

TWO STAGE STOCHASTIC BILEVEL PROGRAMMING USING STOCHASTIC SYSTEMATIC EVOLUTIONARY(SSE) METHOD

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Abstract

Stochastic bilevel programming is a bilevel program having some form of randomness in the problem definition. The main objective is to optimize the leader's (upper level) stochastic programming problem, where the follower's problem is assumed to be satisfied as part of the constraints. Evaluation of the solution requires its corresponding lower level optimal reaction for each leader's action. In addition, due to the existence of randomness property, the problem is computationally so expensive and challenging. In this paper, we considered a two stage stochastic bilevel program and we proposed a new algorithm for solving such kind of problems and the algorithm is checked using constructed test problem. This algorithm is a meta-heuristic type algorithm based on systematical partitioning of upper level decision space for searching the optimal reaction for leader's action from its own decision space and applying particle swarm optimization for searching a better follower's reaction. The result of the numerical simulations of the algorithm is very much promising. The algorithm can be used to solve complex stochastic bilevel programming problem.

Key Words: Two stage stochastic programming, bilevel programming, Stackelberg equilibrium, sample average approximation.

1 Introduction

Stochastic bilevel programming, whose underlying principles have been laid out by Patricsson and Wynter[16] in 1999 for addressing a problem on structural programming, is a bilevel program in which randomness property is involved

in the problem definition. The program finds many application, in transportation network[1], energy market modeling[10], traffic model[15], structural mechanics[6, 16], network pricing[17], energy bidding[8], retail market[5], duopoly price fixation[7], telecommunication[2, 21] and multi product pricing[14]. However, due to the existence of randomness property and its hierarchical structure, the problem is computationally so expensive and challenging even for the linear deterministic bilevel program which is the simplest one among the bilevel programming family. Some authors [3, 19] have shown that bilevel linear program is strongly non-deterministic polynomial-time hard problem.

Some of the methods which are used for solving stochastic bilevel programming are stochastic quasi-gradient algorithm[21], and approximation strategies and genetic algorithm[1]. Those proposed methods are limited to some special assumptions like linearity and convexity property for the objective function of follower's problem. The search for solution methods still continues, especially for non-linearity, non-convexity, non-differentiation and other undesirable property involved in the problem. The main objective of this paper is solving two stage stochastic bilevel programming(TS-SMLP) using our proposed algorithm. The algorithm is derivative free meta heuristic type, which is a combination of systematic sampling at each level's decision space and nature inspired particle swarm optimization. Particle swarm optimization was developed by Kennedy and Eberhart[11, 12] in 1995, based on swarm behavior such as schooling of fish and flock of bird in nature and uses real number randomness and global communication among the swarm particles[9, 23].

The rest part of the paper is organized as follows.

Introduction about two stage stochastic bilevel programming is presented in section 2. In section 3, solution procedures is discussed. In addition, numerical examples are presented in section 4. Finally, some remarks on two stage stochastic bilevel programming are made in section 5.

2 Two Stage Stochastic Bilevel Programming

Consider a probability space $(\Omega, \mathcal{F}, P)$, where Ω is a non empty set of all random events, $\mathcal{F}$ the set of all subsets of Ω and P is a probability measure. In addition, consider a finite realizations ξ of $\omega \in \Omega$ and its support function Ξ . A continuous known probability distribution can be represented by a finite set of independent and identically distributed realization samples ξ , which is identified using sample average approximation or sequential bound approximation. The optimal solution and value of the approximated problem is convergent under a suitable condition like convexity[4, 13, 18].

Two stage stochastic programming problem is mathematically expressed as

$$\begin{aligned} \min_x \{ & F(x) + \mathbb{E}[Q(x, \omega)] \} \\ \text{subject to } & G_i(x) \leq 0, i = 1, \dots, p \end{aligned} \quad (1)$$

where

$$Q(x, \omega) = \begin{cases} \inf_{\bar{x}} & \bar{F}(x, \bar{x}, \omega) \\ \text{subject to} & \bar{G}_{\bar{i}}(x, \bar{x}, \omega) \leq 0, \bar{i} = 1, \dots, \bar{p} \end{cases}$$

with x representing first stage decision variable named as “*here and now variable*” and $\bar{x}$ representing the random second stage decision variable named as “*wait and see variable*” but after the first stage decision, the outcomes of the first stage decision could be affected by random events. So, a second stage decision (recourse) need to be made to compensate for possible violation of constraints and its value function $Q(x, \omega)$ is called the recourse, cost of compensation, or second stage value function.

