

APPROXIMATING SOLUTIONS OF THE SUM OF
MONOTONE MAPPING INCLUSION, SPLIT EQUALITY,
FIXED POINT AND VARIATIONAL INEQUALITY PROBLEMS

by

Getahun Bekele Wega

Reg No. 17100031

Department of Mathematics and Statistical Sciences Botswana International
University of Science and Technology

Email: getahun.wega@studentmail.biust.ac.bw

Tel: (+267)72525430

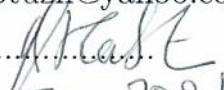
A Thesis Submitted to the College of Science in Fulfilment of the Requirements
for the Award of the Degree of Doctor of Philosophy in Pure and Applied
Mathematics of BIUST

Supervisors:

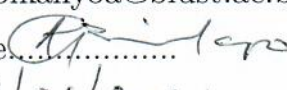
Prof. Habtu Zegeye Hailu
Department of Mathematics
and Statistical Sciences
College of Science, BIUST

Prof. Oganeditse A. Boikanyo
Department of Mathematics
and Statistical Sciences
College of Science, BIUST

Email: habtuzh@yahoo.com

Signature.....
Date...19/Jan/2021

Email: boikanyoa@biust.ac.bw

Signature.....
Date...19/01/2021

January 2021

Declaration and Copyright

I, **Getahun Bekele Wega**, with student ID 17100031, hereby declare that this thesis has been composed by myself and that it has never been submitted for a degree at this or any other university or institution of higher learning. I confirm that the work contained in this thesis is my own original research work under the supervision of Prof. Habtu Zegeye Hailu and Prof. Oganeditse A. Boikanyo. Any work done by others has been acknowledged and referenced accordingly.

Signature.....

Date.....*19 Jan 2021*.....

This thesis is copyright material protected under the Berne Convention, the Copyright Act of 1999 and other international and national enactments, in that behalf, on intellectual property. It must not be reproduced by any means, in full or in part, except in short extracts in fair dealing; for researcher private study, critical scholarly review or discourse with an acknowledgement, without the written permission of the office of the Provost on behalf of both the author and the BIUST.

Certification

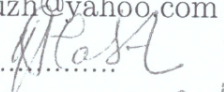
The undersigned certify that advisors have read and hereby recommend for acceptance by the College of Science a thesis titled: "APPROXIMATING SOLUTIONS OF THE SUM OF MONOTONE MAPPING INCLUSION, SPLIT EQUALITY, FIXED POINT AND VARIATIONAL INEQUALITY PROBLEMS", in fulfilment of the requirements for the award of the degree of Doctor of Philosophy in Pure and Applied Mathematics of the BIUST.

Supervisors:

Prof. Habtu Zegeye Hailu
Department of Mathematics
and Statistical Sciences,
College of Science, BIUST

Prof. Oganeditse A. Boikanyo
Department of Mathematics
and Statistical Sciences,
College of Science, BIUST

Email: habtuzh@yahoo.com

Signature.....

Date.....19 Jan 2021

Email: boikanyoa@biust.ac.bw

Signature.....

Date.....19/01/2021

List of Original Contributions

The following scientific papers are our contributions to the scientific knowledge that make up this thesis:

- 1) **Getahun Bekele Wega**, Habtu Zegeye and Oganeditse A. Boikanyo, “Approximating Solutions of the Sum of a Finite Family of Maximally Monotone Mappings in Hilbert Spaces”, *Advances in Operator Theory*, **5(2)** (2020), 359–370.
- 2) **Getahun Bekele Wega** and Habtu Zegeye, “A Method of Approximation for a Zero of the Sum of Maximally Monotone Mappings in Hilbert Spaces”, *Arab Journal of Mathematical Sciences*, (2019), In press).
- 3) **Getahun B. Wega**, Habtu Zegeye and Oganeditse A. Boikanyo, “A strong Convergence Theorems for Zeros of the Sum of a Finite Family of Maximally Monotone Mappings”, *Demonstratio Mathematica*, **53(1)** (2020), 152–166.
- 4) **Getahun Bekele Wega**, Habtu Zegeye, “Convergence Results of Forward-Backward Method for a Zero of the Sum of Maximally Monotone Mappings in Banach Spaces”, *Computational and Applied Mathematics*, **39(3)** (2020), 1–16.
- 5) **Getahun B. Wega**, H. Zegeye, “A Strong Convergence Theorem for Approximation of a Zero of the Sum of Two Maximal Monotone Mappings in Banach Spaces”, *Journal of Fixed Point Theory and Applications*, **22(57)** (2020), 1–24.
- 6) **Getahun B. Wega**, H. Zegeye and Oganeditse A. Boikanyo, “Convergence Results for a Zero of the Sum of a Finite Family of Maximally Monotone Mappings in Banach Spaces”, *Optimization* (2020, 1–30).

-
- 7) **Getahun B. Wega** and Habtu Zegeye, “Split Equality Methods for a Solution of Monotone Inclusion Problems in Hilbert Spaces”, *Linear and Non-linear Analysis*, **5**(3) (2019), 495–516.
 - 8) Habtu Zegeye and **Getahun Bekele Wega**, “Approximation of a Common f -Fixed Point of f -Pseudocontractive Mappings in Banach Spaces”, *Rendiconti del Circolo Matematico di Palermo Series 2* (2020), 1–24.
 - 9) **Getahun Bekele Wega**, Habtu Zegeye, “Convergence Theorem for Approximating a Common Point of Solutions of Variational Inequality and f -Fixed Point Problems in Banach Spaces”, *Applied Set-Valued Analysis and Optimization* (2020, Accepted).

Acknowledgment

I would like to express my special appreciation and thanks to my supervisor Professor Habtu Zegeye Hailu for offering me the opportunity to start my PhD program at Botswana International University of Science and Technology and it is also a great honor for me to be his student. He has shown me all the excellent qualities as an incredible supervisor and his tremendous and continuous guidance throughout my research helped me to grow as a researcher in mathematics. His advice on research as well as on my career have been invaluable. More importantly, he is not only a strict supervisor but also a considerate and friendly father. I am very much fortunate being supervised by him.

I would also like to thank my co-supervisor Professor Oganeditse A. Boikanyo for his brilliant comments and suggestions during my research and the write up of this thesis.

I have to be grateful to Simons Foundation based at Botswana International University of Science and Technology for its financial support towards my PhD. I am also grateful to the funding received from Botswana International University of Science and Technology. Moreover, I would like to express my gratitude to Jimma University for the study leave permit with half salary.

I would also like to thank all the staff and postgraduate students of Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology for their kind assistant and encouragement in various ways during the entire period of this work. Particularly, I would like to thank my friend Chimbola Obias Mulenga for all the time and the fun we have had together in Palapye.

My special appreciation goes to my wife, Mery Chamu for her love, understanding and patience during difficult times she has experienced because of my absence. My gratitude also goes to all those who helped me in one way or the other toward the completion of this work whose names are not mentioned here.

May God Almighty bless you all.

Contents

List of Tables	ix
List of Figures	x
1 Introduction	1
1.1 Background of the Study	1
1.2 Preliminaries	5
1.2.1 Geometry of Banach Spaces	5
1.2.2 Nonlinear Mappings	7
1.2.3 Convex Functions	10
1.2.4 Bregman Distance and Some Related Notions	13
1.3 Statement of the Problem	16
1.4 Research Questions	20
1.5 Objectives	21
2 Review of Literature	23
2.1 Review of Literature	23
2.2 Approximation of Zeros of the Sum of Maximal Monotone Mappings	23
2.3 Solutions of Split Equality Monotone Inclusion Problem for the Sum of Maximal Monotone Mappings	28
2.4 Fixed Points of Nonlinear Mappings in Banach Spaces	31
2.5 Approximation of a Common Solution of Fixed points and Varia- tional Inequality Problems	34
3 Approximating Zeros of Sum of Maximal Monotone Mappings in Hilbert Spaces	38
3.1 Preliminaries	38
3.2 Viscosity Type Douglas-Rachford Splitting Algorithm for Approx- imating a Zero of Sum of a Finite Family of Maximal Monotone Mappings in Hilbert Spaces	41
3.2.1 Application to Convex Minimization Problems	46
3.2.2 Numerical Example	47
3.3 A Strong Convergence Results for Zeros of the Sum of a Finite Family of Maximal Monotone Mappings	49

3.3.1	Application to Convex Minimization Problems	63
3.3.2	Numerical Example	64
4	Convergence Theorems for a Zero of the Sum of Maximal Monotone Mappings in Banach Spaces	66
4.1	Preliminaries	66
4.2	Convergence Results of Forward-Backward Method for a Zero of the Sum of Maximal Monotone Mappings in Banach Spaces	70
4.2.1	Application to Convex Minimization Problems	81
4.2.2	Numerical Example	82
4.3	Convergence Results for a Zero of the Sum of a Finite Family of Maximal Monotone Mappings in Banach Spaces	83
4.3.1	Application to Convex Minimization Problems	102
4.3.2	Numerical Example	103
5	Split Equality Methods for Solutions of Monotone Inclusion Problems	105
5.1	Preliminaries	105
5.2	Iterative Algorithm for Solving Monotone Inclusion Problems	106
5.2.1	Application to Convex Minimization Problem	127
5.2.2	Numerical Example	128
6	Approximation of Common f-Fixed Points of f-Pseudocontractive Mappings in Banach Spaces	130
6.1	Iterative Schemes for a Common f -fixed Point of f -Pseudocontractive Mappings in Banach Spaces	130
6.2	Application to Convex Minimization Problems in Banach Spaces	147
6.2.1	Numerical Example	148
7	Convergence Theorem for Approximating a Common Point of Solutions of Variational Inequality and f-Fixed Point Problems	150
7.1	Preliminaries	150
7.2	Solutions of Variational Inequality and f -Fixed Point Problems	151
8	Discussion, Conclusion and Recommendation	169
8.1	Discussion and Conclusion	169
8.2	Recommendation	172
	References	174

List of Tables

3.1	Convergence of the sequence $\{d_n\}$	48
3.2	Convergence of the first component $\{z_n\}$ of $\{u_n\}$ generated by (3.67)	65
5.1	Convergence of the sequence $\{z_n\}$	129
5.2	Convergence of the sequence $\{\widehat{z}_n\}$	129
6.1	Convergence of the sequence $\{x_n\}$ generated by (6.53)	149

List of Figures

3.1	The graph of $\ d_n - x^*\ $ versus number of iterations with different choices of x_0	49
4.1	Convergence of the sequence $\{u_n\}$ generated by (4.51)	82
4.2	Convergence of the first component $\{z_n\}$ of $\{v_n\}$ generated by (4.97)	104
6.1	The graph of $\ x_n - x^*\ $ versus number of iterations with different choices of x_0	149

Abstract

Many of the most important problems arising in nonlinear analysis reduce to solving inclusions of monotone mappings, split equilibrium, variational inequality or fixed point problems. Analytical methods for finding exact solutions of many nonlinear equations are rare or unknown. Therefore, methods of approximating the solutions of nonlinear inclusion problems are of interest where solutions are known to exist.

