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SOME NEW EXTENSIONS AND GENERALIZATIONS OF LIFETIME DISTRIBUTIONS WITH APPLICATIONS

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A Thesis Submitted to the Faculty of Sciences in Partial Fulfillment of the Requirement
for the Award of the Degree of Doctor of Philosophy in Statistics at BIUST

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2023

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Abstract

In this thesis, families and class of new and generalized distributions called exponentiated half logistic-Kumaraswamy-G (EHL-KUM-G) distribution, gamma odd Burr X-G (GOBX-G) distribution, Topp-Leone odd Burr X-G power series (TLOBX-GPS) distribution, Topp-Leone-exponentiated half logistic odd Burr X-G (TLEHLOBX-G) distribution and Marshall-Olkin-Topp-Leone odd Burr X-G (MOTLOBX-G) distribution are developed and studied in detail. The development of these distributions is motivated by challenges faced in modelling lifetime real-data. In real-life, datasets are sometimes heavy-tailed and skewed, and traditional distributions may not provide a good fit. Additionally, some of the datasets may have outliers which traditional distributions may also not effectively handle. However, researchers have proposed alternative distributions as alternatives but some may suffer the issue of parameter identifiability which may hinder estimation accuracy. Most importantly, real-data often deviate from the assumptions of standard symmetric distributions, hence the need for models that may enhance their flexibility and tail behavior. For that, we developed distributions that will accommodate complex data patterns, and capturing the nuances often encountered in real-world scenarios.

These newly developed families were subjected to detailed exploration of their mathematical and statistical properties. This includes studying their quantile functions, density expansions, moments and generating functions, incomplete moments, Rényi entropy and order statistics. Further, special cases of these family which considered baseline distributions such as Weibull and log-logistic distributions were studied in detail to understand their behavior. Even though other parameter estimation techniques were presented, maximum likelihood estimation (MLE) was employed as the primary method in this study due to its widespread application and strong theoretical foundation. The performance of MLE was further assessed using root mean squared error and absolute bias, which indicated that the technique is consistent and effective.

The special cases within these families were compared against standard distributions as well as various other existing distributions. This evaluation aimed to gauge their adaptability and practicality across a range of real-world datasets with diverse structures. This comparison was carried out using graphical representations and assessments based on several goodness-of-fit statistics, including $-2 \log$ -likelihood, Akaike information criterion, bayesian information criterion, consistent Akaike information criterion, Cramér-Von Mises, Andersen-Darling, Kolmogorov-Smirnov (and its p-value) and the sum of squares. Our comparison results consistently indicated that the special cases derived from these new families consistently excelled in handling real-life datasets. The implications of this finding are profound, suggesting that these newly developed families have the potential to greatly enhance the accuracy and flexibility of statistical analyses in real-world scenarios.

Dedications

In honor of my beloved family and friends, I dedicate this work: To my mother, Ntebatsang Tlhaloganyang, I say thank you for fighting tooth and nail to see that my goal of becoming a doctor came true. I'd want to thank my late grandparents for their advice, their stories, and for helping to shape me into the man I am today. I hope that one day you will emulate my actions and even surpass me in life, champ!, Lerona Nokwane, my son.



Acknowledgements

First and foremost, I would like to praise and thank God, the Almighty, for giving me love, strength, and protection at all times. I also want to extend my sincere gratitude to Prof. Broderick Oluyede, my supervisor, for his inspiration, direction, ongoing support, and patience during my Ph.D. studies. A particular thank also goes out to my family, especially my mother Ntebatsang Tlhaloganyang, for their unending support, love, and encouragement. I also would like to thank the students and members of the scientific faculty at BIUST for their support and encouragement. My buddy Thatayaone Moakofi, who greatly aided me in my studies, deserves a final word of gratitude.



Published, Accepted and Submitted Papers

1. Published Papers

- (i) Tlhaloganyang, B.P., Sengweni, W., & Oluyede, B. (2022). The Gamma Odd Burr X-G Family of Distributions with Applications. *Pakistan Journal of Statistics and Operation Research*, 18(3):721-746.

2. Accepted Papers

- (i) Tlhaloganyang, B.P., Sengweni, W., & Oluyede, B. (2022). The Exponentiated Half Logistic-Kumaraswamy-G Family of Distributions with Properties and Applications. *International Journal of Mathematics in Operational Research*. DOI: 10.1504/IJ-MOR.2022.10048763.
- (ii) Oluyede, B., Tlhaloganyang, B.P., & Sengweni, W. The Topp-Leone Odd Burr X-G Family of Distributions: Properties and Applications. *Statistics, Optimization & Information Computing*.

3. Submitted and Under Review

- (i) Tlhaloganyang, B.P., Oluyede, B., & Sengweni, W. The Marshall-Olkin-Topp-Leone Odd Burr X-G Family of Distributions: Properties and Applications. *Thailand Statistician*.
- (ii) Oluyede, B., Tlhaloganyang, B.P., & Sengweni, W. The Topp-Leone Marshall-Olkin Odd Burr X-G Family of Distributions with Applications to Engineering and Health. *Journal of Statistics & Management Systems*.
- (iii) Tlhaloganyang, B.P., & Oluyede, B. The Topp-Leone Odd Burr X-G Power Series Class of Distributions with Properties and Applications. *Austrian Journal of Statistics*.
- (iv) Tlhaloganyang, B.P., & Oluyede, B. The New Topp-Leone-Exponentiated Half Logistic-Odd Burr X-G Family of Distributions with Properties and Applications. *Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics*.

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List of Abbreviations

ABIAS	Average Bias
AICC	Corrected Akaike Information Criterion
AIC	Akaike Information Criterion
BIC	Bayesian Information Criterion
CDF	Cumulative Distribution Function
EHL-G	Exponentiated Half Logistic-G
EHL-KUM-G ...	Exponentiated Half Logistic-Kumaraswamy-G
EXP-G	Exponentiated-G
GOBX-G	Gamma Odd Burr X-G
GOF	Goodness-Of-Fit
HRF	Hazard Rate Function
IID	Independent And Identically Distributed
KM	Kaplan-Meier
KUM-G	Kumaraswamy-G
MCS	Monte Carlo Simulation
MGF	Moment Generating Function
MLE	Maximum Likelihood Estimation
MO-G	Marshall-Olkin-G
MOTLOBX-G ..	Marshall-Olkin-Topp-Leone Odd Burr X-G
OBX-G	Odd Burr X-G
P-Ps	Probability Probability Plots
PDF	Probability Distribution Function
PSD	Power Series Distribution
PWM	Probability Weighted Moments
QF	Quantile Function
RMSE	Root Mean Square Error
RV	Random Variable
TL-G	Topp Leone-G
TLEHLOBX-G .	Topp-Leone-Exponentiated Half Logistic Odd Burr X-G

TLOBX-GPS ... Topp-Leone Odd Burr X-G Power Series

VAR-COV Variance-Covariance

-2Log(L) -2 Log Likelihood



1. INTRODUCTION

1.1 Introduction

Lifetime data with different characteristics are found in many areas such as engineering, finance, health, economics, insurance and others. Such data needs lifetime statistical distributions that can be used to describe and make predictions based on observed values. For this datasets, there are available classical distributions such Weibull distribution, log-logistic distribution, Lomax distribution, power distribution, Burr III distribution, Burr X distribution and many more which can be readily used or applied. However, in most cases, lifetime data is characterized by complex structures such as skewness, heavy tails, hence making it difficult for standard or classical distributions to provide adequate fit. This increased the need for more flexible distributions that can provide better fits when standard ones fails. For that, researchers seek to develop distributions that accommodate real life datasets which standard distributions does not model adequately. The common strategy is to add one or more parameters to the existing distributions to obtain new flexible families of distributions. The flexibility of these newly developed families of distributions can be seen through the hazard functions which takes several shapes to model datasets with monotone as well as non-monotone shapes. These recent distributions are discussed below together with their extensions.

Using gamma random variables (rvs), Zografos and Balakrishnan (2009) and Ristić and Balakrishnan (2012) respectively developed the Zografos-Balakrishnan-G (gamma type 1) and Ristić-Balakrishnan-G (gamma type 2) families. Their work was further extended to gamma-Weibull-G family by Oluyede (2018), gamma-Kumaraswamy-G family by Arshad et al. (2020), Box-Cox gamma-G family by Al-Babtain et al. (2020), Ristić-Balakrishnan odd log-logistic distributions by Esmaili et al. (2020), gamma-odd Burr-III-G family by Peter et al. (2021), Zografos-Balakrishnan Lindley distribution by Irshad et al. (2021) and gamma Marshall-Olkin-G family by Peter et al. (2022).

Cordeiro et al. (2014) developed the exponentiated half logistic distribution based on the exponentiated half logistic rvs. This family received a lot of attention in literature for being flexible in modeling many complex datasets. For that, other researchers modified it to exponentiated half logistic-exponential distribution (Almarashi et al. 2018), exponentiated half logistic-power generalized Weibull-G family (Oluyede et al. 2021a), exponentiated half logistic odd Weibull-Topp-Leone-G family (Chipepa et al. 2021), exponentiated half-logistic odd Burr III-G family (Oluyede et al. 2022a) and exponentiated half logistic-Topp-Leone-G power series class (Chipepa et al. 2022). These extensions of exponentiated half logistic distribution gave much better fit in real life applications.

Despite having many lifetime distributions to consider in real life applications, attempts have been made to develop other distributions using traditional distributions. From the Burr X

distribution which was part of the twelve distributions developed by Burr (1942), Yousof et al. (2017a) developed the odd Burr X-G (OBX-G) family. Burr III and Burr XII distributions also received attention in literature. Studies which worked around the Burr type X-G family include gamma Burr type X (GBX) distribution (Khaleel et al., 2016), exponentiated Burr X (EBX) distribution (Ahmed et al., 2021), beta Burr X (BBX) distribution (Merovci et al., 2016), exponentiated generalized Burr X (EGBX) distribution (Khaleel et al., 2018) and type I half-logistic Burr X (TIHLBX) distribution (Shrahili et al., 2019).

Using the distribution function of the Topp-Leone rv, Al-Shomrani et al. (2016) developed the Topp-Leone-G (TL-G) family. TL-G family has been extended and some of these extensions include: Topp-Leone odd exponential half logistic-G (TLOEHL-G) family (Chipepa and Oluyede, 2021), Topp-Leone odd Burr III-G (TLOBIII-G) family (Moakofi et al., 2022), Topp-Leone Kumaraswamy-G (TLK-G) family (Sule et al., 2020) and Topp-Leone odd Lindley-G (TLOL-G) family (Reyad et al., 2018). Additional cases of the TL-G family studied include Topp-Leone Weibull (TL-W) distribution (Tuoyo et al., 2021) and Topp-Leone Lomax (TL-L) distribution (Oguntunde et al., 2019).

Other studies have also extended the Kumaraswamy-G family which was developed by by Cordeiro and de Castro (2009). This includes the Topp-Leone-Kumaraswamy-G family (Sule et al., 2020), Marshall Olkin-Kumaraswamy-G family (Handique et al., 2017a), Sine-Kumaraswamy-G family (Chesneau and Jamal 2021), gamma-Kumaraswamy-G family (Arshad et al., 2020) and generalized Kumaraswamy-G family (Nofal et al., 2019).

Some of the generators that have been proposed include Marshall-Olkin-G by Marshall and Olkin (1997), beta-G by Eugene et al. (2002), transform-transformer (T-X) by Alzaghal et al. (2013), logistic-G by Torabi and Montazeri (2014), Weibull-G by Bourguignon et al. (2014), type I half-logistic-G by Cordeiro et al. (2016a) and type II half-logistic-G by Soliman et al. (2017), Marshall–Olkin–Weibull-H family by Afify et al. (2022), truncated Cauchy power Weibull-G family by Alotaibi et al. (2022) and beta Topp-Leone-G by Watthanawisut et al. (2022).

In other studies, researchers compound lifetime distributions with power series distributions to generate new distributions. Some recent power series generalizations in the literature includes generalized Gompertz-G class (Tahmasebi and Jafari, 2016), odd Weibull-Topp-Leone-G power series class (Oluyede et al., 2021b), type II exponentiated half-logistic-Topp-Leone-G power series class (Moakofi et al., 2021), exponentiated half logistic-Topp-Leone-G power series class (Chipepa et al., 2022), Burr-Weibull power series class (Oluyede et al., 2019) and Topp-Leone-G power series class (Makubate et al., 2021a).

In most cases, such as in the study of Zografos and Balakrishnan (2009), Ristić and Balakrishnan (2012) and Cordeiro et al. (2014) among others, the authors derived the pdf, cdf and quantile function for their distributions. They also investigated their statistical properties, such as density expansions, moments, moment generating function, order statistics and entropies. The shapes of the pdfs and hrfs were further studied based on varying model parameter values. The studies further applied different parameter estimation techniques but in most cases they use only one technique. For example Yousof et al. (2017a), Yousof et al. (2017b) Tuoyo et al. (2021) used

maximum likelihood technique only. Drawing parallels with the aforementioned approaches, the utilization of MLE as the primary estimation method in our study finds its rationale. The precedent set by these established studies, where MLE was consistently used as the main method for estimating parameters, serves as a testament to the effectiveness and prominence of this technique in similar contexts. Just as these researchers embraced MLE to ensure consistency and comparability in their analyses, we also opted for MLE as a robust and widely accepted method to facilitate precise estimation and meaningful comparison within the scope of our study. Most studied further proceeded to conduct simulation studies to evaluate the performance of this technique which is what we did in our study.

To demonstrate the flexibility of the proposed models, real life datasets were used in most of the studies that proposed alternative distributions. However, some of the studies used only one real life dataset to show this, for example see [Cordeiro and de Castro \(2009\)](#) and [Al-Shomrani et al. \(2016\)](#). Relying solely on one real-life dataset can limit the robustness, generalizability, and applicability of research findings. To strengthen our research outcomes, our study will use multiple datasets and conducted thorough analyses to gain a more comprehensive and accurate understanding of the proposed distributions.

Further, researchers used different goodness-of-fit statistics and plots to evaluate the performance of the proposed distributions even though some studies used only one statistic or plot. For example, [Cordeiro and de Castro \(2009\)](#) used AIC and histogram whereas [Al-Shomrani et al. \(2016\)](#) used Kolmogorov–Smirnov statistic and histogram only. Using only one goodness-of-fit statistic or plot may not provide a comprehensive assessment of model performance which is the reason why our study will use several goodness-of-fit statistics and plots.

Some studies, such as that of [Al-Shomrani et al. \(2016\)](#), did not compare the proposed models with the existing ones, which may also have limitations in evaluating the performance of the new models. Comparing the newly developed models with existing ones is crucial as it provides a benchmark to assess the goodness of fit, statistical properties, and applicability of the proposed models in real-life applications. Further, some of the studies which compared their proposed models with the existing models only compared them with nested models or models with less number of parameters as compared to the proposed one. For example, [Sule et al. \(2020\)](#) developed Topp Leone Kumaraswamy Exponential distribution with four parameter and only compared it with models with three or less number of parameters of which some were nested models. Comparing a newly proposed model with other models of same number of parameters may be important for proper evaluation, validation, and interpretation. It helps researchers identify the model's strengths and weaknesses which is why our study will use nested models together with models with equal number of parameters.

While some of the developed models significantly showed some flexibilities when applied to real life datasets compared to standard distributions, they may not always provide the best fit for certain types of data. In statistical modelling, there is no a one-size-fits-all solution. Some models could be limited in terms of their performance in specific scenarios where other distributions might be more appropriate which is why our study propose to develop alternative distributions incase existing ones don't perform well.

1.2 Important Generators

Useful generators for developing distributions are discussed in this section of the thesis. Some of the distributions developed using this generators are given in Section 1.1. Let $G(x; \underline{\omega})$ and $g(x; \underline{\omega})$ be the baseline cumulative distribution function (CDF) and probability density function (PDF), respectively, with the vector of baseline parameters $\underline{\omega}$.

1.2.1 Transformed-Transformer Generator

Alzaatreh et al. (2013) introduced a way to develop flexible distributions using the transformed-transformer (T-X) method. For a non-negative continuous rv $T \in [a, b]$, for $-\infty \leq a < b \leq \infty$, the CDF (CDF) of T-X family follows as

$$F(x) = \int_a^{W(G(x; \underline{\omega}))} r(t) dt, \quad x \in \mathbb{R},$$

where $r(t)$ is the PDF of the rv T , $W(G(x; \underline{\omega}))$ represent the function of the CDF $G(x; \underline{\omega})$. $W(G(x; \underline{\omega}))$ should be differentiable, monotonically increasing and satisfy the condition $W(G(x; \underline{\omega})) \rightarrow a$ as $x \rightarrow -\infty$ and $W(G(x; \underline{\omega})) \rightarrow b$ as $x \rightarrow \infty$ (Alzaatreh et al. 2013). $W(G(x; \underline{\omega}))$ can take different forms such as $W(G(x; \underline{\omega})) = G(x; \underline{\omega})/\bar{G}(x; \underline{\omega})$ and $W(G(x; \underline{\omega})) = -\log [\bar{G}(x; \underline{\omega})]$ (Alzaatreh et al. 2013).

1.2.2 Beta Generator

The CDF and PDF of beta-G (B-G) family are given as

$$F_{B-G}(x; m, n, \underline{\omega}) = I_{G(x; \underline{\omega})}(m, n) = \frac{1}{B(m, n)} \int_0^{G(x; \underline{\omega})} v^{m-1} (1-v)^{n-1} dv = \frac{B_{G(x; \underline{\omega})}(m, n)}{B(m, n)}$$

and

$$f_{B-G}(x; m, n, \underline{\omega}) = \frac{1}{B(m, n)} [G(x; \underline{\omega})]^{m-1} [1 - G(x; \underline{\omega})]^{n-1} g(x; \underline{\omega}),$$

respectively, for $m, n > 0$. beta-G family have been used in generating other useful distributions (see Cordeiro et al. (2016b), Handique et al. (2017b) and Yousof et al. (2017b)).

1.2.3 Gamma Generators

The CDFs of Zografos-Balakrishnan-G (ZB-G) family by Zografos and Balakrishnan (2009) and Ristić-Balakrishnan-G (RB-G) family by Ristić and Balakrishnan (2012) are

$$F_{ZB-G}(x; \delta, \underline{\omega}) = \frac{1}{\Gamma(\delta)} \int_0^{-\log(1-G(x; \underline{\omega}))} t^{\delta-1} e^{-t} dt = \frac{\gamma(\delta, -\log(1-G(x; \underline{\omega})))}{\Gamma(\delta)}$$

and

$$F_{RB-G}(x; \delta, \underline{\omega}) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log(G(x; \underline{\omega}))} t^{\delta-1} e^{-t} dt = \frac{\gamma(\delta, -\log(G(x; \underline{\omega})))}{\Gamma(\delta)},$$

respectively, where $\gamma(\delta, z) = \int_0^z t^{\delta-1} e^{-t} dt$. The PDFs of ZB-G, and RB-G family are

$$f_{ZB-G}(x; \delta, \underline{\omega}) = \frac{1}{\Gamma(\delta)} [-\log(1 - G(x; \underline{\omega}))]^{\delta-1} g(x; \underline{\omega})$$

and

$$f_{RB-G}(x; \delta, \underline{\omega}) = \frac{1}{\Gamma(\delta)} [-\log(G(x; \underline{\omega}))]^{\delta-1} g(x; \underline{\omega}),$$

for $\delta > 0$.

1.2.4 Type I and Type II Half-Logistic Generators

The type I half-logistic-G (TIHL-G) and type II half-logistic-G (TIIHL-G) family was developed using half-logistic distribution by [Balakrishnan \(1985\)](#). The general definition of the CDF of the half-logistic-G family CDFs is give by

$$F(x) = \int_0^{W(G(x; \underline{\omega}))} \frac{2\lambda \exp\{-\lambda t\}}{(1 + \exp\{-\lambda t\})} dt. \quad (1.1)$$

In equation [1.1](#), when $W(G(x; \underline{\omega})) = -\ln(1 - G(x; \underline{\omega}))$ and $W(G(x; \underline{\omega})) = -\ln(G(x; \underline{\omega}))$, we get the TIHL-G family by [Cordeiro et al. \(2016a\)](#) and TIIHL-G family by [Soliman et al. \(2017\)](#) with CDFs

$$F_{TIHL-G}(x; \lambda, \underline{\omega}) = \frac{1 - [1 - G(x; \underline{\omega})]^\lambda}{1 + [1 - G(x; \underline{\omega})]^\lambda} \text{ and } F_{TIIHL-G}(x; \lambda, \underline{\omega}) = \frac{2G^\lambda(x; \underline{\omega})}{1 + G^\lambda(x; \underline{\omega})}, \quad (1.2)$$

where $\lambda > 0$. In the first equation [1.2](#), when $\lambda = 1$, we have a reduced type I half-logistic-G (HL-G) family with CDF and PDF given as

$$F(x) = \frac{G(x; \underline{\omega})}{1 + \overline{G}(x; \underline{\omega})} \text{ and } f(x) = \frac{2g(x; \underline{\omega})}{(1 + \overline{G}(x; \underline{\omega}))^2}, \text{ respectively.}$$

1.2.5 Type I and Type II Exponentiated Half-Logistic Generators

[Cordeiro et al. \(2014\)](#) developed the exponentiated half-logistic-G (EHL-G) family using exponentiated half-logistic distribution with $W(G(x; \underline{\omega}))$ set as $W(G(x; \underline{\omega})) = -\ln(1 - G(x; \underline{\omega}))$, which makes it type I EHL-G family. Type II EHL-G family may be obtained by setting $W(G(x; \underline{\omega}))$ as $W(G(x; \underline{\omega})) = -\ln(G(x; \underline{\omega}))$. The CDF and PDF of EHL-G family follows as

$$F(x; \alpha, \lambda, \underline{\omega}) = \left[\frac{1 - [1 - G(x; \underline{\omega})]^\lambda}{1 + [1 - G(x; \underline{\omega})]^\lambda} \right]^\alpha$$

and

$$f(x; \alpha, \lambda, \underline{\omega}) = 2\alpha\lambda g(x; \underline{\omega}) [1 - G(x; \underline{\omega})]^{\lambda-1} \frac{\left\{ 1 - [1 - G(x; \underline{\omega})]^\lambda \right\}^{\alpha-1}}{\left\{ 1 + [1 - G(x; \underline{\omega})]^\lambda \right\}^{\alpha+1}},$$

respectively, where $\lambda, \alpha > 0$. Note that when $\alpha = 1$, EHL-G family reduces to TIHL-G family

and when $\alpha = \lambda = 1$, it reduces to HL-G family given in subsection 1.2.4

1.2.6 Kumaraswamy Generator

From the Kumaraswamy distribution developed by Kumaraswamy (1980), Cordeiro and de Castro (2009) introduced Kumaraswamy-G (Kum-G) family with CDF and PDF given as

$$F_{Kum-G}(x; a, b, \underline{\omega}) = 1 - \psi^b(x; a, \underline{\omega}) \quad \text{and} \quad f_{Kum-G}(x; a, b, \underline{\omega}) = abg(x; \underline{\omega})G(x; \underline{\omega})^{a-1}(x)\psi^{b-1}(x; a, \underline{\omega}),$$

respectively, for $a, b > 0$ and $\psi(x; a, \underline{\omega}) = 1 - G^a(x; \underline{\omega})$. Note that when $b = 1$ in Kum-G family and $n = 1$ in beta-G family given in subsection 1.2.2 then these families are the same if we let $a = m$. Kum-G family has received a lot attention in literature due its tractability compared to the other family like the beta-G family which has beta function ratio and incomplete beta function.

1.2.7 Marshall-Olkin Generator

Marshall and Olkin (1997) obtained a new distribution called Marshall-Olkin-G (MO-G) family with CDF and PDF

$$f_{MO-G}(x; \delta, \underline{\omega}) = \frac{\delta g(x; \underline{\omega})}{[1 - \bar{\delta} \bar{G}(x; \underline{\omega})]^2} \quad \text{and} \quad F_{MO-G}(x; \delta, \underline{\omega}) = 1 - \frac{\delta \bar{G}(x; \underline{\omega})}{1 - \bar{\delta} \bar{G}(x; \underline{\omega})},$$

respectively, where $\delta > 0$ is the tilt parameter and $\bar{\delta} = 1 - \delta$.

1.2.8 Burr III and Burr X Generators

Following the concept of Zografos and Balakrishnan (2009), several authors developed Burr-G generators using the rvs of Burr distributions due to Burr (1942). Among others, Alizadeh et al. (2017) replaced the Burr III rv with $G(x; \underline{\omega})/\bar{G}(x; \underline{\omega})$ when developing odd Burr III-G (OBIII-G) family. Using the Burr X rv T , Yousof et al. (2017a) developed odd Burr X-G (OBX-G) family with CDF and PDF given as

$$F_{OBX-G}(x; \theta, \underline{\omega}) = 2\theta \int_0^{K^{0.5}(x; \underline{\omega})} t \exp(-t^2) [1 - \exp(-t^2)]^{\theta-1} dt = \{1 - \exp[-K(x; \underline{\omega})]\}^\theta$$

and

$$f_{OBX-G}(x; \theta, \underline{\omega}) = \frac{2\theta g(x; \underline{\omega})G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \exp[-K(x; \underline{\omega})] \{1 - \exp[-K(x; \underline{\omega})]\}^{\theta-1},$$

respectively, where $\theta > 0$ and $K(x; \underline{\omega}) = \left(\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})}\right)^2$.

1.2.9 Topp-Leone Generator

Al-Shomrani et al. (2016) developed the Topp-Leone-G (TL-G) family with PDF and CDF given as

$$f_{TL-G}(x; \alpha, \underline{\omega}) = 2\alpha g(x; \underline{\omega}) \bar{G}(x; \underline{\omega}) [1 - \bar{G}^2(x; \underline{\omega})]^{\alpha-1} \quad \text{and} \quad F_{TL-G}(x; \alpha, \underline{\omega}) = [1 - \bar{G}^2(x; \underline{\omega})]^\alpha,$$

respectively, for $\alpha > 0$, $x \geq 0$ and vector of parameters $\underline{\omega}$ from the baseline distribution.

1.2.10 Lehmann Type I and Lehmann Type II Generators

The two Lehmanns' families, commonly known as exponentiated-G (Exp-G) family were introduced by Lehmann (1953). The most common and simplest Exp-G family (Lehmann type I) has the CDF and PDF of the form

$$F_{TIEp-G}(x; \alpha, \underline{\omega}) = G^\alpha(x; \underline{\omega}) \quad \text{and} \quad f_{TIEp-G}(x; \alpha, \underline{\omega}) = \alpha g(x; \underline{\omega}) G^{\alpha-1}(x; \underline{\omega}), \quad (1.3)$$

respectively, where $\alpha > 0$. Some work on Lehmann type I can be seen from (Gupta et al., 1998) who considered baseline distributions as exponential distribution, Pareto distribution, Weibull distribution and gamma distribution. Lehmann type I is usually used in derivation of statistical properties of many new distributions.

Another Exp-G family is called the Lehmann type II distribution which of course has received less attention in literature compared to the type I family (Tahir and Nadarajah, 2015). The Lehmann type II distribution or simply the type II Exp-G family has the CDF and PDF of the form

$$F_{TIIExp-G}(x; \alpha, \underline{\omega}) = 1 - \bar{G}^\alpha(x; \underline{\omega}) \quad \text{and} \quad f_{TIIExp-G}(x; \alpha, \underline{\omega}) = \alpha g(x; \underline{\omega}) \bar{G}^{\alpha-1}(x; \underline{\omega}),$$

respectively, for $\alpha > 0$, and the parameter vector $\underline{\omega}$.

1.2.11 Competing Risk Approach

Consider a system made of components in a series structure and assume that these components has CDFs $G_1(t)$ and $G_2(t)$, with reliability functions $R_1(t) = 1 - G_1(t)$ and $R_2(t) = 1 - G_2(t)$, respectively. Then, the reliability of the system follows as $R(t) = P(T > t) = \prod_{i=1}^2 R_i(t)$. The CDF of the system follows as $G(t) = 1 - R(t)$ and the PDF can be obtained by differentiating $G(t)$ with respect to t . In reliability analysis, a model of this nature is usually termed a competing risk model.

1.2.12 Compounding Approach

A discrete rv N follows a power series distribution if it has a pmf given as:

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)} \quad \text{for} \quad N \in T,$$

where T is a subset of the set of non-negative integers, $\theta > 0$, $a_n > 0$ is a sequence of coefficients in power series and $C(\theta)$ is *positive, finite and differentiable*. The series defining function $C(\theta)$ is given as $C(\theta) = \sum_{n \in T} a_n \theta^n$. Power series is a family of univariate discrete probability distributions which are a member of one parameter exponential distribution (Momeni, 2011).

Let $X_{(1)} = \min(X_1, X_2, X_3, \dots, X_N)$, where X is a continuous rv with PDF, CDF and sf $f(x; \omega)$, $F(x; \omega)$ and $S(x; \omega) = 1 - F(x; \omega)$, respectively. The compounded CDF and PDF follows as

$$F_{X_{(1)}}(x; \theta, \omega) = 1 - \frac{C(\theta S(x; \omega))}{C(\theta)}$$

and

$$f_{X_{(1)}}(x; \theta, \omega) = \frac{\theta f(x; \omega) C'(\theta S(x; \omega))}{C(\theta)},$$

respectively, where $\theta > 0$, $C'(\cdot)$ is the first derivative of $C(\theta)$ with respect to θ , and ω is the baseline vector of parameters.

1.3 Properties of Lifetime Distributions

In this section, we provide some statistical and mathematical properties of lifetime distributions. This includes the HRF, reverse HRF, survival function, QF, ordinary and incomplete moments, order statistics, entropy, stochastic ordering, estimation techniques, model comparison, and assessment techniques. In our study, most of the properties of the functions that we developed depends on the properties of the Exp-G family given in equation (1.3).

1.3.1 Hazard Rate, Reverse Hazard Rate, Survival Functions

The hazard rate function (HRF), sometimes called the instantaneous failure rate function, is very useful in studying the probability that a system or component fails given that it is currently working. Thus, the HRF is made of the conditional probability which seeks the probability of failure for a system given that it has survived sometimes. For the mathematical definition, let the T be a non-negative rv representing the survival time of a unit or the waiting time until the event of interest occurs (time to event) with PDF and CDF given as $f(t)$ and $F(t)$, respectively. Then, the HRF is given as $h(t) = f(t)/\bar{F}(t)$, where $\bar{F}(t) = 1 - F(t)$ is the survival function (SF). Commonly considered structures of HRF shapes are increasing, decreasing, bathtub and upside-down bathtub. In the field of engineering, systems usually have bathtub HRF shapes showing that from an early stage the systems' probability of failure decreases then remains constant for some time but as time increases the probability of failure also increases. Another important function related to HRF is the reverse HRF (RHRF) which focuses on the probability that a system fails in the future given that it once failed (Desai et al., 2011). Mathematically, we can define rHRF in terms of the PDF and CDF as $r(t) = f(t)/F(t)$. Because of the previous failure, the rHRF of a system is always monotonically decreasing with time (Turkson, 2022). Alternatively, we can express RHRF in the form of HRF as $r(t) = h(t)\bar{F}(x)/F(x)$. Using the above functions, we can also obtain the other useful functions such as the cumulative hazard function and mean residual life.

1.3.2 Quantile Function

The quantile function (QF) is defined as the inverse of the distribution function and it is used in generating random samples in most cases. To obtain the inverse of the CDF, the CDF has to be monotonically increasing which is a condition that most of the continuous rvs CDF meet (Siegrist, 2022). The generalized inverse distribution function follows as $F^{-1}(u) = \inf \{x \in \mathbb{R} : F(x) \geq u\}$ for $0 < u < 1$. We can express the QF as $x = Q_X(u) = F^{-1}(u)$. Solving for inverse functions can be complex in some functions especially the ones in open form, for this reason, one can consider the use of computer programs such as R, Python, and Stata. Apart from data generation, QFs can also be used to obtain percentiles, quantiles and quartiles.

1.3.3 Statistical Moments

The shapes of the statistical distributions depend on moments hence we study them. The commonly studied moments are the mean, variance, skewness, and kurtosis. The mean gives the center of the distribution, the standard deviation gives the amount of the spread, skewness gives the amount of asymmetry around the mean and the kurtosis gives the amount of peakedness (Goyal, 2022). For most of the lifetime distributions, the derivation of their moments follows from the linear representation of their PDFs. Thus, if one can successfully expand the PDF of a distribution and write it in a linear form, then its representation can be used to easily derive the moments. In most cases, expansions of these distributions leads to an infinite linear combination of Exp-G family.

Assuming X is a continuous rv with PDF and CDF given as $f(x)$ and $F(x)$, respectively, then the r^{th} ordinary moments also known as the raw moments or moments about zero of X

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx. \quad (1.4)$$

In equation (1.4), when $r = 1$ we obtain the first moment termed as the mean. With the mean and the second to the fourth moment, variance, skewness, and kurtosis can be obtained. The central moments, also known as the moments about zero is given as

$$\mu_r = \int_{-\infty}^{\infty} (x - E(X))^r f(x) dx. \quad (1.5)$$

When $r = 0$ in equation (1.5), $r = 1$ and $r = 2$, we get $\mu_0 = 1$, $\mu_1 = 0$ and the variance, respectively. The third and fourth central moments combine with the variance gives the skewness and kurtosis, respectively. Thus, the mean can only be determined from only the ordinary moment, whereas the variance, skewness and kurtosis are easy to obtain from the central moment. The

moment generating function (MGF) of a continuous rv X is given as

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx. \quad (1.6)$$

Taking the r^{th} derivatives of the $M_X(t)$ obtained from equation (1.6) with respect to t and plug in $t = 0$, we get the r^{th} ordinary moments given in equation (1.4) (Pho et al. 2019).

1.3.4 Order Statistics

Order statistics is important in the application of statistics and reliability. For examples, we need order statistics to get the range, median, percentiles, quantiles and quartiles of the data since they work with the position of the ordered random samples. For that, order statistics has been in many field such as insurance, sport, climate events and where ordering is critical. Let $X_{(i)}$ for $i = 1, 2, 3, \dots, n$ be the order statistics of iid random sample of size n , then the PDF of the i^{th} order statistic follow as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} F^{i-1}(x) \bar{F}^{n-i}(x), \quad (1.7)$$

where $\bar{F}(x) = 1 - F(x)$ and $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function (Elgarhy 2017). If we apply the binomial expansion as $(1 - F(x))^{n-i} = \sum_{t=0}^{n-i} (-1)^t \binom{n-i}{t} F^t(x)$ then we can express equation (1.7) as

$$f_{i:n}(x) = \frac{n!f(x)}{(i-1)!(n-i)!} \sum_{t=0}^{n-i} (-1)^t F^{t+i-1}(x).$$

1.3.5 Entropy Measures

In this section, we look at the two common entropy functions, namely the Shannon entropy by Shannon (1948) and the Rényi entropy by Rényi (1961). The application of entropies is found in many areas such as reliability, life testing, ecology, insurance, and computing. For example, it can be applied in ecology to explore the diversification of species. It can also be applied in predicting earthquakes and exchange rates, and even in studying software and security password reliability (see Maiti et al. 2022 for many more examples). For definitions, let $f(x)$ be a PDF of a continuous rv X , then Rényi entropy is given as

$$I_R(X) = \frac{1}{1-\theta} \log \int_0^{\infty} f^{\theta}(x) dx, \quad (1.8)$$

where $\theta > 0$ and $\theta \neq 1$. Shannon entropy follows as

$$I_S(X) = - \int_0^{\infty} f(x) \log(f(x)) dx.$$

From equation (1.8), we have $\lim_{\theta \rightarrow 1} I_R(X) = I_S(X)$.

1.3.6 Stochastic Ordering

Sometimes in real life, we may be interested in the comparison of some information of the rvs. For example, we may want to compare the lifetimes of bulbs produced by two different companies for the same purpose. Stochastic ordering is very useful in fields that study the reliability and probability of units, especially when rv characteristics (e.g probability distributions) are compared in terms of their largeness. Common distribution function orders include likelihood ratio order, hazard rate order, and regular stochastic order. To define stochastic ordering, let X and Y be two random variables with survival functions $\bar{F}_W(x)$ and $\bar{F}_Z(x)$, where, $\bar{F}_W(x) = 1 - F_W(x)$ and $\bar{F}_Z(x) = 1 - F_Z(x)$ are the survival functions of W and Z , respectively. Assuming $F_W(x) \geq F_Z(x)$ or $\bar{F}_W(x) \leq \bar{F}_Z(x)$ for all x , we say that the rv W is stochastically smaller than the rv Z and it is represented as $W <_{st} Z$. For all x , if $\frac{f_W(x)}{f_Z(x)}$ is decreasing, this is termed likelihood ratio order represented as $W <_{lr} Z$ and if $h_W(x) \geq h_Z(x)$ this is termed hazard rate order represented as $W <_{hr} Z$. In addition, we have that $W <_{lr} Z \Rightarrow W <_{hr} Z \Rightarrow W <_{st} Z$ as proved by Shaked and Shanthikumar (2007).

1.4 Parameter Estimation

There are several parameter estimation techniques such as the maximum likelihood estimation (MLE), Anderson-Darling (AD) estimation, Right-tail Anderson-Darling (RTAD) estimation, percentile estimation (PE) method, maximum product of spacing (MPS) method, Cramér-Von Mises (CVM) estimation, Ordinary Least-Squares (OLS) and Weighted Least-Squares (WLS). In this study, we used mle technique to estimate our parameters. This was because of its easy of derivation and together with its merits in being an all time good estimator in research (Thamer and Zine, 2021; Veena and Lishamol, 2022). Moreover, the gof statistics that we considered for model comparisons in this study depends on the value of the mle hence we look more into it. The aim of estimation is to get good and reasonable estimators of model parameters from a given set of observed random sample.

1.4.1 MLE Method

In mle technique, the point is to get parameter estimates that maximize the likelihood function. Consider n observations which are realization of iid rvs $(X_1, X_2, X_3, \dots, X_n)$. Additionally, let $X_i \sim f(x_i|\underline{\varphi})$, where $i = 1, 2, 3, \dots, n$ and $\underline{\varphi}$ is the unknown vector parameter which governs the function $f(x_i|\underline{\varphi})$. Since the rvs are independent, it follows that the likelihood of the data is the joint probability of the data is

$$L(\underline{\varphi}) = \prod_{i=1}^n f(x_i; \underline{\varphi}). \quad (1.9)$$

Thus, the likelihood of all the data ($L(\underline{\varphi})$) is made up of the product of the likelihood of each data point since they are independent. The log of the likelihood function, $\log(L(\underline{\varphi})) = \ell(\underline{\varphi})$

follows as

$$\ell(\underline{\varphi}) = \sum_{i=1}^n \log(f(x_i; \underline{\varphi})). \quad (1.10)$$

To obtain the maximum likelihood estimator of $\underline{\varphi}$, say $\hat{\underline{\varphi}}$, we get the first derivative of $\ell(\underline{\varphi})$ given in equation (1.10) with respect to $\underline{\varphi}$, then we equate it to 0 and solve for $\underline{\varphi}$. In most cases, the derivatives are not in closed form which requires the use of iterative methods to solve. Interestingly, parameter estimates can be obtained using computer softwares like R, MATLAB and Stata. In our case, we use R software to estimate the parameters using *mle2* function via the **bbmle** package (Ganguli et al., 2022). To derive the confidence interval for model parameters, we use the Fisher information matrix $I(\underline{\varphi}) = -E \left[\frac{\partial^2 \ell(\underline{\varphi})}{\partial \varphi_i \partial \varphi_j} \right]$ for $i, j = 1, 2, 3, \dots$, estimated by the observed information matrix given as $J_n(\hat{\underline{\varphi}}) = - \left[\frac{\partial^2 \ell(\underline{\varphi})}{\partial \varphi_i \partial \varphi_j} \right]$.

The next sections discuss other estimation techniques which have been discussed in detail by Dey et al. (2018) and Anderson and Darling (1952).

1.4.2 AD and RTAD Methods

Let $X_{i:n}$ be the i^{th} order statistics with CDF $F(x_{i:n}; \underline{\omega})$ and survival function $S(x_{i:n}; \underline{\omega}) = 1 - F(x_{i:n}; \underline{\omega})$, where $\underline{\omega}$ is the parameters' vector for this function. The AD estimators minimize the function

$$A(\underline{\omega}) = -n - \frac{1}{n} \sum_{i=1}^n (2i - 1) [\log(F(x_{i:n}; \underline{\omega})) + \log(S(x_{i:n}; \underline{\omega}))].$$

Alternatively, AD estimators can be obtained by solving the equation

$$\sum_{i=1}^n (2i - 1) \left[\frac{\Delta_s(x_{i:n}; \underline{\omega})}{F(x_{i:n}; \underline{\omega})} - \frac{\Delta_s(x_{n+1-i:n}; \underline{\omega})}{S(x_{n+1-i:n}; \underline{\omega})} \right] = 0,$$

where s is the number of parameters to be estimated and

$$\Delta_s(\bullet; \underline{\omega}) = \frac{\partial F(x_{i:n}; \underline{\omega})}{\partial \omega_k}. \quad (1.11)$$

The RTAD estimators minimize the function

$$R(\underline{\omega}) = \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n}; \underline{\omega}) - \frac{1}{n} \sum_{i=1}^n (2i - 1) \log(S(x_{n+1-i:n}; \underline{\omega})).$$

Alternatively, RTAD estimators can be obtained by solving the equation

$$-2 \sum_{i=1}^n \Delta_s(x_{i:n}; \underline{\omega}) + \frac{1}{n} \sum_{i=1}^n (2i - 1) \frac{\Delta_s(x_{n+1-i:n}; \underline{\omega})}{S(x_{n+1-i:n}; \underline{\omega})} = 0,$$

where $\Delta_s(\bullet; \underline{\omega})$ is defined in equation (1.11) and s is the number of parameters to be estimated.

1.4.3 CVM Method

The CVM estimators minimize the function

$$C(\underline{\omega}) = -\frac{1}{12n} + \sum_{i=1}^n \left[F(x_{i:n}; \underline{\omega}) - \frac{2i-1}{2n} \right]^2$$

Alternatively, CVM estimators can be obtained by solving the equation

$$\sum_{i=1}^n \left[F(x_{i:n}; \underline{\omega}) - \frac{2i-1}{2n} \right] \Delta_s(x_{i:n}; \underline{\omega}) = 0,$$

where $\Delta_s(\bullet; \underline{\omega})$ is defined in equation (1.11) and s is the number of parameters to be estimated.

1.4.4 OLS and WLS Methods

The OLS estimators minimize the function

$$V(\underline{\omega}) = \sum_{i=1}^n \left[F(x_{i:n}; \underline{\omega}) - \frac{i}{n+1} \right]^2.$$

Alternatively, OLS estimators can be obtained by solving the equation

$$\sum_{i=1}^n \left[F(x_{i:n}; \underline{\omega}) - \frac{i}{n+1} \right] \Delta_s(x_{i:n}; \underline{\omega}) = 0.$$

The WLS estimators are obtained by minimizing the function

$$W(\underline{\omega}) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-1+1)} \left[F(x_{i:n}; \underline{\omega}) - \frac{i}{n+1} \right]^2.$$

Alternatively, WLS estimators can be obtained by solving the equation

$$\sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-1+1)} \left[F(x_{i:n}; \underline{\omega}) - \frac{i}{n+1} \right] \Delta_s(x_{i:n}; \underline{\omega}) = 0,$$

where $\Delta_s(\bullet; \underline{\omega})$ is defined in equation (1.11) and s is the number of parameters to be estimated.

1.4.5 MPS Method

Let the uniform spacings of the rv $X_1, X_2, \dots, X_n$ be $D_i(x_{i:n}; \underline{\omega}) = F(x_{i:n}; \underline{\omega}) - F(x_{i-1:n}; \underline{\omega})$, where $F(x_{0:n}; \underline{\omega}) = 0$, $F(x_{n+1:n}; \underline{\omega}) = 1$ and $\sum_{i=0}^{n+1} D_i(x_{i:n}; \underline{\omega}) = 1$. The MPS estimators maximize the function

$$S(\underline{\omega}) = \left[\prod_{i=1}^{n+1} D_i(x_{i:n}; \underline{\omega}) \right]^{\frac{1}{n+1}},$$

or equivalently by maximizing the function

$$H(\underline{\omega}) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log(D_i(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})).$$

Alternatively, MPS estimators can be obtained by solving the equation

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(x_i; \underline{\omega})} [\Delta_s(x_{i:n}; \underline{\omega}) - \Delta_s(x_{i-1:n}; \underline{\omega})] = 0,$$

where $\Delta_s(\bullet; \underline{\omega})$ is defined in equation (1.11) and s is the number of parameters to be estimated.

1.4.6 Monte Carlo Simulation Study

Monte Carlo Simulation (MCS) is needed to evaluate the performance of model parameter estimates which were obtained through the mle technique. For different samples sizes, a sequence of n rvs is generated from the quantile function which is done through R software program. From the obtained random samples, we can then obtain the mean parameter estimates, RMSEs and ABIAS for different sample sizes. Considering that N samples were taken, the parameter estimate average is as follows: $\hat{\eta} = \frac{1}{N} \sum_{i=1}^N \hat{\eta}_i$. For an estimated parameter, such as $\hat{\eta}$, the RMSE and ABIAS are provided as $\text{RMSE}(\hat{\eta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\eta}_i - \eta)^2}{N}}$ and $\text{ABIAS}(\hat{\eta}) = \frac{\sum_{i=1}^N \hat{\eta}_i}{N} - \eta$. While the RMSEs and ABIAS should decrease as the sample size increases, consistent maximum likelihood estimators should show that the mean estimates are approaching the correct parameters.

1.4.7 Comparison of Model Performances

To evaluate the performance of model with q estimated parameters from an observed sample size n , we consider the following gof statistics: $-2\log\text{-likelihood}$, $AIC = 2q - 2\ell(\eta)$, $BIC = q\ln(n) - 2\ell(\eta)$, $AICC = AIC + 2\frac{q(q+1)}{n-q-1}$, where $\ell(\eta)$ is the log-likelihood function and q is the number of model parameters. We additionally take into account the W^* and A^* gof statistics (Chen and Balakrishnan, 1995). The Kolmogorov-Smirnov (K-S) statistic and associated p-value are also used to evaluate how closely the stated distribution function and empirical distribution function align (Berger and Zhou, 2014). Sum of Squares (SS) is provided as $SS = \sum_{j=1}^n \left[F(x_{(j)}; \hat{\eta}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2$ in the p-ps where for the ordered observed data $F(x_{j:n}; \hat{\eta})$ represent the CDF associated with parameters' vector $\hat{\eta}$.

Using graphical plots such as fitted densities, empirical CDFs, p-ps, HRF plots, and KM survival charts, model fit was also assessed. As compared to other models, a good fitting model for the provided dataset should have the lowest gof statistics value and the highest K-S p-value. Profile log-likelihood analysis will also be conducted to estimate the confidence intervals for the model parameters. By examining the profile log-likelihood curves, we will be able to determine the ranges of parameter values that are supported by the data and make more robust and informed inferences about the underlying population parameters. If the curves exhibit distinct maxima and well-defined confidence intervals, it suggests that the parameters are identifiable and can be

estimated with precision (Pawitan, 2001).

In addition to assessing the model fit, it is important to estimate the uncertainty associated with the parameter estimates. This is achieved through the estimation of variance-covariance (v-c) matrices and the calculation of confidence intervals (CIs) for the model parameters. The v-c matrix provides information about the variability and correlation among the estimated parameters. It allows us to quantify the uncertainty in the parameter estimates and assess the precision of the model (Cox and Reid, 2000; Verbeke, 1997). The elements of the v-c matrix provide insights into the relationships between the parameters and can be used to calculate standard errors and confidence intervals. Standard errors reflect the uncertainty in the parameter estimates and are typically derived from the square root of the diagonal elements of the v-c matrix. These standard errors are useful for hypothesis testing and assessing the statistical significance of the estimated parameters (McCullagh, 2019).

Additionally, CIs provide a range of plausible values for the true population parameter. They take into account the estimated parameter value, its standard error, and the desired level of confidence. In this study, v-c matrices will be estimated using appropriate statistical methods such as the observed Fisher information matrix technique (Lawless, 2011). Based on the estimated v-c matrix, CIs will be calculated for the model parameters. These CIs will provide valuable information about the uncertainty associated with the parameter estimates (Pawitan, 2001).

1.5 Aims and Objectives

In this study, we construct and develop new generalized families of distributions. Specifically, we consider the gamma-G, odd Burr X-G, Topp-Leone-G, exponentiated half logistic-G, Kumaraswamy-G and Marshall-Olkin-G generators to develop these generalized distributions. Specifically, we aim;

1. To develop the;
 - (a) exponentiated half logistic-Kumaraswamy-G (EHL-KUM-G) family,
 - (b) gamma odd Burr X-G (GOBX-G) family,
 - (c) Topp-Leone odd Burr X-G power series (TLOBX-GPS) class,
 - (d) Topp-Leone-exponentiated half logistic odd Burr X-G (TLEHLOBX-G) family,
 - (e) Marshall-Olkin-Topp-Leone odd Burr X-G (MOTLOBX-G) family.
2. To derive the statistical properties of these newly developed families of distributions.
3. To present some special cases for these distribution.
4. To estimate the parameters of these families of distributions using the Maximum Likelihood Estimation (MLE) technique.
5. To assess, through Monte Carlo simulation experiments, the performance of maximum likelihood estimators.

6. To utilize examples from actual data sets to demonstrate the versatility and applicability of these families of distributions.

1.6 Motivation of the Study

Existing distributions frequently don't offer a good fit to the real data encountered in fields like engineering, finance, biology, health, and economics. With that, researchers have been challenged with developing flexible lifetime distributions from classical ones that can fit heavy-tailed and skewed lifetime data. This has motivated us to develop new families of distributions to enable the generation of new distributions that have different shapes for the PDF and HRF to handle heavy-tailed and skewed real lifetime datasets. Thus, we develop these distributions with the primary goals of producing distributions with symmetric, left-skewed, right-skewed, J and reversed-J shapes, defining special models with a variety of HRFs, including monotonic and non-monotonic shapes, and offering better fits than several other equi-parameter models. The new distributions are expected to enhance the characteristics of traditional distributions such as the skewness and the kurtosis together.

1.7 Outline of the Study

The remaining chapters of the thesis are structured as follows: The exponentiated half logistic-Kumaraswamy-G (EHL-KUM-G) family is discussed in Chapter 2. Chapter 3 contain the gamma odd Burr X-G (GOBX-G) family. In Chapter 4, we present the Topp-Leone odd Burr X-G power Series (TLOBX-GPS) family. Chapter 5 contain the Topp-Leone-exponentiated half logistic-odd Burr X-G (TLEHLOBX-G) family. In Chapter 6, we present the Marshall-Olkin-Topp-Leone odd Burr X-G (MOTLOBX-G) family. The overall conclusions, limitations and further studies are presented in Chapter 7.

1.8 Summary of the Chapter

In this chapter, we have presented literature on some of the works that have been done in the development of statistical distributions. The literature focused more on literature related to the distribution that we are going to work with. Further, a brief discussion on some of the already developed important generators was presented. Moreover, some of the mathematical and statistical properties of lifetime distributions were discussed together with parameter estimation methods. To end the chapter, we discussed the aims and objectives of our thesis together with the motivation for our work.

2. The Exponentiated Half Logistic-Kumaraswamy-G Family

2.1 Introduction

The exponentiated half logistic-Kumaraswamy-G (EHL-KUM-G) family, a novel generalized family of lifespan models, is presented and examined in this chapter. We provide explicit formulations for the structural characteristics, such as the qf, linear representation, ordinary moments, and incomplete moments, MGFs, Rényi entropy, order statistics and pwms. Together with the examination of the forms of their densities and HRFs, special models were also given. A simulation study is undertaken to assess the consistency of maximum likelihood estimators, obtained based on the maximum likelihood estimation approach. The potential usefulness of this new distribution is demonstrated by fitting exponentiated half logistic-Kumaraswamy-Weibull (EHL-KUM-W) distribution to three real data sets.

2.2 The Exponentiated Half Logistic-Kumaraswamy-G Family

The CDF and PDF of the Exponential half logistic-G (EHL-G) distribution by [Cordeiro et al. \(2014\)](#) are given, respectively, as ($x \in \mathbb{R}$):

$$F_{EHL-G}(x; \lambda, \alpha, \underline{\varepsilon}) = \left[\frac{1 - \bar{G}^\lambda(x; \underline{\varepsilon})}{1 + \bar{G}^\lambda(x; \underline{\varepsilon})} \right]^\alpha \quad (2.1)$$

and

$$f_{EHL-G}(x; \lambda, \alpha, \underline{\varepsilon}) = 2\alpha\lambda g(x; \underline{\varepsilon}) \bar{G}^{\lambda-1}(x; \underline{\varepsilon}) \frac{\{1 - \bar{G}^\lambda(x; \underline{\varepsilon})\}^{\alpha-1}}{\{1 + \bar{G}^\lambda(x; \underline{\varepsilon})\}^{\alpha+1}}, \quad (2.2)$$

where $\lambda > 0$ and $\alpha > 0$, $\bar{G}(x; \underline{\varepsilon}) = 1 - G(x; \underline{\varepsilon})$, $G(x; \underline{\varepsilon})$ and $g(x; \underline{\varepsilon})$ are the baseline CDF and PDF, respectively, depending on the parameter vector $\underline{\varepsilon}$. We assume $\lambda = 1$ to avoid over parameterization, hence remaining with only one shape parameter, α . When $\lambda = 1$, Equations [\(2.1\)](#) and [\(2.2\)](#) becomes

$$F_{EHL-G}(x; \alpha, \underline{\varepsilon}) = \frac{G^\alpha(x; \underline{\varepsilon})}{[1 + \bar{G}(x; \underline{\varepsilon})]^\alpha} \quad (2.3)$$

and

$$f_{EHL-G}(x; \alpha, \underline{\varepsilon}) = 2\alpha g(x; \underline{\varepsilon}) \frac{G^{\alpha-1}(x; \underline{\varepsilon})}{[1 + \bar{G}(x; \underline{\varepsilon})]^{\alpha+1}}, \quad (2.4)$$

respectively.

Kumaraswamy-G (KUM-G) distributions' CDF and PDF provided by (Cordeiro and de Castro, 2009) are given as

$$F_{KUM-G}(x; a, b, \underline{\varepsilon}) = 1 - \psi^b(x; a, \underline{\varepsilon}) \quad (2.5)$$

and

$$f_{KUM-G}(x; a, b, \underline{\varepsilon}) = abg(x; \underline{\varepsilon})G^{a-1}(x; \underline{\varepsilon})\psi^{b-1}(x; a, \underline{\varepsilon}), \quad (2.6)$$

where $a > 0$, $b > 0$ and $\psi(x; a, \underline{\varepsilon}) = 1 - G^a(x; \underline{\varepsilon})$. The SF of KUM-G family follows as $\bar{F}(x; a, b, \underline{\varepsilon}) = \psi^b(x; a, \underline{\varepsilon})$. The function $g(x; \underline{\varepsilon})$ is the first derivative of $G(x; \underline{\varepsilon})$ with respect to x .

Substituting Equations (2.5) and (2.6) into Equations (2.3) and (2.4), we obtain exponentiated half logistic-Kumaraswamy-G (EHL-KUM-G) distribution with CDF and PDF taking the form

$$F(x; \underline{\Delta}) = \left[\frac{1 - \psi^b(x; a, \underline{\varepsilon})}{1 + \psi^b(x; a, \underline{\varepsilon})} \right]^\alpha \quad (2.7)$$

and

$$f(x; \underline{\Delta}) = 2ab\alpha g(x; \underline{\varepsilon})G^{a-1}(x; \underline{\varepsilon})\psi^{b-1}(x; a, \underline{\varepsilon}) \frac{[1 - \psi^b(x; a, \underline{\varepsilon})]^{\alpha-1}}{[1 + \psi^b(x; a, \underline{\varepsilon})]^{\alpha+1}}, \quad (2.8)$$

where $\underline{\Delta}$ is a vector of parameters α, a, b and $\underline{\varepsilon}$.

We obtain a qf of EHL-KUM-G family denoted $Q_X(u)$ by inventing

$$u = \left[\frac{1 - \psi^b(x; a, \underline{\varepsilon})}{1 + \psi^b(x; a, \underline{\varepsilon})} \right]^\alpha,$$

for $0 < u < 1$. Expressing the equation in terms of $\psi(x; a, \underline{\varepsilon})$, we get

$$\psi(x; a, \underline{\varepsilon}) = \left[\frac{1 - u^{1/\alpha}}{u^{1/\alpha} + 1} \right]^{1/b}.$$

Replacing $\psi(x; a, \underline{\varepsilon}) = (1 - G^a(x; \underline{\varepsilon}))$ and expressing the equation in terms of $G(x; \underline{\varepsilon})$, we get

$$G(x; \underline{\varepsilon}) = \left\{ 1 - \left[\frac{1 - u^{1/\alpha}}{u^{1/\alpha} + 1} \right]^{1/b} \right\}^{1/a}. \quad (2.9)$$

From Equation (2.9), the quantile values for the EHL-KUM-G family follows as

$$Q_X(u) = G^{-1} \left\{ \left[1 - \left[\frac{1 - u^{1/\alpha}}{u^{1/\alpha} + 1} \right]^{1/b} \right]^{1/a} \right\}, \quad 0 < u < 1,$$

for specified baseline CDF G . By setting $u = 0.5$, we get the median as

$$median = G^{-1} \left\{ \left[1 - \left[\frac{1 - 0.5^{1/\alpha}}{0.5^{1/\alpha} + 1} \right]^{1/b} \right]^{1/a} \right\}.$$

2.2.1 Some Sub-Families of EHL-KUM-G Family

The sub models for EHL-KUM-G family are presented in this section.

- For $\alpha = 1$, the CDF of the reduced EHL-KUM-G family reduces to $F(x; a, b, \underline{\varepsilon}) = \frac{1 - \psi^b(x; a, \underline{\varepsilon})}{1 + \psi^b(x; a, \underline{\varepsilon})}$, where $a, b > 0$ and parameter vector $\underline{\varepsilon}$.
- For $a = 1$, the CDF of the reduced EHL-KUM-G family reduces to $F(x; \alpha, b, \underline{\varepsilon}) = \left[\frac{1 - \bar{G}^b(x; \underline{\varepsilon})}{1 + \bar{G}^b(x; \underline{\varepsilon})} \right]^\alpha$, where $b, \alpha > 0$ and parameter vector $\underline{\varepsilon}$.
- For $b = 1$, the CDF of the reduced EHL-KUM-G family reduces to $F(x; \alpha, a, \underline{\varepsilon}) = \left[\frac{G^a(x; \underline{\varepsilon})}{2 - G^a(x; \underline{\varepsilon})} \right]^\alpha$, where $a, \alpha > 0$ and parameter vector $\underline{\varepsilon}$.
- For $\alpha = a = 1$, the CDF of the reduced EHL-KUM-G family reduces to $F(x; b, \underline{\varepsilon}) = \frac{1 - \bar{G}^b(x; \underline{\varepsilon})}{1 + \bar{G}^b(x; \underline{\varepsilon})}$, where $b > 0$ and parameter vector $\underline{\varepsilon}$.
- For $\alpha = b = 1$, the CDF of the reduced EHL-KUM-G family reduces to $F(x; a, \underline{\varepsilon}) = \frac{G^a(x; \underline{\varepsilon})}{2 - G^a(x; \underline{\varepsilon})}$, where $a > 0$ and parameter vector $\underline{\varepsilon}$.
- For $a = b = 1$, the CDF of the reduced EHL-KUM-G family reduces to $F(x; \alpha, \underline{\varepsilon}) = \left[\frac{G(x; \underline{\varepsilon})}{2 - G(x; \underline{\varepsilon})} \right]^\alpha$, where $\alpha > 0$ and parameter vector $\underline{\varepsilon}$.
- For $\alpha = a = b = 1$, the CDF of the reduced EHL-KUM-G family reduces to $F(x; \underline{\varepsilon}) = \frac{G(x; \underline{\varepsilon})}{2 - G(x; \underline{\varepsilon})}$, with parameter vector $\underline{\varepsilon}$.

2.2.2 Linear Representation

This section offers a helpful linear representation of the PDF of the distribution belonging to the EHL-KUM-G family. Using the series expansion $(1 - z)^n = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} z^k$ which is valid for $|z| < 1$ and $n > 0$, we write Equation (2.8) as

$$\begin{aligned} f(x; \underline{\Delta}) &= 2ab\alpha g(x; \underline{\varepsilon}) G^{a-1}(x; \underline{\varepsilon}) \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} [1 - G^a(x; \underline{\varepsilon})]^{b(i+1)-1} \\ &\quad \times \left[1 + [1 - G^a(x; \underline{\varepsilon})]^b \right]^{-(\alpha+1)}, \end{aligned}$$

which further expand into

$$\begin{aligned}
 f(x; \underline{\Delta}) &= 2ab\alpha g(x; \underline{\varepsilon}) G^{a-1}(x; \underline{\varepsilon}) \sum_{i,j=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \binom{-(\alpha+1)}{j} [1 - G^a(x; \underline{\varepsilon})]^{b(i+j+1)-1} \\
 &= 2ab\alpha \sum_{i,j,k=0}^{\infty} (-1)^{i+k} \binom{\alpha-1}{i} \binom{-(\alpha+1)}{j} \binom{b(i+j+1)-1}{k} g(x; \underline{\varepsilon}) G^{a(k+1)-1}(x; \underline{\varepsilon}) \\
 &= 2ab\alpha \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+k}}{a(k+1)} \binom{\alpha-1}{i} \binom{-(\alpha+1)}{j} \binom{b(i+j+1)-1}{k} \\
 &\quad \times a(k+1) g(x; \underline{\varepsilon}) G^{a(k+1)-1}(x; \underline{\varepsilon}) .
 \end{aligned}$$

As a result, the PDF of the EHL-KUM-G family may be written as

$$f(x; \underline{\Delta}) = \sum_{k=0}^{\infty} w(a, k) h_{a(k+1)}(x; \underline{\varepsilon}) , \quad (2.10)$$

where

$$w(a, k) = 2ab\alpha \sum_{i,j=0}^{\infty} \frac{(-1)^{i+k}}{a(k+1)} \binom{\alpha-1}{i} \binom{-(\alpha+1)}{j} \binom{b(i+j+1)-1}{k} \quad (2.11)$$

and $h_{a(k+1)}(x; \underline{\varepsilon}) = a(k+1)g(x; \underline{\varepsilon})G^{a(k+1)-1}(x; \underline{\varepsilon})$ is the density of Exp-G class with power parameter $a(k+1) > 0$. Thus, the EHL-KUM-G family may be described as an infinite linear combination of Exp-G densities.

2.3 Some Statistical Properties

We outline several statistical properties of the EHL-KUM-G family in this section. This includes the ordinary and incomplete moments, MGF, Rényi entropy, order statistics and pwms.

2.3.1 Ordinary Moments

From Equation (2.10), the r^{th} ordinary moment, denoted as μ'_r for EHL-KUM-G family is given by

$$\mu'_r = E(X^r) = \sum_{k=0}^{\infty} w(a, k) E \left[T_{a(k+1)}^r \right] ,$$

where the rv $T_{a(k+1)}$ with power parameter $a(k+1)$ follows an Exp-G class and $w(a, k)$ is give in Equation (2.11). Results for moments for the EHL-KUM-Weibull (EHL-KUM-W) distribution for various parameter values are shown in Table 2.1

Table 2.1: Moments of the EHL-KUM-W distribution for some parameter values

	(0.8, 0.8, 0.7, 0.5)	(2.0, 2.5, 0.9, 0.5)	(0.5, 2.0, 1.5, 1.5)	(0.9, 5.0, 1.8, 0.5)	(0.8, 4.5, 1.6, 0.4)
$E(X)$	0.1236	0.2299	0.3863	0.1394	0.0865
$E(X^2)$	0.0687	0.1375	0.2284	0.0491	0.0295
$E(X^3)$	0.0474	0.0971	0.1561	0.0259	0.0160
$E(X^4)$	0.0362	0.0748	0.1164	0.0168	0.0107
$E(X^5)$	0.0292	0.0607	0.0918	0.0122	0.0080
$E(X^6)$	0.0245	0.0510	0.0754	0.0095	0.0063
SD	0.0211	0.0440	0.0637	0.0077	0.0052
CV	0.0185	0.0386	0.0550	0.0065	0.0044
CS	0.0165	0.0344	0.0483	0.0056	0.0039
CK	0.0149	0.0311	0.0431	0.0050	0.0034

Plots of the skewness and kurtosis for the EHL-KUM-W distribution for various parameter value limits are shown in Figures 2.1 and 2.2. In Figure 2.1, when a and α are fixed, both skewness and kurtosis increases with increase in b and ϑ . Also, when b and ϑ are maintained constant, smaller values of a and α result in an increase in skewness and a reduction in kurtosis, as seen in Figure 2.2. This demonstrates that all four characteristics affect how skewness and kurtosis vary.

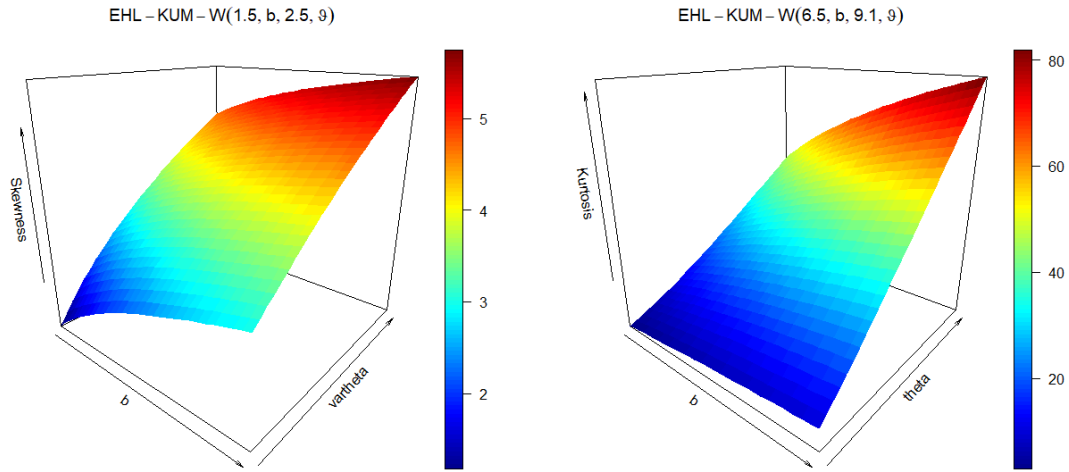


Figure 2.1: Plots of skewness and kurtosis for the EHL-KUM-W distribution for a few chosen parameter values

2.3.2 Incomplete Moments

The study of lifespan data and reliability can make use of various structural aspects of the distribution that can be formulated using incomplete moments. The s^{th} incomplete moment for the EHL-KUM-G family can be expressed from Equation (2.10) as

$$\varphi_s(t) = \int_{-\infty}^t x^s f(x; \underline{\Delta}) dx = \sum_{k=0}^{\infty} w(a, k) \int_{-\infty}^t x^s h_{a(k+1)}(x; \underline{\varepsilon}) dx. \quad (2.12)$$

Notably, the Bonferroni and Lorenz curves, which are helpful in science, economics, and finance, may be derived using $\text{varphi}_1(t)$. Equation (2.12) may be used to generate the generic form of $\text{varphi}_1(t)$ as follows:

$$\varphi_1(t) = \sum_{k=0}^{\infty} w(a, k) J_{a(k+1)}(t),$$

where $J_{a(k+1)}(t) = \int_{-\infty}^t x h_{a(k+1)}(x; \underline{\varepsilon}) dx$ represents the first incomplete moment of the Exp-G class with the power parameter $a(k+1)$, and the definition of $w(a, k)$ is found in Equation (2.11).

