



Faculty of Pure and Applied Sciences
Department of Mathematics & Statistical Sciences

Algorithms for Solving Split Equality Variational Inequality Problems Involving Quasi-monotone Mappings in Banach Spaces

by

Thuso Mpho Mapharing

Reg. No: 18000762

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Botswana International University of Science and Technology (BIUST), Botswana

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Supervisors:

Prof. Habtu Zegeye Hailu
Prof. Oganeditse A. Boikanyo

E-mail: hailuhz@biust.ac.bw
E-mail: boikanyoa@biust.ac.bw

Department of Mathematics & Statistical Sciences,
Faculty of Sciences, BIUST

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Candidate: Thuso M. Mapharing

Student ID: 18000762

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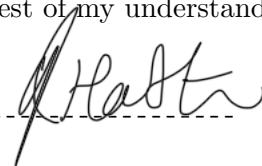
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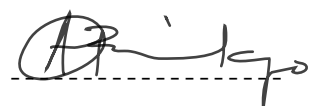
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Abstract

The main objective of this thesis is to introduce a more general problem called the Split Equality Variational Inequality Problems (SEVIP) and the Split Equality Duality Variational Inequality Problems (SEDVIP) and formulate algorithms which solve the SEVIP and SEDVIP involving quasi-monotone mappings in reflexive real Banach spaces. In order to solve this problem, we propose double projection like methods to approximate some solution of the SEVIP and SEDVIP. Using these methods, we prove weak and/or strong convergence of the algorithms to solutions of the variational inequality problems under mild conditions. In addition, we provide some applications of SEVIPs and SEDVIPs. We also present numerical examples to validate the application of the algorithm. The results reached in this thesis generalize and improve a number of results in the literature of this field.



Dedications

What is mathematics? It is only a systematic effort of solving puzzles posed by nature.
— Shakuntala Devi.



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1. Introduction

1.1 Background to the Study

The contributions of this thesis fall under the general field of non-linear functional analysis with particular attention to the variational inequality problem and its generalizations which have recently attracted substantial research interest.

In this section, we begin by giving definitions of mappings and variational inequality problem (VIP) that will be used in this thesis.

Let $\mathbb{E}$ to be a Banach space with norm $\|\cdot\|$, and let $\mathbb{E}^*$ be its dual with norm $\|\cdot\|_*$ and by $\langle \cdot, \cdot \rangle$ we denote the duality pairing. Let C be a non-empty, convex and closed subset of $\mathbb{E}$. A mapping $T : C \rightarrow \mathbb{E}^*$ is called

1. α -strongly monotone on C if there exists $\alpha > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq \alpha \|x - y\|^2, \text{ for all } x, y \in C;$$

2. monotone on C if

$$\langle T(x) - T(y), x - y \rangle \geq 0, \text{ for all } x, y \in C;$$

3. pseudo-monotone on C if for all $x, y \in C$, we have

$$\langle T(y), x - y \rangle \geq 0 \Rightarrow \langle T(x), x - y \rangle \geq 0;$$

4. quasi-monotone on C if for all $y, x \in C$, we have

$$\langle T(y), x - y \rangle > 0 \Rightarrow \langle T(x), x - y \rangle \geq 0;$$

5. Lipschitz continuous on C provided that a constant $L > 0$, called the Lipschitz constant, exists and that

$$\|T(x) - T(y)\|_{\mathbb{E}^*} \leq L \|x - y\|_{\mathbb{E}}, \text{ for all } x, y \in C.$$

where $\|\cdot\|_{\mathbb{E}^*}$ is the norm in $\mathbb{E}^*$ and $\|\cdot\|_{\mathbb{E}}$ is the norm in $\mathbb{E}$. If $L < 1$, then we consider T to be a *contraction mapping* and if $L = 1$, then T is called a *non-expansive mapping*.

Remark 1.1.1.

We note that all α -strongly monotone mappings are monotone, all monotone mappings are pseudo-monotone and all pseudo-monotone mappings are quasi-monotone, however, the converses are not true (see, examples below).

Example 1.1.2. Let $\mathbb{R}$ be the real line with the usual metric. Let $k : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$k(x) = \begin{cases} 1, & x \in (0, \infty) \\ -1, & x \in (-\infty, 0). \end{cases}$$

Then, k is monotone but not α -strongly monotone.

Proof. We first show that k is monotone.

If $y, x \in (0, \infty)$, then we get

$$\langle k(x) - k(y), x - y \rangle = (1 - 1)(x - y) = 0. \quad (1.1.1)$$

If $x \in (0, \infty)$ and $y \in (-\infty, 0)$, then we have $(x - y) > 0$ and hence

$$\langle k(x) - k(y), x - y \rangle = (1 - (-1))(x - y) = 2(x - y) > 0. \quad (1.1.2)$$

If $x, y \in (-\infty, 0)$, then we have

$$\langle k(x) - k(y), x - y \rangle = (-1 - (-1))(x - y) = 0. \quad (1.1.3)$$

If $y \in (0, \infty)$ and $x \in (-\infty, 0)$, then we get that $(x - y) < 0$ and hence

$$\langle k(x) - k(y), x - y \rangle = (-1 - 1)(x - y) = -2(x - y) > 0. \quad (1.1.4)$$

Therefore, we have $\langle k(x) - k(y), x - y \rangle \geq 0$ for all $x, y \in \mathbb{R}$, and hence k is monotone.

To show that the mapping k is not α -strongly monotone, consider $x = 2$ and $y = 1$. Then, $\langle k(2) - k(1), 2 - 1 \rangle = 0 < \alpha \|2 - 1\|^2$, $\forall \alpha > 0$, which is a contradiction. $\square$

Example 1.1.3. Define the mapping $h : (0, \infty) \rightarrow (0, \infty)$ by

$$h(x) = \frac{a}{x + a}, a > 0. \quad (1.1.5)$$

Then, h is pseudo-monotone but not monotone.

Proof. Assume that $\langle h(y), x - y \rangle \geq 0$, as such, $\frac{a(x-y)}{y+a} \geq 0$. Then we get

$$0 \leq \frac{a(x-y)(y+a)}{(y+a)(x+a)} = \frac{a(x-y)}{x+a} = \langle h(x), x - y \rangle.$$

Hence, h is clearly pseudo-monotone.

In order to demonstrate that h is not monotone, take $x, y \in (0, \infty)$ such that $x \neq y$. Then,

$$\begin{aligned}\langle h(x) - h(y), x - y \rangle &= \left\langle \frac{a}{x+a} - \frac{a}{y+a}, x - y \right\rangle \\ &= \frac{-a(x-y)^2}{(x+a)(y+a)} < 0,\end{aligned}$$

since $(x+a)(y+a)$ and a are positive. □

Example 1.1.4. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} -1, & x \in (-\infty, 0) \cup (0, 1) \\ 1, & x \geq 1 \\ 0, & x = 0. \end{cases} \quad (1.1.6)$$

Then, f is quasi-monotone but not pseudo-monotone.

Proof. Assume that $\langle f(y), x - y \rangle > 0$. Then, if $y \in (-\infty, 0) \cup (0, 1)$, then we get that $x < y$. Thus, either $x \in (-\infty, 0) \cup (0, 1)$ or $x = 0$, hence $f(x) = -1$ or $f(x) = 0$. This results in $\langle f(x), x - y \rangle \geq 0$.

If $y \geq 1$, we get $0 < \langle f(y), x - y \rangle = (x - y)$. Hence, $y < x$ which implies that $x \geq 1$ and $f(x) = 1$. Therefore, $\langle f(x), x - y \rangle \geq 0$, which implies that the mapping f is quasi-monotone.

In order to demonstrate that f is not pseudo-monotone, consider $y = 0$ and $x = 0.5$. As such, $f(0)(0.5 - 0) = 0$ and $f(0.5)(0.5 - 0) = -0.5 < 0$. Thus, f is not pseudo-monotone. □

Example 1.1.5. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g(y) = \begin{cases} 2y - 1, & y > 1 \\ y^2, & y \in [-1, 1] \\ 2y + 1, & y < -1. \end{cases}$$

Thus, g is quasi-monotone but neither pseudo-monotone nor monotone.

Proof. Assume that $\langle g(y), x - y \rangle > 0$ for $x, y \in \mathbb{R}$.

If $y > 1$, we get

$$\begin{aligned}0 &< (2y - 1)(x - y) \\ &= 2xy - x - 2y^2 + y \\ &= 2xy - x - 2y^2 + y + 2x^2 - 2x^2 + 2xy - 2xy \\ &= 2x^2 - x - 2xy + y - 2x^2 - 2y^2 + 4xy \\ &= (2x - 1)(x - y) - 2(x - y)^2 \\ &\leq \langle g(x), x - y \rangle.\end{aligned}$$

If $y < -1$, we get

$$\begin{aligned}
 0 &< (2y + 1)(x - y) \\
 &= 2xy + x - 2y^2 - y \\
 &= 2xy + x - 2y^2 - y + 2x^2 - 2x^2 + 2xy - 2xy \\
 &= 2x^2 + x - 2xy - y - 2x^2 - 2y^2 + 4xy \\
 &= (2x + 1)(x - y) - 2(x - y)^2 \\
 &\leq \langle g(x), x - y \rangle.
 \end{aligned}$$

If $y \in [-1, 1]$, then we get that $0 < y^2(x - y)$, which implies $y < x$. Thus, we get

$$x^2y < x^3 \text{ and hence } 0 < x^3 - x^2y = \langle g(x), x - y \rangle. \quad (1.1.7)$$

Therefore, g is quasi-monotone.

To demonstrate that the mapping g is not pseudo-monotone, consider $x = -1$ and $y = 0$. Then $g(0^2)(-1 - 0) = 0 \geq 0$ and $g(-1^2)(-1 - 0) = -1 < 0$. This shows that g is not pseudo-monotone and hence not monotone. $\square$

Let C be a non-empty, convex and closed subset of a real Banach space $\mathbb{E}$. Let $T : \mathbb{E} \rightarrow \mathbb{E}^*$ be a mapping. The Variational Inequality Problem (VIP) is defined as follows:

$$\text{find a vector } x \in C \text{ such that } \langle T(x), y - x \rangle \geq 0, \text{ for all } y \in C. \quad (1.1.8)$$

We denote $VI(C, T)$ as the solution set of the VIP (1.1.8). The following is the formulation of the Dual Variational Inequality Problem (DVIP), often known as the Minty Variational Inequality Problem (MVIP):

$$\text{solve for } x^* \in C \text{ such that } \langle T(y), y - x^* \rangle \geq 0, \text{ for all } y \in C. \quad (1.1.9)$$

The solution set of the DVIP (1.1.9) is denoted by $DVI(C, T)$. Clearly, $DVI(C, T)$ is a closed and convex set. If T is continuous and C is convex, then $DVI(C, T) \subseteq VI(C, T)$ (see, [60]). However, the inclusion $VI(C, T) \subseteq DVI(C, T)$ is not necessarily true when T is continuous and quasi-monotone mapping (see, [57]). In fact, if T is continuous and pseudo-monotone, then we have that $VI(C, T) = DVI(C, T)$ (see, [19]). Since its introduction in the 1960s, variational inequalities theory has gained popularity as an intriguing and exciting area of applied mathematics with a broad range of applications in the financial, economic, and applied sciences as well as the industry.

Due to the dynamic nature of this topic and its rapid expansion in the theory and applications, research methods and issues are derived from a variety of fields. The concepts and methods of variational inequalities are being used in many different scientific fields, and they are effective and creative. As demonstrated, for example, by [1] and the references therein, this theory offers

the most natural, straightforward, simple, coherent, and efficient framework for a broad treatment of a large class of unrelated linear and non-linear problems.

As a result of its usefulness, numerous authors have carried out a great deal of research on VIP's and their abstractions in finite and also infinite dimensional spaces, (see [\[45, 20\]](#)).



Some Applications of Variational Inequality Problems

1. Applications to Optimization Problems: Let y^* be a solution to the optimization problem:

$$\begin{aligned} & \text{Minimize } f(y) \\ & \text{subject to : } y \in C, \end{aligned} \tag{1.1.10}$$

where the mapping $f : C \rightarrow \mathbb{R}$ is convex and continuously Frechet differentiable and C is a closed and convex. Then y^* is considered a solution to the optimization problem (1.1.10) if and only if it is the solution of the variational inequality problem:

$$\langle \nabla f(y^*), y - y^* \rangle \geq 0 \quad \forall y \in C. \tag{1.1.11}$$

where ∇ is the gradient of f . In fact, let $\phi(t) = f(y^* + t(y - y^*))$, for $t \in [0, 1]$. Since ϕ attains its minimum at $t = 0$, we have that $0 \leq \phi'(0) = \nabla f(y^*)^T(y - y^*)$, showing that y^* solves (1.1.11).

2. Applications to Complementary Problems: Numerous classical economic equilibrium issues have been expressed as systems of equations because market clearing circumstances require that the entire supply be equal to the entire amount requested. The following is the formulation of a system of equations for a variational inequality problem.

Lemma 1.1.6. [63] *Let H be a real Hilbert space with $F : H \rightarrow H$ as the given function. The vector $x^* \in H$ solves $VI(H, F)$ if and only if $F(x^*) = 0$.*

3. Applications to Fixed Point Problems: The relationship between a variational inequality and a fixed point problem is as follows.

We note that if C , a subset of the Hilbert space H , is closed and convex, then a mapping $P_C : H \rightarrow C$ is considered a metric projection if $\|x - P_C x\| \leq \|x - y\|, \forall y \in C$. The subsequent property of P_C is known to be satisfied:

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \forall y \in C. \tag{1.1.12}$$

Lemma 1.1.7. [63] *Assume that K , a subset of the Hilbert space H , is closed and convex. Then $x^* \in K$ is a solution to the variational inequality problem VIP if and only if for $\gamma > 0$, we have x^* as a fixed point of the map*

$$P_K(I - \gamma F) : K \rightarrow K,$$

that is,

$$x^* = P_K(x^* - \gamma F(x^*)).$$

1.2 Motivation

Many physical problems in real life can be modelled as non-linear problems which involve different types of mappings. Some of these problems in non-linear analysis which arise from differential equations reduce to variational inequality problems. Since there is no closed method for solving such problems, different approximation methods have been employed to solve them throughout the last few decades.

The Gradient method (GM) is the oldest and most basic projection method for resolving the VIP. However, this methods convergence requires that T be a strongly (or inverse strongly) monotone mapping . To avoid this hypothesis, Korpelevich [30] constructed the so-called Extra-Gradient Method (EGM), which calls for computing the projection on the feasible set twice in each iteration.

It is well known that the EGM has several limitations when used for the VIP [27]. For example, only when the underlying operator is Lipschitz continuous and monotone does the EGM converge weakly. Additionally, the EGM's execution is dependent on the Lipschitz constant as well as the computation of a projection onto a feasible set twice for each iteration . Zheng [13] took his approach by introducing the subgradient extragradient method (SEGM) which replaced the second projection with a projection onto a well constructed half-space. Because this method depends on the Lipschitz condition of the cost operator, it preserves the weakness of the EGM.

The various studies that have been conducted to solve variational inequality problems involved mappings with different continuity and monotonicity conditions. The results obtained from these studies are confined to lower class of mappings either in terms of monotonicity or continuity properties in real Hilbert/ Banach spaces. It should be noted that a number of the results mentioned above were attained in the context of real Hilbert spaces, which uses mainly norm distance and metric projection. Metric projection is non-expansive only when it is in Hilbert space. In [46], Zheng constructed a double projection method with the use of Bregman distance and Bregman projection in Banach spaces for solving variational inequalities.

Motivated and inspired by the works of Tseng in [54], Solodov and Svaiter [46] and Zheng [13], it is therefore natural to introduce the generalization of the above problems and propose methods for approximating the solutions in real reflexive Banach spaces. Hence, the purpose of this thesis is to establish and study double projection like algorithms for solving the split equality variational inequality problem in the setting of reflexive real Banach spaces, where the underlying mappings are quasi-monotone and uniformly continuous.