In this thesis, we combine Douglas-Rachford and Viscosity methods and construct an iterative scheme which converges strongly to a zero of the sum of a finite family of maximal monotone mappings, under suitable conditions, in the setting of Hilbert spaces. Moreover, we construct a Viscosity method to approximate a zero of the sum of maximal monotone mappings and a solution of the split equality monotone inclusion problem for the sum of two maximal monotone mappings in Hilbert spaces. We also establish forward-backward type and Halpern type algorithms and prove strong convergent theorems to zeros of the sum of maximal monotone mappings in the setting of real reflexive Banach spaces.

Furthermore, we introduce a new class of mappings called f -pseudocontractive mappings and construct an algorithm for finding common f -fixed points of those mappings and discuss its strong convergence in reflexive Banach spaces.

Finally, we introduce the concept of a Bregman relatively f -nonexpansive mapping and investigate an algorithm for approximating common element of the set of solutions of variational inequality problems for Lipschitz monotone mappings and the set of f -fixed points of Bregman relatively f -nonexpansive mapping in the setting of Banach spaces.

Chapter 1

Introduction

1.1 Background of the Study

The contribution of this thesis fall within the general area of Nonlinear Functional Analysis and its Applications, an area which has been of active research interest in recent years. We devote our attention to special classes of problems in this area: approximation of solutions to various classes of nonlinear inclusion problems.

Throughout this thesis, unless otherwise specified, E denotes a real Banach space, E^* denotes the dual space of E , that is the space of all bounded linear functionals on E , and 2^{E^*} denotes the collection of all subsets of E^* . We proceed by briefly introducing various classes of nonlinear mappings and various nonlinear mapping inclusion problems which we shall study in this thesis.

Let $A_i : E \rightarrow 2^{E^*}$, $i = 1, \dots, N$, be monotone mappings. Many problems that arises in several branches of applied mathematics can be reduced to finding solutions of the generalized inclusion,

$$0 \in A_1 z + A_2 z + \dots + A_N z. \quad (1.1)$$

This inclusion problem is quite general as it includes variational inequality problems, equilibrium problems, complementary problems, minimization problems, fixed point problems as special cases. Indeed, many nonlinear problems arising in applied areas such as signal processing, image recovery, and machine learning can be mathematically modelled as (1.1) (see, e.g., [54, 120] and the references therein). To be more precise, it includes, the following minimization problem:

$$\min_{x \in E} \{f_1(x) + f_2(x) + \cdots + f_N(x)\}, \quad (1.2)$$

where f_i , $i = 1, 2, \dots, N$, are proper lower semicontinuous convex functions from E to the extended real line $\bar{R} := (-\infty, \infty]$. If, in (1.1), we assume that $A_i = \partial f_i$, for $i = 1, 2, \dots, N$, where ∂f is the subdifferential operator of f in the sense of convex analysis, then (1.2) is equivalent to (1.1). Consequently, considerable research efforts have been devoted to methods of finding approximate solutions (when they exist) of equations of the form (1.1) for a sum of two or more monotone mappings (see, e.g., [7, 47, 54, 83, 95, 106–108, 117]).

This problem is also connected with a split equality monotone inclusion problem of the sum of monotone mappings. The split equality problem was introduced in 2013 by A. Moudafi [72], and it has received much attention due to its application in various disciplines such as medical image reconstruction, game theory, decomposition methods for partial differential equations and radiation therapy treatment planning (see, e.g., [4, 5]).

On the other hand, many of the most important nonlinear problems in applied mathematics reduce to finding the fixed points of certain mappings. Fixed point theorems are applicable in convex optimization, variational analysis, control theory, differential inclusions, economics, graph theory and computer sciences. Consequently, the existence of fixed points and their approximations for nonlinear mappings have been studied by many researchers. A fundamental result in the theory of fixed points is the classical Banach contraction principle which was established by S. Banach in 1922 [8]. It was natural to try to extend such celebrated

result to semi-fixed points in the setting of Banach spaces. The concept of a semi-fixed point was introduced by H. Zegeye [123] and has been studied by many researchers (see, e.g., [35, 37, 64, 99]).

Moreover, fixed point problems are related to one of the very important class of inequality problems called variational inequality problems. This problem was first introduced by P. Hartman and G. Stampacchia [51] in 1966. Since then, it has been extended and generalized in several directions, using new and powerful methods, to study a wide class of problems in a unified and general framework. This general problem formulation covers many problems, such as optimization problems, complementary problems, fixed point problems, etc. These alternative equivalent formulations have played major roles in finding solutions of variational inequality problems using iterative algorithms. In recent years, much attention has been given to constructing efficient iterative schemes for treating approximate solutions of variational inequality problems (see, e.g., [15, 29, 52, 77, 79, 111, 121, 123, 124, 126] and the references therein). Several authors have also studied, iterative algorithms for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for Lipschitz monotone mapping (see, e.g., [58, 77, 89, 107, 108, 121, 125]).

Our purpose in this thesis is, therefore, to introduce and study iterative algorithms for approximating: zeros of the sum of maximal monotone mappings in Hilbert spaces and in Banach spaces more general than Hilbert spaces; solution of the split equality monotone inclusion problems for the sum of two maximal monotone mappings in Hilbert spaces; f -fixed points of nonlinear mappings in Banach spaces and common element of the set of f -fixed points and solutions of variational inequality problems in Banach spaces.

The rest of this thesis is organized as follows. Section 1.2 presents some preliminary definitions of spaces, convex functions, Bregman distance and nonlinear mappings. Section 1.3 to 1.6, respectively, presents statement of the problem,

research questions, objectives and scope of the study. In Chapter two, we give detail literature review on approximation methods of zeros of the sum of maximal monotone mappings, solutions of split equality problems, semi-fixed points of nonlinear mappings in Banach spaces, common element of fixed points and variational inequality problems.

Chapter three deals with the approximation of a zero of the sum of maximal monotone mappings in Hilbert spaces. In Section 3.2, Viscosity type Douglas-Rachford splitting algorithm is constructed and a strong convergence of the scheme to a zero of the sum of a finite family of maximal monotone mappings is obtained. Moreover, in Section 3.3, an algorithm which starts by reformulating (1.1) as the problem of locating a point in a certain extended solution set $S_e(A_1, A_2, \dots, A_m) \subset H \times H^m$ is introduced and its strong convergence to a zero of the sum of a finite family of maximal monotone mappings in Hilbert spaces is proved.

In Chapter four, we prove convergence theorems for approximating zeros of the sum of maximal monotone mappings in Banach spaces. In Section 4.2, the forward-backward scheme is studied and a strong convergence of the method to a zero of the sum of maximal monotone and Lipschitz continuous monotone mappings in real reflexive Banach spaces is obtained. In Section 4.3, a splitting algorithm method associated with a certain extended solution set $S_e(A_1, A_2, \dots, A_m) \subset E \times E^{*^m}$, which converges strongly to a zero of the sum of a finite family of maximal monotone mappings in Banach spaces is proved.

In Chapter five, an algorithm for approximating a solution of the split equality monotone inclusion problem for a sum of two maximal monotone mappings is constructed and a strong convergence of the sequence in a Hilbert space is proved.

Chapter six presents approximation of a common f -fixed point of nonlinear mappings. In this chapter, an algorithm for approximating a common f -fixed point

of f -pseudocontractive mappings is investigated and its strong convergence in reflexive Banach spaces is discussed.

Chapter seven deals with the approximation of a common point of the solution of f -fixed points and variational inequality problems. In this chapter, a strong convergence result to a common point of the set of f -fixed point problem and the set of solutions of variational inequality problem of Lipschitz monotone mappings in real reflexive Banach spaces is obtained.

Finally, in Chapter eight, the discussion, conclusion and highlights of the contributions of our study to the existing knowledge are presented. We also identify and discuss possible areas of future research.

1.2 Preliminaries

In this section, we shall give definitions of some basic terms and concepts, which will play a crucial role in the subsequent chapters.

1.2.1 Geometry of Banach Spaces

Throughout this thesis E will denote a real Banach space with its dual E^* and bidual E^{**} and H will represent a real Hilbert space.

We note that there exists a canonical injection $J : E \rightarrow E^{**}$ defined, for each $x \in E$, by

$$J(x) = \phi_x,$$

where $\phi_x : E^* \rightarrow \mathbb{R}$ is given by $\phi_x(f) = \langle f, x \rangle$, for each $f \in E^*$. Thus $\langle J(x), f \rangle = \langle f, x \rangle$, for each $f \in E^*$ (see, e.g., [60]).

Definition 1.2.1. Let E be a Banach space and $J : E \rightarrow E^{**}$ be the canonical injection from E into E^{**} . The space E is said to be reflexive if J is surjective, that is $J(E) = E^{**}$.

It is well known that any bounded sequence in a reflexive Banach space has a weakly convergent subsequence (see [103]). We remark that Hilbert spaces, ℓ_p and L_p spaces, for $1 < p < \infty$, are reflexive Banach spaces and neither of ℓ_1 , L_1 , ℓ_∞ and L_∞ spaces, are reflexive Banach spaces.

Definition 1.2.2. A Banach space E is said to be strictly convex if for all $x, y \in E$, we have

$$\|x\| = \|y\| = 1, \ x \neq y \Rightarrow \left\| \frac{x+y}{2} \right\| < 1. \quad (1.3)$$

Definition 1.2.3. A function $\delta_E : (0, 2] \rightarrow [0, 1]$ is said to be the modulus of convexity of E if

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| = 1 = \|y\|, \|x-y\| \geq \epsilon \right\}, \quad (1.4)$$

for every $\epsilon \in [0, 2]$.

Definition 1.2.4. A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for each $\epsilon \in (0, 2]$.

It is known that Hilbert spaces, L_p and l_p , for $1 < p < \infty$, are uniformly convex Banach spaces and every uniformly convex Banach space is strictly convex and reflexive (see, e.g., [1]).

Definition 1.2.5. For $p > 1$, a Banach space E is said to be p -uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for every $\epsilon \in [0, 2]$.

It is obvious that a p -uniformly convex Banach space is uniformly convex.

Definition 1.2.6. A function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S_E, \|y\| \leq t \right\}, \quad (1.5)$$

where $S_E = \{x \in E : \|x\| = 1\}$ is the unit sphere of E , is called modulus of smoothness of E .

Definition 1.2.7. For $q > 1$, a Banach space E is said to be q -uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q, t > 0$.

Definition 1.2.8. A Banach space E is said to be smooth if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}, \quad (1.6)$$

exists for all $x, y \in S_E$ and E is said to be uniformly smooth if the limit (1.6) converges uniformly for $x, y \in S_E$.

It is known that E is uniformly smooth if and only if E^* is uniformly convex (see, e.g., [36]).

Lemma 1.2.9. [6] Let $\frac{1}{p} + \frac{1}{q} = 1$ for $p > 1$ and $q > 1$. The Banach space E is q -uniformly smooth if and only if its dual E^* is p -uniformly convex.

