 10^0 \pm 0.19652]$, $\alpha \in [5.5111 \pm 0.8617]$, $\beta \in [3.0809 \times 10^0 \pm 0.0021]$ and $c \in [2.8751 \times 10^{-01} \pm 0.0463]$, respectively.

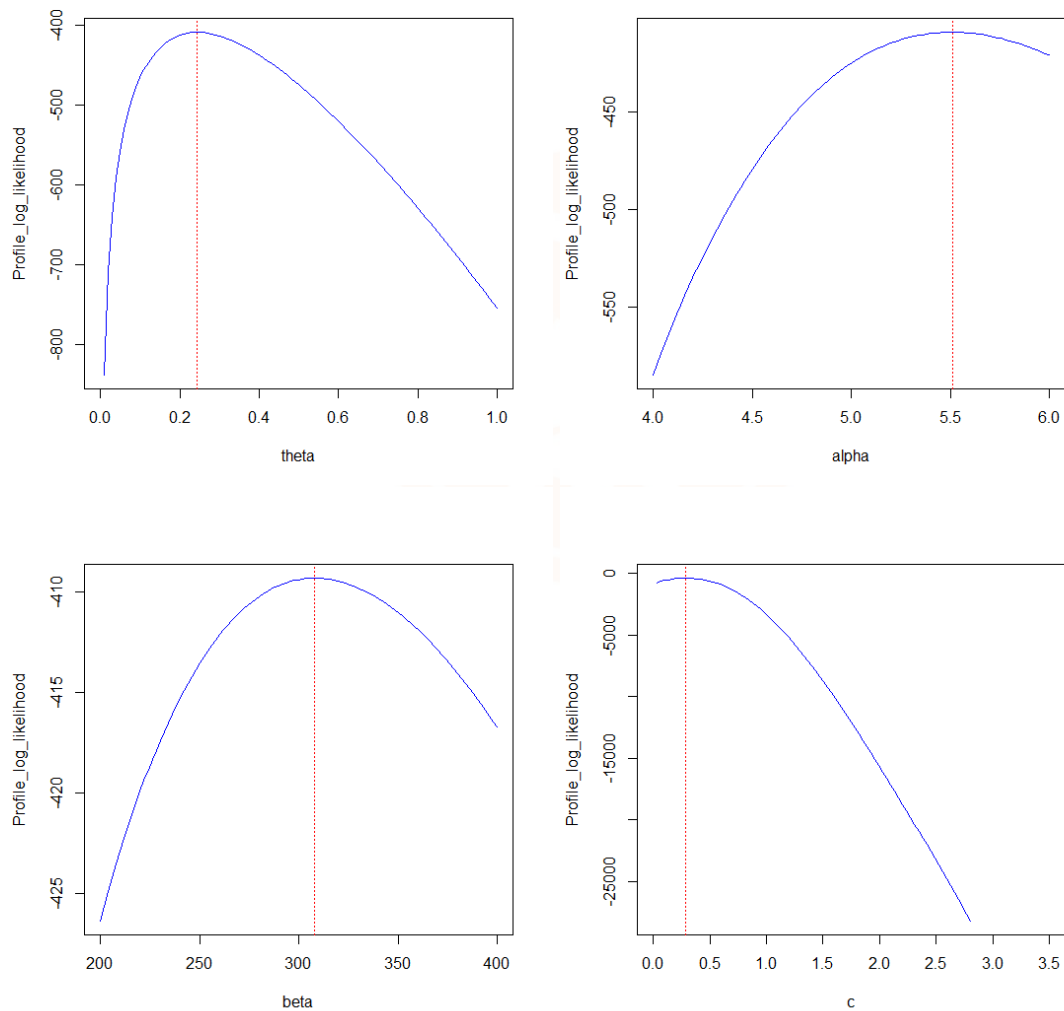


Figure 6.33: Profile log-likelihood function plots for parameters of HT-GEN-EHL-LLoG distribution on the remission times data

Figure 6.33 shows the profile log-likelihood plots for parameters of the HT-GEN-EHL-LLoG distribution on the remission times data. From the profile log-likelihood plots, we can see

that the MLEs for the parameters of the HT-GEN-EHL-LLoG distribution are unique.

Model	Estimates				Statistics							
	θ	a	β	c	$-2\log(L)$	AIC	AICC	BIC	W*	A*	K-S	p-value
HT-GEN-EHL-LLoG	2.4359×10^{-01} (1.0027×10^{-01})	5.5111 (4.3966×10^{-01})	3.0809×10^{02} (1.0777×10^{-03})	2.8751×10^{-01} (2.3635×10^{-02})	818.6002	826.6002	826.9254	838.0084	0.0151	0.1000	0.0291	0.9999
GELLoG	λ 0.0960 (0.0254)	c 0.7279 (0.0909)	a 9.8858 (9.0234)	δ 0.4226 (0.3884)	827.7211	835.7211	836.0463	847.1292	0.1167	0.7369	0.0615	0.7177
APExLLD	a 6.0772×10^{-04} (6.2812×10^{-04})	a 6.4708 (2.9039×10^{-04})	b 61.2460 (1.7677×10^{-06})	c 0.1662 (0.0111)	827.6276	835.6276	835.9528	847.0357	0.1326	0.7957	0.0690	0.5759
TIHT-W	a 1.1030 (0.6049)	θ 1.2551 (3.5043)	β 0.0564 (0.3501)		826.9924	832.9924	833.1859	841.5485	0.1150	0.6940	0.0684	0.5863
HTBPT-W	a 1.0478 (0.0675)	γ 0.0938 (0.0190)	β 0.9999 (0.7280)		828.1738	834.1738	834.3673	842.7298	0.1313	0.7864	0.0700	0.5570
APW-W	a 1.0647×10^{11} (2.2967×10^{-14})	β 0.3034 (0.0187)	λ 1.8514 (0.0745)		850.0939	856.0851	856.2786	864.6412	0.0995	0.6805	0.1768	0.0007

Table 6.32: MLEs and GoF Statistics

In Table 6.32, we observe that the HT-GEN-EHL-LLoG distribution achieves a better fit than the competing distributions for modeling remission times data since it has the smallest values among all goodness-of-fit statistics and the larger p-value of the K-S statistic. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 6.34.

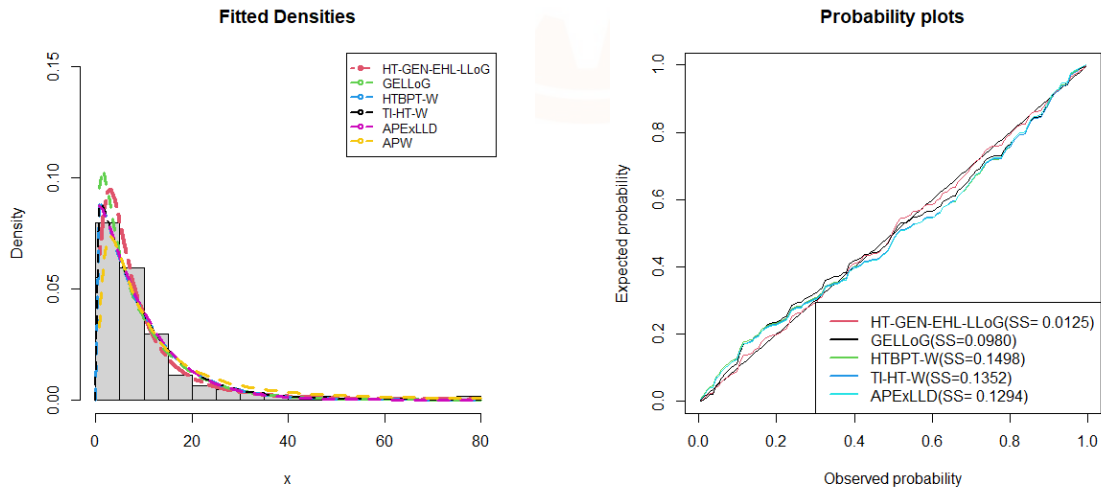


Figure 6.34: Fitted density superimposed on the histogram and observed probability vs expected Probability Plots for the remission times data

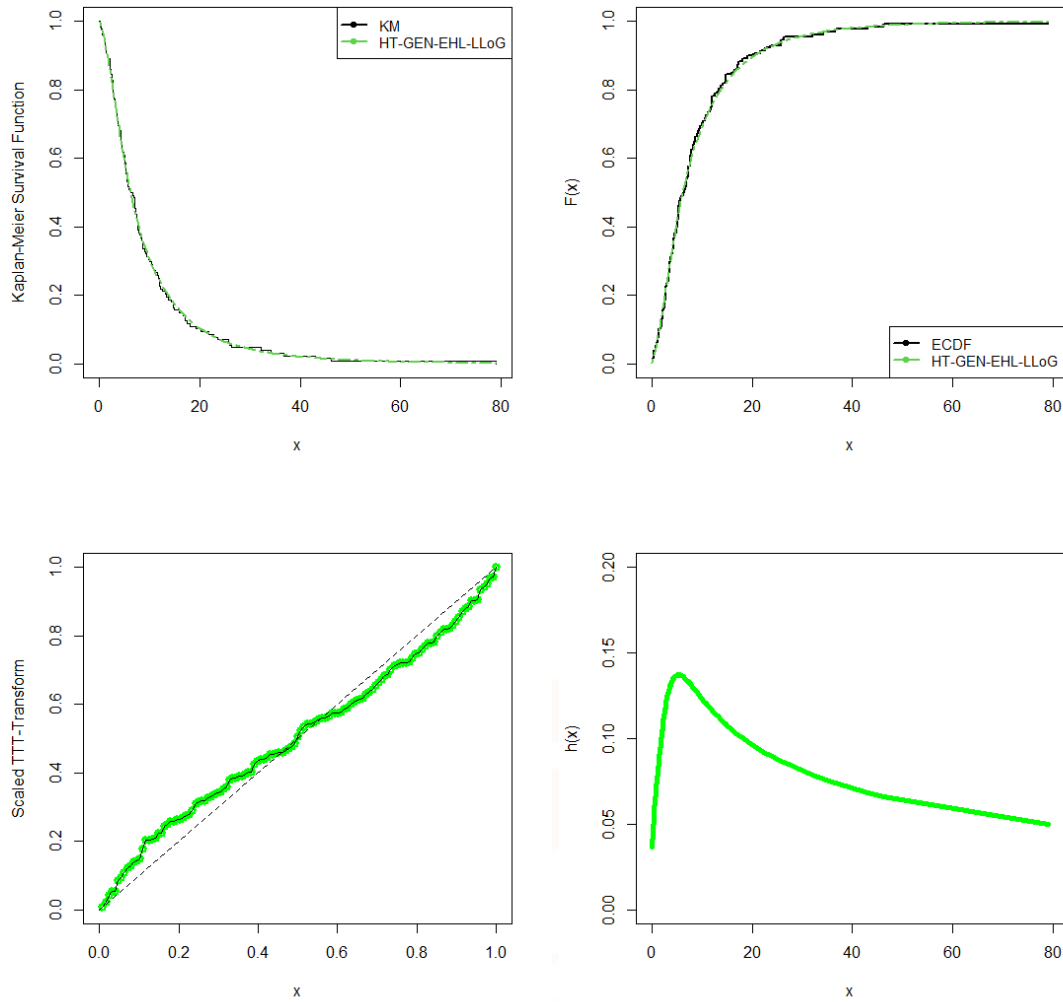


Figure 6.35: Fitted K-M survival curve, ECDF, the TTT scaled plot, and the fitted hrf for the remission times data

Figure 6.35, shows the observed and the fitted K-M survival curves, theoretical and ECDF, TTT scaled plot, and hrf plots for the remission times data. It is quite clear that the estimated survival function and cdf of our model are in agreement with the empirical ones as they are quite close in the presented plots. The TTT scaled plot for the remission times data estimate the hrf to be an upside down shape. The estimated hrf of our model is in agreement with the TTT scaled plot, it is upside down shaped.