2.3.3 Moment Generating Function

Based on the basic definition of MGF, we derive MGF of X , $M_X(t) = E(e^{tX})$ as shown below

$$M_X(t) = \sum_{k=0}^{\infty} w(a, k) M_{a(k+1)}(t),$$

where $M_{a(k+1)}(t)$ is the generating function of $T_{a(k+1)}$, which may be found from the Exp-G.

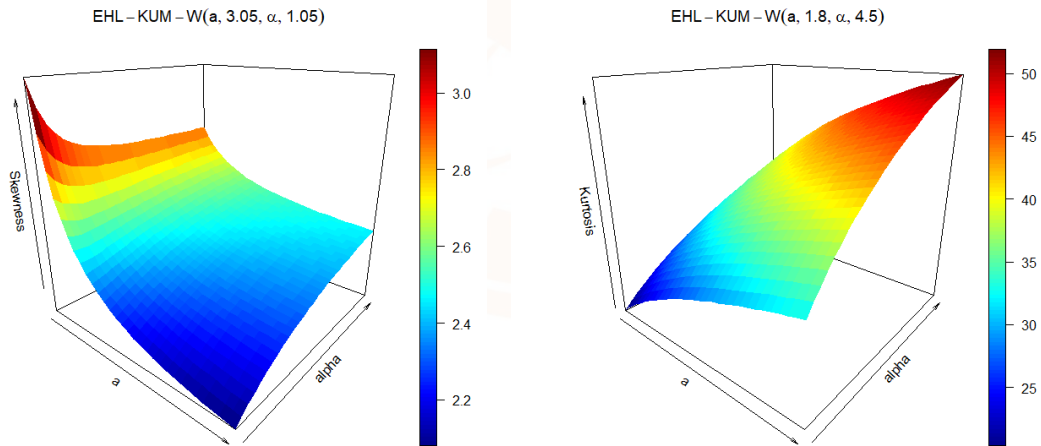


Figure 2.2: Plots of skewness and kurtosis for the EHL-KUM-W distribution for a few chosen parameter values

2.3.4 Entropy

The level of uncertainty's fluctuation is quantified by Rényi entropy. The Rényi entropy for a rv X with the EHL-KUM-G family is defined as

$$I_R(\theta) = \frac{1}{1-\theta} \log \left[\int_0^{\infty} f^{\theta}(x; \underline{\Delta}) dx \right],$$

where $\theta > 0$ and $\theta \neq 1$. Thus, from the EHL-KUM-G PDF in Equation (2.8), $f^\theta(x; \underline{\Delta})$ may be expressed as

$$f^\theta(x; \underline{\Delta}) = (2ab\alpha)^\theta g^\theta(x; \underline{\varepsilon}) G^{\theta(a-1)}(x; \underline{\varepsilon}) [1 - G^a(x; \underline{\varepsilon})]^{\theta(b-1)} \frac{[1 - [1 - G^a(x; \underline{\varepsilon})]^b]^{\theta(\alpha-1)}}{[1 + [1 - G^a(x; \underline{\varepsilon})]^b]^{\theta(\alpha+1)}}.$$

The result of using the generalized binomial series expansion gives

$$\begin{aligned} f^\theta(x; \underline{\Delta}) &= (2ab\alpha)^\theta g^\theta(x; \underline{\varepsilon}) G^{\theta a - \theta}(x; \underline{\varepsilon}) \sum_{i=0}^{\infty} (-1)^i \binom{\theta\alpha - \alpha}{i} [1 - G^a(x; \underline{\varepsilon})]^{b(i+\theta) - \theta} \\ &\quad \times [1 + [1 - G^a(x; \underline{\varepsilon})]^b]^{-\theta\alpha + \theta} \\ &= (2ab\alpha)^\theta g^\theta(x; \underline{\varepsilon}) G^{\theta a - \theta}(x; \underline{\varepsilon}) \sum_{i,j=0}^{\infty} (-1)^i \binom{\theta\alpha - \theta}{i} \binom{-\theta(\alpha + 1)}{j} \\ &\quad \times [1 - G^a(x; \underline{\varepsilon})]^{b(i+j+\theta) - \theta} \\ &= (2ab\alpha)^\theta \sum_{i,j,k=0}^{\infty} (-1)^{i+k} \binom{\theta(\alpha - 1)}{i} \binom{-\theta(\alpha + 1)}{j} \\ &\quad \times \binom{b(i+j+\theta) - \theta}{k} g^\theta(x; \underline{\varepsilon}) G^{a(k+\theta) - \theta}(x; \underline{\varepsilon}). \end{aligned}$$

Hence, the Rényi entropy of the EHL-KUM-G family is given by

$$\begin{aligned} I_R(\theta) &= \frac{1}{1-\theta} \log \left[(2ab\alpha)^\theta \sum_{i,j,k=0}^{\infty} (-1)^{i+k} \binom{\theta(\alpha - 1)}{i} \binom{-\theta(\alpha + 1)}{j} \right. \\ &\quad \times \left. \binom{b(i+j+\theta) - \theta}{k} \int_0^\infty \left[\left(\frac{a(k+\theta)}{\theta} \right) g(x; \underline{\varepsilon}) G^{\frac{a(k+\theta)}{\theta} - 1}(x; \underline{\varepsilon}) \right]^\theta dx \right] \\ &= \frac{1}{1-\theta} \log \left[\sum_{k=0}^{\infty} W_{\frac{a(k+\theta)}{\theta}} \exp[(1-\theta)I_{REG}] \right], \end{aligned}$$

where $I_{REG} = (1-\theta)^{-1} \log \left[\int_0^\infty \left[\left(\frac{a(k+\theta)}{\theta} \right) g(x; \underline{\varepsilon}) G^{\frac{a(k+\theta)}{\theta} - 1}(x; \underline{\varepsilon}) \right]^\theta dx \right]$ denotes Exp-G class Rényi entropy with power parameter $\frac{a(k+\theta)}{\theta}$ and

$$W_{\frac{a(k+\theta)}{\theta}} = (2ab\alpha)^\theta \sum_{i,j=0}^{\infty} \frac{(-1)^{i+k}}{\left(\frac{a(k+\theta)}{\theta} \right)^\theta} \binom{\theta(\alpha - 1)}{i} \binom{-\theta(\alpha + 1)}{j} \binom{b(i+j+\theta) - \theta}{k}.$$

2.3.5 Order Statistics

Allowing $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ to be EHL-KUM-G family n ordered values, we may express the order statistic PDF as

$$f_{l:n}(x; \underline{\Delta}) = \frac{f(x; \underline{\Delta})}{B(l, n-l+1)} \sum_{t=0}^{n-l} (-1)^t \binom{n-l}{t} F^{t+l-1}(x; \underline{\Delta}),$$

where $B(\cdot, \cdot)$ represent the beta function. From Equations (2.7) and (2.8), we get

$$\begin{aligned} f(x; \underline{\Delta}) F^{t+l-1}(x; \underline{\Delta}) &= 2ab\alpha g(x; \underline{\varepsilon}) G^{a-1}(x; \underline{\varepsilon}) [1 - G^a(x; \underline{\varepsilon})]^{b-1} \\ &\times \left[1 - [1 - G^a(x; \underline{\varepsilon})]^b \right]^{\alpha(t+l)-1} \left[1 + [1 - G^a(x; \underline{\varepsilon})]^b \right]^{-\alpha(t+l)-1}. \end{aligned}$$

Applying the generalized binomial series expansion, we obtain

$$\begin{aligned} f(x; \underline{\Delta}) F^{t+l-1}(x; \underline{\Delta}) &= 2ab\alpha g(x; \underline{\varepsilon}) G^{a-1}(x; \underline{\varepsilon}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i \binom{\alpha(t+l)-1}{i} \\ &\times \binom{-\alpha(t+l)-1}{j} [1 - G^a(x; \underline{\varepsilon})]^{b(i+j+1)-1} \\ &= 2ab\alpha g(x; \underline{\varepsilon}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i+k} \binom{\alpha(t+l)-1}{i} \\ &\times \binom{-\alpha(t+l)-1}{j} \binom{b(i+j+1)-1}{k} G^{a(k+1)-1}(x; \underline{\varepsilon}), \end{aligned}$$

which can further be expressed as

$$f(x; \underline{\Delta}) F^{t+l-1}(x; \underline{\Delta}) = \sum_{k=0}^{\infty} C(a, k) h_{a(k+1)}(x; \underline{\varepsilon}),$$

where

$$C(a, k) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{2ab\alpha (-1)^{i+k}}{a(k+1)} \binom{\alpha(t+l)-1}{i} \binom{-\alpha(t+l)-1}{j} \binom{b(i+j+1)-1}{k},$$

and $h_{a(k+1)}(x; \underline{\varepsilon}) = a(k+1)g(x; \underline{\varepsilon})G^{a(k+1)-1}(x; \underline{\varepsilon})$ is the density of the Exp-G class with power parameter $a(k+1) > 0$. In light of this, the l^{th} order statistic's PDF is shown as

$$f_{l:n}(x; \underline{\Delta}) = \sum_{k=0}^{\infty} \sum_{t=0}^{n-l} \frac{(-1)^t \binom{n-l}{t}}{B(l, n-l+1)} C(a, k) h_{a(k+1)}(x; \underline{\varepsilon}).$$

As a result, the Exp-G densities are combined infinitely linearly to form the PDF of the l^{th} order statistic from the EHL-KUM-G family.

2.3.6 Probability Weighted Moments

The $(r, s)^{th}$ probability weighted moments (PMWs) for the EHL-KUM-G family, say $\eta_{(r,s)}$, is given by

$$\eta_{(r,s)} = \int_{-\infty}^{\infty} x^r f(x; \underline{\Delta}) F^s(x; \underline{\Delta}) dx,$$

where

$$\begin{aligned} f(x; \underline{\Delta}) F^s(x; \underline{\Delta}) &= 2ab\alpha g(x; \underline{\varepsilon}) G^{a-1}(x; \underline{\varepsilon}) [1 - G^a(x; \underline{\varepsilon})]^{b-1} \\ &\quad \times \left[1 - [1 - G^a(x; \underline{\varepsilon})]^b \right]^{\alpha(s+1)-1} \left[1 + [1 - G^a(x; \underline{\varepsilon})]^b \right]^{-\alpha(s+1)-1}. \end{aligned}$$

Using the generalized binomial series expansion, we obtain

$$\begin{aligned} f(x; \underline{\Delta}) F^s(x; \underline{\Delta}) &= 2ab\alpha g(x; \underline{\varepsilon}) G^{a-1}(x; \underline{\varepsilon}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i \binom{\alpha(s+1)-1}{i} \\ &\quad \times \binom{-\alpha(s+1)-1}{j} [1 - G^a(x; \underline{\varepsilon})]^{b(i+j+1)-1} \\ &= 2ab\alpha g(x; \underline{\varepsilon}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i+k} \binom{\alpha(s+1)-1}{i} \\ &\quad \times \binom{-\alpha(s+1)-1}{j} \binom{b(i+j+1)-1}{k} G^{a(k+1)-1}(x; \underline{\varepsilon}), \end{aligned}$$

which can further be expressed as

$$f(x; \underline{\Delta}) F^s(x; \underline{\Delta}) = \sum_{k=0}^{\infty} d(a, k) h_{a(k+1)}(x; \underline{\varepsilon}),$$

where

$$d(a, k) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{2ab\alpha (-1)^{i+k}}{a(k+1)} \binom{\alpha(s+1)-1}{i} \binom{-\alpha(s+1)-1}{j} \binom{b(i+j+1)-1}{k}.$$

In light of this, the PMWs of the EHL-KUM-G family are

$$\eta_{(r,s)} = \sum_{k=0}^{\infty} d(a, k) \int_{-\infty}^{\infty} x^r h_{a(k+1)}(x; \underline{\varepsilon}) dx = \sum_{k=0}^{\infty} d(a, k) E \left[T_{a(k+1)}^r \right].$$

2.4 Some Special Models

In this section, four special cases of the EHL-KUM-G family are discussed. For our study, we consider Weibull, Fréchet, Burr III and log-logistic distributions as baseline distributions.

Although, we can generate many lifetime statistical distributions from the EHL-KUM-G family.

2.4.1 Exponentiated Half Logistic-Kumaraswamy-Weibull Distribution

Assuming the baseline distribution for EHL-KUM-G family is Weibull distribution with PDF $g(x; \vartheta) = \vartheta x^{\vartheta-1} e^{-x^\vartheta}$ and CDF $G(x; \vartheta) = 1 - e^{-x^\vartheta}$, for $\vartheta > 0$ and $x > 0$, Exponentiated Half Logistic-Kumaraswamy-Weibull (EHL-KUM-W) distribution is obtained using the provided CDF and PDF below.

$$F(x; a, b, \alpha, \vartheta) = \left[\frac{1 - [1 - \psi_1^a(x; \theta)]^b}{1 + [1 - \psi_1^a(x; \theta)]^b} \right]^\alpha, x > 0, a, b, \alpha, \vartheta > 0$$

and

$$f(x; a, b, \alpha, \vartheta) = 2ab\alpha\vartheta x^{\vartheta-1} \psi_1^{a-1}(x; \theta) e^{-x^\vartheta} [1 - \psi_1^a(x; \theta)]^{b-1} \\ \times \frac{[1 - [1 - \psi_1^a(x; \theta)]^b]^{\alpha-1}}{[1 + [1 - \psi_1^a(x; \theta)]^b]^{\alpha+1}},$$

for $\psi_1(x; \theta) = 1 - e^{-x^\theta}$. Plots of the PDF and HRF for the EHL-KUM-W distribution are given in Figure 2.3.

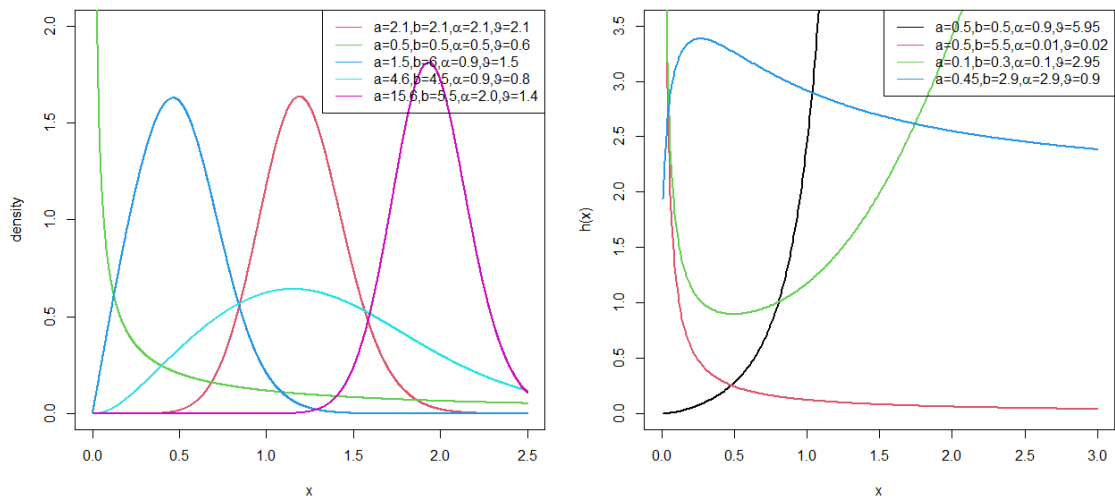


Figure 2.3: EHL-KUM-W distribution's PDF and HRF plots

The PDF and HRF graphs for the EHL-KUM-W distribution for various parameters are displayed in Figure 2.3. The PDF exhibits several forms, including a reverse-J, a unimodal shape, a nearly symmetric shape, and left- and right-skewed shapes. The HRF plots also exhibit increasing, decreasing, bathtub and upside-down bathtub shapes.

2.4.2 Exponentiated Half Logistic-Kumaraswamy-Fréchet Distribution

If the baseline distribution is Fréchet distribution with PDF $g(x; c, \vartheta) = \vartheta c^\vartheta x^{-(\vartheta+1)} \exp[-(\frac{c}{x})^\vartheta]$ and CDF $G(x; c, \vartheta) = \exp[-(\frac{c}{x})^\vartheta]$, for $c, \vartheta > 0$ and $x > 0$, Exponentiated Half Logistic-Kumaraswamy-Fréchet (EHL-KUM-F) distribution is obtained using the provided CDF and PDF below.

$$F(x; a, b, \alpha, c, \vartheta) = \left[\frac{1 - [1 - \psi_2^a(x; c, \vartheta)]^b}{1 + [1 - \psi_2^a(x; c, \vartheta)]^b} \right]^\alpha, x > 0, a, b, \alpha, c, \vartheta > 0$$

and

$$f(x; a, b, \alpha, c, \vartheta) = 2ab\alpha\vartheta c^\vartheta x^{-(\vartheta+1)} \psi_2^a(x; c, \vartheta) [1 - \psi_2^a(x; c, \vartheta)]^{b-1} \times \frac{[1 - [1 - \psi_2^a(x; c, \vartheta)]^b]^{\alpha-1}}{[1 + [1 - \psi_2^a(x; c, \vartheta)]^b]^{\alpha+1}},$$

for $\psi_2(x; c, \vartheta) = \exp\left[-\left(\frac{c}{x}\right)^\vartheta\right]$. Plots of the PDF and HRF for the EHL-KUM-F distribution are given in Figure 2.4.

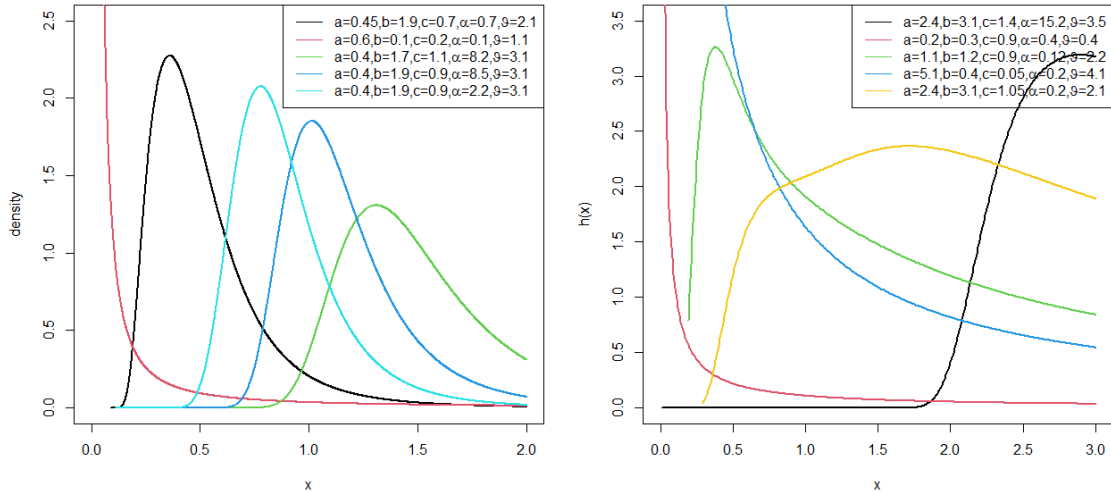


Figure 2.4: EHL-KUM-F distribution's PDF and HRF plots

The PDF and HRF graphs for the EHL-KUM-F distribution for various parameter values are shown in Figure 2.4. The reverse-J, uni-modal, and left-skewed forms, among others, may be found in many data sets that the PDF is applicable to. The EHL-KUM-F HRF graphs exhibit upside-down bathtub, reverse-J, decreasing, increasing, and uni-modal forms.

2.4.3 Exponentiated Half Logistic-Kumaraswamy-Burr III Distribution

If the baseline distribution in EHL-KUM-G family of distributions is Burr III distribution with PDF $g(x; \lambda, \beta) = \lambda\beta x^{-\lambda-1}(1 + x^{-\lambda})^{-\beta-1}$ and CDF $G(x; \lambda, \beta) = (1 + x^{-\lambda})^{-\beta}$, for $\lambda, \beta > 0$ and

$x > 0$, Exponentiated Half Logistic-Kumaraswamy-Burr III (EHL-KUM-BIII) distribution is obtained using the provided CDF and PDF below.

$$F(x; a, b, \alpha, \lambda, \beta) = \left[\frac{1 - \left[1 - \psi_3^{-\beta a}(x; \lambda)\right]^b}{1 + \left[1 - \psi_3^{-\beta a}(x; \lambda)\right]^b} \right]^\alpha, x > 0, a, b, \alpha, \lambda, \beta > 0$$

and

$$f(x; a, b, \alpha, \lambda, \beta) = 2ab\alpha\lambda\beta x^{-\lambda-1}\psi_3^{-\beta a-1}(x; \lambda) \left[1 - \psi_3^{-\beta a}(x; \lambda)\right]^{b-1} \\ \times \frac{\left[1 - \left[1 - \psi_3^{-\beta a}(x; \lambda)\right]^b\right]^{\alpha-1}}{\left[1 + \left[1 - \psi_3^{-\beta a}(x; \lambda)\right]^b\right]^{\alpha+1}},$$

for $\psi_3(x; \lambda) = 1 + x^{-\lambda}$. Figure 2.5 provides plots for the EHL-KUM-BIII distribution's PDF and HRF.

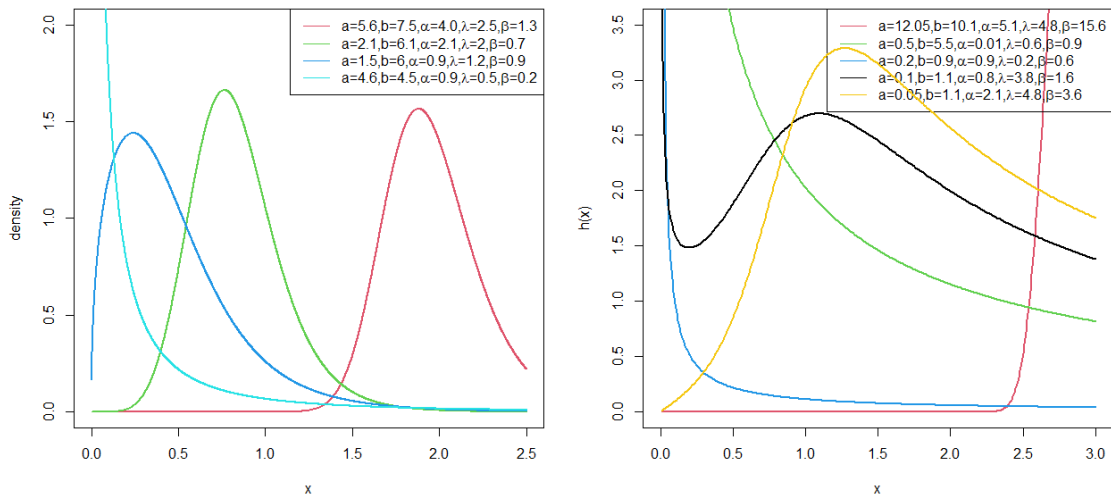


Figure 2.5: EHL-KUM-BIII distribution's PDF and HRF plots

The PDF and HRF graphs for the EHL-KUM-F distribution for various parameter values are shown in Figure 2.5. The PDF can appear in a variety of forms, including the reverse J, uni-modal, almost symmetric, left- and right-skew forms. The EHL-KUM-BIII HRF graphs show forms like a bathtub followed by an upside-down bathtub, as well as shapes like increasing, decreasing, and uni-modal.

2.4.4 Exponentiated Half Logistic-Kumaraswamy-Log Logistic Distribution

Taking the baseline distribution as log-logistic distribution with PDF $g(x; c) = cx^{c-1}(1+x^c)^{-2}$ and CDF $G(x; c) = 1 - (1+x^c)^{-1}$, for $c > 0$ and $x > 0$, Exponentiated Half Logistic-Kumaraswamy-Log Logistic (EHL-KUM-LLo) distribution is obtained using the provided CDF

and PDF below.

$$F(x; a, b, c, \alpha) = \left[\frac{1 - [1 - \psi_4^a(x; c)]^b}{1 + [1 - \psi_4^a(x; c)]^b} \right]^\alpha, x > 0, a, b, c, \alpha > 0$$

and

$$f(x; a, b, c, \alpha) = 2abc\alpha x^{c-1}(1+x^c)^{-2}\psi_4^{a-1}(x; c)[1 - \psi_4^a(x; c)]^{b-1} \\ \times \frac{[1 - [1 - \psi_4^a(x; c)]^b]^{\alpha-1}}{[1 + [1 - \psi_4^a(x; c)]^b]^{\alpha+1}},$$

for $\psi_4(x; c) = 1 - (1 + x^c)^{-1}$. The EHL-KUM-LLo distribution's PDF and HRF plots are shown in Figure 2.6

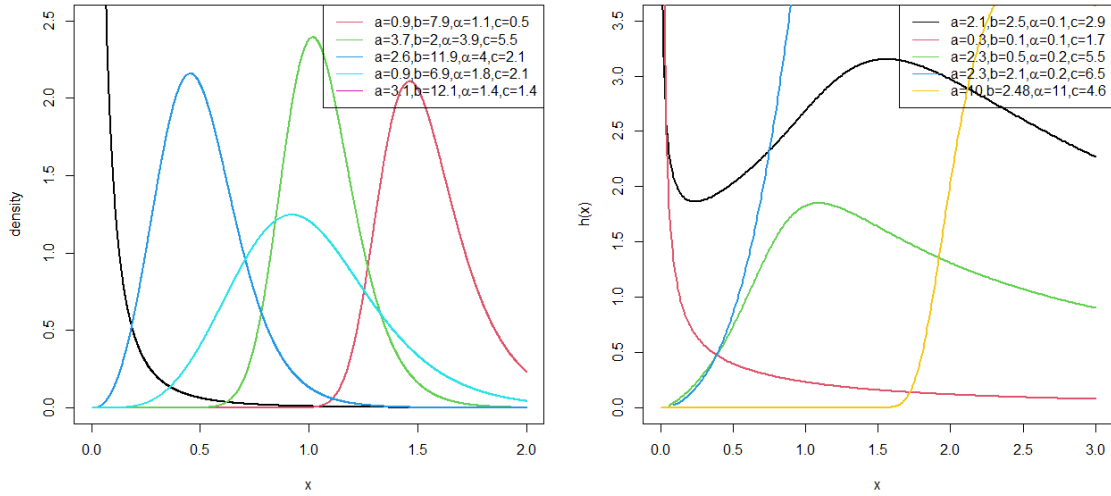


Figure 2.6: EHL-KUM-LLo distribution's PDF and HRF plots

The PDF and HRF graphs for the EHL-KUM-F distribution for various parameter values are shown in Figure 2.6. The PDF can assume a number of forms, such as a reverse-J, a unimodal shape, a nearly symmetric shape, and shapes that are skewed to the left or right. Moreover, the EHL-KUM-LLo HRF graphs show bathtub shapes that are increasing, decreasing, followed by upside-down.

Table 2.2 presents results on quantiles for the EHL-KUM-W distribution for different parameter values.

Table 2.2: Quantiles values of the EHL-KUM-W distribution for some parameter values

u	$(a, b, \alpha, \vartheta)$				
	(0.5,1.2,0.9,1.1)	(2,2.5,0.9,0.8)	(0.6,0.6,1.1,2.5)	(0.9,5,1.8,1.2)	(1.2,4.5,1.2,0.5)
0.1	0.0217	0.2050	0.5031	0.1321	0.0113
0.2	0.0799	0.3575	0.7271	0.1935	0.0311
0.3	0.1670	0.5056	0.8981	0.2460	0.0579
0.4	0.2804	0.6580	1.0452	0.2963	0.0928
0.5	0.4213	0.8220	1.1811	0.3479	0.1383
0.6	0.5963	1.0068	1.3146	0.4039	0.1989
0.7	0.8201	1.2283	1.4547	0.4692	0.2845
0.8	1.1286	1.5208	1.6153	0.5531	0.4181
0.9	1.6381	1.9937	1.8343	0.6843	0.6815

2.5 Parameter Estimation

In the EHL-KUM-G family, let $x_1, x_2, \dots, x_n$ be an observed random sample of size n with parameter vector $\underline{\Delta} = (\alpha, a, b, \underline{\varepsilon})^T$. The log-likelihood of the parameter vector $\underline{\Delta}$, say $\ell(\underline{\Delta})$, can be expressed as

$$\begin{aligned} \ell(\underline{\Delta}) = & n(\ln(2ab\alpha)) + \sum_{j=1}^n \ln(g(x_j; \underline{\varepsilon})) + (a-1) \sum_{j=1}^n \ln(G(x_j; \underline{\varepsilon})) \\ & + (b-1) \sum_{j=1}^n \ln[1 - G^a(x_j; \underline{\varepsilon})] + (\alpha-1) \sum_{j=1}^n \ln \left[1 - [1 - G^a(x_j; \underline{\varepsilon})]^b \right] \\ & - (\alpha+1) \sum_{j=1}^n \ln \left[1 + [1 - G^a(x_j; \underline{\varepsilon})]^b \right]. \end{aligned}$$

The Appendix [B.1](#) contains the first derivatives of the log-likelihood differentiated with respect to each element of the parameter vector $\underline{\Delta}$. We use R software to find solutions to $U(\underline{\Delta}) = 0$ with specified baseline distribution, where $U(\underline{\Delta}) = (U_a(\underline{\Delta}), U_b(\underline{\Delta}), U_\alpha(\underline{\Delta}), U_{\underline{\varepsilon}_k}(\underline{\Delta}))$ is the score vectors. The multivariate normal distribution, $N_p(\mathbf{0}, J(\hat{\underline{\Delta}})^{-1})$, may be used to get model parameter confidence intervals and regions. The mean vector and the observed information matrix are represented by $\mathbf{0} = (0, 0, 0, \mathbf{0})^T$ and $J(\hat{\underline{\Delta}})^{-1}$ evaluated at $\hat{\underline{\Delta}}$. Additionally, $\hat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\Delta})}$, $\hat{a} \pm Z_{\frac{\eta}{2}} \sqrt{I_{aa}^{-1}(\hat{\Delta})}$, $\hat{b} \pm Z_{\frac{\eta}{2}} \sqrt{I_{bb}^{-1}(\hat{\Delta})}$ and $\hat{\underline{\varepsilon}}_k \pm Z_{\frac{\eta}{2}} \sqrt{I_{\underline{\varepsilon}_k \underline{\varepsilon}_k}^{-1}(\hat{\Delta})}$ are the approximate 100(1- η)% two sided confidence intervals for α , a , b and $\underline{\varepsilon}_k$, respectively. The diagonal elements of $I_n^{-1}(\hat{\Delta}) = (nI(\hat{\Delta}))^{-1}$ are given as $I_{\alpha\alpha}^{-1}(\hat{\Delta})$, $I_{aa}^{-1}(\hat{\Delta})$, $I_{bb}^{-1}(\hat{\Delta})$ and $I_{\underline{\varepsilon}_k \underline{\varepsilon}_k}^{-1}(\hat{\Delta})$, whereas $Z_{\frac{\eta}{2}}$ is the standard normal $(\frac{\eta}{2})^{th}$ percentile.

2.6 Simulation Study

This section contains simulation results that are repeated $N = 1000$ times for the sample sizes $n = 25, 50, 100, 200, 400$, and 800 to test the consistency of the maximum likelihood estimators

for α, a, b and ϑ in the EHL-KUM-W distribution. The root mean square errors (RMSEs) and average bias (Abias) should decay to zero as n increases, while the mean estimates for consistent maximum likelihood estimators should be approaching the true parameters which is what we have in Table 2.3 and 2.4

Table 2.3: EHL-KUM-W distribution MCS results

Parameter	Sample Size	(0.6, 0.2, 0.5, 2.3)			(0.5, 0.2, 0.5, 2.5)		
		Mean	RMSE	Abias	Mean	RMSE	Abias
a	25	0.8572	1.4914	0.2572	0.7405	1.7189	0.2405
	50	0.5815	0.6391	-0.0185	0.5121	0.5343	0.0121
	100	0.5814	0.3338	-0.0186	0.5038	0.2841	0.0038
	200	0.5949	0.1928	-0.0051	0.5110	0.1614	0.0110
	400	0.6298	0.1326	0.0298	0.5259	0.0918	0.0259
	800	0.6338	0.0936	0.0338	0.5445	0.0933	0.0445
b	25	0.5124	1.4967	0.3124	0.4478	1.0665	0.2478
	50	0.3486	0.3173	0.1486	0.3281	0.3845	0.1281
	100	0.2836	0.1448	0.0836	0.2772	0.1340	0.0772
	200	0.2542	0.1070	0.0542	0.2502	0.0980	0.0502
	400	0.2100	0.0402	0.0100	0.2050	0.0305	0.0050
	800	0.1979	0.0184	-0.0021	0.2026	0.0277	0.0026
α	25	0.9725	1.0574	0.4725	1.0136	1.1405	0.5136
	50	0.9777	1.0006	0.4777	0.8882	0.8263	0.3882
	100	0.7620	0.6763	0.2620	0.6904	0.4333	0.1904
	200	0.6001	0.2677	0.1001	0.5880	0.2109	0.0880
	400	0.5090	0.0977	0.0090	0.5043	0.0623	0.0043
	800	0.4952	0.0463	-0.0048	0.4943	0.0670	-0.0057
ϑ	25	2.0889	0.5771	-0.2111	2.2666	0.5961	-0.2334
	50	2.0462	0.4616	-0.2538	2.2767	0.4735	-0.2233
	100	2.1361	0.3529	-0.1639	2.3240	0.3598	-0.1760
	200	2.1758	0.2808	-0.1242	2.3746	0.2881	-0.1254
	400	2.2672	0.1281	-0.0328	2.4815	0.1177	-0.0185
	800	2.2871	0.0705	-0.0129	2.4802	0.0855	-0.0198

Table 2.4: EHL-KUM-W distribution MCS results

Parameter	Sample Size	(0.6, 0.5, 0.5, 2.3)			(0.50.50.52.5)		
		Mean	RMSE	Abias	Mean	RMSE	Abias
a	25	1.3154	2.5207	0.7154	0.9689	1.9249	0.4689
	50	0.7949	1.3066	0.1949	0.6692	1.0488	0.1692
	100	0.7334	0.7166	0.1334	0.6828	0.7334	0.1828
	200	0.6862	0.3129	0.0862	0.6144	0.2762	0.1144
	400	0.6879	0.2570	0.0879	0.6442	0.2556	0.1442
	800	0.6889	0.2026	0.0889	0.6230	0.1953	0.1230
b	25	2.0632	7.3247	1.5632	1.6922	7.2205	1.1922
	50	0.9718	3.2134	0.4718	0.9146	3.9493	0.4146
	100	0.5999	1.5968	0.0999	0.5609	1.3656	0.0609
	200	0.4850	0.1715	-0.0150	0.4775	0.1632	-0.0225
	400	0.4712	0.1320	-0.0288	0.4617	0.1305	-0.0383
	800	0.4735	0.1062	-0.0265	0.4654	0.1002	-0.0346
α	25	0.9894	1.4404	0.4894	0.8843	1.3603	0.3843
	50	0.8781	1.0390	0.3781	0.7481	0.7151	0.2481
	100	0.6545	0.6082	0.1545	0.5546	0.4124	0.0546
	200	0.5454	0.3668	0.0454	0.4996	0.2677	-0.0004
	400	0.4980	0.2133	-0.0020	0.4498	0.1709	-0.0502
	800	0.4735	0.1428	-0.0265	0.4419	0.1324	-0.0581
ϑ	25	2.3694	0.9476	0.0694	2.7242	1.0852	0.2242
	50	2.3726	0.7294	0.0726	2.6558	0.8438	0.1558
	100	2.4538	0.6149	0.1538	2.6987	0.6617	0.1987
	200	2.4258	0.4703	0.1258	2.6623	0.5232	0.1623
	400	2.4060	0.3464	0.1060	2.6377	0.3894	0.1377
	800	2.3858	0.2735	0.0858	2.6100	0.2875	0.1100

2.7 Applications

The EHL-KUM-W distribution's adaptability for data fitting is illustrated in this section using three examples from actual data. We take into account the following GOF statistics to assess the effectiveness of our model: GOF statistics $-2\ln(L)$, AIC, BIC, AICC, W^* , A^* , SS K-S and its p-value where the value of the likelihood function is represented with $L = L(\hat{\Delta})$. A good fitting model for the given dataset should have the smallest value of GOF statistics and a large K-S p-value compared to other models.

In addition to GOF tests, graphical presentations have also been provided to demonstrate the data fitting flexibility of the EHL-KUM-W distribution. This includes: density plots, empirical CDF plots, p-ps and HRF plots. EHL-KUM-W distribution is compared with the following models: Kumaraswamy-Weibull (KUM-W) distribution (Cordeiro et al., 2010), beta generalized

Lindley (BGL) distribution (Oluyede and Yang 2015), Weibull Lomax (WL) distribution (Tahir et al., 2015), half logistic (HL) distribution (Johnson et al. 1995), Weibull (W) distribution and some of its sub-models include the EHL-KUM-W($a, b, 1, \vartheta$) and EHL-KUM-W($1, b, 1, \vartheta$) distributions. The PDFs of these distributions are given by

$$f_{KUM-W}(x) = abc\lambda^c x^{c-1} e^{-(\lambda x)^c} (1 - e^{-(\lambda x)^c})^{a-1} \{1 - [1 - e^{-(\lambda x)^c}]^a\}^{b-1}, \quad (2.13)$$

for $a, b, c, \lambda > 0$ and $x > 0$.

$$\begin{aligned} f_{BGL}(x) &= \frac{\alpha\theta^2(1+x)e^{-\theta x}}{B(a, b)(1+\theta)} \left[1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}\right]^{\alpha-1} \\ &\times \left[1 - \left(1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}\right)^a\right]^{b-1}, \end{aligned} \quad (2.14)$$

for $\alpha, \theta, a, b > 0$ and $x > 0$.

$$\begin{aligned} f_{WL}(x) &= \frac{ab\alpha}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{b\alpha-1} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1} \\ &\times \exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^\alpha - 1\right\}\right\}, \end{aligned} \quad (2.15)$$

for $\alpha, \beta, a, b > 0$ and $x > 0$.

$$f_{HL}(x) = \frac{2\lambda e^{-\lambda x}}{(1 + e^{-2\lambda x})^2},$$

for $\lambda > 0$ and $x > 0$.

The first data used to illustrate the flexibility of EHL-KUM-W distribution was analyzed by Fayomi (2019). The data had a minimum, median mean, maximum of 1.1, 1.7, 1.9 and 4.1, respectively, and it is given as: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0. The data has $n = 20$ cases of the relief times (in minutes) of 20 patients receiving an analgesic.

The second data used was analyzed by Eissa and Abdulaziz (2017). The data had a minimum, median mean, maximum of 0.036, 1.414, 3.343 and 15.080, respectively, and it is given as: 0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183, 0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904, 0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228, 3.931, 9.337, 0.086, 0.379, 1.600, 4.073, 10.940, 0.102, 0.381, 2.006, 4.393, 11.020, 0.103, 0.538, 2.054, 4.534, 13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116, 0.574, 3.058, 6.274, 15.080. In the data, there are $n = 50$ instances of 50 component failure times (per 1000 hours).

The third data used was analyzed by Al-Ta'ani and Gharaibeh (2023). The data had a minimum, median mean, maximum of 6.00, 12.00, 13.69 and 26.20, respectively, and it is given as: 12, 10, 8, 10, 10, 8, 7, 10, 6, 10, 16, 10, 12, 11, 11, 18, 18, 12, 9, 16, 15, 11, 16, 19, 22, 16, 23, 25, 26. In the data, there are $n = 29$ observations representing the number of new deaths caused by COVID-19 in Jordan from January 31, 2021 to February 28, 2021.

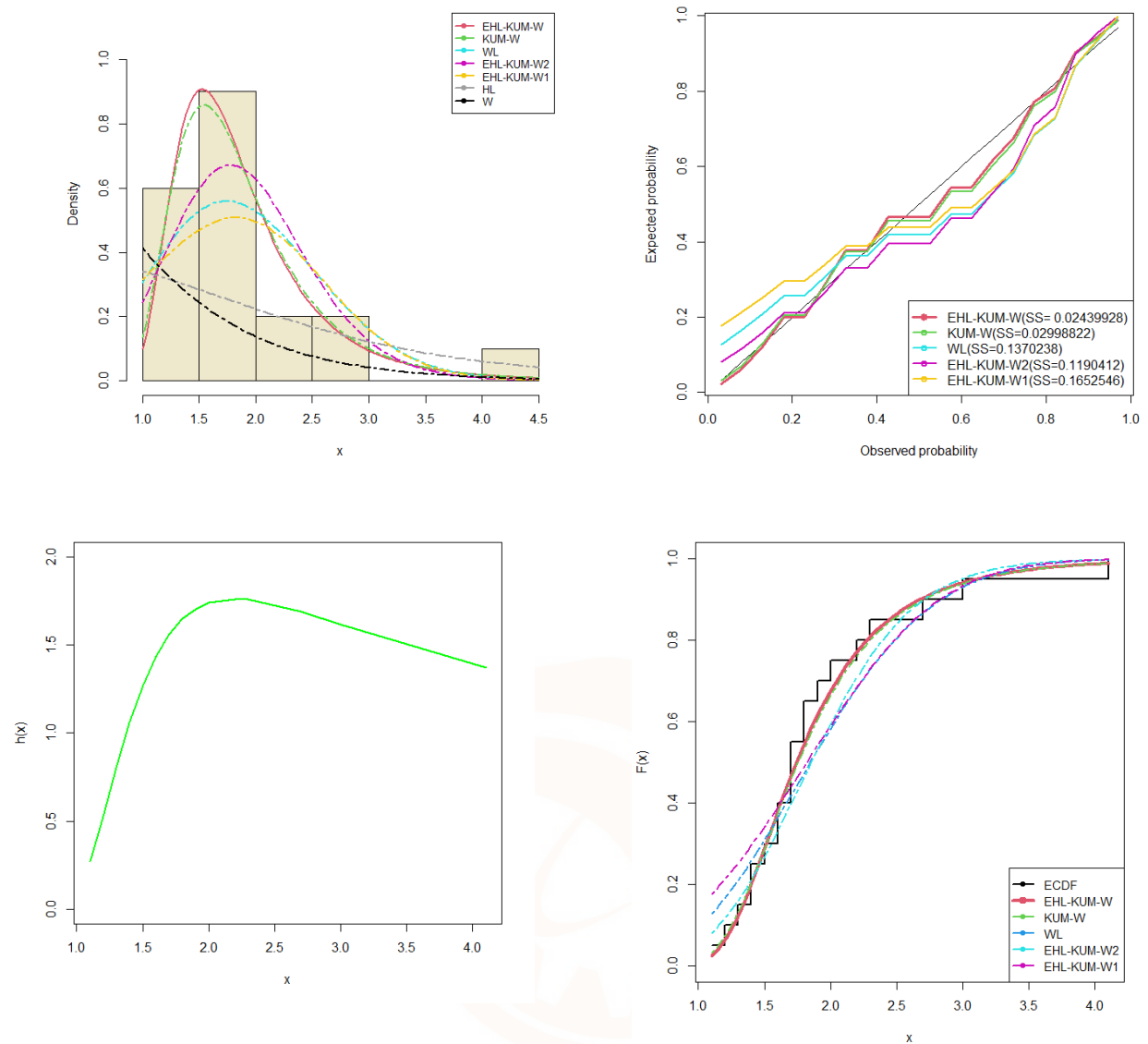


Figure 2.7: Fitted EHL-KUM-W density, probability plot, HRF and empirical CDF plots for relief times dataset

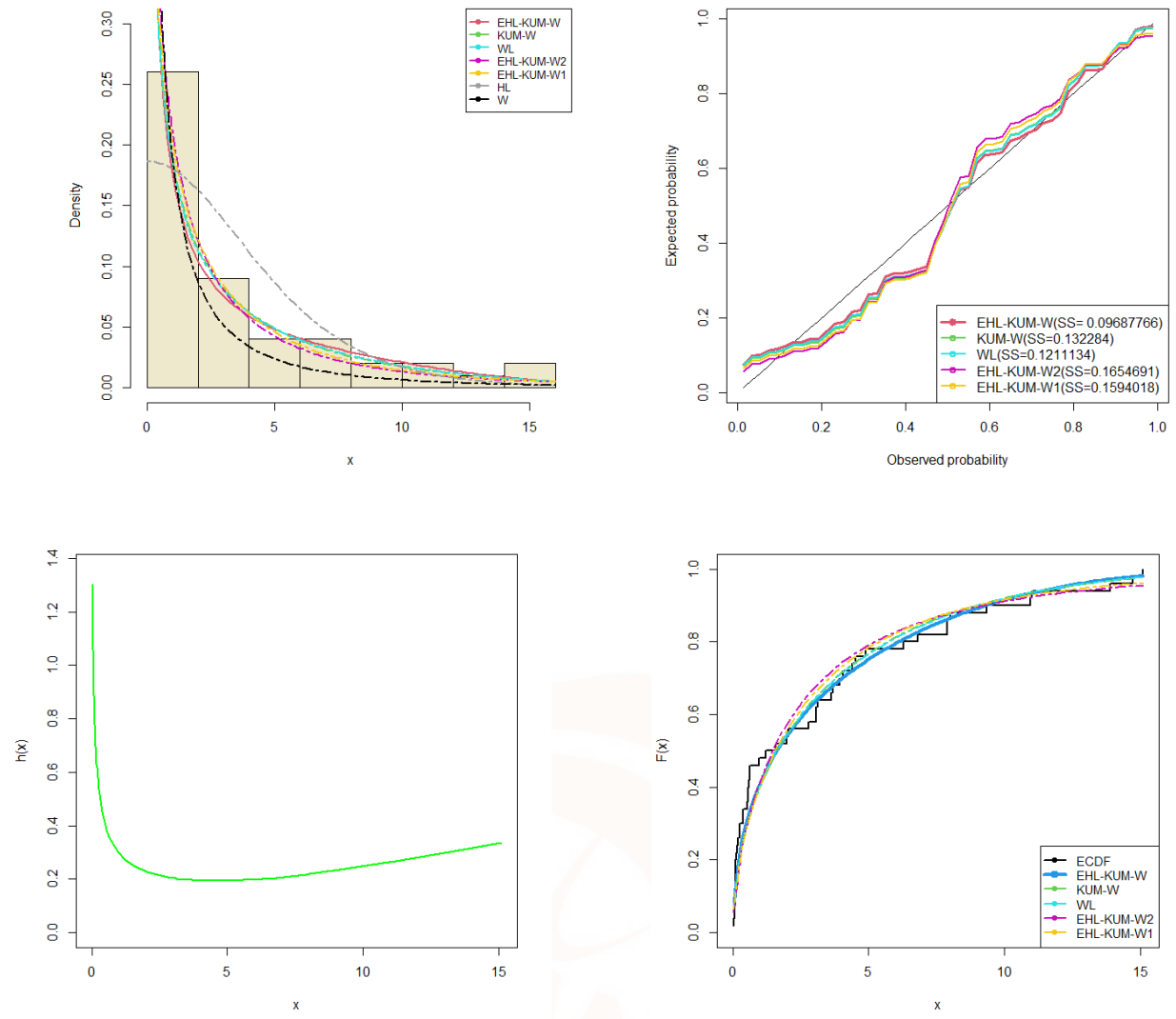


Figure 2.8: Fitted EHL-KUM-W density, probability plot, HRF and empirical CDF plots for failure times dataset

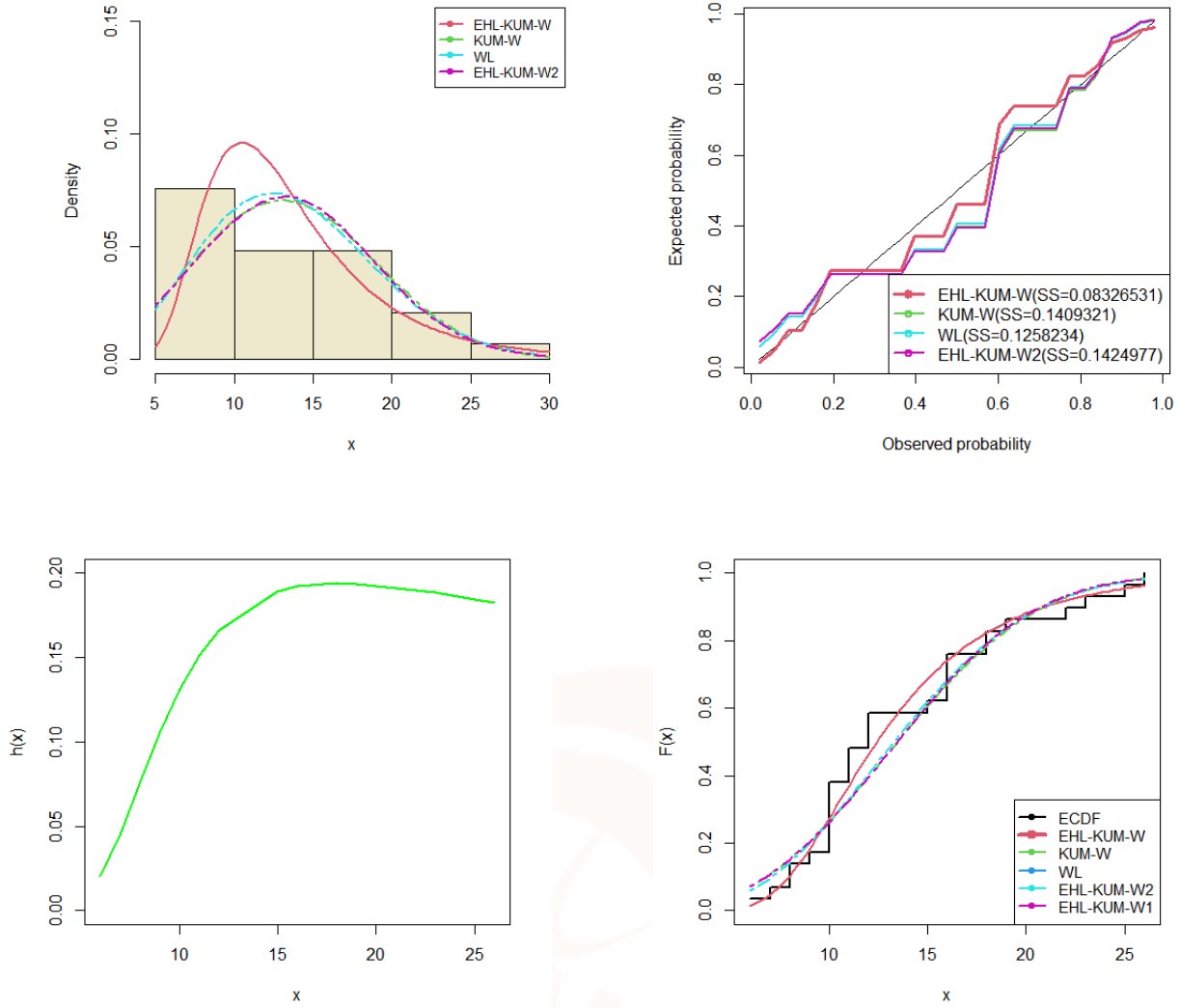


Figure 2.9: Fitted EHL-KUM-W density, probability plot, HRF and empirical CDF plots for COVID-19 dataset

The density, empirical CDF, and p-ps in Figures 2.7, 2.8 and 2.9 shows how best EHL-KUM-W distribution fits skewed data. The HRF plots also show bathtub shape and upside-down bathtubs shapes. The estimated var-cov matrix for the EHL-KUM-W model is given for the COVID-19 data by

$$\begin{bmatrix} 1.81 \times 10^{-3} & 2.47 \times 10^{-3} & -1.04 \times 10^{-3} & -5.87 \times 10^{-1} \\ 2.47 \times 10^{-3} & 3.39 \times 10^{-3} & -1.43 \times 10^{-3} & -8.09 \times 10^{-1} \\ -1.04 \times 10^{-3} & -1.43 \times 10^{-3} & 6.11 \times 10^{-7} & 3.45 \times 10^{-4} \\ -5.87 \times 10^{-1} & -8.09 \times 10^{-1} & 3.45 \times 10^{-4} & 1.95 \times 10^{-1} \end{bmatrix}$$

and the model parameters 95% confidence regions are: $a \in [9.47 \pm 1.80 \times 10^{-1}]$, $b \in [5.24 \times 10^2 \pm 7.44 \times 10^{-4}]$, $\alpha \in [2.44 \times 10^3 \pm 2.29 \times 10^{-5}]$ and $\vartheta \in [8.41 \times 10^{-2} \pm 2.68 \times 10^{-2}]$.

Table 2.5: The parameter estimates of the EHL-KUM-W and competing models' and GOF statistics for the relief times data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
EHL-KUM-W	a	b	α	ϑ	30.9	38.9	41.7	42.9	0.0316	0.1795	0.1054	0.9793	0.0244
	9.47 (9.20×10^{-2})	5.24×10^2 (3.80×10^{-4})	2.44×10^3 (1.17×10^{-5})	8.41×10^{-2} (1.37×10^{-2})									
KUM-W	a	b	c	λ	31.3	39.3	41.9	43.2	0.0355	0.2037	0.1171	0.9469	0.0300
	9.62×10^3 (2.13×10^{-5})	1.31 (4.86×10^{-1})	3.58×10^{-1} (6.13×10^{-3})	2.89×10^2 (7.49×10^{-4})									
BGL	α	θ	a	b	162.5	170.5	173.2	174.5	0.0887	0.5230	0.5123	5.53×10^{-5}	1.5710
	1.20×10^{-1} (1.27×10^{-2})	1.52×10^{-7} (4.34×10^{-6})	3.02×10^{-1} (1.31×10^{-3})	1.01×10^1 (3.49×10^{-5})									
WL	a	b	α	β	38.9	46.9	49.5	50.9	0.1501	0.8869	0.1764	0.5620	0.1370
	3.30×10^3 (5.90×10^{-4})	5.33 (5.36)	1.19×10^{-1} (5.74×10^{-2})	4.90×10^{-1} (1.83)									
EHL-KUM-W ($a, b, 1, \vartheta$)	a	b	1	ϑ	36.6	42.6	44.1	45.6	0.1216	0.7157	0.1881	0.4791	0.1190
	5.44 (2.40)	1.21 (1.38)	-	1.42 (0.62)									
EHL-KUM-W (1, $b, 1, \vartheta$)	1	b	1	ϑ	42.1	46.1	46.8	48.1	0.1936	1.1308	0.1753	0.5703	0.1652
	-	0.29 (0.11)	-	2.25 (0.37)									
HL	λ	-	-	-	59.3	61.3	61.5	62.3	0.1189	0.7037	0.4127	0.0021	0.8226
	0.80 (0.14)												
W	ϑ	-	-	-	75.4	77.4	77.6	78.4	0.1159	0.6866	0.6714	2.95×10^{-8}	3.1462
	1.12 (0.15)												

The estimated var-cov matrix for the EHL-KUM-W model is given for the failure times data by

$$\begin{bmatrix} 5.52 \times 10^1 & -9.85 \times 10^{-1} & -4.41 \times 10^{-1} & 3.45 \times 10^{-1} \\ -9.85 \times 10^{-1} & 1.76 \times 10^{-2} & 7.87 \times 10^{-3} & -6.16 \times 10^{-3} \\ -4.41 \times 10^{-1} & 7.87 \times 10^{-3} & 3.84 \times 10^{-3} & -2.77 \times 10^{-3} \\ 3.45 \times 10^{-1} & -6.16 \times 10^{-3} & -2.77 \times 10^{-3} & 2.33 \times 10^{-3} \end{bmatrix}$$

and the model parameters 95% confidence regions are: $a \in [2.49 \times 10^1 \pm 1.46 \times 10^1]$, $b \in [1.43 \times 10^2 \pm 2.59 \times 10^{-1}]$, $\alpha \in [1.25 \times 10^{-1} \pm 1.21 \times 10^{-1}]$ and $\vartheta \in [2.37 \times 10^{-1} \pm 9.46 \times 10^{-2}]$.

Table 2.6: The parameter estimates of the EHL-KUM-W and competing models' and GOF statistics for the failure times data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
	a	b	α	ϑ									
EHL-KUM-W	2.49×10^1 (7.43)	1.43×10^2 (1.33×10^{-1})	1.25×10^{-1} (6.19×10^{-2})	2.37×10^{-1} (4.82×10^{-2})	200.8	208.8	209.7	216.4	0.1112	0.7489	0.1226	0.4070	0.0969
KUM-W	1.53×10^{-1} (3.09×10^{-1})	3.05 (3.83)	3.67 (6.79)	3.61×10^{-2} (4.36×10^{-2})	203.0	211.0	211.9	218.6	0.1339	0.8669	0.1359	0.2871	0.1323
BGL	1.19×10^{-1} (7.73×10^{-3})	2.84×10^{-7} (2.59×10^{-6})	3.02×10^{-1} (7.09×10^{-4})	1.01×10^1 (2.24×10^{-5})	361.4	369.5	370.4	377.2	0.1592	0.9857	0.4575	4.31×10^{-10}	3.2251
WL	1.90 (2.73×10^{-1})	5.71×10^{-1} (6.29×10^{-2})	1.12×10^4 (3.91×10^{-5})	1.16×10^5 (3.78×10^{-6})	202.5	210.5	211.4	218.1	0.1288	0.8400	0.1315	0.3240	0.1211
EHL-KUM-W ($a, b, 1, \vartheta$)	a 1.77 (2.61)	b 1.50 (2.57)	1 - -	ϑ 4.12×10^{-1} (3.83×10^{-1})	205.7	211.7	212.2	217.4	0.1610	1.0044	0.1314	0.3247	0.1655
EHL-KUM-W ($1, b, 1, \vartheta$)	1 - -	b 8.50×10^{-1} (4.11×10^{-1})	1 - -	ϑ 5.65×10^{-1} (6.82×10^{-2})	206.0	209.9	210.2	218.8	0.1589	1.0012	0.1396	0.2588	0.1594
HL	λ 3.73×10^{-1} (4.64×10^{-2})	-	-	-	233.8	235.8	235.8	237.7	0.1605	1.0455	0.3401	1.15×10^{-5}	1.3162
W	ϑ 5.12×10^{-1} (4.71×10^{-2})	-	-	-	218.3	220.3	220.3	222.2	0.1553	0.9695	0.2567	0.0021	1.0700

The estimated var-cov matrix for the EHL-KUM-W model is given for the COVID-19 data as

$$\begin{bmatrix} 8.47 \times 10^{-3} & -3.49 \times 10^{-5} & 1.08 \times 10^{-6} & 9.34 \times 10^{-4} \\ -3.49 \times 10^{-5} & 1.44 \times 10^{-7} & -4.44 \times 10^{-9} & -3.85 \times 10^{-6} \\ 1.08 \times 10^{-6} & -4.44 \times 10^{-9} & 1.37 \times 10^{-10} & 1.19 \times 10^{-7} \\ 9.34 \times 10^{-4} & -3.85 \times 10^{-6} & 1.19 \times 10^{-7} & 1.87 \times 10^{-4} \end{bmatrix}$$

and the model parameters 95% confidence regions are: $a \in [2.32 \times 10^{-1} \pm 2.63]$, $b \in [1.03 \pm 3.61]$, $\alpha \in [2.13 \times 10^2 \pm 1.53 \times 10^{-3}]$ and $\vartheta \in [6.22 \times 10^{-1} \pm 8.65 \times 10^{-1}]$.

Table 2.7: The parameter estimates of the EHL-KUM-W and competing models' and GOF statistics for the COVID-19 data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
	a	b	α	ϑ									
EHL-KUM-W	2.32×10^{-1} (1.34)	1.03 (1.84)	2.13×10^2 (7.82×10^{-4})	6.22×10^{-1} (4.41×10^{-1})	173.2	181.2	182.9	186.9	0.0831	0.4392	0.1239	0.7652	0.0833
	a	b	c	λ									
KUM-W	3.44 (2.35×10^{-2})	4.44×10^4 (1.08×10^{-8})	8.13×10^{-1} (1.07×10^{-1})	1.45×10^{-3} (7.39×10^{-4})	177.5	185.5	187.2	190.9	0.1590	0.8900	0.1898	0.2471	0.1409
	α	θ	a	b									
BGL	1.11×10^{-1} (9.59×10^{-3})	1.23×10^{-8} (1.86×10^{-6})	3.00×10^{-1} (8.50×10^{-4})	1.01×10^1 (2.51×10^{-5})	340.1	348.4	350.1	353.9	0.0968	0.5009	0.5035	8.25×10^{-7}	2.2398
	a	b	α	β									
WL	2.73×10^1 (5.12×10^{-2})	5.23 (2.79)	2.06×10^{-1} (1.16×10^{-1})	2.21 (4.31)	176.14	184.1	185.80	189.61	0.1417	0.7752	0.1811	0.2974	0.1258
	a	b	1	ϑ									
EHL-KUM-W ($a, b, 1, \vartheta$)	34.05 (3.06)	4.44×10^3 (1.21×10^{-3})	-	1.63×10^{-1} (1.72×10^{-2})	177.9	183.9	184.9	188.0	0.1628	0.9087	0.1907	0.2421	0.1425
	1	b	1	ϑ									
EHL-KUM-W ($1, b, 1, \vartheta$)	-	0.11 (0.04)	-	0.99 (0.14)	201.9	205.9	206.4	208.6	0.1277	0.6817	0.3279	0.0039	0.7645
	λ												
HL	0.11 (0.02)	-	-	-	201.1	203.1	203.3	204.5	0.1284	0.6855	0.3439	0.0021	0.8593
	ϑ												
W	0.31 (0.04)	-	-	-	298.3	203.1	203.3	204.5	0.1284	0.6856	0.3439	2.10×10^{-3}	6.3039

Based on the results in Tables [2.5](#), [2.6](#) and [2.7](#), the statistics for the GOF were lowest for EHL-KUM-W as compared to nested and non-nested models for -2LogL, AIC, AICC, BIC, W^* , A^* , K-S, and SS. Additionally, it had the largest K-S p-value for each dataset. This shows that it provides a better fit to relief times and failure times datasets as compared to other models.

2.8 Concluding Remarks

The exponentiated half logistic Kumaraswamy-G (EHL-KUM-G) distribution is established in this chapter. Quantile function, linear representation, moments, Rényi entropy and order statistics of this family were studied. We considered special cases with some baseline distributions called the Weibull, Fréchet, Burr III and log-logistic distributions. These distributions' parameters were estimated using the maximum likelihood technique. A simulation study is done to evaluate the precision and consistency of maximum likelihood estimates using the Weibull distribution as baseline. Three real world data sets are used to illustrate the flexibility and applicability of exponentiated half logistic-Kumaraswamy-Weibull (EHL-KUM-W) distribution. EHL-KUM-W distribution displayed a good fit to the real life datasets on the plots that are presented. The GOF values for the EHL-KUM-W distribution are the lowest as compared to other competing models.

3. The Gamma Odd Burr X-G Family

3.1 Introduction

In this chapter, a new distribution called gamma odd Burr X-G (GOBX-G) family is introduced. Its structural properties such as the survival function, hazard function, density expansion, QF, moments and generating functions, incomplete moments, PWMs, Rényi entropy and order statistics were derived. Maximum likelihood technique is used to estimate the parameters of this model. The flexibility and applicability of this model is demonstrated using real life datasets.

3.2 The Gamma Odd Burr X-G Family

Yousof et al. (2017a) developed a one parameter odd Burr X-G (OBX-G) family using a one parameter Burr type X distribution introduced by Burr (1942). The CDF and PDF of OBX-G family are given by

$$\begin{aligned} F_{OBX-G}(x; \theta, \xi) &= 2\theta \int_0^{\frac{G(x; \xi)}{\bar{G}(x; \xi)}} t \exp(-t^2) [1 - \exp(-t^2)]^{\theta-1} dt \\ &= \left\{ 1 - \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right] \right\}^{\theta}, x \in \mathbb{R}, \theta > 0 \end{aligned}$$

and

$$f_{OBX-G}(x; \theta, \xi) = \frac{2\theta g(x; \xi) G(x; \xi)}{\bar{G}^3(x; \xi)} \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right] \left\{ 1 - \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right] \right\}^{\theta-1},$$

respectively, where $\bar{G}(x; \xi) = 1 - G(x; \xi)$; $G(x; \xi)$ and $g(x; \xi)$ are the CDF and PDF of any baseline distribution. Ristić and Balakrishnan (2012) developed the gamma-G (G-G) family with CDF and PDF given by

$$F(x; \delta) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log(G(x))} t^{\delta-1} e^{-t} dt, x \in \mathbb{R}, \delta > 0, \quad (3.1)$$

and

$$f(x; \delta) = \frac{1}{\Gamma(\delta)} [-\log(G(x))]^{\delta-1} g(x), x \in \mathbb{R}. \quad (3.2)$$

In Equations (3.1) and (3.2), if we let $G(x) = F_{OBX-G}(x; \theta, \xi)$ and $g(x) = f_{OBX-G}(x; \theta, \xi)$, we get a new distribution called the gamma odd Burr X-G (GOBX-G) distribution with the CDF

and PDF given as

$$\begin{aligned} F(x; \delta, \theta, \xi) &= 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log \left(\left\{ 1 - \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right] \right\}^\theta \right)} t^{\delta-1} e^{-t} dt \\ &= 1 - \frac{\gamma \left(\delta, -\theta \log \left(1 - \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right] \right) \right)}{\Gamma(\delta)} \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} f(x; \delta, \theta, \xi) &= \frac{2\theta g(x; \xi) G(x; \xi)}{\Gamma(\delta) \bar{G}^3(x; \xi)} \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right] \left\{ 1 - \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right] \right\}^{\theta-1} \\ &\times \left[-\theta \log \left(1 - \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right] \right) \right]^{\delta-1}, \quad x \in \mathbb{R}, \delta, \theta > 0 \end{aligned} \quad (3.4)$$

respectively, where ξ being a vector of parameters from the baseline distribution, where $\gamma(\delta, z) = \int_0^z t^{\delta-1} e^{-t} dt$ is the lower incomplete gamma function.