Two stage stochastic bilevel programming is formulated as

$$\begin{aligned} \min_{x_1} & F_1(x_1, x_2) + Q(x_1, x_2) \\ \text{subject to } & G_1(x_1, x_2) \leq 0, \text{ where } x_2 \text{ solve} \\ & \min_{x_2} F_2(x_1, x_2) \\ & \text{subject to } G_2(x_1, x_2) \leq 0 \end{aligned} \quad (2)$$

where $Q(x_1, x_2) = \mathbb{E}_{\xi}[\Phi(x_1, x_2, \xi)], \xi : \Omega \rightarrow R^n$,

$n \in \mathbb{N}$ and, for any outcome $\xi(\omega) \in \Xi$,

$$\Phi(x_1, x_2, \xi(\omega)) = \begin{cases} \min_{\bar{x}_1(\omega)} & \bar{F}_1(\bar{x}_1(\omega), \bar{x}_2(\omega), \omega) \\ \text{subject to } & \bar{G}_1(\bar{x}_1(\omega), \bar{x}_2(\omega), x_1, y_1, \omega) \leq 0, \\ & \text{where } \bar{x}_2 \text{ solve} \\ & \min_{\bar{x}_2(\omega)} & \bar{F}_2(\bar{x}_1(\omega), \bar{x}_2(\omega), \omega) \\ & \text{subject to} \\ & \bar{G}_2(\bar{x}_1(\omega), \bar{x}_2(\omega), x_1, x_2, \omega) \leq 0 \end{cases}$$

with x_1 and x_2 are first stage decision variables which are decided before observing the random outcome at the second-stage, and $\bar{x}_1$ and $\bar{x}_2$ are second stage decision variables which are supposed to be decided after all randomness property has been removed. In problem (2), assuming that $Q(x_1, x_2, \omega)$, $\bar{F}_1(x_1, x_2, \bar{x}_1(\omega), \bar{x}_2(\omega))$, $\bar{F}_2(x_1, x_2, \bar{x}_1(\omega), \bar{x}_2(\omega))$, $\bar{G}_1(x_1, x_2, \bar{x}_1(\omega), \bar{x}_2(\omega))$ and $\bar{G}_2(x_1, x_2, \bar{x}_1(\omega), \bar{x}_2(\omega))$ are measurable functions, then one and two stage stochastic programming problems are equivalent[1]. So, problem (2) can be reformulated as a standard stochastic bilevel program in the following mathematical form

$$\begin{aligned} \min_{x_1, \bar{x}_1} & F_1(x_1, x_2) + \mathbb{E}_{\xi}[\bar{F}_1(\bar{x}_1(\xi), \bar{x}_2(\xi), \xi)] \\ \text{subject to } & G_1(x_1, x_2) \leq 0 \\ & \bar{G}_1(\bar{x}_1(\omega), \bar{x}_2(\omega), x_1, x_2, \omega) \leq 0, \forall \omega \in \Omega, \\ & \text{where } x_2 \text{ and } \bar{x}_2 \text{ solve} \\ & \min_{x_2, \bar{x}_2} F_2(x_1, x_2) + \mathbb{E}_{\xi}[\bar{F}_2(\bar{x}_1(\xi), \bar{x}_2(\xi), \xi)] \\ \text{subject to } & G_2(x_1, x_2) \leq 0 \\ & \bar{G}_2(\bar{x}_1(\omega), \bar{x}_2(\omega), x_1, x_2, \omega) \leq 0, \\ & \forall \omega \in \Omega \end{aligned} \quad (3)$$