1.3 Statement of the Problem

Some significant generalizations of the VIP and DVIP are the Split Equality Variational Inequality Problem (SEVIP) and Split Equality Dual Variational Inequality Problem (SEVIP) respectively. Let C and D be convex and closed subsets of real Banach spaces $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. Let $T : \mathbb{E}_1 \rightarrow \mathbb{E}_1^*$ and $S : \mathbb{E}_2 \rightarrow \mathbb{E}_2^*$ be non-linear mappings and $A : \mathbb{E}_1 \rightarrow \mathbb{E}_3, B : \mathbb{E}_2 \rightarrow \mathbb{E}_3$ be bounded linear mappings, with $\mathbb{E}_3$ as another real Banach space. The SEVIP consists of finding two points:

$$x^* \in VI(C, T) \text{ and } y^* \in VI(D, S) \text{ such that } Ax^* = By^*. \quad (1.3.1)$$

Where as the SEDVIP is to find two points:

$$x^* \in DVI(C, T) \text{ and } y^* \in DVI(D, S) \text{ such that } Ax^* = By^*. \quad (1.3.2)$$

The SEVIP and SEDVIP are broad and they encompasses as special cases, split equality zero point problems (see, [22]), common solutions of the variational inequality problems [14], common zeros of mappings [16], split equality feasibility problems, which have been thoroughly studied by numerous authors and implemented in solving many real world problems such as in sensor networks in computerized tomography and data compression [8, 18], modelling of inverse problems obtained from phase retrieval, and in modelling intensity-modulated radiation therapy treatment planning [10, 11].

1.4 Research Questions

The following questions logically arise from the statement of problem above.

- Can an algorithm which approximates the solutions of SEVIPs and/or SEDVIPs involving non Lipschitz continuous quasi-monotone mappings be established?
- Can we provide applications of our results to a number of problems in reflexive real Banach spaces?
- Is it possible to provide numerical examples to demonstrate the behavior of the proposed algorithm?

1.5 Objectives of the Study

1.5.1 Main Objective of the study

The main objective of this thesis is to propose and investigate algorithms which approximate solutions of SEVIP and/or SEDVIP in reflexive real Banach spaces.

1.5.2 Specific Objectives

The specific objectives of this study are to:

- propose and study algorithms which solve the SEVIP and/or SEDVIP involving quasi-monotone mappings in reflexive real Banach spaces.
- prove weak and/or strong convergence of the algorithms to solutions of the variational inequality problems under mild conditions.
- provide more applications of SEVIPs and/or SEDVIPs.
- provide numerical examples to validate the application of the algorithm.

1.6 Organization of the Thesis

The present thesis is structured in the following manner. The introduction, which is given in Chapter 1 includes the background of the Variational Inequality Problem (VIP), motivation, statement of the problem, research questions along with the objectives of this thesis. We provide preliminaries which includes some lemmas that are of interest in Chapter 2, along with an extensive literature analysis of topics pertaining to the SEVIP and SEDVIP. In Chapter 3, we state iterative algorithms for solving the SEVIP and SEDVIP associated with quasimonotone mappings in reflexive real Banach spaces. We then provide weak and strong convergence results to the solutions of the problems. We next go over the applications of our findings in Chapter 4 and give numerical examples in Chapter 5. Lastly, the conclusion and our contribution to the field are covered in Chapter 6 along with some suggestions for further study.

2. Literature Review

Some well-known lemmas and significant definitions that are important to the work under discussion in this study are stated in this chapter. We will continue acknowledging sources as needed as we elaborate and thoroughly explore the outcomes in this chapter and in the next chapters. We next go over some helpful findings and provide a thorough literature analysis of ideas related to our work.

2.1 Preliminaries

Let $\mathbb{E}$ be a reflexive real Banach space with $\mathbb{E}^*$ as its dual. Let $\{x_n\}$ be a sequence in $\mathbb{E}$. We denote the weak and strong convergence of $\{x_n\}$ to a point $x \in \mathbb{E}$ by $x_n \rightharpoonup x$ and $x_n \rightarrow x$, respectively.

Let $g : \mathbb{E} \rightarrow (-\infty, \infty]$ be a convex function. The domain of g , denoted by $\text{dom } g$, is defined as $\text{dom } g = \{x \in \mathbb{E} : g(x) < \infty\}$. In this case, g is said to be *proper* if $\text{dom } g \neq \emptyset$. If $g : \mathbb{E} \rightarrow (-\infty, \infty]$ is proper, lower semi-continuous and convex, then the *Fenchel conjugate* of g is the real valued function $g^* : \mathbb{E}^* \rightarrow (-\infty, \infty]$ defined by $g^*(x^*) = \sup \{\langle x^*, x \rangle - g(x) : x \in \mathbb{E}\}$ for any $x^* \in \mathbb{E}^*$. The right hand derivative of g at $x \in \text{int}(\text{dom } g)$ in the direction of $y \in \mathbb{E}$ is defined by

$$g^o(x, y) = \lim_{t \downarrow 0} \frac{g(x + ty) - g(x)}{t}. \quad (2.1.1)$$

The function g is said to be *Gâteaux differentiable* at x if the limit in (2.1.1) exists for every $y \in \mathbb{E}$. In this case, the value of the gradient of g at x , denoted by $\nabla g(x)$, is the linear function $\langle \nabla g(x), y \rangle = g^o(x, y)$ for all $y \in \mathbb{E}$. If g is Gâteaux differentiable at every point $x \in \text{int}(\text{dom } g)$, then we say that g is Gâteaux differentiable. If the limit in (2.1.1) is attained uniformly for any $y \in \mathbb{E}$ with $\|y\| = 1$, then g is said to be *Fréchet differentiable* at x and it is said to be *uniformly Fréchet differentiable* on a subset C of $\mathbb{E}$ if the limit (2.1.1) is attained uniformly for $x \in C$ and $\|y\| = 1$. We know that if g is a Fréchet differentiable function, then it is Gâteaux differentiable and if g is Fréchet differentiable, then it is also continuous see ([38], pg. 142). It is also known that if $g : \mathbb{E} \rightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of $\mathbb{E}$, then ∇g is uniformly continuous on bounded subsets of $\mathbb{E}$ (see, e.g., [41]) and it is also uniformly smooth (see, e.g., [58]).

A function $g : \mathbb{E} \rightarrow \mathbb{R}$ is a *Legendre function* if it satisfies the subsequent two conditions:

(L1) g is Gâteaux differentiable, $\text{int}(\text{dom } g) \neq \emptyset$ and $\text{dom } \nabla g = \text{int}(\text{dom } g)$;

(L2) g^* is Gâteaux differentiable, $\text{int}(\text{dom } g^*) \neq \emptyset$ and $\text{dom } \nabla g^* = \text{int}(\text{dom } g^*)$.

Let $B = \{x \in \mathbb{E} : \|x\| = 1\}$. Then $\mathbb{E}$ is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in B$ with $x \neq y$. We say that the space $\mathbb{E}$ is *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \quad (2.1.2)$$

exists for $x, y \in B$.

If $\mathbb{E}$ is a smooth and strictly convex Banach space, then $g(x) = \frac{1}{p}\|x\|^p$ ($1 < p < \infty$) is a typical case of a proper, lower semi-continuous and Legendre function with Fenchel conjugate $g^*(x^*) = \frac{1}{q}\|x^*\|^q$ ($1 < q < \infty$), (see, e.g. [4]), where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, we have $\nabla g = J_p$, where $J_p : \mathbb{E} \rightarrow 2^{\mathbb{E}^*}$ is the multi-valued function defined as follows

$$J_p(x) = \{x^* \in \mathbb{E}^* : \langle x^*, x \rangle = \|x\|^p \text{ and } \|x^*\| = \|x\|^{p-1}\},$$

for all $x \in \mathbb{E}$. The special case when $p = 2$ gives that $J_p = J$, where J is the normalized duality mapping defined by

$$J(x) = \{x^* \in \mathbb{E}^* : \langle x^*, x \rangle = \|x\|^2 \text{ and } \|x^*\| = \|x\|\}.$$

If in addition $\mathbb{E} = \mathbb{H}$, where $\mathbb{H}$ is a real Hilbert space, then $J = I$, where I is the identity mapping on $\mathbb{H}$. We will denote the normalized duality mapping J on $\mathbb{E}$ by $J_{\mathbb{E}}$.

If $g : \mathbb{E} \rightarrow (-\infty, \infty]$ is a Legendre function and $\mathbb{E}$ is a reflexive Banach space, then $\nabla g^* = (\nabla g)^{-1}$ (see, [6]). Moreover, we note that g is a Legendre function if and only if g^* is a Legendre function (see, [4]).

Definition 2.1.1. Let $g : \mathbb{E} \rightarrow (-\infty, \infty]$ be a convex and Gâteaux differentiable function.

1. The real-valued non-negative function $D_g : \text{dom } g \times \text{int}(\text{dom } g) \rightarrow [0, +\infty)$ defined by

$$D_g(y, x) = g(y) - g(x) - \langle \nabla g(x), y - x \rangle, \quad x \in \text{int}(\text{dom } g), \text{ and } y \in \text{dom } g, \quad (2.1.3)$$

is called the *Bregman distance* with respect to g (see, Censor and Lent [15]).

2. The modulus of total convexity of g at the point $x \in \mathbb{E}$ is the function

$v_g : \mathbb{E} \times [0, +\infty) \rightarrow [0, +\infty)$ defined as

$$v_g(x, t) = \inf \{D_g(x, y) : y \in \mathbb{E} \text{ and } \|y - x\| = t\}. \quad (2.1.4)$$

3. The function g is totally convex if the function $v_g(x, t) > 0$ for all $t > 0$ and $x \in \mathbb{E}$.

4. g is β -strongly convex function if there exists $\beta > 0$ such that

$$g(x) - g(y) - \langle \nabla g(y), x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in \mathbb{E}. \quad (2.1.5)$$

Note that if a function g is β -strongly convex, then ∇g^* is $\frac{1}{\beta}$ -Lipschitz continuous (see, [61]).

A function $g: \mathbb{E} \rightarrow (-\infty, \infty]$ is said to be strongly coercive if $\lim_{\|x\| \rightarrow \infty} \left(\frac{g(x)}{\|x\|} \right) = \infty$. If $\mathbb{E}$ is a smooth and 2-uniformly convex Banach space, then the function $f(x) = \frac{1}{2}\|x\|^2$ is strongly coercive, bounded, lower semi-continuous, uniformly Fréchet differentiable and strongly convex with strong convexity constant $\beta \in (0, 1]$ and conjugate $f^*(x^*) = \frac{1}{2}\|x^*\|^2$.

Lemma 2.1.2. [58] *If g is convex and bounded on bounded subsets of $\mathbb{E}$, then the following are equivalent:*

- i. g is uniformly convex and strongly coercive on bounded subsets of $\mathbb{E}$.
- ii. $\text{dom } g^* = \mathbb{E}^*$, g^* is bounded on bounded subsets and uniformly smooth on bounded subsets of $\mathbb{E}^*$.
- iii. $\text{dom } g^* = \mathbb{E}^*$, g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of $\mathbb{E}^*$.

Lemma 2.1.3. [35] *Let $g: \mathbb{E} \rightarrow (-\infty, \infty]$ be a Gâteaux differentiable function which is totally convex on bounded subsets of $\mathbb{E}$. Suppose the sequences $\{x_n\}$ and $\{y_n\}$ are bounded on $\mathbb{E}$. Then, $\lim_{n \rightarrow \infty} D_g(x_n, y_n) = 0$ if and only if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Definition 2.1.4. Let $g: \mathbb{E} \rightarrow (-\infty, \infty]$ be a convex, Gâteaux differentiable function and let $C \subseteq \text{int}(\text{dom } g)$ be a nonempty, closed and convex subset of $\mathbb{E}$. Then, the Bregman projection of $x \in \text{int}(\text{dom } g)$ onto C is the unique vector $\Pi_C^g(x)$ of C with the property

$$D_g(\Pi_C^g(x), x) = \inf \{D_g(y, x) : y \in C\}.$$

The Bregman projection also satisfies the subsequent properties:

$$z = \Pi_C^g(x) \text{ if and only if } \langle \nabla g(x) - \nabla g(z), y - z \rangle \leq 0, \text{ for all } y \in C, \text{ and} \quad (2.1.6)$$

$$D_g(y, \Pi_C^g(x)) + D_g(\Pi_C^g(x), x) \leq D_g(y, x), \text{ for all } x \in \mathbb{E}, y \in C. \quad (2.1.7)$$

Lemma 2.1.5. [40] *Let $g: \mathbb{E} \rightarrow (-\infty, \infty]$ be a convex Gâteaux differentiable function. The Bregman distance $D_g(\cdot, \cdot)$ has the subsequent property called three point identity*

$$D_g(y, z) + D_g(z, x) - D_g(y, x) = \langle \nabla g(z) - \nabla g(x), z - y \rangle, \quad (2.1.8)$$

for any $y \in \text{dom } g$ and $x, z \in \text{int}(\text{dom } g)$.

Lemma 2.1.6. [33] *Let C be a non-empty, closed and convex subset of $\mathbb{E}$ and $g: \mathbb{E} \rightarrow \mathbb{R}$ be a totally convex and Frechet-differentiable function. If ∇g is uniformly continuous on bounded subsets of E and the level sets $D_g(x, \cdot)$ are bounded for all $x \in E$, then $\Pi_C^g: \mathbb{E} \rightarrow C$ maps bounded subset of $\mathbb{E}$ into bounded subset of C .*

Lemma 2.1.7. [35] *Let $\mathbb{E}$ be a real Banach space. The non-negative real valued function $V_g: \mathbb{E} \times \mathbb{E}^* \rightarrow (-\infty, \infty]$ associated with a Gâteaux differentiable Legendre function $g: \mathbb{E} \rightarrow \mathbb{R}$ defined by*

$$V_g(x, x^*) = g(x) + \nabla g'(x^*) - \langle x^*, x \rangle, \forall x \in \mathbb{E}, x^* \in \mathbb{E}^* \quad (2.1.9)$$

satisfies the following two properties

$$V_g(x, x^*) = D_g(x, (\nabla g)^{-1}(x^*)) \quad (2.1.10)$$

and

$$V_g(x, x^*) + \langle y^*, (\nabla g)^{-1}(x^*) - x \rangle \leq V_g(x, x^* + y^*), \forall x \in \mathbb{E}, x^*, y^* \in \mathbb{E}^*. \quad (2.1.11)$$

Lemma 2.1.8. [32] *Let $g : \mathbb{E} \rightarrow \mathbb{R}$ be a totally convex function. The sequence $\{x_k\}$ is bounded if $\{D_g(x_k, x_0)\}$ is bounded for any $x_0 \in \mathbb{E}$.*

Lemma 2.1.9. [46] *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be bounded spaces. Let U be a bounded subset of $\mathbb{E}_1$. $T : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ is bounded on U if T is uniformly continuous on bounded subsets of $\mathbb{E}_1$.*

Lemma 2.1.10. [25] *Let T be a quasimonotone and continuous operator, and let $y \in C$ a subset of $\mathbb{E}$. If for some $x_0 \in C$ we have $\langle T(y), x_0 - y \rangle \geq 0$, then at least one of the following must be true: $\langle T(x_0), x_0 - y \rangle \geq 0$ or $\langle T(y), x - y \rangle \geq 0, \forall x \in C$.*

Lemma 2.1.11. [60] *Let C be a non-empty closed and convex subset of $\mathbb{E}$ and let g be β -strongly convex and continuously differentiable function such that the level sets $D_g(x, \cdot)$ are bounded for all $x \in E$. Define $h : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ by $h(x, v) = \langle T(v), x - v \rangle$ for any given $v \in \mathbb{E}$ and the mapping $T : \mathbb{E} \rightarrow \mathbb{E}^*$ and take $K(v) = \{x \in C : h(x, v) \leq 0\}$. If $K(v) \neq \emptyset$ and $h(x, v)$ is Lipschitz continuous with respect to x on C with modulus $L > 0$, then*

$$D_g(x, y) \geq \frac{\beta}{L^2} h^2(x, v), \forall x \in C \setminus K(v), y \in K(v), v \in \mathbb{E}. \quad (2.1.12)$$

Lemma 2.1.12. [47] *Let $\mathbb{E}$ be a strictly convex and smooth real Banach space. Then the normalized duality mapping on $\mathbb{E}$, denoted by $J_{\mathbb{E}}$ has the following property*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle J_{\mathbb{E}}(x + y), y \rangle, \forall x, y \in \mathbb{E}. \quad (2.1.13)$$

Lemma 2.1.13. [39] *Let $g : \mathbb{E} \rightarrow (-\infty, \infty]$ be a proper, convex, lower semi-continuous and Gâteaux differentiable function. Then, $g^* : \mathbb{E}^* \rightarrow (-\infty, \infty)$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $x \in \mathbb{E}$, we have*

$$D_g \left(x, (\nabla g)^{-1} \left(\sum_{i=1}^N \beta_i \nabla g(x_i) \right) \right) \leq \sum_{i=1}^N \beta_i D_g(x, x_i), \quad (2.1.14)$$

where $\{x_i\} \subseteq \mathbb{E}, \{\beta_i\} \subseteq (0, 1)$ and $N \in \mathbb{N}$ such that $\sum_{i=1}^N \beta_i = 1$.