3.2.1 Series Expansion of the GOBX-G Density Function

A series expansion of the GOBX-G density follows in this section. If we let $y = \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right]$, then we can write Equation (3.4) as

$$f(x; \delta, \theta, \xi) = \frac{2\theta^\delta g(x; \xi) G(x; \xi)}{\Gamma(\delta) \bar{G}^3(x; \xi)} y (1-y)^{\theta-1} [-\log(1-y)]^{\delta-1}.$$

Using series representation $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, for $y \in (0, 1)$, we have

$$[-\log(1-y)]^{\delta-1} = y^{\delta-1} \left[\sum_{m=0}^{\infty} \binom{\delta-1}{m} y^m \left(\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^m \right].$$

Thus, we can write the GOBX-G density as

$$f(x; \delta, \theta, \xi) = \frac{2\theta^\delta g(x; \xi) G(x; \xi)}{\Gamma(\delta) \bar{G}^3(x; \xi)} y^\delta (1-y)^{\theta-1} \left[\sum_{m=0}^{\infty} \binom{\delta-1}{m} y^m \left(\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^m \right].$$

Next, let $a_s = (s+2)^{-1}$ and considering the result of a power series raised to a positive integer we get

$$\left(\sum_{s=0}^{\infty} a_s y^s \right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \quad (3.5)$$

where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^{\infty} [m(l+1) - s] a_l b_{s-l,m}$ and $b_{0,m} = a_0^m$ (Gradshteyn and Ryzhik, 1980). The GOBX-G density function then simplifies to

$$\begin{aligned} f(x; \delta, \theta, \xi) &= \frac{2\theta^\delta g(x; \xi) G(x; \xi)}{\Gamma(\delta) \bar{G}^3(x; \xi)} y^\delta (1-y)^{\theta-1} \left[\sum_{m,s=0}^{\infty} \binom{\delta-1}{m} y^m b_{s,m} y^s \right] \\ &= \frac{2\theta^\delta g(x; \xi) G(x; \xi)}{\Gamma(\delta) \bar{G}^3(x; \xi)} (1-y)^{\theta-1} \left[\sum_{m,s=0}^{\infty} \binom{\delta-1}{m} y^{m+s+\delta} b_{s,m} \right]. \end{aligned}$$

Applying the generalized binomial series representation $(1-z)^n = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} z^i$ which is valid for $|z| < 1$, we can express the GOBX-G density as

$$\begin{aligned} f(x; \delta, \theta, \xi) &= \frac{2\theta^\delta g(x; \xi) G(x; \xi)}{\Gamma(\delta) \bar{G}^3(x; \xi)} \left[\sum_{m,r,s=0}^{\infty} (-1)^r \binom{\theta-1}{r} \binom{\delta-1}{m} y^{m+s+\delta+r} b_{s,m} \right] \\ &= \frac{2\theta^\delta g(x; \xi) G(x; \xi)}{\Gamma(\delta) \bar{G}^3(x; \xi)} \sum_{m,r,s=0}^{\infty} (-1)^r b_{s,m} \binom{\theta-1}{r} \binom{\delta-1}{m} \\ &\quad \times \exp \left[-(m+s+\delta+r) \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right]. \end{aligned}$$

Applying the exponential series representation $\exp(-z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!}$, the density further expand to

$$\begin{aligned} f(x; \delta, \theta, \xi) &= \frac{2\theta^\delta g(x; \xi) G(x; \xi)}{\Gamma(\delta) \bar{G}^3(x; \xi)} \sum_{m,r,s=0}^{\infty} (-1)^r b_{s,m} \binom{\theta-1}{r} \binom{\delta-1}{m} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(-1)^k (m+s+\delta+r)^k}{k!} \frac{G^{2k}(x; \xi)}{\bar{G}^{2k}(x; \xi)} \\ &= \frac{2\theta^\delta}{\Gamma(\delta)} \sum_{k,m,r,s=0}^{\infty} \frac{(-1)^{r+k} b_{s,m} (m+s+\delta+r)^k}{k!} \binom{\theta-1}{r} \binom{\delta-1}{m} \frac{g(x; \xi) G^{2k+1}(x; \xi)}{\bar{G}^{2k+3}(x; \xi)}. \end{aligned}$$

Applying the generalized binomial series representation in $G^{2k+1}(x; \xi) = [1 - \bar{G}(x; \xi)]^{2k+1}$, we get

$$\begin{aligned} f(x; \delta, \theta, \xi) &= \frac{2\theta^\delta}{\Gamma(\delta)} \sum_{j,k,m,r,s=0}^{\infty} \frac{(-1)^{r+j+k} b_{s,m} (m+s+\delta+r)^k}{k!} \\ &\quad \times \binom{\theta-1}{r} \binom{\delta-1}{m} \binom{2k+1}{j} g(x; \xi) \bar{G}^{j-(2k+3)}(x; \xi) \\ &= \frac{2\theta^\delta}{\Gamma(\delta)} \sum_{j,k,m,p,r,s=0}^{\infty} \frac{(-1)^{r+j+k+p} b_{s,m} (m+s+\delta+r)^k}{k!} \binom{\theta-1}{r} \\ &\quad \times \binom{\delta-1}{m} \binom{2k+1}{j} \binom{j-(2k+3)}{p} g(x; \xi) G^p(x; \xi). \end{aligned}$$

The GOBX-G density function can finally be expressed as an infinite linear combination of Exponentiated G (Exp-G) density functions as

$$f(x; \delta, \theta, \xi) = \sum_{p=0}^{\infty} w_{p+1} h_{p+1}(x; \xi), \quad (3.6)$$

where

$$w_{p+1} = \frac{2\theta^\delta}{\Gamma(\delta)} \sum_{j,k,m,r,s=0}^{\infty} \frac{(-1)^{r+j+k+p} b_{s,m}(m+s+\delta+r)^k}{(p+1)k!} \times \binom{\theta-1}{r} \binom{\delta-1}{m} \binom{2k+1}{j} \binom{j-(2k+3)}{p} \quad (3.7)$$

and $h_{p+1}(x; \xi) = (p+1)g(x; \xi)G^{p+1-1}(x; \xi)$ is the Exp-G PDF with power parameter $(p+1)$. Thus, the statistical properties of GOBX-G family are evident from those of the Exp-G distributions.

3.2.2 Quantile Function

This section present the QF for the GOBX-G family. The QF can be derived from the CDF of GOBX-G family in Equation (3.3) by solving the non-linear equation

$$u = 1 - \frac{\gamma \left(\delta, -\theta \log \left(1 - \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] \right) \right)}{\Gamma(\delta)},$$

for $0 < u < 1$. Thus,

$$\gamma \left(\delta, -\theta \log \left(1 - \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] \right) \right) = (1-u)\Gamma(\delta),$$

which follows that

$$-\theta \log \left(1 - \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] \right) = \gamma^{-1}(\delta, (1-u)\Gamma(\delta)).$$

This simplifies to

$$\left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 = -\ln \left[1 - \exp \left(\frac{\gamma^{-1}(\delta, (1-u)\Gamma(\delta))}{-\theta} \right) \right].$$

Replacing $\overline{G}(x; \xi)$ with $1 - G(x; \xi)$, we get

$$\frac{G(x; \xi)}{1 - G(x; \xi)} = \left[-\ln \left[1 - \exp \left(\frac{\gamma^{-1}(\delta, (1-u)\Gamma(\delta))}{-\theta} \right) \right] \right]^{1/2},$$

which finally gives the QF of the GOBX-G family as

$$Q(u) = G^{-1} \left\{ \frac{\left[-\ln \left[1 - \exp \left(\frac{\gamma^{-1}(\delta, (1-u)\Gamma(\delta))}{-\theta} \right) \right] \right]^{1/2}}{1 + \left[-\ln \left[1 - \exp \left(\frac{\gamma^{-1}(\delta, (1-u)\Gamma(\delta))}{-\theta} \right) \right] \right]^{1/2}} \right\}. \quad (3.8)$$

3.3 Special Cases

In this section, GOBX-G family takes on some selected baseline distributions even though any baseline distribution can be considered. For our study, we consider Weibull distribution, log-logistic distribution and Uniform distribution.

3.3.1 Gamma Odd Burr X-Weibull Distribution

Considering the baseline distribution of the GOBX-G family as Weibull distribution with PDF $g(x; \alpha) = \alpha x^{\alpha-1} e^{-x^\alpha}$ and CDF $G(x; \alpha) = 1 - e^{-x^\alpha}$, we respectively obtain the PDF and CDF of the gamma odd Burr X-Weibull (GOBX-W) distribution as

$$f(x; \delta, \theta, \alpha) = \frac{2\theta\alpha x^{\alpha-1} e^{2x^\alpha} (1 - e^{-x^\alpha})}{\Gamma(\delta)} \exp \left[- (e^{x^\alpha} - 1)^2 \right] \left\{ 1 - \exp \left[- (e^{x^\alpha} - 1)^2 \right] \right\}^{\theta-1} \times \left[-\theta \log \left(1 - \exp \left[- (e^{x^\alpha} - 1)^2 \right] \right) \right]^{\delta-1}, x > 0, \delta, \theta, \alpha > 0 \quad (3.9)$$

and

$$F(x; \delta, \theta, \alpha) = 1 - \frac{\gamma \left(\delta, -\theta \log \left(1 - \exp \left[- (e^{x^\alpha} - 1)^2 \right] \right) \right)}{\Gamma(\delta)}.$$

Plots of the PDF and HRF of GOBX-W distribution are given in Figure [3.1](#)

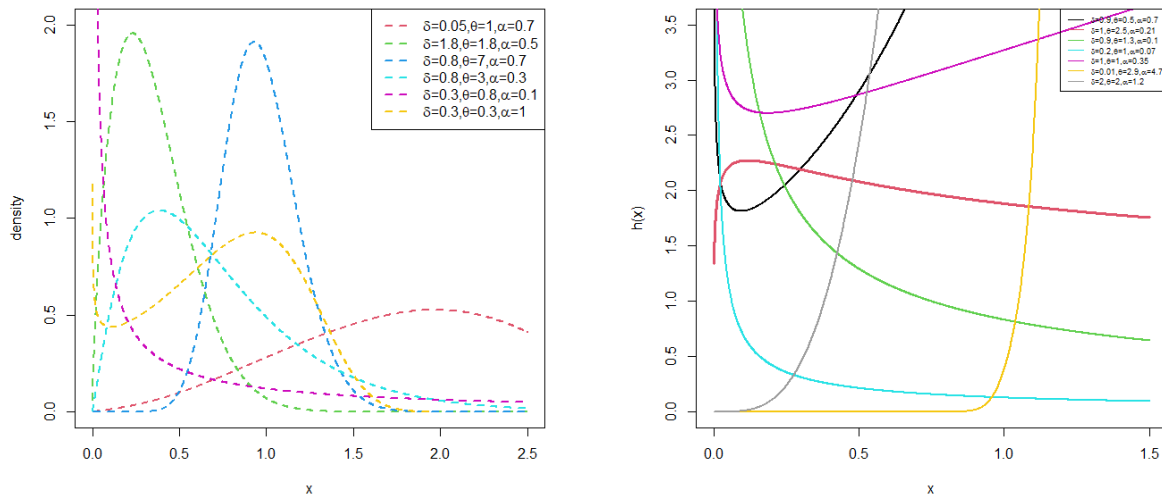


Figure 3.1: Plots of the PDF and HRF for the GOBX-W distribution

Figure 3.1 demonstrate different shapes of GOBX-W distribution for the PDF and HRF at different parameter values. The PDF takes various shapes which includes the reverse-J, left-skew, right-skew and unimodal shapes. The HRF also takes different shapes such as the increasing, decreasing, bathtub and upside down bathtub shapes.

3.3.2 Gamma Odd Burr X-Log-Logistic Distribution

Next, considering baseline distribution of the GOBX-G family as log-logistic distribution with PDF $g(x; \beta) = \beta x^{\beta-1} (1 + x^\beta)^{-2}$ and CDF $G(x; \beta) = 1 - (1 + x^\beta)^{-1}$, we respectively obtain the PDF and CDF of the gamma odd Burr X-log-logistic (GOBX-L) distribution as

$$f(x; \delta, \theta, \beta) = \frac{2\theta\beta x^{\beta-1} (1 + x^\beta) [1 - (1 + x^\beta)^{-1}]}{\Gamma(\delta)} e^{-x^{2\beta}} [1 - e^{-x^{2\beta}}]^{\theta-1} [-\theta \log(1 - e^{-x^{2\beta}})]^{\delta-1},$$

$$x > 0, \delta, \theta, \beta > 0$$
(3.10)

and

$$F(x; \delta, \theta, \beta) = 1 - \frac{\gamma\left(\delta, -\theta \log(1 - e^{-x^{2\beta}})\right)}{\Gamma(\delta)}.$$

Plots of the PDF and HRF of GOBX-L distribution are given in Figure 3.2

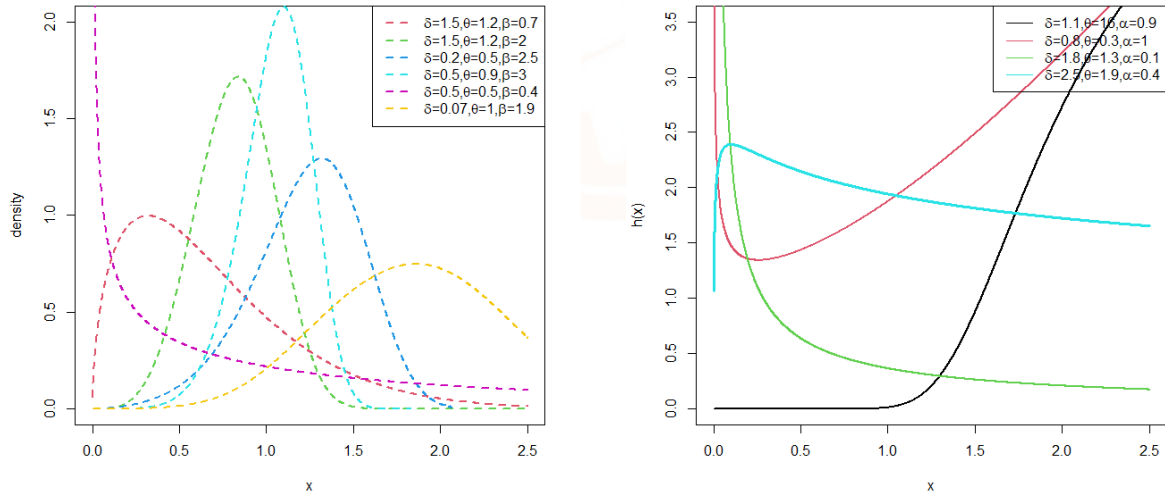


Figure 3.2: Plots of the PDF and HRF for the GOBX-L distribution

Figure 3.2 also shows flexible nature of the GOBX-L PDF and HRF for different underlying parameter values. The PDFs of GOBX-L distribution follow different shapes including unimodal, almost symmetric, reverse-J, left-skewed and right skewed. The HRF also exhibit decreasing, increasing, bathtub and upside down bathtub.

3.3.3 Gamma Odd Burr X-Uniform Distribution

For a uniform baseline distribution with PDF $g(x; \mu) = 1/\mu$ and CDF $G(x; \mu) = x/\mu$, $0 < x < \mu$, we respectively obtain the PDF and CDF of the gamma odd Burr X-Uniform (GOBX-U) distribution as

$$f(x; \delta, \theta, \mu) = \frac{2\theta\mu x}{\Gamma(\delta) (\mu - x)^3} \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \left\{ 1 - \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \right\}^{\theta-1} \times \left[-\theta \log \left(1 - \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \right) \right]^{\delta-1}, \quad x > 0, \delta, \theta, \mu > 0 \quad (3.11)$$

and

$$F(x; \delta, \theta, \mu) = 1 - \frac{\gamma \left(\delta, -\theta \log \left(1 - \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \right) \right)}{\Gamma(\delta)},$$

Plots of the PDF and HRF of GOBX-U distribution are given in Figure 3.3

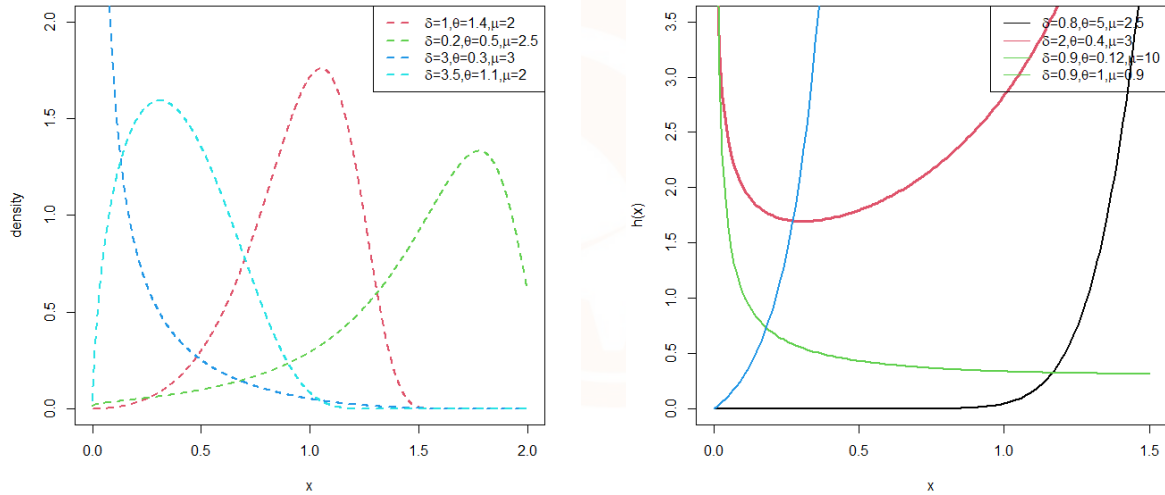


Figure 3.3: Plots of the PDF and HRF for the GOBX-U distribution

Figure 3.3 shows the GOBX-U distribution PDF and HRF. The PDF takes on different shapes which includes left-skew, right-skew, almost symmetric and reverse-J. Additionally, the HRF also show increasing, decreasing and bathtub shapes.

Table 3.1 presents some values generated from the GOBX-W distribution for different values of the parameters δ, θ and α .

Table 3.1: Quantile values for selected parameter values of the GOBX-W distribution

	(δ, θ, α)				
u	(1.5,0.9,1.01)	(2.0,2.5,0.9)	(1.5,.5,1.5)	(0.9,5.0,1.8)	(1.2,4.5,1.2)
0.1	0.1664	0.3579	0.1227	0.8253	0.6916
0.2	0.251	0.4318	0.2066	0.8683	0.751
0.3	0.3214	0.4871	0.2815	0.8986	0.7935
0.4	0.3857	0.5352	0.3521	0.9241	0.8295
0.5	0.4478	0.5805	0.4209	0.9475	0.863
0.6	0.5109	0.626	0.4901	0.9706	0.8961
0.7	0.5782	0.6747	0.5626	0.995	0.9314
0.8	0.6558	0.7314	0.6432	1.0232	0.9722
0.9	0.7594	0.8092	0.7453	1.0614	1.0282

3.4 Other Statistical Properties

In this section, some structural properties of GOBX-G family are presented. These includes moments and generating functions, conditional moments, entropy, distribution of order statistics and PWMs.

3.4.1 Moments and Generating Functions

The r^{th} ordinary moment of GOBX-G family can be derived from Equation (3.6) as

$$\mu'_r = E(X^r) = \sum_{p=0}^{\infty} w_{p+1} E(Y_{p+1}) = \sum_{p=0}^{\infty} w_{p+1} (p+1) \int_0^1 u^{p+1-1} Q_G^r(u; \xi) du, \quad (3.12)$$

where Y_{p+1} is the Exp-G rv with power parameter $(p+1)$ and $Q_G(u; \xi)$ is the QF of the baseline distribution with the CDF $G(x; \xi)$. To obtain the skewness and the kurtosis of GOBX-G family, we use the n^{th} central moment, say M_n , given as

$$\begin{aligned} M_n &= E(X - \mu'_1)^n = \sum_{r=0}^{\infty} \binom{n}{r} (-\mu'_1)^{n-r} E(X^r) \\ &= \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \binom{n}{r} (-\mu'_1)^{n-r} w_{p+1} E(Y_{p+1}). \end{aligned}$$

The MGF of the GOBX-G family can be obtained as

$$M_X(t) = E(e^{tX}) = \sum_{p=0}^{\infty} w_{p+1} M_{p+1}(t),$$

where $M_{p+1}(t)$ is the MGF of the Exp-G rv Y_{p+1} . Figures 3.4 and 3.5 show plots of skewness and kurtosis for the GOBX-W distribution for some selected parameter values. Figure 3.4 shows that, when θ is fixed, skewness increases as α increases whereas kurtosis decrease with a decrease in δ . An increase in θ leads to increase in skewness and kurtosis when α is held constant as shown in Figure 3.5. This shows that all the three parameters play a role in the variation of skewness an kurtosis.

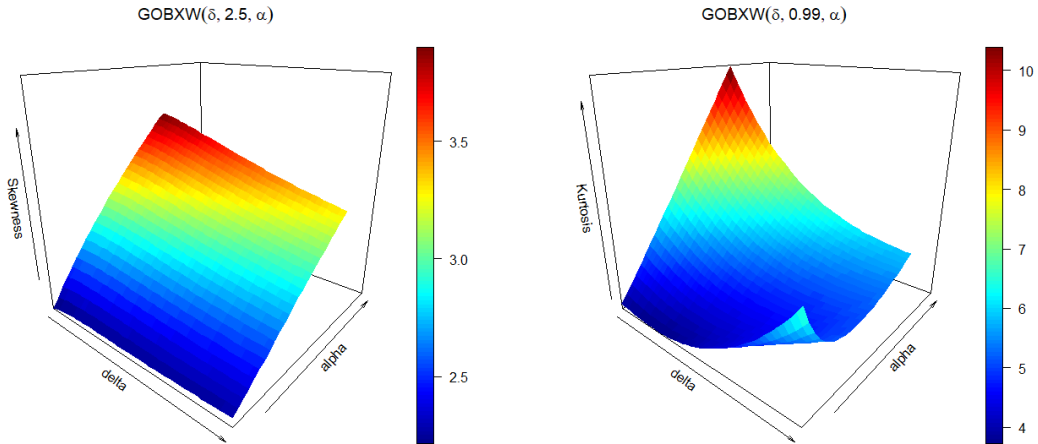


Figure 3.4: Plots of skewness and kurtosis for the GOBX-W distribution for some selected parameter values

3.4.2 Incomplete Moments

The s^{th} ($s > 0$) incomplete moments for the GOBX-G family follow from Equation 3.6 as

$$\varphi_s(t) = \int_{-\infty}^t x^s f(x; \delta, \theta, \xi) dx = \sum_{p=0}^{\infty} w_{p+1} \int_{-\infty}^t x^s h_{p+1}(x; \xi) dx ,$$

where $\int_{-\infty}^t x^s h_{p+1}(x; \xi) dx$ is the s^{th} incomplete moment of the Exp-G rv Y_{p+1} . The incomplete moments are critical as they can be used to derive the mean deviations, Bonferroni, Lorenz and Zenga curves which are usually used in many fields such as demography, engineering and medicine. The mean deviation about the mean $\mu = E(X)$ and median $M = \text{Median}(X)$ can respectively be derived as $E|X - \mu'_1| = 2\mu'_1 F(\mu'_1) - 2\varphi_1(\mu'_1)$ and $E|X - M| = \mu'_1 - 2\varphi_1(M)$, where μ'_1 can be evaluated from Equation 3.12, $M = Q(0.5)$ from Equation 3.8 and $F(\mu'_1)$ from Equation 3.3.

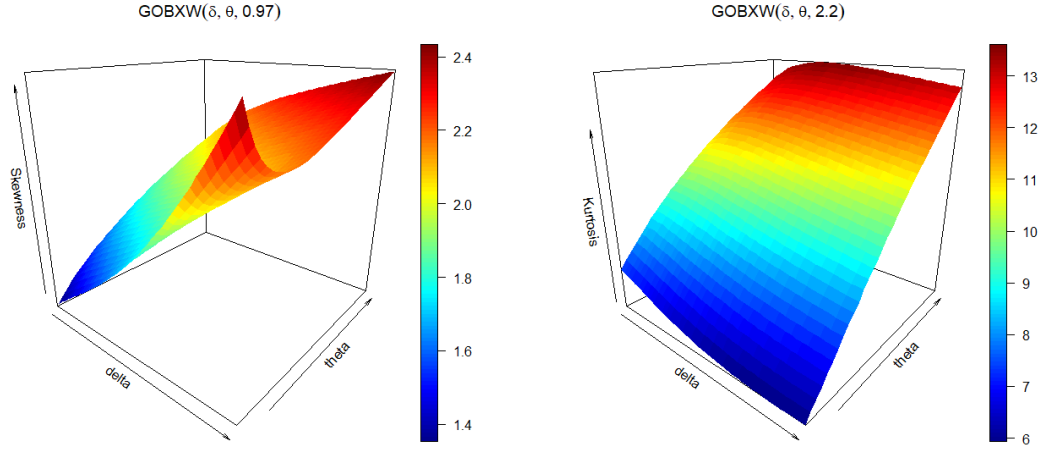


Figure 3.5: Plots of skewness and kurtosis for the GOBX-W distribution for some selected parameter values

3.4.3 Rényi Entropy

This subsection presents the Rényi entropy as a measures of the variation of uncertainty. Rényi entropy is an extension of Shannon entropy and it is defined as

$$I_R(v) = \frac{1}{1-v} \log \left[\int_0^\infty f^v(x; \delta, \theta, \xi) dx \right],$$

for $v > 0$, $v \neq 1$. If we let $y = \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right]$ in the GOBX-G density given in Equation (3.4), then we have

$$f^v(x; \delta, \theta, \xi) = \frac{(2\theta^\delta)^v g^v(x; \xi) G^v(x; \xi)}{[\Gamma(\delta)]^v \bar{G}^{3v}(x; \xi)} y^v (1-y)^{v(\theta-1)} [-\log(1-y)]^{v(\delta-1)}.$$

Using series representation $-\log(1-y) = \sum_{i=0}^\infty \frac{y^{i+1}}{i+1}$, for $y \in (0, 1)$, we have

$$[-\log(1-y)]^{v(\delta-1)} = y^{v(\delta-1)} \left[\sum_{m=0}^\infty \binom{v(\delta-1)}{m} y^m \left(\sum_{s=0}^\infty \frac{y^s}{s+2} \right)^m \right],$$

so that $f^v(x; \delta, \theta, \xi)$ can be written as

$$f^v(x; \delta, \theta, \xi) = \frac{(2\theta^\delta)^v g^v(x; \xi) G^v(x; \xi)}{[\Gamma(\delta)]^v \bar{G}^{3v}(x; \xi)} y^{v(\delta-1)+1} (1-y)^{v(\theta-1)} \left[\sum_{m=0}^\infty \binom{v(\delta-1)}{m} y^m \left(\sum_{s=0}^\infty \frac{y^s}{s+2} \right)^m \right].$$

If we let $c_s = (s + 2)^{-1}$ and considering the result of a power series raised to a positive integer ([Gradshteyn and Ryzhik 1980]), we get

$$\left(\sum_{s=0}^{\infty} c_s y^s \right)^m = \sum_{s=0}^{\infty} d_{s,m} y^s ,$$

where $d_{s,m} = (sc_0)^{-1} \sum_{l=1}^{\infty} [m(l+1) - s] c_l d_{s-l,m}$ and $d_{0,m} = c_0^m$, so that

$$f^v(x; \delta, \theta, \xi) = \frac{(2\theta^\delta)^v g^v(x; \xi) G^v(x; \xi)}{[\Gamma(\delta)]^v \overline{G}^{3v}(x; \xi)} (1-y)^{v(\theta-1)} \left[\sum_{m,s=0}^{\infty} \binom{v(\delta-1)}{m} y^{w_* d_{s,m}} \right] ,$$

where $w_* = m + s + v(\delta - 1) + 1$. Applying the generalized binomial series representation, we get

$$\begin{aligned} f^v(x; \delta, \theta, \xi) &= \frac{(2\theta^\delta)^v g^v(x; \xi) G^v(x; \xi)}{[\Gamma(\delta)]^v \overline{G}^{3v}(x; \xi)} \sum_{m,r,s=0}^{\infty} (-1)^r d_{s,m} \binom{v(\theta-1)}{r} \binom{v(\delta-1)}{m} \\ &\quad \times \exp \left[-(w_* + r) \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] . \end{aligned}$$

Applying the exponential series representation, we have

$$\begin{aligned} f^v(x; \delta, \theta, \xi) &= \frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v} \sum_{k,m,r,s=0}^{\infty} \frac{(-1)^{r+k} d_{s,m} (w_* + r)^k}{k!} \binom{v(\theta-1)}{r} \binom{v(\delta-1)}{m} \\ &\quad \times \frac{g^v(x; \xi) G^{2k+v}(x; \xi)}{\overline{G}^{2k+3v}(x; \xi)} . \end{aligned}$$

Applying the generalized binomial series representation in $G^{2k+v}(x; \xi) = [1 - \overline{G}(x; \xi)]^{2k+v}$, we obtain

$$\begin{aligned} f^v(x; \delta, \theta, \xi) &= \frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v} \sum_{j,k,m,r,s=0}^{\infty} \frac{(-1)^{r+j+k} d_{s,m} (w_* + r)^k}{k!} \\ &\quad \times \binom{v(\theta-1)}{r} \binom{v(\delta-1)}{m} \binom{2k+v}{j} g^v(x; \xi) \overline{G}^{j-(2k+3v)}(x; \xi) \\ &= \frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v} \sum_{j,k,m,p,r,s=0}^{\infty} \frac{(-1)^{r+j+k+p} d_{s,m} (w_* + r)^k}{k!} \binom{v(\theta-1)}{r} \\ &\quad \times \binom{v(\delta-1)}{m} \binom{2k+v}{j} \binom{j-(2k+3v)}{p} g^v(x; \xi) G^p(x; \xi) . \end{aligned}$$

Finally, the Rényi entropy for the GOBX-G family is given as

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left[\frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v} \sum_{j,k,m,p,r,s=0}^{\infty} \frac{(-1)^{r+j+k+p} d_{s,m}(w_*+r)^k}{\left[\frac{2k+j}{v}+2\right]^v k!} \binom{v(\theta-1)}{r} \binom{v(\delta-1)}{m} \right. \\ &\quad \times \binom{2k+v}{j} \binom{j-(2k+3v)}{p} \int_0^\infty \left(\left[\frac{p}{v}+1\right] g(x;\xi) [G(x;\xi)]^{\frac{p}{v}} \right)^v dx \left. \right] \\ &= \frac{1}{1-v} \log \left[\sum_{p=0}^{\infty} W_{\frac{p}{v}+1} \exp[(1-v)I_{REG}] \right], \end{aligned}$$

where $I_{REG} = (1-v)^{-1} \log \left[\int_0^\infty \left(\left[\frac{p}{v}+1\right] g(x;\xi) [G(x;\xi)]^{\frac{p}{v}} \right)^v dx \right]$ is the Rényi entropy for the Exp-G family with power parameter $(\frac{p}{v}+1)$ and

$$\begin{aligned} W_{\frac{p}{v}+1} &= \frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v} \sum_{j,k,m,r,s=0}^{\infty} \frac{(-1)^{r+j+k+p} d_{s,m}(w_*+r)^k}{\left[\frac{p}{v}+1\right]^v k!} \\ &\quad \times \binom{v(\theta-1)}{r} \binom{v(\delta-1)}{m} \binom{2k+v}{j} \binom{j-(2k+3v)}{p}. \end{aligned}$$

3.4.4 Order Statistics

In this subsection, the distribution of the i^{th} order statistics for the GOBX-G family is provided. Let $X_1, X_2, X_3, \dots, X_n$ be iid GOBX-G rvs. The PDF of the i^{th} order statistic, say $X_{i:n}$ is given as

$$\begin{aligned} f_{i:n}(x; \delta, \theta, \xi) &= \frac{n!f(x; \delta, \theta, \xi)}{(i-1)!(n-i)!} \sum_{t=0}^{n-i} (-1)^t \binom{n-i}{t} F^{t+i-1}(x; \delta, \theta, \xi) \\ &= \frac{n!f(x; \delta, \theta, \xi)}{(i-1)!(n-i)!} \sum_{t=0}^{n-i} (-1)^t \binom{n-i}{t} \left[1 - \frac{\gamma \left(\delta, -\theta \log \left(1 - \exp \left[- \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^2 \right] \right) \right)}{\Gamma(\delta)} \right]^{t+i-1}. \end{aligned}$$

Using the binomial theorem, we obtain

$$\begin{aligned} f_{i:n}(x; \delta, \theta, \xi) &= \frac{n!f(x; \delta, \theta, \xi)}{(i-1)!(n-i)!} \sum_{j=0}^{\infty} \sum_{t=0}^{n-i} \frac{(-1)^t}{[\Gamma(\delta)]^j} \binom{n-i}{t} \binom{t+i-1}{j} \\ &\quad \times \left[\gamma \left(\delta, -\theta \log \left(1 - \exp \left[- \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)} \right)^2 \right] \right) \right) \right]^j. \end{aligned} \tag{3.13}$$

Applying the power series $\gamma(\delta, x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+\delta}}{(k+\delta)k!}$ (Abramowitz and Stegun, 1972), we obtain

$$f_{i:n}(x; \delta, \theta, \xi) = \frac{n!f(x; \delta, \theta, \xi)}{(i-1)!(n-i)!} \sum_{j=0}^{\infty} \sum_{t=0}^{n-i} \frac{(-1)^t}{[\Gamma(\delta)]^j} \binom{n-i}{t} \binom{t+i-1}{j} \\ \times \left[\sum_{k=0}^{\infty} \frac{(-1)^k \left(-\theta \log \left(1 - \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] \right) \right)^{k+\delta}}{(k+\delta)k!} \right]^j.$$

Next, let $c_k = \frac{(-1)^k}{(k+\delta)k!}$ and use the result on a power series raised to a positive integer (Gradshteyn and Ryzhik, 1980), then we have

$$\left[\sum_{k=0}^{\infty} c_k \left(-\theta \log \left(1 - \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] \right) \right)^{k+\delta} \right]^j \\ = \sum_{k=0}^{\infty} d_{k,j} \left(-\theta \log \left(1 - \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] \right) \right)^{k+\delta}, \quad (3.14)$$

where $d_{0,j} = c_0^j$ and $d_{k,j} = (kc_0)^{-1} \sum_{l=1}^{\infty} [j(l+1) - k] c_l d_{k-l,j}$. Replacing $f(x; \delta, \theta, \xi)$ in the above expression for $f_{i:n}(x; \delta, \theta, \xi)$, we get

$$f_{i:n}(x; \delta, \theta, \xi) = \frac{2n!g(x; \xi)G(x; \xi)}{(i-1)!(n-i)!\overline{G}^3(x; \xi)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta} (-1)^t}{[\Gamma(\delta)]^{j+1}} d_{k,j} \binom{n-i}{t} \binom{t+i-1}{j} \\ \times \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] \left\{ 1 - \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] \right\}^{\theta-1} \\ \times \left(-\log \left(1 - \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right] \right) \right)^{k+2\delta-1}.$$

Using the previously stated series expansion of $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$ and letting $y = \exp \left[- \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^2 \right]$, we obtain

$$[-\log(1-y)]^{k+2\delta-1} = y^{k+2\delta-1} \left[\sum_{m=0}^{\infty} \binom{k+2\delta-1}{m} y^m \left(\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^m \right], \quad (3.15)$$

which when we consider the previous definition of $b_{s,m}$ in Equation (3.5), we get

$$\begin{aligned}
 f_{i:n}(x; \delta, \theta, \xi) &= \frac{2n!g(x; \xi)G(x; \xi)}{(i-1)!(n-i)!G^3(x; \xi)} \sum_{j,k=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^t}{[\Gamma(\delta)]^{j+1}} d_{k,j} \binom{n-i}{t} \binom{t+i-1}{j} \\
 &\quad \times y^{k+2\delta} (1-y)^{\theta-1} \sum_{m,s=0}^{\infty} \binom{k+2\delta-1}{m} y^m b_{s,m} y^s \\
 &= \frac{2n!g(x; \xi)G(x; \xi)}{(i-1)!(n-i)!G^3(x; \xi)} \sum_{j,k,m,s=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^t}{[\Gamma(\delta)]^{j+1}} d_{k,j} b_{s,m} \\
 &\quad \times \binom{n-i}{t} \binom{t+i-1}{j} \binom{k+2\delta-1}{m} y^{k+2\delta+m+s} (1-y)^{\theta-1} .
 \end{aligned}$$

Further expansion of the last part conducted using binomial theorem gives

$$\begin{aligned}
 f_{i:n}(x; \delta, \theta, \xi) &= \frac{2n!g(x; \xi)G(x; \xi)}{(i-1)!(n-i)!G^3(x; \xi)} \sum_{j,k,m,r,s=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^{t+r}}{[\Gamma(\delta)]^{j+1}} d_{k,j} b_{s,m} \\
 &\quad \times \binom{n-i}{t} \binom{t+i-1}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} y^{v_*} ,
 \end{aligned}$$

where $v_* = k + 2\delta + m + s + r$. Expanding $y^{v_*} = \exp \left[-v_* \left(\frac{G(x; \xi)}{G(x; \xi)} \right)^2 \right]$ based on exponential series representation gives

$$\begin{aligned}
 f_{i:n}(x; \delta, \theta, \xi) &= \frac{2n!}{(i-1)!(n-i)!} \sum_{j,k,m,r,s,u=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^{t+r+u} v_*^u d_{k,j} b_{s,m}}{[\Gamma(\delta)]^{j+1} u!} \\
 &\quad \times \binom{n-i}{t} \binom{t+i-1}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} \frac{g(x; \xi) G^{2u+1}(x; \xi)}{G^{2u+3}(x; \xi)} .
 \end{aligned}$$

Expansion of the series $G^{2u+1}(x; \xi) = [1 - \overline{G}(x; \xi)]^{2u+1}$ gives

$$\begin{aligned}
 f_{i:n}(x; \delta, \theta, \xi) &= \frac{2n!}{(i-1)!(n-i)!} \sum_{j,k,m,p,r,s,u=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^{t+r+u+p} v_*^u d_{k,j} b_{s,m}}{[\Gamma(\delta)]^{j+1} u!} \binom{n-i}{t} \\
 &\quad \times \binom{t+i-1}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} \binom{2u+1}{p} g(x; \xi) \overline{G}^{p-(2u+3)}(x; \xi) \\
 &= \frac{2n!}{(i-1)!(n-i)!} \sum_{j,k,m,p,q,r,s,u=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^{t+r+u+p+q} v_*^u d_{k,j} b_{s,m}}{[\Gamma(\delta)]^{j+1} u!} \binom{n-i}{t} \\
 &\quad \times \binom{t+i-1}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} \binom{2u+1}{p} \binom{p-(2u+3)}{q} g(x; \xi) G^q(x; \xi) .
 \end{aligned}$$

Alternatively, the GOBX-G order statistic can be expressed as

$$f_{i:n}(x; \delta, \theta, \xi) = \sum_{q=0}^{\infty} m_{q+1} h_{q+1}^*(x; \xi) , \quad (3.16)$$

where $h_{q+1}^*(x; \xi) = (q+1)g(x; \xi)G^{q+1-1}(x; \xi)$ is the Exp-G family with power parameter $(q+1)$ and

$$m_{q+1} = \frac{2n!}{(i-1)!(n-i)!} \sum_{j,k,m,p,r,s,u=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta} (-1)^{t+r+u+p+q} v_*^u d_{k,j} b_{s,m}}{(q+1)[\Gamma(\delta)]^{j+1} u!} \binom{n-i}{t} \\ \times \binom{t+i-1}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} \binom{2u+1}{p} \binom{p-(2u+3)}{q}.$$

3.4.5 Probability Weighted Moments

This section present the PWMs with most of the derivations given under the order statistics section. The $(r, s)^{th}$ PWMs for the GOBX-G family is given as

$$\omega_{r,s} = E[X^r F^s(X)] = \int_{-\infty}^{\infty} x^r f(x) F^s(x) dx.$$

With the help of derivations from Equation (3.13) under order statistics section, we get

$$f(x) F^s(x) = f(x) \sum_{j=0}^{\infty} [\Gamma(\delta)]^{-j} \binom{s}{j} [\gamma(\delta, -\theta \log(1-y))]^{-j} \\ = f(x) \sum_{j,k=0}^{\infty} [\Gamma(\delta)]^j \binom{s}{j} d_{j,k} [-\theta \log(1-y)]^{k+\delta},$$

where $y = \exp \left[- \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right)^2 \right]$ and $d_{j,k}$ is defined in Equation (3.14). Plugging $f(x)$, we get

$$f(x) F^s(x) = \frac{2\theta g(x; \xi) G(x; \xi)}{\bar{G}^3(x; \xi)} y(1-y)^{\theta-1} \sum_{j,k=0}^{\infty} [\Gamma(\delta)]^{-(j+1)} \binom{s}{j} d_{j,k} [-\theta \log(1-y)]^{k+2\delta-1}.$$

Further expansion based on Equation (3.15) yields

$$f(x) F^s(x) = \frac{2g(x; \xi) G(x; \xi)}{\bar{G}^3(x; \xi)} \sum_{j,k,m,s=0}^{\infty} [\Gamma(\delta)]^{-(j+1)} \theta^{k+2\delta} d_{j,k} b_{s,m} \\ \times \binom{s}{j} \binom{k+2\delta-1}{m} y^{k+2\delta+m+s} (1-y)^{\theta-1} \\ = \frac{2g(x; \xi) G(x; \xi)}{\bar{G}^3(x; \xi)} \sum_{j,k,m,r,s=0}^{\infty} [\Gamma(\delta)]^{-(j+1)} (-1)^r \theta^{k+2\delta} d_{j,k} b_{s,m} \\ \times \binom{s}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} y^{v_*},$$

where $b_{s,m}$ is defined in Equation (3.5) and $v_* = k + 2\delta + m + s + r$. Series expansion of $y^{v_*} = \exp \left[-v_* \left(\frac{G(x;\xi)}{\bar{G}(x;\xi)} \right)^2 \right]$ gives

$$\begin{aligned}
 f(x)F^s(x) &= 2 \sum_{j,k,m,r,s,u=0}^{\infty} \frac{\theta^{k+2\delta}}{[\Gamma(\delta)]^{(j+1)}} \frac{(-1)^{r+u} v_*^u d_{j,k} b_{s,m}}{u!} \binom{s}{j} \binom{k+2\delta-1}{m} \\
 &\quad \times \binom{\theta-1}{r} \frac{g(x;\xi) G^{2u+1}(x;\xi)}{\bar{G}^{2u+3}(x;\xi)} \\
 &= 2 \sum_{j,k,m,p,r,s,u=0}^{\infty} \frac{\theta^{k+2\delta}}{[\Gamma(\delta)]^{(j+1)}} \frac{(-1)^{r+u+p} v_*^u d_{j,k} b_{s,m}}{u!} \binom{s}{j} \binom{k+2\delta-1}{m} \\
 &\quad \times \binom{\theta-1}{r} \binom{2u+1}{p} g(x;\xi) \bar{G}^{p-(2u+3)}(x;\xi) \\
 &= 2 \sum_{j,k,m,p,q,r,s,u=0}^{\infty} \frac{\theta^{k+2\delta}}{[\Gamma(\delta)]^{(j+1)}} \frac{(-1)^{r+u+p+q} v_*^u d_{j,k} b_{s,m}}{u!} \binom{s}{j} \binom{k+2\delta-1}{m} \\
 &\quad \times \binom{\theta-1}{r} \binom{2u+1}{p} \binom{p-(2u+3)}{q} g(x;\xi) G^q(x;\xi) \\
 &= \sum_{q=0}^{\infty} \eta_{q+1} h_{q+1}^*(x;\xi),
 \end{aligned}$$

where $h_{q+1}^*(x;\xi)$ is defined under Equation (3.16) and

$$\begin{aligned}
 \eta_{q+1} &= 2 \sum_{j,k,m,p,r,s,u=0}^{\infty} \frac{\theta^{k+2\delta}}{[\Gamma(\delta)]^{(j+1)}} \frac{(-1)^{r+u+p+q} v_*^u d_{j,k} b_{s,m}}{u!} \\
 &\quad \times \binom{s}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} \binom{2u+1}{p} \binom{p-(2u+3)}{q}.
 \end{aligned}$$

Consequently, we obtain the PWMs for the GOBX-G family reduces to

$$\omega_{r,s} = \sum_{q=0}^{\infty} \eta_{q+1} \int_{-\infty}^{\infty} x^r h_{q+1}^*(x;\xi) dx.$$

3.5 Maximum Likelihood Estimation

This section present the maximum likelihood technique for estimating the parameters of the GOBX-G family. Maximum likelihood is one of the most reliable estimation technique as it mostly produce accurate and consistent estimators (Zanakis and Kyparisis, 1986). Let $X_1, X_2, X_3, \dots, X_n$ be a random sample of size n from the GOBX-G family and $\Theta = (\delta, \theta, \xi)^T$ be a vector of model parameters, then the log-likelihood function can be expressed as

$$\begin{aligned}
 \ell_n(\Theta) &= n \ln(2) + n \ln(\theta) - n \ln(\Gamma(\delta)) + \sum_{i=1}^n \ln[g(x_i;\xi)] + \sum_{i=1}^n \ln[G(x_i;\xi)] - 3 \sum_{i=1}^n \ln[\bar{G}(x_i;\xi)] \\
 &\quad - \sum_{i=1}^n y_i^2 + (\theta-1) \sum_{i=1}^n \ln(1 - e^{-y_i^2}) + (\delta-1) \sum_{i=1}^n \ln[-\log(1 - e^{-y_i^2})],
 \end{aligned}$$

where $y_i = G(x_i; \xi)/\overline{G}(x_i; \xi)$. The components of the score function $U_n(\Theta) = \left(\frac{\partial \ell_n(\Theta)}{\partial \delta}, \frac{\partial \ell_n(\Theta)}{\partial \theta}, \frac{\partial \ell_n(\Theta)}{\partial \xi_k} \right)^T$ are given in the Appendix C.1 and R software is used to obtain the solution to $U_n(\Theta) = 0$ under a specified baseline distribution. The multivariate normal distribution $N_q(\underline{\mathbf{0}}, J(\hat{\Theta})^{-1})$, where $\underline{\mathbf{0}} = (0, 0, 0)^T$ and $J(\hat{\Theta})^{-1}$ are the mean vector and the observed information matrix evaluated at $\hat{\Theta}$, can be used to construct the confidence intervals and regions for model parameters. The $100(1 - \varphi)\%$ two sided confidence intervals for δ, θ and ε_k are $\hat{\delta} \pm Z_{\frac{\varphi}{2}} \sqrt{I_{\delta\delta}^{-1}(\hat{\Theta})}$, $\hat{\theta} \pm Z_{\frac{\varphi}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\Theta})}$ and $\hat{\varepsilon}_k \pm Z_{\frac{\varphi}{2}} \sqrt{I_{\varepsilon_k \varepsilon_k}^{-1}(\hat{\Theta})}$, respectively, where $I_{\delta\delta}^{-1}(\hat{\Theta})$, $I_{\theta\theta}^{-1}(\hat{\Theta})$ and $I_{\varepsilon_k \varepsilon_k}^{-1}(\hat{\Theta})$ are the diagonal elements of $I_n^{-1}(\hat{\Theta}) = (nI(\hat{\Theta}))^{-1}$ and $Z_{\frac{\varphi}{2}}$ is the standard normal $(\frac{\varphi}{2})^{th}$ percentile.

3.6 Simulation Study

In this section, simulation results are presented for different sample sizes of $n = 100, 200, 400, 600$ and 800 to examine the accuracy and consistency of the maximum likelihood estimators (MLEs) for each of the parameter of the GOBX-W distribution. The simulation was repeated $N = 1000$ times and the mean estimates, average bias (Abias) and the root mean square errors (RMSEs) were evaluated. For consistent MLEs, it is expected that as the sample size n increases, the mean estimates gets closer to the true parameters and, the RSMEs and Abias also decay to zero. Table 3.2 and 3.3 show the mean estimates together with their respective RSMEs and Abias.

Table 3.2: Monte Carlo simulation results for GOBX-W distribution

		(2.6, 1.1, 0.5)			(1.8, 1.2, 1.1)		
Parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
δ	50	3.1189	0.9438	0.5189	2.2192	1.0726	0.4192
	100	2.9600	0.6889	0.3600	2.0007	0.4393	0.2007
	200	2.8260	0.5068	0.2260	1.8990	0.2581	0.0989
	400	2.7332	0.3660	0.1332	1.8302	0.1338	0.0302
	800	2.6926	0.2776	0.0926	1.8171	0.0898	0.0171
	1600	2.6542	0.1934	0.0542	1.8079	0.0642	0.0079
	50	1.4551	2.3365	0.3551	2.2270	9.2460	1.0270
	100	1.2661	1.1842	0.1661	1.3894	1.1417	0.1894
	200	1.1461	0.5660	0.0461	1.2685	0.5166	0.0685
	400	1.0888	0.3916	-0.0112	1.2137	0.3094	0.0137
	800	1.0827	0.2773	-0.0173	1.2016	0.2008	0.0016
	1600	1.0878	0.1994	-0.0122	1.1913	0.1423	-0.0087
θ	50	0.8438	0.6705	0.3438	1.8718	1.7766	0.7718
	100	0.7764	0.5922	0.2764	1.5742	1.3635	0.4742
	200	0.6816	0.4582	0.1816	1.2879	0.7166	0.1879
	400	0.6333	0.3779	0.1333	1.1705	0.3543	0.0705
	800	0.5790	0.2675	0.0790	1.1315	0.1680	0.0315
	1600	0.5430	0.1852	0.0430	1.1217	0.1135	0.0217

Table 3.3: Monte Carlo simulation results for GOBX-W distribution

Parameter	Sample Size	(0.4, 0.4, 1.1)			(1.9, 0.8, 0.5)		
		Mean	RMSE	Abias	Mean	RMSE	Abias
δ	50	0.4239	0.1077	0.0239	2.2691	0.6693	0.3691
	100	0.4148	0.0684	0.0148	2.1402	0.4599	0.2402
	200	0.4104	0.0461	0.0104	2.0366	0.3075	0.1366
	400	0.4044	0.0306	0.0044	1.9694	0.2010	0.0694
	800	0.4032	0.0217	0.0032	1.9521	0.1417	0.0521
	1600	0.4018	0.0153	0.0018	1.9467	0.1002	0.0467
50	50	0.4776	0.3396	0.0776	1.0860	1.4620	0.2860
	100	0.4393	0.1771	0.0393	0.9188	0.6578	0.1188
	200	0.4242	0.1084	0.0242	0.8609	0.3682	0.0609
	400	0.4115	0.0689	0.0115	0.8200	0.2460	0.0200
	800	0.4058	0.0469	0.0058	0.8206	0.1708	0.0206
	1600	0.4005	0.0336	0.0005	0.8134	0.1191	0.0134
50	50	1.1539	0.2750	0.0540	0.7957	0.6224	0.2957
	100	1.1195	0.1776	0.0195	0.7185	0.5234	0.2185
	200	1.1021	0.1210	0.0021	0.6159	0.3603	0.1159
	400	1.0909	0.0750	0.0009	0.5710	0.2694	0.0710
	800	1.0899	0.0520	0.0001	0.5341	0.1802	0.0341
	1600	1.0927	0.0370	0.0000	0.5180	0.0851	0.0180

From the results in Tables 3.2 and 3.3 it is clear that as the sample size increases, the mean estimates gets closer to the true parameters, whereas the respective RSMEs and Abias decay to zero indicating consistent MLEs.

3.7 Applications

This section presents two real data applications to show the flexibility of GOBX-W distribution as compared to other existing distributions. The first data consists of $n = 59$ monthly the actual taxes revenue (in 1000 million Egyptian pounds) in Egypt between January 2006 and November 2010 extracted from Nassar and Nada (2011). The data are: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8. This data have mean, median, minimum and maximum values of 13.49, 10.60, 4.10 and 39.20, respectively.

The second data consists of $n = 46$ active repair times (hours) for an airborne communication transceiver referenced by Sultan and Al-Moisheer (2015). The data are: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5. This data have mean, median, minimum and maximum values of 3.61, 1.75, 0.2 and 24.5, respectively.

The GOBX-W distribution is compared with other existing models with the use of R-software to run model parameter estimates and GOF measures. The GOF measures used for model performance comparison are; $-2\log L$, AIC, BIC, AICC, W^* , (A^*) , SS and K-S together with its corresponding p-value. For all these GOF statistics, smaller values suggests a good fit of the model to the data. Additionally, large p-values also represent a good fit for the model. The new GOBX-W distribution is compared to Weibull (W) distribution, Weibull Lomax (WL) distribution (Jamal et al. 2019), gamma Weibull (GW) distribution (Pogány and Saxena 2010), gamma Burr X (GBX) distribution (Khaleel et al. 2016), exponentiated half logistic (EHL) distribution (Cordeiro et al. 2017) and Burr X (BX) distribution (Khaleel et al. 2016) with the respective PDFs (for $x > 0$):

$$f_{WL}(x) = \alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1} \exp\left(-\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right), a, b, \alpha > 0.$$

$$f_{GW}(x) = \frac{k\lambda^{-k-\beta}x^{\beta+k-1}e^{-\lambda^{-k}x^k}}{\Gamma(1+\frac{\beta}{k})}, k, \beta, \lambda > 0.$$

$$f_{GBX}(x) = \frac{2\theta\lambda^2}{\Gamma(\delta)}xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\theta-1}\left[-\theta\log\left(1-e^{-(\lambda x)^2}\right)\right]^{\delta-1}, \theta, \lambda, \delta > 0.$$

$$f_{EHL}(x) = \frac{2a\lambda e^{-\lambda x}}{1-e^{-2\lambda x}}\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]^a, a, \lambda > 0.$$

$$f_{BX}(x) = 2\theta\alpha^2xe^{-(\alpha x)^2}\left(1-e^{-(\alpha x)^2}\right)^{\theta-1}, \theta, \alpha > 0.$$

3.7.1 Taxes Revenue Data

This subsection contains parameter estimates (standard errors in parenthesis), GOF statistics, plots of the fitted densities, empirical CDF, KM and P-Ps for the taxes revenue data.

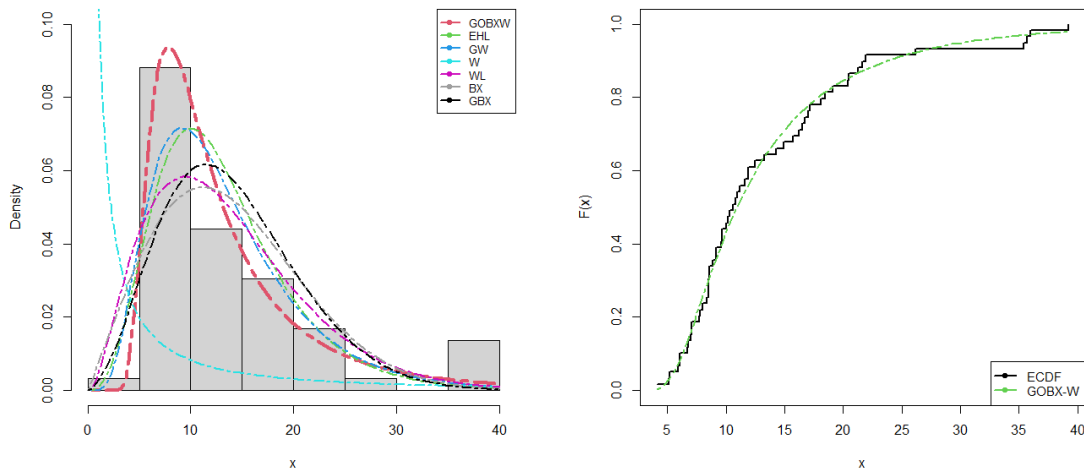


Figure 3.6: Fitted densities and empirical CDF plots for monthly actual taxes revenue data

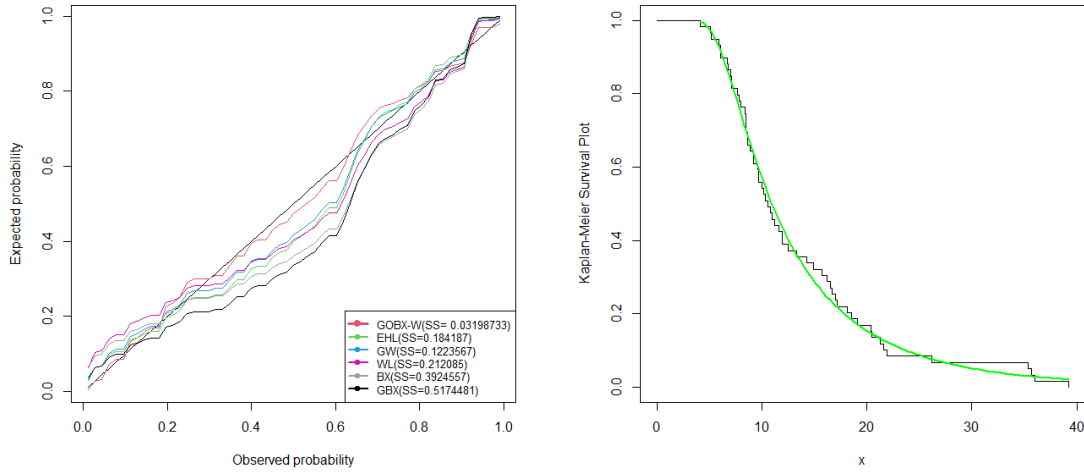


Figure 3.7: P-P and KM plots for monthly actual taxes revenue data

Figures 3.6 and 3.7 illustrate how best the GOBX-W distribution fits the monthly actual taxes revenue data. The fitted density shows that the GOBX-W distribution can accommodate skewed data. The estimated var-cov matrix for GOBX-W model in revenue data is given by

$$\begin{bmatrix} 1.49 \times 10^{-2} & 4.52 \times 10^{-6} & -1.18 \times 10^{-3} \\ 4.52 \times 10^{-6} & 1.38 \times 10^{-9} & -3.58 \times 10^{-7} \\ -1.18 \times 10^{-3} & -3.58 \times 10^{-7} & 1.07 \times 10^{-4} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [3.59 \times 10^{-1} \pm 2.39 \times 10^{-1}]$, $\theta \in [1.02 \times 10^3 \pm 7.27 \times 10^{-5}]$ and $\alpha \in [1.38 \times 10^{-1} \pm 2.03 \times 10^{-2}]$.

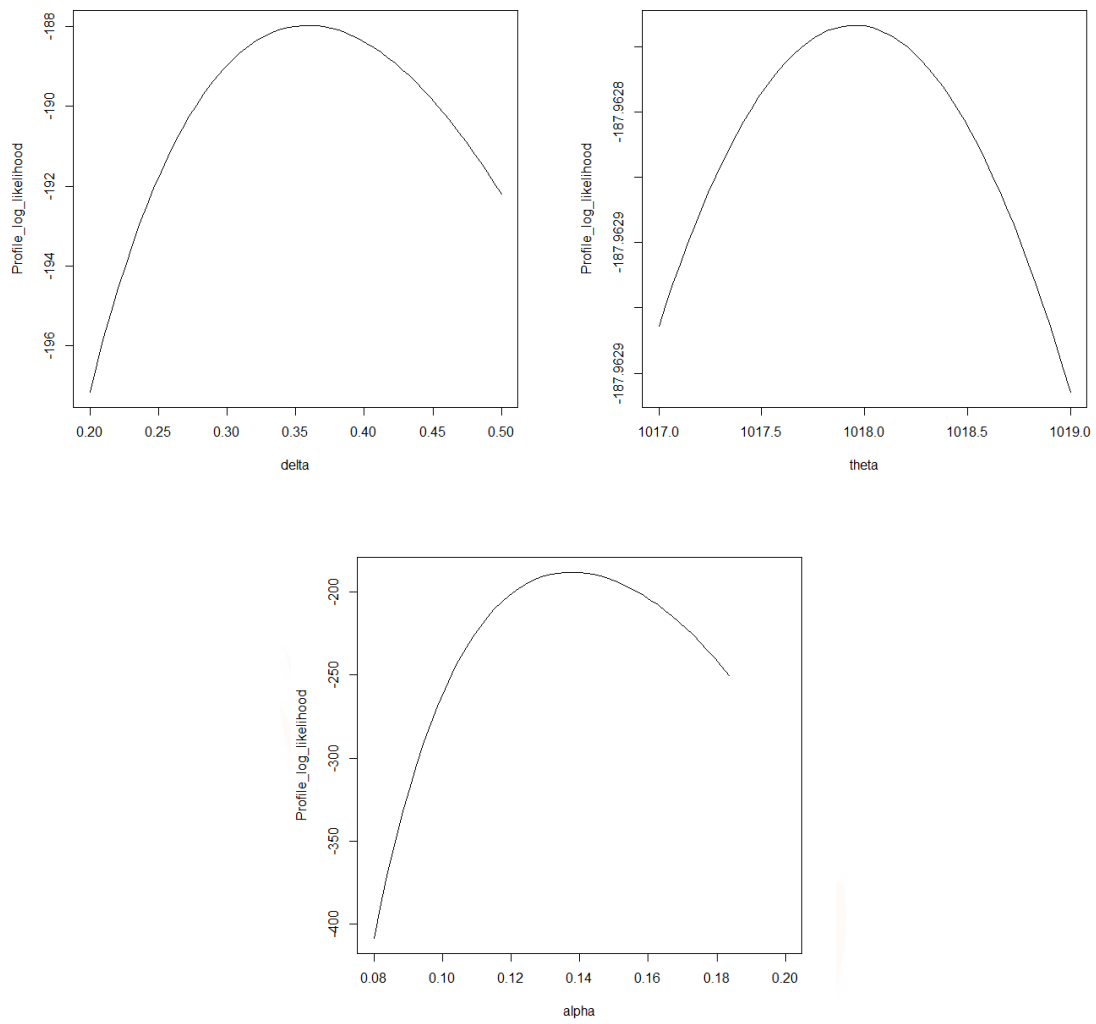


Figure 3.8: LLikelihood profile plots of δ , θ and α for monthly actual taxes revenue data

Figures 3.8 and 3.8 shows the profile plots of the MLEs of δ , θ and α . It can be seen that the parameters attained the absolute maximum for monthly actual taxes revenue data.

Table 3.4: The MLEs and GOF statistics of the GOBX-W model and various models for the revenue data set

Distribution	Estimates			-2LogL	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
	δ	θ	α									
GOBX-W	3.588×10^{-1} (1.220×10^{-1})	1.018×10^3 (3.710×10^{-5})	1.377×10^{-1} (1.034×10^{-2})	375.9	381.9	382.4	388.2	0.03356	0.2240	0.0587	0.9871	0.0320
	a	b	α									
WL	0.2558 (0.1315)	0.9385 (1.3999)	3.9034 (1.6510)	389.1	395.1	395.5	401.3	0.2259	1.4233	0.1332	0.2463	0.2121
	k	β	λ									
GW	3.548×10^{-1} (2.874×10^{-2})	9.9433 (2.871×10^{-6})	9.201×10^{-4} (9.126×10^{-4})	381.5	387.5	387.9	393.7	0.1326	0.7959	0.1060	0.5211	0.1224
	θ	λ	δ									
GBX	1.0741 (0.4418)	0.1412 (0.0562)	0.2269 (0.1836)	394.2	400.2	400.6	406.4	0.2737	1.7546	0.1943	0.0232	0.5174
	a	λ										
EHL	3.4904 (0.8301)	0.1917 (0.0227)	-	384.8	388.8	389.0	392.9	0.1720	1.0405	0.1215	0.3486	0.18418
	θ	α										
BX	1.0309 (0.1844)	0.0644 (0.0056)	-	395.3	399.3	399.6	403.5	0.3112	1.9904	0.1763	0.0510	0.3925
	λ											
W	0.3163 (0.0261)	-	-	594.7	596.7	596.8	598.8	0.1329	0.7973	0.7956	< 0.0001	12.3219

3.7.2 Repair Time Data

This subsection contains parameter estimates (standard errors in parenthesis), GOF statistics, plots of the fitted densities, empirical CDF, KM and P-Ps for the repair time data.

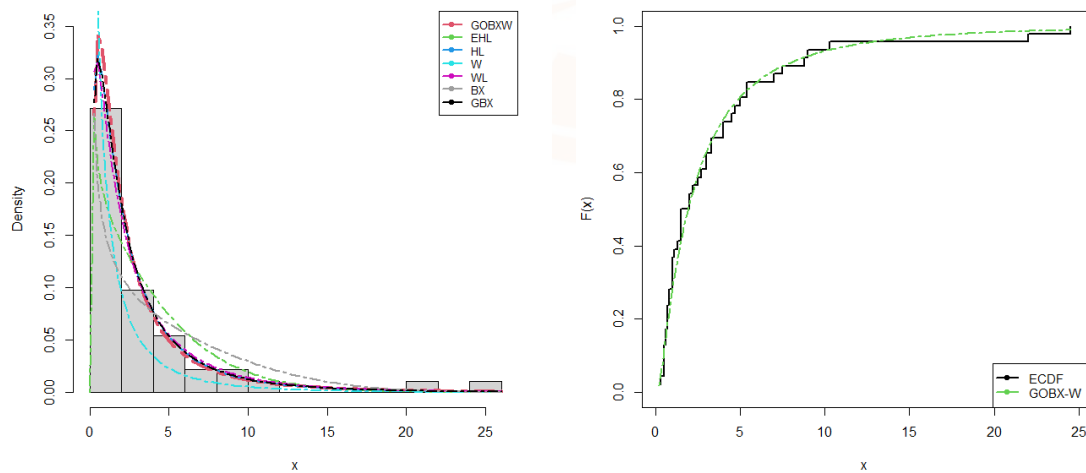


Figure 3.9: Fitted densities and empirical CDF plots for repair time data

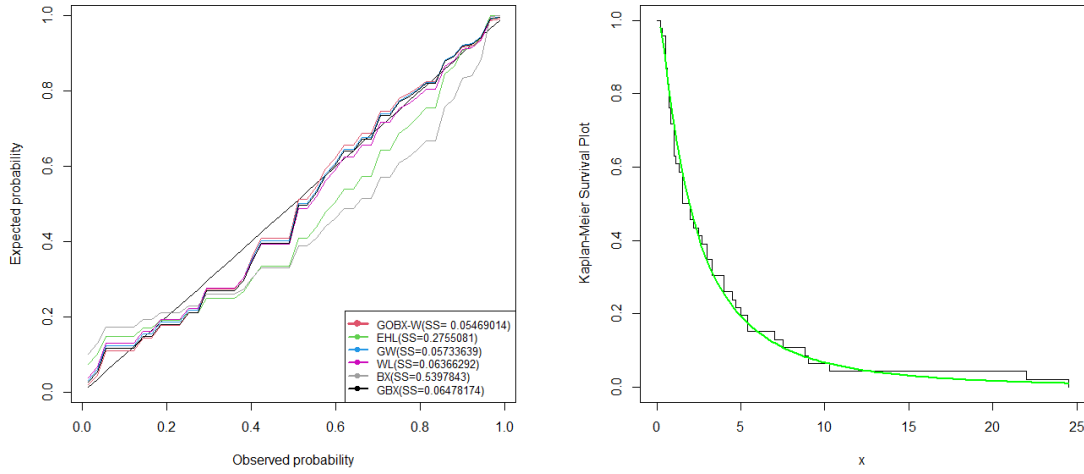


Figure 3.10: P-P and KM plots for repair time data

Figures 3.9 and 3.10 also shows GOBX-W distribution providing a better fit to the repair time data. The fitted density further illustrates the flexibility of GOBX-W distribution with fitting skewed data. The estimated var-cov matrix for GOBX-W model in repair time data is given by

$$\begin{bmatrix} 2.09 \times 10^{-1} & -1.01 \times 10^{-2} & -7.36 \times 10^{-4} \\ -1.01 \times 10^{-2} & 4.91 \times 10^{-4} & 3.56 \times 10^{-5} \\ -7.36 \times 10^{-4} & 3.56 \times 10^{-5} & 1.18 \times 10^{-5} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [8.06 \pm 0.89]$, $\theta \in [1.77 \times 10^2 \pm 4.34 \times 10^{-2}]$ and $\alpha \in [3.16 \times 10^{-2} \pm 6.75 \times 10^{-3}]$.