Therefore, the TS-SBLP in (3) can be rewritten as

$$\begin{aligned} \min_{x_1, \bar{x}_1} & \mathbb{F}_1(x_1, \bar{x}_1, x_2, \bar{x}_2) \\ \text{subject to } & (x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) \in \mathbb{S}_1, \forall \omega \in \Omega \\ & \text{where } x_2, \bar{x}_2 \text{ solve} \\ & \min_{x_2, \bar{x}_2} \mathbb{F}_2(x_1, \bar{x}_1, x_2, \bar{x}_2) \\ \text{subject to } & (x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) \in \mathbb{S}_2, \forall \omega \in \Omega \end{aligned} \quad (4)$$

where $\mathbb{F}_i(x_1, \bar{x}_1, x_2, \bar{x}_2) = F_i(x_1, x_2) + \mathbb{E}_{\xi}[\bar{F}_i(\bar{x}_1(\xi), \bar{x}_2(\xi))]$ and

$$\begin{aligned} \mathbb{S}_i = \{ & (x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) : \\ & G_i(x_1, x_2) \leq 0, \bar{G}_i(x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) \leq 0 \}, \\ & \forall \omega \in \Omega \end{aligned}$$

at i^{th} level of the decision hierarchy, $i \in \{1, 2\}$.

Consider TS-SBLP in (4) and the decision variable space R^m , such that $(x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) \in X_1 \times \bar{X}_1 \times X_2 \times \bar{X}_2 = \mathbb{S}(\neq \emptyset) \subseteq R^m$, where

$$X_i \subseteq R^{m_i}, \bar{X}_i \subseteq R^{\bar{m}_i}, \sum_{i=1}^2 (m_i + \bar{m}_i) = m, \mathbb{S}^i = \prod_{j=i}^2 (X_j \times \bar{X}_j) \text{ and } \mathbb{S}_i \text{ is the } i^{th} \text{ level constraint set.}$$

Assume that decisions or actions are made from first level decision maker who has control over a vector $(x_1, \bar{x}_1(\omega)) \in X_1 \times \bar{X}_1$ and then reaction is made

from the 2nd decision maker, this process continues sequentially until they achieve Stackelberg equilibrium. Now, let us see some important TS-SBLP definitions.

Definition 2.1. *The following definitions are associated to TS-SBLP problem.*

1. The constraint region for problem (4) is given by:

$$\Psi = \{(x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) \in \mathbb{S} : (x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) \in \mathbb{S}_1 \cap \mathbb{S}_2\}, \forall \omega \in \Omega$$

where $\mathbb{S} = X_1 \times \bar{X}_1 \times X_2 \times \bar{X}_2$.

2. For each given vector $(x_1^\circ, \bar{x}_1^\circ(\omega)) \in X_1 \times \bar{X}_1$, the feasible region of lower levels is given by:

$$\Psi(x_1^\circ, \bar{x}_1^\circ(\omega)) = \{(x_2, \bar{x}_2(\omega)) \in X_2 \times \bar{X}_2(\omega) : (x_1^\circ, \bar{x}_1^\circ(\omega), x_2, \bar{x}_2(\omega)) \in \mathbb{S}_2\}, \forall \omega \in \Omega$$

3. Projection of Ψ on to the 1st level decision space is given by:

$$\Psi(X_1, \bar{X}_1) = \{(x_1, \bar{x}_1(\omega)) : \exists(x_2, \bar{x}_2(\omega)), (x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) \in \mathbb{S}\}, \forall \omega \in \Omega$$

4. For each $(x_1^\circ, \bar{x}_1^\circ) \in \Psi(X_1, \bar{X}_1)$, the rational reaction set of the lower order level is given by:

$$\Phi(x_1^\circ, \bar{x}_1^\circ) = \{(x_2^\circ, \bar{x}_2^\circ) \in \text{argmin}\{\mathbb{F}_2(x_1^\circ, \bar{x}_1^\circ, x_2, \bar{x}_2)\} : (x_2^\circ, \bar{x}_2^\circ) \in \Psi(x_1^\circ, \bar{x}_1^\circ(\omega))\}, \forall \omega \in \Omega$$

5. The induced region at 1st level is defined as:

$$\Upsilon = \{(x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) \in \mathbb{S}_1 : (x_1, \bar{x}_1(\omega)) \in X_1 \times \bar{X}_1, (x_2, \bar{x}_2(\omega)) \in \Phi(x_1, \bar{x}_1(\omega))\}, \forall \omega \in \Omega \quad (5)$$

6. Using the set Υ , one can describe TS-SBLP equivalently as a single level problem of the form:

$$\min_{(x_1, \bar{x}_1(\omega), x_2, \bar{x}_2(\omega)) \in \Upsilon} \mathbb{F}_1(x_1, \bar{x}_1, x_2, \bar{x}_2) \quad (6)$$

Therefore, because of this equivalence any optimal solution of problem (6) is called a Stackelberg solution for TS-SBLP.