6.2.23 Silicon Nitride Data

The second data set represents fracture toughness of silicon nitride measured in $\text{MPa } m^{\frac{1}{2}}$ (See [Nadarajah and Kotz \(2007\)](#), and [Ali et al. \(2015\)](#) for details). (See the data in the **Appendix**).

The estimated variance-covariance matrix for HT-GEN-EHL-LLoG distribution on silicon nitride data is given by

$$\begin{bmatrix} 3.6151 \times 10^{-02} & 1.8423 \times 10^{-01} & -9.5071 \times 10^{-07} & 2.5116 \times 10^{-03} \\ 1.8423 \times 10^{-01} & 1.3788 & -7.1124 \times 10^{-06} & 3.3948 \times 10^{-02} \\ -9.5071 \times 10^{-07} & -7.1124 \times 10^{-06} & 3.6689 \times 10^{-11} & -1.7504 \times 10^{-07} \\ 2.5116 \times 10^{-03} & 3.3948 \times 10^{-02} & -1.7504 \times 10^{-07} & 1.2280 \times 10^{-03} \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\theta \in [4.8671 \times 10^{-01} \pm 0.3726]$, $\alpha \in [1.7061 \times 10^{01} \pm 2.3014]$, $\beta \in [1.5264 \times 10^{05} \pm 0.00001]$ and $c \in [5.1723 \times 10^{-01} \pm 0.0686]$, respectively.

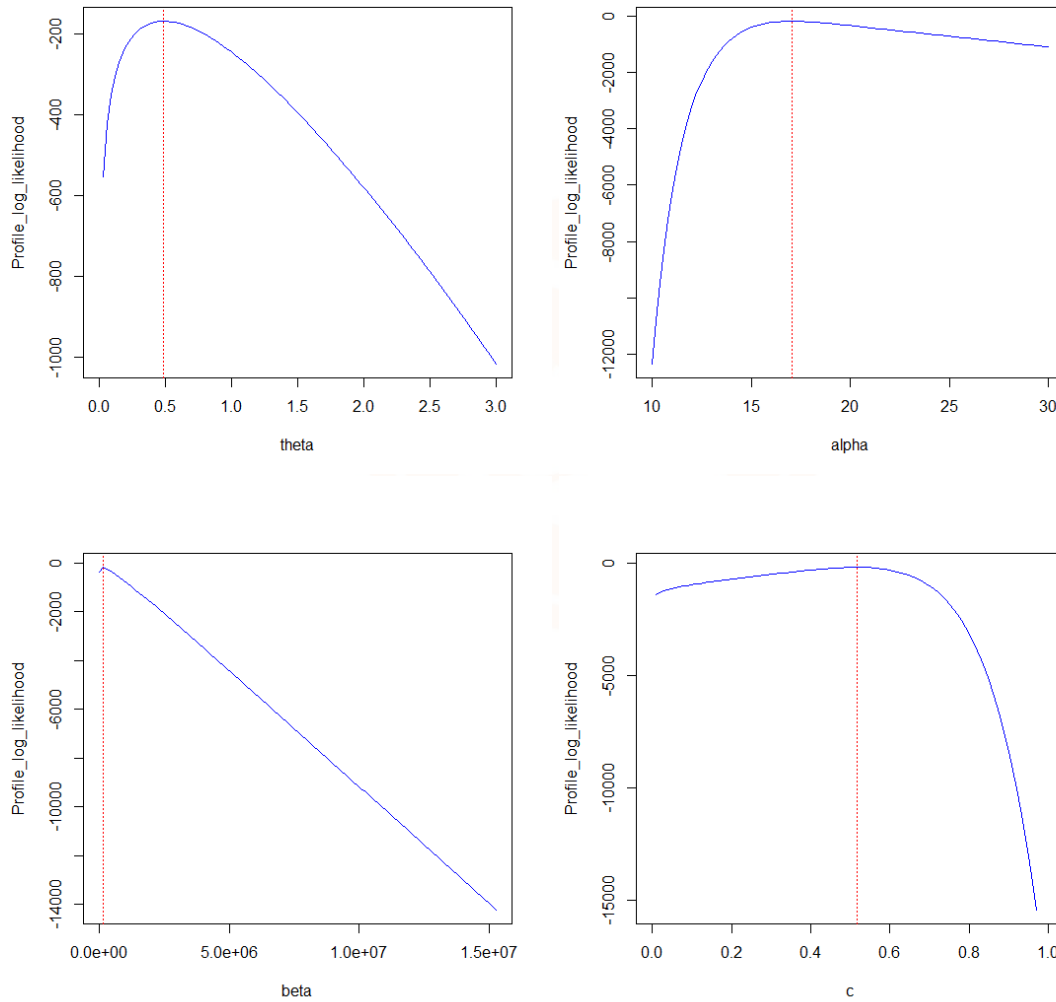


Figure 6.36: Profile log-likelihood function plots for parameters of HT-GEN-EHL-LLoG distribution on the silicon nitride data

The identifiability of model parameters can be assessed by plotting the log-likelihood function. From the profile log-likelihood plots in Figure 6.36, we can see that the MLEs for the HT-GEN-EHL-LLoG distribution are identifiable.

Model	Estimates				Statistics							
	θ	α	β	c	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
HT-GEN-EHL-LLoG	4.8671×10^{-01} (1.9014×10^{-01})	1.7061×10^{01} (1.1742)	1.5264×10^{05} (6.0571×10^{-06})	5.1723×10^{-01} (3.5043×10^{-02})	336.9913	344.9912	345.3421	356.1077	0.0863	0.5526	0.0642	0.7099
GELLoG	λ 3.6262 (6.8189×10^{-01})	c 1.3606×10^{-04} (8.2736×10^{-04})	α 5.0272×10^{-01} (1.1496)	δ 1.5637×10^{01} (4.9117)	353.2454	361.2453	361.5962	372.3618	0.3492	2.1257	0.1276	0.0413
APExLLD	a 1.6283×10^{05} (1.3439×10^{-07})	a 7.2203 (8.9731×10^{-01})	b 4.5279 (2.3004×10^{-01})	c 6.8642×10^{-02} (1.6533×10^{-02})	350.235	358.2352	358.5861	369.3517	0.1989	1.3102	0.0905	0.2830
TIHT-W	α 3.5131 (0.2833)	θ 0.2187 (0.0615)	β 0.0269 (0.0153)		340.9627	346.9627	347.1714	355.3001	0.1274	0.8229	0.0821	0.3973
HTBPT-W	α 2.3699 (0.2306)	γ 0.0689 (0.0271)	β 0.0013 (0.0010)		343.6638	349.6637	349.8724	358.0011	0.1494	0.9760	0.0650	0.6946
APW-W	α 1.0647×10^{11} (2.6089×10^{-14})	β 1.2058 (7.8242×10^{-02})	λ 6.3196×10^{-01} (7.3123×10^{-02})		367.1955	373.194	373.4027	381.5314	0.5801	3.4607	0.1291	0.0377

Table 6.33: MLEs and GoF Statistics

In Table 6.33, the new HT-GEN-EHL-LLoG distribution produced the largest p-value of the K-S statistic and the lowest values of $-2\ln(L)$, AIC , $CAIC$, BIC , W^* , A^* , and K-S when compared with all other models that were fitted to the silicon nitride data. Therefore, we can use the HT-GEN-EHL-LLoG distribution to characterize the silicon nitride data other than the competing models. Figure 6.37 shows the fitted densities, and PP-plot.

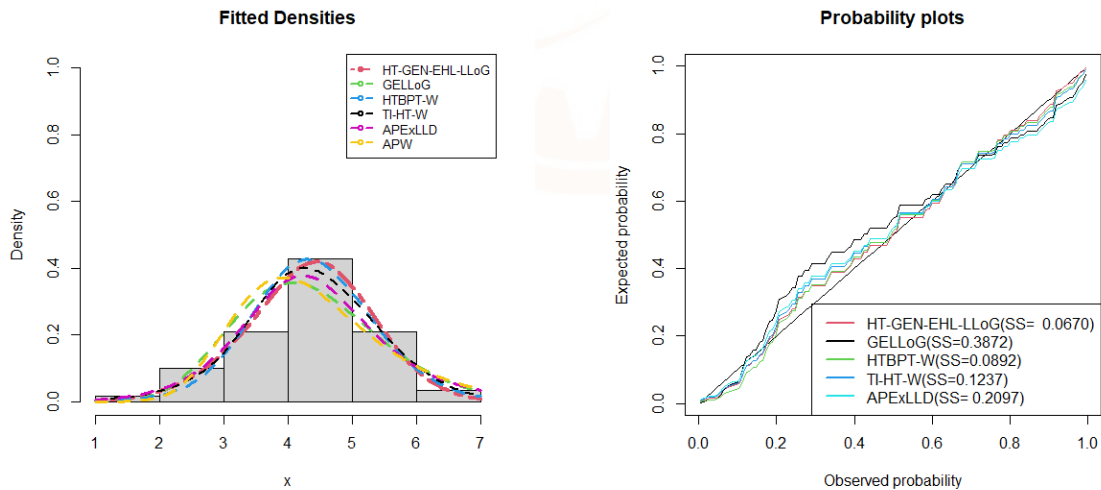


Figure 6.37: Fitted density superimposed on the histogram and observed probability vs expected Probability Plots for the silicon nitride data