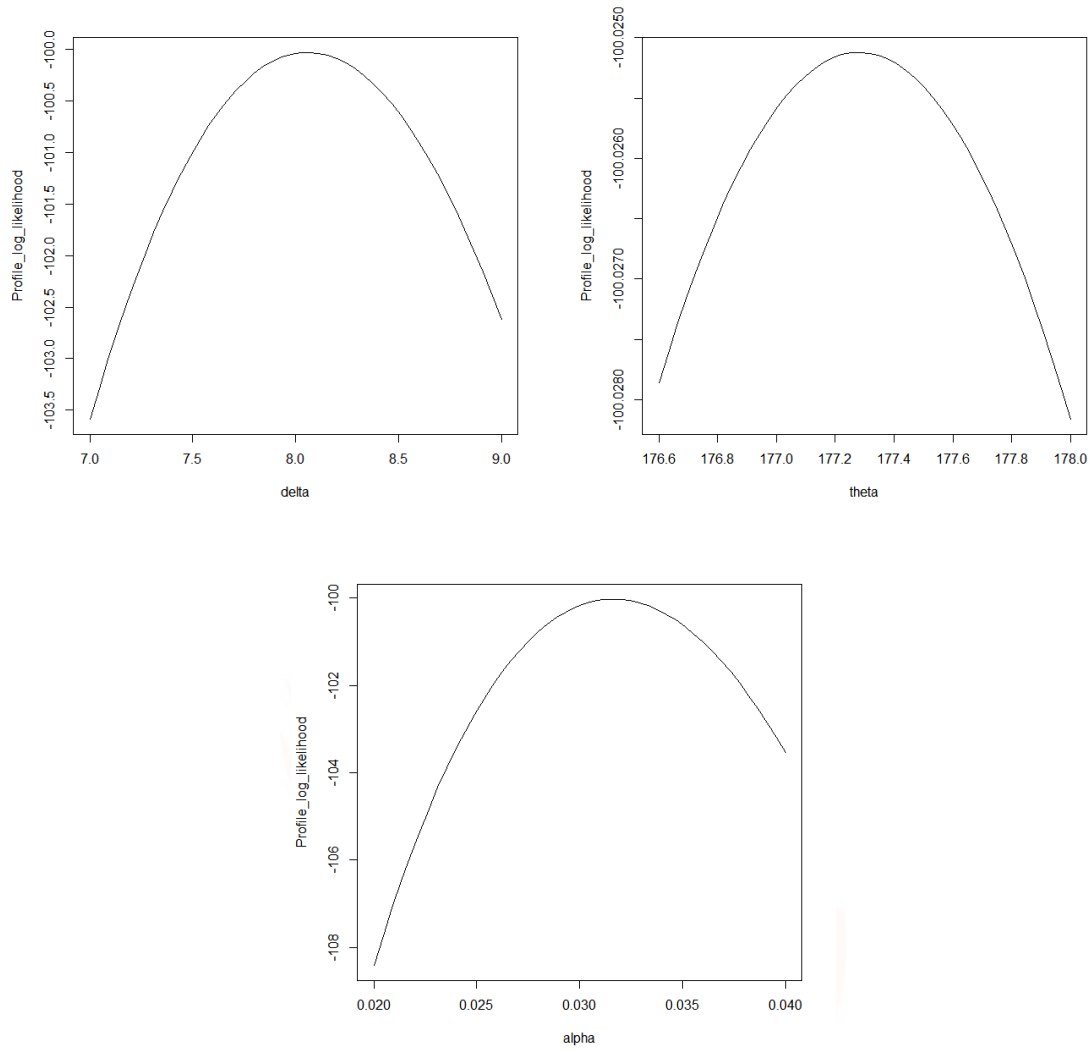


Figure 3.11: Likelihood profile plots of δ , θ and α for the repair time data

Figures 3.11(a), 3.11(b) and 3.11(c) shows the profile plots of the MLEs of δ , θ and α . It can be seen that the parameters attained the absolute maximum for the repair time data.

Table 3.5: The MLEs and GOF statistics of the GOBX-W model and various models for the repair time data set

Distribution	Estimates			-2LogL	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
	δ	θ	α									
GOBX-W	8.057 (4.577×10^{-1})	1.773×10^2 (2.216×10^{-2})	3.165×10^{-2} (3.444×10^{-3})	200.1	206.1	206.6	211.5	0.0523	0.3315	0.0949	0.8017	0.0547
	a	b	α									
WL	0.2222 (0.1094)	7.0310 (11.2110)	2.12279 (0.91009)	202.3	208.3	208.8	213.8	0.0697	0.4745	0.1059	0.6802	0.0637
	k	β	λ									
GW	0.2648 (0.0226)	2.7865 (0.0001)	0.0002 (0.0002)	201.7	207.7	208.2	213.1	0.0661	0.4379	0.0987	0.7619	0.0573
	θ	λ	δ									
GBX	3.5237 (2.086×10^{-9})	7.250×10^{-5} (1.180×10^{-5})	62.530 (1.195×10^{-10})	201.3	207.3	207.9	212.8	0.0631	0.4181	0.1048	0.6923	0.0647
	a	λ										
EHL	0.744 (0.1378)	0.311 (0.0572)	-	215.3	219.3	219.6	223.0	0.2072	1.3926	0.1660	0.1584	0.2755
	θ	α										
BX	0.2906 (0.0482)	0.0991 (0.0145)	-	223.4	227.4	227.7	231.1	0.2782	1.9018	0.1813	0.0973	0.5397
	λ											
W	0.565 (0.0500)	-	-	241.9	243.9	243.9	245.7	0.0977	0.6718	0.4478	< 0.0001	3.1535

From the results in Tables 3.4 and 3.5, the GOBX-W distribution outperforms all the competing non-nested models. It has the lowest values of -2LogL, AIC, AICC, BIC, W^* , A^* , K-S and SS together with largest p-value as compared to competing models in all data sets.

3.8 Concluding Remarks

We develop a new distribution called as the gamma odd Burr X-G (GOBX-G) family. Mathematical and statistical properties of this new family are studied. In addition, different baseline distributions are considered to show the flexibility of this family on real life applications. Simulation study is also carried out using Weibull distribution as the baseline and it showed that its maximum likelihood estimators are accurate and consistent. Using two real data sets, gamma odd Burr X-Weibull (GOBX-W) distribution outperformed several models it is competing with based on the -2LogL, AIC, AICC, BIC, W^* , A^* , K-S and SS statistics which presented very low as compared to non-nested models. Graphical plots are also added to show how best GOBX-W distribution fits real life datasets. Moreover, the HRF can be decreasing, increasing, upside down bathtub and bathtub shaped depending on the selected parameter values.

4. The Topp-Leone Odd Burr X-G Power Series Class

4.1 Introduction

The Topp-Leone odd Burr X-G power series (TLOBX-GPS) class is developed and examined in this chapter. In addition to the QF, density expansion and moments of TLOBX-GPS class, order statistics and Rényi entropy for this class are derived. We demonstrate the utilization of the Weibull and log-logistic distributions as baseline models when dealing with discrete distributions like Poisson and binomial distributions in specific scenarios. Further, the method of maximum likelihood is employed to estimate the parameters and our results are evaluated using a simulation study. Finally, we fit the new Topp-Leone odd Burr X-Weibull Poisson (TLOBX-WP) distribution using four real datasets to illustrate its flexibility and usefulness.

4.2 The Topp-Leone Odd Burr X-G Power Series Distribution

Oluyede et al. (2022b) developed the Topp-Leone odd Burr X-G (TLOBX-G) family which is defined by the following PDF and CDF

$$f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) = \frac{4\alpha\theta g(x; \underline{\omega})G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \exp[-K_G(x; \underline{\omega})] \{1 - \exp[-K_G(x; \underline{\omega})]\}^{\theta-1} \\ \times \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta\right] \left\{1 - \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta\right]^2\right\}^{\alpha-1}, \quad (4.1)$$

$x \in \mathbb{R}, \alpha, \theta > 0$

and

$$F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) = \left\{1 - \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta\right]^2\right\}^\alpha, \quad (4.2)$$

respectively, where $\underline{\omega}$ is the baseline vector of parameters, $K_G(x; \underline{\omega}) = \left(\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})}\right)^2$, and $\bar{G}(x; \underline{\omega}) = 1 - G(x; \underline{\omega})$. The SF and HRF of the TLOBX-G family are given by

$$S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) = 1 - \left\{1 - \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta\right]^2\right\}^\alpha, \quad (4.3)$$

and

$$h(x; \alpha, \theta, \underline{\omega}) = \frac{4\alpha\theta g(x; \underline{\omega})G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \exp[-K_G(x; \underline{\omega})] \{1 - \exp[-K_G(x; \underline{\omega})]\}^{\theta-1} \\ \times \frac{\left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta\right] \left\{1 - \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta\right]^2\right\}^{\alpha-1}}{1 - \left\{1 - \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta\right]^2\right\}^\alpha}, \quad (4.4)$$

respectively. We let rvs $X_1, X_2, X_3, \dots, X_N$ be iid, and suppose each X_i follows the TLOBX-G family. We consider X and N to be independent rvs. Consider the discrete rv N , which has a truncation at zero and follows a power series distribution with pmf

$$P(N = n) = \frac{a_n \lambda^n}{C(\lambda)}, \lambda > 0, n = 1, 2, \dots, \quad (4.5)$$

where $C(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$ and a_n depends only on n . The power series class includes Poisson, binomial, geometric, logarithmic distributions. Now, let $X_{(1)} = \min(X_1, X_2, X_3, \dots, X_N)$, then the conditional CDF of $X_{(1)}|N = n$ is $G_{X_{(1)}|N=n}(x) = 1 - (\bar{F}_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))^n = 1 - (S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))^n$. It follows that the marginal CDF and PDF of $X_{(1)}$ are given as

$$F(x; \alpha, \theta, \lambda, \underline{\omega}) = 1 - \frac{C(\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))}{C(\lambda)}, x \in \mathbb{R}, \alpha, \theta, \lambda > 0 \quad (4.6)$$

and

$$f(x; \alpha, \theta, \lambda, \underline{\omega}) = \frac{\lambda f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) C'(\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))}{C(\lambda)}, \quad (4.7)$$

respectively, where $C'(\cdot)$ is the first derivative of $C(\lambda)$ with respect to λ . Substituting Equations (4.1) and (4.3) in Equations (4.6) and (4.7), we present the PDF and CDF of Topp-Leone odd Burr X-G power series (TLOBX-GPS) class as

$$f(x; \alpha, \theta, \lambda, \underline{\omega}) = \frac{4\lambda\alpha\theta g(x; \underline{\omega})G(x; \underline{\omega})C' \left(\lambda \left[1 - \left\{ 1 - \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta \right]^2 \right\}^\alpha \right] \right)}{\bar{G}^3(x; \underline{\omega})C(\lambda)} \\ \times \exp[-K_G(x; \underline{\omega})] \{1 - \exp[-K_G(x; \underline{\omega})]\}^{\theta-1} \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta \right] \\ \times \left\{ 1 - \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta \right]^2 \right\}^{\alpha-1}, x \in \mathbb{R}, \lambda, \alpha, \theta > 0 \quad (4.8)$$

and

$$F(x; \alpha, \theta, \lambda, \underline{\omega}) = 1 - \frac{C \left(\lambda \left[1 - \left\{ 1 - \left[1 - \{1 - \exp[-K_G(x; \underline{\omega})]\}^\theta \right]^2 \right\}^\alpha \right] \right)}{C(\lambda)}, \quad (4.9)$$

respectively, where $\underline{\omega}$ is the baseline vector of parameters, $K_G(x; \underline{\omega}) = \left(\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})} \right)^2$ and $C'(\cdot)$ is the first derivative of $C(\lambda)$ with respect to λ . The SF of TLOBX-GPS class follows as

$$S_{TLOBX-GPS}(x; \alpha, \theta, \lambda, \underline{\omega}) = \frac{C \left(\lambda \left[1 - \left\{ 1 - \left[1 - \{ 1 - \exp[-K_G(x; \underline{\omega})] \}^\theta \right]^2 \right\}^\alpha \right] \right)}{C(\lambda)}. \quad (4.10)$$

4.2.1 Special Sub-Families

With specified $C(\lambda)$, we obtain special families of TLOBX-GPS class given below.

- If we consider the power series distribution as Poisson distribution with $C(\lambda) = e^\lambda - 1$ and $a_n = (n!)^{(-1)}$, TLOBX-G Poisson class is derived with CDF

$$F(x; \alpha, \theta, \lambda, \underline{\omega}) = 1 - \frac{\exp \left[\lambda \left(1 - \left\{ 1 - \left[1 - \{ 1 - \exp[-K_G(x; \underline{\omega})] \}^\theta \right]^2 \right\}^\alpha \right) \right] - 1}{\exp(\lambda) - 1},$$
 where $\alpha, \theta, \lambda > 0$ and baseline parameter vector $\underline{\omega}$.
- If we consider the power series distribution as binomial distribution with $C(\lambda) = \{1 + \lambda\}^m - 1$ and $a_n = \binom{m}{n}$, TLOBX-G binomial class is derived with CDF

$$F(x; \alpha, \theta, \lambda, \underline{\omega}) = 1 - \frac{\left\{ 1 + \lambda \left(1 - \left\{ 1 - \left[1 - \{ 1 - \exp[-K_G(x; \underline{\omega})] \}^\theta \right]^2 \right\}^\alpha \right) \right\}^m - 1}{\{1 + \lambda\}^m - 1},$$
 where $\alpha, \theta, \lambda > 0$ and baseline parameter vector $\underline{\omega}$.
- If we consider the power series distribution as geometric distribution with $C(\lambda) = \lambda \{1 - \lambda\}^{-1}$ and $a_n = 1$, TLOBX-G geometric class is derived with CDF

$$F(x; \alpha, \theta, \lambda, \underline{\omega}) = 1 - \frac{\{1 - \lambda\} \left(1 - \left\{ 1 - \left[1 - \{ 1 - \exp[-K_G(x; \underline{\omega})] \}^\theta \right]^2 \right\}^\alpha \right)}{1 - \lambda \left(1 - \left\{ 1 - \left[1 - \{ 1 - \exp[-K_G(x; \underline{\omega})] \}^\theta \right]^2 \right\}^\alpha \right)},$$
 where $\alpha, \theta, \lambda > 0$ and baseline parameter vector $\underline{\omega}$.
- If we consider the power series distribution as logarithmic distribution with $C(\lambda) = -\log(1 - \lambda)$ and $a_n = \binom{m}{n}$, TLOBX-G logarithmic class is derived with CDF

$$F(x; \alpha, \theta, \lambda, \underline{\omega}) = 1 - \frac{\log \left\{ 1 - \lambda \left(1 - \left\{ 1 - \left[1 - \{ 1 - \exp[-K_G(x; \underline{\omega})] \}^\theta \right]^2 \right\}^\alpha \right) \right\}}{\log(1 - \lambda)},$$
 where $\alpha, \theta, \lambda > 0, 0 < \lambda < 1$ and baseline parameter vector $\underline{\omega}$.

4.2.2 Quantile Function

Random numbers are produced using the QF. By inverting $F(x; \alpha, \theta, \lambda, \underline{\omega}) = u$ for $0 < u < 1$, the QF may be produced. Consequently, by solving the following non-linear equation, the QF of the TLOBX-GPS class may be found as

$$1 - \frac{C(\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))}{C(\lambda)} = u,$$

which simplifies to

$$S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) = \frac{C^{-1}((1 - u)C(\lambda))}{\lambda},$$

where $C^{-1}(\cdot)$ is the inverse of the function $C(\cdot)$. Note that $S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) = 1 -$

$F(x; \alpha, \theta, \underline{\omega})$, where $F(x; \alpha, \theta, \underline{\omega})$ is given in Equation (4.2). Thus, we can further write

$$F(x; \alpha, \theta, \underline{\omega}) = 1 - \frac{C^{-1}((1-u)C(\lambda))}{\lambda}.$$

If we let $u_* = 1 - \frac{C^{-1}((1-u)C(\lambda))}{\lambda}$, then we can express the QF of the TLOBX-GPS class as

$$G(x; \underline{\omega}) = \frac{\{-\ln[1 - (1 - (1 - u_*^{1/\alpha})^{0.5})^{1/\theta}]\}^{0.5}}{1 + \{-\ln[1 - (1 - (1 - u_*^{1/\alpha})^{0.5})^{1/\theta}]\}^{0.5}}.$$

Finally, the QF of TLOBX-GPS class follows as

$$Q(u) = G^{-1} \left\{ \frac{\{-\ln[1 - (1 - (1 - u_*^{1/\alpha})^{0.5})^{1/\theta}]\}^{0.5}}{1 + \{-\ln[1 - (1 - (1 - u_*^{1/\alpha})^{0.5})^{1/\theta}]\}^{0.5}} \right\}.$$

We use iterative methods in R software to obtain quantiles, from TLOBX-GPS class. Quantiles of the Topp-Leone odd Burr X-Weibull Poisson (TLOBX-WP) distribution are shown in Table 4.1 for various parameter values.

Table 4.1: Quantile values for the TLOBX-WP distribution for some parameter values

	$(\alpha, \theta, \lambda, \beta)$				
u	(0.4,2.5,1.4,3.9)	(1.7,5,2.5,2.5)	(0.4,3,0.5,0.9)	(1.6,0.5,0.5,1.1)	(2.4,1.1,1.9,1.5)
0.1	0.5700	0.9295	0.1437	0.1973	0.6461
0.2	0.6347	0.9606	0.2156	0.2992	0.7233
0.3	0.6804	0.9829	0.2806	0.3870	0.7797
0.4	0.7190	1.0021	0.3456	0.4703	0.8285
0.5	0.7547	1.0204	0.4147	0.5541	0.8751
0.6	0.7900	1.0392	0.4920	0.6428	0.9229
0.7	0.8275	1.0602	0.5841	0.7421	0.9758
0.8	0.8710	1.0862	0.7040	0.8628	1.0407
0.9	0.9298	1.1257	0.8907	1.0353	1.1363

4.3 Some Statistical Properties

We present a few helpful statistical characteristics of the TLOBX-GPS class in this section. The properties includes the density expansion, moments and generating functions, Rényi entropy and distribution of the order statistics.

4.3.1 Density Expansion

The density expansion of the TLOBX-GPS class is shown in this section. Given that $C'(\lambda) = \sum_{n=1}^{\infty} na_n\lambda^{n-1}$, we can write Equation (4.8) as

$$\begin{aligned} f(x; \alpha, \theta, \lambda, \underline{\omega}) &= \frac{\lambda f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) C'(\lambda(1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})))}{C(\lambda)} \\ &= \sum_{n=1}^{\infty} \frac{na_n\lambda^n f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) [1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]^{n-1}}{C(\lambda)}. \end{aligned} \quad (4.11)$$

Using expansion series expansion $(1 - z)^{a-1} = \sum_{i=0}^{\infty} \binom{a-1}{i} (-1)^i z^i$, with the restriction $|z| < 1$, we get

$$f(x; \alpha, \theta, \lambda, \underline{\omega}) = \sum_{h=0}^{\infty} \sum_{n=1}^{\infty} \binom{n-1}{h} \frac{(-1)^h na_n\lambda^n}{C(\lambda)} f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) F_{TLOBX-G}^h(x; \alpha, \theta, \underline{\omega}).$$

The product $f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) F_{TLOBX-G}^h(x; \alpha, \theta, \underline{\omega})$ replaced with its expressions in Equations (4.1) and (4.2) gives an expression similar to the density of TLOBX-G family. Thus, we can write $f(x; \lambda, \alpha, \theta, \underline{\omega})$ in the form of the final density expansion as

$$\begin{aligned} f(x; \alpha, \theta, \lambda, \underline{\omega}) &= 4\alpha\theta \sum_{h,i,j,k,m,p,r=0}^{\infty} \sum_{n=1}^{\infty} \frac{na_n\lambda^n (k+1)^m (-1)^{h+i+j+k+m+r+p} \binom{n-1}{h}}{m!C(\lambda)} \\ &\times \binom{\alpha(h+1)-1}{i} \binom{2i+1}{j} \binom{\theta(j+1)-1}{k} \binom{2m+1}{r} \binom{r-(2m+3)}{p} g(x; \underline{\omega}) G^p(x; \underline{\omega}) \\ &= \sum_{p=0}^{\infty} D_{p+1} h_{p+1}(x; \underline{\omega}), \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} D_{p+1} &= 4\alpha\theta \sum_{h,i,j,k,m,r=0}^{\infty} \sum_{n=1}^{\infty} \frac{na_n\lambda^n (k+1)^m (-1)^{h+i+j+k+m+r+p} \binom{n-1}{h}}{m!C(\lambda)} \\ &\times \binom{\alpha(h+1)-1}{i} \binom{2i+1}{j} \binom{\theta(j+1)-1}{k} \binom{2m+1}{r} \binom{r-(2m+3)}{p} \end{aligned} \quad (4.13)$$

and $h_{p+1}(x; \underline{\omega}) = (p+1)g(x; \underline{\omega})G^{p+1-1}(x; \underline{\omega})$ is the Exp-G density function with power parameter $(p+1)$. Consequently, the density function of the TLOBX-GPS class can be expressed as infinite linear combination of Exp-G densities.

4.3.2 Moments and Generating Functions

In this section, we use Equation (4.12) to derive the moments. Thus, the s^{th} ordinary moment of TLOBX-GPS class, say μ'_s , is given as

$$\mu'_s = E(X^s) = \sum_{q=0}^{\infty} D_{p+1} E(Y_{q+1}), \quad (4.14)$$

where the Exp-G rv Y_{p+1} has the power parameter $p+1$ with D_{p+1} displayed in Equation (4.13). From Equation (4.14), we derive the r^{th} central moment of the TLOBX-GPS class as

$$\mu_r = \sum_{s=0}^r \binom{r}{s} (-\mu'_1)^{r-s} E(X^s) = \sum_{p=0}^{\infty} \sum_{s=0}^r \binom{r}{s} (-\mu'_1)^{r-s} D_{p+1} E(Y_{p+1}),$$

the MGF as

$$M_X(t) = E(e^{tX}) = \sum_{p=0}^{\infty} D_{p+1} M_{p+1}(t),$$

and k^{th} incomplete moment as

$$\eta_k(t) = \int_{-\infty}^t x^k f(x; \lambda, \alpha, \theta, \underline{\omega}) dx = \sum_{p=0}^{\infty} D_{p+1} \int_{-\infty}^t x^k h_{p+1}(x; \underline{\omega}) dx,$$

where the MGF and the k^{th} incomplete moment of the Exp-G are respectively follows as $M_{p+1}(t)$ and $\int_{-\infty}^t x^k h_{p+1}(x; \underline{\omega}) dx$, and D_{p+1} is displayed in (4.13). The Topp-Leone odd Burr X-Weibull Poisson (TLOBX-WP) distribution's skewness and kurtosis plots for various parameter values are shown in Figures 4.1 and 4.2. Holding λ and β constant, both skewness in Figure 4.1 and kurtosis in Figure 4.2 increases with increase in α and θ . Increase in skewness and kurtosis are also observed in Figures 4.1 and 4.2, respectively when parameters are changed.

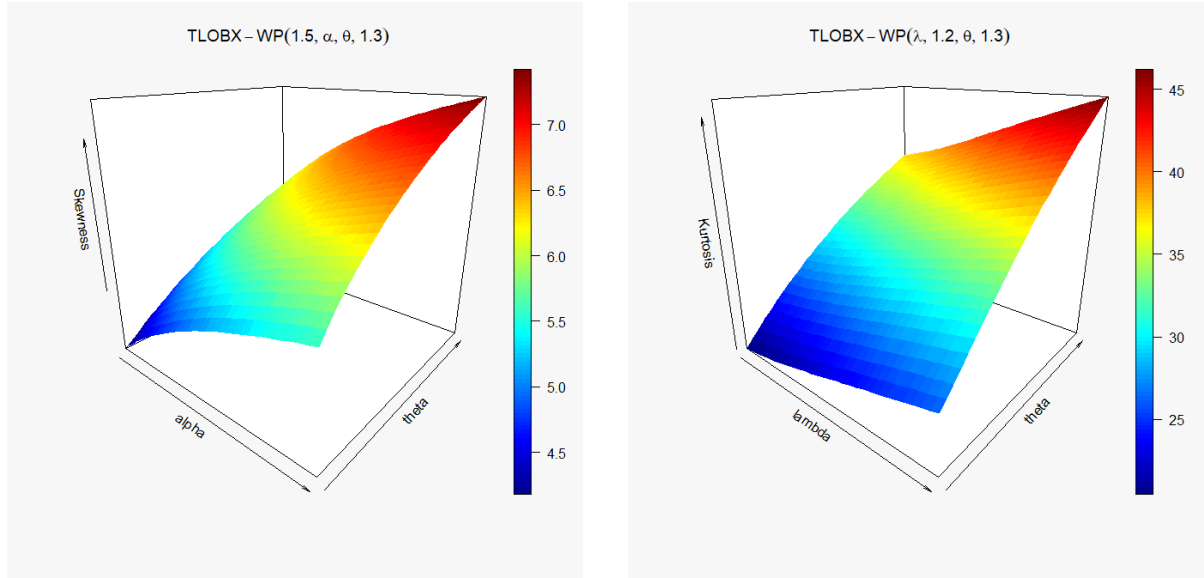


Figure 4.1: Plots of skewness and kurtosis for the TLOBX-WP distribution for a few chosen parameter values

4.3.3 Rényi Entropy

As a measure of a rv's variation, Rényi entropy is defined from

$$E_R(v) = \frac{1}{1-v} \log \left[\int_0^\infty f^v(x; \lambda, \alpha, \theta, \underline{\omega}) dx \right], v > 0, v \neq 1.$$

Using Equation (4.11), we can write $f^v(x; \lambda, \alpha, \theta, \underline{\omega})$ as

$$f^v(x; \alpha, \theta, \lambda, \underline{\omega}) = \left[\frac{\lambda}{C(\lambda)} \right]^v f_{TLOBX-G}^v(x; \alpha, \theta, \underline{\omega}) \{C'[\lambda(1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))]\}^v.$$

Using expansion from Gradshteyn and Ryzhik (1980) given as $(C'(\delta))^m = \sum_{z=0}^\infty d_{z,m} \delta^z$, where $d_{z,m} = (zb_0)^{-1} \sum_{m=1}^z [m(h+1) - z] b_h d_{z-h,m}$ and $d_{0,z} = b_0^z$, we can write

$$\{C'[\lambda(1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))]\}^v = \sum_{s=0}^\infty d_{s,v} \lambda^s (1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))^s,$$

where $d_{s,v} = (sa_0)^{-1} \sum_{l=1}^\infty [v(l+1) - s] a_l d_{s-l,v}$ and $d_{0,v} = a_0^v$. Writing $(1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))^s = \sum_{h=0}^\infty (-1)^h \binom{s}{h} F_{TLOBX-G}^h(x; \alpha, \theta, \underline{\omega})$, we get $f^v(x; \lambda, \alpha, \theta, \underline{\omega})$ as

$$f^v(x; \lambda, \alpha, \theta, \underline{\omega}) = \left[\frac{\lambda}{C(\lambda)} \right]^v \sum_{h,s=0}^\infty d_{s,v} \lambda^s (-1)^h \binom{s}{h} f_{TLOBX-G}^v(x; \alpha, \theta, \underline{\omega}) F_{TLOBX-G}^h(x; \alpha, \theta, \underline{\omega}). \quad (4.15)$$

Following the steps in Appendix [D.1](#) we finally get

$$\begin{aligned}
 E_R(v) &= \frac{1}{1-v} \log \left[\left(\frac{4\alpha\theta\lambda}{C(\lambda)} \right)^v \sum_{h,i,j,k,m,n,p,s=0}^{\infty} \frac{d_{s,v}\lambda^s (-1)^{h+i+j+k+m+n+p} (k+v)^m}{\left[\frac{p}{v}+1\right]^v m!} \right. \\
 &\quad \times \binom{s}{h} \binom{v(\alpha-1)+\alpha h}{i} \binom{2i+v}{j} \binom{\theta(j+v)-v}{k} \binom{2m+v}{n} \binom{n-(2m+3v)}{p} \\
 &\quad \times \int_0^{\infty} \left(\left[\frac{p}{v}+1\right] g(x;\underline{\omega}) G_{\frac{p}{v}}^v(x;\underline{\omega}) \right)^v dx \Bigg] \\
 &= \frac{1}{1-v} \log \left[\sum_{p=0}^{\infty} C_p \exp((1-v)E_{Rp}) \right],
 \end{aligned}$$

where

$$\begin{aligned}
 C_p &= \left[\frac{4\alpha\theta\lambda}{C(\lambda)} \right]^v \sum_{h,i,j,k,m,n,p,s=0}^{\infty} \frac{d_{s,v}\lambda^s (-1)^{h+i+j+k+m+n+p} (k+v)^m}{\left[\frac{p}{v}+1\right]^v m!} \binom{s}{h} \\
 &\quad \times \binom{v(\alpha-1)+\alpha h}{i} \binom{2i+v}{j} \binom{\theta(j+v)-v}{k} \binom{2m+v}{n} \binom{n-(2m+3v)}{p}
 \end{aligned}$$

and the Rényi entropy $E_{Rp} = (1-v)^{-1} \log \left[\int_0^{\infty} \left(\left[\frac{p}{v}+1\right] g(x;\underline{\omega}) G_{\frac{p}{v}}^v(x;\underline{\omega}) \right)^v dx \right]$ with power parameter $\left(\frac{p}{v}+1\right)$ is that of Exp-G family, which indicate that we can derive the entropy of TLOBX-GPS class from that of Exp-G family.

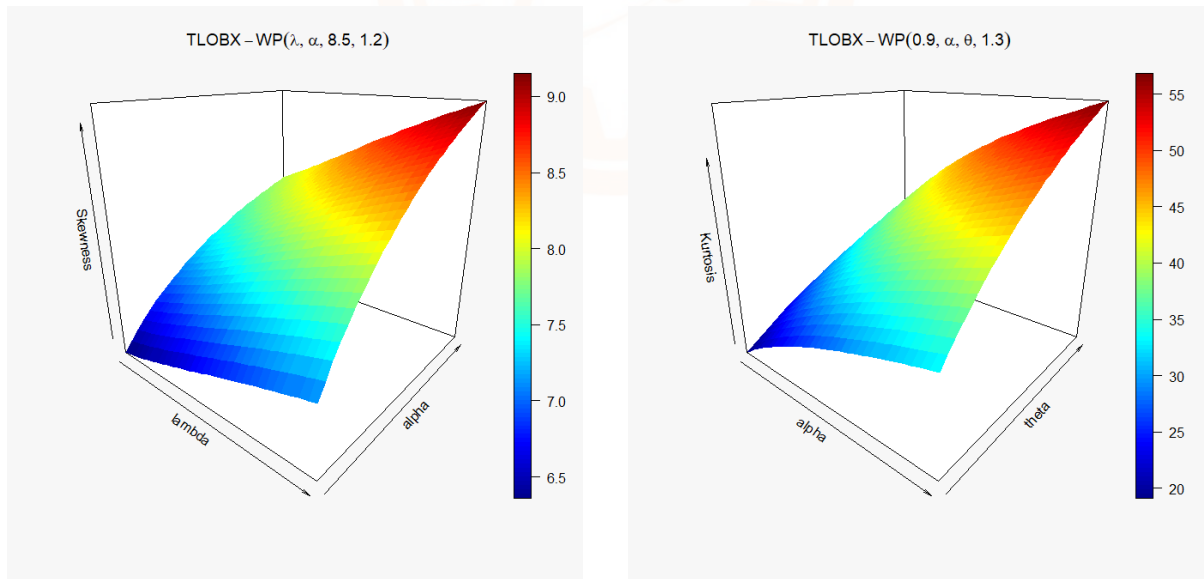


Figure 4.2: Plots of skewness and kurtosis for the TLOBX-WP distribution for a few chosen parameter values

4.3.4 Distribution of Order Statistics

Order statistics is an important concept used in several areas including reliability, insurance and related areas. The i^{th} order statistic's PDF from the TLOBX-GPS class with $X_1, X_2, X_3, \dots, X_n$

representing the random samples from this class, is given as

$$f_{i:n}(x; \alpha, \theta, \lambda, \underline{\omega}) = \frac{n!f(x; \lambda, \alpha, \theta, \underline{\omega})}{(i-1)!(n-i)!} \sum_{t=0}^{n-i} (-1)^t \binom{n-i}{t} F^{t+i-1}(x; \lambda, \alpha, \theta, \underline{\omega}).$$

Plugging in the PDF in Equation (4.11) and the CDF in Equation (4.6), we get

$$f_{i:n}(x; \alpha, \theta, \lambda, \underline{\omega}) = \sum_{t=0}^{n-i} \sum_{s=1}^{\infty} \frac{sn!a_s \lambda^s f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) [1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]^{s-1}}{(i-1)!(n-i)!C(\lambda)} \\ \times (-1)^t \binom{n-i}{t} \left[1 - \frac{C(\lambda(1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})))}{C(\lambda)} \right]^{t+i-1}.$$

Applying expansions as shown in Appendix D.2 we get

$$f_{i:n}(x; \alpha, \theta, \lambda, \underline{\omega}) = \sum_{p=0}^{\infty} D_{p+1}^* h_{p+1}(x; \underline{\omega}),$$

where

$$D_{p+1}^* = \sum_{b+h,j,k,m,q,u=0}^{\infty} \sum_{t=0}^{n-i} \sum_{s=1}^{\infty} \frac{sn!a_s b_{g,h}(k+1)^m \lambda^{s+h} (-1)^{b+j+k+m+p+h+q+t+u}}{(4\alpha\theta)^{-1} m! (i-1)! (n-i)! C^{h+1}(\lambda)} \\ \times \binom{n-i}{t} \binom{t+i-1}{h} \binom{h+s-1}{q} \binom{\alpha(q+1)-1}{u} \binom{2u+1}{j} \binom{\theta(j+1)-1}{k} \\ \times \binom{2m+1}{b} \binom{b-(2m+3)}{p}$$

and the Exp-G density function is given as $h_{p+1}(x; \underline{\omega}) = (p+1)g(x; \underline{\omega})G^{p+1-1}(x; \underline{\omega})$. As a result, it is demonstrated that the TLOBX-GPS class ' i^{th} order statistics' PDF may be written as an unlimited combination of Exp-G densities.

4.4 Some Special Cases

There are several models that can be generated from the TLOBX-GPS class but in this section we present the cases when the power series distributions are Poisson and binomial distributions, and take the baseline distributions as Weibull and log-logistic distributions. For the Poisson distribution is $C(\lambda) = e^\lambda - 1$ and $C'(\lambda) = e^\lambda$, while for the binomial distribution is $C(\lambda) = (1 + \lambda)^m - 1$ and $C'(\lambda) = m(1 + \lambda)^{m-1}$.

4.4.1 Topp-Leone Odd Burr X-Weibull Poisson Distribution

Considering the PDF $g(x; \beta) = \beta x^{\beta-1} e^{-x^\beta}$ and CDF $G(x; \beta) = 1 - e^{-x^\beta}$ of Weibull distribution, Topp-Leone odd Burr X-Weibull Poisson (TLOBX-WP) distribution is obtained with PDF and

CDF given by

$$\begin{aligned}
 f(x; \alpha, \theta, \lambda, \beta) &= \frac{4\lambda\alpha\theta\beta x^{\beta-1} e^{2x^\beta} (1 - e^{-x^\beta}) \{1 - \exp[-K_1(x; \beta)]\}^{\theta-1}}{[e^\lambda - 1] \exp[K_1(x; \beta)]} \\
 &\times \exp \left[\lambda \left(1 - \left\{ 1 - \left[1 - \{1 - \exp[-K_1(x; \beta)]\}^\theta \right]^2 \right\}^\alpha \right) \right] \\
 &\times \left[1 - \{1 - \exp[-K_1(x; \beta)]\}^\theta \right] \left\{ 1 - \left[1 - \{1 - \exp[-K_1(x; \beta)]\}^\theta \right]^2 \right\}^{\alpha-1}, x \in \mathbb{R}, \\
 &\alpha, \theta, \lambda, \beta > 0
 \end{aligned}$$

and

$$F(x; \alpha, \theta, \lambda, \beta) = 1 - \frac{\exp \left[\lambda \left(1 - \left\{ 1 - \left[1 - \{1 - \exp[-K_1(x; \beta)]\}^\theta \right]^2 \right\}^\alpha \right) \right] - 1}{\exp(\lambda) - 1},$$

respectively, where $K_1(x; \beta) = \left(\frac{1 - e^{-x^\beta}}{e^{-x^\beta}} \right)^2$. As shown in Figure 4.3, TLOBX-WP distribution can handle data with reverse-J, left-skewed, right-skewed and almost symmetric shapes. It further shows that the HRF plots can be increasing, decreasing, bathtub and up-side-down bathtub shaped.

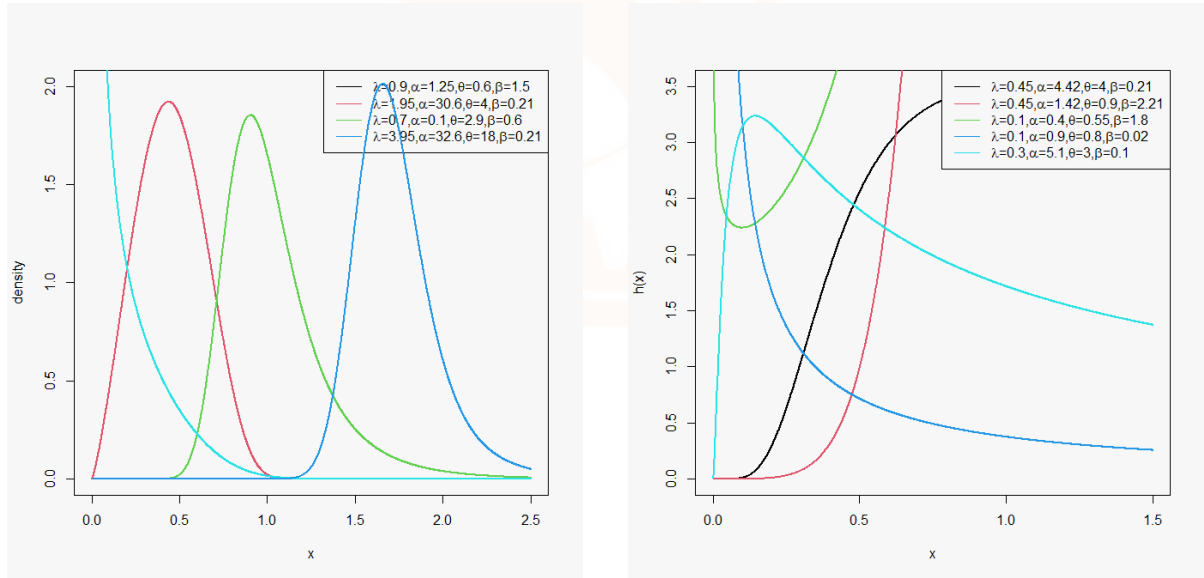


Figure 4.3: TLOBX-WP distribution's PDF and HRF plots

4.4.2 Topp-Leone Odd Burr X-Log-Logistic Poisson Distribution

Considering the PDF $g(x; c) = cx^{c-1}(1+x^c)^{-2}$ and CDF $G(x; c) = 1 - (1+x^c)^{-1}$ of log-logistic distribution, the Topp-Leone odd Burr X-log-logistic Poisson (TLOBX-LP) distribution

is obtained with PDF and CDF given by

$$\begin{aligned}
 f(x; \alpha, \theta, \lambda, c) &= \frac{4\lambda\alpha\theta cx^{c-1}(1+x^c)[1-(1+x^c)^{-1}]}{[e^\lambda - 1]\exp[K_2(x; c)]} \{1 - \exp[-K_2(x; c)]\}^{\theta-1} \\
 &\times \exp\left[\lambda\left(1 - \left\{1 - \left[1 - \{1 - \exp[-K_2(x; c)]\}^\theta\right]^2\right\}^\alpha\right)\right] \\
 &\times \left[1 - \{1 - \exp[-K_2(x; c)]\}^\theta\right] \left\{1 - \left[1 - \{1 - \exp[-K_2(x; c)]\}^\theta\right]^2\right\}^{\alpha-1}, x \in \mathbb{R}, \\
 \alpha, \theta, \lambda, c &> 0
 \end{aligned}$$

and

$$F(x; \alpha, \theta, \lambda, c) = 1 - \frac{\exp\left[\lambda\left(1 - \left\{1 - \left[1 - \{1 - \exp[-K_2(x; c)]\}^\theta\right]^2\right\}^\alpha\right)\right] - 1}{\exp(\lambda) - 1},$$

respectively, where $K_2(x; c) = \left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^2$. Figure 4.4 indicate that the PDF of the TLOBX-LP distribution can take different shapes. Additionally, the HRF shows that it can be used in datasets with increasing, decreasing, bathtub and upside-down bathtub shapes.

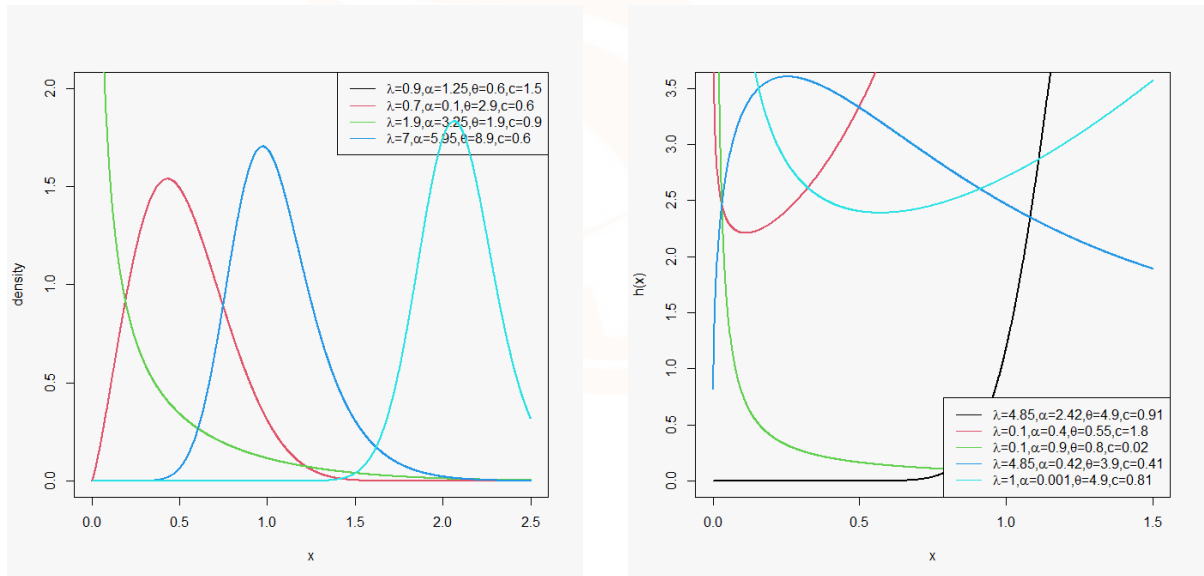


Figure 4.4: TLOBX-LP distribution's PDF and HRF plots

4.4.3 Topp-Leone Odd Burr X-Weibull Binomial Distribution

Considering the combination of Weibull and binomial distributions, we get the PDF and CDF of the Topp-Leone odd Burr X-Weibull Binomial (TLOBX-WB) distribution as

$$f(x; \alpha, \theta, \lambda, m, \beta) = \frac{4\lambda\alpha\theta m\beta x^{\beta-1} e^{2x^\beta} (1 - e^{-x^\beta}) \{1 - \exp[-K_1(x; \beta)]\}^{\theta-1}}{m(1 + \lambda)^{m-1} \exp[K_1(x; \beta)]} \\ \times \left(1 + \left[\lambda \left(1 - \left\{1 - \left[1 - \{1 - \exp[-K_1(x; \beta)]\}^\theta\right]^2\right\}^\alpha\right)\right]\right)^{m-1} \\ \times \left[1 - \{1 - \exp[-K_1(x; \beta)]\}^\theta\right] \left\{1 - \left[1 - \{1 - \exp[-K_1(x; \beta)]\}^\theta\right]^2\right\}^{\alpha-1}, x \in \mathbb{R}, \\ \alpha, \theta, \lambda, m, \beta > 0$$

and

$$F(x; \alpha, \theta, \lambda, m, \beta) = 1 - \frac{\left\{1 + \lambda \left(1 - \left\{1 - \left[1 - \{1 - \exp[-K_1(x; \beta)]\}^\theta\right]^2\right\}^\alpha\right)\right\}^m - 1}{\{1 + \lambda\}^m - 1},$$

respectively, where $K_1(x; \beta) = \left(\frac{1 - e^{-x^\beta}}{e^{-x^\beta}}\right)^2$. TLOBX-WB distribution can model datasets of different skewness and kurtosis as shown in Figure 4.5. The HRF can also adopt increasing, decreasing, bathtub, and upside-down bathtub forms.

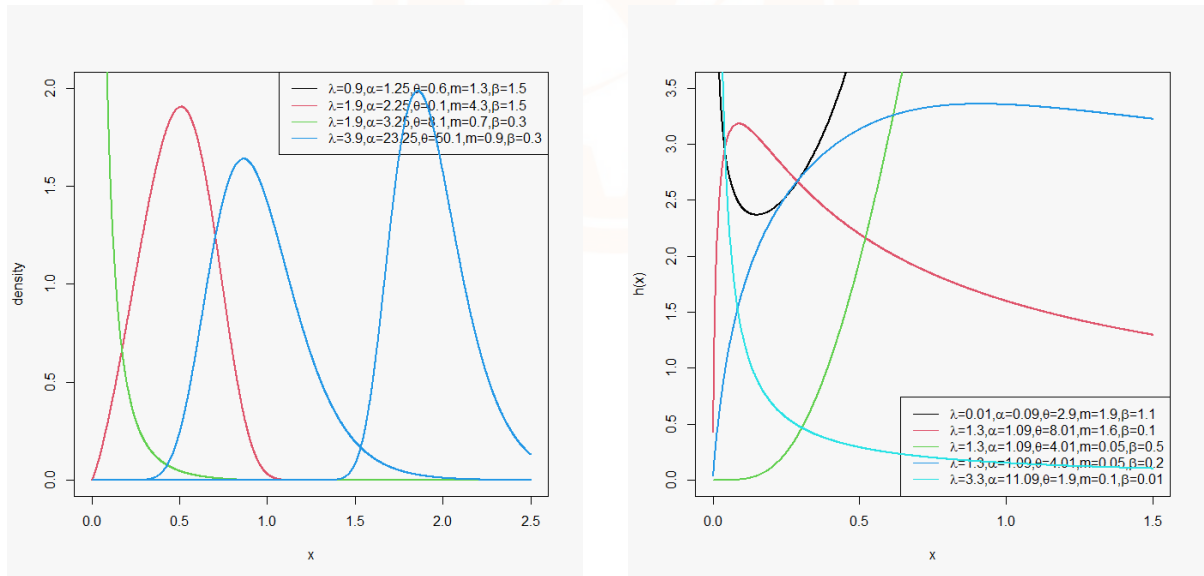


Figure 4.5: TLOBX-WB distribution's PDF and HRF plots

4.4.4 Topp-Leone Odd Burr X-Log-Logistic Binomial Distribution

Considering the combination of log-logistic and binomial distributions, we get the Topp-Leone odd Burr X-log-logistic Binomial (TLOBX-LB) distribution with PDF and CDF given as

$$f(x; \alpha, \theta, \lambda, m, \beta) = \frac{4\lambda\alpha\theta m\beta x^{\beta-1}(1+x^\beta)(1-(1+x^\beta)^{-1})\{1-\exp[-K_2(x; \beta)]\}^{\theta-1}}{m(1+\lambda)^{m-1}\exp[K_2(x; \beta)]} \\ \times \left(1 + \left[\lambda \left(1 - \left\{1 - \left[1 - \{1 - \exp[-K_2(x; \beta)]\}^\theta\right]^2\right\}^\alpha\right)\right]\right)^{m-1} \\ \times \left[1 - \{1 - \exp[-K_2(x; \beta)]\}^\theta\right] \left\{1 - \left[1 - \{1 - \exp[-K_2(x; \beta)]\}^\theta\right]^2\right\}^{\alpha-1}, x \in \mathbb{R}, \\ \alpha, \theta, \lambda, m, \beta > 0$$

and

$$F(x; \alpha, \theta, \lambda, m, \beta) = 1 - \frac{\left\{1 + \lambda \left(1 - \left\{1 - \left[1 - \{1 - \exp[-K_2(x; \beta)]\}^\theta\right]^2\right\}^\alpha\right)\right\}^m - 1}{\{1 + \lambda\}^m - 1}$$

respectively, where $K_2(x; \beta) = \left(\frac{1-(1+x^\beta)^{-1}}{(1+x^\beta)^{-1}}\right)^2$. As shown by PDF plots in Figure 4.6, TLOBX-LB distribution can model skewed datasets and it also have HRF that possess monotonic and non-monotonic shapes.

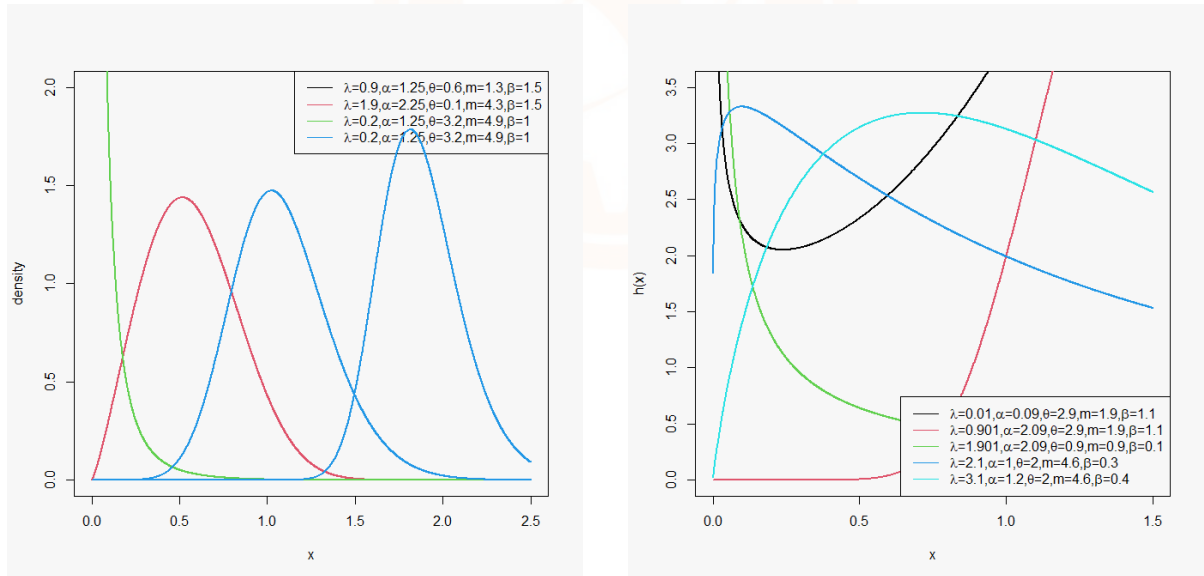


Figure 4.6: TLOBX-LB distribution's PDF and HRF plots

4.5 Parameter Estimation

We present different techniques that can be used to estimate the parameters TLOBX-GPS class within this section. The techniques are; Maximum Likelihood, Anderson-Darling and Right-Tail Anderson Darling, Cramér-Von Mises, Ordinary and Weighted Least Squares and Maximum Product of Spacing.

4.5.1 Maximum Likelihood Method

In this part, we use the maximum likelihood approach to estimate the parameters of the TLOBX-GPS class. Let TLOBX-GPS class have n rvs $X_1, X_2, \dots, X_n$ with parameter vector $\Psi = (\alpha, \theta, \lambda, \underline{\omega})^T$. Then using expression in Equation (4.11), we can express the log-likelihood function as

$$\begin{aligned} \ell_n \equiv \ell_n(\Psi) = & n \log \left(\frac{\lambda}{C(\lambda)} \right) + \sum_{i=1}^{\infty} \log (f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) \\ & + \sum_{i=1}^{\infty} \log (C'(\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))), \end{aligned} \quad (4.16)$$

where $f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})$ and $S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})$ are given in Equations (4.1) and (4.3) respectively. The score vector's components are listed in Appendix D.3. To find solutions to $\lambda, \alpha, \theta, \underline{\omega}$, score vector's components are equated to zero. For such complex equations we obtain solutions using iterative methods. The confidence regions for model parameter derivation rely on large sample theory. Using the theory, we have $\Psi \sim N_4(\underline{0}, I^{-1}(\Psi))$ where $\underline{0} = (0, 0, 0, 0)^T$ and $I(\hat{\Psi})^{-1}$ are the mean vector and the observed Fisher information matrix.

4.5.2 Anderson-Darling and Right-Tail Anderson Darling Methods

Let $x_{(i)}$ be the ordered sample values of size n from random realization associated with the TLOBX-GPS class distribution function $F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})$ given in Equation (4.9). Then, the Anderson Darling (AD) estimators of $\alpha, \theta, \lambda, \underline{\omega}$ minimize the function

$$A(\alpha, \theta, \lambda, \underline{\omega}) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log(F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})) + \log(S(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}))],$$

where $S(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) = 1 - F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})$. The AD estimators can also be derived by solving

$$\sum_{i=1}^n (2i-1) \left[\frac{\Delta_s(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})}{F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})} - \frac{\Delta_s(x_{n+1-i:n}; \alpha, \theta, \lambda, \underline{\omega})}{S(x_{n+1-i:n}; \alpha, \theta, \lambda, \underline{\omega})} \right] = 0, \quad (4.17)$$

where $s = 1, 2, 3, 4$, and

$$\Delta_s(\bullet; \alpha, \theta, \lambda, \underline{\omega}) = \begin{cases} \frac{\partial F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})}{\partial \alpha} \\ \frac{\partial F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})}{\partial \theta} \\ \frac{\partial F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})}{\partial \lambda} \\ \frac{\partial F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})}{\partial \omega_k} \end{cases} \quad (4.18)$$

The Right-Tailed Anderson Darling (RTAD) estimators of $\alpha, \theta, \lambda, \underline{\omega}$ minimize the function

$$R(\alpha, \theta, \lambda, \underline{\omega}) = \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log(S(x_{n+1-i:n}; \alpha, \theta, \lambda, \underline{\omega})),$$

with respect to α, θ, λ , and parameter vector $\underline{\omega}$. The RAD estimators can also be obtained by

solving

$$-2 \sum_{i=1}^n \Delta_s(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) + \frac{1}{n} \sum_{i=1}^n (2i-1) \frac{\Delta_s(x_{n+1-i:n}; \alpha, \theta, \lambda, \underline{\omega})}{S(x_{n+1-i:n}; \alpha, \theta, \lambda, \underline{\omega})} = 0,$$

where $s = 1, 2, 3, 4$, and $\Delta_s(\bullet; \alpha, \theta, \lambda, \underline{\omega})$ is defined in Equation (4.18).

4.5.3 Cramér-Von Mises Method

CVM estimators minimizes the function

$$C(\alpha, \theta, \lambda, \underline{\omega}) = -\frac{1}{12n} \sum_{i=1}^n \left[F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) - \frac{2i-1}{2n} \right]^2,$$

with respect to α, θ, λ , and parameter vector $\underline{\omega}$. The CVM estimators can also be obtained by solving

$$\sum_{i=1}^n \left[F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) - \frac{2i-1}{2n} \right] \Delta_s(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) = 0,$$

where $s = 1, 2, 3, 4$, and $\Delta_s(\bullet; \alpha, \theta, \lambda, \underline{\omega})$ is defined in Equation (4.18).

4.5.4 Ordinary and Weighted Least Squares Methods

Ordinary Least Squares (OLS) estimators are obtained by minimizing the function

$$V(\alpha, \theta, \lambda, \underline{\omega}) = \sum_{i=1}^n \left[F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) - \frac{i}{n+1} \right]^2.$$

The OLS estimators can also be obtained by solving

$$\sum_{i=1}^n \left[F(x_i; \alpha, \theta, \lambda, \underline{\omega}) - \frac{i}{n+1} \right] \Delta_s(x_i; \alpha, \theta, \lambda, \underline{\omega}) = 0,$$

where $s = 1, 2, 3, 4$ and $\Delta_s(\bullet; \alpha, \theta, \lambda, \underline{\omega})$ is defined in Equation (4.18).

To obtain WLS estimators of $\alpha, \theta, \lambda, \underline{\omega}$, we minimize

$$W(\alpha, \theta, \lambda, \underline{\omega}) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-1+1)} \left[F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) - \frac{i}{n+1} \right]^2,$$

with respect to α, θ, λ , and parameter vector $\underline{\omega}$. The OLS estimators can also be obtained by solving

$$\sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-1+1)} \left[F(x_i; \alpha, \theta, \lambda, \underline{\omega}) - \frac{i}{n+1} \right] \Delta_s(x_i; \alpha, \theta, \lambda, \underline{\omega}) = 0,$$

where $s = 1, 2, 3, 4$, and $\Delta_s(\bullet; \alpha, \theta, \lambda, \underline{\omega})$ is defined in Equation (4.18).

4.5.5 Maximum Product of Spacing Method

Let the uniform spacings of the rv $X_1, X_2, \dots, X_n$ following TLOBX-GPS class be $D_i(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) = F(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) - F(x_{i-1:n}; \alpha, \theta, \lambda, \underline{\omega})$, for $i = 1, 2, \dots, n$,

$F(x_0 : n; \alpha, \theta, \lambda, \underline{\omega}) = 0$ and $F(x_{n+1:n}; \alpha, \theta, \lambda, \underline{\omega}) = 1$. From the definition, $\sum_{i=0}^{n+1} D_i(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) = 1$. The MPS estimators can be obtained by maximizing the function

$$S(\alpha, \theta, \lambda, \underline{\omega}) = \left[\prod_{i=1}^{n+1} D_i(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) \right]^{\frac{1}{n+1}},$$

or equivalently by maximizing the function

$$H(\alpha, \theta, \lambda, \underline{\omega}) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log(D_i(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega})),$$

with respect to α, θ, λ , and parameter vector $\underline{\omega}$. Alternatively, MPS estimators can be obtained by solving

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(x_i; \alpha, \theta, \lambda, \underline{\omega})} [\Delta_s(x_{i:n}; \alpha, \theta, \lambda, \underline{\omega}) - \Delta_s(x_{i-1:n}; \alpha, \theta, \lambda, \underline{\omega})] = 0,$$

where $s = 1, 2, 3, 4$, and $\Delta_s(\bullet; \alpha, \theta, \lambda, \underline{\omega})$ is defined in Equation (4.18).

4.6 Simulation Study

In this part, we check the consistency of the parameter estimates for the TLOBX-WP distribution that are derived using maximum likelihood. We conduct simulations for sample sizes $n = 50, 100, 200, 400, 800$ and 1600 , repeated $N = 3000$ times. Tables 4.2 and 4.3 gives mean MLEs together with their respective RMSEs and Abias. We observe that the mean estimates approach the true parameter values as the sample size increases which is an indication of estimates consistency. Moreover, RMSEs and Abias decay to zero as the sample size increases.

Table 4.2: TLOBX-WP distribution simulation results

Parameter	Sample Size	(0.5, 0.5, 1.7, 1.7)			(1.2, 1.2, 1.5, 0.4)		
		Mean	RMSE	Abias	Mean	RMSE	Abias
α	50	1.2418	2.7883	0.7418	1.2838	0.5144	0.0838
	100	0.8226	1.5366	0.3226	1.2605	0.3069	0.0605
	200	0.6347	0.7176	0.1347	1.2534	0.1907	0.0534
	400	0.6199	0.6094	0.1199	1.2387	0.1291	0.0387
	800	0.5566	0.3476	0.0566	1.2273	0.0925	0.0273
	1600	0.5318	0.2608	0.0318	1.2143	0.0652	0.0143
θ	50	0.7701	0.8350	0.2701	1.1448	0.3904	-0.0552
	100	0.7234	0.7178	0.2234	1.1588	0.2161	-0.0412
	200	0.6647	0.5276	0.1647	1.1917	0.1634	-0.0083
	400	0.6058	0.4257	0.1058	1.2090	0.1188	0.0090
	800	0.5629	0.2697	0.0629	1.2140	0.0847	0.0140
	1600	0.5471	0.2196	0.0471	1.2182	0.0653	0.0182
λ	50	1.9734	1.0125	0.2734	1.9376	1.3321	0.4376
	100	1.7684	0.7328	0.0684	1.8565	1.2745	0.3565
	200	1.7233	0.5770	0.0233	1.8017	1.2277	0.3017
	400	1.6711	0.5291	-0.0289	1.6781	0.9429	0.1781
	800	1.6635	0.4378	-0.0365	1.6350	0.8380	0.1350
	1600	1.7010	0.3341	0.0010	1.5614	0.7608	0.0614
β	50	1.8248	0.7800	0.1248	2.8225	12.3992	2.4225
	100	1.7198	0.5984	0.0198	0.6808	2.1651	0.2808
	200	1.6694	0.4833	-0.0306	0.4453	0.4510	0.0453
	400	1.6423	0.3587	-0.0577	0.4028	0.2416	0.0028
	800	1.6344	0.2695	-0.0656	0.4232	0.2095	0.0232
	1600	1.6492	0.1848	-0.0508	0.4419	0.1954	0.0419

Table 4.3: TLOBX-WP distribution simulation results

Parameter	Sample Size	(0.5, 1.7, 1.7, 1.7)			(1.5, 1.3, 0.2, 0.2)		
		Mean	RMSE	Abias	Mean	RMSE	Abias
α	50	2.2059	7.7597	1.7059	1.9493	1.3189	0.4493
	100	0.9916	2.5450	0.4916	1.8576	1.0351	0.3576
	200	0.7802	1.1937	0.2802	1.8107	0.9507	0.3107
	400	0.6410	0.6375	0.1410	1.6786	0.8695	0.1786
	800	0.6435	0.5418	0.1435	1.6458	0.8130	0.1458
	1600	0.6432	0.4378	0.1432	1.5413	0.7857	0.0413
θ	50	1.9048	1.8609	0.2048	1.3291	0.4423	0.0291
	100	1.8859	1.5869	0.1859	1.3016	0.2996	0.0016
	200	1.7947	1.2985	0.0947	1.3018	0.2075	0.0018
	400	1.7690	1.0621	0.0690	1.3059	0.1453	0.0059
	800	1.7152	0.8848	0.0152	1.2980	0.0967	-0.0020
	1600	1.6714	0.7389	-0.0286	1.3001	0.0671	0.0001
λ	50	2.7439	1.7859	1.0439	0.9218	4.2355	0.7218
	100	2.5117	1.4275	0.8117	0.3422	0.8512	0.1422
	200	2.2028	1.1206	0.5028	0.2686	0.2125	0.0686
	400	2.0152	0.9074	0.3152	0.2666	0.1387	0.0666
	800	1.8141	0.7931	0.1141	0.2861	0.1675	0.0861
	1600	1.6697	0.6908	-0.0303	0.2586	0.1074	0.0586
β	50	3.1315	2.4280	1.4315	0.3448	0.4443	0.1448
	100	2.9786	2.1926	1.2786	0.3480	0.4418	0.1480
	200	2.6472	1.8140	0.9472	0.3296	0.4020	0.1296
	400	2.3012	1.3335	0.6012	0.3026	0.3433	0.1026
	800	2.0383	0.9489	0.3383	0.2627	0.2419	0.0627
	1600	1.8285	0.4930	0.1285	0.2283	0.1483	0.0283

4.7 Applications

Using three actual datasets, we demonstrate the adaptability of the TLOBX-WP distribution in this section. Our new model is compared with Marshall-Olkin odd Burr III-logistic (MOOBIII-L) distribution (Affify et al. 2020), generalized Gompertz-logarithmic distribution (Tahmasebi and Jafari, 2016), exponentiated Burr XII Poisson (EBXIIP) distribution (da Silva et al. 2015) and Topp-Leone-Weibull Poisson (TL-WP) distribution (Makubate et al. 2021b). We also compared the TLOBX-WP distribution with its sub-models: TLOBX-WP $(\lambda, 1, \theta, \beta)$ and TLOBX-WP $(\lambda, \alpha, 1, \beta)$ distributions. In Tables 4.4, 4.5 and 4.6, MLEs (SE in parentheses), $-2 \log L$, AIC, BIC, AICC, W^* , (A^*) , SS and K-S together with its corresponding p-value are also presented. We also provide graphical plots such as the fitted densities, P-Ps, HRFs and KM survival plots.

4.7.1 Taxes Revenue Data

The first dataset with $n = 59$ observations given in Appendix D.5 comes from Nassar and Nada (2011) and it includes monthly real tax revenue (in 1000 million Egyptian pounds) for Egypt between January 2006 and November 2010.

With data on tax revenue, the estimated var-cov matrix for the TLOBX-WP distribution is provided by

$$\begin{bmatrix} 2.94 & -5.28 \times 10^{-3} & -3.78 \times 10^{-1} & -1.26 \times 10^{-2} \\ -5.28 \times 10^{-3} & 4.19 \times 10^{-4} & 2.61 \times 10^{-2} & 1.01 \times 10^{-4} \\ -3.78 \times 10^{-1} & 2.61 \times 10^{-2} & 1.63 & 6.52 \times 10^{-3} \\ -1.26 \times 10^{-2} & 1.01 \times 10^{-4} & 6.52 \times 10^{-3} & 7.28 \times 10^{-5} \end{bmatrix}.$$

The 95% confidence intervals for the TLOBX-WP model parameters are given by $\alpha \in [2.71 \times 10^2 \pm 4.01 \times 10^{-2}]$, $\theta \in [8.16 \pm 2.50]$, $\lambda \in [2.53 \times 10^{-1} \pm 3.]$ and $\beta \in [6.73 \times 10^{-2} \pm 1.67 \times 10^{-2}]$.