Lemma 2.1.14. [31] *Let $\{a_n\}$ be a sequence of non-negative real numbers. If $\{a_{n_i}\}$ is a sub-sequence of $\{a_n\}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \in \mathbb{N}$, then there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ with $\lim_{k \rightarrow \infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max \{n \leq k : a_n < a_{n+1}\}$.

Lemma 2.1.15. [55] If $\{a_k\}$ is a sequence of non-negative real numbers such that

$$a_{k+1} \leq (1 - \alpha_k) a_k + \alpha_k d_k, \quad (2.1.15)$$

where $\{\alpha_k\} \subset (0, 1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\{d_k\}$ is a sequence of real numbers with $\limsup_{k \rightarrow \infty} d_k \leq 0$, then $\lim_{k \rightarrow \infty} a_k = 0$.



2.2 Literature Review on Split Equality Variational Inequality Problem

Let $T : C \rightarrow \mathbb{E}$ be a mapping. We recall the variational inequality problem (VIP) involves:

$$\text{finding a vector } x^* \in C \text{ such that } \langle T(x^*), y - x^* \rangle \geq 0, \text{ for all } y \in C. \quad (2.2.1)$$

The solution set of the VIP (2.2.1) is denoted $VI(C, T)$. Since it was initially independently investigated by Stampacchia [48] and Fichera [24] in 1964, the theory of variational inequalities has been thoroughly researched and is still a hot topic. One of the main causes of this is the wide range of problems that may be expressed as variational inequalities, including those emerging in the domains of optimization and control, engineering science, mechanics, game theory, elasticity, physics, economics, transportation equilibrium, etc. (see, e.g. [3, 5]).

It was noted in [29] that the Minty problem has practical implications. A dynamical system associated with T has been shown to have solutions of a DVIP associated with a single-valued continuous mapping T defined on an open convex domain, which can be viewed as the subset of stable equilibria within the set of all equilibria (represented by the Stampacchia variational inequality solutions).

Many authors have been driven to create and generalize many forms of variational inequalities in many different directions utilizing fresh and inventive methodologies, as evidenced by the need to frame and research these kinds of problems (see, e.g., [2, 3, 23] and the references therein).

To solve variational inequalities and associated optimization problems, numerous numerical iterative methods have been developed (see, e.g., [13, 43]). Authors have examined a variety of projection type algorithms to solve the variational inequality problem. The gradient method (GM) is the first projection technique for solving the VIP and is provided by

$$x_{n+1} = P_C(x_n - \lambda T x_n), \quad (2.2.2)$$

where P_C is the projection operator on the convex and closed subset C of a Hilbert space $\mathbb{H}$ and $\lambda > 0$ is a suitable step-size. With only one projection onto the feasible set, the gradient method (GM) is the most straightforward approach for solving the VIP. The approach, however, only converges when the cost operator T is α -strongly monotone and L -Lipschitz continuous with $\lambda \in (0, \frac{2\alpha}{L^2})$. These stringent conditions significantly limit the scope of applications of the GM (2.2.2).

To avoid the notion of strong monotonicity, Korpelevich [30] introduced the extra-gradient method (EGM), described below, which is an algorithm for resolving variational inequalities in

a Hilbert space $\mathbb{H}$.

$$\begin{cases} y_n = P_C(x_n - \lambda T x_n), \\ x_{n+1} = P_C(x_n - \lambda T y_n), \end{cases} \quad (2.2.3)$$

where $T : C \rightarrow \mathbb{H}$ is monotone and L -Lipschitz continuous and $\lambda \in (0, \frac{1}{L})$ and $C \subseteq \mathbb{H}$ is closed and convex. He established the weak convergence of (2.2.3) to a solution of the VIP (2.2.1) in Hilbert spaces if T is monotone and Lipschitz.

The algorithm (2.2.3) has a number of important limitations, as demonstrated by the above technique. It requires that the mapping have the properties of Lipschitz continuity, monotonicity, and known Lipschitz constant. In addition, each iteration of this algorithm requires the computation of projections twice to the feasible set C , which complicates the algorithm's calculations and significantly lowers its efficiency. Therefore, under the heading of guaranteeing algorithm convergence, two key areas of variational inequality problems research directions are “weakening the mapping” and “accelerating the algorithm's convergence rate.”

The EGM has garnered the interest of scholars in recent times, who have made numerous improvements to it (see, e.g., [21, 36, 42]) and the references therein. The method's reduction in the number of projections onto the feasible set C per iteration is one of its main areas of improvement. The first attempt was made by Censor *et al.* [13], who changed the EGM and substituted a projection onto a half-space for the second projection. This approach, which is called the sub-gradient extra-gradient technique (SEGM), only necessitates a single projection onto the feasible set C . The SEGM is presented as follows:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda T x_n), \\ T_n = \{z \in H : \langle x_n - \lambda T x_n - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda T y_n). \end{cases} \quad (2.2.4)$$

where $\lambda \in (0, \frac{1}{L})$. As seen, the projection for x_{n+1} in the scheme (2.2.4) is computed on a half-space T_n which is inherently explicit. Using the same assumptions as the EGM (2.2.3), Censor *et al.* [13] attained a weak convergence result for the SEGM (2.2.4). The SEGM can be viewed as an improvement over the EGM as projecting onto a half-space can be computed explicitly. We note, however, that each iteration of the SEGM still necessitates computing two projections onto the closed convex sets C and T_n . This may potentially be a significant obstacle to the SEGM's implementation.

The second attempt is the following Tseng extra-gradient method: choose $x_0 \in \mathbb{H}$, for each $n \geq 0$, compute:

$$\begin{cases} y_n = P_C(x_n - \lambda T x_n), \\ x_{n+1} = y_n - \lambda(T y_n - T x_n), \end{cases} \quad (2.2.5)$$

where $\lambda \in (0, \frac{1}{L})$. One may view the scheme (2.2.5) as the outcome of Tseng's modified forward-

backward splitting technique, which was first presented in [54]. He proved that the sequence $\{x_n\}$ weakly converges to a solution of the VIP (2.2.1). Tseng's approach has an advantage over Korpelevich's in that it only needs one projection for each iteration. It still necessitates two evaluations of the mapping T at each iteration. Several researchers have examined modifications to Tseng's algorithm (see, e.g., [36, 37, 43, 44, 42, 59]).

We note that the above methods make use the norm distance and metric projections. Attempting to apply this theory to Banach spaces presents certain challenges, as certain helpful properties of non-expansive operators in Hilbert space (such as the metric projection P_K onto a non-empty, closed, and convex subset K of $\mathbb{H}$) are no longer non-expansive in Banach spaces. There are numerous approaches to overcoming these challenges. Using the Bregman distance, which does not require symmetry or triangle inequality features, is one of them. Instead of using metric projections, which are less flexible and broad, researchers in this case employ Bregman distance and Bregman projection. Therefore, the Bregman distance and projection methods for approximating the solutions of the VIP are valuable in the analysis of numerous problems.

In 1999, Solodov and Svaiter [46] studied a double projection method with the use of the Bregman distance and Bregman projection in Euclidean spaces to solving variational inequalities.

Algorithm: Choose $x_0 \in \mathbb{H}$, and $\gamma, \sigma \in (0, 1)$. Take $k = 0$. Calculate $\{x_k\}$ as follows:

Step 1: Compute

$$z_k = (\nabla g)^{-1} [\nabla g(x_k) - T(x_k)]; \quad (2.2.6)$$

If $x_k = \Pi_C^g(z_k)$, then stop; else go to Step 2.

Step 2: Compute

$$m_k = \min \{m \in \mathbb{N} : \langle T(x_k) - T(y_m), x_k - \Pi_C^g(z_k) \rangle \leq \sigma D_g(\Pi_C^g(z_k), x_k)\}; \quad (2.2.7)$$

where

$$y_m = \gamma^m \Pi_C^g(z_k) + (1 - \gamma^m)x_k; \quad (2.2.8)$$

Let

$$y_k = \gamma^{m_k} \Pi_C^g(z_k) + (1 - \gamma^{m_k})x_k; \quad (2.2.9)$$

Step 3: Compute

$$x_{k+1} = \Pi_{C \cap H_k}^g(x_k); \quad (2.2.10)$$

where

$$H_k = \{v \in \mathbb{H} : \langle T(y_k), v - y_k \rangle \leq 0\}; \quad (2.2.11)$$

Step 4: Let $k = k + 1$ and return to **Step 1**.

In 2018, Zheng [13] extended the results from Euclidean spaces to Banach spaces. He proved that

the sequence generated from the proposed algorithm strongly converges to the solution of the variational inequalities in reflexive real Banach spaces under the assumptions that $VI(C, T) \neq \emptyset$ and T is uniformly continuous and quasi-monotone mapping. In our opinion, there is a gap in the proof of the strong convergence theorem.

In the next chapter, we introduce a more general problem called the SEVIP and provide a method for approximating its solution. Our results provides positive answers to the questioned raised above.



3. Split Equality Variational Inequality Problem

Let C and D be convex and closed subsets of real Banach spaces E_1 and E_2 , respectively. Let $T : E_1 \rightarrow E_1^*$ and $S : E_2 \rightarrow E_2^*$ be non-linear mappings and $A : E_1 \rightarrow E_3$, $B : E_2 \rightarrow E_3$ be bounded linear mappings, where E_3 is another real Banach space. The SEVIP consists of finding two points:

$$x^* \in VI(C, T) \text{ and } y^* \in VI(D, S) \text{ such that } Ax^* = By^*. \quad (3.0.1)$$

On the other hand the SEDVIP consists of finding two points:

$$x^* \in DVI(C, T) \text{ and } y^* \in DVI(D, S) \text{ such that } Ax^* = By^*. \quad (3.0.2)$$

3.1 Main Results

In this section, we state our algorithms and discuss their convergence results under the subsequent conditions:

- (H₁) Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be reflexive real Banach spaces with duals $\mathbb{E}_1^*$ and $\mathbb{E}_2^*$, respectively, and let C and D be non-empty, closed and convex subsets of $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively.
- (H₂) Let $T : \mathbb{E}_1 \rightarrow \mathbb{E}_1^*$ and $S : \mathbb{E}_2 \rightarrow \mathbb{E}_2^*$ be quasi-monotone mappings which are uniformly continuous on bounded subsets of C and D , respectively, satisfying

$$\|T(x)\| \leq \liminf_{k \rightarrow \infty} \|T(x_k)\| \text{ and } \|S(y)\| \leq \liminf_{k \rightarrow \infty} \|S(y_k)\|, \quad (3.1.1)$$

whenever $\{x_k\}$ and $\{y_k\}$ are sequences in $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively, such that $x_k \rightharpoonup x$ and $y_k \rightharpoonup y$.

- (H₃) Let $A : \mathbb{E}_1 \rightarrow \mathbb{E}_3$ and $B : \mathbb{E}_2 \rightarrow \mathbb{E}_3$, where $\mathbb{E}_3$ is another real Banach space, be bounded linear mappings with adjoints A^* and B^* , respectively.
- (H₄) Let the set $\Omega = \{(x^*, y^*) \in DVI(C, T) \times DVI(D, S) : Ax^* = By^*\}$ be non-empty.
- (H₅) Let the proper lower semi-continuous functions $g : \mathbb{E}_1 \rightarrow \mathbb{R}$ and $f : \mathbb{E}_2 \rightarrow \mathbb{R}$ be strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and β -strongly convex on bounded subsets of $\mathbb{E}_1$ and $\mathbb{E}_2$ respectively.
- (H₆) Let the sequence $\{\alpha_k\} \subset (0, 1)$ be such that $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\lim_{k \rightarrow \infty} \alpha_k = 0$.

Algorithm 3.1 Choose $x_0 \in \mathbb{E}_1$, $y_0 \in \mathbb{E}_2$ and $\gamma, \sigma \in (0, 1)$. Take $k = 1$. Calculate $\{x_k\}$ and $\{y_k\}$ as follows:

Step 1: Compute

$$a_k = \Pi_C^g(\nabla g)^{-1} [\nabla g(x_k) - \gamma_k A^* (J_{\mathbb{E}_3}(Ax_k - By_k))]; \quad (3.1.2)$$

$$b_k = \Pi_D^f(\nabla f)^{-1} [\nabla f(y_k) - \gamma_k B^* (J_{\mathbb{E}_3}(By_k - Ax_k))],$$

where $0 < \rho \leq \gamma_k \leq \rho_k$ with

$$\rho_k = \min \left\{ \rho + 1, \frac{\beta \|Ax_k - By_k\|^2}{2[\|A^* J_{\mathbb{E}_3}(Ax_k - By_k)\|^2 + \|B^* J_{\mathbb{E}_3}(By_k - Ax_k)\|^2]} \right\}, \quad (3.1.3)$$

for $k \in \Upsilon = \{m \in \mathbb{N} : Ax_m - By_m \neq 0\}$, otherwise $\gamma_k = \rho$.

Step 2: Compute

$$z_k = (\nabla g)^{-1} [\nabla g(a_k) - \beta_k T(a_k)]; \quad (3.1.4)$$

$$u_k = (\nabla f)^{-1} [\nabla f(b_k) - \beta_k S(b_k)],$$

where $\{\beta_k\} \subset (0, 1)$ for all $k \geq 1$.

Step 3: Compute

$$m_k = \min \{m \in \mathbb{N} : \langle T(a_k) - T(d_m), a_k - \Pi_C^g(z_k) \rangle \leq \sigma D_g(\Pi_C^g(z_k), a_k)\}; \quad (3.1.5)$$

$$l_k = \min \{l \in \mathbb{N} : \langle S(b_k) - S(e_l), b_k - \Pi_D^f(u_k) \rangle \leq \sigma D_f(\Pi_D^f(u_k), b_k)\}, \quad (3.1.6)$$

where

$$d_m = \gamma^m \Pi_C^g(z_k) + (1 - \gamma^m) a_k; \quad (3.1.7)$$

$$e_l = \gamma^l \Pi_D^f(u_k) + (1 - \gamma^l) b_k.$$

Let

$$d_k = \gamma^{m_k} \Pi_C^g(z_k) + (1 - \gamma^{m_k}) a_k; \quad (3.1.8)$$

$$e_k = \gamma^{l_k} \Pi_D^f(u_k) + (1 - \gamma^{l_k}) b_k.$$

Step 4: Compute

$$x_{k+1} = \Pi_C^g \cap H_k(a_k); \quad (3.1.9)$$

$$y_{k+1} = \Pi_D^f \cap R_k(b_k);$$

where

$$H_k = \{x \in \mathbb{E}_1 : \langle T(d_k), x - d_k \rangle \leq 0\}; \quad (3.1.10)$$

$$R_k = \{y \in \mathbb{E}_2 : \langle S(e_k), y - e_k \rangle \leq 0\}.$$

Step 5 : Let $k = k + 1$ and return to **Step 1**.