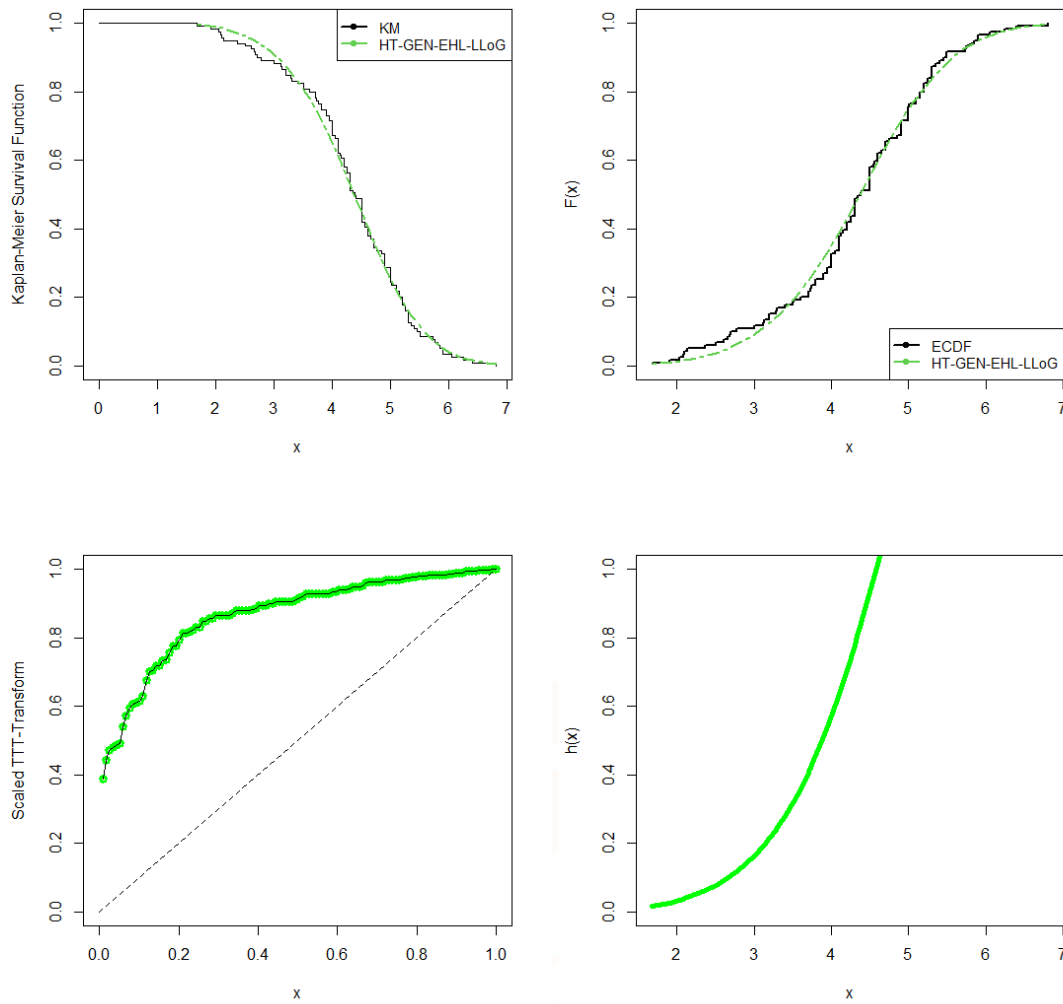


Figure 6.38: Fitted K-M survival curve, ECDF, the TTT scaled plot, and the fitted hrf for the silicon nitride data

Figure 6.38 shows the observed and the fitted K-M survival curves, ECDF, TTT scaled plot and theoretical hrf plots. We can see that the HT-GEN-EHL-LLoG distribution follows the ECDF, and K-M survival curves very closely. The TTT scaled plot and estimated hrf of HT-GEN-EHL-LLoG distribution shows an increasing shape.

6.2.24 Failure Times Data

In this application, we have used the real data set from a test that involved accelerated life for 59 conductors performed by Nelson and Doganaksoy (1995). Electro-migration, or the movement of atoms within the conductors of a circuit, is a cause of failures in microcircuits. Failure times are given in hours. (See the data in the Appendix).

The estimated variance-covariance matrix for HT-GEN-EHL-LLoG distribution on failure times data is given by

$$\begin{bmatrix} 2.4285 \times 10^{-02} & 0.1331 & 1.8703 \times 10^{-05} & -0.0046 \\ 1.3314 \times 10^{-01} & 3.7107 & 5.2797 \times 10^{-04} & 0.0838 \\ 1.8703 \times 10^{-05} & 0.0005 & 7.5125 \times 10^{-08} & 0.00001 \\ -4.6224 \times 10^{-03} & 0.0838 & 1.2032 \times 10^{-05} & 0.0049 \end{bmatrix}$$

and the corresponding 95% confidence intervals for the model parameters are given by $\theta \in [2.5928 \times 10^{-01} \pm 0.3054]$, $\alpha \in [1.5811 \times 10^{01} \pm 3.7756]$, $\beta \in [1.4632 \times 10^{03} \pm 0.0005]$ and $c \in [7.6760 \times 10^{-01} \pm 0.1385]$, respectively.

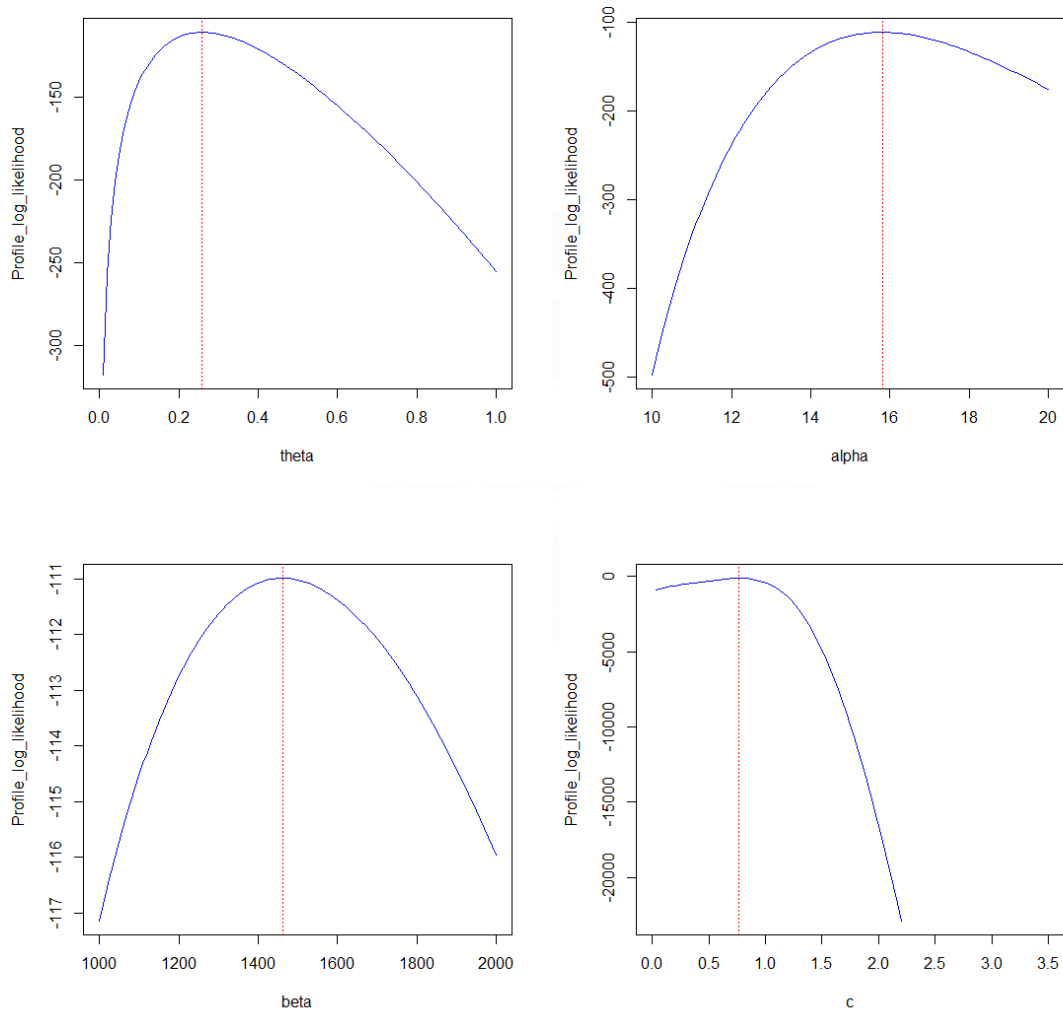


Figure 6.39: Profile log-likelihood function plots for parameters of HT-GEN-EHL-LLoG distribution on the failure time data

The profile log-likelihood function plot is used to check if the MLEs obtained are the true ones. It is quite clear from Figure 6.39 that the MLEs of the parameters of HT-GEN-EHL-LLoG distribution are archived at different points. This establishes the uniqueness of the

MLEs of the parameters of the HT-GEN-EHL-LLoG distribution.

Model	Estimates				Statistics							
	θ	α	β	c	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
HT-GEN-EHL-LLoG	2.5928×10^{-01} (1.5584×10^{-01})	1.5811×10^{01} (1.9263)	1.4632×10^{03} (2.7409×10^{-04})	7.6760×10^{-01} (7.0701×10^{-02})	221.9756	229.9756	230.7163	238.2857	0.0232	0.1370	0.0536	0.9924
GELLoG	λ 2.2501 (0.7202)	c 0.1223 (0.9530)	a 4.3796 (12.6541)	δ 13.3164 (7.4908)	224.0603	232.0603	232.801	240.3705	0.0441	0.2647	0.0777	0.8405
APExLLD	a 3.8226×10^{05} (3.2050×10^{-08})	a 7.1020 (1.2314)	b 6.7394 (6.1715×10^{-01})	c 8.8323×10^{-02} (3.7546×10^{-02})	224.2004	232.2004	232.9412	240.5106	0.0257	0.1762	0.0596	0.9767
TIHT-W	a 4.6993 (1.4346×10^{-09})	θ 1.0004 (1.0156×10^{-09})	β 7.1957×10^{-05} (9.3688×10^{-06})		224.9937	230.9936	231.4300	237.2263	0.0821	0.4618	0.0956	0.6183
HTBPT-W	a 4.6988 (1.4388×10^{-09})	γ 7.2066×10^{-05} (9.3821×10^{-06})	β 9.9399×10^{-01} (2.0688×10^{-12})		224.9946	230.9946	231.431	237.2272	0.0821	0.4619	0.0957	0.6175
APW-W	a 1.0647×10^{11} (2.5553×10^{-14})	β 1.3442 (1.2421×10^{-01})	λ 2.7335×10^{-01} (6.5786×10^{-02})		225.9431	231.9433	232.3797	238.1759	0.0638	0.3948	0.0798	0.8167

Table 6.34: MLEs and GoF Statistics

From the results in Table 6.34, it is quite clear that among the fitted models, the HT-GEN-EHL-LLoG distribution has the smallest $-2\ln(L)$, AIC , $CAIC$, BIC , W^* , A^* , and K-S values and largest K-S p-value. This indicates that the HT-GEN-EHL-LLoG distribution seems to be a distribution to be considered in modeling the failure times data set as opposed to other considered models. The fitted densities, histogram and PP plots are offered in Figure 6.40.