Table 4.4: The MLEs of the TLOBX-WP and competing models and GOF statistics for the failure times data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
	α	θ	λ	β									
TLOBX-WP	2.71×10^2 (2.04×10^{-2})	8.16 (1.27)	2.53×10^{-1} (1.71)	6.73×10^{-2} (8.53×10^{-3})	376.5	384.5	385.2	392.8	0.0412	0.2519	0.0066	0.9614	0.0411
		θ	λ	β									
TLOBX-WP(1, θ , λ , β)	1	8.92 (8.31×10^{-1})	9.43×10^{-9} (9.93×10^{-3})	2.49×10^{-2} (2.71×10^{-3})	725.8	731.8	732.3	738.1	0.1094	0.6452	0.9194	< 0.0001	16.4269
	α		λ	β									
TLOBX-WP(α , 1, λ , β)	3.90×10^1 (5.08)	1	8.54×10^{-9} (9.76×10^{-3})	2.19×10^{-2} (2.50×10^{-3})	772.9	778.9	779.4	785.2	0.1050	0.6169	0.9438	< 0.0001	17.3383
	δ	α	β	λ									
MOOBII-L	7.08×10^2 (5.25×10^{-7})	1.42×10^1 (1.39)	3.33×10^{-2} (1.22×10^{-2})	5.55×10^1 (3.58×10^{-1})	407.6	415.6	416.3	423.9	0.4286	2.5919	0.1328	0.2488	0.2763
	θ	α	β	γ									
GG-L	1.89×10^1 (6.09)	2.27×10^{-1} (8.15×10^{-2})	1.58×10^{-1} (3.62×10^{-2})	5.78×10^{-10} (1.23×10^{-2})	385.1	393.1	393.8	401.4	0.1702	1.0368	0.1090	0.4838	0.1376
	c	s	α	λ									
EBXHIP	1.01 (1.53×10^{-1})	1.83 (4.49×10^{-1})	2.93 (4.32×10^{-1})	9.95×10^{-10} (4.46×10^{-3})	456.3	464.3	465.0	472.6	0.0695	0.3987	0.4065	< 0.0001	2.8251
	α	β	b	θ									
TL-WP	2.76×10^{-1} (7.48×10^{-2})	4.64×10^{-1} (7.59×10^{-2})	2.39 (4.21×10^{-1})	6.39×10^{-9} (7.94×10^{-3})	457.5	465.5	466.3	473.8	0.1343	0.8065	0.3995	< 0.0001	2.5148

Table 4.4 indicates that the TLOBX-WP distribution outperforms the other competing models for taxes revenue data since it have the smallest values of all the GOF statistics together with the largest p-value. Figures 4.7 and 4.8 show how well TLOBX-WP distribution fit the taxes revenue data. TLOBX-WP distribution can handle datasets with non-monotonic HRF shapes as shown in Figure 4.8

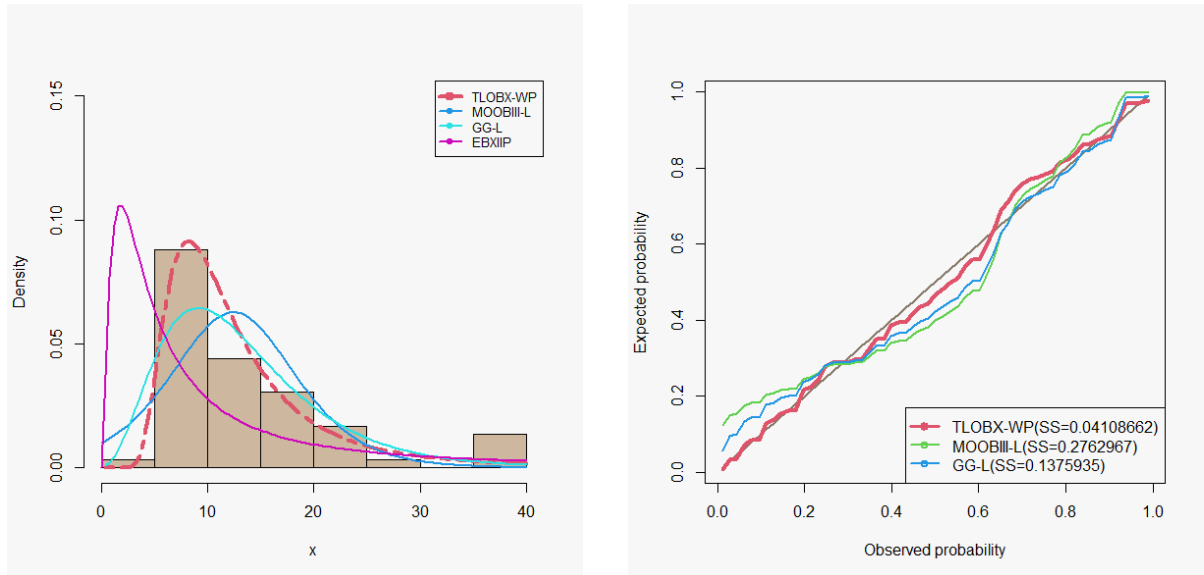


Figure 4.7: Fitted TLOBX-WP densities and P-Ps for taxes revenue dataset

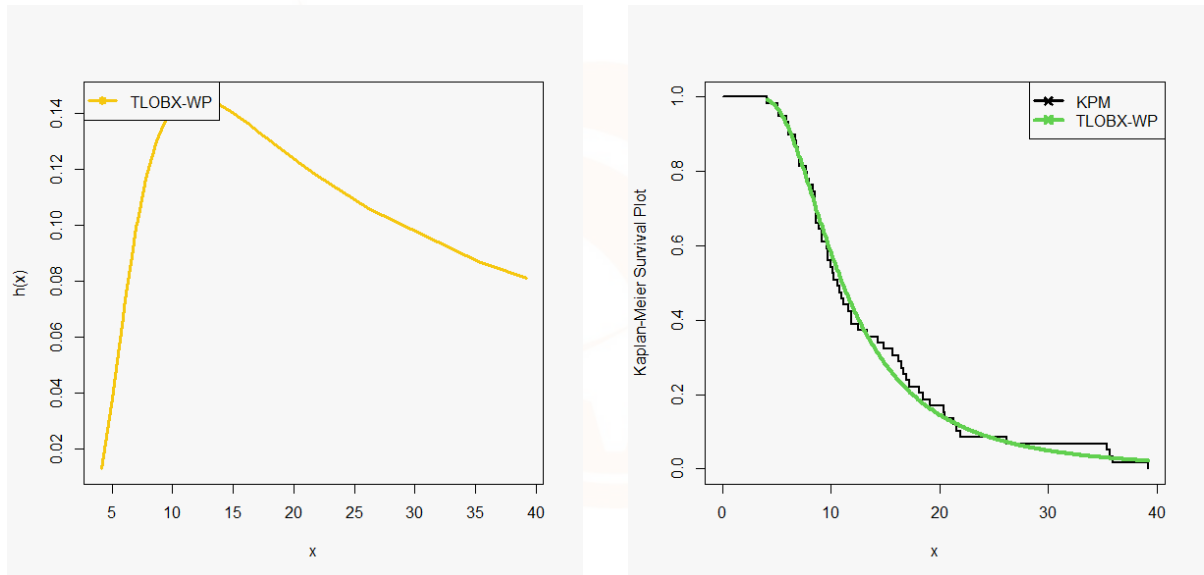
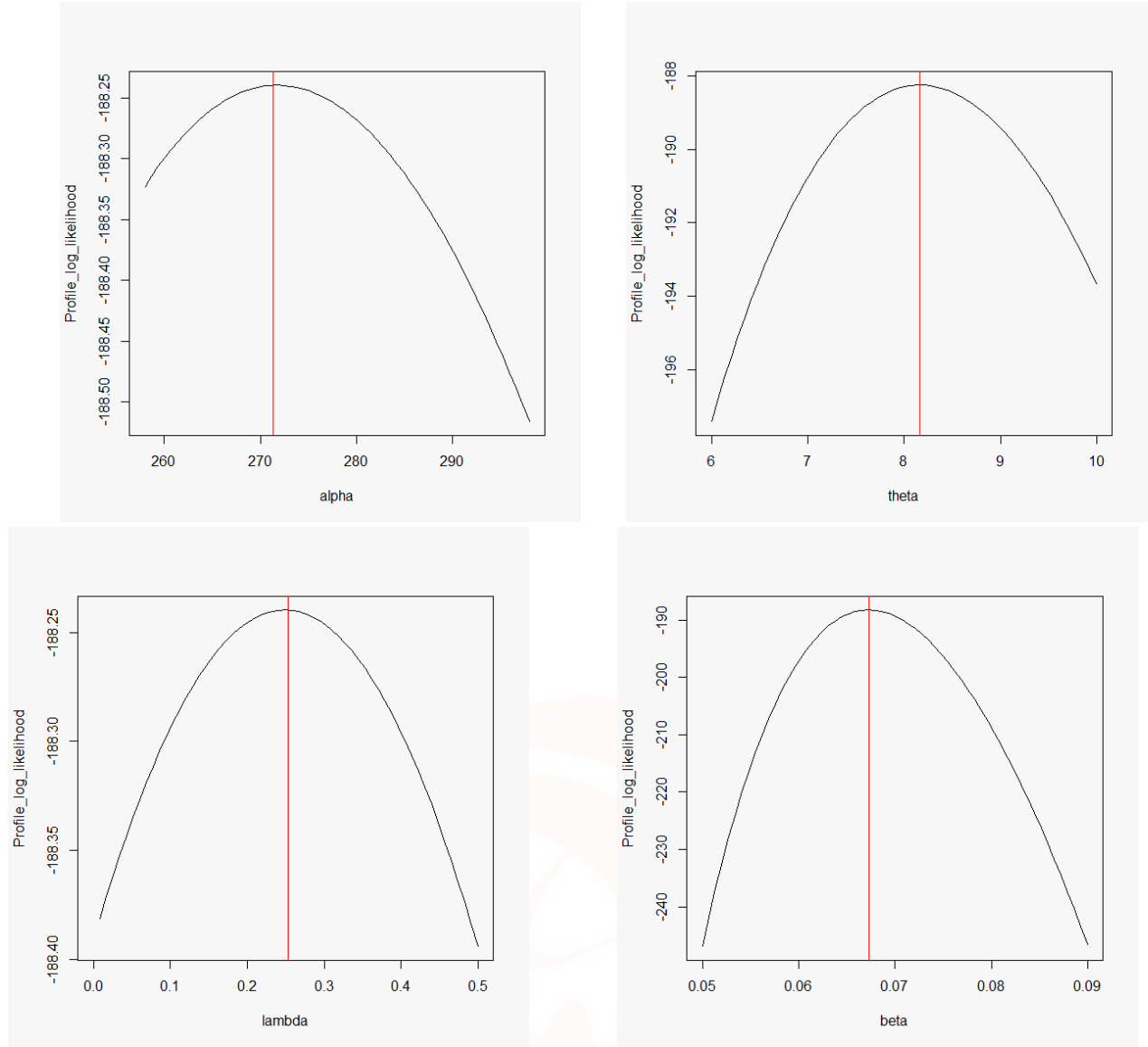


Figure 4.8: TLOBX-WP HRF and KM survival plots for taxes revenue dataset

Figure 4.9: Likelihood profile plots for α, θ, λ and β in taxes revenue data

As shown in Figure 4.9 which gives profile log-likelihood function for taxes revenue data, TLOBX-WP distribution estimates obtained maximizes the likelihood function.

4.7.2 Waiting Times Data

The second dataset with $n = 100$ observations given in Appendix D.5 comes from Ruidas (2020) and it includes waiting times (in minutes) before service of 100 bank customers.

With data on waiting times, the estimated var-cov matrix for the TLOBX-WP model is provided by

$$\begin{bmatrix} 1.21 \times 10^{-5} & -6.92 \times 10^{-4} & -1.43 \times 10^{-3} & -3.35 \times 10^{-6} \\ -6.92 \times 10^{-4} & 3.97 \times 10^{-2} & 8.19 \times 10^{-2} & 1.92 \times 10^{-4} \\ -1.43 \times 10^{-3} & 8.19 \times 10^{-2} & 1.69 \times 10^{-1} & 3.97 \times 10^{-4} \\ -3.35 \times 10^{-6} & 1.92 \times 10^{-4} & 3.97 \times 10^{-4} & 1.16 \times 10^{-6} \end{bmatrix}.$$

The 95% confidence intervals for the TLOBX-WP model parameters are given by $\alpha \in [2.17 \times 10^1 \pm 3.91 \times 10^{-1}]$, $\theta \in [1.64 \times 10^1 \pm 8.06 \times 10^{-1}]$, $\lambda \in [1.14 \times 10^2 \pm 6.81 \times 10^{-3}]$ and $\beta \in [1.69 \times 10^{-2} \pm 2.11 \times 10^{-3}]$.

Table 4.5: The MLEs of the TLOBX-WP and competing models and GOF statistics for the failure times data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
	α	θ	λ	β									
TLOBX-WP	2.17×10^1 (1.99×10^{-1})	1.64×10^1 (4.11×10^{-1})	1.14×10^2 (3.47×10^{-3})	1.69×10^{-2} (1.08×10^{-3})	634.0	642.0	642.5	652.5	0.0181	0.1310	0.0363	0.9994	0.0175
TLOBX-WP($1, \theta, \lambda, \beta$)	1	9.27 (6.64×10^{-1})	4.34×10^{-9} (5.73×10^{-3})	2.94×10^{-2} (2.44×10^{-3})	1094.9	1100.9	1101.2	1108.8	0.0457	0.3489	0.8434	< 0.0001	26.3461
TLOBX-WP($\alpha, 1, \lambda, \beta$)	1.68×10^1 (1.68)	1	3.97×10^{-9} (5.98×10^{-3})	2.56×10^{-2} (2.26×10^{-3})	1349.6	1355.6	1355.9	1363.5	0.0488	0.3713	0.9502	< 0.0001	31.1892
	δ	α	β	λ									
MOOBII-L	7.09×10^2 (1.25×10^{-7})	1.70×10^1 (1.25)	1.81×10^{-2} (4.39×10^{-3})	6.11×10^1 (3.49×10^{-1})	676.7	684.7	685.1	695.1	0.4566	2.7883	0.1163	0.1340	0.2693
	θ	α	β	γ									
GG-L	1.49×10^1 (6.79)	1.55×10^{-1} (8.22×10^{-2})	1.61×10^{-1} (3.42×10^{-2})	3.27×10^{-9} (1.49×10^{-2})	635.1	643.1	643.5	653.5	0.0289	0.1879	0.0409	0.9961	0.0263
	c	s	α	λ									
EBXIIIP	1.16 (1.61×10^{-1})	1.99 (3.34×10^{-1})	2.41 (3.02×10^{-1})	3.62×10^{-9} (6.45×10^{-3})	682.9	690.9	691.4	701.4	0.1502	1.0472	0.2499	< 0.0001	1.9711
	α	β	b	θ									
TL-WP	6.53×10^{-1} (1.98×10^{-1})	2.79×10^{-1} (6.49×10^{-2})	1.51 (1.91×10^{-1})	6.90×10^{-9} (7.12×10^{-3})	892.7	900.7	901.1	911.1	0.02786	0.2196	0.6589	< 0.0001	16.6213

From Table 4.5, the TLOBX-WP distribution have the smallest values of all the GOF statistics together with the largest p-value corresponding to K-S statistic indicating that it outperformed the other competing models for waiting times data. Figures 4.10 and 4.11 show how well TLOBX-WP distribution fit the waiting times dataset. From Figure 4.11, we can see that TLOBX-WP distribution can be handle datasets with non-monotonic HRF shapes.

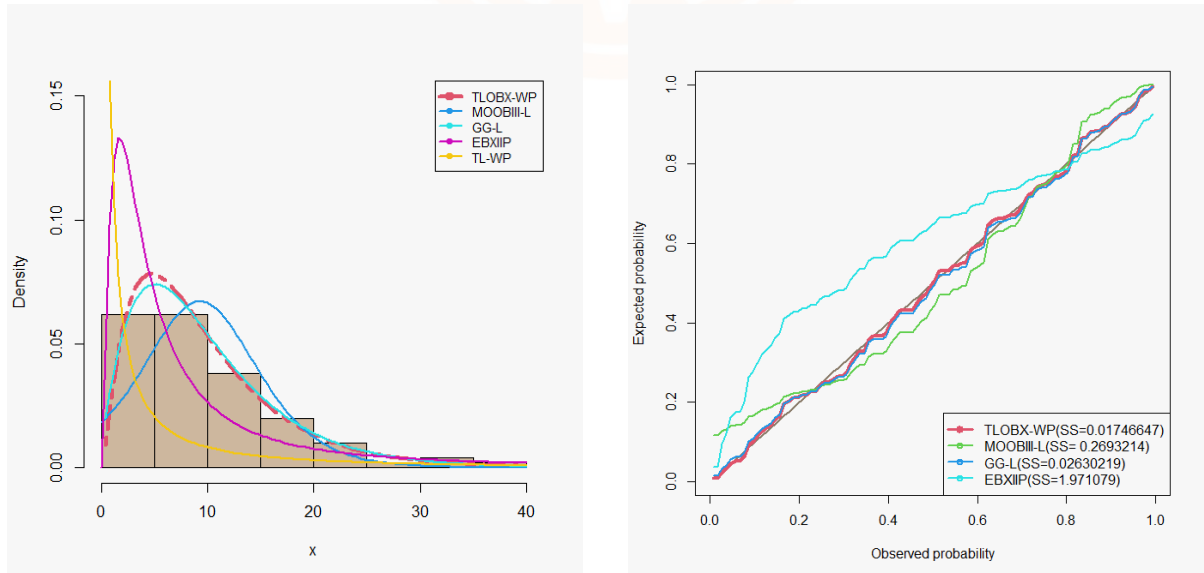


Figure 4.10: Fitted TLOBX-WP densities and P-Ps for waiting times dataset

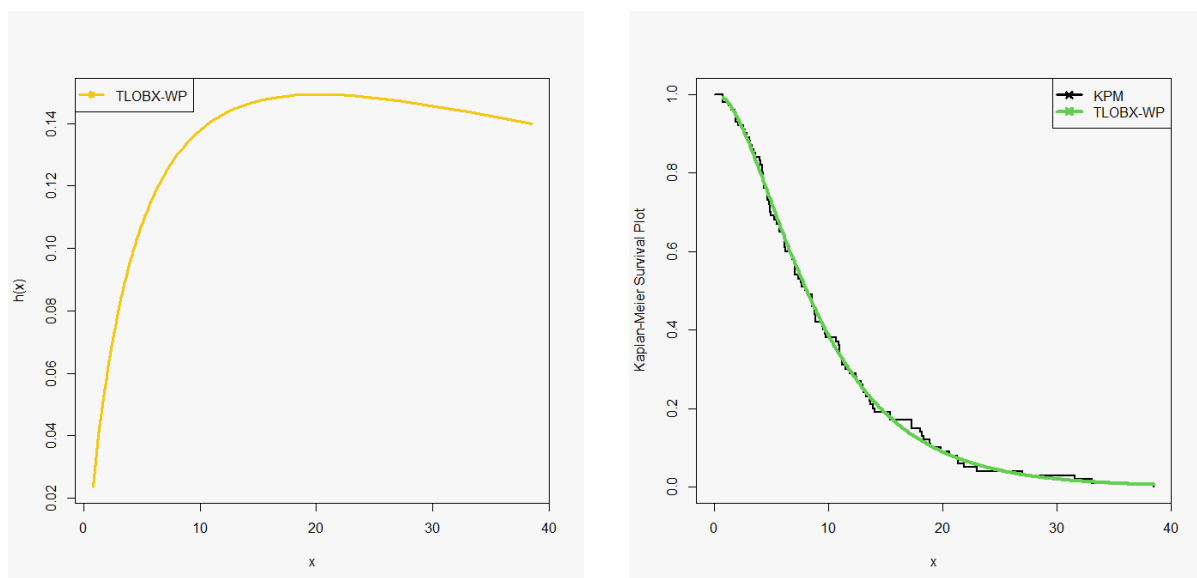


Figure 4.11: TLOBX-WP HRF and KM survival plots for waiting times dataset



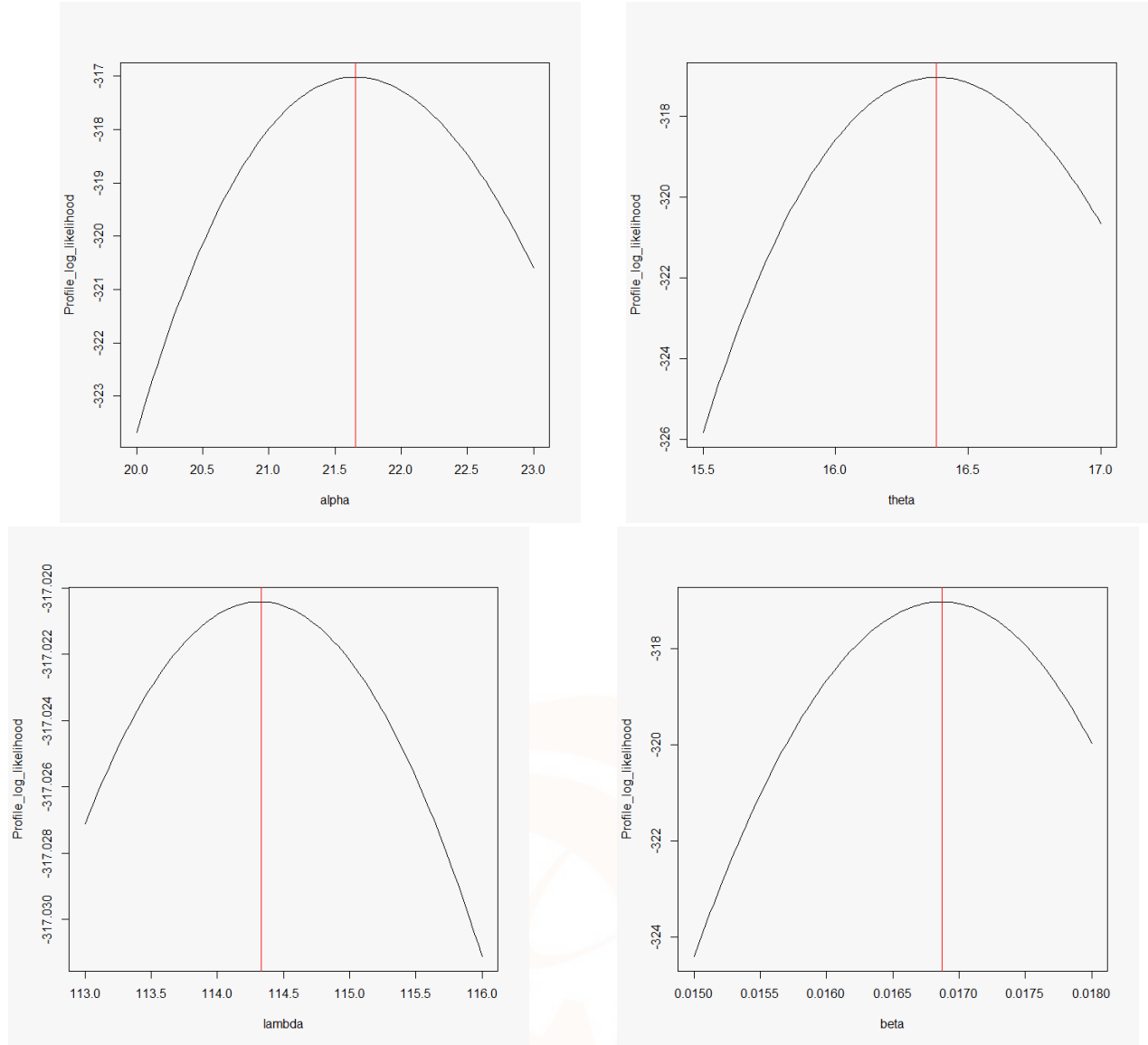


Figure 4.12: Likelihood profile plots for α, θ, λ and β in waiting times data

Figure 4.12 also show that the profile log-likelihood function of the parameters of the TLOBX-WP distribution in waiting times data maximizes the likelihood function.

4.7.3 Mortality Rates Data

The third dataset with $n = 106$ observations given in Appendix D.5 comes from Zhou et al. (2023) and it includes mortality rates of the patients infected by the COVID-19 pandemic in Mexico between March 4, 2020, and July 20, 2020.

With data on mortality rates, the estimated var-cov matrix for the TLOBX-WP distribution is

provided by

$$\begin{bmatrix} 5.29 \times 10^{-7} & -1.96 \times 10^{-6} & -8.19 \times 10^{-5} & -8.93 \times 10^{-7} \\ -1.96 \times 10^{-6} & 7.30 \times 10^{-6} & 3.05 \times 10^{-4} & 3.32 \times 10^{-6} \\ -8.19 \times 10^{-5} & 3.05 \times 10^{-4} & 1.27 \times 10^{-2} & 1.39 \times 10^{-4} \\ -8.93 \times 10^{-7} & 3.32 \times 10^{-6} & 1.39 \times 10^{-4} & 2.40 \times 10^{-6} \end{bmatrix}.$$

The 95% confidence intervals for the TLOBX-WP model parameters are given by $\alpha \in [1.17 \times 10^2 \pm 5.29 \times 10^{-3}]$, $\theta \in [5.04 \pm 2.21 \times 10^{-1}]$, $\lambda \in [6.01 \times 10^1 \pm 1.42 \times 10^{-3}]$ and $\beta \in [2.39 \times 10^{-2} \pm 3.04 \times 10^{-3}]$.

Table 4.6: The MLEs of the TLOBX-WP and competing models and GOF statistics for the failure times data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
	α	θ	λ	β									
TLOBX-WP	1.17×10^2 (2.70×10^{-3})	5.04 (1.13×10^{-1})	6.01×10^1 (7.27×10^{-4})	2.39×10^{-2} (1.55×10^{-3})	376.6	384.6	384.9	395.2	0.0538	0.3030	0.0694	0.6871	0.0552
TLOBX-WP($1, \theta, \lambda, \beta$)	1	9.93 (6.89×10^{-1})	8.77×10^{-9} (7.80×10^{-3})	6.23×10^{-2} (4.87×10^{-3})	744.5	750.5	750.8	758.5	0.0605	0.3389	0.7847	< 0.0001	25.1426
TLOBX-WP($\alpha, 1, \lambda, \beta$)	1.06×10^3 (2.95×10^{-3})	1	4.75×10^{-3} (5.76×10^{-1})	7.85×10^{-2} (7.43×10^{-3})	385.7	391.7	391.9	399.7	0.1418	0.8852	0.0883	0.3796	0.1444
MOOBII-L	7.09×10^2 (2.47×10^{-6})	8.08×10^1 (2.65)	3.45×10^{-2} (9.62×10^{-3})	7.02×10^1 (3.05)	405.4	413.4	413.7	424.0	0.3049	1.9654	0.0896	0.3619	0.2003
GG-L	1.28×10^{-8} (8.25×10^{-3})	1.69 (3.48×10^{-1})	2.95×10^{-1} (7.97×10^{-2})	2.12×10^{-1} (6.67×10^{-2})	384.9	392.9	393.3	403.6	0.1241	0.8116	0.0703	0.6719	0.1081
EBXIIP	2.88 (5.67×10^{-1})	2.44 (6.84×10^{-1})	1.06 (5.42×10^{-1})	1.06×10^{-8} (1.11×10^{-2})	382.3	390.3	390.7	400.9	0.1101	0.6261	0.0699	0.6769	0.0875
TL-WP	1.83×10^{-1} (3.68×10^{-2})	β 1.09 (1.03×10^{-1})	b 1.32 (1.91×10^{-1})	θ 3.11×10^{-9} (5.52×10^{-3})	406.4	414.4	414.8	425.1	0.0576	0.3370	0.1724	0.0036	0.7625

We infer from the GOF statistics in Table 4.6 that the TLOBX-WP distribution outperforms the other competing models in mortality rates data since it has the largest p-value for the K-S statistic and the smallest values for GOF statistics. Figures 4.13 and 4.14 demonstrate that the TLOBX-WP distribution offers a superior match in mortality rates data than the non-nested distributions. The HRF plot in Figure 4.14 also show that TLOBX-WP model can model data with increasing HRF shapes.

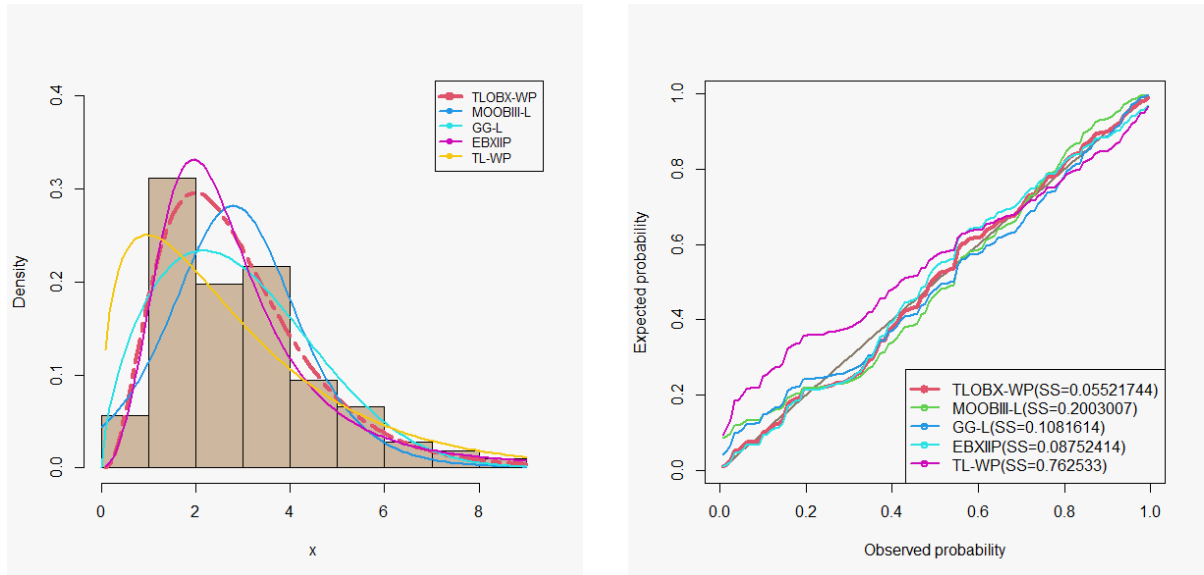


Figure 4.13: Fitted TLOBX-WP densities and P-Ps for mortality rates datasets

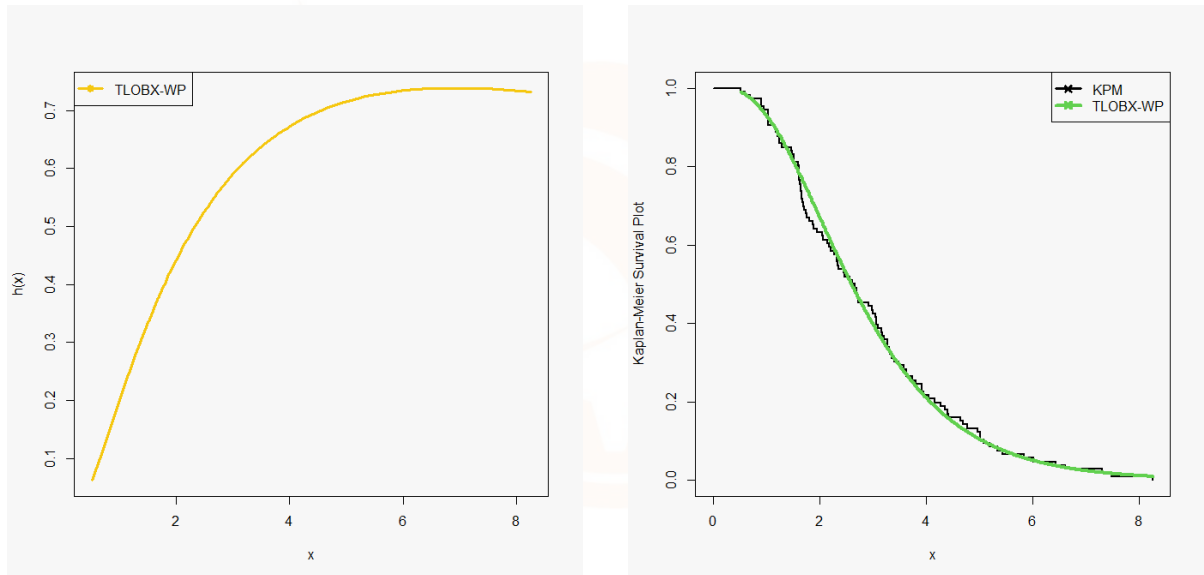


Figure 4.14: TLOBX-WP HRF and KM survival plots for mortality rates dataset

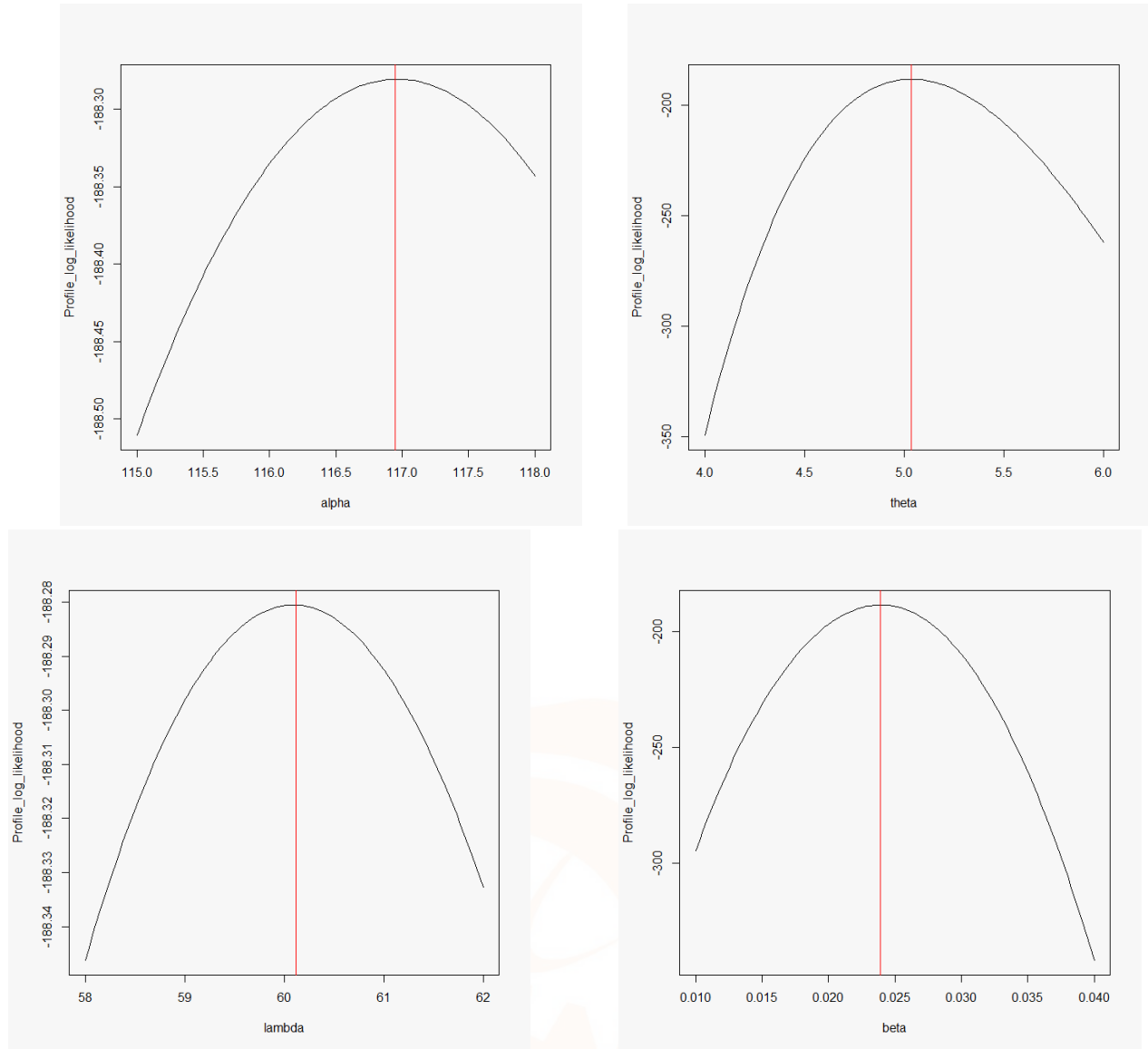


Figure 4.15: Likelihood profile plots for α , θ , λ and β in mortality rates data

Figure 4.15 show the profile log-likelihood function of the parameters of the TLOBX-WP distribution for the mortality rates data. Clearly, the maximum values of the estimated parameters can be seen from the plots.

4.8 Concluding Remarks

This study established and introduced a new class known as the Topp-Leone odd Burr X-G power series (TLOBX-GPS) family. Mathematical and statistical properties of this new class are derived together with estimation of the parameters using maximum likelihood technique. This class' statistical characteristics are defined, and the parameters are estimated using the maximum likelihood approach. To evaluate the consistency of the estimators, a simulation experiment is carried out and the results showed that the mean estimates approached the true parameters when the sample size increases, whereas the RMSEs and Abias decay to zero. To illustrate the usefulness of this class, we applied the special case of Topp-Leone odd Burr X-Weibull Poisson

(TLOBX-WP) distribution to three real life datasets and TLOBX-WP distribution had the lowest values of GOF statistics together with the highest p-value for the K-S statistic across all datasets.



5. The Marshall-Olkin-Topp-Leone Odd Burr X-G Family

5.1 Introduction

In this chapter, we develop the Marshall-Olkin-Topp-Leone odd Burr X-G family. Statistical properties of this new family are also presented. To estimate model parameters, we employ the MLE technique. A simulation experiment is used to evaluate the performance of MLE technique. The usefulness and flexibility of Marshall-Olkin-Topp-Leone odd Burr X-log-logistic distribution is illustrated using three real data examples.

5.2 The Marshall-Olkin-Topp-Leone Odd Burr X-G Family

The Marshall-Olkin-G (MO-G) family was developed by [Marshall and Olkin \(1997\)](#) with PDF and CDF given as

$$f_{MO-G}(x; \delta, \underline{\omega}) = \frac{\delta g(x; \underline{\omega})}{M_G^2(x; \delta, \underline{\omega})}, x \in \mathbb{R}, \delta > 0 \quad (5.1)$$

and

$$F_{MO-G}(x; \delta, \underline{\omega}) = 1 - \frac{\delta \bar{G}(x; \underline{\omega})}{M_G(x; \delta, \underline{\omega})}, \quad (5.2)$$

where $\bar{\delta} = 1 - \delta$, δ is the tilt parameter, $M_G(x; \delta, \underline{\omega}) = 1 - \bar{\delta} \bar{G}(x; \underline{\omega})$, $\underline{\omega}$ is vector of parameters from the baseline distribution. [Oluyede et al. \(2022b\)](#) developed the Topp-Leone odd Burr X-G (TLOBX-G) family with PDF and CDF

$$\begin{aligned} f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) &= \frac{4\alpha\theta g(x; \underline{\omega})G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta-1} \\ &\times \exp[-O_G^2(x; \underline{\omega})] \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right] \\ &\times \left[1 - \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right]^2\right]^{\alpha-1}, x \in \mathbb{R}, \alpha, \theta > 0 \end{aligned} \quad (5.3)$$

and

$$F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) = \left[1 - \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right]^2\right]^\alpha, \quad (5.4)$$

where $O_G(x; \underline{\omega}) = \frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})}$ and $\underline{\omega}$ is the baseline vector of parameters.

5.3 The Marshall-Olkin-Topp-Leone Odd Burr X-G Family

In this section, we present a new family called Marshall-Olkin-Topp-Leone odd Burr X-G (MOTLOBX-G) family. Plugging Equations (5.3) and (5.4) into Equations (5.1) and (5.2) yields the MOTLOBX-G family with PDF and CDF given by

$$\begin{aligned}
 f(x; \alpha, \theta, \delta, \underline{\omega}) &= \frac{4\delta\alpha\theta g(x; \underline{\omega})G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \exp[-O_G^2(x; \underline{\omega})] \\
 &\times \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta-1} \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right] \\
 &\times \left[1 - \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right]^2\right]^{\alpha-1} \\
 &\times \left[1 - \bar{\delta} \left\{1 - \left[1 - \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right]^2\right]^\alpha\right\}\right]^{-2}, x \in \mathbb{R}, \alpha, \theta, \delta > 0
 \end{aligned} \tag{5.5}$$

and

$$F(x; \alpha, \theta, \delta, \underline{\omega}) = 1 - \frac{\delta \left\{1 - \left[1 - \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right]^2\right]^\alpha\right\}}{1 - \bar{\delta} \left\{1 - \left[1 - \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right]^2\right]^\alpha\right\}}, \tag{5.6}$$

respectively, where $\bar{\delta} = 1 - \delta$ and $\underline{\omega}$ is vector of parameters from the baseline distribution.

5.3.1 Sub-Families

The sub-families of the MOTLOBX-G family are presented in this subsection.

- With $\delta = 1$, a CDF of the reduced TLOBX-G family as

$$F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) = 1 - \left\{1 - \left[1 - \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right]^2\right]^\alpha\right\},$$

for $\alpha, \theta, x > 0$ and vector of baseline parameters $\underline{\omega}$.

- With $\theta = 1$, a CDF of the reduced MOTLOBX-G family as

$$F(x; \alpha, \delta, \underline{\omega}) = 1 - \frac{\delta \{1 - [1 - \exp[-2O_G^2(x; \underline{\omega})]]^\alpha\}}{1 - \bar{\delta} \{1 - [1 - \exp[-2O_G^2(x; \underline{\omega})]]^\alpha\}},$$

for $\alpha, \delta, x > 0$ and vector of baseline parameters $\underline{\omega}$.

- With $\alpha = 1$, a CDF of the reduced MOTLOBX-G family as

$$F(x; \theta, \delta, \underline{\omega}) = 1 - \frac{\delta \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right]^2}{1 - \bar{\delta} \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^\theta\right]^2},$$

for $\theta, \delta, x > 0$ and vector of baseline parameters $\underline{\omega}$.

- With $\alpha = \delta = 1$, a CDF of the reduced MOTLOBX-G family as

$$F(x; \theta, \underline{\omega}) = 1 - \left[1 - \left\{ 1 - \exp \left[-O_G^2(x; \underline{\omega}) \right] \right\}^\theta \right]^2,$$

for $\theta, x > 0$ and vector of baseline parameters $\underline{\omega}$.

- With $\theta = \delta = 1$, a CDF of the reduced MOTLOBX-G family as

$$F(x; \alpha, \underline{\omega}) = 1 - \left\{ 1 - \left[1 - \exp \left[-2O_G^2(x; \underline{\omega}) \right] \right]^\alpha \right\},$$

for $\alpha, x > 0$ and vector of baseline parameters $\underline{\omega}$.

- With $\alpha = \theta = 1$, a CDF of the reduced MOTLOBX-G family as

$$F(x; \delta, \underline{\omega}) = 1 - \frac{\delta \left\{ \exp \left[-2O_G^2(x; \underline{\omega}) \right] \right\}}{1 - \bar{\delta} \left\{ \exp \left[-2O_G^2(x; \underline{\omega}) \right] \right\}},$$

for $\delta, x > 0$ and vector of baseline parameters $\underline{\omega}$.

- With $\alpha = \theta = \delta = 1$, a CDF of the reduced MOTLOBX-G family as

$$F(x; \underline{\omega}) = 1 - \exp \left[-2O_G^2(x; \underline{\omega}) \right],$$

with the vector of baseline parameters $\underline{\omega}$.

5.4 Some Structural Properties

This section provides the QF, density expansion and the moments of the MOTLOBX-G family together with the Rényi entropy and order statistics. To derive the statistical properties of MOTLOBX-G family, we use the general results of MO-G family from [Barreto-Souza et al. \(2013\)](#).

5.4.1 Quantile Function

We invert the CDF of the MOTLOBX-G family as follows to obtain the QF:

$$1 - \frac{\delta \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[-O_G^2(x; \underline{\omega}) \right] \right\}^\theta \right]^2 \right]^\alpha \right\}}{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[-O_G^2(x; \underline{\omega}) \right] \right\}^\theta \right]^2 \right]^\alpha \right\}} = p$$

for $0 < u < 1$. Replacing $O_G(x; \underline{\omega}) = \frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})}$ with, note that we get

$$\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})} = \left[-\ln \left[1 - \left[1 - \left[1 - \left[1 - \frac{1-p}{\bar{\delta}(1-p) + \delta} \right]^{1/\alpha} \right]^{1/2} \right]^{1/\theta} \right] \right]^{1/2},$$

which implies that

$$G(x; \underline{\omega}) = \frac{\left[-\ln \left[1 - \left[1 - \left[1 - \left[1 - \frac{1-p}{\delta(1-p)+\delta} \right]^{1/\alpha} \right]^{1/2} \right]^{1/\theta} \right] \right]^{1/2}}{1 + \left[-\ln \left[1 - \left[1 - \left[1 - \left[1 - \frac{1-p}{\delta(1-p)+\delta} \right]^{1/\alpha} \right]^{1/2} \right]^{1/\theta} \right] \right]^{1/2}}.$$

Finally, the QF is given as

$$Q_X(p) = G^{-1} \left\{ \frac{\left[-\ln \left[1 - \left[1 - \left[1 - \left[1 - \frac{1-p}{\delta(1-p)+\delta} \right]^{1/\alpha} \right]^{1/2} \right]^{1/\theta} \right] \right]^{1/2}}{1 + \left[-\ln \left[1 - \left[1 - \left[1 - \left[1 - \frac{1-p}{\delta(1-p)+\delta} \right]^{1/\alpha} \right]^{1/2} \right]^{1/\theta} \right] \right]^{1/2}} \right\}.$$

Table 5.1 presents quantiles of the Marshall-Olkin-Topp-Leone odd Burr X-log-logistic (MOTLOBX-L) distribution for various parameter values. We use the QF in simulation.

Table 5.1: Quantiles values for the MOTLOBX-L distribution for some parameter values

	$(\alpha, \theta, \delta, \beta)$				
p	(1,1.5,0.9,8.5)	(0.5,1,0.7,2.5)	(1.5,0.8,0.8,5)	(1.5,1.5,0.6,1.8)	(1,1.5,0.4,3.5)
0.1	0.8902	0.3043	0.7511	0.6477	0.6977
0.2	0.9186	0.4074	0.8028	0.7229	0.7540
0.3	0.9373	0.4869	0.8378	0.7778	0.7934
0.4	0.9522	0.5565	0.8664	0.8254	0.8265
0.5	0.9654	0.6217	0.8919	0.8707	0.8572
0.6	0.9777	0.6862	0.9164	0.9168	0.8879
0.7	0.9903	0.7535	0.9416	0.9670	0.9205
0.8	1.0042	0.8290	0.9696	1.0268	0.9585
0.9	1.0219	0.9265	1.0059	1.1109	1.0101

5.4.2 Expansion of Density Function

Using TLOBX-G family PDF in Equation (5.3) and CDF in Equation (5.4) in Equation (5.5) we get

$$f(x; \alpha, \theta, \delta, \underline{\omega}) = \frac{\delta f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})}{[1 - \delta \bar{F}_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]^2}.$$

From the results of Barreto-Souza et al. (2013), we have

$$f(x; \alpha, \theta, \delta, \underline{\omega}) = \frac{f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})}{\delta [1 - \frac{\delta-1}{\delta} F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]^2}. \quad (5.7)$$

If $0 < \delta < 1$ and we apply the series expansion given as

$$(1 - z)^{-v} = \sum_{u=0}^{\infty} \frac{\Gamma(u+v)}{\Gamma(v)u!} z^u$$

which is valid for $|z| < 1$ and $v > 0$, and write Equation (5.7) as

$$f(x; \alpha, \theta, \delta, \underline{\omega}) = \sum_{u=0}^{\infty} \sum_{v=0}^u w_{u,v} f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) F_{TLOBX-G}^{u-v}(x; \alpha, \theta, \underline{\omega}), \quad (5.8)$$

where $w_{u,v} = w_{u,v}(\delta) = \delta(u+1)(1-\delta)^u(-1)^{u-v} \binom{u}{v}$. Simplifying Equation (5.8) as shown in Appendix E.1 for $\delta \in (0, 1)$, we get

$$f(x; \alpha, \theta, \delta, \underline{\omega}) = \sum_{p=0}^{\infty} W_{p+1} h_{p+1}(x; \underline{\omega}), \quad (5.9)$$

where

$$W_{p+1} = 4\alpha\theta \sum_{i,j,k,m,n,u=0}^{\infty} \sum_{v=0}^u \frac{w_{u,v}(-1)^{i+j+k+m+n+p}(k+1)^m}{(p+1)m!} \binom{\gamma^*}{i} \binom{2i+1}{j} \times \binom{\theta(j+1)-1}{k} \binom{2m+1}{n} \binom{n-(2m+3)}{p}, \quad (5.10)$$

$\gamma^* = \alpha(u-v+1) - 1$ and $h_{p+1}(x; \underline{\omega}) = (p+1)g(x; \underline{\omega})G^{p+1-1}(x; \underline{\omega})$.

Furthermore, for $\delta > 1$, Equation (5.7) can be expressed as

$$f(x; \alpha, \theta, \delta, \underline{\omega}) = \sum_{u=0}^{\infty} v_u f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) F_{TLOBX-G}^u(x; \alpha, \theta, \underline{\omega}), \quad (5.11)$$

where $v_u = v_u(\delta) = \frac{(u+1)(1-1/\delta)}{\delta}$. Following steps in Appendix E.1, we can write Equation (5.11) as

$$f(x; \alpha, \theta, \delta, \underline{\omega}) = \sum_{p=0}^{\infty} V_{p+1} h_{p+1}(x; \underline{\omega}), \quad (5.12)$$

where

$$V_{p+1} = 4\alpha\theta \sum_{i,j,k,m,n,u=0}^{\infty} \frac{v_u(-1)^{i+j+k+m+n+p}(k+1)^m}{(p+1)m!} \binom{2i+1}{j} \times \binom{\alpha(u+1)-1}{i} \binom{\theta(j+1)-1}{k} \binom{2m+1}{n} \binom{n-(2m+3)}{p}. \quad (5.13)$$

As a result, the PDF of the MOTLOBX-G family may be written as an infinite linear combination of the Exp-G densities with power parameter $(p+1)$.

5.4.3 Moments and Generating Functions

The r^{th} moment for the rv X following the MOTLOBX-G family is obtained using Equations (5.9) and (5.12). For $0 < \delta < 1$,

$$E(X^r) = \sum_{p=0}^{\infty} W_{p+1} E(Y_{p+1}),$$

where W_{p+1} is given in Equation (5.10) and r^{th} moment $E(Y_{p+1})$ with power parameter $p+1$ follows Exp-G family. For $\delta > 1$,

$$E(X^r) = \sum_{p=0}^{\infty} V_{p+1} E(Y_{p+1}),$$

where V_{p+1} is defined in Equation (5.13). For $0 < \delta < 1$, the s^{th} incomplete moments is given as

$$I_X(t) = \int_0^t x^s f(x; \alpha, \theta, \delta, \underline{\omega}) dx = \sum_{p=0}^{\infty} W_{p+1} \int_0^t x^s h_{p+1}(x; \underline{\omega}) dx,$$

where W_{p+1} is given in Equation (5.10) and the s^{th} incomplete moment $\int_0^t x^s h_{p+1}(x; \underline{\omega}) dx$ of the rv Y_{p+1} follows Exp-G density.

For $0 < \delta < 1$, the s^{th} incomplete moment is given as

$$I_X(t) = \int_0^t x^s f(x; \alpha, \theta, \delta, \underline{\omega}) dx = \sum_{p=0}^{\infty} V_{p+1} \int_0^t x^s h_{p+1}(x; \underline{\omega}) dx,$$

where V_{p+1} is given in Equation (5.13).

The MGF of X for $0 < \delta < 1$ is given as

$$M_X(t) = \sum_{p=0}^{\infty} W_{p+1} E(e^{tY_{p+1}}),$$

where W_{p+1} is given in Equation (5.10) and $E(e^{tY_{p+1}})$ is the MGF of the Exp-G rv.

The MGF of X for $\delta > 1$ is given as

$$M_X(t) = \sum_{p=0}^{\infty} V_{p+1} E(e^{tY_{p+1}}),$$

where V_{p+1} is given in Equation (5.13). Figures (5.1) and (5.2) gives the skewness and kurtosis plots of MOTLOBX-L distribution for different parameter values which are obtained using R

software.

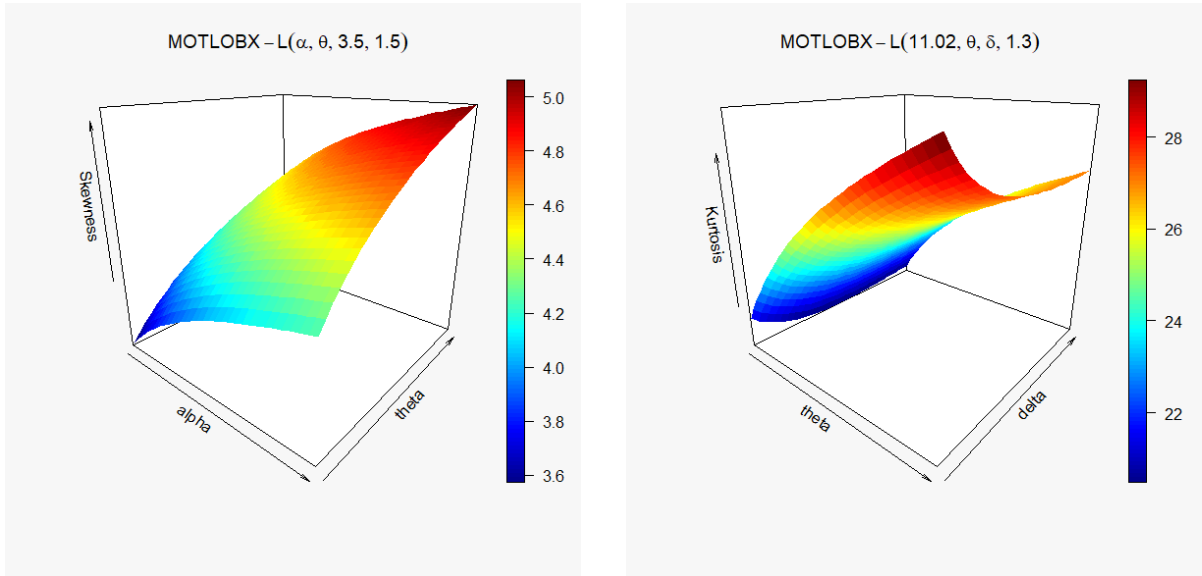


Figure 5.1: Plots of skewness and kurtosis for the MOTLOBX-L distribution for some parameter values

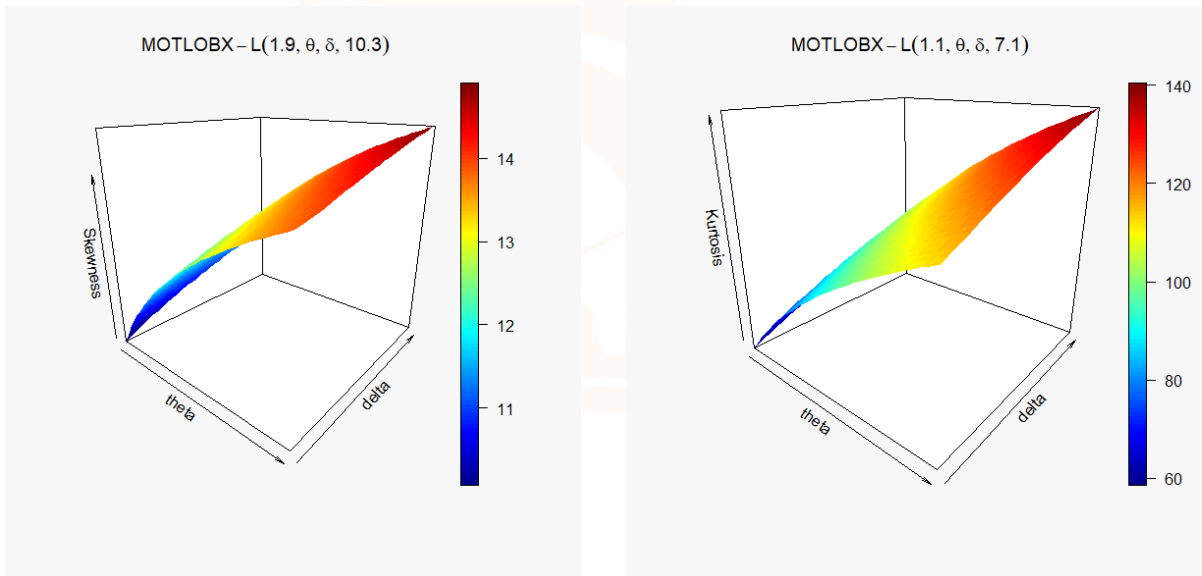


Figure 5.2: Plots of skewness and kurtosis for the MOTLOBX-L distribution for some parameter values

Holding δ and β constant as shown in [5.1](#), skewness increases with increase of both α and θ . Holding α and β in [Figure 5.1](#) skewness depends on both θ and δ . As for the kurtosis in [Figure 5.2](#) it increases with increase of both θ and δ .

5.4.4 Rényi Entropy

The two most used entropy metrics are Shannon and Rényi entropies, with Shannon entropy being a particular case of Rényi entropies (see [Rényi \(1961\)](#)). Entropies are used to quantify

the degree of uncertainty associated with a rv. As stated by Rényi, entropy is defined as

$$E_R(v) = \frac{1}{1-v} \log \left[\int_0^\infty f^v(x; \alpha, \theta, \delta, \underline{\omega}) dx \right], \quad (5.14)$$

where $v > 0$ and $v \neq 1$. For $0 < \delta < 1$, $f^v(x; \alpha, \theta, \delta, \underline{\omega})$ is given as

$$f^v(x; \alpha, \theta, \delta, \underline{\omega}) = f_{TLOBX-G}^v(x; \alpha, \theta, \underline{\omega}) \sum_{u=0}^{\infty} e_u [1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]^u,$$

where $e_u = e_u(\delta) = \frac{\delta^v(1-\delta)^u \Gamma(2v+u)}{\Gamma(2v)u!}$, $f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})$ and $F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})$ are as given in Equations (5.3) and (5.4), respectively. Expanding $f^v(x; \alpha, \theta, \delta, \underline{\omega})$ as shown in Appendix E.2 and plugging in Equation (5.14), we get

$$I_R(v) = \frac{1}{1-v} \log \left[\sum_{p=0}^{\infty} M_p \exp [(1-v)E_{Rp}] \right],$$

where $E_{Rp} = (1-v)^{-1} \log \left[\int_0^\infty \left(\left[\frac{p}{v} + 1 \right] g(x; \underline{\omega}) G^{\frac{p}{v}}(x; \underline{\omega}) \right)^v dx \right]$ is the Rényi entropy of Exp-G family with power parameter $\left(\frac{p}{v} + 1 \right)$ and

$$M_p = (4\alpha\theta)^v \sum_{h,i,j,k,m,n,u=0}^{\infty} \frac{e_u (-1)^{h+i+j+k+m+n+p} (k+v)^m}{\left[\frac{p}{v} + 1 \right]^v m!} \\ \times \binom{v(\alpha-1) + \alpha h}{i} \binom{2i+v}{j} \binom{\theta(j+v) - v}{k} \binom{2m+v}{n} \binom{n - (2m+3v)}{p}.$$

For $\delta > 1$, $f^v(x; \alpha, \theta, \delta, \underline{\omega})$ is given as

$$f^v(x; \alpha, \theta, \delta, \underline{\omega}) = f_{TLOBX-G}^v(x; \alpha, \theta, \underline{\omega}) \sum_{u=0}^{\infty} h_u F^u(x; \alpha, \theta, \underline{\omega}),$$

where $h_u = h_u(\delta) = \frac{(\delta-1)^u \Gamma(2v+u)}{\delta^v \Gamma(2v)u!}$, $f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})$ and $F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})$ are as given in Equations (5.3) and (5.4), respectively. That is, for $\delta > 1$, $f^v(x; \alpha, \theta, \delta, \underline{\omega})$ is given as

$$f^v(x; \alpha, \theta, \delta, \underline{\omega}) = \frac{(4\alpha\theta)^v g^v(x; \underline{\omega}) G^v(x; \underline{\omega})}{\bar{G}^{3v}(x; \underline{\omega})} \exp [-vO_G^2(x; \underline{\omega})] \\ \times \{1 - \exp [-O_G^2(x; \underline{\omega})]\}^{v(\theta-1)} \sum_{i,u=0}^{\infty} h_u (-1)^i \binom{v(\alpha-1) + \alpha u}{i} \\ \times \left[1 - \{1 - \exp [-O_G^2(x; \underline{\omega})]\}^\theta \right]^{2i+v}.$$

Applying generalized binomial series and exponential series representation (see Appendix E.2),

Equation (5.14) becomes

$$I_R(v) = \frac{1}{1-v} \log \left[\sum_{p=0}^{\infty} N_p \exp [(1-v)E_{Rp}] \right],$$

where $E_{Rp} = (1-v)^{-1} \log \left[\int_0^{\infty} \left(\left[\frac{p}{v} + 1 \right] g(x; \underline{\omega}) G_{\frac{p}{v}}^{\frac{p}{v}}(x; \underline{\omega}) \right)^v dx \right]$ is the Rényi entropy of Exp-G family with power parameter $\left(\frac{p}{v} + 1 \right)$ and

$$N_p = (4\alpha\theta)^v \sum_{i,j,k,m,n,u=0}^{\infty} \frac{h_u(-1)^{i+j+k+m+n+p}(k+v)^m}{\left[\frac{p}{v} + 1 \right]^v m!} \\ \times \binom{v(\alpha-1) + \alpha u}{i} \binom{2i+v}{j} \binom{\theta(j+v)-v}{k} \binom{2m+v}{n} \binom{n-(2m+3v)}{p}.$$

5.4.5 Order Statistics

Let MOTLOBX-G rvs be defined as X_i for $i = 1, 2, 3, \dots, n$, then the PDF of the i^{th} PDF of the order statistic is given as

$$f_{i:n}(x; \alpha, \theta, \delta, \underline{\omega}) = \frac{\delta n! f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})}{(i-1)!(n-i)!} \sum_{t=0}^{n-i} \frac{(-1)^t F^{t+i-1}(x; \alpha, \theta, \underline{\omega})}{[1 - \delta F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]^{t+i-1}}.$$

For $0 < \delta < 1$,

$$f_{i:n}(x; \alpha, \theta, \delta, \underline{\omega}) = f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) \sum_{u=0}^{\infty} \sum_{v=0}^u \sum_{t=0}^{n-i} b_{u,v,t} F_{TLOBX-G}^{u+t-v+i-1}(x; \alpha, \theta, \underline{\omega})$$

where

$$b_{u,v,t} = b_{u,v,t}(\delta) = \frac{\delta n! (-1)^t (1-\delta)^u (-1)^{u-v}}{(i-1)!(n-i)!} \binom{u}{v} \binom{t+i+u}{u}.$$

Expanding $f_{i:n}(x; \alpha, \theta, \delta, \underline{\omega})$ as shown in Appendix E.3 we get

$$f_{i:n}(x; \alpha, \theta, \delta, \underline{\omega}) = \sum_{p=0}^{\infty} C_{p+1} h_{p+1}(x; \underline{\omega}),$$

where

$$C_{p+1} = 4\alpha\theta \sum_{h,j,k,m,q,u=0}^{\infty} \sum_{v=0}^u \sum_{t=0}^{n-i} \frac{b_{u,v,t} (-1)^{h+j+k+m+p+q} (k+1)^m}{(p+1)m!} \binom{\varphi^*}{h} \binom{2h+1}{j} \\ \times \binom{\theta(j+1)-1}{k} \binom{2m+1}{q} \binom{q-(2m+3)}{p},$$

$\varphi^* = \alpha(u+t-v+i) - 1$ and the Exp-G family $h_{p+1}(x; \underline{\omega}) = (p+1)g(x; \underline{\omega})G^{p+1-1}(x; \underline{\omega})$ is of

the power parameter $p + 1$. For $\delta > 1$,

$$f_{i:n}(x; \alpha, \theta, \delta, \underline{\omega}) = f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) \sum_{u=0}^{\infty} \sum_{t=0}^{n-i} c_{u,t} F_{TLOBX-G}^{u+t+i-1}(x; \alpha, \theta, \underline{\omega})$$

where $c_{u,t} = c_{u,t}(\delta) = \frac{n!(-1)^t(1-\delta)^u}{\delta^{t+u+i}(i-1)!(n-i)!} \binom{t+i+u}{u}$. Consequently, the i^{th} order statistic for $\delta > 1$ is given as

$$f_{i:n}(x; \alpha, \theta, \delta, \underline{\omega}) = \sum_{p=0}^{\infty} D_{p+1} h_{p+1}(x; \underline{\omega}),$$

where

$$D_{p+1} = 4\alpha\theta \sum_{h,j,k,m,q,u=0}^{\infty} \sum_{t=0}^{n-i} \frac{c_{u,t}(-1)^{h+j+k+m+p+q}(k+1)^m}{(p+1)m!} \binom{\omega^*}{h} \binom{2h+1}{j} \\ \times \binom{\theta(j+1)-1}{k} \binom{2m+1}{q} \binom{q-(2m+3)}{p},$$

$\omega^* = \alpha(u+t+i) - 1$ and Exp-G family $h_{p+1}(x; \underline{\omega}) = (p+1)g(x; \underline{\omega})G^{p+1-1}(x; \underline{\omega})$ with the power parameter $(p+1)$.

5.4.6 Stochastic Ordering

Let X_1 and X_2 be two independent rvs following MOTLOBX-G($x; \alpha_1, \theta, \delta, \underline{\omega}$) and MOTLOBX-G($x; \alpha_2, \theta, \delta, \underline{\omega}$). Then $\frac{f_1(x; \alpha_1, \theta, \delta, \underline{\omega})}{f_2(x; \alpha_2, \theta, \delta, \underline{\omega})}$ is decreasing in x if $\alpha_1 \leq \alpha_2$. To show this, we have the ratio of these two PDFs as

$$\frac{f_1(x; \alpha_1, \theta, \delta, \underline{\omega})}{f_2(x; \alpha_2, \theta, \delta, \underline{\omega})} = \frac{\alpha_1}{\alpha_2} \left[1 - \left[1 - \left\{ 1 - \exp[-O_G^2(x; \underline{\omega})] \right\}^\theta \right]^2 \right]^{\alpha_1 - \alpha_2} \\ \times \left[\frac{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp[-O_G^2(x; \underline{\omega})] \right\}^\theta \right]^2 \right]^{\alpha_2} \right\}}{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp[-O_G^2(x; \underline{\omega})] \right\}^\theta \right]^2 \right]^{\alpha_1} \right\}} \right]^2 \right]^2.$$

The derivative of the above ratio with respect to x follows as

$$\frac{d}{dx} \left(\frac{f_1(x; \alpha_1, \theta, \delta, \underline{\omega})}{f_2(x; \alpha_2, \theta, \delta, \underline{\omega})} \right) = \frac{\alpha_1(\alpha_1 - \alpha_2)}{\alpha_2} \\ \times \left[1 - \left[1 - \left\{ 1 - \exp[-O_G^2(x; \underline{\omega})] \right\}^\theta \right]^2 \right]^{\alpha_1 - \alpha_2 - 1} \\ \times \frac{d}{dx} \left\{ \left[\frac{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp[-O_G^2(x; \underline{\omega})] \right\}^\theta \right]^2 \right]^{\alpha_2} \right\}}{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp[-O_G^2(x; \underline{\omega})] \right\}^\theta \right]^2 \right]^{\alpha_1} \right\}} \right]^2 \right\} \right\}.$$

Thus, if $\alpha_1 \leq \alpha_2$, then $\frac{d}{dx} \left(\frac{f_1(x; \alpha_1, \theta, \delta, \underline{\omega})}{f_2(x; \alpha_2, \theta, \delta, \underline{\omega})} \right)$ is negative which means $\frac{f_1(x; \alpha_1, \theta, \delta, \underline{\omega})}{f_2(x; \alpha_2, \theta, \delta, \underline{\omega})}$ is decreasing in x . Based on the findings of Shaked and Shanthikumar (2007), we can conclude that $X_1 <_{hr} X_2$ and $X_1 <_{st} X_2$ since $X_1 <_{lr} X_2$.