Lemma 3.1.1. *Let conditions $(H_1) - (H_5)$ be satisfied. Then the line search rules (3.1.5) and (3.1.6) are well defined.*

Proof. Suppose that $a_k = \Pi_C^g(z_k)$, then (3.1.5) is satisfied for $m = 0$. Let $a_k \neq \Pi_C^g(z_k)$ and assume on the contrary that

$$\langle T(a_k) - T(d_m), a_k - \Pi_C^g(z_k) \rangle > \sigma D_g(\Pi_C^g(z_k), a_k) \quad \forall m \geq 1. \quad (3.1.11)$$

Given that $\gamma \in (0, 1)$, we have $\lim_{m \rightarrow \infty} d_m = \lim_{m \rightarrow \infty} \gamma^m (\Pi_C^g(z_k)) + (1 - \gamma^m) a_k = a_k$.

Letting $m \rightarrow \infty$ in (3.1.11) and by the continuity of T , we get

$0 = \lim_{m \rightarrow \infty} \langle T(a_k) - T(d_m), a_k - \Pi_C^g(z_k) \rangle \geq \sigma D_g(\Pi_C^g(z_k), a_k)$, and the fact that $\sigma > 0$, we get

$$D_g(\Pi_C^g(z_k), a_k) \leq 0. \quad (3.1.12)$$

But as g is strongly convex and $a_k \neq \Pi_C^g(z_k)$, we have

$$D_g(\Pi_C^g(z_k), a_k) > 0, \quad (3.1.13)$$

which is a contradiction. Thus, (3.1.5) is well defined. Similarly, the same can be shown for (3.1.6). $\square$

Lemma 3.1.2. *Assume the conditions $(H_1), (H_2)$ and (H_5) hold. Let T and S be continuous mappings on C and D , respectively. For all $x \in C$ and $y \in D$, we have*

$$\left\langle T(x), x - \Pi_C^g \left[(\nabla g)^{-1} (\nabla g(x) - \beta_k T(x)) \right] \right\rangle \geq D_g \left(\Pi_C^g \left[(\nabla g)^{-1} (\nabla g(x) - \beta_k T(x)) \right], x \right), \quad (3.1.14)$$

and

$$\left\langle S(y), y - \Pi_D^f \left[(\nabla f)^{-1} (\nabla f(y) - \beta_k S(y)) \right] \right\rangle \geq D_f \left(\Pi_D^f \left[(\nabla f)^{-1} (\nabla f(y) - \beta_k S(y)) \right], y \right). \quad (3.1.15)$$

Proof: Lemma 2.5 of [53].

Remark 3.1.3. *Note from Algorithm 3.1 and Lemma 3.1.2 the following hold.*

$$\langle T(d_k), a_k - \Pi_C^g(z_k) \rangle \geq (1 - \sigma) D_g(\Pi_C^g(z_k), a_k), \quad (3.1.16)$$

and

$$\langle S(e_k), b_k - \Pi_D^f(u_k) \rangle \geq (1 - \sigma) D_f(\Pi_D^f(u_k), b_k). \quad (3.1.17)$$

Proof: From Step 3 of the algorithm we have

$$\langle T(d_m), a_k - \Pi_C^g(z_k) \rangle \geq \langle T(a_k), a_k - \Pi_C^g(z_k) \rangle - \sigma D_g(\Pi_C^g(z_k), a_k). \quad (3.1.18)$$

From Lemma 3.1.2, we get

$$\langle T(a_k), a_k - \Pi_C^g(z_k) \rangle \geq D_g(\Pi_C^g(z_k), a_k). \quad (3.1.19)$$

Hence,

$$\langle T(d_m), a_k - \Pi_C^g(z_k) \rangle \geq (1 - \sigma) D_g(\Pi_C^g(z_k), a_k). \quad (3.1.20)$$

Therefore, for $m = m_k$, we obtain

$$\langle T(d_k), a_k - \Pi_C^g(z_k) \rangle \geq (1 - \sigma) D_g(\Pi_C^g(z_k), a_k). \quad (3.1.21)$$

Similarly, the relation (3.1.17) holds.

Lemma 3.1.4. *Assume the conditions $(H_1) - (H_5)$ hold. If $\{a_k\}$ and $\{b_k\}$ are sequences generated by Algorithm 3.1, we get*

- (i) $DVI(C, T) \subseteq C \cap H_k$ and $DVI(D, S) \subseteq D \cap R_k$, for all $k \geq 1$;
- (ii) $h_k(a_k) \geq c_k(1 - \sigma) D_g(\Pi_C^g(z_k), a_k)$ and $w_k(b_k) \geq t_k(1 - \sigma) D_g(\Pi_D^g(u_k), b_k)$ for $c_k = \gamma^{m_k}$, $t_k = \gamma^{l_k}$, $h_k(v) = \langle Td_k, v - d_k \rangle$ for $v \in C$ and $w_k(u) = \langle S(e_k), u - e_k \rangle$, for $u \in D$, for all $k \geq 1$.

Proof. Let $(x^*, y^*) \in \Omega$. Then $\langle T(d_k), x^* - d_k \rangle \leq 0$ from which it follows that $x^* \in C \cap H_k$ and so $DVI(C, T) \subseteq C \cap H_k$, for all $k \geq 1$. Similarly, we get $DVI(D, S) \subseteq D \cap R_k$, for all $k \geq 1$.

(ii) From Step 4 and Remark 3.1.3, we obtain that

$$\begin{aligned} h_k(a_k) &= \langle T(d_k), a_k - d_k \rangle \\ &= \langle T(d_k), a_k - (c_k \Pi_C^g(z_k) + (1 - c_k) a_k) \rangle \\ &= c_k \langle T(d_k), a_k - \Pi_C^g(z_k) \rangle \\ &\geq c_k(1 - \sigma) D_g(\Pi_C^g(z_k), a_k), \end{aligned}$$

which yields that the first part of (ii) holds. Similarly, we obtain that the second part of (ii) holds. $\square$

Lemma 3.1.5. *Assume the conditions $(H_1) - (H_5)$ hold. Let $(x^*, y^*) \in \Omega$. Then, the sequences $\{x_k\}$, $\{a_k\}$, $\{y_k\}$ and $\{b_k\}$ generated by Algorithm 3.1 satisfy the following relations.*

$$1. D_g(x_{k+1}, a_k) \leq D_g(x^*, x_k) - D_g(x^*, x_{k+1}) - \gamma_k \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x^* \rangle;$$

$$D_f(y_{k+1}, b_k) \leq D_f(y^*, y_k) - D_f(y^*, y_{k+1}) - \gamma_k \langle B^* J_{\mathbb{E}_3}(By_k - Ax_k), t_k - y^* \rangle.$$

$$\text{where } p_k = (\nabla g)^{-1} [\nabla g(x_k) - \gamma_k A^*(J_{\mathbb{E}_3}(Ax_k - By_k))]$$

$$\text{and } t_k = (\nabla f)^{-1} [\nabla f(y_k) - \gamma_k B^*(J_{\mathbb{E}_3}(By_k - Ax_k))].$$

2. $\{x_k\}$ and $\{y_k\}$ are bounded.

Proof. 1) From the three point identity, taking $z = x_{k+1}$, $x = a_k$, Lemma 2.1.5 and (2.1.6) we get that:

$$\begin{aligned} D_g(y, x_{k+1}) + D_g(x_{k+1}, a_k) - D_g(y, a_k) &= \langle \nabla g(x_{k+1}) - \nabla g(a_k), x_{k+1} - y \rangle \\ &\leq 0, \forall y \in C \cap H_k. \end{aligned} \quad (3.1.22)$$

Let $y = x^*$ in (3.1.22), then

$$D_g(x_{k+1}, a_k) \leq D_g(x^*, a_k) - D_g(x^*, x_{k+1}). \quad (3.1.23)$$

Let $p_k = (\nabla g)^{-1} [\nabla g(x_k) - \gamma_k A^*(J_{\mathbb{E}_3}(Ax_k - By_k))]$ and $t_k = (\nabla f)^{-1} [\nabla f(f_k) - \gamma_k B^*(J_{\mathbb{E}_3}(Ay_k - Bx_k))]$. From (2.1.7), (2.1.10), (2.1.11) and (3.1.2) we have

$$\begin{aligned} D_g(x^*, a_k) &\leq D_g(x^*, p_k) \\ &= D_g(x^*, (\nabla g)^{-1} [\nabla g(x_k) - \gamma_k A^*(J_{\mathbb{E}_3}(Ax_k - By_k))]) \\ &= V_g(x^*, \nabla g(x_k) - \gamma_k A^*(J_{\mathbb{E}_3}(Ax_k - By_k))) \\ &\leq V_g(x^*, \nabla g(x_k)) - \gamma_k \langle A^*(J_{\mathbb{E}_3}(Ax_k - By_k)), p_k - x^* \rangle \\ &= D_g(x^*, x_k) - \gamma_k \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x^* \rangle. \end{aligned} \quad (3.1.24)$$

Combining (3.1.23) and (3.1.24), we get

$$D_g(x_{k+1}, a_k) \leq D_g(x^*, x_k) - D_g(x^*, x_{k+1}) - \gamma_k \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x^* \rangle. \quad (3.1.25)$$

Following the same approach, we arrive at

$$D_f(y_{k+1}, b_k) \leq D_f(y^*, y_k) - D_f(y^*, y_{k+1}) - \gamma_k \langle B^* J_{\mathbb{E}_3}(By_k - Ax_k), t_k - y^* \rangle. \quad (3.1.26)$$

2) From the fact that $D_g(x_{k+1}, a_k) \geq 0$, $D_f(y_{k+1}, b_k) \geq 0$, (3.1.25), (3.1.26) and $Ax^* = By^*$, we have

$$\begin{aligned} D_g(x_{k+1}, a_k) + D_f(y_{k+1}, b_k) &\leq D_g(x^*, x_k) - D_g(x^*, x_{k+1}) - \gamma_k \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x^* \rangle \\ &\quad + D_f(y^*, y_k) - D_f(y^*, y_{k+1}) - \gamma_k \langle B^* J_{\mathbb{E}_3}(By_k - Ax_k), t_k - y^* \rangle \\ &= D_g(x^*, x_k) - D_g(x^*, x_{k+1}) + D_f(y^*, y_k) - D_f(y^*, y_{k+1}) \\ &\quad - \gamma_k \langle J_{\mathbb{E}_3}(Ax_k - By_k), Ap_k - Bt_k \rangle. \end{aligned} \quad (3.1.27)$$

Furthermore, we have

$$\begin{aligned}
 -\langle J_{\mathbb{E}_3}(Ax_k - By_k), Ap_k - Bt_k \rangle &= -\langle J_{\mathbb{E}_3}(Ax_k - By_k), Ax_k - By_k \rangle \\
 &\quad -\langle J_{\mathbb{E}_3}(Ax_k - By_k), Ap_k - Ax_k \rangle \\
 &\quad -\langle J_{\mathbb{E}_3}(Ax_k - By_k), By_k - Bt_k \rangle \\
 &= -\|Ax_k - By_k\|^2 - \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x_k \rangle \\
 &\quad - \langle B^* J_{\mathbb{E}_3}(Ax_k - By_k), y_k - t_k \rangle \\
 &\leq -\|Ax_k - By_k\|^2 + \|p_k - x_k\| \|A^* J_{\mathbb{E}_3}(Ax_k - By_k)\| \\
 &\quad + \|y_k - t_k\| \|B^* J_{\mathbb{E}_3}(Ax_k - By_k)\|. \tag{3.1.28}
 \end{aligned}$$

Using the Lipschitz continuity of $(\nabla g)^{-1}$, we get

$$\begin{aligned}
 \|p_k - x_k\| &= \|(\nabla g)^{-1}[\nabla g(x_k) - \gamma_k A^* J_{\mathbb{E}_3}(Ax_k - By_k)] - (\nabla g)^{-1}(\nabla g(x_k))\| \\
 &\leq \frac{\gamma_k}{\beta} \|A^* J_{\mathbb{E}_3}(Ax_k - By_k)\|. \tag{3.1.29}
 \end{aligned}$$

Similarly, the Lipschitz continuity of $(\nabla f)^{-1}$ yields

$$\|t_k - y_k\| \leq \frac{\gamma_k}{\beta} \|B^* J_{\mathbb{E}_3}(By_k - Ax_k)\|. \tag{3.1.30}$$

Combining (3.1.28), (3.1.29), (3.1.30) and making use of (3.1.3), we obtain

$$\begin{aligned}
 -\gamma_k \langle J_{\mathbb{E}_3}(Ax_k - By_k), Ap_k - Bt_k \rangle &\leq -\gamma_k \|Ax_k - By_k\|^2 + \frac{\gamma_k^2}{\beta} \|A^* J_{\mathbb{E}_3}(Ax_k - By_k)\|^2 \\
 &\quad + \frac{\gamma_k^2}{\beta} \|B^* J_{\mathbb{E}_3}(By_k - Ax_k)\|^2 \\
 &\leq -\frac{\rho}{2} \|Ax_k - By_k\|^2 - \frac{\gamma_k}{2} \|Ax_k - By_k\|^2 \\
 &\quad + \frac{\gamma_k}{2} \left(\frac{2\gamma_k}{\beta} [\|A^* J_{\mathbb{E}_3}(Ax_k - By_k)\|^2] \right) \\
 &\quad + \frac{\gamma_k}{2} \left(\frac{2\gamma_k}{\beta} [\|B^* J_{\mathbb{E}_3}(By_k - Ax_k)\|^2] \right) \\
 &\leq -\frac{\rho}{2} \|Ax_k - By_k\|^2. \tag{3.1.31}
 \end{aligned}$$

As such, from (3.1.27) and (3.1.31), we obtain

$$\begin{aligned}
 D_g(x_{k+1}, a_k) + D_f(y_{k+1}, b_k) &\leq D_g(x^*, x_k) - D_g(x^*, x_{k+1}) - D_f(y^*, y_{k+1}) \\
 &\quad + D_f(y^*, y_k) - \frac{\rho}{2} \|Ax_k - By_k\|^2. \tag{3.1.32}
 \end{aligned}$$

Hence,

$$D_g(x^*, x_{k+1}) + D_f(y^*, y_{k+1}) \leq D_g(x^*, x_k) + D_f(y^*, y_k) - \frac{\rho}{2} \|Ax_k - By_k\|^2. \tag{3.1.33}$$

We get that $\{D_g(x^*, x_k) + D_f(y^*, y_k)\}$ is decreasing and bounded below. Thus, $\{D_g(x^*, x_k)\}$ and $\{D_f(y^*, y_k)\}$ are bounded which implies that $\{x_k\}$ and $\{y_k\}$ are bounded. $\square$

Lemma 3.1.6. *Assume that conditions $(H_1) - (H_5)$ hold. Then, the sequences $\{x_k\}$, $\{a_k\}$, $\{y_k\}$ and $\{b_k\}$ generated by Algorithm 3.1 satisfy the following relations.*

1. $\lim_{k \rightarrow \infty} D_g(x_{k+1}, a_k) = 0$ and $\lim_{k \rightarrow \infty} D_f(y_{k+1}, b_k) = 0$.
2. $D_g(x_{k+1}, a_k) \geq \frac{\alpha c_k^2}{L_1^2} (1 - \sigma)^2 D_g(\Pi_C^g(z_k), a_k)$ and
 $D_f(y_{k+1}, b_k) \geq \frac{\alpha t_k^2}{L_2^2} (1 - \sigma)^2 D_f(\Pi_D^f(u_k), b_k)$.