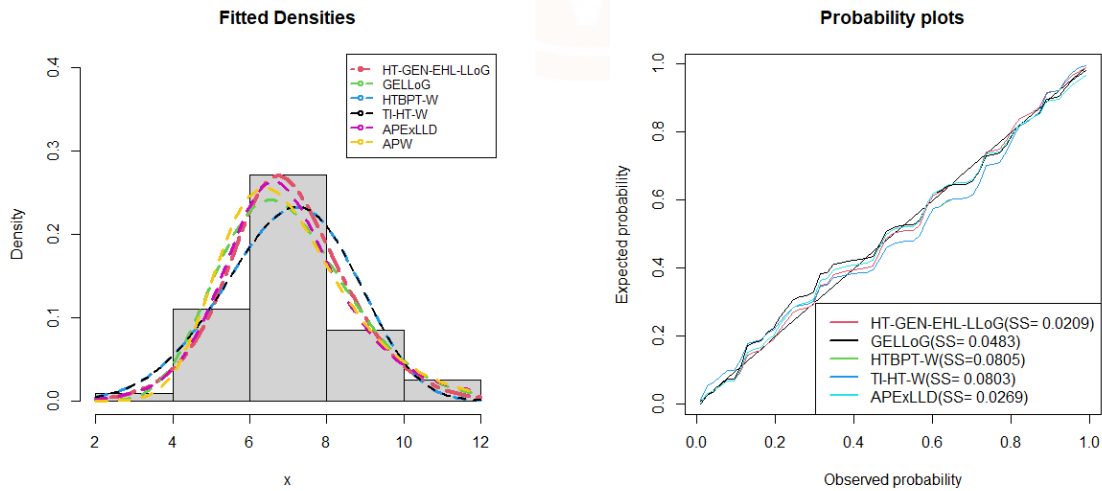


Figure 6.40: Fitted density superimposed on the histogram and observed probability vs expected Probability Plots for the failure times data

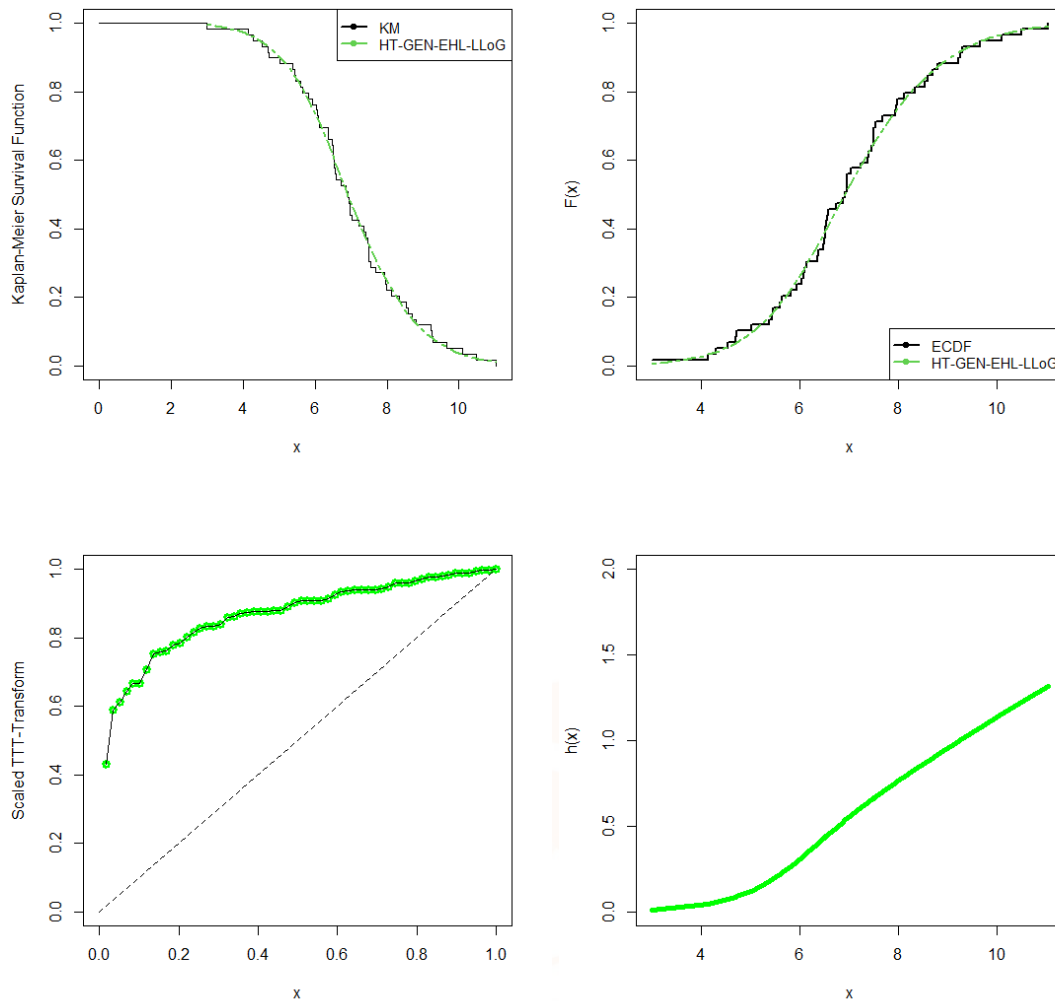


Figure 6.41: Fitted K-M survival curve, ECDF, the TTT scaled plot, and the fitted hrf for the failure times data

The empirical and theoretical K-M survival plots, empirical and theoretical cdf, TTT scaled plots and estimated hrf of HT-GEN-EHL-LLoG distribution on the failure times data are given in Figure 6.41. We observe that the fitted empirical and theoretical plots are in agreement hence we conclude that our model provides the better fit for the data. Both the TTT scaled plot and hrf plot describe the failure times data to have an increasing hazard rate structure.

6.2.25 Conclusions

The heavy-tailed generalized exponential half logistic-G (HT-GEN-EHL-G) is a family of heavy-tailed distributions that we have proposed and investigated in this study. Several statistical characteristics were established, including the quantile function, linear representation of the density function, moments, moment generating function, distribution of order statistics, and Rényi entropy. Through a simulation study, the consistency of the maximum likelihood estimates were assessed. Additionally, actuarial metrics (risk measures)

for the suggested distribution were developed. The actuarial metrics were numerically compared to other distributions. Three special cases of the HT-GEN-EHL-G were showcased along side their respective plots. Using four actual data sets from various domains and the special case of HT-GEN-EHL-LLoG distribution, the superiority and significance of the HT-GEN-EHL-G GFPD were finally demonstrated.



7. Summary and Areas of Further Studies

We developed and studied in detail the following distributions, the type II exponentiated half-logistic Topp-Leone-Marshall-Olkin-G (TIIEHL-TL-MO-G) family of distributions, the type II exponentiated half-logistic-Gompertz-G (TIIEHL-Gom-G) family of distributions, Topp-Leone odd Burr III-G (TL-OBII-G) family of distributions, the type II exponentiated half-logistic Topp-Leone-G power Series (TIIEHL-TL-GPS) class of distributions, the type I heavy-tailed odd power generalized Weibull-G (TI-HT-OPGW-G) family of distributions and the heavy-tailed generalised exponentiated half logistic-G (HT-GEN-EHL-G) family of distributions. Several properties of these distributions have been discussed and computed. The maximum likelihood estimation method has been used to estimate the unknown parameter across all the new distributions and the consistency property of the maximum likelihood estimates have been examined via the Monte Carlo simulation study. The importance, adaptability and flexibility of the proposed distributions was assessed by application to real life datasets.

In the future, we will seek to use these new proposed distributions in censored experiments by applying them to different real data sets like type-II progressive hybrid censored data, type-II double censored data or step-stress acceleration data. These new distribution can also be generalized using bivariate and multivariate extensions. Also, further research on these proposed distributions can be estimating unknown parameters using other classical estimation methods including Anderson-Darling, Ordinary Least Squares, Weighted Least Squares, Cramér-Von Mises, Maximum Product of Spacing, quantile-based procedures and Bayesian estimation paradigm. Furthermore, the importance and flexibility of new distributions can be verified by using goodness-of-fit statistics that do not depend on the likelihood function, for example by using the minimum entropy criterion or Kullback-Leibler divergence criterion.

A. Appendix

Chapter 2 Appendix

Click the link below to access the appendix.

https://drive.google.com/file/d/1qb3Us-pbn1TZdbj9Fg1_9mGhjcC2HS0A/view?usp=sharing

Chapter 3 Appendix

Click the link below to access the appendix.

<https://drive.google.com/file/d/1c6Z2RKEhWVQjC1A9wAg9zuqxqVwRRe52/view?usp=sharing>

Chapter 4 Appendix

Click the link below to access the appendix.

https://drive.google.com/file/d/12i3dzZdufPZ43I_2mcfePE04Ab7CNoNq/view?usp=sharing

Chapter 5 Appendix

Click the link below to access the appendix.

<https://drive.google.com/file/d/1GhPO14SljphR3ymDqVwvBvUJ07Cg3I12/view?usp=sharing>

Chapter 6 Appendix

Click the link below to access the appendix.

https://drive.google.com/file/d/1SmI_EcPUzLbyUJS13bMroFAUgfDzZRuu/view?usp=sharing *****

Bibliography

- Abbas, S., Taqi, S. A., Mustafa, F., Murtaza, M., and Shahbaz, M. Q. , 2017. Topp-Leone Inverse Weibull Distribution: Theory and Application, *European Journal of Pure And Applied Mathematics*, vol. 10, no. 5, 1005-1022.
- Abouelmagd, T. H. M., Hamed, M. S., and Ebraheim, A. E. H. N. (2017). The Poisson-G Family of Distributions with Applications. *Pakistan Journal of Statistics and Operation Research*, 13(2), 313-326.
- Afify, A. Z., Al-Mofleh, H., and Dey, S. (2021). Topp–Leone Odd Log-Logistic Exponential Distribution: Its Improved Estimators and Applications. *Anais da Academia Brasileira de Ciências*, 93(4), e20190586, 1-14, DOI: <https://doi.org/10.1590/0001-3765202120190586>
- Afify, A. Z., Altun, E., Alizadeh, M., Ozel, G., and Hamedani, G. G., (2017). The Odd Exponentiated Half-Logistic-G Family: Properties, Characterizations and Applications. *Chilean Journal of Statistics*, 8(2), 65-91.
- Afify, A. Z., Alizadeh, M., Zayed, M., Ramires, T. G. and Louzada, F. (2018). The Odd Log-Logistic Exponentiated Weibull Distribution: Regression Modelling, Properties, and Applications. *Iranian Journal of Science and Technology*, 42(4), pp. 2273-2288.
- Afify, A. Z., and Alizadeh, M., (2020). The Odd Dagum Family of Distributions: Properties and Applications. *Journal of Applied Probability and Statistics*, 15(1), 45-72.
- Afify, A. Z., Cordeiro G., Jamal, F., Elgarhy, M., and Nasir, M. (2019). The Marshall-Olkin Odd Burr III-G Family of Distributions: Theory, Estimation and Applications, hal-02376067.
- Afify, A.Z., Gemeay, A.M. and Ibrahim, N.A., (2020). The Heavy-Tailed Exponential Distribution: Risk Measures, Estimation, and Application to Actuarial Data. *Mathematics*, 8(8), p.1276.
- Ahmad, Z., Elgarhy, M., and Hamedani, G. G., (2018). A New Weibull-X Family of Distributions: Properties, Characterizations and Applications. *Journal of Statistical Distributions and Applications*, 5(1), 1-18.
- Ahmad, Z., (2020). The Zubair-G Family of Distributions: Properties and Applications. *Annals of Data Science*, 7(2), 195-208.
- Ahn, S., Kim, J.H. and Ramaswami, V., (2012). A New Class of Models for Heavy-Tailed Distributions in Finance and Insurance Risk. *Insurance: Mathematics and Economics*, 51(1), 43-52.