5.5 Maximum Likelihood Method

Let $X \sim \text{MOTOBX-G}(\alpha, \theta, \delta, \underline{\omega})$ with vector of parameters $\Psi = (\alpha, \theta, \delta, \underline{\omega})^T$, then the log-likelihood function of a random size n , $\ell = \ln(L(\Psi))$, is given as

$$\begin{aligned} \ell = & n \ln(4\delta\alpha\theta) + \sum_{i=1}^n \ln[g(x_i; \underline{\omega})] + \sum_{i=1}^n \ln[G(x_i; \underline{\omega})] - 3 \sum_{i=1}^n \ln[\overline{G}(x_i; \underline{\omega})] \\ & - \sum_{i=1}^n Z_i^2 + (\theta - 1) \sum_{i=1}^n \ln(1 - e^{-Z_i^2}) + \sum_{i=1}^n \ln \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right] \\ & + (\alpha - 1) \sum_{i=1}^n \ln \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2 \right] \\ & - 2 \sum_{i=1}^n \ln \left[1 - \delta \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2 \right]^\alpha \right\} \right], \end{aligned}$$

where $Z_i = \frac{G(x_i; \underline{\omega})}{\overline{G}(x_i; \underline{\omega})}$. The elements of the score vector given as $(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \omega_j})$ are provided in Appendix E.4. We may use the multivariate normal distribution together with the mean vector $J(\hat{\Psi})^{-1}$ and the observed Fisher information matrix $\hat{\Psi}$ to generate the confidence region for the model parameters $N_{p+3}(\underline{\mathbf{Q}}, J(\hat{\Psi})^{-1})$. We use R and a given baseline distribution to get numerical solutions to these partial derivatives.

5.6 Some Special Cases

We present some special cases for the MOTLOBX-G family in this section by considering log-logistic distribution, Weibull distribution, Uniform distribution, Burr III and Burr XII as baseline distributions.

5.6.1 Marshall-Olkin-Topp-Leone Odd Burr X-Log-Logistic Distribution

Taking the PDF $g(x; \beta) = \beta x^{\beta-1} (1+x^\beta)^{-2}$ and CDF $G(x; \beta) = 1 - (1+x^\beta)^{-1}$ of log-logistic distribution, respectively, for $\beta > 0$, then the Marshall-Olkin-Topp-Leone odd Burr X-log-logistic (MOTLOBX-L) distribution is obtained with PDF and CDF given by

$$\begin{aligned}
f(x; \alpha, \theta, \delta, \beta) &= \frac{4\alpha\theta\delta\beta x^{\beta-1}(1+x^\beta)(1-(1+x^\beta)^{-1})}{\exp[x^{2\beta}]} \left\{1 - \exp[-x^{2\beta}]\right\}^{\theta-1} \\
&\times \left[1 - \left\{1 - \exp[-x^{2\beta}]\right\}^\theta\right] \left[1 - \left[1 - \left\{1 - \exp[-x^{2\beta}]\right\}^\theta\right]^2\right]^{\alpha-1} \\
&\times \left[1 - \bar{\delta} \left\{1 - \left[1 - \left[1 - \left\{1 - \exp[-x^{2\beta}]\right\}^\theta\right]^2\right]^\alpha\right\}\right]^{-2}, x \in \mathbb{R}, \alpha, \theta, \delta, \beta > 0
\end{aligned}$$

and

$$F(x; \alpha, \theta, \delta, \beta) = 1 - \frac{\delta \left\{1 - \left[1 - \left[1 - \left\{1 - \exp[-x^{2\beta}]\right\}^\theta\right]^2\right]^\alpha\right\}}{1 - \bar{\delta} \left\{1 - \left[1 - \left[1 - \left\{1 - \exp[-x^{2\beta}]\right\}^\theta\right]^2\right]^\alpha\right\}},$$

respectively.

The plots of the PDFs and HRFs of the MOTLOBX-L distribution are displayed in Figure 5.3. The PDF can apply to reverse-J, left-skewed, right-skewed and almost symmetric data sets. The HRF may be shaped into decreasing, increasing, upside-down, and bathtub forms.

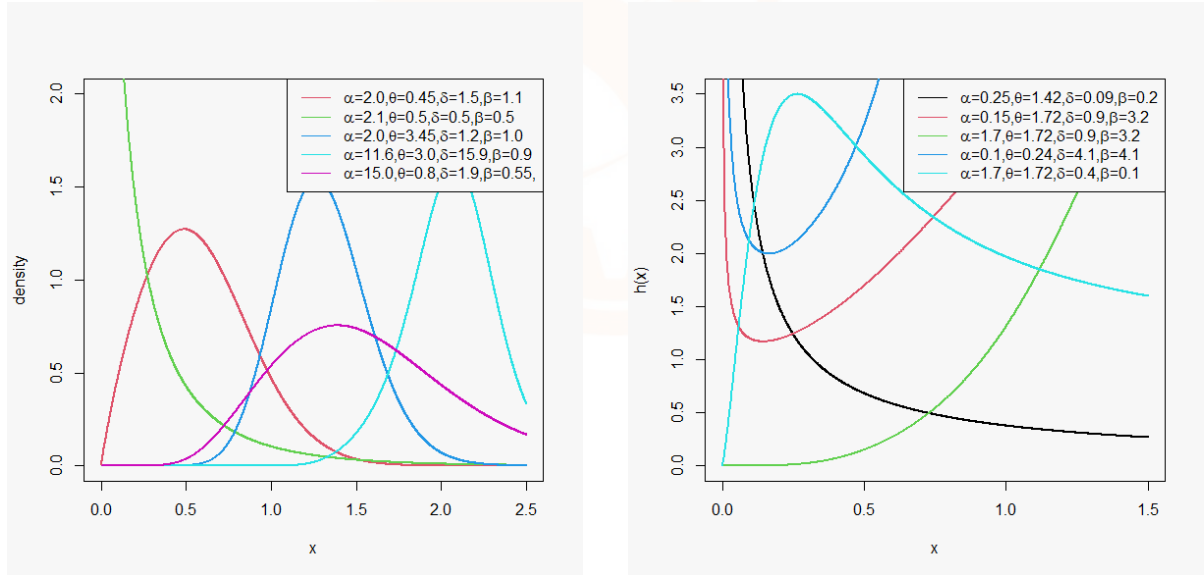


Figure 5.3: MOTLOBX-L distribution's PDF and HRF plots

5.6.2 Marshall-Olkin-Topp-Leone Odd Burr X-Weibull Distribution

If Weibull distribution is assumed to be the MOTLOBX-G family's baseline distribution with PDF $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$ and CDF $G(x; \lambda) = 1 - e^{-x^\lambda}$, respectively, for $\lambda > 0$, then the Marshall-Olkin-Topp-Leone odd Burr X-Weibull (MOTLOBX-W) distribution is obtained with

PDF and CDF given by

$$f(x; \alpha, \theta, \delta, \lambda) = \frac{4\alpha\theta\delta\lambda x^{\lambda-1} e^{-2x^\lambda} (1 - e^{-x^\lambda})}{\exp \left[(e^{x^\lambda} - 1)^2 \right]} \left\{ 1 - \exp \left[- (e^{x^\lambda} - 1)^2 \right] \right\}^{\theta-1} \\ \times \left[1 - \left\{ 1 - \exp \left[- (e^{x^\lambda} - 1)^2 \right] \right\}^\theta \right] \left[1 - \left[1 - \left\{ 1 - \exp \left[- (e^{x^\lambda} - 1)^2 \right] \right\}^\theta \right]^2 \right]^{\alpha-1} \\ \times \left[1 - \bar{\delta} \left\{ 1 - \left[1 - \left\{ 1 - \exp \left[- (e^{x^\lambda} - 1)^2 \right] \right\}^\theta \right]^2 \right]^\alpha \right\}^{-2}, x \in \mathbb{R}, \alpha, \theta, \delta, \lambda > 0$$

and

$$F(x; \alpha, \theta, \delta, \lambda) = 1 - \frac{\delta \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[- (e^{x^\lambda} - 1)^2 \right] \right\}^\theta \right]^2 \right]^\alpha \right\}}{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[- (e^{x^\lambda} - 1)^2 \right] \right\}^\theta \right]^2 \right]^\alpha \right\}},$$

respectively.

The plots of the PDFs and HRFs of the MOTLOBX-W distribution are shown in Figure 5.4. The PDF is applicable to virtually symmetric and reverse-J data sets with varied kurtosis. The HRF might be shaped like a bathtub, increasing, or decreasing.

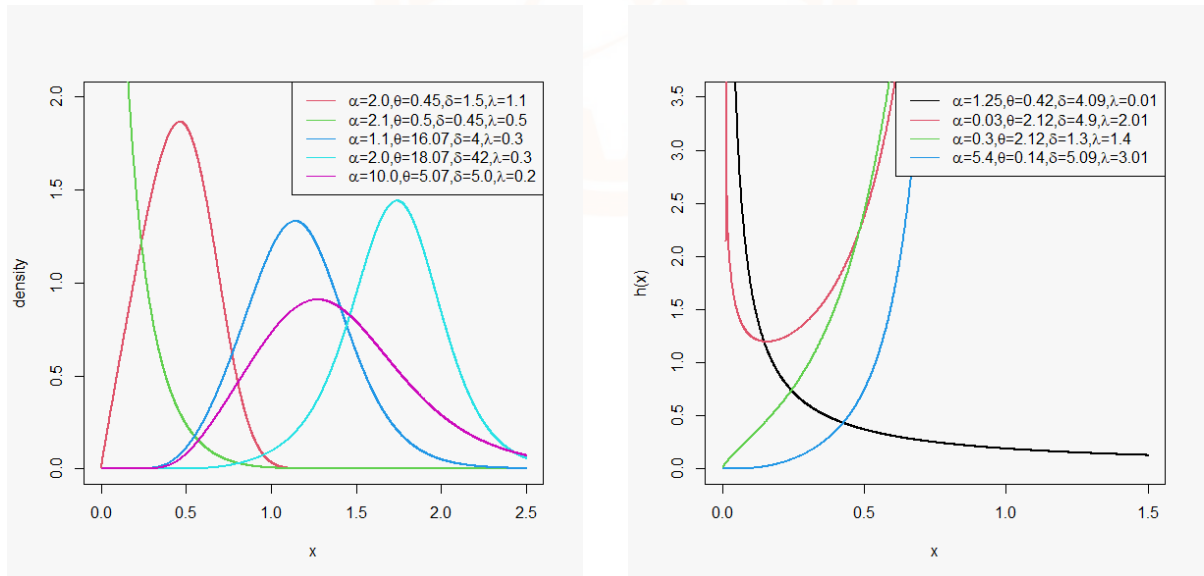


Figure 5.4: MOTLOBX-W distribution's PDF and HRF plots

5.6.3 Marshall-Olkin-Topp-Leone Odd Burr X-Uniform Distribution

If Uniform distribution is assumed to be the MOTLOBX-G family's baseline distribution with PDF $g(x; \mu) = 1/\mu$ and CDF $G(x; \mu) = x/\mu$, respectively, for $x \leq \mu$, then the Marshall-Olkin-Topp-Leone odd Burr X-Uniform (MOTLOBX-U) distribution is obtained with PDF and CDF

given by

$$f(x; \alpha, \theta, \delta, \mu) = \frac{4\alpha\theta\delta\mu x}{(x - \mu)^3} \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \left\{ 1 - \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \right\}^{\theta-1} \\ \times \left[1 - \left\{ 1 - \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \right\}^\theta \right] \left[1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \right\}^\theta \right]^{2\gamma} \right]^{\alpha-1} \\ \times \left[1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \right\}^\theta \right]^{2\gamma} \right]^\alpha \right\} \right]^{-2}, x \in \mathbb{R}, \alpha, \theta, \delta, \mu > 0$$

and

$$F(x; \alpha, \theta, \delta, \mu) = 1 - \frac{\delta \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \right\}^\theta \right]^{2\gamma} \right]^\alpha \right\}}{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{x}{\mu - x} \right)^2 \right] \right\}^\theta \right]^{2\gamma} \right]^\alpha \right\}},$$

respectively.

The plots of the PDFs and HRFs of the MOTLOBX-U distribution are shown in Figure 5.5. The PDF is applicable to kurtosis-varying heavy-tailed data sets. The HRF selects forms that are decreasing, increasing, upside-down, and bathtub-style.

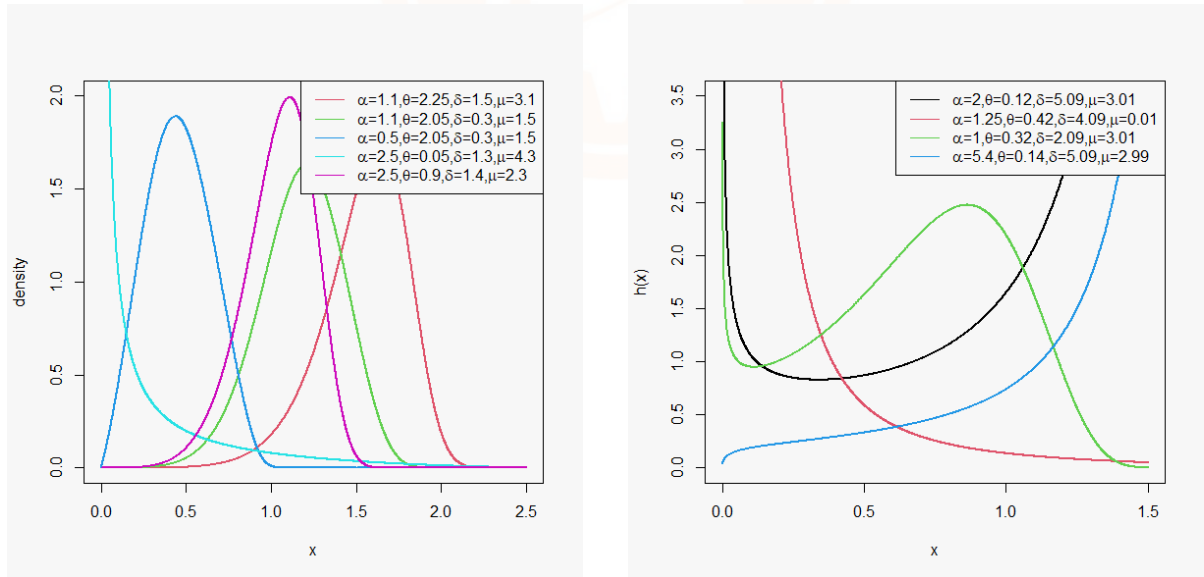


Figure 5.5: MOTLOBX-U distribution's PDF and HRF plots

5.6.4 Marshall-Olkin-Topp-Leone-Odd Burr X Burr III Distribution

If Burr III distribution is assumed to be the MOTLOBX-G family's baseline distribution with PDF $g(x; \lambda, \beta) = \lambda\beta x^{-\lambda-1}(1 + x^{-\lambda})^{-\beta-1}$ and CDF $G(x; \lambda, \beta) = (1 + x^{-\lambda})^{-\beta}$, respectively, for

$\lambda, \beta > 0$, the Marshall-Olkin-Topp-Leone-Odd Burr X Burr III (MOTLOBX-BIII) distribution is obtained using the below provided PDF and CDF.

$$\begin{aligned}
 f(x; \alpha, \theta, \delta, \lambda, \beta) &= \frac{4\delta\alpha\theta\lambda\beta x^{-\lambda-\beta}(1+x^{-\lambda})^{-2\beta-1}}{(1-(1+x^{-\lambda})^{-\beta})^3} \exp \left[- \left(\frac{(1+x^{-\lambda})^{-\beta}}{1-(1+x^{-\lambda})^{-\beta}} \right)^2 \right] \\
 &\times \frac{\left[1 - \left\{ 1 - \exp \left[- \left(\frac{(1+x^{-\lambda})^{-\beta}}{1-(1+x^{-\lambda})^{-\beta}} \right)^2 \right] \right\}^\theta \right]}{\left\{ 1 - \exp \left[- \left(\frac{(1+x^{-\lambda})^{-\beta}}{1-(1+x^{-\lambda})^{-\beta}} \right)^2 \right] \right\}^{1-\theta}} \\
 &\times \left[1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{(1+x^{-\lambda})^{-\beta}}{1-(1+x^{-\lambda})^{-\beta}} \right)^2 \right] \right\}^\theta \right]^2 \right]^{\alpha-1} \\
 &\times \left[1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{(1+x^{-\lambda})^{-\beta}}{1-(1+x^{-\lambda})^{-\beta}} \right)^2 \right] \right\}^\theta \right]^2 \right]^\alpha \right\} \right]^{-2}, x \in \mathbb{R}, \\
 &\alpha, \theta, \delta, \lambda, \beta > 0
 \end{aligned}$$

and

$$F(x; \alpha, \theta, \delta, \lambda, \beta) = 1 - \frac{\delta \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{(1+x^{-\lambda})^{-\beta}}{1-(1+x^{-\lambda})^{-\beta}} \right)^2 \right] \right\}^\theta \right]^2 \right]^\alpha \right\}}{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{(1+x^{-\lambda})^{-\beta}}{1-(1+x^{-\lambda})^{-\beta}} \right)^2 \right] \right\}^\theta \right]^2 \right]^\alpha \right\}},$$

respectively.

Figure 5.6 shows the plots of the PDFs and HRFs for the MOTLOBX-BIII distribution. The PDF may be used with data sets that have varied kurtosis and are left, right, or virtually symmetrically skewed. The HRF may be shaped into decreasing, increasing, upside-down, and bathtub forms.

5.6.5 Marshall-Olkin-Topp-Leone Odd Burr X-Burr XII Distribution

Now, considering Burr XII distribution as the MOTLOBX-G family's baseline distribution with PDF $g(x; \lambda, \beta) = \lambda\beta x^{\lambda-1}(1+x^\lambda)^{-\beta-1}$ and CDF $G(x; \lambda, \beta) = 1 - (1+x^\lambda)^{-\beta}$, respectively, for $\lambda, \beta > 0$, then the Marshall-Olkin-Topp-Leone Odd Burr X-Burr XII (MOTLOBX-BXII) dis-

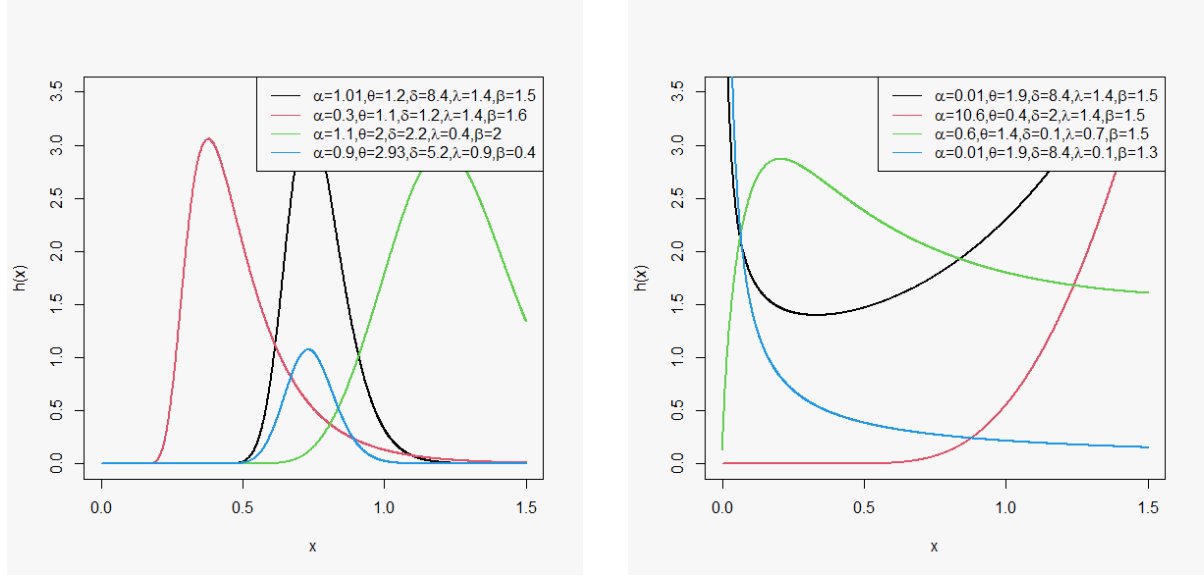


Figure 5.6: MOTLOBX-BIII distribution's PDF and HRF plots

tribution is obtained with PDF and CDF given by

$$\begin{aligned}
 f(x; \alpha, \theta, \delta, \lambda, \beta) &= \frac{4\delta\alpha\theta\lambda\beta x^{\lambda-1}(1+x^\lambda)^{-1} [1 - (1+x^\lambda)^{-\beta}]}{\exp \left[\left(\frac{1-(1+x^\lambda)^{-\beta}}{(1+x^\lambda)^{-\beta}} \right)^2 \right]} \\
 &\times \left\{ 1 - \exp \left[- \left(\frac{1 - (1+x^\lambda)^{-\beta}}{(1+x^\lambda)^{-\beta}} \right)^2 \right] \right\}^{\theta-1} \left[1 - \left\{ 1 - \exp \left[- \left(\frac{1 - (1+x^\lambda)^{-\beta}}{(1+x^\lambda)^{-\beta}} \right)^2 \right] \right\}^\theta \right] \\
 &\times \left[1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{1 - (1+x^\lambda)^{-\beta}}{(1+x^\lambda)^{-\beta}} \right)^2 \right] \right\}^\theta \right]^\alpha \right]^{\alpha-1} \\
 &\times \left[1 - \bar{\delta} \left\{ 1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{1 - (1+x^\lambda)^{-\beta}}{(1+x^\lambda)^{-\beta}} \right)^2 \right] \right\}^\theta \right]^\alpha \right\} \right]^{-2}, x \in \mathbb{R}, \\
 &\alpha, \theta, \delta, \lambda, \beta > 0
 \end{aligned}$$

and

$$F(x; \alpha, \theta, \delta, \lambda, \beta) = 1 - \frac{\delta \left\{ 1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{1-(1+x^\lambda)^{-\beta}}{(1+x^\lambda)^{-\beta}} \right)^2 \right] \right\}^\theta \right]^\alpha \right\}}{1 - \bar{\delta} \left\{ 1 - \left[1 - \left\{ 1 - \exp \left[- \left(\frac{1-(1+x^\lambda)^{-\beta}}{(1+x^\lambda)^{-\beta}} \right)^2 \right] \right\}^\theta \right]^\alpha \right\}},$$

respectively.

Figure 5.7 shows the PDFs and HRFs visualizations associated with MOTLOBX-BXII distribution. The PDF may be used with data sets with various levels of skewness and kurtosis. The HRF accepts decreasing, bathtub, and bathtub-turned-upside-down forms.

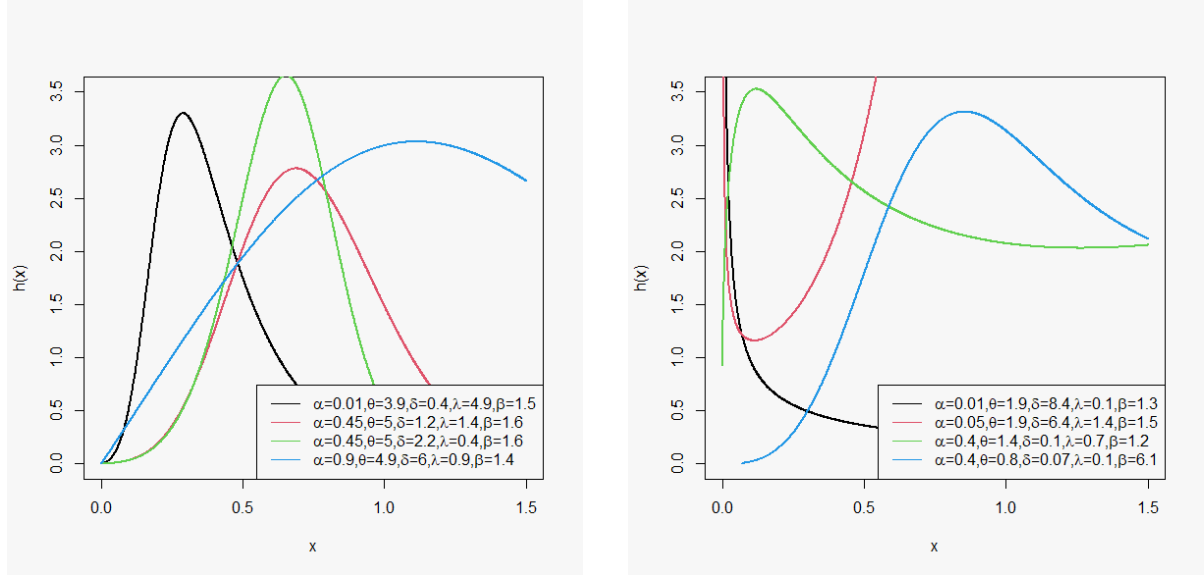


Figure 5.7: MOTLOBX-BXII distribution's PDF and HRF plots

5.7 Simulation Study

We present simulation experiment results for the Marshall-Olkin-Topp-Leone odd Burr X-log-logistic (MOTLOBX-L) distribution in this section. The simulation is run $N = 3000$ times with sample sizes of 50, 100, 200, 400, 800, and 1600. Mean estimates, RMSEs and Abias are also presented to assess estimators' accuracy and consistency. As shown in Table 5.2 and 5.3, as the sample size increases, the mean values get closer to the underlying parameter values. Additionally, RMSE and Abias also decreases with increasing sample size which is an indication that the maximum likelihood estimates provided by the MOTLOBX-L model are reliable.

Table 5.2: MOTLOBX-L distribution simulation experiment results: Mean, RSME, Abias

(0.2, 1.3, 1.3, 1.4)					(0.5, 0.5, 1.0, 2.0)		
Parameter	n	Mean	RMSE	Abias	Mean	RMSE	Abias
α	50	1.7215	4.1194	1.5215	2.9436	6.8803	2.4436
	100	1.0598	2.6824	0.8598	2.2481	5.0734	1.7481
	200	0.5999	1.4995	0.3999	1.4850	2.9316	0.9850
	400	0.3327	0.4955	0.1327	0.9878	1.8993	0.4878
	800	0.2499	0.1291	0.0499	0.7425	0.8543	0.2425
	1600	0.2191	0.0510	0.0191	0.5850	0.3916	0.0850
θ	50	1.0016	0.8308	-0.2984	0.6438	0.6350	0.1438
	100	1.0068	0.6768	-0.2932	0.5765	0.4699	0.0765
	200	1.0185	0.5353	-0.2815	0.5506	0.3809	0.0506
	400	1.0999	0.4026	-0.2001	0.5403	0.3045	0.0403
	800	1.1858	0.2697	-0.1142	0.5179	0.2396	0.0179
	1600	1.2441	0.1563	-0.0559	0.5164	0.1691	0.0164
δ	50	3.1330	5.5285	1.8330	2.0429	3.8188	1.0429
	100	2.3821	2.9202	1.0821	1.7153	1.9411	0.7153
	200	1.8824	1.7618	0.5824	1.4883	1.4403	0.4883
	400	1.5349	0.8556	0.2349	1.2484	0.8903	0.2484
	800	1.3734	0.3296	0.0734	1.1680	0.6493	0.1680
	1600	1.3094	0.1317	0.0094	1.0484	0.3608	0.0484
β	50	1.4347	0.5038	0.0347	2.2270	0.8590	0.2270
	100	1.3806	0.3269	-0.0194	2.0698	0.5623	0.0698
	200	1.3392	0.2207	-0.0608	2.0000	0.4052	0.0000
	400	1.3441	0.1508	-0.0559	1.9897	0.2961	-0.0103
	800	1.3510	0.1061	-0.0490	1.9675	0.2290	-0.0325
	1600	1.3733	0.0612	-0.0267	1.9667	0.1525	-0.0333

Table 5.3: MOTLOBX-L distribution simulation experiment results: Mean, RSME, Abias

Parameter	n	(0.5, 0.5, 0.5, 2.0)			(0.2, 1.3, 0.8, 1.4)		
		Mean	RMSE	Abias	Mean	RMSE	Abias
α	50	2.7654	6.2118	2.2654	1.5496	3.8831	1.3496
	100	2.2456	4.8333	1.7456	1.1310	2.7969	0.9310
	200	1.6754	3.2420	1.1754	0.6567	1.5409	0.4567
	400	1.0693	1.8300	0.5693	0.3822	0.7117	0.1822
	800	0.8057	1.1216	0.3057	0.2385	0.1029	0.0385
	1600	0.6289	0.6244	0.1289	0.2169	0.0503	0.0169
θ	50	0.5975	0.5991	0.0975	1.0353	0.7663	-0.2647
	100	0.5707	0.4653	0.0706	0.9946	0.6290	-0.3054
	200	0.5363	0.3658	0.0363	1.0237	0.5296	-0.2763
	400	0.5358	0.2934	0.0358	1.1118	0.3891	-0.1882
	800	0.5202	0.2301	0.0202	1.2123	0.2239	-0.0877
	1600	0.5062	0.1543	0.0062	1.2586	0.1386	-0.0414
δ	50	0.9555	1.3506	0.4555	1.7017	2.8324	0.9017
	100	0.8353	0.8952	0.3353	1.3845	1.5476	0.5845
	200	0.7536	0.7116	0.2536	1.1664	1.1182	0.3664
	400	0.6366	0.4613	0.1366	0.9414	0.5652	0.1414
	800	0.5888	0.3462	0.0888	0.8277	0.1522	0.0277
	1600	0.5350	0.2061	0.0350	0.7950	0.0783	-0.0050
β	50	2.1405	0.7894	0.1405	1.4042	0.4986	0.0042
	100	1.9890	0.5550	-0.0110	1.3450	0.3383	-0.0550
	200	1.9343	0.4167	-0.0657	1.3313	0.2523	-0.0687
	400	1.9321	0.3047	-0.0679	1.3417	0.1727	-0.0583
	800	1.9274	0.2373	-0.0726	1.3511	0.1122	-0.0489
	1600	1.9484	0.1493	-0.0516	1.3746	0.0617	-0.0254

5.8 Applications

Three real data examples are presented in this section to show the flexibility and applicability of Marshall-Olkin-Topp-Leone odd Burr X-log-logistic (MOTLOBX-L) distribution. Model performance is assessed using the GOF statistics $-2\ln(L)$, AIC, BIC, AICC, W^* , A^* , K-S and its p-value where the value of the likelihood function is represented with $L = L(\hat{\Psi})$. A smaller GOF statistic value and a higher p-value for the K-S statistic should characterize the ideal model.

MOTLOBX-L distribution is compared with MOTLOBX-W distribution and the following models given in Appendix E.5: Weibull Lomax (WL) distribution (Tahir et al., 2015), Marshall-Olkin generalized-Lomax (MOG-L) distribution (Yousof et al., 2018), Marshall-Olkin odd Burr III-

log-logistic (MOOBIII-Lo) distribution (Afify et al., 2020), Marshall-Olkin odd Burr III-logistic (MOOBIII-L) distribution (Afify et al., 2020), Marshall-Olkin inverted Nadarajah-Haghighi (MOINH) distribution (Raffiq et al., 2020), Marshall-Olkin log-logistic (MO-L) distribution (Alshangiti et al., 2016) and Topp-Leone-odd-Burr-X-log-logistic (TLOBX-L) distribution (Oluyede et al., 2022b) .

5.8.1 Fibre Strength Data

The first data with $n = 63$ observations represent the strength of glass fibre of length 1.5cm from National Physical Laboratory in England (Alizadeh et al., 2015). The data points are: 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.

The estimated var-cov matrix of MOTLOBX-L model based on fibre strength data is

$$\begin{bmatrix} 6.94 \times 10^{-7} & -1.80 \times 10^{-7} & 9.16 \times 10^{-13} & 3.31 \times 10^{-5} \\ -1.80 \times 10^{-7} & 8.66 \times 10^{-8} & -5.91 \times 10^{-14} & -1.60 \times 10^{-5} \\ 9.16 \times 10^{-13} & -5.91 \times 10^{-14} & 2.00 \times 10^{-18} & 1.09 \times 10^{-11} \\ 3.31 \times 10^{-5} & -1.60 \times 10^{-5} & 1.09 \times 10^{-11} & 2.94 \times 10^{-3} \end{bmatrix}$$

and for the MOTLOBX-L parameters, the 95% confidence regions are provided by $\alpha \in [2.62 \times 10^{-3} \pm 1.63 \times 10^{-3}]$, $\theta \in [1.21 \times 10^1 \pm 5.77 \times 10^{-4}]$, $\delta \in [1.06 \times 10^3 \pm 2.77 \times 10^{-9}]$ and $\beta \in [1.13 \pm 1.06 \times 10^{-1}]$.

Table 5.4: Models' MLE and GOF statistics: Fibre strength data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
	α	θ	δ	β									
MOTLOBX-L	2.62×10^{-3} (8.33×10^{-4})	1.21×10^1 (2.94×10^{-4})	1.06×10^3 (1.41×10^{-9})	1.13 (5.42×10^{-2})	24.4	32.4	33.1	41.0	0.0639	0.3929	0.0891	0.6998	0.0581
MOTLOBX-W	5.71×10^{-3} (2.01×10^{-3})	8.45×10^1 (6.09×10^{-6})	8.12×10^2 (1.28×10^{-8})	3.61×10^{-1} (2.07×10^{-2})	25.1	33.1	33.8	41.7	0.1076	0.6042	0.0971	0.5928	0.0840
WL	3.29×10^{-1} (6.38×10^{-2})	3.97 (3.85×10^{-1})	7.24×10^3 (6.30×10^{-5})	1.41×10^4 (3.25×10^{-5})	29.1	37.1	37.8	45.6	0.1962	1.0892	0.1425	0.1544	0.1813
MOG-L	7.20×10^3 (1.34×10^{-4})	6.52 (1.48×10^{-1})	8.26×10^3 (1.21×10^{-5})	6.46×10^3 (1.49×10^{-4})	32.0	40.0	40.7	48.6	0.1616	0.2596	1.4306	0.1254	0.1694
MOOBIII-Lo	4.51×10^2 (4.34×10^{-6})	1.96 (1.71×10^{-1})	9.20×10^{-2} (4.17×10^{-2})	4.45 (7.51×10^{-2})	40.4	48.4	49.1	57.0	0.3724	2.0787	0.1412	0.1619	0.2428
MOOBIII-L	7.08×10^2 (1.74×10^{-1})	1.07×10^2 (3.24×10^{-1})	1.14×10^1 (1.14×10^1)	1.84×10^1 (1.89)	32.0	40.0	40.7	48.6	0.2639	1.4517	0.1257	0.2727	0.1704
MOINH	7.85×10^1 (2.61×10^{-5})	2.62×10^{-5} (2.18×10^{-5})	6.17×10^{-1} (1.83×10^{-2})		59.2	65.2	65.6	71.6	0.7141	3.9321	0.1780	0.0369	0.4530
MO-L	1.95 (3.69)	7.93 (8.74×10^{-1})	1.40 (4.08×10^{-2})		45.6	51.6	51.9	58.0	0.4969	2.7489	0.1537	0.1019	0.3005
TLOBX-L	3.17×10^{-1} (1.56×10^{-1})	1.99×10^1 (8.88)	9.13×10^{-1} (6.17×10^{-2})		42.3	48.3	48.7	54.7	0.4676	2.5657	0.2280	0.0029	0.6164

Table 5.4 shows that MOTLOBX-L and MOTLOBX-W distributions fits the fibre strength data better than other models since they have the lowest values of all GOF statistics and the largest p-values for the K-S statistic.

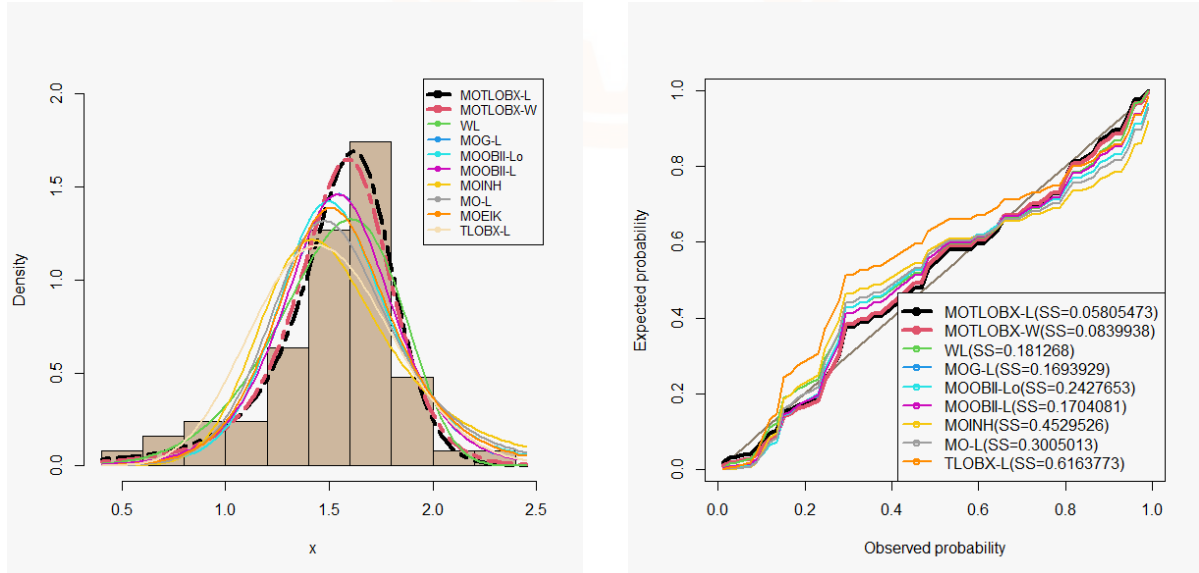


Figure 5.8: Fitted densities and P-Ps for fibre strength data

Also, from the fitted density, P-Ps and empirical CDF in Figures 5.8 and 5.9, the MOTLOBX-L and MOTLOBX-W distributions offer a superior match to the data on fiber strength compared to competing models. MOTLOBX-L and MOTLOBX-W distributions can model data with monotonic HRF shapes as shown in Figure 5.9.

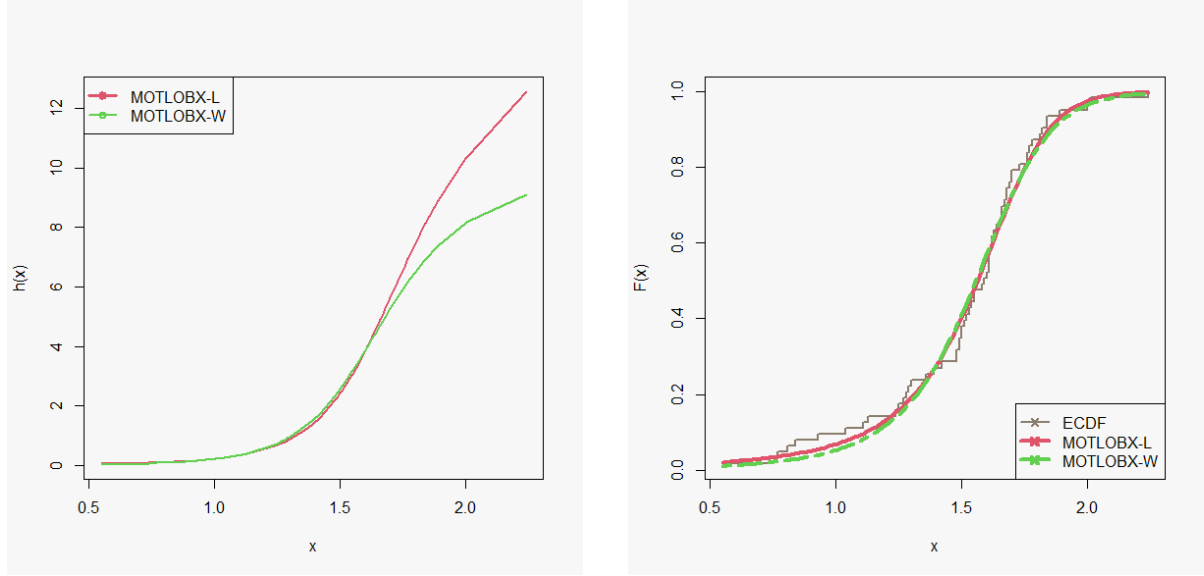


Figure 5.9: Hrf and empirical CDF plots for fibre strength data

5.8.2 Tumor Data

The second data with $n = 155$ observations represent the ages of 155 breast tumor patients who entered the unit of breast tumor early detection at Benha Hospital University from June until November, 2014 (Al-Noor et al., 2021). The data points are: 46, 32, 50, 46, 44, 42, 69, 31, 25, 29, 40, 42, 24, 17, 35, 48, 49, 50, 60, 26, 36, 56, 65, 48, 66, 44, 45, 30, 28, 40, 40, 50, 41, 39, 36, 63, 40, 42, 45, 31, 48, 36, 18, 24, 35, 30, 40, 48, 50, 60, 52, 47, 50, 49, 38, 30, 52, 52, 12, 48, 50, 45, 50, 50, 50, 53, 55, 38, 40, 42, 42, 32, 40, 50, 58, 48, 32, 45, 42, 36, 30, 28, 38, 54, 90, 80, 60, 45, 40, 50, 50, 40, 50, 50, 50, 60, 39, 34, 28, 18, 60, 50, 20, 40, 50, 38, 38, 42, 50, 40, 36, 38, 38, 50, 50, 31, 59, 40, 42, 38, 40, 38, 50, 50, 50, 40, 65, 38, 40, 38, 58, 35, 60, 90, 48, 58, 45, 35, 38, 32, 35, 38, 34, 43, 40, 35, 54, 60, 33, 35, 36, 43, 40, 45, 56.

The estimated var-cov matrix of MOTLOBX-L model based on tumor data is

$$\begin{bmatrix} 2.32 \times 10^{-7} & 1.84 \times 10^{-4} & 1.89 \times 10^{-7} & 2.72 \times 10^{-6} \\ 1.84 \times 10^{-4} & 1.47 \times 10^{-1} & 1.50 \times 10^{-4} & 2.16 \times 10^{-3} \\ 1.89 \times 10^{-7} & 1.50 \times 10^{-4} & 1.54 \times 10^{-7} & 2.21 \times 10^{-6} \\ 2.72 \times 10^{-6} & 2.16 \times 10^{-3} & 2.21 \times 10^{-6} & 3.36 \times 10^{-5} \end{bmatrix}$$

and for the MOTLOBX-L parameters, the 95% confidence regions are provided by $\alpha \in [5.19 \times 10^2 \pm 9.44 \times 10^{-4}]$, $\theta \in [1.30 \pm 7.50 \times 10^{-1}]$, $\delta \in [6.00 \times 10^2 \pm 7.69 \times 10^{-4}]$ and $\beta \in [2.51 \times 10^{-1} \pm 1.14 \times 10^{-2}]$.

Table 5.5: Models' MLE and GOF statistics: Tumor data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
	α	θ	δ	β									
MOTLOBX-L	5.19×10^2 (4.82×10^{-4})	1.30 (3.83×10^{-1})	6.00×10^2 (3.92×10^{-4})	2.51×10^{-1} (5.80×10^{-3})	1198.0	1206.0	1206.2	1218.1	0.1348	0.7468	0.0778	0.3055	0.1309
MOTLOBX-W	9.04×10^2 (1.96×10^{-9})	1.42×10^1 (2.49×10^{-7})	1.26×10^3 (1.41×10^{-9})	9.18×10^{-2} (5.01×10^{-4})	1198.5	1206.5	1206.7	1218.6	0.1372	0.7722	0.0778	0.3052	0.1304
WL	2.56×10^5 (1.19×10^{-5})	3.61 (1.92×10^{-1})	8.38 (1.46)	1.28×10^4 (9.15×10^{-4})	1221.2	1229.2	1229.4	1249.3	0.3242	1.9897	0.13604	0.0065	0.4102
MOG-L	3.59×10^3 (1.62×10^{-6})	5.42×10^{-1} (1.08×10^{-2})	7.87×10^2 (1.02×10^{-6})	1.03×10^3 (5.67×10^{-6})	1200.3	1208.3	1208.5	1220.5	0.1608	0.8676	0.0844	0.2198	0.1447
MOOBIII-Lo	4.41×10^3 (5.86×10^{-12})	5.87×10^2 (7.29×10^{-10})	2.70×10^3 (9.56×10^{-12})	7.43×10^{-3} (5.75×10^{-5})	1232.7	1240.7	1240.9	1252.8	0.2129	1.3271	0.1417	0.0040	0.8774
MOOBIII-L	3.85×10^1 (4.00×10^1)	1.38×10^1 (1.26×10^1)	1.63×10^1 (1.26×10^1)	9.22×10^1 (1.59×10^{-1})	1199.4	1207.4	1207.6	1219.6	0.1529	0.8429	0.0815	0.2541	0.1418
MOINH	5.79×10^2 (5.42×10^{-6})	8.28×10^{-4} (2.99×10^{-4})	7.75×10^{-1} (1.60×10^{-2})		1228.9	1234.9	1235.0	1244.0	0.4447	2.8030	0.0991	0.0950	0.2921
MO-L	6.93 (4.41×10^{-1})	6.48 (4.38×10^{-1})	3.16×10^1 (6.28×10^{-1})		1203.1	1209.1	1209.2	1218.2	0.1743	1.0530	0.0801	0.2719	0.2921
TLOBX-L	5.97×10^{-1} (9.54×10^{-2})	5.23×10^2 (8.88×10^{-5})	2.33×10^{-1} (3.15×10^{-3})		1226.7	1232.7	1232.8	1241.8	0.4122	2.6016	0.1285	0.0119	0.4770

Because of the lowest values of all GOF statistics and the highest p-values for the K-S statistic, the MOTLOBX-L and MOTLOBX-W distributions match the tumor data more accurately than other competing models, as shown by the Table 5.5.

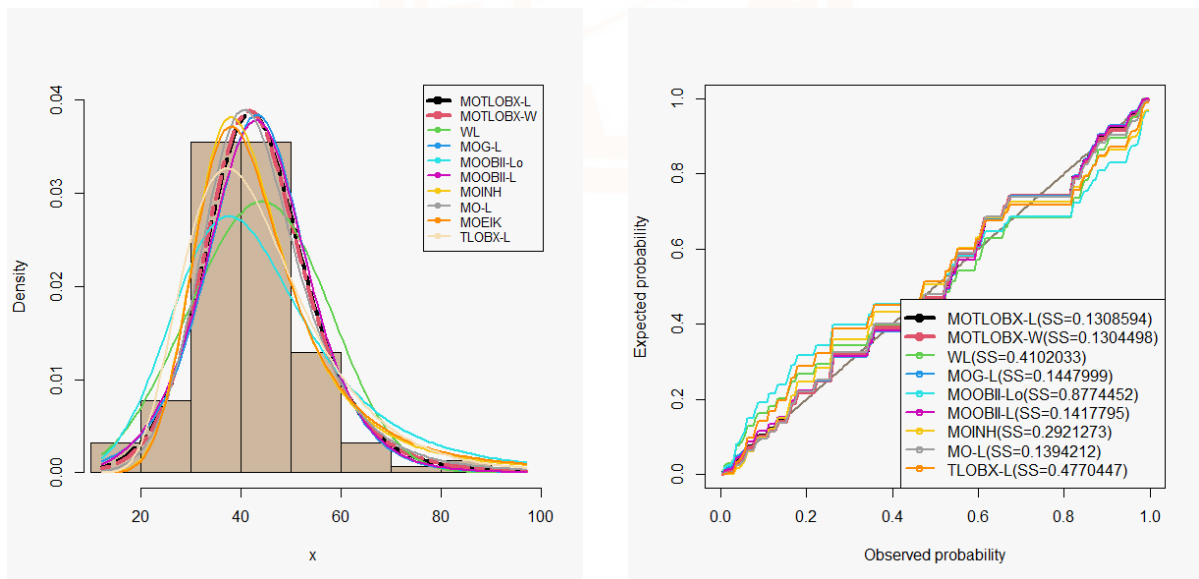


Figure 5.10: Fitted densities and P-Ps for tumor data

Also, from the fitted density, P-Ps and empirical CDF in Figures 5.10 and 5.11, MOTLOBX-L and MOTLOBX-W distributions offered a superior match to the tumor data compared to other models. MOTLOBX-L and MOTLOBX-W distributions can model data with non-monotonic HRF shapes as shown in Figure 5.11.

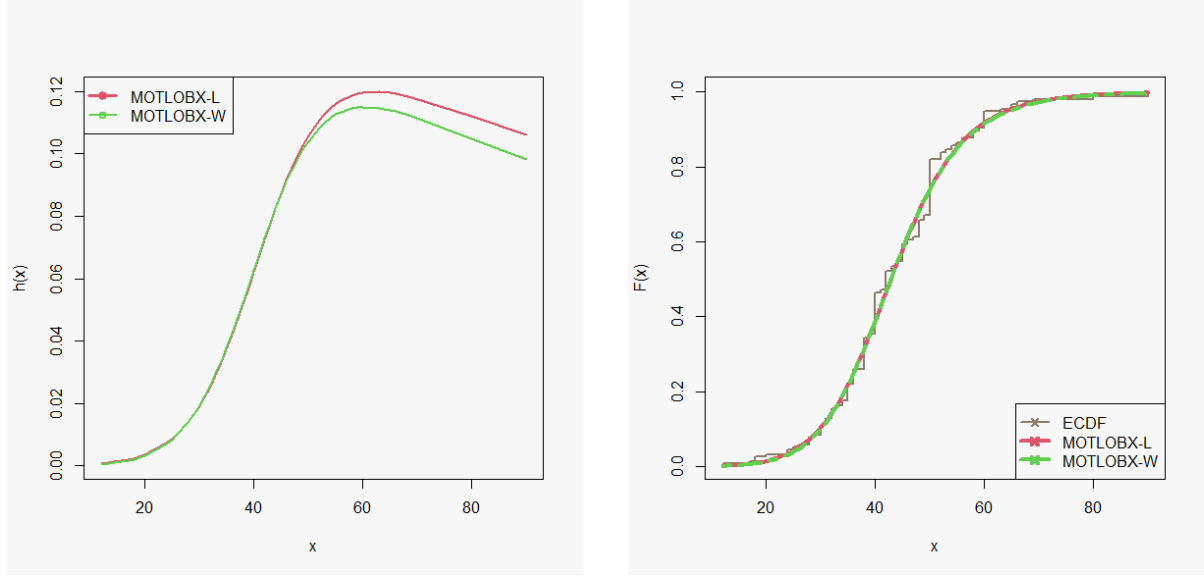


Figure 5.11: Hrf and empirical CDF plots for tumor data

5.8.3 Covid-19 Data

The third data with $n = 108$ observations represent the COVID-19 mortality rate data in Mexico for 108 days, from 4/3/2020 to 20/7/2020 studied by Baharith (2022). The data points are: 8.826, 6.105, 10.383, 7.267, 13.220, 6.015, 10.855, 6.122, 10.685, 10.035, 5.242, 7.630, 14.604, 7.903, 6.327, 9.391, 14.962, 4.730, 3.215, 16.498, 11.665, 9.284, 12.878, 6.656, 3.440, 5.854, 8.813, 10.043, 7.260, 5.985, 4.424, 4.344, 5.143, 9.935, 7.840, 9.550, 6.968, 6.370, 3.537, 3.286, 10.158, 8.108, 6.697, 7.151, 6.560, 2.988, 3.336, 6.814, 8.325, 7.854, 8.551, 3.228, 3.499, 3.751, 7.486, 6.625, 6.140, 4.909, 4.661, 1.867, 2.838, 5.392, 12.042, 8.696, 6.412, 3.395, 1.815, 3.327, 5.406, 6.182, 4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120, 3.922, 3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027, 2.352, 1.205, 2.077, 3.778, 3.218, 2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, 1.923.

The estimated var-cov matrix of MOTLOBX-L model based on Covid-19 data is

$$\begin{bmatrix} 2.05 \times 10^{-2} & -6.37 & -2.40 \times 10^{-2} & -2.26 \times 10^{-2} \\ -6.37 & 2.14 \times 10^3 & 1.82 & 6.47 \times 10^{-1} \\ -2.40 \times 10^{-2} & 1.82 & 2.93 \times 10^{-1} & 7.63 \times 10^{-3} \\ -2.26 \times 10^{-2} & 6.47 \times 10^{-1} & 7.63 \times 10^{-3} & 4.68 \times 10^{-4} \end{bmatrix}$$

and for the MOTLOBX-L parameters, the 95% confidence regions are provided by $\alpha \in [1.82 \times 10^{-1} \pm 2.81 \times 10^{-1}]$, $\theta \in [6.22 \times 10^1 \pm 9.07 \times 10^1]$, $\delta \in [9.85 \times 10^{-1} \pm 1.06]$ and $\beta \in [3.01 \times 10^{-1} \pm 4.24 \times 10^{-2}]$.

Table 5.6: Models' MLE and GOF statistics: Covid-19 data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
	α	θ	δ	β									
MOTLOBX-L	1.82×10^{-1} (1.43×10^{-1})	6.22×10^1 (4.62×10^1)	9.85×10^{-1} (5.41×10^{-1})	3.01×10^{-1} (2.16×10^{-2})	531.6	539.7	540.1	550.4	0.0520	0.2857	0.0655	0.7438	0.0534
MOTLOBX-W	1.81×10^{-1} (4.92×10^{-2})	4.97×10^2 (1.33×10^{-5})	1.81 (1.04)	1.02×10^{-1} (5.77×10^{-3})	531.8	539.8	540.2	550.6	0.0521	0.2879	0.0630	0.7845	0.0489
WL	a 4.18 (9.65)	b 3.73 (2.66)	α 2.25×10^{-1} (1.71×10^{-1})	β 7.01×10^{-1} (1.96)	533.3	541.3	541.7	552.1	0.0678	0.4055	0.0729	0.6149	0.0619
MOG-L	λ 8.32×10^{-1} (2.39×10^{-1})	β 3.12×10^{-2} (3.75×10^{-3})	δ 8.39×10^2 (1.18×10^{-4})	a 1.10×10^2 (2.27×10^{-5})	538.8	546.8	547.1	557.5	0.1151	0.6579	0.0765	0.5511	0.0930
MOOBIII-Lo	δ 2.63×10^1 (2.86×10^1)	α 4.19 (9.79×10^{-3})	β 3.53 (2.83)	λ 6.78×10^{-1} (6.12×10^{-1})	537.4	545.4	545.7	556.1	0.1081	0.6169	0.0690	0.6827	0.0896
MOOBIII-L	δ 4.91×10^{-1} (3.69×10^{-1})	α 2.96 (4.51×10^{-1})	β 8.49 (1.43)	λ 7.69 (1.73×10^{-1})	538.9	546.9	547.3	557.6	0.1084	0.6527	0.0879	0.3745	0.1230
MOINH	β 2.30×10^4 (6.18×10^{-9})	γ 8.25×10^{-5} (4.30×10^{-5})	α 2.76×10^{-1} (5.82×10^{-3})		543.0	549.0	549.2	557.1	0.1658	0.9765	0.0837	0.4362	0.1253
MO-L	α 1.69 (1.22×10^{-1})	β 2.93 (2.31×10^{-1})	γ 4.15 (1.46×10^{-1})		538.2	544.2	544.4	552.2	0.1151	0.6510	0.07367	0.6100	0.0957
TLOBX-L	α 2.64×10^3 (1.60×10^{-7})	θ 1.12×10^{-1} (7.31×10^{-3})	β 2.23×10^{-1} (1.05×10^{-2})		537.6	543.6	543.8	551.6	0.1109	0.6688	0.0839	0.4326	0.1132

Table 5.6 also shows that MOTLOBX-L and MOTLOBX-W distributions have the lowest values of all GOF statistics and the highest p-values for the K-S statistic when compared to other competing models, implicating that they are best appropriate for COVID-19 data.

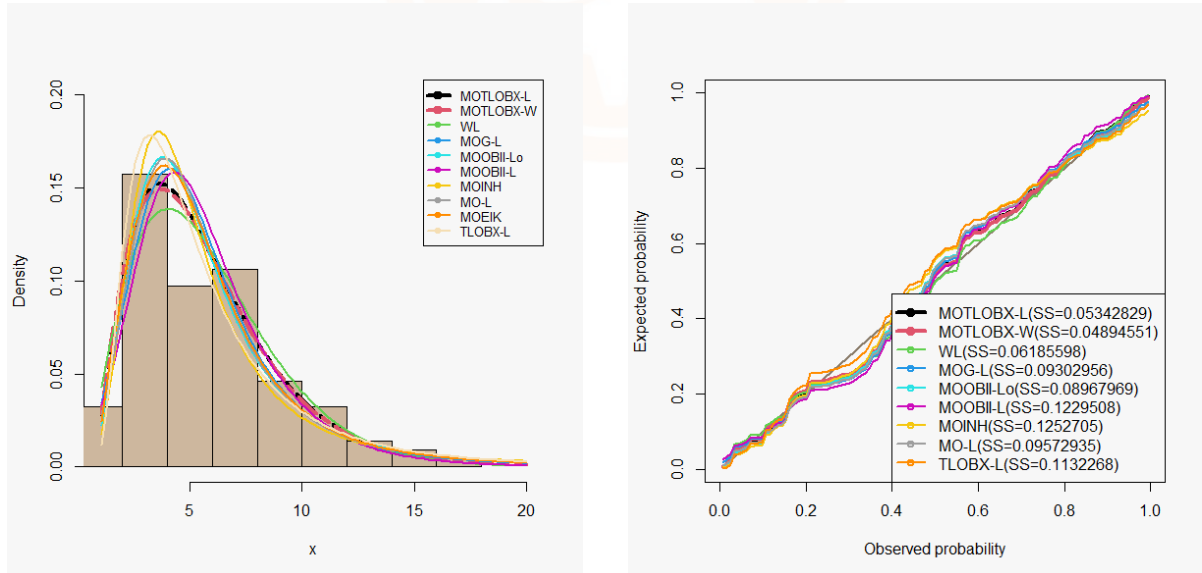


Figure 5.12: Fitted densities and P-Ps for Covid-19 data

A superior match to the data is also provided by the MOTLOBX-L and MOTLOBX-W distributions, as demonstrated by the fitted density, P-Ps, and empirical CDF in Figures 5.12 and 5.13. MOTLOBX-L and MOTLOBX-W distributions can also model data with non-monotone HRF shapes as shown in Figure 5.13.

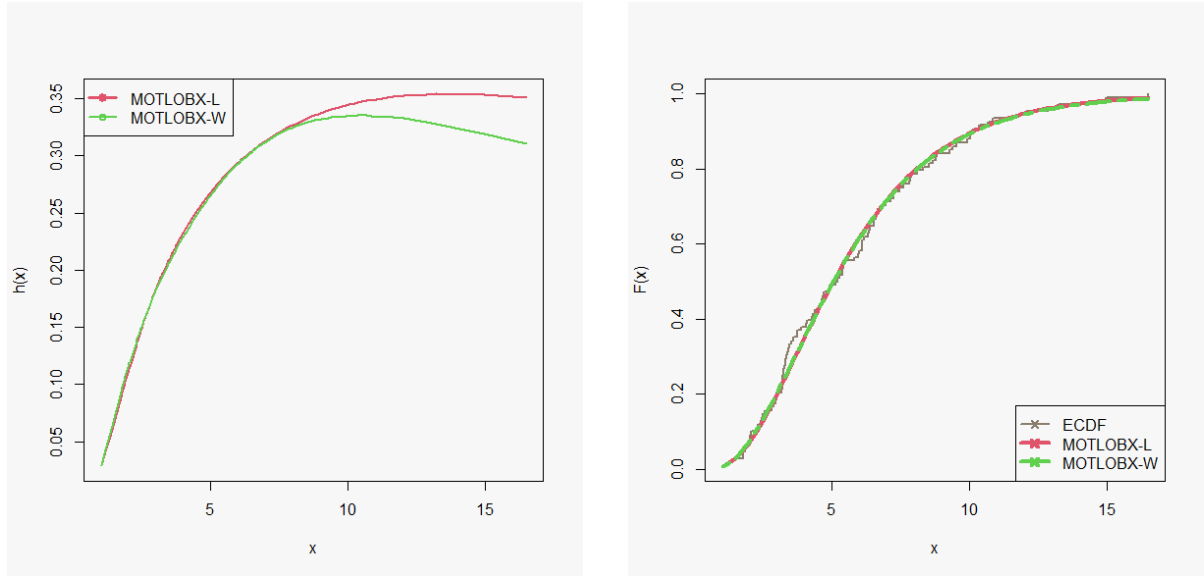


Figure 5.13: Hrf and empirical CDF plots for Covid-19 data

5.9 Concluding Remarks

The Marshall-Olkin-Topp-Leone odd Burr X-G (MOTLOBX-G) family is developed in this chapter. The statistical characteristics of the MOTLOBX-G family, which consists the QF, density expansion, ordinary moments, central moments, moments and generating function, Rényi entropy, order statistics and stochastic ordering are provided. We use MLE technique to derive the parameters of Marshall-Olkin-Topp-Leone odd Burr X-log-logistic (MOTLOBX-L) distribution. Subsequent simulation investigation shows the consistency of the maximum likelihood estimates. To demonstrate the usability and adaptability of the MOTLOBX-G family, three real data sets are applied to the MOTLOBX-L and Marshall-Olkin-Topp-Leone odd Burr X-Weibull (MOTLOBX-W) special instances. From the results in Tables 5.4, 5.5 and 5.6 in contrast to other competing models, the MOTLOBX-L distribution often had the lowest GOF statistics values and the largest p-value for the K-S statistic. Second place went to MOTLOBX-W distribution. Moreover, it is demonstrated that the MOTLOBX-L and MOTLOBX-W distributions can represent datasets with monotonic and non-monotonic HRF forms. There is evidence that the MOTLOBX-G family may function effectively in practical situations.

6. The New Topp-Leone-Exponentiated Half Logistic-Odd Burr X-G Family

6.1 Introduction

The Topp-Leone-exponentiated half logistic-odd Burr X-G (TLEHLOBX-G) family is introduced and studied in this chapter. Some of the statistical properties of this family are derived. The model parameters are obtained using the maximum likelihood approach, and a simulation experiment is used to assess the consistency of the MLE. The usefulness and versatility of the TLEHLOBX-G family are demonstrated by applications to three actual data sets from different fields.

6.2 The Topp-Leone-Exponentiated Half Logistic-Odd Burr X-G Family

The Topp-Leone-exponentiated half logistic-odd Burr X-G (TLEHLOBX-G) family and the QF are presented in this section. The Topp-Leone-G (TL-G) family developed by [Al-Shomrani et al. \(2016\)](#) has the PDF and CDF given as

$$f_{TL-G}(x; b, \underline{\Psi}) = 2bg(x; \underline{\Psi})\bar{G}(x; \underline{\Psi}) [1 - \bar{G}^2(x; \underline{\Psi})]^{b-1}, x \in \mathbb{R}, b \quad (6.1)$$

and

$$F_{TL-G}(x; b, \underline{\Psi}) = [1 - \bar{G}^2(x; \underline{\Psi})]^b, \quad (6.2)$$

respectively, where $G(x; \underline{\Psi})$ and $g(x; \underline{\Psi})$ are the baseline CDF and PDF with the baseline vector of parameters $\underline{\Psi}$. The exponentiated half logistic-G (EHL-G) family developed by [Cordeiro et al. \(2014\)](#) has the PDF and CDF given as

$$f_{EHL-G}(x; \gamma, \alpha, \underline{\Psi}) = 2\alpha\gamma g(x; \underline{\Psi}) [1 - G(x; \underline{\Psi})]^{\gamma-1} \frac{\{1 - [1 - G(x; \underline{\Psi})]^\gamma\}^{\alpha-1}}{\{1 + [1 - G(x; \underline{\Psi})]^\gamma\}^{\alpha+1}}, x \in \mathbb{R}, \gamma, \alpha > 0 \quad (6.3)$$

and

$$F_{EHL-G}(x; \gamma, \alpha, \underline{\Psi}) = \left[\frac{1 - [1 - G(x; \underline{\Psi})]^\gamma}{1 + [1 - G(x; \underline{\Psi})]^\gamma} \right]^\alpha, \quad (6.4)$$

respectively, where $\underline{\Psi}$ is the baseline distribution vector of parameters. [Yousof et al. \(2017a\)](#)

gave the PDF and CDF of odd Burr X-G (OBX-G) family as

$$f_{OBX-G}(x; \beta, \underline{\Psi}) = \frac{2\beta g(x; \underline{\Psi}) G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi})} \exp(-K_G(x; \underline{\Psi})) [1 - \exp(-K_G(x; \underline{\Psi}))]^{\beta-1}, x \in \mathbb{R}, \beta > 0 \quad (6.5)$$

and

$$F_{OBX-G}(x; \beta, \underline{\Psi}) = [1 - \exp(-K_G(x; \underline{\Psi}))]^\beta, \quad (6.6)$$

respectively, where $K_G(x; \underline{\Psi}) = \left(\frac{G(x; \underline{\Psi})}{\bar{G}(x; \underline{\Psi})}\right)^2$, $\bar{G}(x; \underline{\Psi}) = 1 - G(x; \underline{\Psi})$, $x \geq 0$ and $\underline{\Psi}$ is the baseline vector of parameters. If we let $\gamma = 1$ in Equations (6.3) and (6.4), we get the exponentiated half logistic-odd Burr X-G (EHLOBX-G) family with PDF and CDF

$$\begin{aligned} f_{EHLOBX-G}(x; \alpha, \beta, \underline{\Psi}) &= 2\alpha f_{OBX-G}(x; \beta, \underline{\Psi}) \frac{F_{OBX-G}^{\alpha-1}(x; \beta, \underline{\Psi})}{[1 + \bar{F}_{OBX-G}(x; \beta, \underline{\Psi})]^{\alpha+1}} \\ &= \frac{4\alpha\beta g(x; \underline{\Psi}) G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi})} \exp(-K_G(x; \underline{\Psi})) [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha\beta-1} \\ &\quad \times [2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta]^{-\alpha-1}, x \in \mathbb{R}, \alpha, \beta > 0 \end{aligned} \quad (6.7)$$

and

$$\begin{aligned} F_{EHLOBX-G}(x; \alpha, \beta, \underline{\Psi}) &= \left[\frac{F_{OBX-G}(x; \beta, \underline{\Psi})}{1 + \bar{F}_{OBX-G}(x; \beta, \underline{\Psi})} \right]^\alpha \\ &= \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha, \end{aligned} \quad (6.8)$$

respectively, where $K_G(x; \underline{\Psi}) = \left(\frac{G(x; \underline{\Psi})}{\bar{G}(x; \underline{\Psi})}\right)^2$ and baseline vector of parameters $\underline{\Psi}$. Plugging Equations (6.7) and (6.8) into Equations (6.1) and (6.2), we get the TLEHLOBX-G family with the PDF and CDF as

$$\begin{aligned} f_{TLEHLOBX-G}(x; b, \alpha, \beta, \underline{\Psi}) &= \frac{8b\alpha\beta g(x; \underline{\Psi}) G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi}) \exp(K_G(x; \underline{\Psi}))} [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha\beta-1} \\ &\quad \times [2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta]^{-\alpha-1} \left\{ 1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right\} \\ &\quad \times \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right]^2 \right\}^{b-1}, x \in \mathbb{R}, b, \alpha, \beta > 0 \end{aligned} \quad (6.9)$$

and

$$F_{TLEHLOBX-G}(x; b, \alpha, \beta, \underline{\Psi}) = \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right]^2 \right\}^b, \quad (6.10)$$

respectively, where $K_G(x; \underline{\Psi}) = \left(\frac{G(x; \underline{\Psi})}{\bar{G}(x; \underline{\Psi})} \right)^2$ and $\underline{\Psi}$ is the baseline vector of parameters.