1.2.2 Nonlinear Mappings

Definition 1.2.10. Let $A : E \rightarrow 2^{E^*}$ be a mapping.

- i) The domain of A denoted by $Dom(A)$ is defined by $Dom(A) = \{x \in E : Ax \neq \emptyset\}$.
- ii) The range of A denoted by $ran(A)$ is defined by $ran(A) = \{Ax : x \in Dom(A)\}$.
- iii) The graph of A denoted by $Gph(A)$ is defined by $Gph(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$.
- iv) The zero of A denoted by $A^{-1}(0)$ is defined by $A^{-1}(0) = \{x \in E : 0 \in Ax\}$.

Definition 1.2.11. A mapping $A : E \rightarrow 2^{E^*}$ is called monotone if for all $(x, x^*), (y, y^*) \in Gph(A)$, we have

$$\langle x^* - y^*, x - y \rangle \geq 0.$$

Definition 1.2.12. A monotone mapping $A : \text{Dom}(A) \subseteq E \rightarrow 2^{E^*}$ is called maximal monotone if $\text{Gph}(A)$ is not properly contained in the graph of any other monotone mapping, that is, A is maximal monotone if for $(x, y^*) \in E \times E^*$ such that $\langle x - y, Ax - y^* \rangle \geq 0$, for every $x \in \text{Dom}(A)$ implies that $y \in \text{Dom}(A)$ and $Ay = y^*$.

Definition 1.2.13. A mapping $A : E \rightarrow E^*$ is said to be:

- i) β -strongly monotone if there exists a real constant $\beta > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in E.$$

- ii) α -inverse strongly monotone if there exists a real constant $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in E.$$

Definition 1.2.14. The generalized duality mapping $J_p : E \rightarrow 2^{E^*}$ is defined by

$$J_p(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^p, \|y^*\| = \|x\|^{p-1}\}, \quad (1.7)$$

for all $x \in E$. From the Hahn-Banach theorem, it follows that $J_p x \neq \emptyset$, for each $x \in E$. It is well known that if E^* is strictly convex, then J_p is single-valued. If $p = 2$, then we write $J_2 = J$ and call it normalized duality mapping and $J_p x = \|x\|^{p-2} Jx$. If $E = H$, a real Hilbert space, then $J = I$, where I is the identity mapping on H .

Definition 1.2.15. The extended solution set of problem (1.1) is defined by:

$$S_e(A_1, \dots, A_m) = \{(z, w_1, \dots, w_m) \in V : z \in E, w_k \in A_k(z), k = 1, 2, \dots, m\},$$

where $V = \{(z, w_1, \dots, w_m) \in E \times E^{*m} : w_1 + \dots + w_m = 0\}$.

Now, we give the definitions of nonlinear mappings that are of interest to this study.

Definition 1.2.16. Let C be a nonempty subset of E . A mapping $T : C \rightarrow E$ is called Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in C. \quad (1.8)$$

- i) If $L < 1$, then T is Called a contraction mapping.
- ii) If $L = 1$, then T is called a nonexpansive mapping.

Definition 1.2.17. Let C be a nonempty, closed and convex subset of E . Let $T : C \rightarrow E$ be a mapping. A point $x \in C$ is called a fixed point of T if $Tx = x$. We denote the set of fixed point of T by $F(T)$.

Definition 1.2.18. Let C be a nonempty subset of E . A mapping $T : C \rightarrow E$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - Tp\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$.

Definition 1.2.19. Let C be a nonempty subset of E . A mapping $T : C \rightarrow E$ is called demiclosed at 0 if for any sequence $\{x_n\}$ in C which converges weakly to x and $\|(I - T)x_n\| \rightarrow 0$ implies $(I - T)x = 0$ (where I is the identity mapping on E).

Definition 1.2.20. Let C be a nonempty, closed and convex subset of E . Let $T : C \rightarrow E$ be a mapping. A point $x \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to x and $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. We denote the set of asymptotic fixed points of T by $\hat{F}(T)$.

Definition 1.2.21. [123] Let C be a nonempty, closed and convex subset of E . Let $T : C \rightarrow E$ be a mapping. A point $x \in C$ is called a semi-fixed point of T if $Jx = Tx$, where J is normalized duality mapping. We denote the set of semi-fixed points of T by $F_s(T)$, that is, $F_s(T) := \{x \in H : Jx = Tx\}$.

Definition 1.2.22. [123] A mapping $T : E \rightarrow E^*$ is called semi-pseudocontractive if for any $x, y \in E$, we have

$$\langle x - y, Tx - Ty \rangle \leq \langle x - y, Jx - Jy \rangle. \quad (1.9)$$

We observe that A is monotone if and only if $T := J - A$ is semi-pseudocontractive and thus a zero of A is a semi-fixed point of T .

Definition 1.2.23. [66] Let C be a nonempty, closed and convex subset of E . A mapping $T : C \rightarrow E$ is called relatively nonexpansive if

- i) $F(T) \neq \emptyset$;
- ii) $\phi(p, Tx) \leq \phi(p, x) \forall x \in C, p \in F(T)$, where

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$
;
- iii) $\widehat{F}(T) = F(T)$.

Definition 1.2.24. [39] Let C be a nonempty, closed and convex subset of E and let $T : C \rightarrow E^*$ be a mapping. A point $p \in C$ is called an asymptotic semi-fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n \rightarrow \infty} \|Tx_n - Jx_n\| = 0$. We denote the set of asymptotic semi-fixed point by $\widehat{F}_s(T)$.

Definition 1.2.25. [39] Let C be a nonempty, closed and convex subset of E . A mapping $T : C \rightarrow E^*$ is called relatively semi-nonexpansive if

- i) $F_s(T) \neq \emptyset$;
- ii) $\phi(p, J^{-1}Ty) \leq \phi(p, y) \forall y \in C, p \in \widehat{F}_s(T)$;
- iii) $\widehat{F}_s(T) = F_s(T)$.

1.2.3 Convex Functions

We give the following important definitions which are used in the subsequent chapters.

Let E be a real Banach space and $f : E \rightarrow (-\infty, \infty]$ be a proper convex and lower semicontinuous function. We denote the domain of f by $Dom f = \{x \in E : f(x) < \infty\}$.

Definition 1.2.26. Let $f : E \rightarrow (-\infty, +\infty]$ be a function. Then the conjugate of f , denoted by f^* , is the function $f^* : E^* \rightarrow (-\infty, +\infty]$ defined by $f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}$, for $x^* \in E^*$. The right-hand derivative of f at $x \in \text{int dom } f$ in the direction of y is defined by

$$f'(x, y) = \lim_{t \rightarrow 0^+} \frac{f(x + ty) - f(x)}{t}. \quad (1.10)$$

Definition 1.2.27. [122] Let C be a convex subset of a vector space E and $f : C \rightarrow (-\infty, +\infty]$ be a mapping. Then,

i) f is called proper if there exists at least one $x \in C$ such that

$$f(x) \neq +\infty;$$

ii) f is convex if for all $x, y \in E$ and $t \in [0, 1]$, we have

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y);$$

iii) f is uniformly convex function with modulus ϕ , if for each x, y in its domain and $t \in [0, 1]$, we have

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - t(1 - t)\phi(\|x - y\|),$$

where ϕ is a function that is increasing and vanishes only at 0;

iv) $f : \text{Dom } f \rightarrow (-\infty, +\infty]$ is lower semicontinuous at $x \in \text{Dom } f$ if

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n),$$

for each sequence $\{x_n\}$ in $\text{Dom } f$ such that $\lim_{n \rightarrow \infty} x_n = x$;

v) f is called Gâteaux differentiable at $x \in \text{int dom } f$ if for any $y \in E$, we have

$$\lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t}, \quad (1.11)$$

exists. In this case, $f'(x, y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x ;

vi) f is Gâteaux differentiable if it is Gâteaux differentiable at every point $x \in \text{int dom } f$;

vii) f is Fréchet differentiable at x if the limit in (1.11) is attained uniformly for $\|y\| = 1$;

viii) f is uniformly Fréchet differentiable on a subset C of E , if the limit in (1.11) is attained uniformly for $x \in C$ and $\|y\| = 1$;

ix) f is strongly coercive if $\lim_{\|x\| \rightarrow \infty} \frac{f(x)}{\|x\|} = +\infty$.

Definition 1.2.28. The function $\rho_r : [0, \infty) \rightarrow [0, \infty)$ given by

$$\rho_r(t) = \inf_{x, y \in B_r, \|x-y\|=t, \lambda \in (0,1)} \frac{\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y)}{\lambda(1-\lambda)},$$

where $B_r = \{x \in E : \|x\| \leq r\}$ for all $r > 0$ is called the gauge of uniform convexity of f .

Definition 1.2.29. Let $f : E \rightarrow (-\infty, +\infty]$ be a convex function. The subdifferential of f at $x \in E$ is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle, \quad \forall y \in E\}. \quad (1.12)$$

Note that when the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$ (see, e.g., [82]).

Definition 1.2.30. A function $f : E \rightarrow (-\infty, +\infty]$ is said to be Legendre if it satisfies the following conditions:

- i) the interior of the domain of f , $\text{int dom } f$, is nonempty, f is Gâteaux differentiable and $\text{dom } \nabla f = \text{int dom } f$;
- ii) the interior of the domain of f^* , $\text{int dom } f^*$, is nonempty, f^* is Gâteaux differentiable and $\text{dom } \nabla f^* = \text{int dom } f^*$.

Note that if $f : E \rightarrow (-\infty, +\infty]$ is a Legendre function and E is a reflexive Banach space, then $\nabla f^* = (\nabla f)^{-1}$ (see, e.g., [21]).

Remark 1.2.31. If E is in addition, a smooth and strictly convex Banach space, then examples of Legendre functions are $f(x) = \frac{1}{p}\|x\|^p$ ($1 < p < \infty$) with the conjugate functions $f^*(x^*) = \frac{1}{q}\|x^*\|^q$ ($1 < q < \infty$), (see, e.g., [9, 10]), where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, the gradient ∇f coincides with the generalized duality mapping, J_p , of E , that is, $\nabla f = J_p$.

Definition 1.2.32. [18] A Gâteaux differentiable function f is called strongly convex with constant $\mu > 0$ if the following inequality holds:

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2, \quad (1.13)$$

for all points x, y in its domain, or equivalently (see, e.g., [78])

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|^2. \quad (1.14)$$

Remark 1.2.33. Note that a strongly convex function is uniformly convex with $\phi(t) = \frac{\mu}{2}t^2$ and hence the class of uniformly convex functions contains the class of strongly convex functions.

Example 1.2.34. Let E in addition, a smooth and strictly convex Banach space, then the function $f(x) = \frac{1}{2}\|x\|^2$ is strongly convex function with parameter $0 < \mu \leq 1$.

1.2.4 Bregman Distance and Some Related Notions

Definition 1.2.35. Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable convex function. The function $D_f : \text{dom } f \times \text{int dom } f \rightarrow [0, \infty)$ defined by

$$D_f(y, x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle, \quad x, y \in E \quad (1.15)$$

is called the Bregman distance with respect to f (see, e.g., [30]).