- Akaike, H., (1974). A New Look at the Statistical Model Identification. *IEEE Transactions on Automatic Control*, 19(6), 716–723.
- Al-Babtain, A. A., Elbatal, I., Al-Mofleh, H., Gemeay, A. M., Afify, A. Z., and Sarg, A. M., (2021). The Flexible Burr X-G Family: Properties, Inference, and Applications in Engineering Science. *Symmetry*, 13, 474.
- Al-Babtain, A. A., Elbatal, I., Chesneau, C., and Elgarhy, M., (2020). Sine Topp-Leone-G Family of Distributions: Theory and Applications. *Open Physics*, 18(1), 574-593.
- Al-Aqtash, R., Lee, C., and Famoye, F., (2014). Gumbel-Weibull Distribution: Properties and Applications. *Journal of Modern Applied Statistical Methods*, 13(2), 11.
- Aldahlan, M. and Afify, A. Z., 2018. The odd exponentiated half-logistic Burr XII distribution. *Pakistan Journal of Statistics and Operation Research*, 14(2), 305-317.
- Aldahlan, M. A., Jamal, F., Chesneau, C., Elbatal, I., and Elgarhy, M., (2020). Exponentiated Power Generalized Weibull Power Series Family of Distributions: Properties, Estimation and Applications. *PLoS ONE* 15(3): e0230004. <https://doi.org/10.1371/journal.pone.0230004>.
- Aldahlan, M. A., Afify, A. Z. and Ahmed, A. N., (2019). The Odd Inverse Pareto-G Class: Properties and Applications. *Journal of Non-linear Sciences and Applications*, 12, 278-290.
- Alfaer, N. M., Gemeay, A. M., Aljohani, H. M., and Afify, A. Z., (2021). The Extended Log-Logistic Distribution: Inference and Actuarial Applications. *Mathematics*, 9(12), 1386.
- Ali, A., Hasnain, S.A. and Ahmad, M., (2015). Modified Burr XII Distribution, Properties and Applications. *Pakistan Journal of Statistics*, 31(6), 697-708
- Alizadeh, M., Cordeiro, G. M., Pinho, L. G. B., and Ghosh, I., (2017). The Gompertz-G Family of Distributions. *Journal of Statistical Theory and Practice*, 11(1), 179-207.
- Alizadeh, M., Tahir, M. H., Cordeiro, G. M., Mansoor, M., Zubair, M., and Hamedani, G., (2015). The Kumaraswamy Marshal-Olkin Family of Distributions. *Journal of the Egyptian Mathematical Society*, 23(3), 546-557.
- Alizadeh, M., Cordeiro, G. M., De Brito, E., and Demétrio, C. G. B., (2015). The Beta Marshall-Olkin Family of Distributions. *Journal of Statistical Distributions and Applications*, 2(1), 1-18.
- Alizadeh, M., Lak, F., Rasekhi, M., Ramires, T. G., Yousof, H. M., and Altun, E., (2018). The Odd Log-Logistic Topp–Leone G Family of Distributions: Heteroscedastic Regression Models and Applications. *Computational Statistics*, 33(3), 1217-1244.
- Alizadeh, M., Tahmasebi, S. and Haghbin, H., (2018). The Exponentiated Odd Log-logistic Family of Distributions: Properties and Applications. *Journal of Statistical Modeling: Theory and Applications*, 1(2), 1-24.

- Alizadeh, M., Bagheri, S. F., Bahrami, S. E., Ghobadi, S., and Nadarajah, S., (2017). Exponentiated Power Lindley Power Series Class of Distributions: Theory and Applications. *Communications in Statistics-Simulation and Computation*, DOI: 10.1080/03610918.2017.1350270.
- Alghamdi, A. S., and Abd El-Raouf, M. M., (2023). A New Alpha Power Cosine-Weibull Model with Applications to Hydrological and Engineering Data. *Mathematics*, 11(3), 673.
- Alfaer, N. M., Bandar, S. A., Kharazmi, O., Al-Mofleh, H., Ahmad, Z., and Afify, A. Z., (2023). Classical and Bayesian Estimation for Type-I Extended-F Family with an Actuarial Application. *Plos one*, 18(2), e0275430.
- AL-Kazrajy, A. A, (2001). Comparative Study of Estimation Methods of Reliability with Complete Data using Simulation With Application. MSc Thesis, Mosul University, Iraq.
- Al-Marzouki, S., Jamal, F., Chesneau, C., and Elgarhy, M., (2020). Topp-Leone Odd Fréchet Generated Family of Distributions with Applications to Covid-19 Datasets. *Computational Modelling in Engineering and Sciences*, 125, 437-458.
- Al-Mofleh, H., Elgarhy, M., Afify, A., and Zannon, M., (2020). Type II Exponentiated Half Logistic Generated Family of Distributions with Applications. *Electronic Journal of Applied Statistical Analysis*, 13(2), 536-561.
- Alotaibi, N., Elbatal, I., Almetwally, E.M., Alyami, S.A., Al-Moisheer, A.S. and Elgarhy, M., (2022). Truncated Cauchy power Weibull-G class of distributions: Bayesian and non-Bayesian inference modelling for COVID-19 and carbon fiber data. *Mathematics*, 10(9), p.1565.
- Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. , and Shahbaz, M.Q., (2016). Topp Leone Family of Distributions: Some Properties and Application, *Pakistan Journal of Statistics and Operation Research*, 12, 3, 443-451.
- Altun, E., Yousof, H.M., Chakraborty, S. and Handique, L., (2018). Zografos-Balakrishnan Burr XII distribution: regression modeling and applications. *International Journal of Mathematics and Statistics*, 19(3), 46-70.
- Alyami, S.A., Babu, M.G., Elbatal, I., Alotaibi, N. and Elgarhy, M., (2022). Type II half-logistic odd Fréchet class of distributions: Statistical Theory and Applications. *Symmetry*, 14(6), p.1222.
- Alzaghal, A., Famoye, F. and Lee, C., (2013). Exponentiated T-X Family of Distributions With Some Applications. *International Journal of Probability and Statistics*, 2, 31-49.
- Alzaatreh, A., Lee, C., and Famoye, F., (2013). A New Method for Generating Families of Continuous Distributions. *Metron*, 71(1), 63-79.
- Ampadu, C. B., (2020). The Zeid-G Family of Distributions. *Biomedical Journal of Scientific and Technical Research*, 27(4), 20940-20942.

- Anwar, A., and Bibi, A., (2018). The Half-Logistic Generalized Weibull Distribution. *Journal of Probability and Statistics*, Volume 2018, Article ID 8767826, 12 pages, <https://doi.org/10.1155/2018/8767826>
- Arif, M., Khan, D. M., Khosa, S. K., Aamir, M., Aslam, A., Ahmad, Z., and Gao, W., (2021). Modelling Insurance Losses with a new Family of Heavy-Tailed Distributions. *Computers, Materials & Continua*, 66(1), 537-550.
- Arshad, R. M. I., Tahir, M. H., Chesneau, C., and Jamal, F., (2020). The Gamma Kumaraswamy-G Family of Distributions: Theory, Inference and Applications. *Statistics in Transition new series*, 21, 5, 17–40.
- Aryal, G. R., Ortega, E. M., Hamedani, G. G., and Yousof, H. M., (2017). The Topp-Leone Generated Weibull Distribution: Regression Model, Characterizations and Applications. *International Journal of Statistics and Probability*, 6(1): 126-141.
- Aslam, M., Hussain, Z., and Asghar, Z., (2018). Cubic Transmuted-G Family of Distributions and its Properties. *Stochastics and Quality Control*, 33(2), 103-112.
- Awodutire, P. O., Balogun, O. S., Olapade, A. K., and Nduka, E. C., (2021). The Modified Beta Transmuted Family of Distributions with Applications using the Exponential Distribution. *Plos one*, 16(11), e0258512.
- Balogun, O. S., Arshad, M. Z., Iqbal, M. Z., and Ghamkhar, M., (2021). A New Modified Lehmann Type-II G Class of Distributions: Exponential Distribution With Theory, Simulation, and Applications to Engineering Sector. *F1000Research*, 10(483), 483.
- Bantan, R. A., Jamal, F., Chesneau, C., and Elgarhy, M., (2020). Type II Power Topp-Leone Generated Family of Distributions with Statistical Inference and Applications, *Symmetry*, 12(1), 75.
- Barlow, R. E., Toland, R. H., and Freeman, T., (1984). A Bayesian Analysis of Stress-rupture Life of Kevlar 49/epoxy Spherical Pressure Vessels. In *Proc. Conference on Applications of Statistics*, Marcel Dekker, New York.
- Barreto-Souza, W., Santos, A. H. S. and Cordeiro, G. M., (2010). The Beta Generalized Exponential Distribution. *Journal of Statistical Computation and Simulation*, 80(2), 159–172.
- Batsidis, A., and Lemonte, A. J., (2015). On the Harris Extended Family of Distributions. *Statistics*, 49(6), 1400-1421.
- Benkhelifa, L., (2022). Alpha power Topp-Leone Weibull Distribution: Properties, Characterizations, Regression Modeling and Applications. *Journal of Statistics and Management Systems*, 25(8), 1-26.
- Bekker, A., Roux, J. J. J., and Mosteit, P. J., (2000). A Generalization of the Compound Rayleigh Distribution: using a Bayesian Method on Cancer Survival Times. *Communications in Statistics-Theory and Methods*, 29(7), 1419-1433.