Proof. 1) From (3.1.32), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} D_g(x_{k+1}, a_k) &\leq \sum_{k=0}^{\infty} D_g(x_{k+1}, a_k) + D_f(y_{k+1}, b_k) \\ &\leq \sum_{k=0}^{\infty} (D_g(x^*, x_k) - D_g(x^*, x_{k+1}) - D_f(y^*, y_{k+1}) + D_f(y^*, y_k)) \\ &\leq D_g(x^*, x_0) + D_f(y^*, y_0), \end{aligned} \quad (3.1.34)$$

which implies that

$$\lim_{k \rightarrow \infty} D_g(x_{k+1}, a_k) = 0. \quad (3.1.35)$$

Again with a similar approach, we have

$$\lim_{k \rightarrow \infty} D_f(y_{k+1}, b_k) = 0.$$

2) From Lemma 3.1.5, (3.1.24) and (3.1.35), we have that $\{a_k\}$ is bounded. Similarly, $\{b_k\}$ is bounded. Using (H_5) and the fact that T is uniformly continuous on bounded subsets of $\mathbb{E}_1$, we get that $\{z_k\}$ is bounded. Also by Lemma 2.1.6, we get that $\{\Pi_C^g(z_k)\}$ is bounded. Thus, we get that $\{d_k\}$ is bounded and hence $\{T(d_k)\}$ is bounded. Therefore, there exists $L_1 > 0$ such that

$$\|T(d_k)\| \leq L_1, \forall k \geq 1.$$

Since $h_k(v) = \langle T(d_k), v - d_k \rangle$, we get

$$\begin{aligned} |h_k(a_k) - h_k(x_{k+1})| &= |\langle T(d_k), a_k - x_{k+1} \rangle| \\ &\leq \|T(d_k)\| \|a_k - x_{k+1}\| \\ &\leq L_1 \|a_k - x_{k+1}\|, \end{aligned}$$

which implies that h_k is Lipschitz continuous on C . Using Lemmas 3.1.4 and 2.1.11, we obtain

$$D_g(x_{k+1}, a_k) \geq \frac{\alpha c_k^2}{L_1^2} (1 - \sigma)^2 D_g(\Pi_C^g(z_k), a_k).$$

Similarly, we get

$$D_f(y_{k+1}, b_k) \geq \frac{\alpha t_k^2}{L_2^2} (1 - \sigma)^2 D_f^2(\Pi_D^f(u_k), b_k). \quad (3.1.36)$$

□

Lemma 3.1.7. *Assume that conditions $(H_1) - (H_5)$ hold. If $\{a_k\}$ and $\{b_k\}$ are sequences generated by Algorithm 3.1, then $VI(C, T)$ and $VI(D, S)$ contain weak accumulation points of $\{a_k\}$ and $\{b_k\}$, respectively.*

Proof: From Lemma 3.1.6 (2), we have

$$\lim_{k \rightarrow \infty} D_g(x_{k+1}, a_k) \geq \lim_{k \rightarrow \infty} \frac{\alpha c_k^2}{L_1^2} (1 - \sigma)^2 D_g^2(\Pi_C^g(z_k), a_k). \quad (3.1.37)$$

Since $\alpha > 0$, inequality (3.1.37) and equation (3.1.35) imply that

$$\lim_{k \rightarrow \infty} c_k D_g(\Pi_C^g(z_k), a_k) = 0. \quad (3.1.38)$$

Let $\{a_{k_i}\}$ be a sub-sequence of $\{a_k\}$ which converges weakly to a^* . If there exists $N > 0$ such that $a_{k_i} = \Pi_C^g z_{k_i}$ for all $i \geq N$, then $Ta_{k_i} = 0$, for all $i \geq N$ and hence by (H_2) , we get that $0 \leq \|Ta^*\| \leq \liminf_{i \rightarrow \infty} \|Ta_{k_i}\| = 0$, which implies that $a^* \in VI(C, T)$. Otherwise, we can take a subsequence of $\{a_{k_i}\}$, with no loss of generality, which could still be denoted $\{a_{k_i}\}$ such that $a_{k_i} \neq \Pi_C^g z_{k_i}$ for all $i \geq 1$. In this case, the equality in (3.1.38) yields

$$\lim_{i \rightarrow \infty} c_{k_i} D_g(\Pi_C^g(z_{k_i}), a_{k_i}) = 0. \quad (3.1.39)$$

Now, we show that $a^* \in VI(T, C)$ under two cases.

Case 1. If $\limsup_{i \rightarrow \infty} c_{k_i} > 0$, then there exists a sub-sequence, with no loss of generality, still denoted by $\{a_{k_i}\}$, and constant $\theta > 0$, such that $c_{k_i} > \theta$, for $i \geq N$ and for some $N > 0$. From (3.1.39), we get

$$\lim_{i \rightarrow \infty} D_g(\Pi_C^g(z_{k_i}), a_{k_i}) = 0. \quad (3.1.40)$$

So by Lemma 2.1.3, we have

$$\lim_{i \rightarrow \infty} \|\Pi_C^g(z_{k_i}) - a_{k_i}\| = 0. \quad (3.1.41)$$

On the other hand, (2.1.6) implies that

$$\langle \nabla g(z_{k_i}) - \nabla g(\Pi_C^g(z_{k_i})), y - \Pi_C^g(z_{k_i}) \rangle \leq 0, \forall y \in C. \quad (3.1.42)$$

Now, from the definition of z_{k_i} and (3.1.42), we obtain

$$\langle \nabla g(a_{k_i}) - \nabla g(\Pi_C^g(z_{k_i})), y - \Pi_C^g(z_{k_i}) \rangle \leq \beta_k \langle T(a_{k_i}), y - \Pi_C^g(z_{k_i}) \rangle. \quad (3.1.43)$$

Consequently, we get

$$\langle \nabla g(a_{k_i}) - \nabla g(\Pi_C^g(z_{k_i})), y - \Pi_C^g(z_{k_i}) \rangle - \beta_k \langle T(a_{k_i}), a_{k_i} - \Pi_C^g(z_{k_i}) \rangle \leq \beta_k \langle T(a_{k_i}), y - a_{k_i} \rangle. \quad (3.1.44)$$

From the uniform continuity of ∇g , the fact that $\beta_k < 1$ for all $k \geq 1$, (3.1.41), boundedness of $\{a_{k_i}\}$ and $\{\Pi_C^g(z_{k_i})\}$, (see Lemma 3.1.6 (2)), and letting $i \rightarrow \infty$ in (3.1.44), we get that

$$\liminf_{i \rightarrow \infty} \langle T(a_{k_i}), y - a_{k_i} \rangle \geq 0. \quad (3.1.45)$$

Hence, for any $\epsilon > 0$, $\exists N > 0$ such that for $i \geq N$, we have

$$\langle T(a_{k_i}), y - a_{k_i} \rangle + \epsilon \geq 0. \quad (3.1.46)$$

Since $a_{k_i} \neq \Pi_C^g(z_{k_i})$, we have $T(a_{k_i}) \neq 0$. Take $v_{k_i} \in \mathbb{E}_1$ such that $\langle T(a_{k_i}), v_{k_i} \rangle = 1$, then

$$\langle T(a_{k_i}), y + \epsilon v_{k_i} - a_{k_i} \rangle \geq 0, \quad \forall i \geq N. \quad (3.1.47)$$

So, by Lemma 2.1.10, either

$$\langle T(y + \epsilon v_{k_i}), y + \epsilon v_{k_i} - a_{k_i} \rangle \geq 0, \quad \forall i \geq N \quad (3.1.48)$$

or

$$\langle T(a_{k_i}), z - a_{k_i} \rangle \geq 0, \quad \forall z \in C, \quad \forall i \geq N. \quad (3.1.49)$$

However, the inequality in (3.1.49) suggests that $a_{k_i} = \Pi_C^g(z_{k_i})$, which contradicts the assumption that $a_{k_i} \neq \Pi_C^g(z_{k_i})$. So, we get (3.1.48) that can be expressed as

$$\langle T(y), y - a_{k_i} \rangle \geq \langle T(y) - T(y + \epsilon v_{k_i}), y + \epsilon v_{k_i} - a_{k_i} \rangle - \epsilon \langle T(y), v_{k_i} \rangle, \quad \forall i \geq N. \quad (3.1.50)$$

Taking $\epsilon \rightarrow 0$ and using the continuity of T and boundedness of $\{a_{k_i}\}$, we get

$$\langle T(y), y - a_{k_i} \rangle \geq 0, \quad \forall y \in C. \quad (3.1.51)$$

Hence, letting $i \rightarrow \infty$, we get

$$\langle T(y), y - a^* \rangle \geq 0, \quad \forall y \in C. \quad (3.1.52)$$

Therefore, $a^* \in DVI(C, T)$ and hence $a^* \in VI(C, T)$.

Case 2. If $\limsup_{i \rightarrow \infty} c_{k_i} = 0$, then we get $\lim_{i \rightarrow \infty} c_{k_i} = 0$. Now, we show that

$$\lim_{i \rightarrow \infty} D_g(\Pi_C^g(z_{k_i}), a_{k_i}) = 0. \quad (3.1.53)$$

Take

$$\bar{d}_{k_i} = \frac{c_{k_i}}{\gamma} \Pi_C^g(z_{k_i}) + \left(1 - \frac{c_{k_i}}{\gamma}\right) a_{k_i}$$

or

$$\bar{d}_{k_i} - a_{k_i} = \frac{c_{k_i}}{\gamma} (\Pi_C^g(z_{k_i}) - a_{k_i}).$$

From the boundedness of $\{\Pi_C^g(z_{k_i}) - a_{k_i}\}$ and the fact that $\lim_{i \rightarrow \infty} c_{k_i} = 0$, we get

$$\lim_{i \rightarrow \infty} \|\bar{d}_{k_i} - a_{k_i}\| = 0. \quad (3.1.54)$$

Hence, from the definition of c_{k_i} and Step 3 of the Algorithm 3.1, we obtain

$$\langle T(a_{k_i}) - T(\bar{d}_{k_i}), a_{k_i} - \Pi_C^g(z_{k_i}) \rangle > \sigma D_g(\Pi_C^g(z_{k_i}), a_{k_i}). \quad (3.1.55)$$

Applying the fact that T is uniformly continuous on bounded subsets of $\mathbb{E}_1$, $\sigma > 0$ and boundedness of $\{\Pi_C^g(z_{k_i})\}$ and $\{a_{k_i}\}$, we have

$$\lim_{i \rightarrow \infty} D_g(\Pi_C^g(z_{k_i}), a_{k_i}) = 0. \quad (3.1.56)$$

Then following the argument in Case 1, we arrive at the desired conclusion.

Remark 3.1.8. Note that, if in Lemma 3.1.7, we assume that $T(x) \neq 0$ for all $x \in C$ and $S(y) \neq 0$ for all $y \in D$, then the method of proof of Lemma 3.1.7 provides that $DVI(C, T)$ and $DVI(D, S)$ contain the weak accumulation points of $\{a_k\}$ and $\{b_k\}$, respectively.

In the sequel, we shall use the following lemma.

Lemma 3.1.9. Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be real Banach spaces and $g : \mathbb{E}_1 \rightarrow (-\infty, \infty]$, $f : \mathbb{E}_2 \rightarrow (-\infty, \infty]$ be proper and strictly convex Gâteaux differentiable functions such that ∇f and ∇g are weakly sequentially continuous mappings. Suppose the sequence $\{(x_k, y_k)\}$ in $\mathbb{E}_1 \times \mathbb{E}_2$ converges weakly to $(\bar{x}, \bar{y})$. Then

$$\limsup_{k \rightarrow \infty} [D_g(\bar{x}, x_k) + D_f(\bar{y}, y_k)] < \limsup_{k \rightarrow \infty} [D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)] \quad (3.1.57)$$

for any point $(\hat{x}, \hat{y}) \in \mathbb{E}_1 \times \mathbb{E}_2$ such that $(\hat{x}, \hat{y}) \neq (\bar{x}, \bar{y})$.

Proof. Using the Bregman distance definition, we get that

$$\begin{aligned}
 & D_g(\bar{x}, x_k) + D_f(\bar{y}, y_k) - [D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)] \\
 &= g(\bar{x}) - g(x_k) - \langle \bar{x} - x_k, \nabla g(x_k) \rangle + f(\bar{y}) - f(y_k) - \langle \bar{y} - y_k, \nabla f(y_k) \rangle \\
 &\quad - [g(\hat{x}) - g(x_k) - \langle \hat{x} - x_k, \nabla g(x_k) \rangle + f(\hat{y}) - f(y_k) - \langle \hat{y} - y_k, \nabla f(y_k) \rangle] \\
 &= g(\bar{x}) - g(\hat{x}) - \langle \bar{x} - \hat{x}, \nabla g(x_k) \rangle + f(\bar{y}) - f(\hat{y}) - \langle \bar{y} - \hat{y}, \nabla f(y_k) \rangle \\
 &= -[g(\hat{x}) - g(\bar{x}) - \langle \hat{x} - \bar{x}, \nabla g(x_k) \rangle + f(\hat{y}) - f(\bar{y}) - \langle \hat{y} - \bar{y}, \nabla f(y_k) \rangle] \\
 &= -[g(\hat{x}) - g(\bar{x}) - \langle \hat{x} - \bar{x}, \nabla g(x_k) + \nabla g(\bar{x}) - \nabla g(\bar{x}) \rangle + f(\hat{y}) - f(\bar{y}) \\
 &\quad - \langle \hat{y} - \bar{y}, \nabla f(y_k) - \nabla f(\bar{y}) + \nabla f(\bar{y}) \rangle] \\
 &= -[g(\hat{x}) - g(\bar{x}) - \langle \hat{x} - \bar{x}, \nabla g(\bar{x}) \rangle - \langle \hat{x} - \bar{x}, \nabla g(x_k) - \nabla g(\bar{x}) \rangle + f(\hat{y}) - f(\bar{y}) - \langle \hat{y} - \bar{y}, \nabla f(\bar{y}) \rangle] \\
 &\quad - \langle \hat{y} - \bar{y}, \nabla f(y_k) - \nabla f(\bar{y}) \rangle] \\
 &= -D_g(\hat{x}, \bar{x}) + \langle \hat{x} - \bar{x}, \nabla g(x_k) - \nabla g(\bar{x}) \rangle - D_f(\hat{y}, \bar{y}) + \langle \hat{y} - \bar{y}, \nabla f(y_k) - \nabla f(\bar{y}) \rangle. \tag{3.1.58}
 \end{aligned}$$

Now, using the fact that ∇f and ∇g are weakly sequentially continuous we get that

$$\limsup_{k \rightarrow \infty} [D_g(\bar{x}, x_k) + D_f(\bar{y}, y_k) - [D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)]] = -D_g(\hat{x}, \bar{x}) - D_f(\hat{y}, \bar{y}).$$

Consequently,

$$\begin{aligned}
 \limsup_{k \rightarrow \infty} [D_g(\bar{x}, x_k) + D_f(\bar{y}, y_k)] &\leq \limsup_{k \rightarrow \infty} [D_g(\bar{x}, x_k) + D_f(\bar{y}, y_k) - D_g(\hat{x}, x_k) - D_f(\hat{y}, y_k)] \\
 &\quad + \limsup_{k \rightarrow \infty} [D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)] \\
 &= -D_g(\hat{x}, \bar{x}) - D_f(\hat{y}, \bar{y}) + \limsup_{k \rightarrow \infty} [D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)] \\
 &< \limsup_{k \rightarrow \infty} [D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)]. \tag{3.1.59}
 \end{aligned}$$

Therefore, the conclusion of the lemma holds. $\square$

Theorem 3.1.10. Assume that conditions $(H_1) - (H_5)$ hold. In addition, let ∇g and ∇f be weakly sequentially continuous mappings. The sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 converges weakly to a point in Ω^* , where $\Omega^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : Ax^* = By^*\}$.

Proof. Let $(x^*, y^*) \in \Omega^*$. From Lemma 3.1.5, we obtain that

$$\Omega_{k+1}^* \leq \Omega_k^* - \frac{\rho}{2} \|Ax_k - By_k\|^2, \tag{3.1.60}$$

where $\Omega_k^* = D_g(x^*, x_k) + D_f(y^*, y_k)$. Hence, $\{\Omega_k^*\}$ is a decreasing and hence convergent sequence. Thus (3.1.60) implies $\|Ax_k - By_k\| \rightarrow 0$ as $k \rightarrow \infty$. Since $\{(x_k, y_k)\}$ is bounded by Lemma 3.1.5, a sub-sequence $\{(x_{k_i}, y_{k_i})\}$ of $\{(x_k, y_k)\}$ such that $(x_{k_i}, y_{k_i}) \rightharpoonup (\hat{x}, \hat{y})$ exists. This implies that $x_{k_i} \rightharpoonup \hat{x}$ and $y_{k_i} \rightharpoonup \hat{y}$. By Lemmas 3.1.6 and 2.1.8, we get $a_{k_i-1} \rightharpoonup \hat{x}$ and $b_{k_i-1} \rightharpoonup \hat{y}$ as $i \rightarrow \infty$. Thus, by Lemma 3.1.7, we get that $\hat{x} \in VI(C, T)$ and $\hat{y} \in VI(D, S)$. Moreover, from (3.1.60) and the fact that A and B are bounded linear mappings and $\{\Omega_k\}$ is convergent, we have $A\hat{x} = B\hat{y}$. Hence, $(\hat{x}, \hat{y}) \in \Omega^*$.