We remark that the Bregman distance is not a distance in the usual sense because it is not symmetric and does not satisfy the triangle inequality. However, it has the following important properties (see, e.g., [28, 90, 92]):

- i) $D_f(y, x) \geq 0 \ \forall x, y \in E$ and $D_f(x, y) = 0$ if and only if $x = y$.
- ii) Three point identity: for any $x \in \text{dom } f$ and $y, z \in \text{int } \text{dom } f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle. \quad (1.16)$$

- iii) Four point identity: for any $y, w \in \text{dom } f$ and $x, z \in \text{int } \text{dom } f$,

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle. \quad (1.17)$$

In addition, if $f : E \rightarrow \mathbb{R}$ is a Gâteaux differentiable, convex and lower semicontinuous function, then we have (see, e.g., [92])

$$D_f(x, y) + D_f(y, x) = \langle x - y, \nabla f(x) - \nabla f(y) \rangle, \ \forall x, y \in E. \quad (1.18)$$

Let $f : E \rightarrow \mathbb{R}$ be a Legendre function. Let $V_f : E \times E^* \rightarrow [0, +\infty)$, associated with f , be defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + \nabla f^*(x^*), \ \forall x \in E, x^* \in E^*. \quad (1.19)$$

Then, V_f is nonnegative function which satisfies

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \ \forall x \in E, x^* \in E^*, \quad (1.20)$$

and

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*), \ \forall x \in E, x^*, y^* \in E^*, \quad (1.21)$$

(see, e.g., [96]).

Definition 1.2.36. Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable convex function. The Bregman projection with respect to f on $x \in \text{int dom } f$ onto a nonempty, closed and convex subset $C \subset \text{int dom } f$ is denoted by $P_C^f(x) \in C$ and satisfies

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}. \quad (1.22)$$

Remark 1.2.37. Some of the special cases of the Bregman projection are the following:

- i) If, in (1.22), C is a subset of a reflexive Banach space E and $f(x) = \|x\|^2$, then the Bregman projection $P_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$, which is defined by

$$\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x).$$

- ii) If, in (1.22), E is a Hilbert space and $f(x) = \|x\|^2$, then the Bregman projection $P_C^f(x)$ reduces to the metric projection of x onto C .

Definition 1.2.38. Let C be a nonempty, closed and convex subset of E . A mapping $T : C \rightarrow C$ with $\widehat{F}(T) \neq \emptyset$ is called

- i) Bregman strongly nonexpansive ([20, 90]) with respect to $\widehat{F}(T)$, if

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in \widehat{F}(T)$$

and, if whenever $\{x_n\} \subseteq C$ is bounded, $p \in \widehat{F}(T)$, and

$$\lim_{n \rightarrow \infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

then

$$\lim_{n \rightarrow \infty} D_f(x_n, Tx_n) = 0.$$

ii) Bregman firmly nonexpansive (see, e.g., [91]) if for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle.$$

Note that if T is a Bregman firmly nonexpansive mapping and f is a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E , then it is known that $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex (see [91]). It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Definition 1.2.39. Let $f : E \rightarrow (-\infty, +\infty]$ be a Gâteaux differentiable function. The Modulus of total convexity of f at $x \in \text{int dom } f$ is the function $\nu_f(.,.) : E \times \mathbb{R}^+ \rightarrow \mathbb{R}$ defined as

$$\nu_f(x, t) = \inf_{\{y \in E : \|x - y\| = t\}} D_f(y, x). \quad (1.23)$$

The mapping f is called totally convex if $\nu_f(x, t) > 0$ for all $x \in E$ and $t > 0$.

Remark 1.2.40. The function $f : E \rightarrow (-\infty, +\infty]$ is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets of E (see [25], Theorem 2.10, page 9).

1.3 Statement of the Problem

In this section, we discuss the existing literature on approximation of solutions of some inclusion problems and indicate the open problems.

Let $A, B : E \rightarrow 2^{E^*}$ be monotone mappings. Consider the problem of finding:

$$x^* \in E \text{ such that } 0 \in Ax^* + Bx^*. \quad (1.24)$$

This problem has been studied by many authors in Hilbert spaces (see, e.g., [12, 43, 44, 75, 80] and the references therein). In 2008, Eckstein and Svaiter

[47] constructed a new approach to splitting algorithms which starts by reformulating (1.24) as the problem of locating a point in a certain extended solution set $S_e(A, B) \subset H \times H$ and proved weak convergence results provided that H has finite dimension or $A + B$ is maximal monotone mapping.

In connection with the strong convergence, several authors have studied different iterative schemes (see, e.g., [83, 106–108, 117] and the references therein) for approximating a zero of the sum of monotone mappings A and B in Hilbert spaces under certain restrictions.

In 2012, Takahashi *et al.* [106] proved a strong convergence theorem of the Halpern-type iteration in a Hilbert space H , which is defined by the following manner: for any $x_1 \in H$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J_{r_n B}(x_n - r_n A x_n)), \forall n \geq 1, \quad (1.25)$$

where, $J_{rB} = (I + rB)^{-1}$, $u \in H$ is a fixed vector and A is an α -inverse strongly monotone mapping on H and B is a maximal monotone mapping on H . Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.25) converges strongly to a zero point of $A + B$. Several researchers have also studied and obtained similar results in Hilbert (see, e.g., [14, 85, 97]). We observe that the strong convergence results available for the zero of the sum of monotone mappings A and B , are either one of the mappings is α -inverse strongly monotone or single-valued.

For results in Banach spaces more general than Hilbert spaces, Iusem *et al.* [53] studied a splitting algorithm which approximates a solution via the extended solution set $S_e(A, B) \subset E \times E^*$ and proved weak convergence results of their scheme to a solution of problem (1.24) in reflexive Banach spaces.

More recently, Kimura and Nakajo [56] studied the following algorithm in uniformly smooth and 2-uniformly convex Banach spaces: for any $x_1 \in E$, let

$$x_{n+1} = \Pi_C J_{\lambda_n A}(\gamma_n J u + (1 - \gamma_n) J x_n - \lambda_n B x_n), \quad n \geq 1, \quad (1.26)$$

where $u \in E$, Π_C is the generalized projection of E onto C , $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences in $\mathbb{R}$ satisfying certain conditions, A is a maximal monotone mapping in $E \times E^*$, B is an α -inverse strongly monotone mapping from C into E^* and $J : E \rightarrow E^*$ is the normalized duality mapping. Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.26) converges strongly to a zero point of $A + B$. We remark that the convergence is strong, unlike the result of Iusem *et al.* [53], which is more convenient in applications. However, the result is restricted to uniformly smooth and 2-uniformly convex Banach spaces and B is α -inverse strongly monotone and single valued.

With regard to the approximation of a zero of the sum of a finite family of maximal monotone mappings, Svaiter [100] introduced a splitting algorithm method associated with a certain extended solution set $S_e(A_1, A_2, \dots, A_m) \subseteq H \times H^m$ which approximates a solution of the sum of a finite family of maximal monotone mappings and proved weak convergence results provided that H has finite dimension or $A_1 + A_2 + \dots + A_m$ is a maximal monotone mapping. However, obtaining a scheme which converges strongly to a zero of the sum of a finite family of maximal monotone mappings in Hilbert spaces and Banach spaces more general than Hilbert spaces is still an open problem.

On the other hand, Moudafi [72] introduced the split equality problem (SEP) which can be mathematically formulated as the problem of finding x and y with the property:

$$x \in C, y \in Q, \text{ such that } Ax = By. \quad (1.27)$$

Several authors have studied SEP with $C = F(T)$ and $Q = F(S)$, where $F(T)$ and $F(S)$ denote the sets of fixed points of T and S , respectively, for T and S nonexpansive mappings or their generalizations in Banach spaces (see, e.g., [34, 73]).

In 2015, Chang *et al.* [34] introduced the following iterative algorithm for the class of quasi-pseudo contractive mappings

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)[(1 - \xi_n)I + \xi_n T[(1 - \eta_n)I + \eta_n T]]u_n, \\ v_n = y_n - \gamma_n B^*(By_n - Ax_n), \\ y_{n+1} = \alpha_n y_n + (1 - \alpha_n)[(1 - \xi_n)I + \xi_n S[(1 - \eta_n)I + \eta_n S]]v_n, \quad n \geq 0, \end{cases} \quad (1.28)$$

and proved that under certain assumptions the algorithm above converges weakly to a solution of the split equality fixed point problem. In addition they showed that the strong convergence is guaranteed if the mappings T and S are semi-compact. However, it is worth to mention that, as far as we know, there is no convergent result available to a solution of the split equality monotone inclusion problems for the sum of two maximal monotone mappings in Hilbert spaces.

With regard to the approximation of semi-fixed points of nonlinear mappings in Banach spaces, many researchers have established and studied different algorithms (see, e.g., [35, 37, 64, 99, 123] and the references therein).

Recently, Chidume and Idu [37] studied the following iterative process for a semi-fixed point of a bounded semi-pseudocontractive mapping T from E into E^* :

$$x_{n+1} = J^{-1}[(1 - \lambda_n)Jx_n + \lambda_n Tx_n - \lambda_n \theta_n(Jx_n - Ju)], n \geq 1, \quad (1.29)$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying appropriate conditions. They proved that under certain assumptions, the algorithm in (1.29) converges to a semi-fixed point of T in p -uniformly convex and q -uniformly smooth real Banach spaces. Whether this can be extended to a more general class of mappings

is an open problem.

On the other hand, with regard to a common point of solutions of fixed points of certain nonlinear mappings and variational inequality problems, several authors have studied algorithms for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for Lipschitz monotone mappings (see, e.g., [58, 77, 89, 107, 108, 121, 125] and the references therein).

Recently, Tufa and Zegeye [113] studied an algorithm for approximating a common point of solutions of fixed point problem of a relatively nonexpansive mapping $T : C \rightarrow E$ and variational inequality problems for Lipschitz monotone mappings $A_1, A_2 : C \rightarrow E^*$ in uniformly smooth and 2-uniformly convex real Banach spaces. They proved that the sequence generated by: for $x_0, u \in C$

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 x_n), \\ y_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 x_n), \\ w_n = a_n Jx_n + b_n JT x_n + c_n J(u_n) + d_n J(v_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1 - \alpha_n)w_n), \quad n \geq 0, \end{cases} \quad (1.30)$$

where, J is the normalized duality mapping, $u_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 y_n)$, $v_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 z_n)$, $\{\gamma_n\} \subset [a, b] \subset (0, \frac{1}{\mu L})$, for $L = \max\{L_1, L_2\}$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subset [e, 1] \subset (0, 1)$ such that $a_n + b_n + c_n + d_n = 1$, converges strongly to $x^* \in F(T) \cap VI(C, A_1) \cap VI(C, A_2)$. We remark that the convergence is restricted to a uniformly smooth and 2-uniformly convex real Banach spaces.

1.4 Research Questions

The aforementioned research gaps lead us to pose the following research questions.