3 SSE Method for TS-SBLP

Assumption 3.1. Assume $\mathbb{F}_1$ and $\mathbb{F}_2$ are continuous and measurable in ω , $\mathbb{S}_1$ is convex with respect to x_1 and $\bar{x}_1$, and $\mathbb{S}_2$ is convex with respect to x_2 and $\bar{x}_2$, respectively. In addition, the decision variables are bounded.

From Assumptions 3.1, $\mathbb{F}_1$ has corresponding to any choice $(x_1, \bar{x}_1(\omega))$ of the leader there is at least one reaction $(x_2^\circ, \bar{x}_2^\circ(\omega))$. But, we can possibly select one of the elements from a set Φ in a unique way, say $\Phi^\circ(x_1, \bar{x}_1(\omega))$. Hence, $\forall(x_1, \bar{x}_1(\omega)) \in X_1 \times \bar{X}_1$, $\Phi^\circ(x_1, \bar{x}_1(\omega))$ contain a unique point and then problem (6) is equivalent to

$$\min_{x_1, \bar{x}_1} \mathbb{F}_1(x_1, \bar{x}_1, \Phi^\circ(x_1, \bar{x}_1(\omega))) \quad (7)$$

subject to $(x_1, \bar{x}_1(\omega)) \in X_1 \times \bar{X}_1$
 $\forall \omega \in \Omega$

Therefore, due to continuity assumption in Assumption 3.1 over compact set and Weierstrass Theorem, there exist a solution for problem (7).

The randomness property which is involved in the TS-SBLP problem is approximated using sample average approximation method. Then, the next step is partitioning of first level decision space x_1 and $\bar{x}_1(\omega)$ using a step size value β into $a_1^j = \frac{d_1^j}{\beta}$, $\forall j \in \{1, \dots, m_1\}$, and $\bar{a}_1^j = \frac{\bar{d}_1^j}{\beta}$, $\forall j \in \{1, \dots, \bar{m}_1\}$, where d_1^j and $\bar{d}_1^j$ are the range of decision variables on the first level dimension space for x_1 and $\bar{x}_1$. Then, merge the first odd $(a_1^j(1)$ or $\bar{a}_1^j(1))$ and even $(a_1^j(2)$ or $\bar{a}_1^j(2))$ partition together as $b_1^j(1)$ for decision space x_1 and $\bar{b}_1^j(1)$ for decision space $\bar{x}_1$ and also merge the remaining partition as $b_i^j(p) = a_i^j(2p-1) + a_i^j(2p)$ and $\bar{b}_i^j(p) = \bar{a}_i^j(2p-1) + \bar{a}_i^j(2p)$, where p indicates partition number. In addition, partition each cell $a_1^j(1)$ and $\bar{a}_1^j(1)$ into maximum number of iteration (MaxIter) using a step size value β° for selecting one distinct point at each iteration to avoid repetition, where

$$\beta^\circ = \frac{\beta}{\text{MaxIter}} \quad (8)$$

Here, select γ^{th} first sub-interval and $(\text{MaxIter} - \gamma + 1)^{th}$ second sub-interval from the first partition $(b_1^j(1)$ or $\bar{b}_1^j(1))$ of j^{th} dimension leader's decision space and distribute it to the remaining first level decision space using systematic sampling approach with a step size β , where γ is a unique random natural number between 1 and MaxIter in each iteration. Then, after representatives are chosen, leader's action from his/her own space $X_1 \times \bar{X}_1$ is sent to search good reactions from $X_2 \times \bar{X}_2$ using particle swarm optimization.

Let $(x_2^\lambda, \bar{x}_2^\lambda)$ and $(v_2^\lambda, \bar{v}_2^\lambda)$ be the position vector and velocity for particle λ , where $\lambda \in \{1, \dots, q\}$. Each particle is attracted towards current global best position $(x_2^G, \bar{x}_2^G)$ and its own best location $(x_2^P, \bar{x}_2^P)$ in history, while at the same time it has a tendency to move randomly. Therefore, the new velocity vector is determined by

$$(v_2^\lambda, \bar{v}_2^\lambda)^{t+1} = w(v_2^\lambda, \bar{v}_2^\lambda)^t + \alpha_1 r_1((x_2^G, \bar{x}_2^G) - (x_2^\lambda, \bar{x}_2^\lambda)^t) + \alpha_2 r_2((x_2^P, \bar{x}_2^P) - (x_2^\lambda, \bar{x}_2^\lambda)^t) \quad (9)$$