Now we prove that $(x_k, y_k) \rightharpoonup (\hat{x}, \hat{y})$ as $k \rightarrow \infty$.

Suppose to the contrary that this is not the case. Then there exists a sub-sequence $\{(x_{k_j}, y_{k_j})\}$ of $\{(x_k, y_k)\}$ such that $(x_{k_j}, y_{k_j}) \rightharpoonup (\bar{x}, \bar{y})$ as $j \rightarrow \infty$, where $(\bar{x}, \bar{y}) \neq (\hat{x}, \hat{y})$. This implies that $x_{k_j} \rightharpoonup \bar{x}$ and $y_{k_j} \rightharpoonup \bar{y}$ as $j \rightarrow \infty$. Note that one may also show that $(\bar{x}, \bar{y}) \in \Omega^*$. Now, using the fact that $\{\Omega_k^*\}$ is convergent and the property in Lemma 3.1.9, we get

$$\begin{aligned}
 \lim_{k \rightarrow \infty} [D_g(\hat{x}, x_{k_i}) + D_f(\hat{y}, y_{k_i})] &= \limsup_{i \rightarrow \infty} [D_g(\hat{x}, x_{k_i}) + D_f(\hat{y}, y_{k_i})] \\
 &< \limsup_{i \rightarrow \infty} [D_g(\bar{x}, x_{k_i}) + D_f(\bar{y}, y_{k_i})] \\
 &= \lim_{k \rightarrow \infty} [D_g(\bar{x}, x_k) + D_f(\bar{y}, y_k)] \\
 &= \limsup_{j \rightarrow \infty} [D_g(\bar{x}, x_{k_j}) + D_f(\bar{y}, y_{k_j})] \\
 &< \limsup_{j \rightarrow \infty} [D_g(\hat{x}, x_{k_j}) + D_f(\hat{y}, y_{k_j})] \\
 &= \lim_{k \rightarrow \infty} [D_g(\hat{x}, x_{k_i}) + D_f(\hat{y}, y_{k_i})], \tag{3.1.61}
 \end{aligned}$$

which is a contradiction. This yields that $(\hat{x}, \hat{y}) = (\bar{x}, \bar{y})$. Therefore, the whole sequence $\{(x_k, y_k)\}$ converges weakly to $(\hat{x}, \hat{y}) \in \Omega^*$ as $k \rightarrow \infty$ as desired. $\square$

Next, we propose Halpern-type double projection algorithm method for approximating a solution of variational problems in Banach spaces.

Algorithm 3.2 Choose $x_0 \in \mathbb{E}_1$, $y_0 \in \mathbb{E}_2$ and two parameters: $\gamma, \sigma \in (0, 1)$. Take $k = 0$. For arbitrary points $u \in C$ and $v \in D$, calculate $\{x_k\}$ and $\{y_k\}$ as follows:

Step 1: Compute

$$a_k = \Pi_C^g(\nabla g)^{-1} [\nabla g(x_k) - \gamma_k A^*(J_{\mathbb{E}_3}(Ax_k - By_k))], \tag{3.1.62}$$

$$b_k = \Pi_D^f(\nabla f)^{-1} [\nabla f(y_k) - \gamma_k B^*(J_{\mathbb{E}_3}(By_k - Ax_k))], \tag{3.1.63}$$

where $0 < \rho \leq \gamma_k \leq \rho_k$ with

$$\rho_k = \min \left\{ \rho + 1, \frac{\beta \|Ax_k - By_k\|^2}{2[\|A^*J_{\mathbb{E}_3}(Ax_k - By_k)\|^2 + \|B^*J_{\mathbb{E}_3}(By_k - Ax_k)\|^2]} \right\},$$

for $k \in \Upsilon = \{m \in \mathbb{N} : Ax_m - By_m \neq 0\}$, otherwise $\gamma_k = \rho$.

Step 2: Compute

$$z_k = (\nabla g)^{-1} [\nabla g(a_k) - \beta_k T(a_k)],$$

$$u_k = (\nabla f)^{-1} [\nabla f(b_k) - \beta_k S(b_k)],$$

where $\{\beta_k\} \subset (0, 1)$ for all $k \geq 1$.

Step 3 : Compute

$$m_k = \min \left\{ m \in \mathbb{N} : \langle T(a_k) - T(d_m), a_k - \Pi_C^g(z_k) \rangle \leq \sigma D_g(\Pi_C^g(z_k), a_k) \right\},$$

$$l_k = \min \left\{ l \in \mathbb{N} : \langle S(b_k) - S(e_l), b_k - \Pi_D^f(u_k) \rangle \leq \sigma D_f(\Pi_D^f(u_k), b_k) \right\},$$

where

$$d_m = \gamma^m \Pi_C^g(z_k) + (1 - \gamma^m) a_k,$$

$$e_l = \gamma^l \Pi_D^f(u_k) + (1 - \gamma^l) b_k.$$

Let

$$d_k = \gamma^{m_k} \Pi_C^g(z_k) + (1 - \gamma^{m_k}) a_k,$$

$$e_k = \gamma^{l_k} \Pi_D^f(u_k) + (1 - \gamma^{l_k}) b_k.$$

Step 4 : Compute

$$x_{k+1} = \alpha_k u + (1 - \alpha_k) \Pi_{C \cap H_k}^g(a_k),$$

$$y_{k+1} = \alpha_k v + (1 - \alpha_k) \Pi_{D \cap R_k}^f(b_k),$$

where

$$H_k = \{x \in \mathbb{E}_1 : \langle T(d_k), x - d_k \rangle \leq 0\} \text{ and } R_k = \{y \in \mathbb{E}_2 : \langle S(e_k), y - e_k \rangle \leq 0\}.$$

Step 5 : Let $k = k + 1$ and return to **Step 1**.

Lemma 3.1.11. *Assume that conditions $(H_1) - (H_6)$ hold. Then, the sequences $\{x_k\}$, $\{a_k\}$, $\{y_k\}$ and $\{b_k\}$ generated by Algorithm 3.2 are bounded.*

Proof. Let $(\bar{x}, \bar{y}) \in \Omega$. Then from (3.1.24), (2.1.11) and Lemma 2.1.13, we obtain that

$$\begin{aligned} D_g(\bar{x}, x_{k+1}) &= D_g(\bar{x}, (\alpha_k u + (1 - \alpha_k) \Pi_{C \cap H_k}^g(a_k))) \\ &\leq \alpha_k D_g(\bar{x}, u) + (1 - \alpha_k) D_g(\bar{x}, \Pi_{C \cap H_k}^g(a_k)) \\ &\leq \alpha_k D_g(\bar{x}, u) + (1 - \alpha_k) D_g(\bar{x}, a_k) \\ &\leq \alpha_k D_g(\bar{x}, u) + (1 - \alpha_k) (D_g(\bar{x}, x_k) - \gamma_k \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - \bar{x} \rangle). \end{aligned} \quad (3.1.64)$$

Similarly, we get

$$D_f(\bar{y}, y_{k+1}) \leq \alpha_k D_f(\bar{y}, v) + (1 - \alpha_k) (D_f(\bar{y}, y_k) - \gamma_k \langle B^* J_{\mathbb{E}_3}(By_k - Ax_k), t_k - \bar{y} \rangle). \quad (3.1.65)$$

Denote: $\Omega_k = D_g(\bar{x}, x_k) + D_f(\bar{y}, y_k)$.

Then, from (3.1.64), (3.1.65) and (3.1.31) we obtain that

$$\begin{aligned}
 \Omega_{k+1} &\leq (1 - \alpha_k)\Omega_k + \alpha_k [D_g(\bar{x}, u) + D_f(\bar{y}, v)] - \frac{\rho}{2} \|Ax_k - By_k\|^2 \\
 &\leq (1 - \alpha_k)\Omega_k + \alpha_k [D_g(\bar{x}, u) + D_f(\bar{y}, v)] \\
 &\leq \max \{\Omega_k, D_g(\bar{x}, u) + D_f(\bar{y}, v)\} \\
 &\quad \vdots \\
 &\leq \max \{\Omega_0, D_g(\bar{x}, u) + D_f(\bar{y}, v)\}, \forall k \geq 0.
 \end{aligned} \tag{3.1.66}$$

Thus, the sequence $\{\Omega_k\}$ is bounded and hence the sequences $\{D_g(\bar{x}, x_k)\}$ and $\{D_f(\bar{y}, y_k)\}$ are bounded. Therefore, by Lemma 2.1.8, we have $\{x_k\}$ and $\{y_k\}$ are bounded. $\square$

Theorem 3.1.12. Assume that conditions $(H_1) - (H_6)$ hold. Let $T(x) \neq 0$, for all $x \in C$ and $S(y) \neq 0$, for all $y \in D$. Then, the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2 converges strongly to $(x^*, y^*) = \Pi_{\Omega}^h(u, v)$, where $h(x, y) = (g(x), f(y))$.

Proof. Let $(x^*, y^*) = \Pi_{\Omega}^h(u, v)$. Then, from (2.1.6) we get that

$$\langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (s, t) - (x^*, y^*) \rangle \leq 0 \quad \forall (s, t) \in \Omega. \tag{3.1.67}$$

From (2.1.10), (2.1.11), Lemma 2.1.13 and (3.1.25), we have

$$\begin{aligned}
 D_g(x^*, x_{k+1}) &= D_g(x^*, (\nabla g)^{-1}(\alpha_k \nabla g(u) + (1 - \alpha_k) \nabla g(\Pi_{C \cap H_k}^g(a_k)))) \\
 &= V_g(x^*, (\alpha_k \nabla g(u) + (1 - \alpha_k) \nabla g(\Pi_{C \cap H_k}^g(a_k)))) \\
 &\leq V_g(x^*, (1 - \alpha_k) \nabla g(\Pi_{C \cap H_k}^g(a_k)) + \alpha_k \nabla g(x^*)) + \alpha_k \langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x^* \rangle \\
 &= D_g(x^*, (\nabla g)^{-1}((1 - \alpha_k) \nabla g(\Pi_{C \cap H_k}^g(a_k)) + \alpha_k \nabla g(x^*)) + \alpha_k \nabla g(x^*)) \\
 &\quad + \alpha_k \langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x^* \rangle \\
 &\leq \alpha_k D_g(x^*, x^*) + (1 - \alpha_k) D_g(x^*, \Pi_{C \cap H_k}^g(a_k)) + \alpha_k \langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x^* \rangle \\
 &\leq (1 - \alpha_k) D_g(x^*, \Pi_{C \cap H_k}^g(a_k)) + \alpha_k \langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x^* \rangle
 \end{aligned} \tag{3.1.68}$$

$$\leq (1 - \alpha_k) D_g(x^*, a_k) + \alpha_k \langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x^* \rangle \tag{3.1.69}$$

$$\begin{aligned}
 &\leq (1 - \alpha_k) D_g(x^*, x_k) - (1 - \alpha_k) \gamma_k \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x^* \rangle \\
 &\quad + \alpha_k \langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x^* \rangle.
 \end{aligned} \tag{3.1.70}$$

Similarly,

$$\begin{aligned}
 D_f(y^*, y_{k+1}) &\leq (1 - \alpha_k) D_f(y^*, \Pi_{D \cap R_k}^f(y_k)) + \alpha_k \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y^* \rangle \\
 &\leq (1 - \alpha_k) D_f(y^*, b_k) + \alpha_k \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y^* \rangle \\
 &\leq (1 - \alpha_k) D_f(y^*, y_k) - (1 - \alpha_k) \gamma_k \langle B^* J_{\mathbb{E}_3}(By_k - Ax_k), t_k - y^* \rangle \\
 &\quad + \alpha_k \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y^* \rangle.
 \end{aligned} \tag{3.1.71}$$

Let $\Omega_k = D_g(x^*, x_k) + D_f(y^*, y_k)$, then from (3.1.70), (3.1.71) and (3.1.31), we get

$$\begin{aligned} \Omega_{k+1} &\leq (1 - \alpha_k)\Omega_k - (1 - \alpha_k)\frac{\rho}{2}\|Ax_k - By_k\|^2 \\ &\quad + \alpha_k [\langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x^* \rangle + \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y^* \rangle] \end{aligned} \quad (3.1.72)$$

$$\begin{aligned} &\leq (1 - \alpha_k)\Omega_k + \alpha_k [\langle \nabla g(u) - \nabla g(x^*), x_k - x^* \rangle + \langle \nabla f(v) - \nabla f(y^*), y_k - y^* \rangle] \\ &\quad + \alpha_k [\langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x_k \rangle + \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y_k \rangle] \\ &\leq (1 - \alpha_k)\Omega_k + \alpha_k [\langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_k, y_k) - (x^*, y^*) \rangle] \\ &\quad + \alpha_k [\langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x_k \rangle + \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y_k \rangle]. \end{aligned} \quad (3.1.73)$$

We now consider two cases on the sequence $\{\Omega_k\}$.

Case I. Suppose there exists a natural number N such that $\Omega_{k+1} \leq \Omega_k$ for all $k \geq N$, then we have by the Monotone Convergence Theorem that the sequence $\{\Omega_k\}$ converges. Taking the limit as $k \rightarrow \infty$ in (3.1.72), we obtain

$$\lim_{k \rightarrow \infty} \|Ax_k - By_k\| = 0. \quad (3.1.74)$$

Moreover, since $\{(x_k, y_k)\}$ is bounded in $\mathbb{E}_1 \times \mathbb{E}_2$ which is reflexive, then there exists a subsequence $\{(x_{k_i}, y_{k_i})\}$ of $\{(x_k, y_k)\}$ such that $(x_{k_i}, y_{k_i}) \rightharpoonup (\bar{x}, \bar{y}) \in \mathbb{E}_1 \times \mathbb{E}_2$ and

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_k, y_k) - (x^*, y^*) \rangle \\ &= \lim_{i \rightarrow \infty} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{k_i}, y_{k_i}) - (x^*, y^*) \rangle \\ &= \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (\bar{x}, \bar{y}) - (x^*, y^*) \rangle. \end{aligned} \quad (3.1.75)$$

Consequently, we have $x_{k_i} \rightharpoonup \bar{x}$ and $y_{k_i} \rightharpoonup \bar{y}$. Now, we show that $(\bar{x}, \bar{y}) \in \Omega$.