- 1) Can we establish strong convergence theorems for approximating zeros of the sum of maximal monotone mappings in Hilbert spaces and Banach spaces more general than Hilbert spaces?
- 2) Can an iterative algorithm be introduced which converges strongly to a solution of the split equality monotone inclusion problem for the sum of two maximal monotone mappings in Hilbert spaces?
- 3) Can we extend the class of semi-pseudocontractive mappings to a more general class of mappings with a more general notion of fixed points?
- 4) Is it possible to construct an algorithm which converges strongly to a common point of the set of fixed points of a mapping more general than Bregman relatively nonexpansive mapping and the set of solutions of variational inequality problems for Lipschitz monotone mappings in spaces more general than uniformly smooth and 2-uniformly convex real Banach spaces?

1.5 Objectives

The main objectives of this thesis are to:

- 1) combine Douglas-Rachford and Viscosity methods and construct an iterative scheme which converges strongly to a zero of the sum of a finite family of maximal monotone mappings under suitable conditions in Hilbert spaces,
- 2) construct a viscosity type algorithm for finding a zero of the sum of a finite family of maximal monotone mappings via the extended solution set $S_e(A_1, A_2, \dots, A_m) \subset H \times H^m$ and discuss its strong convergence in Hilbert spaces,
- 3) develop a forward-backward type algorithm which approximates a zero of the sum of maximal monotone and Lipschitz continuous monotone mappings in real reflexive Banach spaces,

- 4) establish Halpern type algorithms for approximating zeros of the sum of a finite family of maximal monotone mappings via the extended solution set and discuss their strong convergence in real reflexive Banach spaces,
- 5) introduce an iterative scheme for approximating a solution of the split equality monotone inclusion problem for the sum of two maximal monotone mappings in Hilbert spaces and discuss its convergence,
- 6) introduce and develop a new class of mappings called f -pseudocontractive mappings and propose a scheme that estimates a common f -fixed point of those mappings and discuss its strong convergence in reflexive Banach spaces,
- 7) establish the concept of a Bregman relatively f -nonexpansive mapping and construct an algorithm for approximating a common element of the set of f -fixed points of Bregman relatively f -nonexpansive mapping and the set of solutions of variational inequality problems for Lipschitz monotone mappings in the setting of real reflexive Banach spaces.

Chapter 2

Review of Literature

2.1 Review of Literature

In this chapter, we discuss literature review of some past and recent results related to our work.

2.2 Approximation of Zeros of the Sum of Maximal Monotone Mappings

In this section, we present some research findings that have been obtained in approximating a zero of the sum of maximal monotone mappings.

For approximating a solution of problem (1.24) several authors have studied different iterative schemes in the setting of Hilbert spaces (see, e.g., [43, 44, 75, 80, 84, 112] and the references therein).

In 1979, Passty [80] introduced the forward-backward splitting method which defines a sequence $\{x_n\}$ by

$$x_{n+1} = (I + r_n B)^{-1}(I - r_n A)x_n, n \geq 1, \quad (2.1)$$

where x_1 is chosen arbitrarily in H , $\{r_n\}$ is a sequence of positive numbers, A and B are maximal monotone mappings with $Dom(A) \subset Dom(B)$, and A is single valued. This method is essentially a generalization of the classical gradient method for constrained convex optimization and monotone variational inequalities, and therefor inherited restrictions similar to those methods such as A is single valued. In general, this method provides weak convergence even with the restriction that A being single-valued. It fails to provide strong convergence results to the zero of the sum of maximal monotone mappings.

In 1979, Lions and Mercier [63] introduced the Peaceman-Rachford splitting method whose $(n + 1)^{th}$ iteration is given by

$$\begin{aligned} y_n &= (2(I + \lambda B)^{-1} - I)x_n, \\ z_n &= (2(I + \lambda A)^{-1} - I)y_n, \\ x_{n+1} &= (1 - \rho_n)x_n + \rho_n z_n, n \geq 1, \end{aligned} \quad (2.2)$$

where x_1 is chosen arbitrarily in H , $\lambda > 0$ is a fixed scalar, and $\{\rho_n\} \subset (0, 1]$ is a sequence of relaxation parameters. They proved weak convergence of the sequence $\{x_n\}$ to the solution of problem (1.24) under certain conditions.

In 2003, Moudafi and Oliny [71] proposed the following inertial proximal point algorithm for finding zeros of the sum of two monotone mappings in real Hilbert space H : for any $x_1 \in H$,

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \gamma_n B)^{-1}(y_n - \gamma_n A x_n), n \geq 1, \end{cases} \quad (2.3)$$

where $A : H \rightarrow H$ is a single valued α -inverse strongly monotone and $B : H \rightarrow 2^H$ is a maximal monotone mappings. They obtained weak convergence theorem provided that $\gamma_n < 2/L$ where, L is the Lipschitz constant of A and $\{\theta_n\} \subset [0, 1)$ with $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$.

In 2015, Lorenz and Pock [62] proposed the following inertial forward-backward algorithm for monotone mappings which is basically a modification of the forward-backward splitting algorithm in Hilbert spaces: For any $x_1 \in H$,

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \gamma_n B)^{-1}(y_n - \gamma_n A y_n), n \geq 1, \end{cases} \quad (2.4)$$

where $\{\gamma_n\}$ is a positive real sequence, $A : H \rightarrow H$ and $B : H \rightarrow 2^H$ are α -inverse strongly monotone and maximal monotone mappings, respectively. They obtained weak convergence of the sequence $\{x_n\}$ to a point in $(A + B)^{-1}(0)$ under certain appropriate conditions.

The other well known algorithm for solving problem (1.24) is the so-called Douglas-Rachford splitting method given by $x_1 \in H$ and

$$x_{n+1} = 2J_{\gamma}^A(2J_{\gamma}^B - I)x_n + (I - 2J_{\gamma}^B)x_n, n \geq 1, \quad (2.5)$$

where $J_{\gamma}^B = (I + \gamma B)^{-1}$ for some $\gamma \in (0, 1)$. This method was initially proposed in [63] for finding a zero of the sum of two maximal monotone mappings and has been well studied from various perspectives in the literature (see, e.g., [7, 42, 54, 63, 80]).

In [42], Combettes established weak convergence of the asymptotic behavior of the Douglas-Rachford splitting-type algorithm to a zero of the sum of two monotone mappings in real Hilbert spaces: For any $x_1 \in H$,

$$\begin{cases} y_n = J_{\gamma}^B z_n + b_n, \\ z_{n+1} = z_n + \lambda_n(J_{\gamma}^A(2y_n - z_n) + a_n - y_n), n \geq 1, \end{cases} \quad (2.6)$$

where $A : H \rightarrow 2^H$ and $B : H \rightarrow 2^H$ are maximal monotone mappings and $\{a_n\}$ and $\{b_n\}$ are sequences in H satisfying certain conditions. He proved that $\{y_n\}$ converges weakly to a point in $(A + B)^{-1}(0)$ under certain conditions imposed on $\{\lambda_n\}$, $\{a_n\}$ and $\{b_n\}$.

In 2018, in order to solve problem (1.24), Svaiter [101] studied the following Algorithm: For any $\lambda > 0, x_0, b_0 \in H$,

Step 1. Compute, y_n, a_n such that $a_n \in A(y_n), \lambda a_n + y_n = x_{n-1} - \lambda b_{n-1}$,

Step 2. Compute, x_n, b_n such that $b_n \in B(x_n), \lambda b_n + x_n = y_n + \lambda b_{n-1}, n \geq 1$.

They proved that $\{(x_n, b_n)\}$ and $\{(y_n, -a_n)\}$ converge weakly to a point (z, w) in $S_e(A, B)$, where $z \in (A + B)^{-1}(0)$.

We remark that the algorithms mentioned above are weakly convergent. However, in some applied disciplines, the norm convergence is more desirable than the weak convergence [14].

With regard to strong convergence, several authors have studied different iterative schemes (see, e.g., [14, 22, 83, 98, 106–108, 117] and the references therein) to approximate the zero of the sum of monotone mappings A and B , where $A : H \rightarrow H$ is an α -inverse strongly monotone mapping and $B : H \rightarrow 2^H$ is a maximal monotone mapping under certain conditions.

Recently, Dadashi and Postolache [45] studied the following forward-backward splitting algorithm in a Hilbert space setting: For any $x_1 \in H$,

$$\begin{cases} w_n = J_\gamma^B(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + \beta_n w_n + \gamma_n e_n, \quad n \geq 1, \end{cases} \quad (2.7)$$

where A is an α -inverse strongly monotone and single valued mapping on H , B is a maximal monotone mapping of H into 2^H , e_n is an error vector, $\alpha_n, \beta_n, \gamma_n \in (0, 1)$

such that $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$. They proved that the sequence $\{x_n\}$ given by (2.7) converges strongly to a point $x \in (A + B)^{-1}(0)$ provided that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy appropriate conditions.

Concerning the approximation of the solution of problem (1.24) in Banach spaces more general than Hilbert spaces, several authors have studied different iterative schemes (see, e.g., [53, 56, 97] and the references therein).

More recently, Shehu [97] established a strong convergence theorem of forward-backward splitting algorithm in uniformly smooth and 2-uniformly convex real Banach spaces to approximate a solution of problem (1.24), where B is maximal monotone mapping and A is a Lipschitz continuous monotone mapping.

Several researchers have also studied and obtained similar results in Hilbert and Banach spaces more general than Hilbert spaces (see, e.g., [61, 85, 97]). However, we observe that the strong convergence results available for the zero of the sum of monotone mappings A and B , are only when either one of the mappings is α -inverse strongly monotone or single-valued.

From the above discussions, it is natural to ask the following general questions. Can we have strong convergence results for approximating:

- (1) zeros of the sum of two maximal monotone mappings without the assumption that one of the mappings is single valued and/or α -inverse strongly monotone?
- (2) zeros of the sum of a finite family of maximal monotone mappings via the extended solution set of the sum of maximal monotone mappings?

Answers to these questions are given in Chapter three and Chapter four in the affirmative.

2.3 Solutions of Split Equality Monotone Inclusion Problem for the Sum of Maximal Monotone Mappings

In this section, we present some research findings that have been obtained for approximating solutions of the split equality monotone inclusion problems.

Let H_1 and H_2 be real Hilbert spaces. Let $C \subset H_1$ and $Q \subset H_2$ be two nonempty, closed and convex subsets; let $F_1 : H_1 \rightarrow H_1$ and $F_2 : H_2 \rightarrow H_2$ be two given mappings and $A : H_1 \rightarrow H_2$ be a bounded linear mapping. The split variational inequality problem (SVIP) introduced by Censor, Gibali and Reich [31] can mathematically be formulated as the problem of finding:

$$\begin{cases} x^* \in C \text{ such that } \langle F_1(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C, \\ \text{and} \\ y^* = Ax^* \in Q \text{ solves } \langle F_2(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q. \end{cases} \quad (2.8)$$

The SVIP is quite general and permits split minimization between two spaces so that the image of a minimizer of a given function, under a bounded linear operator, is a minimizer of another function. Other special cases are, for instance, split zero problems (SZP) and the split feasibility problem (SFP), which had already been used in practice as a model in the intensity-modulation radiation therapy (IMRT) treatment planning (see, e.g., [32]).