where r_1 and r_2 are two random numbers, and each entry taking the values between 0 and 1, α_1 and α_2 are called acceleration coefficients and $\alpha_1 \approx \alpha_2 \approx 2$ [12]. In addition, $w(v_2^\lambda, \bar{v}_2^\lambda)^t$ is said to be inertia term, $\alpha_1 r_1((x_2^G, \bar{x}_2^G) - (x_2^\lambda, \bar{x}_2^\lambda)^t)$ is social component and $\alpha_2 r_2((x_2^P, \bar{x}_2^P) - (x_2^\lambda, \bar{x}_2^\lambda)^t)$ is cognitive component. So, the new position should be updated by

$$(x_2^\lambda, \bar{x}_2^\lambda)^{t+1} = (x_2^\lambda, \bar{x}_2^\lambda)^t + (v_2^\lambda, \bar{v}_2^\lambda)^{t+1} \quad (10)$$

If a particle find a location that is better than any previously found locations, then update it as the new current best for particle λ . The aim is to find global best among all the current best solutions until a certain number of sub-iterations, then send $(x_2^\circ, \bar{x}_2^\circ(\omega)) := (x_2^G, \bar{x}_2^G)$ as an optimal response for the upper level problem.

Finally, for each distinct and selected leader's action there is optimal reaction which is in the induced region and this continues until we find a better solution based on leader's problem using equation (7). However, we may not be sure that the current solution for a given iteration is the best. Due to this we should formulate a new solution in each iteration and use the following two definitions which are taken from [22] for comparison of results at each iteration.

Definition 3.2. A given point $(x_1^\circ, \bar{x}_1^\circ, x_2^\circ, \bar{x}_2^\circ)$ is an ϵ -approximate Stackelberg solution in problem (4) if

$$\mathbb{F}_i(x_1^\circ, \bar{x}_1^\circ, x_2^\circ, \bar{x}_2^\circ) \leq \mathbb{F}_i(x_1^*, \bar{x}_1^*, x_2^*, \bar{x}_2^*) + \epsilon_i(1 + |\mathbb{F}_i(x_1^\circ, \bar{x}_1^\circ, x_2^\circ, \bar{x}_2^\circ)|)$$

where, $\epsilon = (\epsilon_1, \epsilon_2)$, $(x_2, \bar{x}_2(\omega)) \in \Phi(x_1, \bar{x}_1(\omega))$, $\forall i \in \{1, 2\}$ and $\forall \omega \in \Omega$.

Definition 3.3. A point $(x_1^\circ, \bar{x}_1^\circ, x_2^\circ, \bar{x}_2^\circ) \in \Psi$ is said to be an ϵ -better solution in problem (4) than $(x_1^\nabla, \bar{x}_1^\nabla, x_2^\nabla, \bar{x}_2^\nabla)$ if

1. $(x_1^\nabla, \bar{x}_1^\nabla, x_2^\nabla, \bar{x}_2^\nabla) \notin \Psi$, or if the following two statements hold true.
2. $\forall \delta = (\delta_1, \delta_2)$ such that $(x_1^\nabla, \bar{x}_1^\nabla, x_2^\nabla, \bar{x}_2^\nabla)$ is a δ -approximate Stackelberg solution, $\exists \epsilon = (\epsilon_1, \epsilon_2)$ such that $\epsilon_i \leq \delta_i$, $\forall i \in \{1, 2\}$ but at least one component is $\epsilon_i < \delta_i$ and $(x_1^\circ, \bar{x}_1^\circ, x_2^\circ, \bar{x}_2^\circ)$ is an ϵ -approximate Stackelberg solution.

If number of iterations increases to infinity, then ϵ -approximate Stackelberg solution approaches to Stackelberg solution. The following pseudo code is the algorithm for SSE method, to solve TS-SBLP.

- Step 1: Consider TS-SBLP with bounded decision variables.
 Input: $m_i, \bar{m}_i, l_i^j, u_i^j, \bar{l}_i^j, \bar{u}_i^j, S_i, \mathbb{F}_i, d_1^j, \bar{d}_1^j, \xi \in \Xi$, where $\xi(\omega) \rightarrow R^N$, $\{\xi_i\}_{i=1}^{100}, \forall j \in \{1, \dots, m_j\}$

and $\forall \bar{j} \in \{1, \dots, \bar{m}_j\}$.