From the definition of a_k , (2.1.11), (2.1.10), (2.1.11), the property of the Bregman projection and the Cauchy Schwarz inequality, we have

$$\begin{aligned} D_g(x_k, a_k) &\leq D_g(x_k, (\nabla g)^{-1}[\nabla g(x_k) - \gamma_k A^*(J_{\mathbb{E}_3}(Ax_k - By_k))]) \\ &= V_g(x_k, \nabla g(x_k) - \gamma_k A^*(J_{\mathbb{E}_3}(Ax_k - By_k))) \\ &\leq V_g(x_k, \nabla g(x_k)) - \gamma_k \langle A^*(J_{\mathbb{E}_3}(Ax_k - By_k)), p_k - x_k \rangle \\ &= D_g(x_k, x_k) - \gamma_k \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x_k \rangle \\ &\leq \gamma_k \|A^* J_{\mathbb{E}_3}(Ax_k - By_k)\| \cdot \|p_k - x_k\|. \end{aligned} \quad (3.1.76)$$

Substituting (3.1.29) into (3.1.76), taking the limit on both sides and making use of the inequality

in (3.1.74), we obtain

$$\begin{aligned}
 0 &\leq \lim_{k \rightarrow \infty} D_g(x_k, a_k) \\
 &\leq \lim_{k \rightarrow \infty} \left(\frac{\gamma_k^2}{\beta} \|A\|^2 \|J_{\mathbb{E}_3}(Ax_k - By_k)\|^2 \right) \\
 &\leq \lim_{k \rightarrow \infty} \left(\frac{\gamma_k^2}{\beta} \|A\|^2 \|Ax_k - By_k\|^2 \right) \\
 &= 0.
 \end{aligned} \tag{3.1.77}$$

This implies that $\lim_{k \rightarrow \infty} D_g(x_k, a_k) = 0$ and hence by Lemma 2.1.3, we get

$$\lim_{k \rightarrow \infty} \|x_k - a_k\| = 0. \tag{3.1.78}$$

Similarly, we have that

$$\lim_{k \rightarrow \infty} \|y_k - b_k\| = 0. \tag{3.1.79}$$

Hence, we obtain that $a_{k_i} \rightarrow \bar{x}$ and $b_{k_i} \rightarrow \bar{y}$. Therefore, by Remark 3.1.8, we get $\bar{x} \in DVI(C, T)$ and $\bar{y} \in DVI(D, S)$. Moreover, since A and B are bounded linear maps, we obtain that $Ax_{k_i} \rightarrow A\bar{x}$ and $By_{k_i} \rightarrow B\bar{y}$ as $i \rightarrow \infty$ and this with (3.1.74) imply that $A\bar{x} = B\bar{y}$. Therefore, we have $(\bar{x}, \bar{y}) \in \Omega$. Thus, from (3.1.67) and (3.1.75), we obtain that

$$\begin{aligned}
 &\limsup_{k \rightarrow \infty} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_k, y_k) - (x^*, y^*) \rangle \\
 &= \lim_{i \rightarrow \infty} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{k_i}, y_{k_i}) - (x^*, y^*) \rangle \\
 &= \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (\bar{x}, \bar{y}) - (x^*, y^*) \rangle \leq 0.
 \end{aligned} \tag{3.1.80}$$

Let $v_k = \Pi_C^g \cap H_k(a_k)$ and $u_k = \Pi_D^f \cap R_k(b_k)$. Then, from the convergence of Ω_k , (3.1.68), (3.1.83), (3.1.70) and (3.1.71) we get

$$\lim_{k \rightarrow \infty} [D_g(x^*, a_k) + D_f(y^*, b_k)] = \lim_{k \rightarrow \infty} [D_g(x^*, v_k) + D_f(y^*, u_k)] = \lim_{k \rightarrow \infty} [D_g(x^*, x_{k+1}) + D_f(y^*, y_{k+1})]. \tag{3.1.81}$$

Moreover, using (2.1.7) and (3.1.81) we get

$$\begin{aligned}
 \lim_{k \rightarrow \infty} [D_g(v_k, a_k) + D_g(u_k, b_k)] &= \lim_{k \rightarrow \infty} [D_g(x^*, a_k) - D_g(x^*, v_k) + D_f(y^*, b_k) - D_f(y^*, u_k)] \\
 &= \lim_{k \rightarrow \infty} [D_g(x^*, a_k) + D_f(y^*, b_k) - (D_g(x^*, x_{k+1}) + D_g(y^*, y_{k+1}))] \\
 &\quad + \lim_{k \rightarrow \infty} [D_g(x^*, x_{k+1}) + D_g(y^*, y_{k+1}) - (D_g(x^*, v_k) + D_g(y^*, u_k))] \\
 &\rightarrow 0 \text{ as } k \rightarrow \infty,
 \end{aligned} \tag{3.1.82}$$

and hence $D_g(v_k, a_k) \rightarrow 0$ and $D_g(u_k, b_k) \rightarrow 0$ as $k \rightarrow \infty$. This, along with Lemma 2.1.3, provides that $v_k - a_k \rightarrow 0$ and $u_k - b_k \rightarrow 0$ as $k \rightarrow \infty$. Thus, from this and Step 4 of Algorithm

3.2, we obtain

$$x_{k+1} - x_k = \alpha_k(u - x_k) + (1 - \alpha_k)(v_k - x_k) \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (3.1.83)$$

Similarly,

$$y_{k+1} - y_k \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (3.1.84)$$

Now, from (3.1.73), (3.1.80), (3.1.83), (3.1.84) and Lemma 2.1.15, we obtain that $\Omega_k \rightarrow 0$ as $k \rightarrow \infty$, and hence $D_g(x^*, x_k) \rightarrow 0$ and $D_f(y^*, y_k) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, by Lemma 2.1.3, we obtain that $x_k \rightarrow x^*$ and $y_k \rightarrow y^*$ as $k \rightarrow \infty$.

Case II. Suppose there exists a subsequence $\{\Omega_{k_i}\}$ of $\{\Omega_k\}$ with $\Omega_{k_i} < \Omega_{k_{i+1}}$ for all $i \geq 0$. Then, by Lemma 2.1.14, there exists a non-decreasing sequence $\{m_l\}$ of positive integers such that $\lim_{l \rightarrow \infty} m_l = \infty$ and

$$\Omega_{m_l} \leq \Omega_{m_l+1} \text{ and } \Omega_k \leq \Omega_{m_l+1}, \quad (3.1.85)$$

for all positive integers l . We have from (3.1.73) that

$$\begin{aligned} \frac{\rho}{2} \|Ax_{m_l} - By_{m_l}\|^2 &\leq (1 - \alpha_l)\Omega_{m_l} - \Omega_{m_l+1} + \alpha_{m_l} [\langle (\nabla g(u) - \nabla g(x^*)), x_{m_l+1} - x^* \rangle \\ &\quad + \langle (\nabla f(v) - \nabla f(y^*)), y_{m_l+1} - y^* \rangle]. \end{aligned} \quad (3.1.86)$$

Taking the limit as $l \rightarrow \infty$ on both sides of (3.1.86) and taking (3.1.85) and the property of α_{m_l} into account, we obtain that $\lim_{l \rightarrow \infty} \|Ax_{m_l} - By_{m_l}\| = 0$. Moreover, following similar methods used in Case I, we obtain

$$\limsup_{l \rightarrow \infty} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{m_l}, y_{m_l}) - (x^*, y^*) \rangle \leq 0. \quad (3.1.87)$$

Thus, the inequalities in (3.1.73) and (3.1.85) imply that

$$\begin{aligned} \alpha_{m_l}\Omega_{m_l} &\leq \Omega_{m_l} - \Omega_{m_l+1} + \alpha_{m_l} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{m_l}, y_{m_l}) - (x^*, y^*) \rangle \\ &\quad + \alpha_{m_l} [\langle (\nabla g(u) - \nabla g(x^*)), x_{m_l+1} - x_{m_l} \rangle + \langle \nabla f(v) - \nabla f(y^*), y_{m_l+1} - y_{m_l} \rangle]. \end{aligned} \quad (3.1.88)$$

which yields

$$\begin{aligned} \Omega_{m_l} &\leq \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{m_l}, y_{m_l}) - (x^*, y^*) \rangle \\ &\quad + [\langle (\nabla g(u) - \nabla g(x^*)), x_{m_l+1} - x_{m_l} \rangle + \langle \nabla f(v) - \nabla f(y^*), y_{m_l+1} - y_{m_l} \rangle]. \end{aligned} \quad (3.1.89)$$

Taking the limsup as $l \rightarrow \infty$ on both sides of (3.1.89) and using (3.1.87), (3.1.83) and (3.1.84), we obtain that $\lim_{l \rightarrow \infty} \Omega_{m_l} = 0$. This together with (3.1.72) imply that $\lim_{k \rightarrow \infty} \Omega_{m_l+1} = 0$, which implies by (3.1.85) that $\lim_{k \rightarrow \infty} \Omega_k = 0$. As a consequence, we get $\lim_{k \rightarrow \infty} D_f(x^*, x_k) = \lim_{k \rightarrow \infty} D_g(y^*, y_k) = 0$ and hence, we obtain by Lemma 2.1.3 that $\lim_{k \rightarrow \infty} x_k = x^*$ and $\lim_{k \rightarrow \infty} y_k = y^*$. Therefore, we conclude from Cases I and II that the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Omega^h(\hat{x}, \hat{y})$, and hence the proof is complete. $\square$

We notice that if the mappings T and S in Theorem 3.1.10 and Theorem 3.1.12 are assumed to be Lipschitz continuous and quasimonotone, then we get the following corollaries,

Corollary 3.1.13. *Assume that (H_1) and $(H_3) - (H_5)$ hold. In addition, let ∇f and ∇g be weakly sequentially continuous mappings. If $T : \mathbb{E}_1 \rightarrow \mathbb{E}_1^*$ and $S : \mathbb{E}_2 \rightarrow \mathbb{E}_2^*$ are quasimonotone mappings that are Lipschitz continuous on C and D , respectively, satisfying (3.1.1), then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 converges weakly to $(x^*, y^*) \in \Omega^*$, where $\Omega^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : Ax^* = By^*\}$.*

Corollary 3.1.14. *Assume that (H_1) and $(H_3) - (H_6)$ hold. Let $T : \mathbb{E}_1 \rightarrow \mathbb{E}_1^*$ and $S : \mathbb{E}_2 \rightarrow \mathbb{E}_2^*$ be quasimonotone mappings that are Lipschitz continuous on C and D , respectively, satisfying (3.1.1). Let $T(x) \neq 0$, for all $x \in C$ and $S(y) \neq 0$, for all $y \in D$. The sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2 converges strongly to (x^*, y^*) , where $(x^*, y^*) = \Pi_\Omega^h(u, v)$.*

If we assume that $\mathbb{E}_1$ and $\mathbb{E}_2$ are smooth and 2-uniformly convex Banach spaces and take $g(\cdot) = \frac{1}{2} \|\cdot\|^2$ and $f(\cdot) = \frac{1}{2} \|\cdot\|^2$, then $\nabla g = J_{\mathbb{E}_1}$, $\nabla f = J_{\mathbb{E}_2}$, $(\nabla g)^{-1} = J_{\mathbb{E}_1^*}^{-1}$ and $(\nabla f)^{-1} = J_{\mathbb{E}_2^*}^{-1}$. Thus, we get the following corollaries.

Corollary 3.1.15. *Let C and D be nonempty, closed and convex subsets of the smooth and 2-uniformly convex Banach spaces $\mathbb{E}_1$ and $\mathbb{E}_2$, and assume that conditions $(H_2) - (H_4)$ are satisfied. In addition, let $\nabla g = J_{\mathbb{E}_1}$, $\nabla f = J_{\mathbb{E}_2}$, $(\nabla g)^{-1} = J_{\mathbb{E}_1^*}^{-1}$ and $(\nabla f)^{-1} = J_{\mathbb{E}_2^*}^{-1}$ be weakly sequentially continuous mappings. Then, the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 converges weakly to $(x^*, y^*) \in \Omega^*$, where $\Omega^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : Ax^* = By^*\}$.*

Corollary 3.1.16. *Let C and D be nonempty, closed and convex subsets of the smooth and 2-uniformly convex Banach spaces $\mathbb{E}_1$ and $\mathbb{E}_2$, and assume that the conditions $(H_2) - (H_4)$ and (H_6) are satisfied. Let $T(x) \neq 0$, for all $x \in C$ and $S(y) \neq 0$, for all $y \in D$. Then, the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2 with $\nabla g = J_{\mathbb{E}_1}$, $\nabla f = J_{\mathbb{E}_2}$, $(\nabla g)^{-1} = J_{\mathbb{E}_1^*}^{-1}$ and $(\nabla f)^{-1} = J_{\mathbb{E}_2^*}^{-1}$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = \Pi_\Omega^h(u, v)$.*

If we assume, in Theorem 3.1.10 and Theorem 3.1.12 that $\mathbb{E}_1$, $\mathbb{E}_2$ and $\mathbb{E}_3$ are real Hilbert spaces, $g(\cdot) = \frac{1}{2} \|\cdot\|^2$ and $f(\cdot) = \frac{1}{2} \|\cdot\|^2$, then $\nabla g = I_{\mathbb{E}_1}$, $\nabla f = I_{\mathbb{E}_2}$, $(\nabla g)^{-1} = I_{\mathbb{E}_1^*}^{-1}$ and $(\nabla f)^{-1} = I_{\mathbb{E}_2^*}^{-1}$ and hence we obtain the following corollaries.

Corollary 3.1.17. *Let $\mathbb{E}_1, \mathbb{E}_2$ and $\mathbb{E}_3$ be real Hilbert spaces and let C and D be nonempty, closed and convex subsets of $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. Assume that conditions $(H_2) - (H_4)$ are satisfied. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 with $\nabla g = I_{\mathbb{E}_1}$, $\nabla f = I_{\mathbb{E}_2}$, $(\nabla g)^{-1} = I_{\mathbb{E}_1^*}^{-1}$ and $(\nabla f)^{-1} = I_{\mathbb{E}_2^*}^{-1}$, converges weakly to $(x^*, y^*) \in \Omega^*$, where $\Omega^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : Ax^* = By^*\}$.*

Corollary 3.1.18. *Let $\mathbb{E}_1, \mathbb{E}_2$ and $\mathbb{E}_3$ be real Hilbert spaces and let C and D be nonempty, closed and convex subsets of $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. Assume that conditions $(H_2) - (H_4)$ and*

(H_6) are satisfied. Let $T(x) \neq 0$, for all $x \in C$ and $S(y) \neq 0$, for all $y \in D$. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2 with $\nabla g = I_{\mathbb{E}_1}, \nabla f = I_{\mathbb{E}_2}, (\nabla g)^{-1} = I_{\mathbb{E}_1}^{-1}$ and $(\nabla f)^{-1} = I_{\mathbb{E}_2}^{-1}$, converges strongly to $(x^*, y^*) = P_{\Omega}(u, v)$.



4. Applications

In this section, we apply our main result to solve some specific problems.

4.0.1 Split Variational Inequality Problems (SVIP)

Let C and D be nonempty, closed and convex subsets of real Banach spaces $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. Let $T : \mathbb{E}_1 \rightarrow \mathbb{E}_1^*$ and $S : \mathbb{E}_2 \rightarrow \mathbb{E}_2^*$ be two non-linear mappings. Let $A : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ be a bounded linear mapping with A^* as its adjoint. The SVIP (see, e.g., Censor et al. [12]) is to:

$$\text{find } x^* \in VI(C, T), y^* \in VI(D, S) \text{ such that } y^* = Ax^*, \quad (4.0.1)$$

where as SDVIP is to:

$$\text{find } x^* \in DVI(C, T), y^* \in DVI(D, S) \text{ such that } y^* = Ax^*. \quad (4.0.2)$$

Censor et al. [12] introduced the SVIP in 2012. It is widely known to have a number applications such as in phase retrieval, image and signal processing, data compression, among others (see, [28, 52, 56] and the references therein). Denote

$$\begin{aligned} \Gamma &= \{(x^*, y^*) \in DVI(C, T) \times DVI(D, S) : Ax^* = y^*\} \text{ and} \\ \Gamma^* &= \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : Ax^* = y^*\} \end{aligned}$$

Then, we have the following corollaries.

Corollary 4.0.1. *Assume that conditions $(H_1) - (H_3)$ and (H_5) , with $\mathbb{E}_2 = \mathbb{E}_3$ and $B = I_{\mathbb{E}_2}$ hold. Let ∇f and ∇g be weakly sequentially continuous mappings. If $\Gamma \neq \emptyset$, then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 converges weakly to $(x^*, y^*) \in \Gamma^*$.*

Corollary 4.0.2. *Assume that conditions $(H_1) - (H_3)$ and $(H_5) - (H_6)$ with $\mathbb{E}_2 = \mathbb{E}_3$ and $B = I_{\mathbb{E}_2}$ hold. If $\Gamma \neq \emptyset$, then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2, with $B = I_{\mathbb{E}_2}$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = \Pi_{\Gamma}^h(u, v)$.*

4.0.2 Common Solutions of Variational Inequality Problems

Let C and D be nonempty, closed and convex subsets of a real Banach space $\mathbb{E}$ with its dual $\mathbb{E}^*$. Let $T, S : \mathbb{E} \rightarrow \mathbb{E}^*$ be two nonlinear mappings. The Common Solutions of Variational Inequality Problem is to:

$$\text{find a point } x^* \in C \text{ such that } x^* \in VI(C, T) \cap VI(C, S), \quad (4.0.3)$$

where as the Common Solutions of Dual Variational Inequality Problem is to:

$$\text{find a point } x^* \in C \text{ such that } x^* \in DVI(C, T) \cap DVI(C, S), \quad (4.0.4)$$

Denote $\Psi = \{(x^*, y^*) \in DVI(C, T) \times DVI(D, S) : x^* = y^*\}$ and $\Psi^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : x^* = y^*\}$. Now, we have the following corollaries.