For solving the SVIP problem given in (2.8), Censor, Gibali and Reich [31] proposed the following algorithm: For $\lambda > 0$ and $x_0 \in H_1$, let

$$x_{n+1} = U(x_n + \gamma A^*(T - I)Ax_n), \quad (2.9)$$

where $\gamma \in (0, \frac{1}{\|A\|^2})$, A^* is the adjoint operator of A , $T = P_Q(I - \lambda F_2)$, $U = P_C(I - \lambda F_1)$, where P_Q and P_C are metric projections of H onto Q and C , respectively. They proved weak convergence of the sequence (2.9) to the solution of problem (2.8) under certain conditions provided that F_1 and F_2 are α -inverse strongly monotone mappings.

In 2011, Moudafi [74] introduced the following split monotone inclusion problem (SMIP) given by

$$\begin{cases} x^* \in H_1 \text{ such that } 0 \in F_1(x^*) + G_1(x^*), \\ \text{and} \\ y^* = Ax^* \in H_2 \text{ solves } 0 \in F_2(y^*) + G_2(y^*); \end{cases} \quad (2.10)$$

where $G_1 : H_1 \rightarrow 2^{H_1}$, $G_2 : H_2 \rightarrow 2^{H_2}$ are multi-valued maximal monotone mappings, $F_1 : H_1 \rightarrow H_1$ and $F_2 : H_2 \rightarrow H_2$ are single valued and α -inverse strongly monotone mappings, and $A : H_1 \rightarrow H_2$ is a bounded linear operator. It is worth mentioning that by taking $G_1 = N_C, G_2 = N_Q$ normal cones of the closed and convex sets C and Q , respectively, we recover the split variational inequality problem (2.8).

For solving the SMIP, Moudafi [73] introduced the following algorithm: For $\lambda > 0$ and $x_0 \in H_1$, let

$$x_{n+1} = U(x_n + \gamma A^*(T - I)Ax_n), \quad (2.11)$$

where $\gamma \in (0, \frac{1}{\|A\|^2})$, A^* is the adjoint operator of A , $T = J_{\lambda G_2}(I - \lambda F_2)$ and $U = J_{\lambda G_1}(I - \lambda F_1)$, with $\lambda \in (0, 2\alpha)$, for $\alpha := \min\{\alpha_1, \alpha_2\}$. Under suitable conditions, some weak convergence theorems are proved which can be seen as an important generalization of an iterative method given by Censor *et al.* [31] for split variational inequality problem, provided that the solution set of the problem is nonempty.

Let H_1, H_2 and H_3 be real Hilbert spaces. Let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed and convex subsets; let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be bounded linear mappings. Moudafi [72] introduced the following split equality problem (SEP) which can be mathematically formulated as the problem of finding x and y with the property:

$$x \in C, y \in Q, \text{ such that } Ax = By. \quad (2.12)$$

This kind of split equality problem allows asymmetric and partial relations between the variables x and y . If, in (2.12), we consider $H_2 = H_3$ and $B = I$, then the split equality problem (2.12) reduces to the split feasibility problem (originally introduced in Censor and Elfving [33]) which is to find:

$$x \in C \text{ such that } Ax \in Q.$$

Several authors have studied split equality problem with $C = F(T)$ and $Q = F(S)$, where $F(T)$ and $F(S)$ denote the sets of fixed points of T and S , respectively, for T and S nonexpansive mappings or their generalizations in Banach spaces (see, e.g., [73]).

Now, we consider the following inclusion problem: Let $F_1, G_1 : H_1 \rightarrow 2^{H_1}$ and $F_2, G_2 : H_2 \rightarrow 2^{H_2}$ be maximal monotone mappings. Let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be bounded linear mappings with adjoints A^* and B^* , respectively. The split equality monotone inclusion problem (SEMIP) for the sum of two maximal monotone mappings can mathematically be formulated as the problem of finding $x^* \in H_1$ and $y^* \in H_2$ such that

$$0 \in F_1 x^* + G_1 x^* \text{ and } 0 \in F_2 y^* + G_2 y^* \text{ with } Ax^* = By^*. \quad (2.13)$$

If, in (2.13), we take $H_2 = H_3$ and $B = I$, then we recover the split monotone inclusion problem (SMIP) given in (2.10). If, in (2.13), we take $C = \{x^* : 0 \in F_1 x^* + G_1 x^*\}$ and $Q = \{y^* : 0 \in F_2 y^* + G_2 y^*\}$, then we recover the split equality

problem (2.12).

Motivated by the above results, one of our aims in this thesis is to introduce an algorithm for finding a point which is a solution of the split equality inclusion problem for the sum of maximal monotone mappings in Hilbert spaces and discuss its convergence.

2.4 Fixed Points of Nonlinear Mappings in Banach Spaces

In this section, we present some research findings on iterative methods for approximating fixed point of pseudocontractive mappings and semi-fixed points of semi-pseudocontractive mappings in Banach spaces.

It was Picard who first applied the technique of successive approximations to solve certain problems which can be formulated as fixed point problems of nonlinear mappings. This method of Picard was precisely formulated by Banach [8] in 1922. This fundamental result is stated as follows.

Theorem 2.4.1. [8] *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a contraction mapping. Then T has a unique fixed point in X . Moreover, the Picard scheme, $x_{n+1} = Tx_n$, $n \geq 0$, converges to a unique fixed point of T .*

Theorem 2.4.1 is referred to as the “Contraction Mapping Principle” and it is the most known and very important tool in the theory of fixed point problems.

The generalization of this principle has been obtained either by relaxing the property on the mapping or by generalizing the domain of the mapping (see, e.g., [3, 16, 17, 23, 48, 49]).

One of the generalizations of the Banach contraction principle is the fixed point

theorems for pseudocontractive mappings. The concept of pseudocontractive mappings introduced by Browder [24] in 1967 and has been studied by many researchers (see, e.g., [2, 37, 40, 46, 94] and the references therein).

Pseudocontractive mappings in Hilbert spaces are related with the important class of nonlinear monotone mappings. We note that T is pseudocontractive if and only if $A = I - T$, where I is the identity mapping on H , is monotone and hence the set of fixed points of T is the set of zero points of A . Consequently, considerable research efforts have been devoted, especially within the past 30 years or so, to methods of finding approximate solutions (when they exist) of equation $Ax = 0$, where A is monotone (see, e.g., [2, 94]) or fixed points of pseudocontractive mappings (see, e.g., [40, 46, 123]).

In 2008, Zegeye [123] introduced the class of mapping called semi-pseudocontractive and the notion of semi-fixed points in Banach spaces which are extensions of the class of pseudocontractive mappings and the notion of fixed points in Hilbert spaces, respectively. In 2012, Cheng et al. [35], Liu [64], and Su and Xu [99] studied and called duality fixed points. The notion was later, in 2016, studied and called J -fixed point by Chidume and Idu [37].

Interest in semi-pseudocontractive mappings stems mainly from their firm connection with the important class of nonlinear monotone mappings in Banach spaces more general than Hilbert spaces. We note that T is semi-pseudocontractive if and only if $A = J - T$, where J is the normalized duality mapping, is monotone and hence a semi-fixed point of T is a zero of A .

These broad applications caught the attention of many researchers, and eventually led to various research output for the methods of finding a semi-fixed point of a semi-pseudocontractive or a zero of a monotone mapping (see, e.g., [35, 37, 64, 99, 123]).

In 2008, Zegeye [123], studied a Halpern-type approximation method for finding a semi-fixed point of a Lipschitz semi-pseudocontractive mapping $T : E \rightarrow E^*$ in a uniformly smooth and 2-uniformly convex real Banach space. He proved that under certain assumptions, the sequence $\{x_n\}$ generated by

$$x_{n+1} = J^{-1}[\beta_n Ju + (1 - \beta_n)((1 - \alpha_n)Jx_n + \alpha_n Tx_n)], \quad n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ satisfying certain conditions, converges strongly to $x^* \in F_s(T)$, where $F_s(T)$ is the set of semi-fixed points of T , under some mild conditions.

In 2012, Su and Xu [99], also studied an algorithm for approximating a semi-fixed point of a contractive type Lipschitz mapping $T : E \rightarrow E^*$ in 2-uniformly convex and uniformly smooth real Banach spaces. They proved that under certain assumptions, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)J^{-1}Tx_n, \quad n \geq 1,$$

where $0 < \frac{2L}{c^2} \leq 1$, for some constant $c > 0$, $\{\alpha_n\}$ is real sequences in $(0, 1)$ satisfying certain conditions, converges strongly to $x^* \in F_s(T)$.

In 2012, Liu [64] proposed the following iterative process for approximating a semi-fixed point of a γ -strictly semi-pseudocontractive mapping T from E into

E^* :

$$\left\{ \begin{array}{l} x_0 \in E, \text{ chosen arbitrarily,} \\ y_n = J^{-1}((1 - \lambda_n)Jx_n + \lambda_nTx_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ C_0 = \{z \in E : \phi(z, y_0) \leq \phi(z, x_0)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ Q_0 = E, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \quad n \geq 1, \end{array} \right. \quad (2.14)$$

where $\{\lambda_n\}$ is sequence in $(0, 1)$ satisfying appropriate conditions. He proved that the sequence $\{x_n\}$ generated by (2.14) converges to a semi-fixed point of T in uniformly smooth and 2-uniformly convex real Banach spaces.

Motivated and inspired by the above results, one of our aim in this thesis is to introduce, a new class of mappings called f -pseudocontractive mappings from E into E^* with the notion of f -fixed points and construct an algorithm for finding a common f -fixed point of those mappings and discuss its strong convergence in reflexive Banach spaces.

2.5 Approximation of a Common Solution of Fixed points and Variational Inequality Problems

In this section, we present some literature review on iterative methods for approximating a common solution of fixed points and variational inequality problems.

Let $A : C \rightarrow E$ be a monotone mapping, where C is a nonempty, closed and convex subset of E . The variational inequality problem (VIP) is formulated as

finding a point

$$x^* \in C \text{ such that } \langle Ax^*, x - x^* \rangle \geq 0, \forall x \in C. \quad (2.15)$$

We denote the solution set of (2.15) by $VI(C, A)$. This problem was first introduced by Hartman and Stampacchia [51] in 1966 and has been studied by several authors (see, e.g., [15, 29, 52, 77, 79, 111, 113, 121, 123, 124, 126] and the references therein).

In [77], Nadezhkina and Takahashi studied the following algorithm in a finite dimensional Hilbert space setting: for any $x_0 \in C$,

$$\begin{cases} y_n = P_C(x_n - \beta_n Ax_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) P_C(x_n - \beta_n Ay_n), \quad n \geq 0, \end{cases} \quad (2.16)$$

where A is an L -Lipschitz continuous monotone mapping and P_C is a metric projection from H onto C . They proved that the sequence generated by (2.16), converges weakly to a solution of problem (2.15) under certain conditions imposed on the control sequences $\{\beta_n\}$ and $\{\alpha_n\}$.