- Bhatti F. A., Hamedani, G. G., Korkmaz, M. C., Cordeiro, G. M., Yousof, H. M., and Ahmad, M., (2019). On Burr III Marshal-Olkin Family: Development, Properties, Characterizations and Applications. *Journal of Statistical Distributions and Applications*, vol. 6, no. 12.
- Bourguignon, M., Silva, R. B. and Cordeiro, G. M., (2014). The Weibull-G Family of Probability Distributions. *Journal of Data Science*, 12, 53-68.
- Bozdogan, H., (1987). Model selection and Akaike's information criterion (AIC): The General Theory and its Analytical Extensions. *Psychometrika*, 52(3), 345-370.
- Chakravarti, I.M., Laha, R.G. and Roy, J., (1967). *Handbook of Methods of Applied Statistics*. Wiley Series in Probability and Mathematical Statistics, 1, 392-394.
- Chambers, J., Cleveland, W., Kleiner, B., and Tukey, J., (1983). *Graphical Methods for Data Analysis*, Chapman and Hall, London.
- Chen, G., and Balakrishnan, N., (1995). A General Purpose Approximate Goodness-of-fit Test. *Journal of Quality Technology*, 27, 154-161.
- Chesneau, C., and El Achi, T., (2020). Modified Odd Weibull Family of Distributions: Properties and Applications. *Journal of the Indian Society for Probability and Statistics*, 21, 1, 259-286.
- Chhikara R. S., and Folks J. I., (1977). The Inverse Gaussian Distribution as a Lifetime Model, *Technometrics*, 19(4), 461-468.
- Chipepa, F., and Oluyede, B., (2021). The Marshall-Olkin-Gompertz-G Family of Distributions: Properties and Applications. *Journal of Nonlinear Sciences and Applications*, 14(4).
- Chipepa, F., Oluyede, B., and Peter, P. O., (2021). The Burr III-Topp-Leone-G Family of Distributions with Applications. *Heliyon*, 7(4), e06534.
- Chipepa, F., Oluyede, B., Wanduku, D., and Moakofi, T., (2022). The Exponentiated Half Logistic-Topp-Leone-G Power Series Class of Distributions: Model, Properties and Applications. In *Methods of Mathematical Modelling and Computation for Complex Systems*, 341-374. Springer, Cham.
- Chipepa, F., Oluyede, B. and Wanduku, D., (2020). The Exponentiated Half Logistic Odd Weibull-Topp-Leone-G Family of Distributions: Model, Properties and Applications. *Journal of Statistical Modelling: Theory and Applications*, 2(1), 15-38.
- Chipepa, F., Oluyede, B. and Makubate, B., (2020). The Odd Generalized Half-Logistic Weibull-G Family of Distributions: Properties and Applications. *Journal of Statistical Modeling: theory and Applications*, 1(1), 65-89.

- Chipepa, F., Oluyede, B. and Makubate, B., (2020). The Topp-Leone-Marshall-Olkin-G Family of Distributions with Applications. *International Journal of Statistics and Probability*, 9(4), 15-32. doi:10.5539/ijsp.v9n4p15
- Chipepa, F., Oluyede, B. and Makubate, B., (2019). A New Generalized Family of Odd Lindley-G Distributions with Application. *International Journal of Statistics and Probability*, 8(6), pp. 1-22. doi:10.5539/ijsp.v8n6p1
- Cordeiro, G. M., Afify, A. Z., Ortega, E. M. M., Suzuki, A. K. and Mead, M. E., (2019). The Odd Lomax Generator of Distributions: Properties, Estimation and Applications. *Journal of Computational and Applied Mathematics*, 347, 222-237.
- Cordeiro, G. M., Afify, A. Z., Yousof, H. M., Pescim, R. R., and Arya, G. R., (2017). The Exponentiated Weibull-H Family of Distributions: Theory and Applications. *Mediterranean Journal of Mathematics*, 14, 155, DOI 10.1007/s00009-017-0955-1.
- Cordeiro, G. M., Alizadeh, M., Nascimento, A. D., and Rasekhi, M., (2016). The Exponentiated Gompertz Generated Family of Distributions: Properties and Applications. *Chilean Journal of Statistics*, 7(2).
- Cordeiro, G. M., Ortega, E. M. M., and Ramires T. G., (2015). A New Generalized Weibull Family of Distributions: Mathematical Properties and Applications. *Journal of Statistical Distributions and Applications*, 2, 13.
- Cordeiro, G. M. and Silva, R. B., (2014). The Complementary Extended Weibull Power Series Class of Distributions. *Ciência e Natura*, 36(3).
- Cordeiro, G. M., and Lemonte, A. J., (2011). The β -Birnbau–Saunders Distribution: An Improved Distribution for Fatigue Life Modeling. *Computational Statistics and Data Analysis*, 55(3), 1445-1461.
- Cordeiro, G. M. Ortega, E. M. M. and Nadarajaah, S., (2010). The Kumaraswamy Weibull Distribution with Application to Failure Data. *Journal of the Franklin Institute*, 347, 1399 -1429.
- Cordeiro, G. M., Ortega, E. M. M., and da Cunha, D. C. C., (2013). The Exponentiated Generalized Class of Distributions. *Journal of Data Science*, 11, 1-27.
- Cordeiro, G. M., Alizadeh, M., and Marinho, P. R. D., (2016). The TypeI Half-Logistic Family of Distributions. *Journal of Statistical Computation and Simulation*, 86(4), 707-728.
- Cordeiro, G.M. and de Castro, M., (2011). A New Family of Generalized Distributions. *Journal of statistical computation and simulation*, 81(7), 883-898.
- Cordeiro, G.M., Aristizábal, W.D., Suárez, D.M. and Lozano, S., (2015). The Gamma Modified Weibull Distribution. *Chilean Journal of Statistics*, 6(1), 37-48.
- Dhungana, G.P. and Kumar, V., (2022). Exponentiated Odd Lomax Exponential Distribution with Application to COVID-19 Death Cases of Nepal. *PloS one*, 17(6), p.e0269450.

- Doostmoradi, A., Zadkarami, M. R., and Roshani Sheykhabad, A., (2014). A New Modified Weibull Distribution and Its Application. *Journal of Statistical Research and Training Center, Iran* vol. 11, pp. 97–118.
- Eghwerido, J. T., Oguntunde, P. E., and Agu, F. I., (2021). The Alpha Power Marshall-Olkin-G Distribution: Properties, and Applications. *Sankhya A*, 1-26.
- Eghwerido, J.T. and Agu, F.I., (2021). The shifted Gompertz-G Family of Distributions: Properties and Applications. *Mathematica Slovaca*, 71(5), pp.1291-1308.
- Elbatal, I., (2011). Exponentiated Modified Weibull Distribution. *Economic Quality Control*. 26, 189-200.
- Elgarhy, M., Nasir, A., Jamal, F., and Kadilar, G. O., (2018). Type II Topp-Leone Generated Family of Distributions: Properties and Applications. *Journal of Statistics and Management Systems*, 21(8), 1529-1551.
- El-Gohary, A., Alshamrani, A., Al-Otaibi, A. N., (2013). The Generalized Gompertz Distribution. *Applied Mathematical Modelling*, 37(1-2), 13-24.
- El-Morshedy, M., El-Faheem, A. A., El-Dawoody, M., (2020). Kumaraswamy Inverse Gompertz Distribution: Properties and Engineering Applications to Complete, Type-II Right Censored and Upper Record Data. *Plos one*, 15(12), e0241970.
- El-Morshedy, M., Eliwa, M. S., Tahir, M. H., Alizadeh, M., El-Desokey, R., Al-Bossly, A., Alqifari, H., (2023). A Bivariate Extension to Exponentiated Inverse Flexible Weibull Distribution: Shock Model, Features, and Inference to Model Asymmetric Data. *Symmetry*, 15(2), 411.
- El-Sherpieny, E. S. A., and Elsehetry, M. M., (2019). Type II Kumaraswamy Half-Logistic Family of Distributions with Applications to Exponential Model. *Annals of Data Science*, 6(1), 1-20.
- Eugene, N., Lee, C., and Famoye, F., (2002). Beta-Normal Distribution and its Applications. *Communications in Statistics: Theory and Methods*, 31(4), 497-512.
- Fagbamigbe, A.F., Melamu, P., Oluyede, B.O. and Makubate, B., (2018). The Ristić and Balakrishnan Lindley-Poisson Distribution: Model, Theory and Application. *Afrika Statistika*, 13(4), 1837-1864.
- Famoye, F., Lee, C., and Olumolade, O., (2005). The Beta Weibull Distribution. *Journal of Statistical Theory and Applications*, 4, 2, 121-136.
- Figuerola-Zúñiga, J., Niklitschek-Soto, S., Leiva, V., & Liu, S. (2022). Modeling Heavy-Tailed Bounded Data by the Trapezoidal Beta Distribution with Applications. *REVSTAT-Statistical Journal*, 20(3), 387-404.

- Flores, J., Borges, P., Cancho, V. G., and Louzada, F., (2013). The Complementary Exponential Power Series Distribution. *Brazilian Journal of Probability and Statistics*, 27(4), 565-584.
- Gabanakgosi, M., Moakofi, T., Oluyede, B. and Makubate, B., (2021). The Gamma Odd Power Generalized Weibull-G Family of Distributions with Applications. *Journal of Statistical Modelling: Theory and Applications*, 2(2), 79-101.
- Gay, D.M., (1990). Usage Summary for Selected Optimization Routines. *Computing Science Technical Report*, 153(153), 1-21.
- Gomes-Silva, F., de Andrade, T. A. N. and Bourguignon, M., (2018). Ristić-Balakrishnan Extended Exponential Distribution. *Acta Scientiarum. Technology*, 40, pp.e34963-e34963.
- Gradshteyn, I. S., and Ryzhik, I. M., (2000). *Tables of Integrals, Series and Products*. Sixth Edition, Academic Press, San Diego.
- Greenwood, J.A., Landwehr, J.M., Matalas, N.C. and Wallis, J.R., (1979). Probability Weighted Moments: Definition and Relation to Parameters of Several Distributions Expressible in Inverse Form. *Water Resources Research*, 15(5), 1049-1054.
- Gupta, R.C., Gupta, P.L. and Gupta, R.D., (1998). Modeling Failure Time Data by Lehman Alternatives. *Communications in Statistics-Theory and methods*, 27(4), 887-904.
- Gupta, R.C. and Kundu, D., (2011). Weighted Inverse Gaussian—a Versatile Lifetime Model. *Journal of Applied Statistics*, 38(12), 2695-2708.
- Hamedani, G. G., Rasekhi, M., Najibi, S., Yousof, H. M., and Alizadeh, M., (2019). Type II General Exponential Class of Distributions. *Pakistan Journal of Statistics and Operation Research*, 15(2), 503-523.
- Handique, L., Ahsan, A.L. and Chakraborty, S., (2020). Generalized Modified Exponential-G Family of Distributions: its Properties and Applications. *International Journal of Mathematics and Statistics*, 21, 1-17.
- Hassan, A. S., and Nassr, S. G., (2019). Power Lindley-G Family of Distributions. *Annals of Data Science*, 6(2), 189-210.
- Hassan, A. S., Elgarhy, M., and Ahmad, Z., (2019). Type II Generalized Topp-Leone Family Of Distributions: Properties and Applications. *Journal of data science*, 17(4).
- Hosseini, B., Afshari, M., and Alizadeh, M., (2018). The Generalized Odd Gamma-G Family of Distributions: Properties and Applications. *Austrian Journal of Statistics*, 47(2), 69-89.
- Hussein, M., Elsayed, H. and Cordeiro, G.M., (2022). A New family of Continuous Distributions: Properties and Estimation. *Symmetry*, 14(2), 276.