6.2.1 Quantile Function

Equating Equation (6.10) to u , for $0 \leq u \leq 1$, we have

$$\left\{ 1 - \left[1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x; \underline{\Psi})}{\bar{G}(x; \underline{\Psi})} \right)^2 \right)^\beta \right)^\alpha}{2 - \left(1 - \exp \left(- \left(\frac{G(x; \underline{\Psi})}{\bar{G}(x; \underline{\Psi})} \right)^2 \right)^\beta \right)} \right]^\alpha \right]^2 \right\}^b = u.$$

After a few simplifications, we obtain

$$G(x; \underline{\Psi}) = \left[\left(-\log \left[1 - \left(\frac{2 \left[1 - \left(1 - u^{\frac{1}{b}} \right)^{\frac{1}{2}} \right]^{\frac{1}{\alpha}}}{1 + \left[1 - \left(1 - u^{\frac{1}{b}} \right)^{\frac{1}{2}} \right]^{\frac{1}{\alpha}} \right)} \right]^{\frac{1}{\beta}} \right)^{\frac{-1}{2}} + 1 \right]^{-1},$$

which finally gives the QF of TLEHLOBX-G family as

$$Q_x(u) = G^{-1} \left(\left[\left(-\log \left[1 - \left(\frac{2 \left[1 - \left(1 - u^{\frac{1}{b}} \right)^{\frac{1}{2}} \right]^{\frac{1}{\alpha}}}{1 + \left[1 - \left(1 - u^{\frac{1}{b}} \right)^{\frac{1}{2}} \right]^{\frac{1}{\alpha}} \right)} \right]^{\frac{1}{\beta}} \right)^{\frac{-1}{2}} + 1 \right]^{-1} \right).$$

With specified model parameters of TLEHLOBX-G family, we can use the above function to generate random numbers which are needed for simulation study. In Table 6.1 we present quantiles for specified values of u and parameters of Topp-Leone-exponentiated half logistic-odd Burr X-log-logistic (TLEHLOBX-L) distribution.

Table 6.1: Quantiles values for the TLEHLOBX-L distribution

u	$(b, \alpha, \beta, \lambda)$				
	(1,0.5,0.9,1.5)	(0.5,1,0.7,2.5)	(1.5,0.5,0.5,1)	(1.5,1.5,0.6,1.5)	(1,1.5,0.4,3.5)
0.1	0.2324	0.3430	0.0429	0.6070	0.6883
0.2	0.3953	0.4991	0.1064	0.7186	0.7947
0.3	0.5484	0.6229	0.1831	0.7982	0.8696
0.4	0.7029	0.7325	0.2737	0.8650	0.9323
0.5	0.8666	0.8364	0.3810	0.9263	0.9898
0.6	1.0486	0.9405	0.5114	0.9866	1.0463
0.7	1.2633	1.0513	0.6769	1.0499	1.1057
0.8	1.5407	1.1800	0.9050	1.1227	1.1743
0.9	1.9731	1.3565	1.2837	1.2218	1.2680

6.3 Some Statistical Properties

In this section, we present the density expansion, moments and MGF for the TLEHLOBX-G family together with the other characteristics such as the Rényi Entropy, pwms and distribution of the order statistics.

6.3.1 Linear Representation

The linear presentation of the density is very important as it can help in deriving other properties of the distribution. We use the following generalized binomial and Taylor series expansions

$$(1+z)^{-\beta} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta+k-1}{k} z^k \quad \text{for } |z| < 1, \beta > 0, \quad \text{and} \quad e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!},$$

respectively. Applying these expansions as shown in Appendix [F.1](#) we finally get the expansion of TLEHLOBX-G density as

$$\begin{aligned}
f(x; b, \alpha, \beta, \underline{\Psi}) &= \sum_{i,j,k,l,m,s,u,w=0}^{\infty} \binom{2i+1}{j} \binom{b-1}{i} \binom{\alpha(j+1)+k}{k} \binom{\alpha(\beta+j)+k+\beta l-1}{m} \\
&\times \binom{k}{l} \binom{2s+1}{u} \binom{u-(3+2s)}{w} (-1)^{i+j+l+m+s+u+w} \\
&\times \frac{8b\alpha\beta(m+1)^s g(x; \underline{\Psi}) (w+1)}{s! (w+1)} G^w(x; \underline{\Psi}) \\
&= \sum_{w=0}^{\infty} \phi_{w+1} g_{w+1}(x; \underline{\Psi}),
\end{aligned}$$

where

$$\begin{aligned} \phi_{w+1} = & \sum_{i,j,k,l,m,s,u=0}^{\infty} \binom{2i+1}{j} \binom{b-1}{i} \binom{\alpha(j+1)+k}{k} \binom{k}{l} \binom{\alpha(\beta+j)+k+\beta l-1}{m} \\ & \times \binom{2s+1}{u} \binom{u-(3+2s)}{w} (-1)^{i+j+l+m+s+u+w} \frac{8b\alpha\beta(m+1)^s}{s!(w+1)}, \end{aligned} \quad (6.11)$$

and

$$g_{w+1}(x; \underline{\Psi}) = (w+1)g(x; \underline{\Psi})G^{w+1-1}(x; \underline{\Psi}). \quad (6.12)$$

Thus, the PDF of the TLEHLOBX-G family can be expressed as an infinite linear combination of the Exp-G densities with power parameter $(w+1)$.

6.3.2 Moments and Generating Functions

Let Y_{w+1} be a rv with the Exp-G family and power parameter $(w+1)$. The TLEHLOBX-G family's n^{th} moment may thus be expressed as

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_{TLEHLOBX-G}(x; b, \alpha, \beta, \underline{\Psi}) dx = \sum_{w=0}^{\infty} \phi_{w+1} E(Y_{w+1}^n),$$

where $E(Y_{w+1}^n)$ is the n^{th} moment of the Exp-G family with power parameter $(w+1)$ and ϕ_{w+1} is given in Equation (6.11). Furthermore, the MGF for $h < 1$ is

$$M_X(h) = \sum_{w=0}^{\infty} \phi_{w+1} M_{w+1}(h),$$

where $M_{w+1}(h)$ is the MGF of Y_{w+1} and ϕ_{w+1} is defined in Equation (6.11).

6.3.3 Incomplete Moments

The TLEHLOBX-G family's n^{th} incomplete moment may thus be expressed as

$$\omega_n(t) = \int_{-\infty}^t x^n f_{TLEHLOBX-G}(x; b, \alpha, \beta, \underline{\Psi}) dx = \sum_{w=0}^{\infty} \phi_{w+1} \int_{-\infty}^t x^n g_{w+1}(x; \underline{\Psi}) dx, \quad (6.13)$$

where the Exp-G family PDF $g_{w+1}(x; \underline{\Psi})$ has the power parameter $(w+1)$ given in Equation (6.12). We may get the TLEHLOBX-G family's first incomplete moment by setting $n = 1$ in Equation (6.13).

6.3.4 Conditional Moments

The TLEHLOBX-G family's n^{th} conditional moment may thus be expressed as

$$\begin{aligned} E(X^n | X \geq a) &= \frac{1}{\bar{F}_{TLEHLOBX-G}(a; b, \alpha, \beta, \underline{\Psi})} \int_a^\infty x^n f_{TLEHLOBX-G}(x; b, \alpha, \beta, \underline{\Psi}) dx \\ &= \frac{1}{\bar{F}_{TLEHLOBX-G}(a; b, \alpha, \beta, \underline{\Psi})} \sum_{w=0}^\infty \phi_{w+1} E(Y_{w+1}^n I_{\{Y_{w+1}^n \geq a\}}), \end{aligned}$$

where

$$\begin{aligned} E(Y_{w+1}^n I_{\{Y_{w+1}^n \geq a\}}) &= \int_a^\infty y^n g_{w+1}(y; \underline{\Psi}) dy \\ &= (w+1) \int_{G(a, \underline{\Psi})}^1 [Q_G(u; \underline{\Psi})]^n u^{r+1} du, \end{aligned}$$

for parameter vector $\underline{\Psi}$.

6.3.5 Rényi Entropy

Rényi entropy for a given TLEHLOBX-G rv X is defined to be

$$E_Q(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f(x; b, \alpha, \beta, \underline{\Psi})]^v dx \right), v \neq 1, v > 0.$$

Following derivations in Appendix [F.2](#), we get the entropy as

$$\begin{aligned} E_Q(v) &= \frac{1}{1-v} \log \left[\sum_{i,j,k,l,m,s,u,w=0}^\infty \binom{\alpha(j+v) + v + k - 1}{k} (-1)^{i+j+l+m+s+u+w} \binom{k}{l} \binom{vb-v}{i} \right. \\ &\quad \times \binom{2i+v}{j} \binom{v+2s}{u} \binom{\alpha(\beta v + j) - v + \beta l}{m} \binom{u - (3v+2s)}{w} \frac{(8b\alpha\beta)^v (v+m)^s}{s!} \\ &\quad \times \left. \int_0^\infty \left(\left[1 + \frac{w}{v} \right] (G(x; \underline{\Psi}))^{\frac{w}{v}} (g(x; \underline{\Psi})) \right)^v dx \right] \\ &= \frac{1}{1-v} \log \left[\sum_{w=0}^\infty b_w \exp((1-v)E_{Qw}) \right], \end{aligned}$$

for $v > 0$, $v \neq 1$, where $E_{Qw} = \frac{1}{1-v} \log \left[\int_0^\infty \left(\left[1 + \frac{w}{v} \right] (G(x; \underline{\Psi}))^{\frac{w}{v}} (g(x; \underline{\Psi})) \right)^v dx \right]$ is the Rényi entropy of Exp-G family with power parameter $(\frac{w}{v} + 1)$ and

$$\begin{aligned} b_w &= \sum_{i,j,k,l,m,s,u=0}^\infty \binom{vb-v}{i} \binom{2i+v}{j} \binom{\alpha(j+v) + v + k - 1}{k} (-1)^{i+j+l+m+s+u+w} \binom{k}{l} \\ &\quad \times \binom{v+2s}{u} \binom{\alpha(\beta v + j) - v + \beta l}{m} \binom{u - (3v+2s)}{w} \frac{(8b\alpha\beta)^v (v+m)^s}{s!} \frac{1}{\left[1 + \frac{w}{v} \right]^v}. \end{aligned}$$

6.3.6 Probability Weighted Moments

Generally, a rv's PWMs follows as

$$\vartheta_{z,\eta} = E(X^z (F(X))^\eta) = \int_{-\infty}^{\infty} x^z F(x)^\eta f(x) dx.$$

Assuming the above PDF and CDF are that of the TLEHLOBX-G family, we get

$$\begin{aligned} (F(x))^\eta f(x) &= \frac{8b\alpha\beta g(x; \underline{\Psi}) G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi})} [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha\beta-1} \\ &\times \exp(-K_G(x; \underline{\Psi})) \left[2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right]^{-\alpha-1} \\ &\times \left\{ 1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right\} \\ &\times \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right]^2 \right\}^{b(\eta+1)-1}. \end{aligned}$$

Using the same procedures used in obtaining Equation (6.11), we get

$$F(x)^\eta f(x) = \sum_{w=0}^{\infty} \tau_{w+1} g_{w+1}(x; \underline{\Psi}).$$

Thus, the PMWs reduces to

$$\vartheta_{z,\eta} = E(X^z (F(X))^\eta) = \sum_{w=0}^{\infty} \tau_{w+1} \int_{-\infty}^{\infty} x^z g_{w+1}(x; \underline{\Psi}) dx,$$

where

$$\begin{aligned} \tau_{w+1} &= \sum_{i,j,k,l,m,s,u=0}^{\infty} \binom{2i+1}{j} \binom{b(\eta+1)-1}{i} \binom{\alpha(j+1)+k}{k} \binom{k}{l} \binom{\alpha(\beta+j)+k+\beta l-1}{m} \\ &\times \binom{2s+1}{u} \binom{u-(3+2s)}{w} (-1)^{i+j+l+m+s+u+w} \frac{8b\alpha\beta(m+1)^s}{s!(w+1)}, \end{aligned}$$

and Exp-G family's PDF $g_{w+1}(x; \underline{\Psi})$ is of power parameter $(w+1)$ and it is given in Equation (6.12).

6.3.7 Order Statistics

Let X represent TLEHLOBX-G rv, then the r^{th} order statistic's PDF from the TLEHLOBX-G family is given as

$$f_{r:n}(x) = \frac{n!f(x)}{(r-1)!(n-r)!} \sum_{z=0}^{n-r} (-1)^z \binom{n-r}{z} [F(x)]^{z+r-1}.$$

Note that

$$\begin{aligned}
 f(x)[F(x)]^{z+r-1} &= \frac{8b\alpha\beta g(x; \underline{\Psi})G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi})} [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha\beta-1} \\
 &\times \exp(-K_G(x; \underline{\Psi})) \left[2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right]^{-\alpha-1} \\
 &\times \left\{ 1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right\} \\
 &\times \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right]^2 \right\}^{b(z+r)-1}.
 \end{aligned}$$

Now, following the same steps leading to Equation (6.11), we obtain

$$f(x)[F(x)]^{z+r-1} = \sum_{w=0}^{\infty} C_{w+1} g_{w+1}(x; \underline{\Psi}).$$

which finally gives

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{z,w=0}^{n-r} (-1)^z \binom{n-r}{z} C_{w+1} g_{w+1}(x; \underline{\Psi}),$$

where

$$\begin{aligned}
 C_{w+1} &= \sum_{i,j,k,l,m,s,u=0}^{\infty} \binom{b(z+r)-1}{i} \binom{\alpha(j+1)+k}{k} \binom{k}{l} \binom{\alpha(\beta+j)+k+\beta l-1}{m} \\
 &\times \binom{2i+1}{j} \binom{2s+1}{u} \binom{u-(3+2s)}{w} (-1)^{i+j+l+m+s+u+w} \frac{8b\alpha\beta(m+1)^s}{s!(w+1)},
 \end{aligned}$$

and the Exp-G family's PDF $g_{w+1}(x, \underline{\Psi})$ is of the power parameter $(w+1)$ and given in Equation (6.12).

6.4 Some Special Cases

In this section, we some present the special cases for our new distribution by considering the baseline distributions as log-logistic, Weibull and Burr III distributions.

6.4.1 Topp-Leone-Exponentiated Half Logistic-Odd Burr X-Log-Logistic Distribution

Assuming log-logistic distribution with PDF $g(x; \lambda) = \lambda x^{\lambda-1} (1+x^\lambda)^{-2}$ and CDF $G(x; \lambda) = 1 - (1+x^\lambda)^{-1}$ is the baseline, we obtain the PDF and CDF of Topp-Leone-exponentiated half

logistic-odd Burr X-log-logistic (TLEHLOBX-L) distribution as

$$f_{TLEHLOBX-L}(x; b, \alpha, \beta, \lambda) = \frac{8b\alpha\beta\lambda x^{\lambda-1}(1+x^\lambda)}{(1-(1+x^\lambda)^{-1})^{-1}\exp(K_1(x; \lambda))} [1 - \exp(-K_1(x; \lambda))]^{\alpha\beta-1} \\ \times \left[2 - (1 - \exp(-K_1(x; \lambda)))^\beta \right]^{-\alpha-1} \left\{ 1 - \left[\frac{(1 - \exp(-K_1(x; \lambda)))^\beta}{2 - (1 - \exp(-K_1(x; \lambda)))^\beta} \right]^\alpha \right\} \\ \times \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_1(x; \lambda)))^\beta}{2 - (1 - \exp(-K_1(x; \lambda)))^\beta} \right]^\alpha \right]^2 \right\}^{b-1}, x \in \mathbb{R}, b, \beta, \alpha, \lambda > 0$$

and

$$F_{TLEHLOBX-L}(x; b, \beta, \alpha, \lambda) = \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_1(x; \lambda)))^\beta}{2 - (1 - \exp(-K_1(x; \lambda)))^\beta} \right]^\alpha \right]^2 \right\}^b,$$

respectively, where $K_1(x; \lambda) = \left(\frac{1-(1+x^\lambda)^{-1}}{(1+x^\lambda)^{-1}} \right)^2$. Figure 6.1 shows the plots of PDFs and HRFs' plots of TLEHLOBX-L distribution, respectively. The PDF can take several shapes including almost symmetric, right-skewed, left-skewed, reverse-J and unimodal shapes. The HRF displays increasing, decreasing, bathtub and upside-down bathtub shapes.

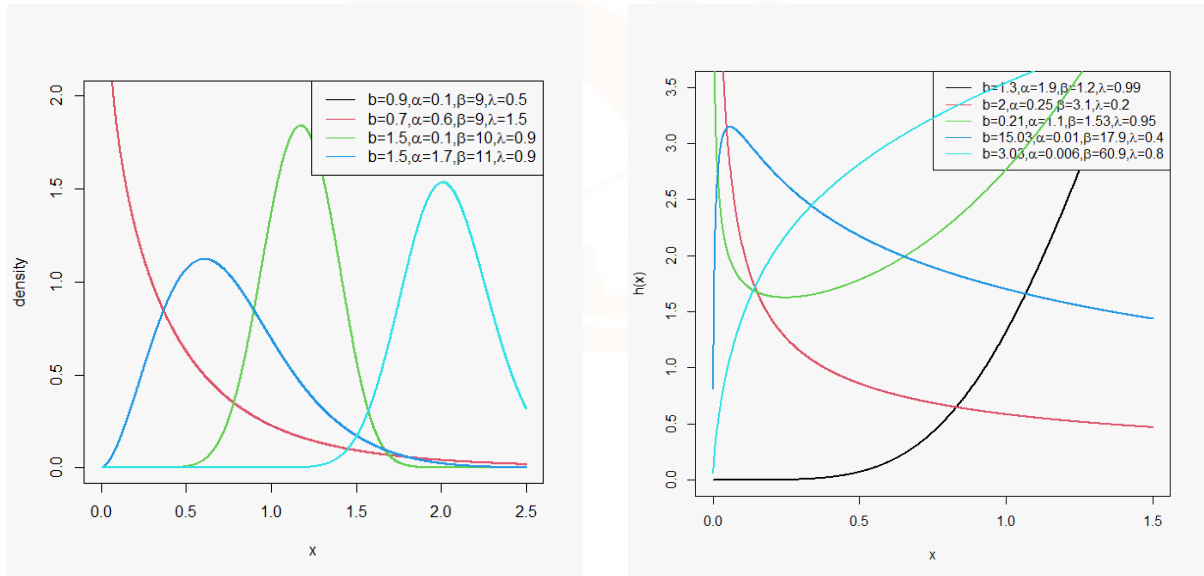


Figure 6.1: TLEHLOBX-L distribution's PDF and HRF plots

6.4.2 Topp-Leone-Exponentiated Half Logistic-Odd Burr X-Weibull Distribution

Taking Weibull distribution as the baseline associated with the PDF $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$ and CDF $G(x; \lambda) = 1 - e^{-x^\lambda}$, we get the Topp-Leone-exponentiated half logistic-odd Burr X-Weibull

(TLEHLOBX-W) PDF and CDF as

$$f_{TLEHLOBX-W}(x; b, \alpha, \beta, \lambda) = \frac{8b\alpha\beta\lambda x^{\lambda-1}e^{2x^\lambda}}{\exp(K_2(x; \lambda))} [1 - \exp(-K_2(x; \lambda))]^{\alpha\beta-1} \\ \times \left[2 - (1 - \exp(-K_2(x; \lambda)))^\beta \right]^{-\alpha-1} \left\{ 1 - \left[\frac{(1 - \exp(-K_2(x; \lambda)))^\beta}{2 - (1 - \exp(-K_2(x; \lambda)))^\beta} \right]^\alpha \right\} \\ \times \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_2(x; \lambda)))^\beta}{2 - (1 - \exp(-K_2(x; \lambda)))^\beta} \right]^\alpha \right]^2 \right\}^{b-1}, x \in \mathbb{R}, b, \alpha, \beta, \lambda > 0$$

and

$$F_{TLEHLOBX-W}(x; b, \alpha, \beta, \lambda) = \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_2(x; \lambda)))^\beta}{2 - (1 - \exp(-K_2(x; \lambda)))^\beta} \right]^\alpha \right]^2 \right\}^b,$$

respectively, where $K_2(x; \lambda) = \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^2$. Figure 6.2 shows the plots of PDFs and HRFs' plots of TLEHLOBX-W distribution, respectively. The PDF includes coverage for the reverse-J, unimodal shapes together with left and right-skewed datasets. Increasing, decreasing, bathtub, and upside-down bathtub forms can be seen in the HRF.

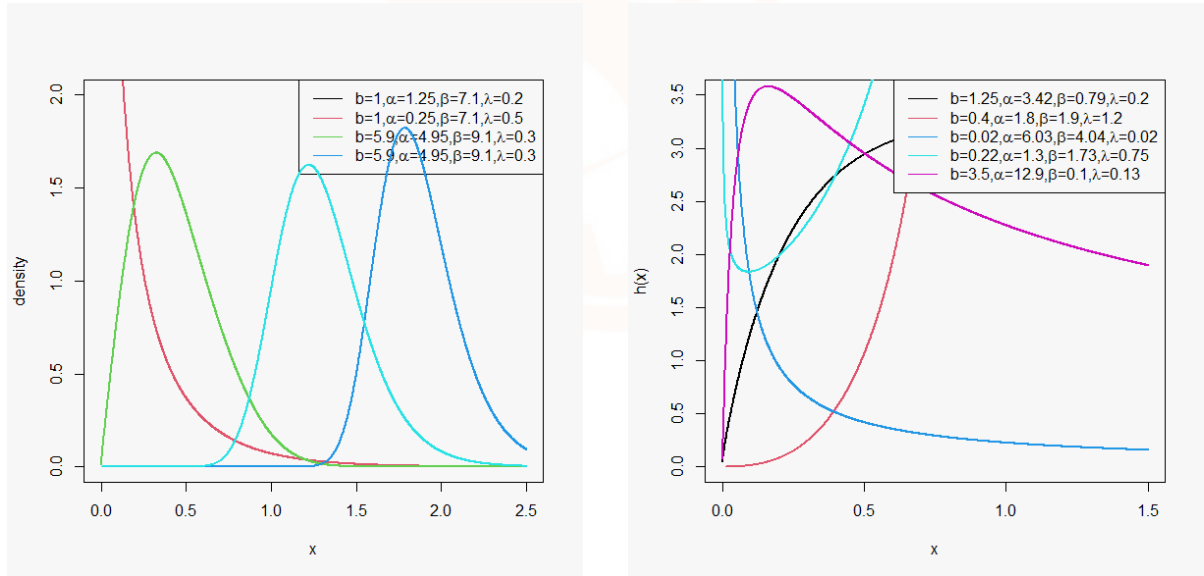


Figure 6.2: TLEHLOBX-W distribution's PDF and HRF plots

6.4.3 Topp-Leone-Exponentiated Half Logistic-Odd Burr X-Burr III Distribution

Taking the Burr III distribution as the baseline associated with PDF $g(x; \lambda, \delta) = \lambda \delta x^{-\lambda-1} (1 + x^{-\lambda})^{-\delta-1}$ and CDF $G(x; \lambda, \delta) = (1 + x^{-\lambda})^{-\delta}$, we obtain the Topp-Leone-exponentiated half

logistic-odd Burr X-Burr III (TLEHLOBX-B) PDF and CDF as

$$f_{TLEHLOBX-B}(x; b, \alpha, \beta, \lambda, \delta) = \frac{8b\alpha\beta\lambda\delta x^{-\lambda-1}(1+x^{-\lambda})^{-2\delta-1}}{(1-(1+x^{-\lambda})^{-\delta})^3 \exp(K_3(x; \lambda, \delta))} \\ \times \left[2 - (1 - \exp(-K_3(x; \lambda, \delta)))^\beta \right]^{-\alpha-1} \left\{ 1 - \left[\frac{(1 - \exp(-K_3(x; \lambda, \delta)))^\beta}{2 - (1 - \exp(-K_3(x; \lambda, \delta)))^\beta} \right]^\alpha \right\} \\ \times [1 - \exp(-K_3(x; \lambda, \delta))]^{\alpha\beta-1} \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_3(x; \lambda, \delta)))^\beta}{2 - (1 - \exp(-K_3(x; \lambda, \delta)))^\beta} \right]^\alpha \right]^2 \right\}^{b-1}, \\ x \in \mathbb{R}, b, \alpha, \beta, \lambda, \delta > 0$$

and

$$F_{TLEHLOBX-B}(x; b, \alpha, \beta, \lambda, \delta) = \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_3(x; \lambda, \delta)))^\beta}{2 - (1 - \exp(-K_3(x; \lambda, \delta)))^\beta} \right]^\alpha \right]^2 \right\}^b,$$

respectively, where $K_3(x; \lambda, \delta) = \left(\frac{(1+x^{-\lambda})^{-\delta}}{1-(1+x^{-\lambda})^{-\delta}} \right)^2$. Figure 6.2 shows the plots of PDFs and HRFs' plots of TLEHLOBX-B distribution, respectively. The PDF applies to datasets with different shapes of skewness and kurtosis. Increasing and decreasing shapes together with the bathtub, and upside-down bathtub forms can be seen in the HRF.

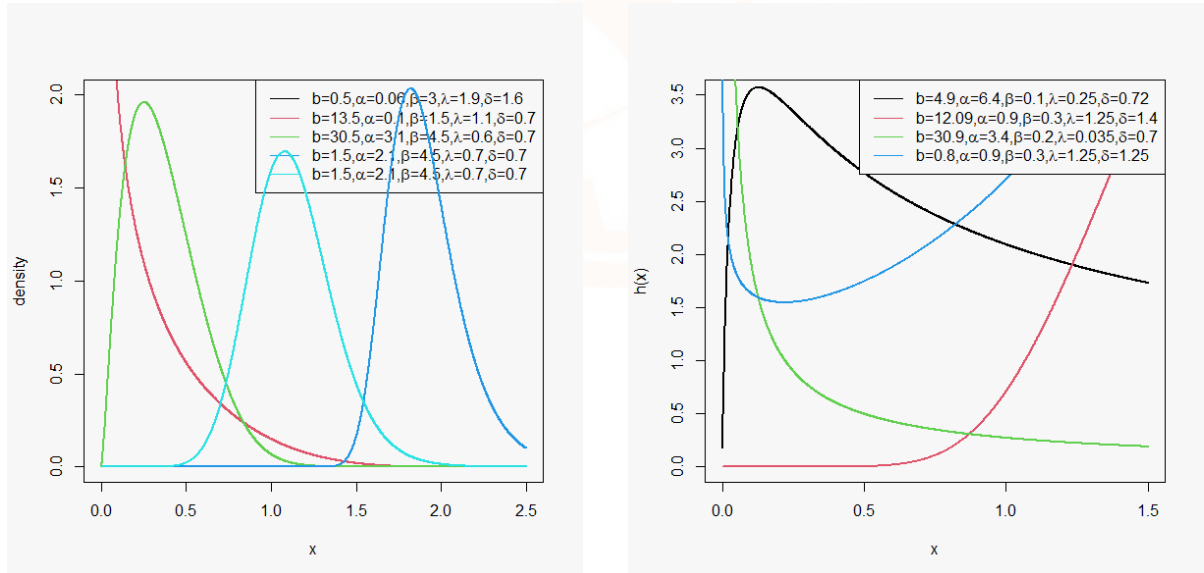


Figure 6.3: TLEHLOBX-B distribution's PDF and HRF plots

6.5 Maximum Likelihood Estimation

For the observed values $x_1, x_2, x_3, \dots, x_n$ of size n , we may obtain the log-likelihood function as

$$\begin{aligned} \ell_n(\Delta) = & n \ln(8b\alpha\beta) + \sum_{i=1}^n (-K(x_i; \Psi)) \\ & + (\alpha\beta - 1) \sum_{i=1}^n \log [1 - \exp(-K(x_i; \Psi))] \\ & + (-\alpha - 1) \sum_{i=1}^n \log \left[2 - (1 - \exp(-K(x_i; \Psi)))^\beta \right] + \sum_{i=1}^n \log(g(x_i; \Psi)) \\ & + \sum_{i=1}^n \log \left\{ 1 - \left[\frac{(1 - \exp(-K(x_i; \Psi)))^\beta}{2 - (1 - \exp(-K(x_i; \Psi)))^\beta} \right]^\alpha \right\} \\ & + (b - 1) \sum_{i=1}^n \log \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K(x_i; \Psi)))^\beta}{2 - (1 - \exp(-K(x_i; \Psi)))^\beta} \right]^\alpha \right]^2 \right\} \\ & + \sum_{i=1}^n \log(G(x; \Psi)) - 3 \sum_{i=1}^n \log(\bar{G}(x; \Psi)), \end{aligned}$$

where $K_G(x_i; \Psi) = \left(\frac{G(x_i; \Psi)}{\bar{G}(x_i; \Psi)} \right)^2$ and $\Delta = (b, \alpha, \beta, \Psi)^T$ is the vector of parameters of the TLEHLOBX-G family. The Appendix [F.3](#) contains the score vector components. We use iterative methods from R software to obtain these MLEs. Under regularity conditions, MLEs are unbiased and normal ([Mahmood et al. 2019](#)). For a given covariance matrix $J(\hat{\Delta})^{-1}$, then we can approximate the confidence interval for b, α, β and Ψ_k as $\hat{b} \pm Z_{\frac{\varphi}{2}} \sqrt{J_{bb}^{-1}(\hat{\Delta})}$, $\hat{\alpha} \pm Z_{\frac{\varphi}{2}} \sqrt{J_{\alpha\alpha}^{-1}(\hat{\Delta})}$, $\hat{\beta} \pm Z_{\frac{\varphi}{2}} \sqrt{J_{\beta\beta}^{-1}(\hat{\Delta})}$ and $\hat{\Psi}_k \pm Z_{\frac{\varphi}{2}} \sqrt{J_{\Psi_k \Psi_k}^{-1}(\hat{\Delta})}$, where $Z_{\frac{\varphi}{2}}$ is the $(\frac{\varphi}{2})^{th}$ percentile of a standard normal distribution. The diagonal elements are represented with $J_{bb}^{-1}(\hat{\Delta})$, $J_{\alpha\alpha}^{-1}(\hat{\Delta})$, $J_{\beta\beta}^{-1}(\hat{\Delta})$ and $J_{\Psi_k \Psi_k}^{-1}(\hat{\Delta})$.

6.6 Simulation Study

The TLEHLOBX-L distribution's simulation research findings are reported in this section. For $N = 3000$ repetitions for each sample, we take into account sample sizes of $n = 50, 100, 200, 400, 800$, and 1600 . We further obtain the mean estimates, root mean square errors (RMSEs) and average bias (Abias).

From Table [6.2](#) and [6.3](#), the mean estimates approximate the true parameter values, and RMSE and Abias also decays to zero for all parameter values with increasing sample size, which indicates that our model gives consistent estimates.

Table 6.2: TLEHLOBX-L distribution simulation experiment results: Mean, RSME, Abias

Parameter	n	(1.0, 2.5, 1.0, 0.5)			(0.6, 0.9, 0.9, 0.9)		
		Mean	RMSE	Abias	Mean	RMSE	Abias
b	50	1.9541	4.7853	0.9541	1.7369	4.8830	1.1369
	100	1.3545	2.3618	0.3545	0.8957	1.3103	0.2957
	200	1.0878	0.8731	0.0878	0.6885	0.5556	0.0885
	400	0.9758	0.5264	-0.0242	0.6100	0.3101	0.0100
	800	0.9813	0.4178	-0.0187	0.6048	0.2281	0.0048
	1600	0.9757	0.3478	-0.0243	0.5898	0.1795	-0.0102
α	50	2.7934	0.9262	0.2934	0.9621	0.2788	0.0621
	100	2.6557	0.5717	0.1557	0.9321	0.1730	0.0321
	200	2.5912	0.3988	0.0912	0.9145	0.1205	0.0145
	400	2.5640	0.2922	0.0640	0.9127	0.0946	0.0127
	800	2.5261	0.2241	0.0261	0.9013	0.0709	0.0013
	1600	2.5215	0.1753	0.0215	0.9052	0.0568	0.0052
β	50	1.5418	1.5336	0.5418	1.2597	1.2720	0.3597
	100	1.4067	1.3403	0.4067	1.1748	1.0534	0.2748
	200	1.3831	1.2119	0.3831	1.1857	1.0371	0.2857
	400	1.3797	1.1515	0.3797	1.1865	1.0058	0.2865
	800	1.2786	0.9230	0.2786	1.0750	0.7559	0.1750
	1600	1.2664	0.8724	0.2664	1.1091	0.7225	0.2091
λ	50	0.9497	1.4208	0.4497	1.4801	1.7789	0.5801
	100	0.9001	1.2568	0.4001	1.5523	1.8086	0.6523
	200	0.7825	0.9663	0.2825	1.4075	1.5327	0.5075
	400	0.6935	0.8079	0.1935	1.2503	1.2804	0.3503
	800	0.6063	0.4918	0.1063	1.1132	0.9359	0.2132
	1600	0.5419	0.2704	0.0419	0.9388	0.3769	0.0388

Table 6.3: TLEHLOBX-L distribution simulation experiment results: Mean, RSME, Abias

(0.9, 0.9, 0.6, 0.9)					(2.1, 2.0, 1.3, 0.3)		
Parameter	n	Mean	RMSE	Abias	Mean	RMSE	Abias
b	50	2.5768	7.4996	1.6768	3.7815	8.5169	1.6815
	100	1.2097	2.8979	0.3097	2.6277	3.7412	0.5277
	200	0.9052	0.7826	0.0052	2.2539	2.0302	0.1539
	400	0.8281	0.4529	-0.0719	2.1271	1.3645	0.0271
	800	0.8358	0.3185	-0.0642	2.1587	1.0502	0.0587
	1600	0.8544	0.2450	-0.0456	2.1361	0.8430	0.0361
α	50	0.9979	0.3132	0.0979	2.2067	0.6260	0.2067
	100	0.9692	0.2132	0.0692	2.1091	0.4057	0.1091
	200	0.9469	0.1475	0.0469	2.0587	0.2751	0.0587
	400	0.9373	0.1151	0.0373	2.0318	0.1994	0.0318
	800	0.9195	0.0837	0.0195	2.0035	0.1511	0.0035
	1600	0.9140	0.0632	0.0140	2.0030	0.1163	0.0030
β	50	0.9725	1.0048	0.3725	2.1689	2.4254	0.8689
	100	0.9710	0.9815	0.3710	1.9936	2.0039	0.6936
	200	0.9751	0.9851	0.3751	1.8849	2.1140	0.5849
	400	0.9864	1.0203	0.3864	1.7041	1.4229	0.4041
	800	0.8806	0.8382	0.2806	1.5898	1.1940	0.2898
	1600	0.8205	0.7048	0.2205	1.5368	0.9517	0.2368
λ	50	1.5266	1.9211	0.6266	0.6701	1.1882	0.3701
	100	1.5663	1.8804	0.6663	0.5869	0.9351	0.2869
	200	1.4584	1.7009	0.5584	0.5063	0.7310	0.2063
	400	1.2466	1.2819	0.3466	0.4247	0.4950	0.1247
	800	1.0736	0.8365	0.1736	0.3717	0.3030	0.0717
	1600	0.9319	0.4333	0.0319	0.3300	0.1572	0.0300

6.7 Applications

To illustrate the flexibility and usefulness of the TLEHLOBX-L distribution, we apply it to three real life datasets in this section. We provide the GOF statistics to assess how this new model is performing in real life as shown in Tables 6.4, 6.5 and 6.6. These GOF includes: $-2\log L$, AIC, BIC, AICC, W^* , (A^*) , SS and K-S together with the p-value of K-S statistic. Also included in Figures 6.4, 6.5, 6.7, 6.8, 6.10 and 6.11 are the fitted densities, P-Ps, HRFs, and KM survival graphs.

Our model is compared with the Topp-Leone-odd Burr III-log-logistic (TLOBIII-L) distribution (Moakofi et al., 2022), odd generalized half-logistic Weibull-Weibull (OGHLW-W) distribution (Chipepa et al., 2020a), Topp-Leone-Marshall-Olkin Burr XII (TLMO-BXII) distribution (Chipepa et al., 2020b), exponentiated Burr XII Poisson (EBXIIP) distribution (da Silva et al., 2015), half logistic generalized Weibull (HLGW) distribution (Anwar and Bibi, 2018) and Heavy-

tailed beta-power transformed-Weibull (HTBPT-W) distribution (Zhao et al., 2021). The PDFs of these distributions are given in Appendix F.4

6.7.1 Waiting Times Data

The first data, extracted from Klakattawi (2019), represents the waiting times between 65 consecutive eruptions of a blowhole, called the Kiama Blowhole. See the data in Appendix F.5

Table 6.4: The MLE for the TLEHLOBX-L and competing models and GOF statistics: Waiting times data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
TLEHLOBX-L	b	α	β	λ	586.7	594.7	595.4	603.4	0.1015	0.7524	0.0949	0.6122	0.0376
	1.31×10^{-1} (2.32×10^{-2})	8.89×10^1 (3.17×10^{-5})	1.09×10^1 (2.42×10^{-4})	6.04×10^{-2} (2.49×10^{-3})									
TLOBIII-L	α	β	b	λ	590.6	598.6	599.3	607.3	0.1691	1.174	0.0953	0.6067	0.1815
	1.01 (2.51×10^1)	6.87×10^{-1} (1.16×10^{-1})	1.79×10^2 (5.72×10^{-4})	7.18×10^{-1} (1.79×10^1)									
OGHLW-W	α	β	λ	γ	602.7	610.7	611.4	619.4	0.2207	1.428	0.1398	0.1639	0.1815
	2.66×10^3 (4.99×10^3)	5.18×10^{-1} (2.43×10^{-1})	6.54 (4.11×10^{-3})	1.64×10^{-1} (4.74×10^2)									
TLMO-BXII	b	δ	λ	β	749.2	757.2	757.9	765.8	0.1199	0.8704	0.5750	< 0.0001	0.4999
	4.80×10^{-1} (6.84×10^2)	2.04 (9.05×10^{-1})	11.0 (7.17)	1.65×10^2 (1.08×10^{-2})									
EBXIIP	c	s	α	λ	675.2	683.2	683.9	691.8	0.1205	0.8789	0.4553	< 0.0001	2.2481
	6.77×10^{-1} (8.59×10^{-2})	1.92 (4.89×10^{-1})	2.34 (3.17×10^{-1})	8.43×10^{-9} (1.28×10^{-2})									
HLGW	w	λ	γ		664.7	670.7	671.1	677.2	0.1272	0.8924	0.4447	< 0.0001	0.9452
	8.92×10^2 (5.97×10^{-8})	2.50×10^{-1} (2.89×10^{-2})	4.61×10^{-4} (5.81×10^{-5})										
HTBPT-W	α	γ	β		593.8	599.8	600.2	606.3	0.1471	1.008	0.1113	0.406	0.9452
	1.27 (1.20×10^{-1})	8.23×10^{-3} (4.22×10^{-3})	1.00×10^{-1} (7.69×10^{-5})										

From the GOF statistics in Table 6.4, TLEHLOBX-L model compared to its competitors clearly offers the greatest match for waiting times data. Figures 6.4 and 6.5 also shows that our model provides a better fit to waiting times data. It is also evident that TLEHLOBX-L distribution can model data with upside-down bathtub HRF shapes as shown in Figure 6.5

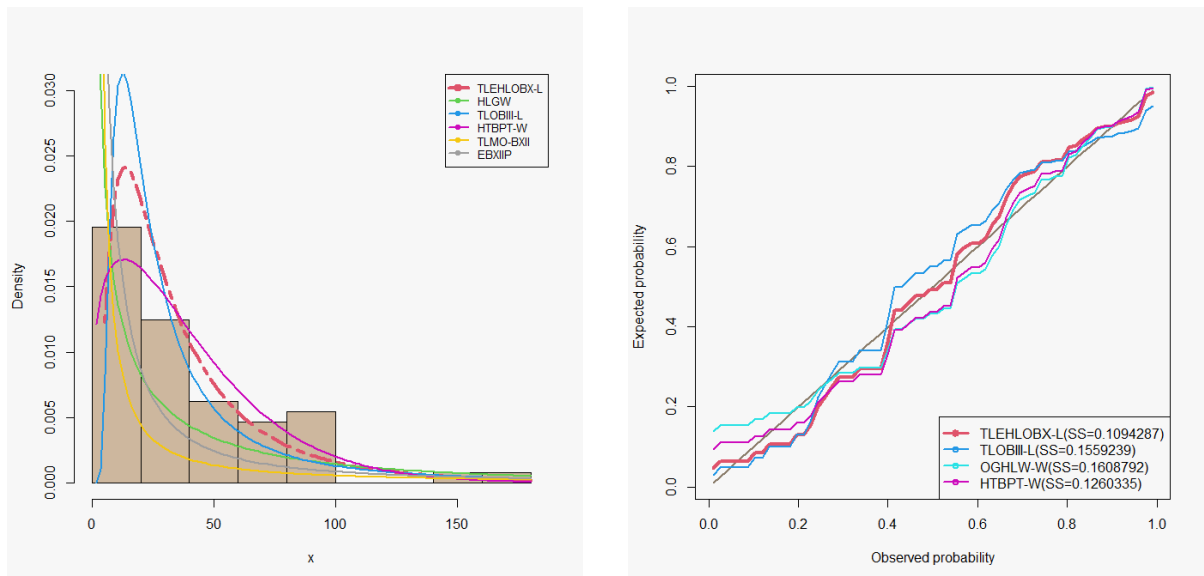


Figure 6.4: Waiting times data fitted density and probability graphs

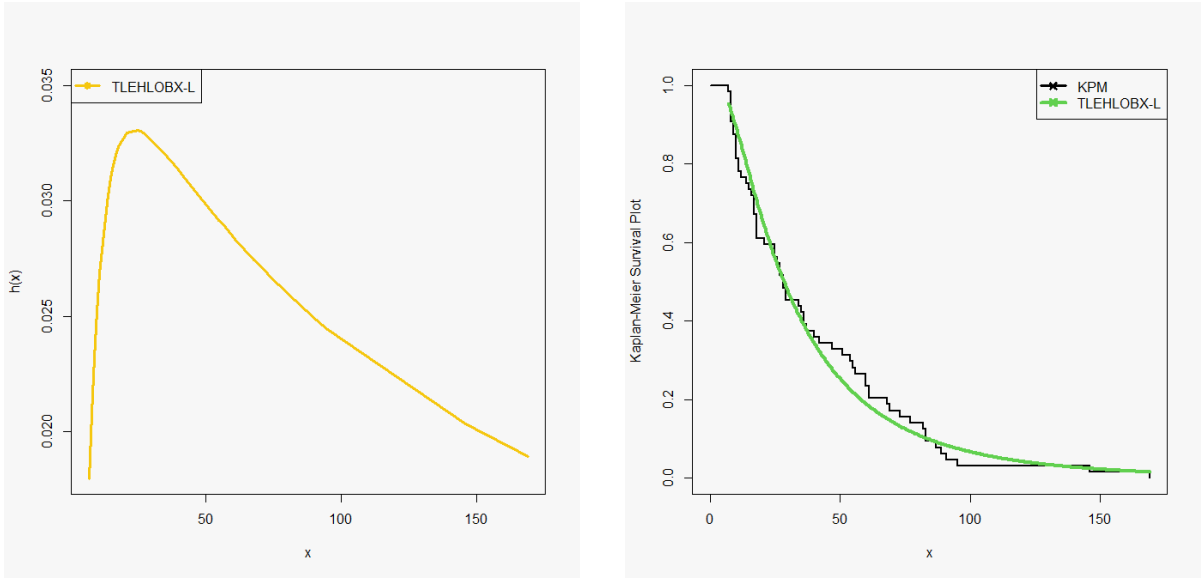


Figure 6.5: Waiting times datas' HRF and KM survival graphs



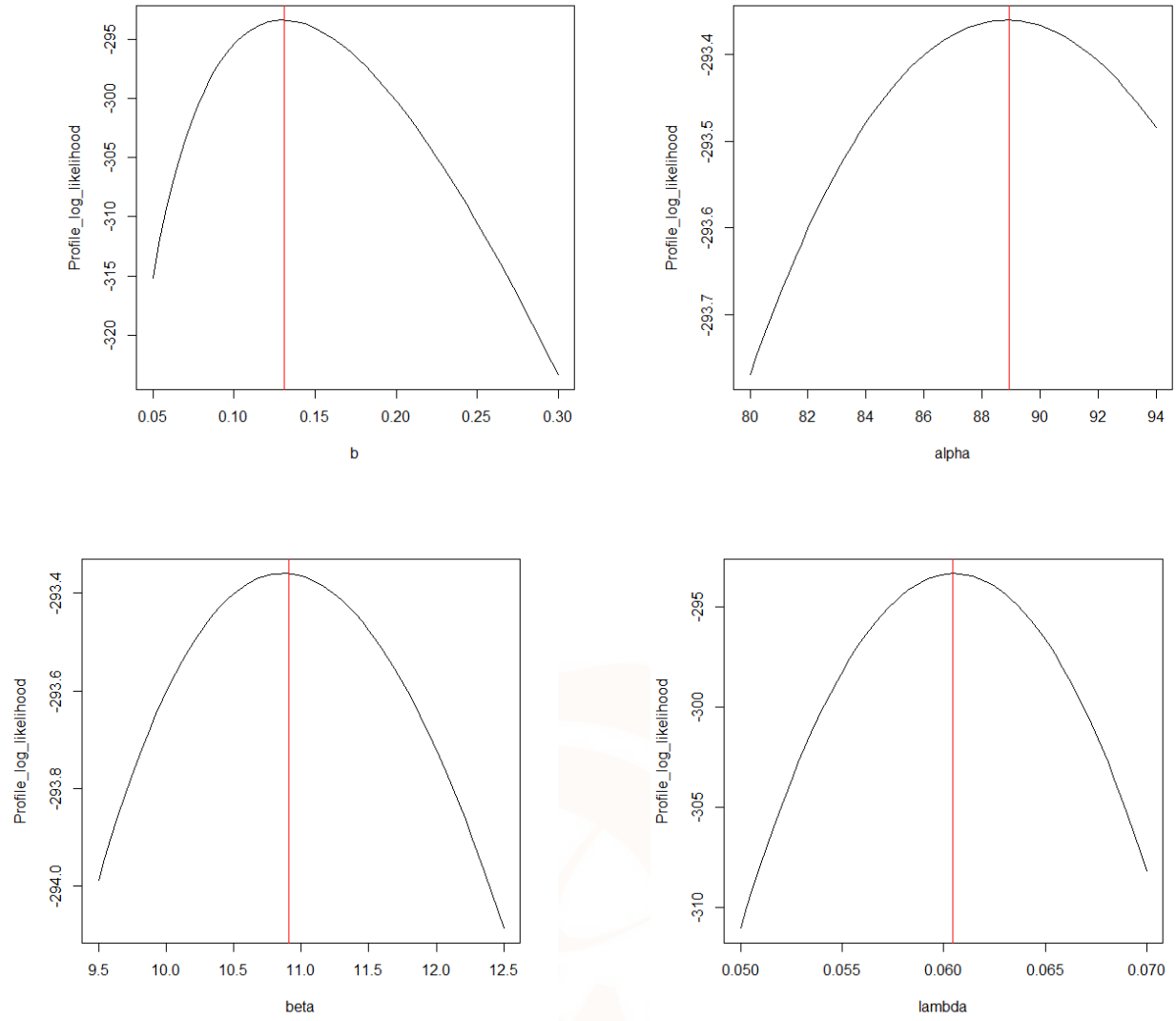


Figure 6.6: Likelihood Profile Plots for b, α, β and λ in waiting times data

Figure 6.6 shows the profile log-likelihood function of the parameters of the TLEHLOBX-L distribution for waiting times data. The plots clearly show that the log-likelihood is maximized at a single point for all the parameters which it is identifiable.

6.7.2 Remission Times Data

The second data, extracted from Chamunorwa et al. (2022), represents remission times of a random sample of 128 bladder cancer patients. See the data in Appendix F.5.

Table 6.5: The MLE for the TLEHLOBX-L and competing models and GOF statistics: remission times data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
	b	α	β	λ									
TLEHLOBX-L	3.38×10^{-1} (1.63×10^{-1})	10.3 (4.95)	10.09 (4.59)	6.44×10^{-2} (3.97×10^{-3})	823.4	831.4	831.8	842.9	0.0545	0.3817	0.0539	0.8504	0.0376
	α	β	b	λ									
TLOBIII-L	1.0478 (3.633)	9.705 (5.718)	4.889×10^{-1} (3.259×10^{-1})	8.144×10^{-1} (2.824)	838.1	846.1	846.5	857.5	0.2073	1.369	0.0830	0.3406	0.1815
	α	β	λ	γ									
OGHLW-W	2.142×10^{-4} (3.501×10^{-5})	4.946×10^{-1} (7.893×10^{-4})	14.34 (2.719×10^{-5})	9.676×10^{-2} (5.086×10^{-3})	840.4	1440.0	1440.3	1451.4	0.0956	0.6537	0.5546	< 0.0001	0.1815
	b	δ	λ	β									
TLMO-BXII	2.567×10^{-1} (2.334×10^{-2})	3.678 (3.867×10^{-1})	11.01 (1.390)	3.925×10^{-2} (1.450×10^{-3})	1025.6	1039.9	1040.2	1051.3	1.625	9.399	0.5483	< 0.0001	0.4999
	c	s	α	λ									
EBXIIP	1.301 (2.147×10^{-1})	2.737 (1.438)	2.063 (9.387×10^{-1})	2.990×10^{-8} (1.573×10^{-2})	836.4	844.5	844.8	855.9	0.2012	1.325	0.0693	0.57	2.2481
	w	λ	γ										
HLGW	6.297 (9.940)	5.000×10^{-1} (7.769×10^{-2})	5.169×10^{-2} (8.042×10^{-2})		852.0	858.0	858.2	866.5	0.2699	1.592	0.16478	0.0019	0.9452
	α	γ	β										
HTBPT-W	1.048 (6.758×10^{-2})	9.389×10^{-2} (1.908×10^{-2})	1.00×10^{-1} (7.280×10^{-1})		827.5	835.5	835.8	846.9	0.1314	0.7865	0.0700	0.557	0.9452

As shown in Tables [6.5](#) TLEHLOBX-L model has the smallest GOF statistics and the largest p-value of the K-S statistic. This indicates that it performs better than its competitors in remission times data. The fitted density in Figures [6.7](#) and [6.8](#) further shows that TLEHLOBX-L model can work well in right-skewed datasets with upside-down HRF shapes.

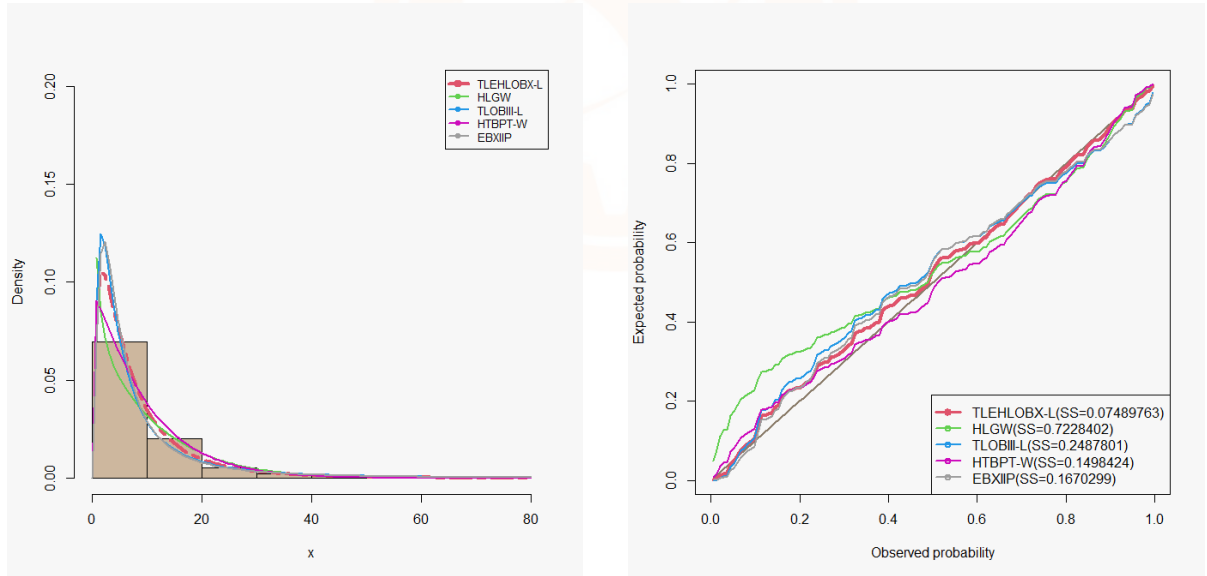


Figure 6.7: Remission times data fitted density and probability graphs

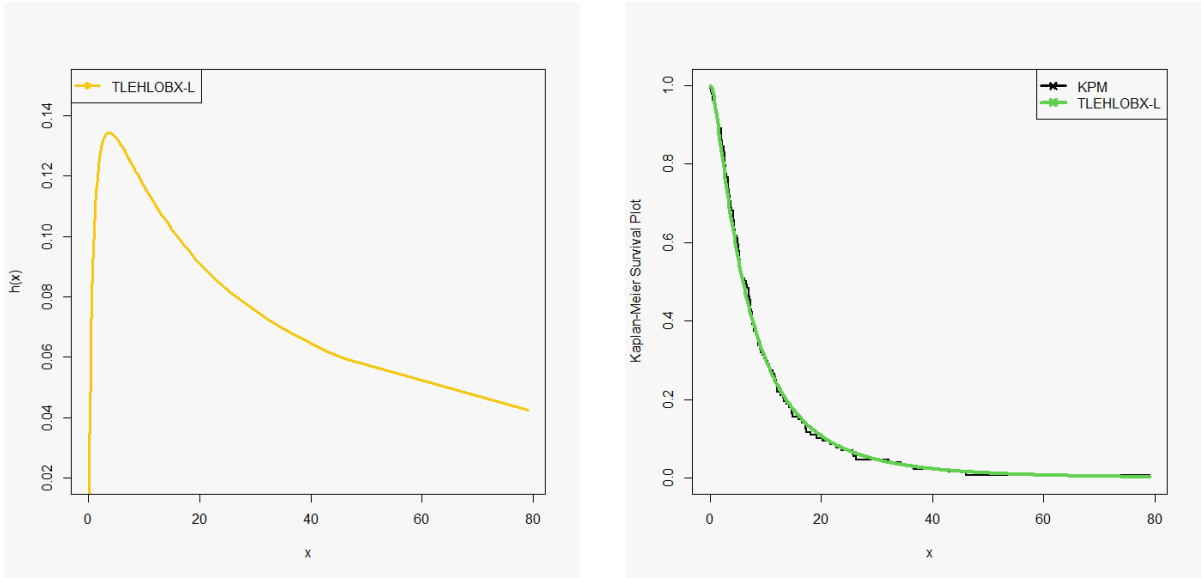


Figure 6.8: Remission times datas' HRF and KM survival graphs



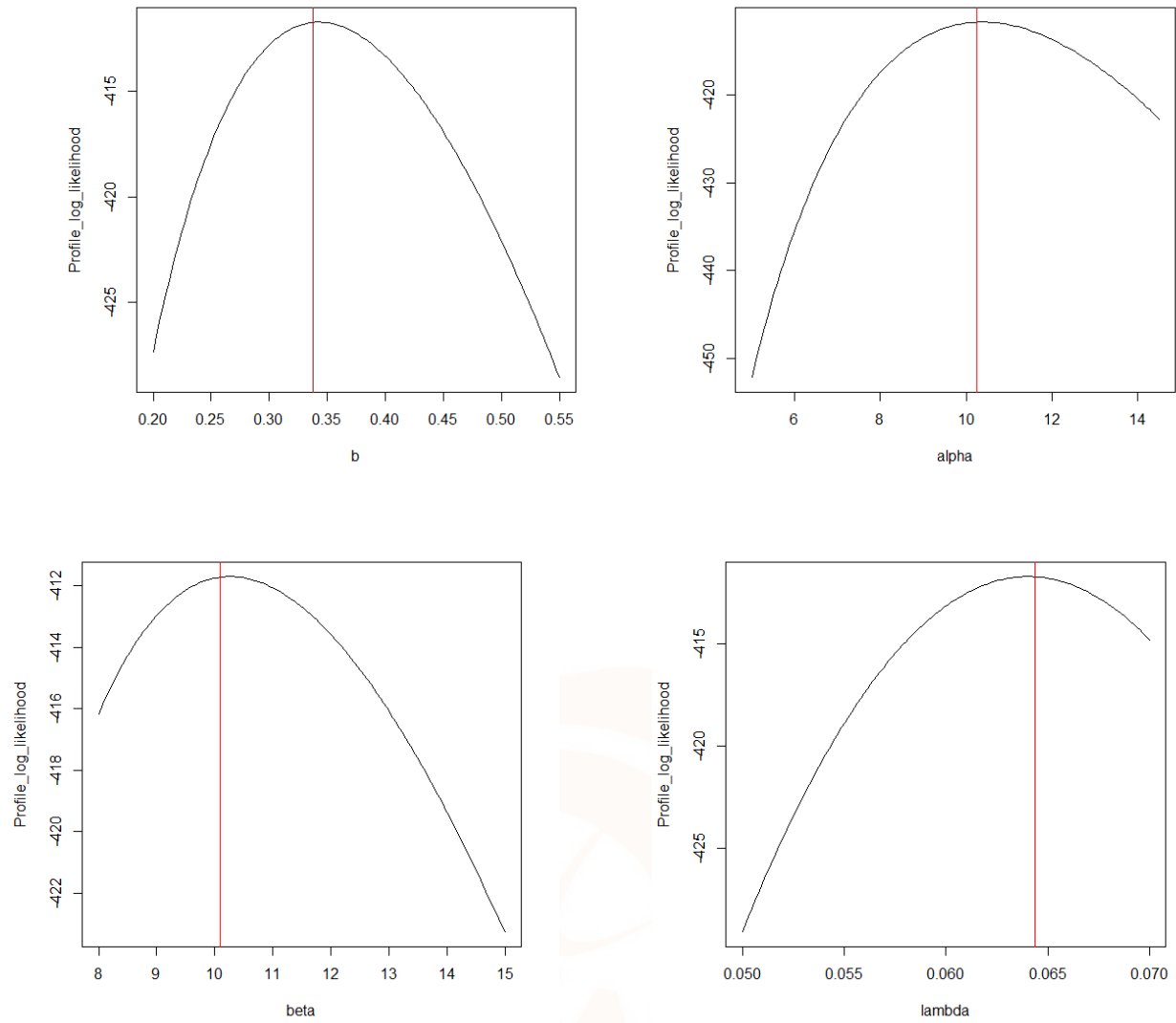


Figure 6.9: Likelihood Profile Plots for b, α, β and λ in remission times data

Figure 6.9 show the profile log-likelihood plots for different parameters of TLEHLOBX-L distribution in remission times data. Clearly, the parameter estimates maximize the likelihood function.

6.7.3 Mortality Rates Data

The third data, extracted from Zhou et al. (2023), represents the mortality rates of the 106 patients infected by the COVID-19 pandemic in Mexico recorded from March 4, 2020 to July 20, 2020. See the data in Appendix F.5.

Table 6.6: The MLE for the TLEHLOBX-L and competing models and gof statistics: Mortality rates data

Distribution	Estimates				-2Log(L)	AIC	AICC	BIC	W*	A*	K-S	P-value	SS
	b	α	β	λ									
TLEHLOBX-L	1.970×10^{-1} (9.586×10^{-2})	10.727 (5.009)	15.972 (8.360)	1.248×10^{-1} (8.141×10^{-3})	376.1	384.3	384.7	394.9	0.0500	0.2783	0.05671	0.8849	0.0376
	β	b	λ										
TLOBIII-L	1.040 (1.408)	16.10 (19.04)	2.765×10^{-1} (3.482×10^{-1})	1.658 (2.245)	382.4	497.9	498.3	508.6	0.0804	0.4681	0.4370	< 0.0001	0.1815
	α	β	λ	γ									
OGHLW-W	1.948×10^{-6} (9.59×10^{-2})	4.946×10^{-1} (5.01)	1.345 (8.36)	11.83×10^{-1} (8.14×10^{-3})	389.2	1440.0	1440.3	1451.4	0.0956	0.6537	0.5546	< 0.0001	0.1815
	b	δ	λ	β									
TLMO-BXII	2.441 (1.242)	3.720×10^2 (2.940×10^{-3})	4.653×10^{-1} (1.269×10^{-1})	6.353 (9.961×10^{-1})	379.3	387.3	387.6	397.9	0.0784	0.4395	0.0691	0.6917	0.4999
	c	s	α	λ									
EBXIIP	2.888 (5.120×10^{-1})	2.442 (5.972×10^{-1})	1.048 (4.660×10^{-1})	5.645×10^{-8} (3.184×10^{-2})	382.3	390.3	390.7	400.9	0.1094	0.6217	0.0696	0.6832	2.2481
	w	λ	γ										
HLGW	7.333×10^2 (2.445×10^{-7})	5.001×10^{-1} (4.357×10^{-2})	7.7592×10^{-4} (5.820×10^{-5})		454.3	460.3	460.6	468.3	0.0934	0.5941	0.3164	< 0.0001	0.9452
	α	γ	β										
HTBPT-W	1.920 (1.408×10^{-1})	1.012×10^{-1} (2.266×10^{-2})	1.000 (6.598×10^{-1})		381.6	387.6	387.9	395.6	0.1022	0.6570	0.0692	0.6905	0.9452

Table 6.6 further shows that TLEHLOBX-L model fits the mortality rates data better than its competitors as it has the smallest GOF statistics and the largest p-value of the K-S statistic. The TLEHLOBX-L model also has the smallest value of SS when compared to its competitors as shown in Figure 6.10. Figure 6.11 also indicates that TLEHLOBX-L model can handle data with upside down bathtub HRF shape.

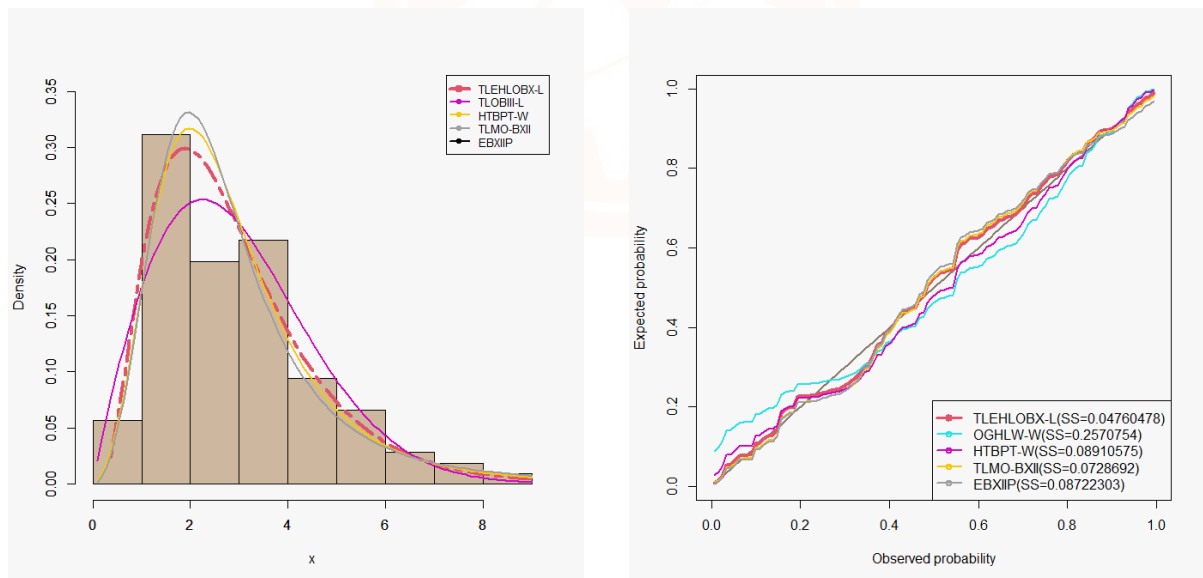


Figure 6.10: Mortality rates data fitted density and probability graphs

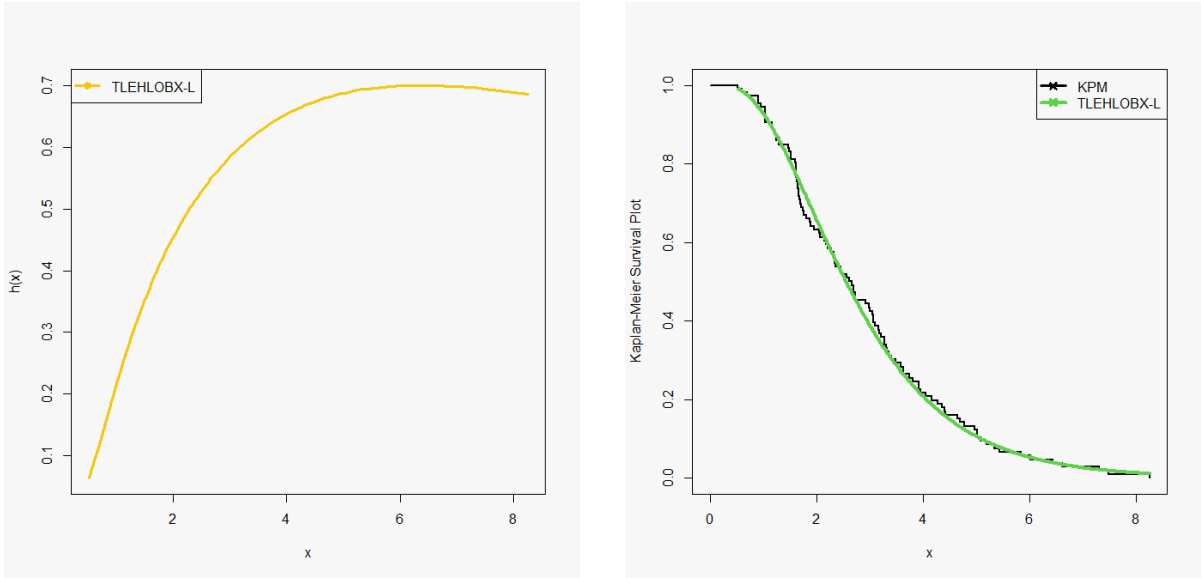


Figure 6.11: Mortality rates datas' HRF and KM survival graphs



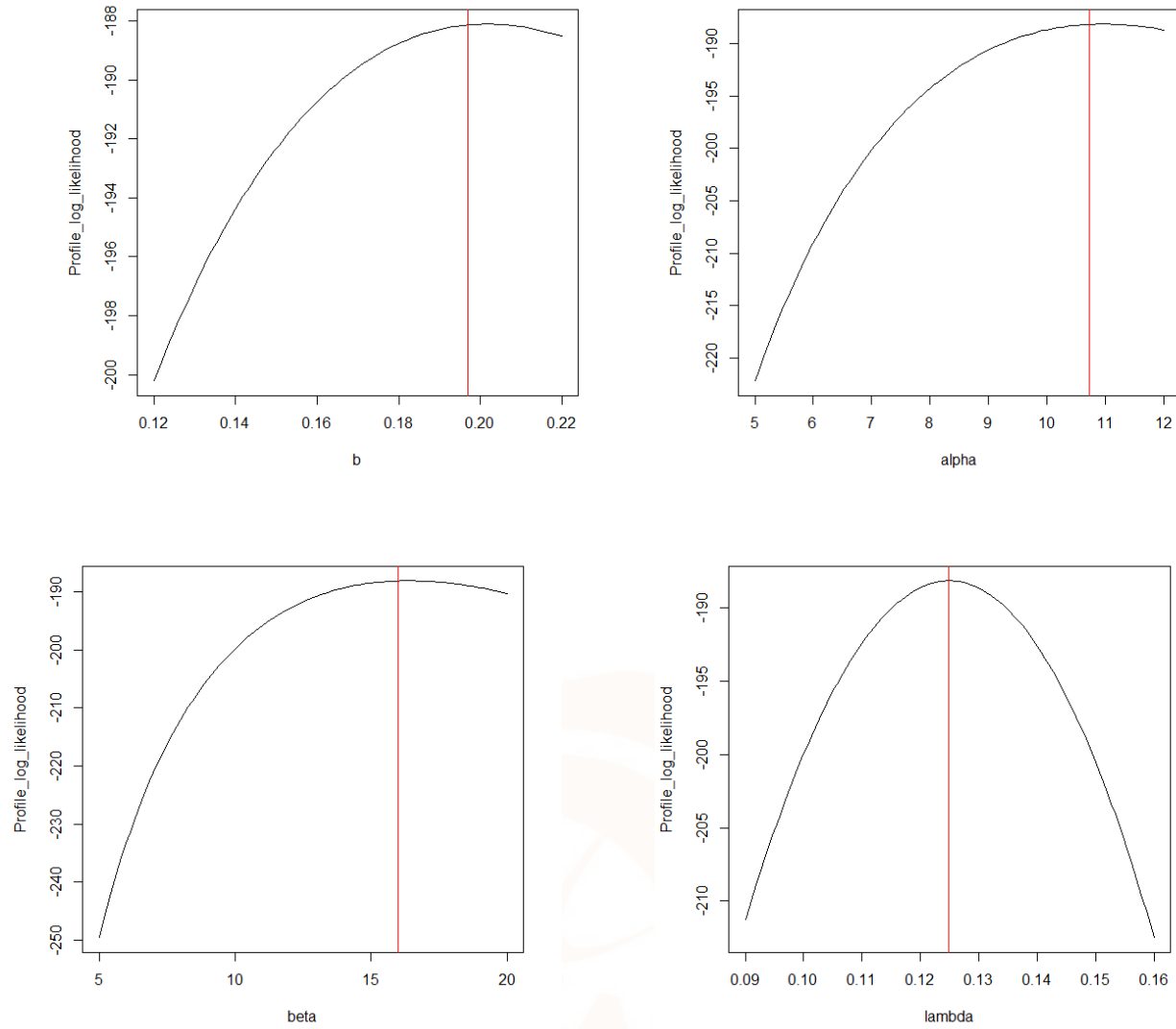
Figure 6.12: Likelihood Profile Plots for b, α, β and λ in mortality rates data

Figure 6.12 shows the profile log-likelihood function of the different parameters of TLEHLOBX-L distribution in mortality rates data. Clearly, the parameter estimates obtained gives maximized likelihood function.