Algorithm parameters: $a_1^j = 2 \cdot \text{ceil}(\frac{d_1^j}{2\beta})$, $\bar{a}_1^j = 2 \cdot \text{ceil}(\frac{\bar{d}_1^j}{2\beta})$, $b_1^j, \bar{b}_1^j, \epsilon, \beta^\circ = \frac{\beta}{\text{MaxIter}}$, k and $\text{perm} = \text{randperm}(\text{MaxIter})$

- Step 2: Set iteration $IT = 1$
- Step 3: Partitioning of random space and decision space
 - Step 3.1: Partition the random space using the probability distribution

$$\xi = \mu + \text{rand}(1, n)\sigma$$

where μ is mean, σ is variance and n is the number of realization, and then use sample average approximation.

- Step 3.2 Partition the space of the “here and now variables” on the interval $[l_1^j, u_1^j]^{m_1}$ for leader's problem

$$x_1^j(1) = l_1^j + (\text{perm}(IT) - 1)\beta^\circ + \text{rand}(1)\beta^\circ$$

$$x_1^j(2) = l_1^j + \beta + (\text{MaxIter} - \text{perm}(IT) + 1)\beta^\circ + \text{rand}(1)\beta^\circ$$
 for $q = 3 : 2 : 2b_1 - 1$

$$x_1^j(q) = x_1^j(q - 2) + 2\beta$$

$$x_1^j(q + 1) = x_1^j(q - 1) + 2\beta$$
 end for
 do Step 3.2 again for the space of the “wait and see variables” on the interval $[\bar{l}_1^j, \bar{u}_1^j]^{\bar{m}_1}$ for leader's problem

- Step 4: Find optimal reaction $(x_2^\circ, \bar{x}_2^\circ) \in X_2 \times \bar{X}_2$, for each $(x_1^*, \bar{x}_1^*) \in X_1 \times \bar{X}_1$, using particle swarm optimization from the second level decision space

- Step 4.1: Input: q number of particles, maximum number of iterations, α_1, α_2, w
- Step 4.2: Initialize locations $(x_2^\lambda, \bar{x}_2^\lambda)$ and velocity $(v_2^\lambda, \bar{v}_2^\lambda)$ for each λ , where $\lambda \in \{1, \dots, q\}$
- Step 4.3: Find $(x_2^P, \bar{x}_2^P)^{t=0}$ from $\text{argmin}\{\mathbb{F}_2(x_1^*, \bar{x}_1^*, x_2^\lambda, \bar{x}_2^\lambda)\}$
- Step 4.4: For loop($t = t + 1$ pseudo time or iteration counter)
 for loop over all q particles and $m_i + \bar{m}_i$ dimensions
 Generate new velocity $(v_2^\lambda, \bar{v}_2^\lambda)^{t+1}$ using equation (9)
 Calculate new locations $(x_2^\lambda, \bar{x}_2^\lambda)^{t+1}$ using equation (10)

Evaluate objective functions at new locations $(x_2^\lambda, \bar{x}_2^\lambda)^{t+1}$

Find the current best for each particle $(x_2^P, \bar{x}_2^P)$
 end for

Find the current global best $(x_2^G, \bar{x}_2^G)$
 end for

- Step 4.5: Output the final results $(x_2^G, \bar{x}_2^G)$ and let $(x_2^\circ, \bar{x}_2^\circ) := (x_2^G, \bar{x}_2^G)$

- Step 5: Select approximate solution using leader's problem from equation (7),

$$\mathbb{F}_1(x_1^\circ, \bar{x}_1^\circ, x_2^\circ, \bar{x}_2^\circ) \leq \mathbb{F}_1(x_1, \bar{x}_1, x_2^\circ, \bar{x}_2^\circ)$$

and then set a solution, $(x_1, \bar{x}_1, x_2, \bar{x}_2)$ for iteration IT .