- Irshad, M.R., D'cruz, V. and Maya, R., (2021). The Zografos-Balakrishnan Lindley Distribution: Properties and Applications. *Statistica*, 81(1), 45-64.
- Jamal, F., Chesneau, C., and Elgarhy, M., (2020). Type II General Inverse Exponential Family of Distributions. *Journal of Statistics and Management Systems*, 23(3), 617-641.
- Jamal, F., Chesneau, C., Saboor, A., Aslam, M., Tahir, M. H., and Mashwani, W. K., (2022). The U Family of Distributions: Properties and Applications. *Mathematica Slovaca*, 72(1), 217-240.
- Jamal, F., Reyad, H. M., Nasir, M. A., Chesneau, C., Shah, M. A. A. and Ahmed S. O., (2019). Topp-Leone Weibull-Lomax Distribution: Properties, Regression Model and Applications. hal-02270561.
- Jamal, F., Nasir, M. A., Tahir, M. H., and Montazeri, N. H., (2017). The Odd Burr-III Family of Distributions. *Journal of Statistics Applications and Probability*, 6(1), 105-122.
- Johnson, N. L., Kotz, S., and Balakrishnan, N., (1994). Continuous Distributions. Volume 1, John Wiley & Sons, New York, NY.
- Jorgensen, B., (2012). Statistical properties of the generalized inverse Gaussian distribution (Vol. 9). Springer Science & Business Media.
- Karakaya, K. and Tanis, C., (2020). Estimating the Parameters of X-Gamma Weibull Distribution. *Adiyaman University Journal of Science*, 10(2), 557-571.
- Karakaya, K. and Tanis, C., (2020). Different Methods of Estimation for the one Parameter Akash Distribution. *Cumhuriyet Science Journal*, 41(4), 994-950.
- Khaleel, M. A., Oguntunde, P. E., Al Abbasi, J. N., Ibrahim, N. A., and AbuJarad, M. H., (2020). The Marshall-Olkin Topp Leone-G Family of Distributions: A Family for Generalizing Probability Models. *Scientific African*, 8, e00470.
- Korkmaz, M. C., Alizadeh, M., Yousof, H. M. and Butt, N. S., (2018). The Generalized Odd Weibull Generated Family of Distributions: Statistical Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, XIV, 3, 541-556.
- Korkmaz, M.Ç., (2020). A New Heavy-Tailed Distribution Defined on the Bounded Interval: the Logit Slash Distribution and its Application. *Journal of Applied Statistics*, 47(12), 2097-2119.
- Kumagai, S., Matsunaga, I., Sugimoto, K., Kusaka, Y. and Shirakawa, T., (1989). Assessment of Occupational Exposures to Industrial Hazardous Substances. (III) On the Frequency Distribution of Daily Exposure Averages (8 hr TWA). *Japanese Journal of Industrial Health* 31, 216-226.
- Kundu, D. and Gupta, R.D., (2011). An Extension of the Generalized Exponential Distribution. *Statistical Methodology*, 8(6), 485-496.

- Kunjiratanachot, N., Bodhisuwan, W., and Volodin, A., (2018). The Topp-Leone Generalized Exponential Power Series Distribution with Applications. *Journal of Probability and Statistical Science*, 16(2), 197–208.
- Lee, E. T., & Wang, J., (2003). *Statistical Methods for Survival Data Analysis* (Vol. 476). John Wiley & Sons.
- Leiva V., Barros M., and Paula G. A., (2009). Generalized Birnbaum-Sanders Models using R. *Brazilian Statistical Association*, 53, 1511-1528.
- Lehmann, E. L., (1953). The Power of Rank Tests. *The Annals of Mathematical Statistics*, 23-43.
- Lima, S. R., and Cordeiro, G. M. (2017). The Extended Log-Logistic Distribution: Properties and Application. *Anais da Academia Brasileira de Ciências*, 89(1), 3-17.
- Makubate, B., Moakofi, T., and Oluyede, B., (2021). A New Generalized Lindley-Weibull Class of Distributions: Theory, Properties and Applications. *Mathematica Slovaca*, 71(1), 211–234.
- Marshall, A.W. and Olkin, I., (1997). A New Method for Adding a Parameter to a Family of Distributions with Application to the Exponential and Weibull Families. *Biometrika*, 84(3), 641-652.
- Mead, M. E., (2016). On Five-parameter Lomax Distribution: Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, 185-199.
- Méndez-González, L. C., Rodríguez-Picón, L. A., Pérez-Olguín, I. J. C., and Vidal Portilla, L. R. (2023). An Additive Chen Distribution with Applications to Lifetime Data. *Axioms*, 12(2), 118.
- Moakofi, T., Oluyede, B. Chipepa, F., (2021). Type II Exponentiated Half-Logistic Topp-Leone Marshall-Olkin-G Family of Distributions with Applications. *Heliyon*, 7, <https://doi.org/10.1016/j.heliyon.2021.e08590>.
- Moakofi, T., Oluyede, B., and Makubate, B., (2020). A New Gamma Generalized Lindley-Log Logistic Distribution with Applications. *Afrika Statistika*, 15(4), 2451-2481.
- Moakofi, T., Oluyede, B. and Chipepa, F., (2021). Type II Exponentiated Half-Logistic-Topp-Leone-G Power Series Class of Distributions with Applications. *Pakistan Journal of Statistics and Operation Research*, 17(4), 885-909.
- Moakofi, T., Oluyede, B. and Gabanakgosi, M., (2022). The Topp-Leone Odd Burr III-G Family of Distributions: Model, Properties and Applications. *Statistics, Optimization & Information Computing*, 10(1), 236-262.
- Moakofi, T., Oluyede, B., Chipepa, F and Makubate, B., (2021). Odd Power Generalized Weibull-G Family of Distributions: Properties and Applications. *Journal of Statistical Modelling: Theory and Applications*, 2(1), 121-142.

- Moakofi, T., and Oluyede, B., (2023). The Type I Heavy-Tailed Odd Power Generalized Weibull-G Family of Distributions with Applications. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics (To appear).
- Mohammed, H. F., and Yahia N., (2019). On Type II Topp-Leone Inverse Rayleigh Distribution. Applied Mathematical Sciences, 13(13), 607-615.
- Mohamed, R. A. H., Elgarhy, M., Alabdulhadi, M. H., Almetwally, E. M., and Radwan, T., (2023). Statistical Inference of Truncated Cauchy Power-Inverted Topp-Leone Distribution under Hybrid Censored Scheme with Applications. Axioms, 12(2), 148.
- Morais, A. L. and Barreto-Souza, W., (2011). A Compound Class of Weibull and Power Series Distributions. Computational Statistics and Data Analysis, 55(3), 1410-1425.
- Mozafari, M., Afshari, M., Alizadeh, M. and Karamikabir, H., (2019). The Zografos-Balakrishnan Odd Log-Logistic Generalized Half-Normal Distribution with Mathematical Properties and Simulations. Statistics, Optimization and Information Computing, 7(1), 211-234.
- Murthy, D. P., Xie, M., and Jiang, R., (2004). Weibull Models. New York: John Wiley and Sons.
- Nadarajah, S., Cordeiro, G. M., and Ortega, E. M., (2015). The Zografos–Balakrishnan-G Family of Distributions: Mathematical Properties and Applications. Communications in Statistics-Theory and Methods, 44(1), 186-215.
- Nadarajah, S., and Kotz, S., (2005). On Some Recent Modifications of Weibull Distribution. IEEE Transactions on Reliability, 54(4), 561–562.
- Nadarajah, S., and Kotz, S., (2007). On The Alternative to the Weibull Function. Engineering Fracture Mechanics, 74(3), 451-456.
- Nascimento, A. D., Silva, K. F., Cordeiro, G. M., Alizadeh, M., Yousof, H. M., and Hamedani, G. G., (2019). The Odd Nadarajah-Haghighi Family of Distributions: Properties and Applications. Studia Scientiarum Mathematicarum Hungarica, 56(2), 185-210.
- Nasir, M. A., Tahir, M. H., Chesneau, C., Jamal, F., and Shah, M. A. A., (2020). The Odds Generalized Gamma-G Family of Distributions: Properties, Regressions and Applications. Statistica, 80(1), 3-38.
- Nassar, M., Kumar, D., Dey, S., Cordeiro, G. M., and Afify, A. Z., (2019). The Marshall-Olkin Alpha Power Family of Distributions with Applications. Journal of Computational and Applied Mathematics, 351, 41-53.
- Nassar, M., Alzaatreh, A., Mead, M., and Abo-Kasem, O. (2017). Alpha power Weibull Distribution: Properties and Applications. Communications in Statistics-Theory and Methods, 46(20), 10236-10252.

- Nelson, W., and Doganaksoy, N., (1995). Statistical Analysis of Life or Strength Data from Specimens of Various Sizes using the Power-(log) Normal Model. *Recent Advances in Life-Testing and Reliability*, CRC Press Boca Raton, 377-408
- Nichols, M. D. and Padgett, W. J. A., (2006). A Bootstrap Control Chart for Weibull Percentiles. *Quality and Reliability Engineering International*, 22, 141-151.
- Nofal, Z. M., Afify, A. Z., Yousof, H. M., and Cordeiro G. M., (2016). The Generalized Transmuted-G Family of Distributions. *Communications in Statistics Theory and Methods*, 46, 4119-4136.
- Oguntunde, P. E., Balogun, O. S., Okagbue, H. I. and Bishop, S. A., (2015). The Weibull-Exponential Distribution: Its Properties and Applications. *Journal of Applied Sciences*, 15(11), 1305-1311.
- Oluyede, B., and Moakofi, T., (2022). Type II Exponentiated Half-Logistic-Gompertz Topp-Leone-G Family of Distributions with Applications. *Central European Journal of Economic Modelling and Econometrics*, 14(4), 415-461.
- Oluyede, B., Pu, S., Makubate, B., Qiu, Y., 2018. The Gamma-Weibull-G Family of Distributions with Applications. *Austrian Journal of Statistics*, 47(1), 45-76.
- Oluyede, B., Chipepa, F. and Wanduku, D., (2021). The Odd Weibull-Topp-Leone-G Power Series Family of Distributions: Model, Properties and Applications. *Journal of Nonlinear Sciences and Applications*, 14(4), 268-286.
- Oluyede, B., Fagbamigbe, A., Mashabe, B., Makubate, B., and Wanduku, D., (2020). The Exponentiated Generalized Power Series Family of distributions: Theory Properties and Applications. *Heliyon*, 6(8), e04653
- Oluyede, B. O., Mdlongwa, P., Makubate, B., and Huang, S., (2019). The Burr-Weibull Power Series Class of Distributions. *Austrian Journal of Statistics*, 48, 1-13.
- Oluyede, B., Moakofi, T., Chipepa, F and Makubate, B., (2021). A New Power Generalized Weibull-G Family of Distributions: Properties and Applications. *Journal of Statistical Modelling: Theory and Applications*, 1(2), 167-191.
- Oluyede, B., Chipepa, F. and Wanduku, D., (2020). The Exponentiated Half Logistic-Power Generalized Weibull-G Family of Distributions: Model, Properties and Applications. *Eurasian Bulletin of Mathematics*, 3(3), 134-161.
- Oluyede, B., and Chipepa, F., (2022). The Marshall-Olkin Odd Exponential Half Logistic-G Family of Distributions: Properties and Applications. *Statistics, Optimization and Information Computing* (Accepted).
- Oluyede, B. O, and Yang, T., (2015). A New Class of Generalized Lindley Distributions with Applications. *Journal of Statistical Computation and Simulation*, 85(10), 2072-2100.