6.8 Concluding Remarks

We introduce the Topp-Leone-exponentiated half logistic-odd Burr X-G (TLEHLOBX-G) family in this chapter. It has been demonstrated that this new family can be expressed as an infinite linear combination of Exp-G family. Using the MLE approach, the model parameters were computed. Some statistical characteristics are obtained. The Topp-Leone-exponentiated half logistic-odd Burr X-log-logistic (TLEHLOBX-L) model was used as a special case and applied to three datasets. The results show that the new suggested model performs better than numerous other models.

7. Conclusions, Limitations and Future Research

7.1 Conclusions

In this thesis, five new families are developed. These families are exponentiated half logistic-Kumaraswamy-G (EHL-KUM-G), gamma odd Burr X-G (GOBX-G), Topp-Leone odd Burr X-G power series (TLOBX-GPS), Topp-Leone-exponentiated half logistic odd Burr X-G (TLEHLOBX-G) and Marshall-Olkin-Topp-Leone odd Burr X-G (MOTLOBX-G) families. We derived the statistical properties of these families which including the QF, density expansion, ordinary moments, central moments, moments and generating function, Rényi entropy, order statistics and stochastic ordering. The parameters of these distributions are estimated using the maximum likelihood estimation approach. Further, simulation studies were conducted and they showed that maximum likelihood estimates are consistent. Special cases of these family of distributions are applied on real data sets to show their applicability. From the assessment of the results in which we used -2LogL , AIC, AICC, BIC together with W^* , A^* , SS and K-S GOF statistics together with the K-S p-value, the special cases performed better than existing models. The PDFs of special cases can be symmetric, left-skewed, right-skewed, J and reversed-J shapes, whereas the HRFs can be monotonic and non-monotonic shapes which makes them flexible for modelling datasets. Thus, the distributions that we proposed can be considered as alternatives in real life modelling. For that, we hope that these distributions will receive researchers' attention especially those working on reliability and survival analysis.

7.2 Limitations

In this study, the developed families of distributions represent a notable advancement in the field of statistical distributions; however, there exists some certain limitations. While we initially planned to explore various parameter estimation techniques, we ended up using MLE technique only. Practical constraints such as computational resources and time limitations influenced this decision. The decision to employ MLE as the primary parameter estimation technique was rooted by many factors such as its robustness, widespread acceptance, and proven effectiveness across various research domains especially when dealing with complex distributions.

While MLE is a widely accepted method, its implementation can sometimes be computationally intensive, particularly for complex distribution families or in high-dimensional parameter spaces. This could result in longer processing times and resource requirements. Furthermore, the application scope of the proposed distribution families is noteworthy. While the special cases showcased promising results across various datasets, their performance may be influenced by the unique characteristics of the data employed in this study. For a comprehensive assessment of

the generality and reliability of these distributions, future research demands rigorous validation across a broader spectrum of real-world datasets.

Deriving statistical properties for distribution families that combine multiple distributions might become increasingly complex. This complexity could impede the development of closed-form expressions for certain properties. The equations studied may become more complex and might involve special functions, integrals that cannot be solved analytically.

7.3 Future Research

- We presented several special cases for the families of distributions that are developed and some are not applied in real life datasets. These special cases can be applied to real life data in several areas in the future.
- Different baseline distributions can also be considered to further develop other new special cases of distributions, particularly to deal with censored data.
- Other methods of parameter estimation including Bayesian technique can be used and their performances can be assessed via Monte Carlo simulations.
- Future work will also focus on the introduction of covariates/concomitant variables to the developed models via regression analysis.
- The researcher plans to develop R packages for the end users of the proposed new lifetime distributions.

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A. Appendices for Introduction Chapter

A.1 Power Series distributions

In [Table A.1](#) we provide we have the $C(\theta)$, $C'(\theta)$, $C''(\theta)$, $C^{-1}(\theta)$, a_n and parameter space for binomial, Poisson, geometric and logarithmic distributions.

Table A.1: Useful Quantities for Some Power Series Distributions

Distribution	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	a_n	Parameter Space
Binomial	$(1 + \theta)^m - 1$	$m(1 + \theta)^{m-1}$	$m(m-1)(1 + \theta)^{m-2}$	$(\theta - 1)^{1/m} - 1$	$\binom{m}{n}$	$(0, 1)$
Poisson	$e^\theta - 1$	e^θ	e^θ	$\log(1 + \theta)$	$(n!)^{-1}$	$(0, \infty)$
Geometric	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(1 + \theta)^{-1}$	1	$(0, 1)$
Logarithmic	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	n^{-1}	$(0, 1)$

A.2 Standard Lifetime Distributions

In this section, we provide statistical properties of some common lifetime standard distributions that are usually considered in real-life applications. In the generalized distribution, baseline distributions are set as standard distributions. Researchers also generalize these standard distributions to obtain other families of distributions. [Table A.2](#) gives a summary of some of these distributions.

Table A.2: Some Important Standard Distributions

Distribution	Density function (pdf)	Distribution function (cdf)	Parameter space	Random variable space
Weibull	$g(x; \beta) = \beta x^{\beta-1} e^{-x^\beta}$	$G(x; \beta) = 1 - e^{-x^\beta}$	$\beta > 0$	$\mathbb{R}$
log-logistic	$g(x; c) = cx^{c-1}(1+x^c)^{-2}$	$G(x; c) = 1 - (1+x^c)^{-1}$	$c > 0$	$\mathbb{R}$
Burr III	$g(x; \lambda, \beta) = \lambda \beta x^{-\lambda-1} (1+x^{-\lambda})^{-\beta-1}$	$G(x; \lambda, \beta) = (1+x^{-\lambda})^{-\beta}$	$\lambda, \beta > 0$	$\mathbb{R}$
Burr X	$g(x; \alpha) = 2\alpha x e^{-x^2} (1 - e^{-x^2})^{\alpha-1}$	$G(x; \alpha) = (1 - e^{-x^2})^\alpha$	$\lambda > 0$	$\mathbb{R}$
Burr XII	$g(x; \lambda, \beta) = \lambda \beta x^{\beta-1} (1+x^\beta)^{-\lambda-1}$	$G(x; \lambda, \beta) = 1 - (1+x^\beta)^{-\lambda}$	$\lambda, \beta > 0$	$\mathbb{R}$
Power	$g(x; \lambda, \beta) = \lambda \beta^\lambda x^{\lambda-1}$	$G(x; \lambda, \beta) = (\beta x)^\lambda$	$\lambda, \beta > 0$	$(0, 1/\beta)$
Fréchet	$g(x; c, \vartheta) = \vartheta c^\vartheta x^{-(\vartheta+1)} \exp[-(\frac{c}{x})^\vartheta]$	$G(x; c, \vartheta) = \exp[-(\frac{c}{x})^\vartheta]$	$c, \vartheta > 0$	$\mathbb{R}$
Uniform	$g(x; \theta) = 1/\theta$	$G(x; \theta) = x/\theta$	$\theta > 0$	$(0, \theta)$
Kumaraswamy	$g(x; a, c) = acx^{c-1}(1-x^c)^{a-1}$	$G(x; a, c) = 1 - (1-x^c)^a$	$a, c > 0$	$\mathbb{R}$
Lindley	$g(x; \alpha) = \frac{\alpha^2(1+x)e^{-\alpha x}}{1+\alpha}$	$G(x; \alpha) = 1 - \frac{1+\alpha+\alpha x}{1+\alpha} e^{-\alpha x}$	$\alpha > 0$	$\mathbb{R}$
Normal	$g(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$	$G(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^x \exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) dt$	$-\infty < \mu < \infty, \sigma > 0$	$\mathbb{R}$
Exponential	$g(x; \alpha) = \alpha e^{-\alpha x}$	$G(x; \alpha) = 1 - e^{-\alpha x}$	$\alpha > 0$	$\mathbb{R}$

B. Appendices for EHL-KUM-G Family of Distributions

B.1 Components of the Score Vector

The components of the score vector for the EHL-KUM-G distribution are:

$$\begin{aligned}
 U_a(\underline{\Delta}) &= \frac{n}{a} + \sum_{j=1}^n \ln[G(x_j; \underline{\varepsilon})] - (b-1) \sum_{j=1}^n \frac{G^a(x_j; \underline{\varepsilon}) \ln(G(x_j; \underline{\varepsilon}))}{1 - G^a(x_j; \underline{\varepsilon})} \\
 &\quad + b(\alpha-1) \sum_{j=1}^n \frac{G^a(x_j; \underline{\varepsilon}) [1 - G^a(x_j; \underline{\varepsilon})]^{b-1} \ln(G(x_j; \underline{\varepsilon}))}{1 - [1 - G^a(x_j; \underline{\varepsilon})]^b} \\
 &\quad + b(\alpha+1) \sum_{j=1}^n \frac{G^a(x_j; \underline{\varepsilon}) [1 - G^a(x_j; \underline{\varepsilon})]^{b-1} \ln(G(x_j; \underline{\varepsilon}))}{1 + [1 - G^a(x_j; \underline{\varepsilon})]^b},
 \end{aligned}$$

$$\begin{aligned}
 U_b(\underline{\Delta}) &= \frac{n}{b} + \sum_{j=1}^n \ln[1 - G^a(x_j; \underline{\varepsilon})] - (\alpha-1) \sum_{j=1}^n \frac{[1 - G^a(x_j; \underline{\varepsilon})]^b \ln[1 - G^a(x_j; \underline{\varepsilon})]}{1 - [1 - G^a(x_j; \underline{\varepsilon})]^b} \\
 &\quad - (\alpha+1) \sum_{j=1}^n \frac{[1 - G^a(x_j; \underline{\varepsilon})]^b \ln[1 - G^a(x_j; \underline{\varepsilon})]}{1 + [1 - G^a(x_j; \underline{\varepsilon})]^b},
 \end{aligned}$$

$$U_\alpha(\underline{\Delta}) = \frac{n}{\alpha} + \sum_{j=1}^n \ln[1 - [1 - G^a(x_j; \underline{\varepsilon})]^b] - \sum_{j=1}^n \ln[1 + [1 - G^a(x_j; \underline{\varepsilon})]^b],$$

and

$$\begin{aligned}
 U_{\underline{\varepsilon}_k}(\underline{\Delta}) &= \sum_{j=1}^n \frac{[\dot{g}(x_j; \underline{\varepsilon})]_{\underline{\varepsilon}_k}}{g(x_j; \underline{\varepsilon})} + (a-1) \sum_{j=1}^n \frac{[\dot{G}(x_j; \underline{\varepsilon})]_{\underline{\varepsilon}_k}}{G(x_j; \underline{\varepsilon})} - a(b-1) \sum_{j=1}^n \frac{G^{a-1}(x_j; \underline{\varepsilon}) [\dot{G}(x_j; \underline{\varepsilon})]_{\underline{\varepsilon}_k}}{1 - G^a(x_j; \underline{\varepsilon})} \\
 &\quad + ab(\alpha-1) \sum_{j=1}^n \frac{G^{a-1}(x_j; \underline{\varepsilon}) [1 - G^a(x_j; \underline{\varepsilon})]^{b-1} [\dot{G}(x_j; \underline{\varepsilon})]_{\underline{\varepsilon}_k}}{1 - [1 - G^a(x_j; \underline{\varepsilon})]^b} \\
 &\quad + ab(\alpha+1) \sum_{j=1}^n \frac{G^{a-1}(x_j; \underline{\varepsilon}) [1 - G^a(x_j; \underline{\varepsilon})]^{b-1} [\dot{G}(x_j; \underline{\varepsilon})]_{\underline{\varepsilon}_k}}{1 + [1 - G^a(x_j; \underline{\varepsilon})]^b},
 \end{aligned}$$

where $[\dot{g}(x_j; \underline{\varepsilon})]_{\underline{\varepsilon}_k} = \frac{dg(x_j; \underline{\varepsilon})}{d\underline{\varepsilon}_k}$, $[\dot{G}(x_j; \underline{\varepsilon})]_{\underline{\varepsilon}_k} = \frac{dG(x_j; \underline{\varepsilon})}{d\underline{\varepsilon}_k}$ and $\underline{\varepsilon}_k$ is the k^{th} element of the vector of baseline parameters $\underline{\varepsilon}$ in $\underline{\Delta}$.

C. Appendices for GOBX-G Family

C.1 Components of the Score Vector

The components of the score vector for the GOBX-G distribution are:

$$\frac{\partial \ell_n(\Theta)}{\partial \delta} = n \ln(\theta) - n\psi(\delta) + \sum_{i=1}^n \ln \left[-\log \left(1 - e^{-y_i^2} \right) \right] ,$$

$$\frac{\partial \ell_n(\Theta)}{\partial \theta} = \frac{n\delta}{\theta} + \sum_{i=1}^n \ln \left(1 - e^{-y_i^2} \right)$$

and

$$\begin{aligned} \frac{\partial \ell_n(\Theta)}{\partial \xi_k} &= \sum_{i=1}^n \frac{g'(x; \xi)}{g(x; \xi)} + \sum_{i=1}^n \frac{G'(x; \xi)}{G(x; \xi)} - 3 \sum_{i=1}^n \frac{\overline{G}'(x; \xi)}{\overline{G}(x; \xi)} - 2 \sum_{i=1}^n y_i \frac{\partial y_i}{\partial \xi_k} \\ &\quad + (\theta - 1) \sum_{i=1}^n \frac{2y_i e^{-y_i^2}}{1 - e^{-y_i^2}} \frac{\partial y_i}{\partial \xi_k} + (\delta - 1) \sum_{i=1}^n \frac{2y_i e^{-y_i^2}}{(1 - e^{-y_i^2}) \log(1 - e^{-y_i^2})} \frac{\partial y_i}{\partial \xi_k} , \end{aligned}$$

where $\psi(\delta) = \partial \Gamma(\delta) / \partial \delta$, $g'(x; \xi) = \partial g(x; \xi) / \partial \xi_k$, $G'(x; \xi) = \partial G(x; \xi) / \partial \xi_k$, $\overline{G}'(x; \xi) = \partial \overline{G}(x; \xi) / \partial \xi_k$, ξ_k is the k^{th} element of the vector of parameters ξ and $y_i = G(x_i; \xi) / \overline{G}(x_i; \xi)$.

D. Appendices for TLOBX-GPS Class

D.1 Rényi Entropy Proofs

Note that from equation (4.15), if we plug in $f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})$ and $F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})$, we can write $f^v(x; \lambda, \alpha, \theta, \underline{\omega})$ as

$$\begin{aligned}
 f^v(x; \alpha, \theta, \lambda, \underline{\omega}) &= \left[\frac{4\alpha\theta\lambda}{C(\lambda)} \right]^v \sum_{h,i,s=0}^{\infty} d_{s,v} \lambda^s (-1)^{h+i} \binom{v(\alpha-1) + \alpha h}{i} \frac{g^v(x; \underline{\omega}) G^v(x; \underline{\omega})}{\bar{G}^{3v}(x; \underline{\omega})} \\
 &\quad \times \binom{s}{h} \exp \left[-v \left(\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})} \right)^2 \right] \left\{ 1 - \exp \left[- \left(\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})} \right)^2 \right] \right\}^{v(\theta-1)} \\
 &\quad \times \left[1 - \left\{ 1 - \exp \left[- \left(\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})} \right)^2 \right] \right\}^{\theta} \right]^{2i+v} \\
 &= \left[\frac{4\alpha\theta\lambda}{C(\lambda)} \right]^v \sum_{h,i,j,s=0}^{\infty} d_{s,v} \lambda^s (-1)^{h+i+j} \binom{v(\alpha-1) + \alpha h}{i} \frac{g^v(x; \underline{\omega}) G^v(x; \underline{\omega})}{\bar{G}^{3v}(x; \underline{\omega})} \\
 &\quad \times \binom{s}{h} \binom{2i+v}{j} \exp \left[-v \left(\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})} \right)^2 \right] \left\{ 1 - \exp \left[- \left(\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})} \right)^2 \right] \right\}^{\theta(j+v)-v} \\
 &= \left[\frac{4\alpha\theta\lambda}{C(\lambda)} \right]^v \sum_{h,i,j,k,s=0}^{\infty} d_{s,v} \lambda^s (-1)^{h+i+j+k} \binom{v(\alpha-1) + \alpha h}{i} \frac{g^v(x; \underline{\omega}) G^v(x; \underline{\omega})}{\bar{G}^{3v}(x; \underline{\omega})} \\
 &\quad \times \binom{s}{h} \binom{2i+v}{j} \binom{\theta(j+v)-v}{k} \exp \left[-(k+v) \left(\frac{G(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})} \right)^2 \right] \\
 &= \left[\frac{4\alpha\theta\lambda}{C(\lambda)} \right]^v \sum_{h,i,j,k,m,s=0}^{\infty} \frac{d_{s,v} \lambda^s (-1)^{h+i+j+k+m} (k+v)^m}{m!} \binom{v(\alpha-1) + \alpha h}{i} \\
 &\quad \times \binom{s}{h} \binom{2i+v}{j} \binom{\theta(j+v)-v}{k} \frac{g^v(x; \underline{\omega}) G^{2m+v}(x; \underline{\omega})}{\bar{G}^{2m+3v}(x; \underline{\omega})}.
 \end{aligned}$$

Writing $G^{2m+v}(x; \underline{\omega})$ as $[1 - \bar{G}(x; \underline{\omega})]^{2m+v}$ and apply the generalized binomial series representation, we get

$$\begin{aligned}
 f^v(x; \alpha, \theta, \lambda, \underline{\omega}) &= \left[\frac{4\alpha\theta\lambda}{C(\lambda)} \right]^v \sum_{h,i,j,k,m,n,s=0}^{\infty} \frac{d_{s,v} \lambda^s (-1)^{h+i+j+k+m+n} (k+v)^m}{m!} \binom{v(\alpha-1) + \alpha h}{i} \\
 &\quad \times \binom{s}{h} \binom{2i+v}{j} \binom{\theta(j+v)-v}{k} \binom{2m+v}{n} g^v(x; \underline{\omega}) \bar{G}^{n-(2m+3v)}(x; \underline{\omega}) \\
 &= \left[\frac{4\alpha\theta\lambda}{C(\lambda)} \right]^v \sum_{h,i,j,k,m,n,p,s=0}^{\infty} \frac{d_{s,v} \lambda^s (-1)^{h+i+j+k+m+n+p} (k+v)^m}{m!} \binom{v(\alpha-1) + \alpha h}{i} \\
 &\quad \times \binom{s}{h} \binom{2i+v}{j} \binom{\theta(j+v)-v}{k} \binom{2m+v}{n} \binom{n-(2m+3v)}{p} g^v(x; \underline{\omega}) G^p(x; \underline{\omega}).
 \end{aligned}$$

D.2 Order Statistics

Applying the identity $(1 - z)^{a-1} = \sum_{i=0}^{\infty} \binom{a-1}{i} (-1)^i z^i$, we can write $f_{i:n}(x; \alpha, \theta, \lambda, \underline{\omega})$ as

$$f_{i:n}(x; \alpha, \theta, \lambda, \underline{\omega}) = \sum_{h=0}^{\infty} \sum_{t=0}^{n-i} \sum_{s=1}^{\infty} \frac{sn! a_s \lambda^s f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) [1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]^{s-1}}{(i-1)!(n-i)! C^{h+1}(\lambda)} \\ \times (-1)^{t+h} \binom{n-i}{t} \binom{t+i-1}{h} C^h[\lambda(1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}))].$$

Applying the result on power series raised to a positive integer given as $C^m(\delta) = \sum_{z,m=0}^{\infty} d_{z,m} \delta^z$ where $d_{z,m} = (zb_0)^{-1} \sum_{m=1}^z [m(h+1) - z] b_h d_{z-h,m}$ and $d_{0,z} = b_0^z$ (see Gradshteyn and Ryzhik (50)), we get

$$f_{i:n}(x; \alpha, \theta, \lambda, \underline{\omega}) = \sum_{h=0}^{\infty} \sum_{t=0}^{n-i} \sum_{s=1}^{\infty} \frac{sn! a_s b_{g,h} \lambda^{s+h} f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})}{(i-1)!(n-i)! C^{h+1}(\lambda)} \\ \times (-1)^{t+h} \binom{n-i}{t} \binom{t+i-1}{h} [1 - F_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]^{h+s-1}$$

which further expands into

$$f_{i:n}(x; \alpha, \theta, \lambda, \underline{\omega}) = \sum_{h,q=0}^{\infty} \sum_{t=0}^{n-i} \sum_{s=1}^{\infty} \frac{sn! a_s b_{g,h} \lambda^{s+h} (-1)^{t+h+q} \binom{n-i}{t}}{(i-1)!(n-i)! C^{h+1}(\lambda)} \\ \times \binom{t+i-1}{h} \binom{h+s-1}{q} f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) F_{TLOBX-G}^q(x; \alpha, \theta, \underline{\omega}).$$

Oluyede et al. (81) derived the expansion of $f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) F_{TLOBX-G}^q(x; \alpha, \theta, \underline{\omega})$ under the pwms section to show that

$$f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) F_{TLOBX-G}^q(x; \alpha, \theta, \underline{\omega}) = 4\alpha\theta \sum_{b,j,k,m,p,u=0}^{\infty} \frac{(-1)^{b+j+k+m+p+u} (k+1)^m}{m!} \\ \times \binom{\alpha(q+1)-1}{u} \binom{2u+1}{j} \binom{\theta(j+1)-1}{k} \binom{2m+1}{b} \binom{n-(2m+3)}{p} g(x; \underline{\omega}) G^p(x; \underline{\omega}).$$

That is,

$$f_{i:n}(x; \alpha, \theta, \lambda, \underline{\omega}) = \sum_{b,h,j,k,m,p,q,u=0}^{\infty} \sum_{t=0}^{n-i} \sum_{s=1}^{\infty} \frac{sn! a_s b_{g,h} (k+1)^m \lambda^{s+h} (-1)^{b+j+k+m+p+h+q+t+u}}{(4\alpha\theta)^{-1} m! (i-1)!(n-i)! C^{h+1}(\lambda)} \\ \times \binom{n-i}{t} \binom{t+i-1}{h} \binom{h+s-1}{q} \binom{\alpha(q+1)-1}{u} \binom{2u+1}{j} \binom{\theta(j+1)-1}{k} \\ \times \binom{2m+1}{b} \binom{b-(2m+3)}{p} g(x; \underline{\omega}) G^p(x; \underline{\omega}).$$

D.3 Elements of the Score Vector

The elements of the score vector for the TLOBX-GPS class are:

$$\frac{\partial \ell_n}{\partial \alpha} = \frac{\partial(f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \alpha}{f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})} + \frac{\lambda C'''[\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})] \partial(S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \alpha}{C'[\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]},$$

$$\frac{\partial \ell_n}{\partial \theta} = \frac{\partial(f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \theta}{f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})} + \frac{\lambda C'''[\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})] \partial(S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \theta}{C'[\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]},$$

$$\frac{\partial \ell_n}{\partial \lambda} = \frac{n}{\lambda} - n \frac{C'(\lambda)}{C(\lambda)} + \sum_{i=1}^{\infty} S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega}) \frac{C'''[\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]}{C'[\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]},$$

and

$$\frac{\partial \ell_n}{\partial \omega_k} = \frac{\partial(f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \omega_k}{f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})} + \frac{\lambda C'''[\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})] \partial(S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \omega_k}{C'[\lambda S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})]},$$

where the first derivatives $\partial(f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \alpha$, $\partial(f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \theta$, $\partial(f_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \omega_k$, $\partial(S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \alpha$, $\partial(S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \theta$ and $\partial(S_{TLOBX-G}(x; \alpha, \theta, \underline{\omega})) / \partial \omega_k$ have been derived by Oluyede et al. (81).

D.4 Competing Distributions

The non-nested distributions used for comparisons are:

$$f_{MOOBII-L}(x; \delta, \alpha, \beta, \lambda) = \frac{\delta \alpha, \beta \lambda^{-1} \exp(-x/\lambda) (1 + \exp(-x/\lambda))^{\alpha-1}}{\left[1 - (1-\delta) \left(1 - \left\{ 1 + \left[\frac{(1+\exp(-x/\lambda))^{-1}}{1-(1+\exp(-x/\lambda))^{-1}} \right]^{-\alpha} \right\}^{-\beta} \right) \right]^2} \\ \times (1 - (1 + \exp(-x/\lambda))^{-1})^{\alpha-1} \left\{ 1 + \left[\frac{(1 + \exp(-x/\lambda))^{-1}}{1 - (1 + \exp(-x/\lambda))^{-1}} \right]^{-\alpha} \right\}^{-\beta-1}$$

for $\delta, \alpha, \beta, \lambda > 0$ and $x > 0$,

$$f_{GG-L}(x; \alpha, \beta, b, \theta) = \frac{\theta \alpha \beta e^{\gamma x} (1-t) t^{\alpha-1}}{(\theta t^{\alpha} - 1) \log(1-\theta)}$$

for $0 < \theta < 1$, $\alpha, \beta, \gamma > 0$, $t = 1 - \exp(-\beta/\gamma(\exp(\gamma x) - 1))$ and $x > 0$,

$$f_{\text{EBXIIP}}(x; c, s, \alpha, \lambda) = \frac{cs^{-c}\alpha\lambda}{1 - e^{-\lambda}} x^{c-1} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-2} \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-1}\right\}^{\alpha-1} \\ \times \exp\left\{-\lambda \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-1}\right\}^{\alpha}\right\}$$

for $c, s, \alpha, \lambda > 0$ and $x > 0$, and

$$f_{\text{TL-WP}}(x; \alpha, \beta, b, \theta) = \frac{2b\theta\alpha\beta x^{\beta-1} e^{-2\alpha x^{\beta}} \left(1 - e^{-2\alpha x^{\beta}}\right)^{b-1} e^{\theta \left[1 - \left(1 - e^{-2\alpha x^{\beta}}\right)^b\right]}}{e^{\theta} - 1}$$

for $\alpha, \beta, b, \theta > 0$ and $x > 0$.

D.5 Datasets

Taxes revenue data: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

Waiting times data: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

Mortality rates data: 4.4130, 3.0525, 4.6955, 7.4810, 5.1915, 3.6335, 6.6100, 8.2490, 5.8325, 3.0075, 5.4275, 3.0610, 3.3280, 1.7200, 2.9270, 5.3425, 5.0175, 2.6210, 2.1720, 2.5715, 3.8150, 7.3020, 3.9515, 3.1850, 1.7685, 3.1635, 2.3650, 1.6075, 4.6420, 6.4390, 4.4065, 5.0215, 3.6300, 2.9925, 3.2060, 1.6975, 2.2120, 4.9675, 3.9200, 4.7750, 1.7495, 1.8755, 3.4840, 1.6430, 5.0790, 4.0540, 3.3485, 3.5755, 3.2800, 1.0385, 1.8890, 1.4940, 1.6680, 3.4070, 4.1625, 3.9270, 4.2755, 1.6140, 3.7430, 3.3125, 3.0700, 2.4545, 2.3305, 2.6960, 6.0210, 4.3480, 0.9075, 1.6635, 2.7030, 3.0910, 0.5205, 0.9000, 2.4745, 2.0445, 1.6795, 1.0350, 1.6490, 2.6585, 2.7210, 2.2785, 2.1460, 1.2500, 3.2675, 2.3240, 2.3485, 2.7295, 2.0600, 1.9610, 1.6095, 0.7010, 1.2190, 1.6285, 1.8160, 1.6165, 1.5135, 1.1760, 0.6025, 1.6090, 1.4630, 1.3005, 1.0325, 1.5145, 1.0290, 1.1630, 1.2530, 0.9615.

E. Appendices for MOTLOBX-G Family

E.1 Series Expansion Proofs

Note that for $\delta \in (0, 1)$,

$$\begin{aligned} f(x; \alpha, \theta, \delta, \underline{\omega}) &= \frac{4\alpha\theta g(x; \underline{\omega})G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta-1} \\ &\times \exp[-O_G^2(x; \underline{\omega})] \sum_{u=0}^{\infty} \sum_{v=0}^u w_{u,v} \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta}\right] \\ &\times \left[1 - \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta}\right]^2\right]^{\alpha(u-v+1)-1}. \end{aligned}$$

Applying the generalized binomial series representation $(1 - z)^n = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} z^i$ which is valid for $|z| < 1$ and letting $\gamma^* = \alpha(u - v + 1) - 1$ we get

$$\begin{aligned} f(x; \alpha, \theta, \delta, \underline{\omega}) &= \frac{4\alpha\theta g(x; \underline{\omega})G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \exp[-O_G^2(x; \underline{\omega})] \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta-1} \\ &\times \sum_{i,u=0}^{\infty} \sum_{v=0}^u w_{u,v} (-1)^i \binom{\gamma^*}{i} \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta}\right]^{2i+1} \\ &= \frac{4\alpha\theta g(x; \underline{\omega})G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \exp[-O_G^2(x; \underline{\omega})] \sum_{i,j,u=0}^{\infty} \sum_{v=0}^u w_{u,v} (-1)^{i+j} \binom{\gamma^*}{i} \\ &\times \binom{2i+1}{j} \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta(j+1)-1} \\ &= \frac{4\alpha\theta g(x; \underline{\omega})G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \sum_{i,j,k,u=0}^{\infty} \sum_{v=0}^u w_{u,v} (-1)^{i+j+k} \binom{\gamma^*}{i} \binom{2i+1}{j} \binom{\theta(j+1)-1}{k} \\ &\times \exp[-(k+1)O_G^2(x; \underline{\omega})]. \end{aligned}$$

Application of the series expansion $\exp(-z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!}$ gives

$$\begin{aligned} f(x; \alpha, \theta, \delta, \underline{\omega}) &= 4\alpha\theta \sum_{i,j,k,m,u=0}^{\infty} \sum_{v=0}^u \frac{w_{u,v} (-1)^{i+j+k+m} (k+1)^m}{m!} \binom{\gamma^*}{i} \\ &\times \binom{2i+1}{j} \binom{\theta(j+1)-1}{k} \frac{g(x; \underline{\omega})G^{2m+1}(x; \underline{\omega})}{\bar{G}^{2m+3}(x; \underline{\omega})}. \end{aligned}$$

Rewriting $G^{2m+1}(x; \underline{\omega})$ as $[1 - \bar{G}(x; \underline{\omega})]^{2m+1}$ and applying previous binomial series expansion

gives

$$\begin{aligned}
 f(x; \alpha, \theta, \delta, \underline{\omega}) &= 4\alpha\theta \sum_{i,j,k,m,n,u=0}^{\infty} \sum_{v=0}^u \frac{w_{u,v}(-1)^{i+j+k+m+n}(k+1)^m}{m!} \binom{\gamma^*}{i} \binom{2i+1}{j} \\
 &\quad \times \binom{\theta(j+1)-1}{k} \binom{2m+1}{n} g(x; \underline{\omega}) \bar{G}^{n-(2m+3)}(x; \underline{\omega}) \\
 &= 4\alpha\theta \sum_{i,j,k,m,n,p,u=0}^{\infty} \sum_{v=0}^u \frac{w_{u,v}(-1)^{i+j+k+m+n+p}(k+1)^m}{m!} \binom{\gamma^*}{i} \binom{2i+1}{j} \\
 &\quad \times \binom{\theta(j+1)-1}{k} \binom{2m+1}{n} \binom{n-(2m+3)}{p} g(x; \underline{\omega}) G^p(x; \underline{\omega}).
 \end{aligned}$$

E.2 Rényi Entropy Proofs

Note that if $\delta \in (0, 1)$ and we apply generalized binomial series representation $(1 - z)^n = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} z^i$ which is valid for $|z| < 1$, we get

$$\begin{aligned}
 f^v(x; \alpha, \theta, \delta, \underline{\omega}) &= \sum_{h,u=0}^{\infty} e_u(-1)^h \binom{u}{h} f_{TLOBX-G}^v(x; \alpha, \theta, \underline{\omega}) F_{TLOBX-G}^h(x; \alpha, \theta, \underline{\omega}) \\
 &= \frac{(4\alpha\theta)^v g^v(x; \underline{\omega}) G^v(x; \underline{\omega})}{\bar{G}^{3v}(x; \underline{\omega})} \exp[-vO_G^2(x; \underline{\omega})] \\
 &\quad \times \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{v(\theta-1)} \sum_{h,i,u=0}^{\infty} e_u(-1)^{i+h} \binom{u}{h} \binom{v(\alpha-1) + \alpha h}{i} \\
 &\quad \times \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta}\right]^{2i+v} \\
 &= \frac{(4\alpha\theta)^v g^v(x; \underline{\omega}) G^v(x; \underline{\omega})}{\bar{G}^{3v}(x; \underline{\omega})} \exp[-vO_G^2(x; \underline{\omega})] \sum_{h,i,j,u=0}^{\infty} e_u(-1)^{h+i+j} \binom{u}{h} \\
 &\quad \times \binom{v(\alpha-1) + \alpha h}{i} \binom{2i+v}{j} \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta(j+v)-v} \\
 &= \frac{(4\alpha\theta)^v g^v(x; \underline{\omega}) G^v(x; \underline{\omega})}{\bar{G}^{3v}(x; \underline{\omega})} \exp[-vO_G^2(x; \underline{\omega})] \sum_{h,i,j,k,u=0}^{\infty} e_u(-1)^{h+i+j+k} \binom{u}{h} \\
 &\quad \times \binom{v(\alpha-1) + \alpha h}{i} \binom{2i+v}{j} \binom{\theta(j+v)-v}{k} \exp[-(k+v)O_G^2(x; \underline{\omega})] \\
 &= (4\alpha\theta)^v \sum_{h,i,j,k,m,u=0}^{\infty} \frac{e_u(-1)^{h+i+j+k+m}(k+v)^m}{m!} \binom{u}{h} \binom{v(\alpha-1) + \alpha h}{i} \\
 &\quad \times \binom{2i+v}{j} \binom{\theta(j+v)-v}{k} \frac{g^v(x; \underline{\omega}) G^{2m+v}(x; \underline{\omega})}{\bar{G}^{2m+3v}(x; \underline{\omega})}.
 \end{aligned}$$

Rewriting $G^{2m+v}(x; \underline{\omega})$ as $[1 - \bar{G}(x; \underline{\omega})]^{2m+v}$ and applying binomial series expansion gives

$$\begin{aligned}
 f^v(x; \alpha, \theta, \delta, \underline{\omega}) &= (4\alpha\theta)^v \sum_{h,i,j,k,m,n,u=0}^{\infty} \frac{e_u(-1)^{h+i+j+k+m+n}(k+v)^m}{m!} \binom{v(\alpha-1) + \alpha h}{i} \\
 &\quad \times \binom{2i+v}{j} \binom{\theta(j+v) - v}{k} \binom{2m+v}{n} g^v(x; \underline{\omega}) \bar{G}^{n-(2m+3v)}(x; \underline{\omega}) \\
 &= (4\alpha\theta)^v \sum_{h,i,j,k,m,n,p,u=0}^{\infty} \frac{e_u(-1)^{h+i+j+k+m+n+p}(k+v)^m}{m!} \binom{v(\alpha-1) + \alpha h}{i} \\
 &\quad \times \binom{2i+v}{j} \binom{\theta(j+v) - v}{k} \binom{2m+v}{n} \binom{n-(2m+3v)}{p} g^v(x; \underline{\omega}) G^p(x; \underline{\omega}).
 \end{aligned}$$

E.3 Distribution of Order Statistics Proofs

For $\delta \in (0, 1)$, let $\varphi^* = \alpha(u + t - v + i) - 1$, then

$$\begin{aligned}
 f_{i:n}(x; \alpha, \theta, \delta, \underline{\omega}) &= \frac{4\alpha\theta g(x; \underline{\omega}) G(x; \underline{\omega})}{\bar{G}^3(x; \underline{\omega})} \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta-1} \\
 &\quad \times \exp[-O_G^2(x; \underline{\omega})] \sum_{u=0}^{\infty} \sum_{v=0}^u \sum_{t=0}^{n-i} b_{u,v,t} \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta}\right] \\
 &\quad \times \left[1 - \left[1 - \{1 - \exp[-O_G^2(x; \underline{\omega})]\}^{\theta}\right]^2\right]^{\varphi^*}.
 \end{aligned}$$

Applying generalized binomial series and exponential series representation as it was done in Appendix [E.1](#) we get

$$\begin{aligned}
 f(x; \alpha, \theta, \delta, \underline{\omega}) &= 4\alpha\theta \sum_{h,j,k,m,q,u=0}^{\infty} \sum_{v=0}^u \sum_{t=0}^{n-i} \frac{b_{u,v,t}(-1)^{h+j+k+m+q}(k+1)^m}{m!} \binom{\varphi^*}{h} \binom{2h+1}{j} \\
 &\quad \times \binom{\theta(j+1) - 1}{k} \binom{2m+1}{q} g(x; \underline{\omega}) \bar{G}^{q-(2m+3)}(x; \underline{\omega}) \\
 &= 4\alpha\theta \sum_{h,j,k,m,p,q,u=0}^{\infty} \sum_{v=0}^u \sum_{t=0}^{n-i} \frac{b_{u,v,t}(-1)^{h+j+k+m+p+q}(k+1)^m}{m!} \binom{\varphi^*}{h} \binom{2h+1}{j} \\
 &\quad \times \binom{\theta(j+1) - 1}{k} \binom{2m+1}{q} \binom{q-(2m+3)}{p} g(x; \underline{\omega}) G^p(x; \underline{\omega}).
 \end{aligned}$$

E.4 Components of the Score Vector

The elements of the score vector for the MOTLOBX-G family are given as:

$$\begin{aligned}
 \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \ln \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2 \right] \\
 &\quad - 2 \sum_{i=1}^n \frac{\bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2 \right]^\alpha \right\} \ln \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2 \right]}{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2 \right]^\alpha \right\}}, \\
 \\
 \frac{\partial \ell}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^n \ln \left(1 - e^{-Z_i^2} \right) - \sum_{i=1}^n \frac{\left\{ 1 - e^{-Z_i^2} \right\}^\theta \ln \left\{ 1 - e^{-Z_i^2} \right\}}{1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta} \\
 &\quad + (\alpha - 1) \sum_{i=1}^n \frac{2 \left\{ 1 - e^{-Z_i^2} \right\}^\theta \ln \left\{ 1 - e^{-Z_i^2} \right\} \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]}{1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2} \\
 &\quad - 4 \sum_{i=1}^n \frac{\bar{\delta} \alpha \ln \left[1 - e^{-Z_i^2} \right] \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right] \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2 \right]^{\alpha-1}}{\left\{ 1 - e^{-Z_i^2} \right\}^\theta \left\{ 1 - \bar{\delta} \left\{ 1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^{-\theta} \right]^2 \right]^\alpha \right\}}}, \\
 \\
 \frac{\partial \ell}{\partial \bar{\delta}} &= \frac{n}{\bar{\delta}} - 2 \sum_{i=1}^n \frac{1 - \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2 \right]^\alpha}{1 - \bar{\delta} \left\{ 1 - \left[1 - \left[1 - \left\{ 1 - e^{-Z_i^2} \right\}^\theta \right]^2 \right]^\alpha \right\}},
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \ell}{\partial \underline{\omega}_j} &= \sum_{i=1}^n \frac{g'(x; \underline{\omega})}{g(x; \underline{\omega})} + \sum_{i=1}^n \frac{G'(x; \underline{\omega})}{G(x; \underline{\omega})} - 3 \sum_{i=1}^n \frac{\bar{G}'(x; \underline{\omega})}{\bar{G}(x; \underline{\omega})} - 2 \sum_{i=1}^n Z_i \frac{\partial Z_i}{\partial \underline{\omega}_j} \\
 &+ (\theta - 1) \sum_{i=1}^n \frac{2Z_i e^{-Z_i^2}}{1 - e^{-Z_i^2}} \frac{\partial Z_i}{\partial \underline{\omega}_j} - \frac{2\theta Z_i e^{-Z_i^2} \{1 - e^{-Z_i^2}\}^{\theta-1}}{1 - \{1 - e^{-Z_i^2}\}^\theta} \frac{\partial Z_i}{\partial \underline{\omega}_j} \\
 &+ \frac{4\theta Z_i e^{-Z_i^2} \{1 - e^{-Z_i^2}\}^{\theta-1} \left[1 - \{1 - e^{-Z_i^2}\}^\theta\right]}{1 - \left[1 - \{1 - e^{-Z_i^2}\}^\theta\right]^2} \frac{\partial Z_i}{\partial \underline{\omega}_j} \\
 &+ \sum_{i=1}^n \frac{8\alpha\theta\bar{\delta}Z_i e^{-Z_i^2} \{1 - e^{-Z_i^2}\}^{\theta-1}}{1 - \left[1 - \{1 - e^{-Z_i^2}\}^\theta\right]^2} \\
 &\times \left[1 - \{1 - e^{-Z_i^2}\}^\theta\right] \left[1 - \left[1 - \{1 - e^{-Z_i^2}\}^\theta\right]^2\right]^{\alpha-1} \frac{\partial Z_i}{\partial \underline{\omega}_j},
 \end{aligned}$$

where $Z_i = G(x_i; \underline{\omega})/\bar{G}(x_i; \underline{\omega})$, $g'(x; \underline{\omega}) = \partial g(x; \underline{\omega})/\partial \underline{\omega}_j$, $G'(x; \underline{\omega}) = \partial G(x; \underline{\omega})/\partial \underline{\omega}_j$ and $\bar{G}'(x; \underline{\omega}) = \partial \bar{G}(x; \underline{\omega})/\partial \underline{\omega}_j$ for $\underline{\omega}_j$ representing the j^{th} element of the vector of baseline parameters $\underline{\omega}$.

E.5 Competing Distributions

The pdf of the models used for comparisons are:

$$\begin{aligned}
 f_{WL}(x) &= \frac{ab\alpha}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{b\alpha-1} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1} \\
 &\times \exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^\alpha - 1\right\}\right\},
 \end{aligned}$$

for $\alpha, \beta, a, b > 0$ and $x > 0$,

$$f_{MOG-L}(x; \lambda, \beta, \delta, a) = \frac{\delta a \alpha (1 + (x/\beta))^{-(\alpha a + 1)}}{\beta [1 - (1 - \delta)(1 + (x/\beta))^{-\alpha a}]^2}$$

for $\lambda, \beta, \delta, a > 0$ and $x > 0$,

$$\begin{aligned}
 f_{MOOBIII-Lo}(x; \delta, \alpha, \beta, \lambda) &= \frac{\delta \alpha \beta \lambda x^{\lambda-1} (1 + x^\lambda)^{-\alpha-1}}{(1 - (1 + x^\lambda)^{-1})^{\alpha+1}} \left\{1 + \left[\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}}\right]^{-\alpha}\right\}^{-\beta-1} \\
 &\times \left[1 - (1 - \delta) \left(1 - \left\{1 + \left[\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}}\right]^{-\alpha}\right\}^{-\beta}\right)\right]^{-2}
 \end{aligned}$$

for $\delta, \alpha, \beta, \lambda > 0$ and $x > 0$,

$$f_{\text{MOOBIII-L}}(x; \delta, \alpha, \beta, \lambda) = \frac{\delta \alpha, \beta \lambda^{-1} \exp(-x/\lambda) (1 + \exp(-x/\lambda))^{\alpha-1}}{\left[1 - (1 - \delta) \left(1 - \left\{ 1 + \left[\frac{(1 + \exp(-x/\lambda))^{-1}}{1 - (1 + \exp(-x/\lambda))^{-1}} \right]^{-\alpha} \right\}^{-\beta} \right) \right]^2} \\ \times (1 - (1 + \exp(-x/\lambda))^{-1})^{\alpha-1} \left\{ 1 + \left[\frac{(1 + \exp(-x/\lambda))^{-1}}{1 - (1 + \exp(-x/\lambda))^{-1}} \right]^{-\alpha} \right\}^{-\beta-1}$$

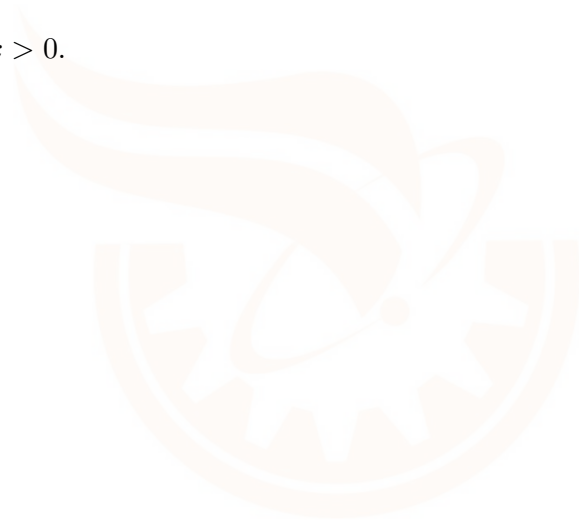
for $\delta, \alpha, \beta, \lambda > 0$ and $x > 0$,

$$f_{\text{MOINH}}(x; \beta, \gamma, \alpha) = \frac{\alpha \beta \gamma x^{-2} (1 + \beta x^{-1})^{\alpha-1} \exp [1 - (1 + \beta x^{-1})^\alpha]}{[1 - (1 - \gamma) (1 - \exp [1 - (1 + \beta x^{-1})^\alpha])]^2}$$

for $\beta, \gamma, \alpha > 0$ and $x > 0$,

$$f_{\text{MO-L}}(x; \alpha, \beta, \gamma) = \frac{\alpha (\beta/\gamma) (x/\gamma)^{\beta-1}}{[\alpha + (x/\gamma)^\beta]^2}$$

for $\gamma, \alpha > 0, \beta > 1$ and $x > 0$.



F. Appendices for TLEHLOBX-G Family

F.1 Density Expansion Proofs

In the TLEHLOBX-G pdf $f(x; b, \alpha, \beta, \underline{\Psi})$, if we let $Q = \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right]^2 \right\}^{b-1}$, then we can write

$$Q = \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i \left[1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right]^{2i},$$

and the pdf of TLEHLOBX-G family reduces to

$$\begin{aligned} f(x; b, \alpha, \beta, \underline{\Psi}) &= \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i \frac{8b\alpha\beta g(x; \underline{\Psi}) G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi})} \exp(-K_G(x; \underline{\Psi})) \\ &\quad \times [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha\beta-1} \\ &\quad \times [2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta]^{-\alpha-1} \\ &\quad \times \left\{ 1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right\}^{2i+1}. \end{aligned}$$

Now, considering $R = \left\{ 1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right\}^{2i+1}$ written as

$$R = \sum_{j=0}^{\infty} \binom{2i+1}{j} (-1)^j \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^{\alpha j},$$

then the pdf of TLEHLOBX-G family reduces to

$$\begin{aligned} f(x; b, \alpha, \beta, \underline{\Psi}) &= \sum_{i,j=0}^{\infty} \binom{2i+1}{j} \binom{b-1}{i} (-1)^{i+j} \frac{8b\alpha\beta g(x; \underline{\Psi}) G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi})} \\ &\quad \times \exp(-K_G(x; \underline{\Psi})) [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha(\beta+j)-1} \\ &\quad \times [2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta]^{-\alpha(j+1)-1}. \end{aligned}$$

Now, letting

$$\begin{aligned} P &= [2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta]^{-\alpha(j+1)-1} \\ &= [1 - [1 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta]]^{-\alpha(j+1)-1}, \end{aligned}$$

we obtain

$$P = \sum_{k=0}^{\infty} \binom{\alpha(j+1)+k}{k} \left[1 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right]^k.$$

Then the pdf of TLEHLOBX-G family reduces to

$$\begin{aligned} f(x; b, \alpha, \beta, \underline{\Psi}) &= \sum_{i,j,k=0}^{\infty} \binom{2i+1}{j} \binom{b-1}{i} \binom{\alpha(j+1)+k}{k} (-1)^{i+j} \frac{8b\alpha\beta g(x; \underline{\Psi}) G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi})} \\ &\times \exp(-K_G(x; \underline{\Psi})) [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha(\beta+j)+k-1} \\ &\times \left[1 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right]^k \\ &= \sum_{i,j,k,l=0}^{\infty} \binom{2i+1}{j} \binom{b-1}{i} \binom{\alpha(j+1)+k}{k} \binom{k}{l} (-1)^{i+j+l} \frac{8b\alpha\beta g(x; \underline{\Psi}) G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi})} \\ &\times \exp(-K_G(x; \underline{\Psi})) [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha(\beta+j)+k+\beta l-1} \\ &= \sum_{i,j,k,l,m=0}^{\infty} \binom{2i+1}{j} \binom{b-1}{i} \binom{\alpha(j+1)+k}{k} \binom{k}{l} \binom{\alpha(\beta+j)+k+\beta l-1}{m} \\ &\times (-1)^{i+j+l+m} \frac{8b\alpha\beta g(x; \underline{\Psi}) G(x; \underline{\Psi})}{\bar{G}^3(x; \underline{\Psi})} \exp(-(m+1)K_G(x; \underline{\Psi})) \\ &= \sum_{i,j,k,l,m,s=0}^{\infty} \binom{2i+1}{j} \binom{b-1}{i} \binom{\alpha(j+1)+k}{k} \binom{k}{l} \binom{\alpha(\beta+j)+k+\beta l-1}{m} \\ &\times (-1)^{i+j+l+m+s} \frac{8b\alpha\beta(m+1)^s g(x; \underline{\Psi}) G^{2s+1}(x; \underline{\Psi})}{s! \bar{G}^{3+2s}(x; \underline{\Psi})} \\ &\sum_{i,j,k,l,m,s,u=0}^{\infty} \binom{2i+1}{j} \binom{b-1}{i} \binom{\alpha(j+1)+k}{k} \binom{k}{l} \binom{\alpha(\beta+j)+k+\beta l-1}{m} \\ &\times \binom{2s+1}{u} (-1)^{i+j+l+m+s+u} \frac{8b\alpha\beta(m+1)^s g(x; \underline{\Psi})}{s!} \bar{G}^{u-(3+2s)}(x; \underline{\Psi}) \\ &= \sum_{i,j,k,l,m,s,u,w=0}^{\infty} \binom{2i+1}{j} \binom{b-1}{i} \binom{\alpha(j+1)+k}{k} \binom{k}{l} \binom{\alpha(\beta+j)+k+\beta l-1}{m} \\ &\times \binom{2s+1}{u} \binom{u-(3+2s)}{w} (-1)^{i+j+l+m+s+u+w} \frac{8b\alpha\beta(m+1)^s g(x; \underline{\Psi})}{s!} G^w(x; \underline{\Psi}). \end{aligned}$$

F.2 Rényi Entropy Proofs

Note that

$$\begin{aligned}
 [f(x; b, \alpha, \beta, \underline{\Psi})]^v &= \frac{(8b\alpha\beta)^v g^v(x; \underline{\Psi}) G^v(x; \underline{\Psi})}{\bar{G}^{3v}(x; \underline{\Psi})} \exp(-vK_G(x; \underline{\Psi})) \\
 &\times [1 - \exp(-K_G(x; \underline{\Psi}))]^{v\alpha\beta-v} \left[2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right]^{-v\alpha-v} \\
 &\times \left\{ 1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right\}^v \\
 &\times \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right]^2 \right\}^{vb-v}.
 \end{aligned}$$

Using the following generalized binomial and Taylor series expansions

$$(1+z)^{-\beta} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta+k-1}{k} z^k \quad \text{for } |z| < 1, \beta > 0, \quad \text{and} \quad e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!},$$

respectively, and if we let $Q = \left\{ 1 - \left[1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right]^2 \right\}^{vb-v}$, then we can write

$$Q = \sum_{i=0}^{\infty} \binom{vb-v}{i} (-1)^i \left[1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right]^{2i},$$

and we have

$$\begin{aligned}
 [f(x; b, \alpha, \beta, \underline{\Psi})]^v &= \sum_{i=0}^{\infty} \binom{vb-v}{i} \frac{(-1)^i (8b\alpha\beta)^v g^v(x; \underline{\Psi}) G^v(x; \underline{\Psi})}{\bar{G}^{3v}(x; \underline{\Psi})} \exp(-vK_G(x; \underline{\Psi})) \\
 &\times [1 - \exp(-K_G(x; \underline{\Psi}))]^{v\alpha\beta-v} \left[2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right]^{-v\alpha-v} \\
 &\times \left\{ 1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right\}^{2i+v}.
 \end{aligned}$$

Now, considering $R = \left\{ 1 - \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^\alpha \right\}^{2i+v}$

$$R = \sum_{j=0}^{\infty} \binom{2i+v}{j} (-1)^j \left[\frac{(1 - \exp(-K_G(x; \underline{\Psi})))^\beta}{2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta} \right]^{\alpha j},$$

then we have

$$\begin{aligned}
 [f(x; b, \alpha, \beta, \underline{\Psi})]^v &= \sum_{i,j=0}^{\infty} \binom{vb-v}{i} \binom{2i+v}{j} (-1)^{i+j} \frac{(8b\alpha\beta)^v g^v(x; \underline{\Psi}) G^v(x; \underline{\Psi})}{\bar{G}^{3v}(x; \underline{\Psi})} \\
 &\times \exp(-vK_G(x; \underline{\Psi})) [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha(\beta v+j)-v} \\
 &\times \left[2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right]^{-\alpha(j+v)-v}.
 \end{aligned}$$

Now, letting

$$\begin{aligned}
 P &= \left[2 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right]^{-\alpha(j+v)-v} \\
 &= \left[1 - \left[1 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right] \right]^{-\alpha(j+v)-v},
 \end{aligned}$$

we obtain

$$P = \sum_{k=0}^{\infty} \binom{\alpha(j+v) + v + k - 1}{k} \left[1 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta \right]^k.$$

Then we can write

$$\begin{aligned}
 [f(x; b, \alpha, \beta, \underline{\Psi})]^v &= \sum_{i,j,k=0}^{\infty} \binom{vb-v}{i} \binom{2i+v}{j} (-1)^{i+j} \frac{(8b\alpha\beta)^v g^v(x; \underline{\Psi}) G^v(x; \underline{\Psi})}{\bar{G}^{3v}(x; \underline{\Psi})} \\
 &\times \binom{\alpha(j+v) + v + k - 1}{k} [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha(\beta v + j) - v} \\
 &\times \exp(-vK_G(x; \underline{\Psi})) [1 - (1 - \exp(-K_G(x; \underline{\Psi})))^\beta]^k \\
 &= \sum_{i,j,k,l=0}^{\infty} \binom{vb-v}{i} \binom{2i+v}{j} \binom{\alpha(j+v) + v + k - 1}{k} (-1)^{i+j+l} \binom{k}{l} \\
 &\times \frac{(8b\alpha\beta)^v g^v(x; \underline{\Psi}) G^v(x; \underline{\Psi})}{\bar{G}^{3v}(x; \underline{\Psi})} \exp(-vK_G(x; \underline{\Psi})) \\
 &\times [1 - \exp(-K_G(x; \underline{\Psi}))]^{\alpha(\beta v + j) - v + \beta l} \\
 &= \sum_{i,j,k,l,m=0}^{\infty} \binom{vb-v}{i} \binom{2i+v}{j} \binom{\alpha(j+v) + v + k - 1}{k} (-1)^{i+j+l+m} \binom{k}{l} \\
 &\times \binom{\alpha(\beta v + j) - v + \beta l}{m} \frac{(8b\alpha\beta)^v g^v(x; \underline{\Psi}) G^v(x; \underline{\Psi})}{\bar{G}^{3v}(x; \underline{\Psi})} \exp(-(v+m)K_G(x; \underline{\Psi})) \\
 &= \sum_{i,j,k,l,m,s=0}^{\infty} \binom{vb-v}{i} \binom{2i+v}{j} \binom{\alpha(j+v) + v + k - 1}{k} (-1)^{i+j+l+m+s} \binom{k}{l} \\
 &\times \binom{\alpha(\beta v + j) - v + \beta l}{m} \frac{(8b\alpha\beta)^v (v+m)^s g^v(x; \underline{\Psi}) G^{v+2s}(x; \underline{\Psi})}{s! \bar{G}^{3v+2s}(x; \underline{\Psi})} \\
 &= \sum_{i,j,k,l,m,s,u=0}^{\infty} \binom{vb-v}{i} \binom{2i+v}{j} \binom{\alpha(j+v) + v + k - 1}{k} (-1)^{i+j+l+m+s+u} \binom{k}{l} \\
 &\times \binom{v+2s}{u} \binom{\alpha(\beta v + j) - v + \beta l}{m} \frac{(8b\alpha\beta)^v (v+m)^s g^v(x; \underline{\Psi}) \bar{G}^{u-(3v+2s)}(x; \underline{\Psi})}{s!} \\
 &= \sum_{i,j,k,l,m,s,u,w=0}^{\infty} \binom{vb-v}{i} \binom{2i+v}{j} \binom{\alpha(j+v) + v + k - 1}{k} (-1)^{i+j+l+m+s+u+w} \\
 &\times \binom{k}{l} \binom{v+2s}{u} \binom{\alpha(\beta v + j) - v + \beta l}{m} \binom{u-(3v+2s)}{w} \frac{(8b\alpha\beta)^v (v+m)^s}{s!} \\
 &\times g^v(x; \underline{\Psi}) G^w(x; \underline{\Psi}).
 \end{aligned}$$

F.3 Elements of the Score Vector

The elements of the score vector for the TLEHLOBX-G family are:

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^n \log \left\{ 1 - \left[1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right)^\beta \right)} \right]^\alpha \right]^2 \right\},$$

$$\begin{aligned}
 \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \beta \sum_{i=1}^n \log \left[1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right] \\
 &- \sum_{i=1}^n \log \left[2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \right] \\
 &- \sum_{i=1}^n \frac{\left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \ln \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]}{\left\{ 1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \right\}} \\
 &+ 2(b-1) \sum_{i=1}^n \frac{\left[1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \right]^2}{\left\{ 1 - \left[1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \right]^2 \right\}} \\
 &\times \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \ln \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right],
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \alpha \sum_{i=1}^n \log \left[1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right] \\
 &- (-\alpha - 1) \sum_{i=1}^n \frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \ln \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)}{\left[2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \right]} \\
 &- 2\alpha \sum_{i=1}^n \frac{\left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^{\alpha-1}}{\left\{ 1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \right\} \left[2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \right]^2} \\
 &\times \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \ln \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right) \\
 &+ 2\alpha(b-1) \sum_{i=1}^n \frac{\left[1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \right] \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^{\alpha-1}}{\left\{ 1 - \left[1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \right]^2 \right\}} \\
 &\times \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \ln \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right) \\
 &\times \left[2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \right]^{-2},
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \ell}{\partial \underline{\Psi}_k} = & -2 \sum_{i=1}^n \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right) \frac{\partial G(x_i; \underline{\Psi})}{\partial \underline{\Psi}_k} \\
 & + 2(\alpha\beta - 1) \sum_{i=1}^n \frac{\exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right) \frac{\partial G(x_i; \underline{\Psi})}{\partial \underline{\Psi}_k}}{\left[1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right]} \\
 & - \beta(-\alpha - 1) \sum_{i=1}^n \frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^{\beta-1} \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right) \frac{\partial G(x_i; \underline{\Psi})}{\partial \underline{\Psi}_k}}{\left[2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \right]} \\
 & + 2\alpha\beta \sum_{i=1}^n \frac{\left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^{\alpha-1} \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^{\beta-1}}{\left\{ 1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \right\}} \\
 & \times \frac{\exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right) \frac{\partial G(x_i; \underline{\Psi})}{\partial \underline{\Psi}_k}}{\left[2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \right]^2} \\
 & + 2(b-1) \sum_{i=1}^n \frac{\left[1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \right] \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^{\alpha-1}}{\left\{ 1 - \left[1 - \left[\frac{\left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta}{2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta} \right]^\alpha \right]^2 \right\}} \\
 & \times \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^{\beta-1} \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right) \frac{\partial G(x_i; \underline{\Psi})}{\partial \underline{\Psi}_k} \\
 & \times \left[2 - \left(1 - \exp \left(- \left(\frac{G(x_i; \underline{\Psi})}{\bar{G}(x_i; \underline{\Psi})} \right)^2 \right) \right)^\beta \right]^{-2} \\
 & + \sum_{i=1}^n \frac{\frac{\partial G(x_i; \underline{\Psi})}{\partial \underline{\Psi}_k}}{(G(x; \underline{\Psi}))} - 3 \sum_{i=1}^n \frac{\frac{\partial G(x_i; \underline{\Psi})}{\partial \underline{\Psi}_k}}{(\bar{G}(x; \underline{\Psi}))} + \sum_{i=1}^n \frac{\frac{\partial g(x_i; \underline{\Psi})}{\partial \underline{\Psi}_k}}{(g(x_i; \underline{\Psi}))}.
 \end{aligned}$$

F.4 Competing Distributions

The pdf of existing distributions that we compared our model with are given below:

$$f_{TLOBIII-L}(x; \alpha, \beta, b, \lambda) = \left[1 - \left(1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-\alpha} \right)^{-\beta} \right)^2 \right]^{b-1} \\ \times \left(1 - \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-\alpha} \right)^{-\beta} \right) \left(1 + \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-\alpha} \right)^{-\beta-1} \\ \times 2\alpha\beta b \lambda x^{\lambda-1} \left(\frac{1 - (1 + x^\lambda)^{-1}}{(1 + x^\lambda)^{-1}} \right)^{-\alpha-1}$$

for $\alpha, \beta, b, \lambda > 0$, and $x \geq 0$,

$$f_{OGHLW-W}(x; w, \lambda, \gamma) = \frac{2w\lambda\gamma x^{\lambda-1} (1 + \gamma x^\lambda)^{w-1} \exp(1 - (1 + \gamma x^\lambda)^w)}{[1 + \exp(1 - (1 + \gamma x^\lambda)^w)]^2},$$

for $w, \lambda, \gamma > 0$, and $x \geq 0$,

$$f_{TLMO-BXII}(x; b, \delta, \xi) = \frac{2b\delta^2\lambda\beta x^{\lambda-1}(1 + x^\lambda)^{-\beta-1}(1 + x^\lambda)^{-\beta}}{[1 - \bar{\delta}(1 + x^\lambda)^{-\beta}]^3} \left[1 - \frac{\delta^2((1 + x^\lambda)^{-\beta})^2}{(1 - \bar{\delta}(1 + x^\lambda)^{-\beta})^2} \right]^{b-1}$$

for $b, \delta, \lambda, \beta > 0$ and $x \geq 0$,

$$f_{HLGW}(x; w, \lambda, \gamma) = \frac{2w\lambda\gamma x^{\lambda-1} (1 + \gamma x^\lambda)^{w-1} \exp(1 - (1 + \gamma x^\lambda)^w)}{[1 + \exp(1 - (1 + \gamma x^\lambda)^w)]^2},$$

for $w, \lambda, \gamma > 0$, and $x \geq 0$, and

$$f_{HTBPT-W}(x; \alpha, \gamma, \beta) = \alpha\gamma x^{\alpha-1} e^{-\gamma x^\alpha} \left(\beta - \log(\beta) \beta^{e^{-\gamma x^\alpha}} \right),$$

for $\alpha, \gamma, \beta > 0$, $\beta \neq 1$ and $x \geq 0$.

F.5 Datasets

Waiting times data: 83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

Remission times data: 0.08, 4.98, 25.74, 3.7, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.5, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.4, 11.98, 6.54, 6.93, 6.94, 0.4, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26, 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26,

17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.5, 2.02, 0.2, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26 ,4.23, 17.12, 2.87, 18.1, 8.37, 6.76 ,3.52, 13.8, 2.54, 7.32 ,0.9 ,5.41, 46.12 ,5.62, 1.46, 12.02 ,12.07.

Mortality rates data: 4.4130, 3.0525, 4.6955, 7.4810, 5.1915, 3.6335, 6.6100, 8.2490, 5.8325, 3.0075, 5.4275, 3.0610, 3.3280, 1.7200, 2.9270, 5.3425, 5.0175, 2.6210, 2.1720, 2.5715, 3.8150, 7.3020, 3.9515, 3.1850, 1.7685, 3.1635, 2.3650, 1.6075, 4.6420, 6.4390, 4.4065, 5.0215, 3.6300, 2.9925, 3.2060, 1.6975, 2.2120, 4.9675, 3.9200, 4.7750, 1.7495, 1.8755, 3.4840, 1.6430, 5.0790, 4.0540, 3.3485, 3.5755, 3.2800, 1.0385, 1.8890, 1.4940, 1.6680, 3.4070, 4.1625, 3.9270, 4.2755, 1.6140, 3.7430, 3.3125, 3.0700, 2.4545, 2.3305, 2.6960, 6.0210, 4.3480, 0.9075, 1.6635, 2.7030, 3.0910, 0.5205, 0.9000, 2.4745, 2.0445, 1.6795, 1.0350, 1.6490, 2.6585, 2.7210, 2.2785, 2.1460, 1.2500, 3.2675, 2.3240, 2.3485, 2.7295, 2.0600, 1.9610, 1.6095, 0.7010, 1.2190, 1.6285, 1.8160, 1.6165, 1.5135, 1.1760, 0.6025, 1.6090, 1.4630, 1.3005, 1.0325, 1.5145, 1.0290, 1.1630, 1.2530, 0.9615.