In 2017, Thong and Hieu [111] established the following algorithm in a Hilbert space setting: for arbitrary $x_0 \in H$,

$$\begin{cases} y_n = P_C[x_n - \gamma_n Ax_n], \\ z_n = y_n - \gamma_n (Ay_n - Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n, \quad n \geq 0, \end{cases} \quad (2.17)$$

where, A is L -Lipschitz continuous on H and $f : H \rightarrow H$ is a contraction mapping. They proved that the sequence generated by (2.17), converges strongly to $x^* = P_{VI(C,A)} f(x^*)$ provided that the control sequences $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy appropriate conditions.

Several authors have also studied, algorithms for finding a common element of the set of fixed points of nonexpansive mapping and the set of solutions of variational inequality problem for Lipschitz monotone mapping (see, e.g., [58, 77, 109, 110, 121, 125] and the references therein).

In 2003, Takahashi and Toyoda [109] introduced the following scheme for finding a point in $VI(C, A) \cap F(T)$ in a Hilbert space setting: for arbitrary $x_0 \in C$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C(x_n - \gamma_n A x_n), \quad n \geq 0, \quad (2.18)$$

where $A : C \rightarrow H$ is a α -inverse strongly monotone mapping, $T : C \rightarrow H$ is nonexpansive, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\gamma_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $VI(C, A) \cap F(T)$ is nonempty, then the sequence generated by (2.18) converges weakly to some $x^* \in VI(C, A) \cap F(T)$.

In 2005, Iiduka and Takahashi [52] established the following algorithm: for arbitrary $x_0, x \in C$,

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)TP_C(x_n - \gamma_n A x_n), \quad n \geq 0, \quad (2.19)$$

where $A : C \rightarrow H$ is α -inverse strongly monotone mapping and $T : C \rightarrow H$ is nonexpansive mapping. They proved that the sequence generated by (2.19), converges strongly to $x^* = P_{VI(C, A) \cap F(T)}x$ provided that the control sequences $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy appropriate conditions.

Recently, Zhang and Yuan [127] investigated an iterative method for approximating a common point of the set of fixed points of nonexpansive mapping and solutions of variational inequality problems for a finite family of α -inverse strongly monotone mappings in the setting of Hilbert spaces: for arbitrary $x_1 \in C$

$$\begin{cases} y_{n,i} = P_C[x_n - \gamma_i A_i x_n], \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T \sum_{i=1}^n \eta_i y_{n,i}, \quad n \geq 1, \end{cases} \quad (2.20)$$

where $\|y_{n,i} - P_C(x_n - \gamma_n A_i x_n)\| \leq e_{n,i}$, and $\lim_{n \rightarrow \infty} e_{n,i} = 0$, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = \sum_{i=1}^n \eta_i = 1$, $\{\gamma_i\}$ is a real sequence in $(0, 2\mu_i)$, $f : C \rightarrow C$ is α -contractive mapping, $T : C \rightarrow C$ is nonexpansive and $A_i : C \rightarrow H$ is μ_i -inverse strongly monotone mapping for $1 \leq i \leq n$. They proved that the sequence generated by (2.20) converges strongly to $x^* \in \cap_{i=1}^n (VI, A_i) \cap F(T)$ provided that the control sequences satisfy certain conditions.

Motivated and inspired by the above results, one of our aims in this thesis is to introduce the concept of a Bregman relatively f -nonexpansive mapping and construct an algorithm for approximating a common element of the set of f -fixed points of a Bregman relatively f -nonexpansive mapping and the set of solutions of variational inequality problems for Lipschitz monotone mappings in the setting of reflexive real Banach spaces.

Chapter 3

Approximating Zeros of Sum of Maximal Monotone Mappings in Hilbert Spaces

In this chapter, we present some preliminary results that will be useful to prove our main theorems of the chapter and study the method of approximation for a zero of the sum of a finite family of maximal monotone mappings using Viscosity type Douglas-Rachford splitting algorithm and prove some strong convergence theorems of the proposed algorithm. In addition, we introduce an algorithm which starts by reformulating (1.1) as the problem of locating a point in a certain extended solution set $S_e(A_1, A_2, \dots, A_m) \subset H \times H^m$ and prove strong convergence of the proposed algorithm under suitable conditions.

3.1 Preliminaries

Lemma 3.1.1. *[42] Let H be a real Hilbert space. Let $A, B : H \rightarrow 2^H$ be two maximal monotone mappings and $\gamma \in (0, \infty)$. Set $T = R_{\gamma A}R_{\gamma B}$, where $R_{\gamma A} =$*

$2J_{\gamma A} - I$ and $R_{\gamma B} = 2J_{\gamma B} - I$. Then,

$$(A + B)^{-1}(0) = J_{\gamma B}(F(T)).$$

Furthermore, if $A = N_D$, the normal cone of D , where D is closed affine subspace of H , then $x \in F(T)$ if and only if $P_D x = J_{\gamma B} x$, where P_D is the metric projection of H onto D .

Lemma 3.1.2. [13] Let H be a real Hilbert space. Let N be an integer such that $N \geq 2$, set $I = \{1, 2, \dots, N\}$, and let $(A_i)_{i \in I}$ be maximal monotone mapping, from H into 2^H . Let $\mathbb{H}$ be the real Hilbert space obtained by endowing the Cartesian product H^N with the scalar product $\langle X, Y \rangle = \sum_{i=1}^N \langle x_i, y_i \rangle$, where $X = (x_1, x_2, \dots, x_N), Y = (y_1, y_2, \dots, y_N) \in \mathbb{H}$. Suppose

$$\begin{cases} \mathbb{D} = \{(x, x, \dots, x) \in \mathbb{H} : x \in H\}, \\ \Pi : H \rightarrow \mathbb{D} : x \rightarrow (x, x, \dots, x), \\ \mathbb{B} = \times_{i \in I} A_i. \end{cases}$$

Then, for $X \in \mathbb{H}$, the following holds:

- i) $N_{\mathbb{D}} X = \begin{cases} \{U \in \mathbb{H} : \sum_{i \in I} u_i = 0\}, & \text{if } X \in \mathbb{D} \\ \emptyset, & \text{otherwise;} \end{cases}$
- ii) $J_{\gamma \mathbb{A}} X = P_{\mathbb{D}} X = \Pi(\frac{1}{N} \sum_{i \in I} x_i)$, where $\mathbb{A} = N_{\mathbb{D}}$;
- iii) $J_{\gamma \mathbb{B}} X = (J_{\gamma A_i} x_i)_{i \in I}$.

Lemma 3.1.3. [38] Let $x, y \in H$. If H is a real Hilbert space, then the following inequality holds:

- i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$,
- ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$.

Lemma 3.1.4. [57] (Demiclosedness principle) Let H be a Hilbert space, C be a closed and convex subset of H , and $T : C \rightarrow C$ be a nonexpansive operator with

$F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converges to $x \in C$ and $\{(I - T)x_n\}$ converges strongly to $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in F(T)$.

Lemma 3.1.5. [100] Finding a point in $S_e(A_1, \dots, A_m)$ is equivalent to solving (1.1) in the sense that

$$0 \in A_1(z) + \dots + A_m(z) \Leftrightarrow \exists w_1, \dots, w_m \in H : (z, w_1, \dots, w_m) \in S_e(A_1, \dots, A_m).$$

Lemma 3.1.6. [100] If the monotone operators $A_1, \dots, A_m$ are maximal monotone, the corresponding extended solution set $S_e(A_1, \dots, A_m)$ is closed and convex in H^{m+1} .

Lemma 3.1.7. [100] Given $(x_k, y_k) \in \text{Gph}(A_k), k = 1, \dots, m$, define $\varphi : V \rightarrow \mathbb{R}$ via

$$\varphi(z, w_1, \dots, w_m) = \sum_{k=1}^m \langle z - x_k, y_k - w_k \rangle. \quad (3.1)$$

Then, for any $(z, w_1, \dots, w_m) \in S_e(A_1, \dots, A_m)$, one has $\varphi(z, w_1, \dots, w_m) \leq 0$, that is

$$S_e(A_1, \dots, A_m) \subseteq \{(z, w_1, \dots, w_m) \in V : \varphi(z, w_1, \dots, w_m) \leq 0\}.$$

In addition, φ is affine on V , with

$$\nabla \varphi = \left(\sum_{k=1}^m y_k, x_1 - \bar{x}, x_2 - \bar{x}, \dots, x_m - \bar{x} \right), \text{ where } \bar{x} = \frac{1}{m} \sum_{k=1}^m x_k, \quad (3.2)$$

and

$$\begin{aligned} \nabla \varphi = 0 &\Leftrightarrow (x_1, y_1, \dots, y_m) \in S_e(A_1, \dots, A_m), x_1 = x_2 = \dots = x_m \\ &\Leftrightarrow \varphi(z, w_1, \dots, w_m) = 0, \forall (z, w_1, \dots, w_m) \in V. \end{aligned} \quad (3.3)$$

The function φ in Lemma 3.1.7 is called *decomposable separators*.

Lemma 3.1.8. [118] *Let C be a closed and convex subset of a real Hilbert space H and $x \in H$ be given. The metric projection of H onto C , P_C , is characterized by the following:*

- i) $P_C x \in C$ if and only if $\langle x - P_C x, P_C x - z \rangle \geq 0$, for all $z \in C$;
- ii) P_C is firmly nonexpansive and hence nonexpansive.

Lemma 3.1.9. [119] *If a nonnegative real sequence $\{a_n\}$ satisfies the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, \quad n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n \delta_n| < \infty$, then $\lim_{n \rightarrow \infty} a_n = 0$.

3.2 Viscosity Type Douglas-Rachford Splitting Algorithm for Approximating a Zero of Sum of a Finite Family of Maximal Monotone Mappings in Hilbert Spaces

In this section, we study the method of approximation of a zero of the sum of a finite family of maximal monotone mappings using Viscosity type Douglas-Rachford splitting algorithm and prove some strong convergence theorems of the proposed algorithm under suitable conditions.

For the rest of this section, let $\gamma \in (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ or $\lim_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Now, we state and prove the convergence theorem for a zero of the sum of maximal monotone mappings.

Theorem 3.2.1. *Let H be a real Hilbert space. Let $A, B : H \rightarrow 2^H$ be maximal monotone mappings satisfying $\Omega := \{x \in H : 0 \in Ax + Bx\} \neq \emptyset$. Let $g : H \rightarrow H$ be a contraction mapping with constant α . For arbitrary $x_0 \in H$ define an iterative algorithm $\{x_n\}$ by*

$$\begin{cases} d_n = J_{\gamma B}x_n, \\ y_n = 2J_{\gamma A}(2J_{\gamma B} - I)x_n - 2J_{\gamma B}x_n + x_n, \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)y_n, \forall n \geq 0. \end{cases} \quad (3.4)$$

Then, $x_n \rightarrow x^* = P_{F(R_{\gamma A}R_{\gamma B})}(g(x^*))$, as $n \rightarrow \infty$, such that $J_{\gamma B}x^* \in \Omega$ and $d_n = J_{\gamma B}x_n \rightarrow J_{\gamma B}x^* \in \Omega$, where $R_{\gamma A} = 2J_{\gamma A} - I$ and $R_{\gamma B} = 2J_{\gamma B} - I$.