Corollary 4.0.3. *Assume that conditions $(H_1) - (H_3)$ and (H_5) , with $\mathbb{E}_1 = \mathbb{E}_2 = \mathbb{E}_3 = \mathbb{E}$ hold. Let ∇f and ∇g be weakly sequentially continuous mappings. If $\Psi \neq \emptyset$, then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1, with $A = B = I_{\mathbb{E}}$, converges weakly to $(x^*, x^*) \in \Psi^*$.*

Corollary 4.0.4. *Assume that conditions $(H_1) - (H_3)$ and $(H_5) - (H_6)$ with $\mathbb{E}_1 = \mathbb{E}_2 = \mathbb{E}_3 = \mathbb{E}$ hold. If $\Psi \neq \emptyset$, then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2, with $A = B = I_{\mathbb{E}}$, converges strongly to (x^*, x^*) , where $(x^*, x^*) = \Pi_{\Psi}^h(u, v)$.*



5. Numerical Example

In this section, we give a numerical example with illustrations to demonstrate the applicability of the algorithms.

Example 5.0.1. Let $\mathbb{E} = \mathbb{E}_1 = \mathbb{E}_2 = \mathbb{E}_3 = \mathbb{R}^2$ with the usual norm. Let

$$C = \{(x, y) \in \mathbb{E} : (x - 2)^2 + y^2 \leq 1\}$$

$$D = \{(x, y) \in \mathbb{E} : x^2 + (y - 2)^2 \leq 1\}.$$

Clearly, C and D are closed and convex subsets of $\mathbb{E}$. Define the mappings $T, S: \mathbb{E} \rightarrow \mathbb{E}$ by $T(x, y) = \left(\frac{x^2}{1+x^2}, 0\right)$ and $S(x, y) = \left(0, \frac{y^2}{1+y^2}\right)$. Clearly, the mappings T and S are uniformly continuous on bounded subsets of C and D , respectively. It remains to show that the mappings T and S are quasimonotone on $\mathbb{E}$. Indeed, suppose that for $(x_1, x_2), (y_1, y_2) \in \mathbb{E}$, we have $\langle T(x_1, x_2), (y_1, y_2) - (x_1, x_2) \rangle > 0$, that is,

$$\left\langle \left(\frac{x_1^2}{1+x_1^2}, 0\right), (y_1, y_2) - (x_1, x_2) \right\rangle = \frac{x_1^2}{1+x_1^2}(y_1 - x_1) > 0.$$

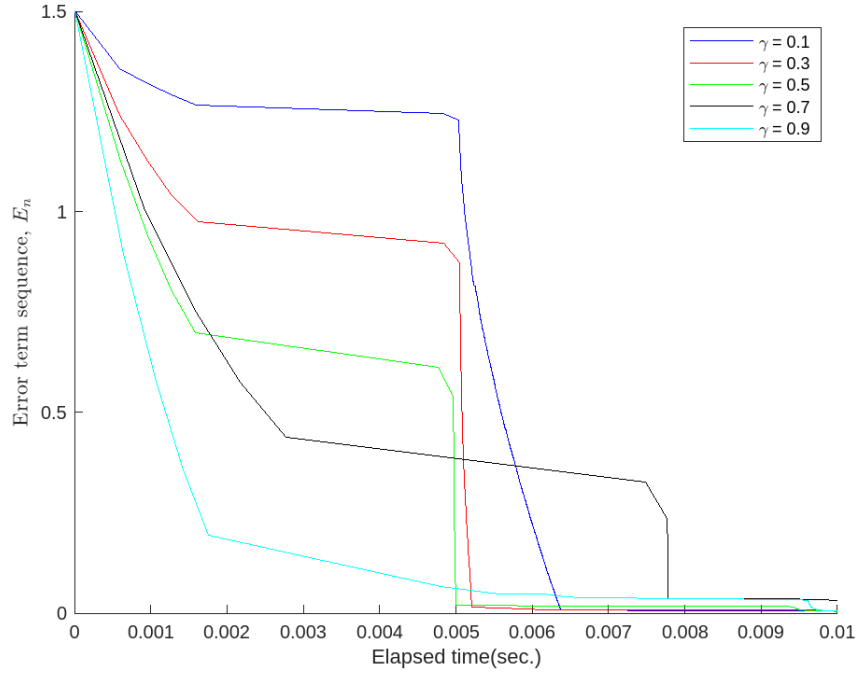
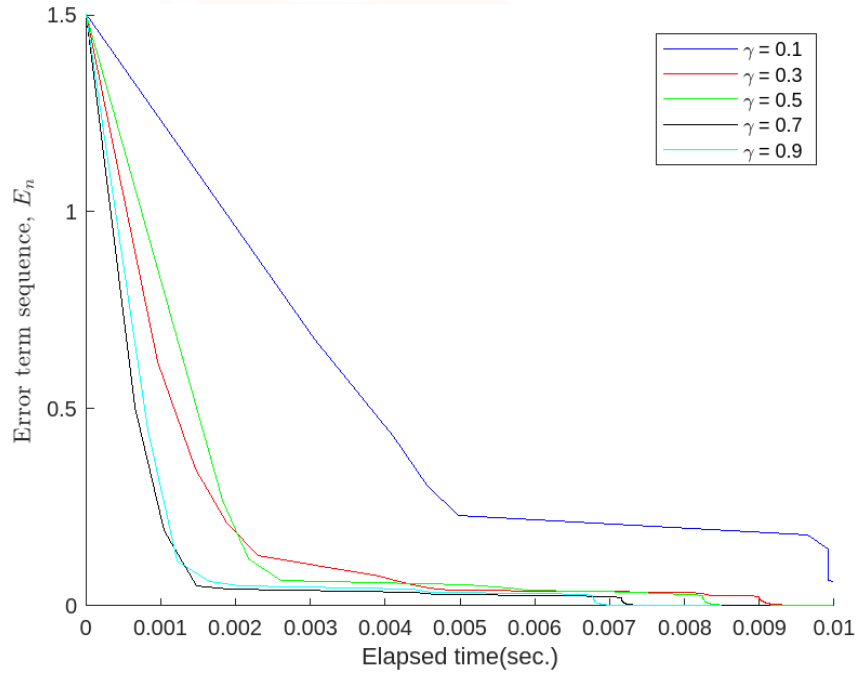
Then, this implies that $y_1 - x_1 > 0$, and hence

$$\langle T(y_1, y_2), (y_1, y_2) - (x_1, x_2) \rangle = \left\langle \left(\frac{y_1^2}{1+y_1^2}, 0\right), (y_1, y_2) - (x_1, x_2) \right\rangle = \frac{y_1^2}{1+y_1^2}(y_1 - x_1) \geq 0.$$

This yields that T is quasimonotone. However, if we take $(x_1, x_2) = (0, 0)$ and $(y_1, y_2) = \left(\frac{-1}{2}, 0\right)$, then $\langle T(x_1, x_2), (y_1, y_2) - (x_1, x_2) \rangle = 0$, but $\langle T(y_1, y_2), (y_1, y_2) - (x_1, x_2) \rangle < 0$, which implies that T is not pseudomonotone. Similarly, one can show that S is uniformly continuous on bounded subsets of D and quasimonotone. Moreover, $DVI(C, T) = VI(C, T) = \{(1, 0)\}$ and $DVI(D, S) = VI(D, S) = \{(0, 1)\}$. Let $A: \mathbb{E} \rightarrow \mathbb{E}$ and $B: \mathbb{E} \rightarrow \mathbb{E}$ be defined by $A(x_1, x_2) = \left(0, \frac{x_2}{2}\right)$ and $B(x_1, x_2) = (3x_1, 0)$. Clearly, A and B are bounded linear mappings with adjoints $A^*(x_1, x_2) = \left(0, \frac{x_2}{2}\right)$ and $B^*(x_1, x_2) = (3x_1, 0)$, respectively. In addition, we have $A(1, 0) = (0, 0) = B(0, 1)$. Therefore, we have that

$$(x^*, y^*) = ((1, 0), (0, 1)) \in \Omega = \{(x, y) \in VI(C, T) \times VI(D, S) : Ax = By\}.$$

Define $g: \mathbb{E} \rightarrow (-\infty, +\infty]$ and $f: \mathbb{E} \rightarrow (-\infty, +\infty]$ by $g(x) = \frac{1}{2}\|x\|^2$ and $f(y) = \frac{1}{2}\|y\|^2$. Then, $\nabla g(x) = (\nabla g)^{-1}x = I_{\mathbb{E}}x = x$ and $\nabla f(y) = (\nabla f)^{-1}y = I_{\mathbb{E}}y = y$. Take $\alpha_n = \frac{1}{n+1}$, $\sigma = 0.9$, $\beta_n = 0.01 + \frac{1}{n+1}$. Then the conditions $(H_1) - (H_6)$ are satisfied. We have conducted the numerical experiments to demonstrate that the error term sequence $E_n = \{|| (x_n, y_n) - (x^*, y^*) ||\}$, $n \geq 1$, of Algorithm 3.1 and 3.2 converges to zero for different parameters.


 Figure 5.1: Rate of convergence of **Algorithm 3.1** for different choices γ .

 Figure 5.2: Convergence of **Algorithm 3.2** for different choices of γ .

In Figure 5.1 and Figure 5.2, the convergence of the error term sequence $E_n = \{||(x_n, y_n) - (x^*, y^*)||\}$ with the aforementioned parameters is indicated for various values of γ and initial value $(x_0, y_0) = ((2, 0), (0, 2))$ for Algorithm 3.1 and 3.2, respectively.

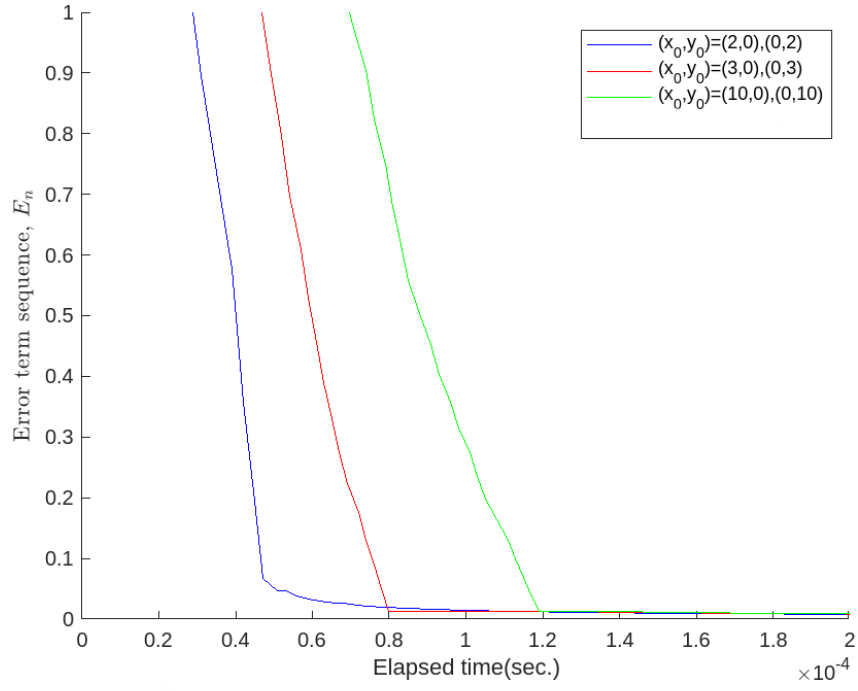


Figure 5.3: Convergence of **Algorithm 3.1** for $\gamma = 0.9$ and varying initial points.

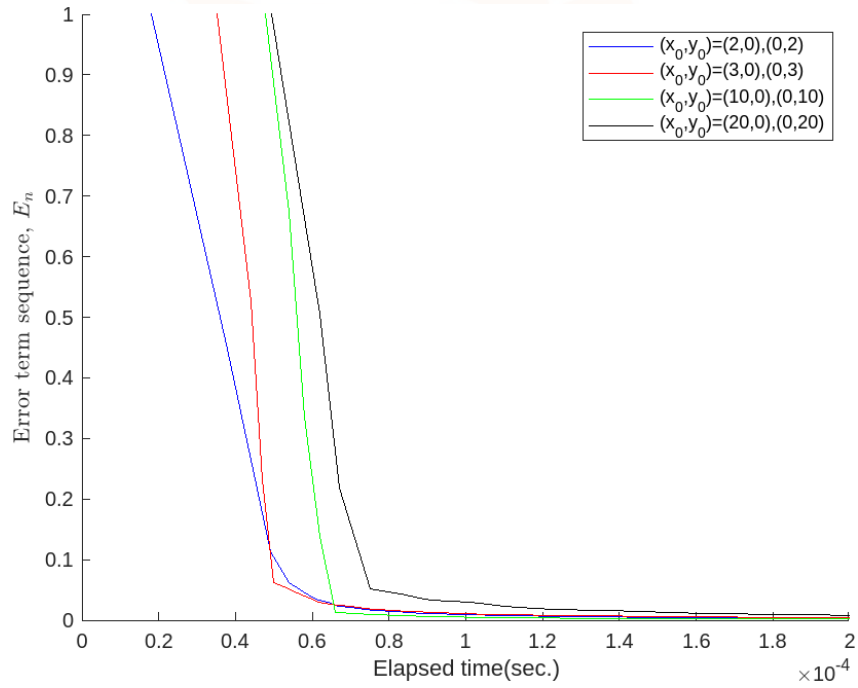


Figure 5.4: Convergence of **Algorithm 3.2** for $\gamma = 0.9$ and varying initial points.

It is clear that Figure 5.3 and Figure 5.4 illustrate how $E_n = \{||(x_n, y_n) - (x^*, y^*)||\}$ converges for the inertial parameter $\gamma = 0.9$ and different initial values $(x_0, y_0) = ((2, 0), (0, 2))$, $(x_0, y_0) = ((3, 0), (0, 3))$, $(x_0, y_0) = ((10, 0), (0, 10))$ and $(x_0, y_0) = ((20, 0), (0, 20))$ for Algorithms 3.1 and 3.2, respectively.

Remark 5.0.2. *The numerical experiment's results, which can be seen in Figure 5.1 and figure 5.2, demonstrate that $E_n = ||(x_n, y_n) - (x^*, y^*)||$ converges to zero; that is, for both algorithms, the sequence $\{(x_n, y_n)\}$ converges strongly to the point $(x^*, y^*) = ((1, 0), (0, 1))$ of Ω .*

As we can see from figure 5.3 and figure 5.4, the rate of convergence for both algorithms eventually becomes the same regardless of the beginning locations we choose.



6. Conclusion and Future Research Problems

6.1 Conclusion

In this thesis, we proposed and studied an iterative method for solving SEVIP in reflexive real Banach spaces. Weak and strong convergence theorems are established for Algorithm 3.1 and Algorithm 3.2, respectively, under the assumption that the underlying mappings are quasi-monotone and uniformly continuous. Our main result has been supplemented with specific applications. In addition, a numerical example has been provided to demonstrate the effectiveness of our method.

This thesis main result generalizes most of the results found in the literature because it deals with a more general split equality problem in the setting of Banach spaces, which is broader than Hilbert spaces. Specifically, it extends the works in [60] from a VIP to a more general SEVIP and from weak convergence to strong convergence results.

6.2 Future Research Problems

The following questions will be at the core of future research work.

- Can the assumption of uniform continuity on T and S be relaxed to a more general class of mappings?
- Can we relax some of the assumptions of the main results and still obtain strong convergence theorems for the solution of SEVIP's?
- Can we alter the algorithms to simpler and more efficient ones in the sense of rate of convergence?

These questions will surely stipulate further research problems in this direction.

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