- Oluyede, B.O., Makubate, B., Wanduku, D., Elbatal, I. and Sherina, V., (2017). The Gamma-Generalized Inverse Weibull Distribution with Applications to Pricing and Life-time Data. *Journal of Computations & Modelling*, 7(2), 1-28.
- Osatohanmwun, P., Oyegun, F. O., and Ogbonmwun, S. M., (2020). The T-R Y Power Series Family of Probability Distributions. *Journal of the Egyptian Mathematical Society*, 28(1), 1-18.
- Peter, P. O., Oluyede, B., Bindele, H. F., Ndwapi, N. and Mabikwa, O., 2021. The Gamma Odd Burr III-G Family of Distributions: Model, Properties and Applications. *Revista Colombiana de Estadística*, 44(2), 331-368.
- Pinho, L. G. B., Cordeiro, G. M., and Nobre, J. S., (2015). On the Harris-G Class of Distributions: General Results and Application. *Brazilian Journal of Probability and Statistics*, 29(4), 813-832.
- Pinho, L.G.B., Cordeiro, G.M. and Nobre, J.S., (2012). The Gamma-Exponentiated Weibull Distribution. *Journal of Statistical Theory and Applications*, 11(4), 379-395.
- Rannona, K., Oluyede, B., Chipepa, F. and Makubate, B., (2022). The Marshall-Olkin-Exponentiated odd Exponential Half Logistic-G Family of Distributions with Applications. *Eurasian Bulletin of Mathematics (ISSN: 2687-5632)*, 4(3), 134-161.
- Ramzan, Q., Amin, M., Elhassanein, A., and Ikram, M., (2021). The Extended Generalized Inverted Kumaraswamy Weibull Distribution: Properties and Applications. *AIMS Mathematics*, 6(9), 9955-9980.
- R Development Core Team, (2011). *A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria.
- Rezaei, S., Sadr, B. B., Alizadeh, M., and Nadarajah, S., (2017). Topp-Leone Generated Family of Distributions: Properties and Applications. *Communications in Statistics-Theory and Methods*, 46(6), 2893-2909.
- Reyad, H., Alizadeh, M., Jamal, F. and Othman, S., (2018). The Topp-Leone Odd Lindley-G Family of Distributions: Properties and Applications. *Journal of Statistics and Management Systems*, 21(7), 1273-1297.
- Reyad, H., Selim, M. A., and Othman, S., (2019). The Nadarajah Haghighi Topp Leone-G Family of Distributions with Mathematical Properties and Application. *Pakistan Journal of Statistics and Operation Research*, XV(IV), 849-866.
- Reyad, H. M., Alizadeh M., Jamal, F., Othman, S., and Hamedani, G., (2019). The Exponentiated Generalized Topp Leone-G Family of Distributions: Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, XV(1), 1-24.
- Rényi, A., (1961). On measures of Entropy and Information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, The Regents of the University of California, 1, 547 - 561.

- Ristic, M. and Balakrishnan, N., (2011). The Gamma-Exponentiated Exponential Distribution, *Journal of Statistical Computation and Simulations*, 82(8), 1191-1206.
- Rosaiah, K., Kantam, R. R. L., and Kumar, S., (2007). Exponentiated Log-Logistic Distribution-An Economic Reliability Test Plan. *Pakistan Journal of Statistics*, 23(2).
- Salahuddin, N., Khalil, A., Mashwani, W. K., Shah, H., Jomsri, P., and Panityakul, T., (2021). A Novel Generalized Family of Distributions for Engineering and Life Sciences Data Applications. *Mathematical Problems in Engineering*, <https://doi.org/10.1155/2021/9949999>.
- Sangsanit, Y., and Bodhisuwan, W., (2016). The Topp-Leone Generator of Distributions: Properties and Inferences. *Songklanakarin Journal of Science & Technology*, 38(5).
- Santos Silva, J. M. C., and Tenreyro, S., (2010). On the Existence of Maximum Likelihood Estimates in Poisson Regression. *Economics Letters*, 107, 2, 310-312.
- Sanusi, A. A., Doguwa, S. I. S., Audu, I., and Baraya, Y. M., (2020). Burr X Exponential-G Family of Distributions: Properties and Application. *Asian Journal of Probability and Statistics*, 7(3), 58-75.
- Schwarz, G. E., (1978). Estimating the dimension of a model. *Annals of Statistics*, 6(2), 461-464.
- Sengweni, W., Oluyede, B., and Makubate, B., (2021). The Exponentiated Half-Logistic Odd Lindley-G Family of Distributions with Applications. *Journal of Nonlinear Sciences and Applications*, 14(2021), 287-309.
- Seregin, A., (2010). Uniqueness of the Maximum Likelihood Estimator for k-Monotone Densities. *Proceedings of the American Mathematical Society*, 138(12), 4511-4515.
- Shannon, C. E., (1951). Prediction and Entropy of Printed English. *The Bell System Technical Journal*, 30(1), 50-64.
- Shaked, M. and Shanthikumar, J.G. eds., 1994. *Stochastic orders and their applications*. Academic press.
- Silva, R. B., Bourguignon, M., Dias, C. R. B., and Cordeiro, G. M., (2013). The Compound Class of Extended Weibull Power Series Distributions. *Computational Statistics and Data Analysis*, 58, 352-367.
- Silva, R. B., and Cordeiro, G. M., (2015). The Burr-XII Power Series Distributions: A New Compounding Family. *Brazilian Journal of Probability and Statistics*, 29(3), 565-589.
- Szekli, R., (2012). *Stochastic Ordering and Dependence in Applied Probability* (Vol. 97). Springer Science & Business Media.
- Sobhi, A. L., and Mashail, M., (2020). The Inverse-Power Logistic-Exponential Distribution: Properties, Estimation Methods, and Application to Insurance Data. *Mathematics*, 8(11), 2060.

- Soliman, A. H., Elgarhy, M. A. E., and Shakil, M., (2017). Type II Half Logistic Family of Distributions with Applications. *Pakistan Journal of Statistics and Operation Research*, 2, 245-264.
- Tahir, M. H., Hussain, M. A., Cordeiro, G. M., El-Morshedy, M., and Eliwa, M. S., (2020). A New Kumaraswamy Generalized Family of Distributions with Properties, Applications, and Bivariate Extension. *Mathematics*, 8(11), 1989.
- Tahir, M. H., Cordeiro, G. M., Mansoor, M. and Zubair, M., (2015). The Weibull Lomax Distribution: Properties and Applications. *Haceteppe Journal of Mathematics and Statistics*, 44(2), 461-480.
- Tanis, C., and Karakaya, K., (2021). On Estimating Parameters of Lindley-Geometric Distribution. *Eskisehir Technical University Journal of Science and Technology A-Applied Sciences and Engineering*, 22(2), 160-167.
- Tanis, C., (2021). On Estimating Transmuted Power Function Distribution: Characterization, Risk Measures, and Estimation. *Journal of New Theory*, (34), 72-81.
- Teamah, A.E.A., Elbanna, A.A. and Gemeay, A.M., (2021). Heavy-Tailed Log-Logistic Distribution: Properties, Risk Measures and Applications. *Statistics, Optimization & Information Computing*, 9(4), 910-941.
- Wenhao, G., (2013). Marshall-Olkin Extended Log-Logistic Distribution and its Application in Minification Processes. *Applied Mathematical Sciences*, 7(80), 3947-3961.
- Xia, J., Mi, J., and Zhou, Y. Y., (2009). On the Existence and Uniqueness of the Maximum Likelihood Estimators of Normal and Log-normal Population Parameters with Grouped Data. *Journal of Probability and Statistics*, Article id 310575, 16.
- Xu, K., Xie, M., Tang, L.C., and Ho, S. L., (2003). Application of Neural Networks in Forecasting Engine Systems Reliability. *Applied Software Computing*, 2(4), 255-268.
- Yousof, H. M., Rasekhi, M., Afify, A. Z., Ghosh, I., Alizadeh, M., and Hamedani, G. G., (2017). The Beta Weibull-G Family of Distributions: Theory, Characterizations and Applications. *Pakistan Journal of Statistics*, 33(2).
- Yousof, H. M., Afify, A. Z., Nadarajah, S., Hamedani, G., and Aryal, G. R., (2018). The Marshall-Olkin Generalized-G Family of Distributions with Applications. *Statistica*, anno LXXVIII, 3.
- Zaidi, S. M., Sobhi, M. M. A., El-Morshedy, M., and Afify, A. Z., (2021). A New Generalized Family of Distributions: Properties and Applications. *Aims Math*, 6, 456-476.
- ZeinEldin, R. A., Hashmi, S., Elsehety, M., and Elgarhy, M., (2020). Type II Half-Logistic Kumaraswamy Distribution with Applications. *Journal of Function Spaces*, 2020, Article ID 1343596, 15 pages.

- ZeinEldin, R. A., Chesneau, C., Jamal, F., Elgarhy, M., Almarashi, A. M., and Al-Marzouki, S., (2021). Generalized Truncated Fréchet Generated Family Distributions and Their Applications. *Computer Modelling in Engineering and Sciences*, 126(2), 791-819.
- ZeinEldin, R. A., Ahsan ul Haq, M., Hashmi, S., and Elsehety, M. (2020). Alpha Power Transformed Inverse Lomax Distribution with Different Methods of Estimation and Applications. *Complexity*, 2020, 1-15.
- Zhao, W., Khosa, S.K., Ahmad, Z., Aslam, M. and Afify, A.Z., (2020). Type-I Heavy-Tailed Family with Applications in Medicine, Engineering and Insurance. *PloS one*, 15(8), p.e0237462.
- Zhao, J., Ahmad, Z., Mahmoudi, E., Hafez, E.H. and Mohie El-Din, M.M., (2021). A New Class of Heavy-Tailed Distributions: Modeling and Simulating Actuarial Measures. *Complexity*, 2021, <https://doi.org/10.1155/2021/5580228>
- Zhou, C., (2009). Existence and Consistency of the Maximum Likelihood Estimator for the Extreme Index. *Journal of Multivariate Analysis*, 100, 794-815.
- Zografos, K. and Balakrishnan, N., (2009). On Families of Beta and Generalized Gamma-Generated Distributions and Associated Inference. *Statistical Methodology*, 6, 344-362.