- Step 6: If $IT > 1$, update ϵ -approximate Stackelburg solution using Definition 3.2 and equation

$$\epsilon_i(IT) = \frac{1}{IT} \cdot \frac{k(4\beta^2)^{\frac{1}{2}}}{1 + |\mathbb{F}_i(x_1^\circ, \bar{x}_1^\circ, x_2^\circ, \bar{x}_2^\circ)|}$$

- Step 7: Terminate, if $IT = \text{MaxIter}$, else set $IT = IT + 1$ and go to step 3.
- Step 8: Out put: $(x_1, \bar{x}_1, x_2, \bar{x}_2)$

solution (9.9994, 9.9994) is appeared at 6th iteration. Whereas the optimal value and solution on the literature[20] are $\mathbb{F}_1 = 100.0001$ and $\mathbb{F}_2 = 5.5e^{-011}$, with optimal solution (10.0009, 9.9993). Therefore, according to equation (5), the solution method SSE is competitive and very promising.

The following example is for TS-SBLP in normal (N) probability distribution.

Example 4.2.

$$\begin{aligned} & \min_{x_1, \bar{x}_1} x_1^2 + (x_2 - 10)^2 + \mathbb{E}[\omega \bar{x}_1 + (\bar{x}_2 + \omega)^2] \\ & \text{subject to } S_1 = \{-x_1 + x_2 \leq 60, x_1^2 + \bar{x}_1 + 2\bar{x}_2 - \omega \leq 20\}, \\ & \quad \text{where } x_2 \text{ and } \bar{x}_2 \text{ solves} \\ & \min_{x_2, \bar{x}_2} (x_1 + 2x_2 - 30)^2 + \mathbb{E}[2\omega \bar{x}_1 + \omega \bar{x}_2] \\ & \text{subject to } S_2 = \{2x_1 + x_2 \leq 20, \\ & \quad x_1 + \bar{x}_1 + 2\bar{x}_2 - \omega \leq 10\} \\ & \quad x_1 \in X_1 = [0, 15], x_2 \in X_2 = [0, 20] \\ & \quad \bar{x}_1 \in \bar{X}_1 = [0, 10], \bar{x}_2 \in \bar{X}_2 = [0, 6] \\ & \quad \forall \omega \in \Omega, \omega \sim N(2, 3) \end{aligned}$$

After implementing the algorithm the optimal values are $\mathbb{F}_1 = 33.8243$ and $\mathbb{F}_2 = 0.47757$, with optimal solution (0.408675, 0.401272, 14.7957, 0) which is appeared at 18th iteration, where $\beta = 0.5$.

4 Simulation Results

TS-SBLP problem (4) can be reduced to a deterministic problem or a finite number of deterministic problems when either the random variables which are provided in the problem are fixed by a single or finite number of realization ξ . In any case, we will check the effectiveness and competitiveness of our proposed algorithm using deterministic bench mark Stackelberg problem from literature[20] in Example 4.1. Moreover, we construct TS-SBLP problem which is in Example 4.2 for testing our new algorithm.

Example 4.1. [20].

$$\begin{aligned} & \min_{x_1} x_1^2 + (x_2 - 10)^2 \\ & \text{subject to } S_1 = \{(x_1, x_2) : -x_1 + x_2 \leq 0\}, \\ & \quad \text{where } x_2 \text{ solves} \\ & \min_{x_2} (x_1 + 2x_2 - 30)^2 \\ & \text{subject to } S_2 = \{(x_1, x_2) : x_1 + x_2 \leq 20\} \\ & \quad x_1 \in X_1 = [0, 15], x_2 \in X_2 = [0, 20] \end{aligned}$$

After implementing the SSE algorithm using MATLAB©software, we let it run 10 independent times, where each run has 50 iteration and $\beta = 0.25$. So, we obtain optimal values to be $\mathbb{F}_1 = 99.989$ and $\mathbb{F}_2 = 2.7439e^{-006}$, with optimal

5 Conclusion

In this paper TS-SBLP is considered and a new type of derivative free meta-heuristic algorithm so-called SSE method is proposed for solving our problem. The algorithm is constructed based on systematic partitioning of upper level decision space for searching the optimal reaction for each leader's action on its own decision space and applying particle swarm optimization for searching a better response from lower level reaction. These updating responses for each leader's action will continue iteratively using an ϵ -approximate Stackelberg solution definition. The proposed algorithm is compared with other methods using bench mark problem from literature. The result of the numerical simulations of the algorithm is very much promising. The algorithm can be used also to solve complex stochastic multilevel programming. In addition, it is a useful new addition topic for stochastic multilevel program.

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