Proof. Note that, taking into account the definition of the reflected resolvent the iterative scheme in (3.4) can be equivalently given by

$$\begin{aligned} x_{n+1} &= \alpha_n g(x_n) + (1 - \alpha_n) \left[2J_{\gamma A}(2J_{\gamma B} - I)x_n - 2J_{\gamma B}x_n + x_n \right], \\ &= \alpha_n g(x_n) + (1 - \alpha_n)Tx_n, \forall n \geq 0. \end{aligned} \quad (3.5)$$

where $T = R_{\gamma A}R_{\gamma B}$. We note that since $R_{\gamma A}$ and $R_{\gamma B}$ are nonexpansive we have T is nonexpansive. Let $x \in \Omega$. Then, by Lemma 5.2.5 there exists $z \in F(T)$ such that $x = J_{\gamma B}(z)$ and hence $F(T) \neq \emptyset$. Thus, by Theorem 3.2 of [119] we obtain that

$$\|x_n - Tx_n\| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (3.6)$$

and $\{x_n\}$ converges strongly to the point $x^* = P_{F(T)}(g(x^*)) = P_{F(R_{\gamma A}R_{\gamma B})}(g(x^*))$. Thus, by Lemma 5.2.5 we have $J_{\gamma B}x^* \in \Omega$. Moreover, the fact that $J_{\gamma B}$ is continuous provides $d_n = J_{\gamma B}x_n \rightarrow J_{\gamma B}x^* \in \Omega$. This completes the proof. $\square$

By using the product space technique we obtain the following theorem from Theorem 3.2.1 for a zero of the sum of a finite family of maximal monotone mappings in Hilbert spaces.

Theorem 3.2.2. *Let H be a real Hilbert space. Let $A_i : H \rightarrow 2^H$, for $i \in I = \{1, 2, \dots, N\}$ and $N \geq 2$, be maximal monotone mappings satisfying $\Omega := \{x \in H : 0 \in A_1x + A_2x + \dots + A_Nx\} \neq \emptyset$. Let $g : H \rightarrow H$ be a contraction mapping with constant α . For arbitrary $x_{i,0} \in H$, $i \in I$, define an iterative algorithm by*

$$\begin{cases} p_n = \frac{1}{N} \sum_{i=1}^N x_{i,n}, \\ d_{i,n} = J_{\gamma A_i} x_{i,n}, \forall i \in I, \\ q_n = \frac{1}{N} \sum_{i=1}^N d_{i,n}, \\ y_{i,n} = x_{i,n} + 2(2q_n - p_n - d_{i,n}), \forall i \in I, \\ x_{i,n+1} = \alpha_n g(x_{i,n}) + (1 - \alpha_n) y_{i,n}, \forall i \in I, \forall n \geq 0. \end{cases} \quad (3.7)$$

Then, $\{p_n\}$ and $\{q_n\}$ converge strongly to $x^* \in \Omega$ as $n \rightarrow \infty$.

Proof. Let $\mathbb{H}$ be the real Hilbert space obtained by endowing the Cartesian product H^N with the scalar product $\langle X, Y \rangle = \sum_{i=1}^N \langle x_i, y_i \rangle$, where $X = (x_1, x_2, \dots, x_N)$, $Y = (y_1, y_2, \dots, y_N) \in \mathbb{H}$. Let

$$\begin{cases} \mathbb{D} := \{(x, x, \dots, x) \in \mathbb{H} : x \in H\}, \\ \mathbb{B} := \otimes A_{i=1}^N = A_1 \times A_2 \times \dots \times A_N. \end{cases} \quad (3.8)$$

Define $\Pi : H \rightarrow \mathbb{D}$ by $\Pi(x) = (x, x, \dots, x)$ and $\mathbb{A} = N_{\mathbb{D}}$. Then, from Example 20.41 and Proposition 23.16 of [13] we have $\mathbb{B}$ and $\mathbb{A}$ are maximal monotone mappings. In addition, from Lemma 3.1.2 we obtain the following:

$$P_{\mathbb{D}}X = \Pi\left(\frac{1}{N} \sum_{i=1}^N x_i\right) = J_{\gamma \mathbb{A}}X, J_{\gamma \mathbb{B}}X = (J_{\gamma A_i}x_i)_{i=1,2,\dots,N}, \quad (3.9)$$

for any $X \in \mathbb{H}$ and

$$N_{\mathbb{D}}X = \begin{cases} \mathbb{D}^\perp = \{U \in \mathbb{H} : \sum_{i=1}^N u_i = 0\}, & \text{if } X \in \mathbb{D}, \\ \emptyset, & \text{otherwise.} \end{cases} \quad (3.10)$$

Moreover, for any $z \in H$ such that $0 \in \sum_{i=1}^N A_i z$ and making use of equation (3.10) we obtain

$$\begin{aligned} & \exists U = (u_1, u_2, \dots, u_N) \in \otimes A_{i=1}^N z \text{ such that } \sum_{i=1}^N u_i = 0 \\ \Leftrightarrow & \exists U \in \mathbb{B}\Pi(z) \text{ such that } -U \in \mathbb{D}^\perp = N_{\mathbb{D}}\Pi(z) \\ \Leftrightarrow & \Pi(z) \in \text{zer}(N_{\mathbb{D}} + \mathbb{B}) \subset \mathbb{D}, \end{aligned} \quad (3.11)$$

and hence

$$\Pi\left(\text{zero} \sum_{i=1}^N A_i\right) = \text{zer}(\mathbb{A} + \mathbb{B}). \quad (3.12)$$

Now, let

$$\left\{ \begin{array}{l} Y_n = (y_{1,n}, y_{2,n}, \dots, y_{N,n}), \\ X_n = (x_{1,n}, x_{2,n}, \dots, x_{N,n}), \\ P_n = (p_n, p_n, \dots, p_n), \\ D_n = (d_{1,n}, d_{2,n}, \dots, d_{N,n}), \\ Q_n = (q_n, q_n, \dots, q_n), \\ G(X_n) = (g(x_{1,n}), g(x_{2,n}), \dots, g(x_{N,n})), \forall n \geq 0. \end{array} \right. \quad (3.13)$$

From (3.13) and the fact that $J_{\gamma\mathbb{A}} = P_{\mathbb{D}}$, we see that equation (3.7) can be rewritten as

$$\left\{ \begin{array}{l} D_n = J_{\gamma\mathbb{B}}X_n, \\ Q_n = P_{\mathbb{D}}D_n, \\ P_n = J_{\gamma\mathbb{A}}X_n, \\ Y_n = X_n + 2(2Q_n - P_n - D_n), \\ X_{n+1} = \alpha_n G(X_n) + (1 - \alpha_n)Y_n, \forall n \geq 0. \end{array} \right. \quad (3.14)$$

Therefore, by Theorem 3.2.1 we get that $\{X_n\}$ converges strongly to the point $X^* = P_{F(\mathbb{T})}(G(X^*))$ and $J_{\gamma\mathbb{B}}X^* \in \text{zer}(\mathbb{A} + \mathbb{B})$, where $\mathbb{T} = R_{\gamma\mathbb{A}}R_{\gamma\mathbb{B}}$. Moreover,

from (3.9) and the fact that $P_{\mathbb{D}}$ is continuous and affine we have

$$X^* = \mathbb{T}X^* = 2P_{\mathbb{D}}(2J_{\gamma\mathbb{B}} - I)X^* + X^* - 2J_{\gamma\mathbb{B}}X^*,$$

which implies that

$$\begin{aligned} J_{\gamma\mathbb{B}}X^* &= 2P_{\mathbb{D}}(J_{\gamma\mathbb{B}}X^*) + (1 - 2)P_{\mathbb{D}}X^* \in \mathbb{D}, \\ \Leftrightarrow P_{\mathbb{D}}(J_{\gamma\mathbb{B}}X^*) &= J_{\gamma\mathbb{B}}X^* = 2P_{\mathbb{D}}(J_{\gamma\mathbb{B}}X^*) - P_{\mathbb{D}}X^*, \end{aligned}$$

and hence

$$P_{\mathbb{D}}X^* = J_{\gamma\mathbb{B}}X^*. \quad (3.15)$$

Thus, from (3.9) and the continuities of $J_{\gamma\mathbb{A}}$ we have $P_n = J_{\gamma\mathbb{A}}X_n \rightarrow J_{\gamma\mathbb{A}}X^* = P_{\mathbb{D}}(X^*) = \Pi(x^*) \in \text{zer}(\mathbb{A} + \mathbb{B})$, where $x^* \in \Omega$. Hence, from (3.7) and the definition of Π we obtain $p_n = \Pi^{-1}(P_n) \rightarrow x^*$ as $n \rightarrow \infty$.

Furthermore, from (3.14), (3.15), continuities of $P_{\mathbb{D}}$ and $J_{\gamma\mathbb{B}}$ we have $Q_n = P_{\mathbb{D}}D_n = P_{\mathbb{D}}J_{\gamma\mathbb{B}}X_n \rightarrow P_{\mathbb{D}}J_{\gamma\mathbb{B}}X^* = P_{\mathbb{D}}X^* = \Pi(x^*)$ and hence $q_n = \Pi^{-1}(Q_n) \rightarrow x^*$, as $n \rightarrow \infty$, where $x^* \in \Omega$. The proof is complete. $\square$

Remark 3.2.3. At this point, we note that the limit point of the sequence $\{X_n\}$ in (3.14) is the unique solution $X^* = (x_1^*, x_2^*, \dots, x_N^*) \in F(\mathbb{T})$ satisfying the variational inequality

$$\langle (G - I)X^*, X^* - X \rangle \geq 0, \forall X \in F(\mathbb{T}), \quad (3.16)$$

where $P_{\mathbb{D}}X^* = \Pi(x^*)$ and $x^* = \frac{1}{N} \sum_{i=1}^N x_i^* \in \Omega$.

Remark 3.2.4. We observe from the proof of Theorem 3.2.2 that, since $P_n = P_{\mathbb{D}}X_n = P_{\mathbb{D}}(x_{1,n}, x_{2,n}, \dots, x_{N,n}) = \Pi(\frac{1}{N} \sum_{i=1}^N x_{i,n}) \rightarrow P_{\mathbb{D}}X^* = P_{\mathbb{D}}(x_1^*, x_2^*, \dots, x_N^*) = \Pi(\frac{1}{N} \sum_{i=1}^N x_i^*) = \Pi(x^*)$, we obtain that $\frac{1}{N} \sum_{i=1}^N x_{i,n} \rightarrow \frac{1}{N} \sum_{i=1}^N x_i^* = x^* \in \Omega$ as $n \rightarrow \infty$.

If, in Theorem 3.2.1, we assume that $B = 0$, then we get the following corollary for a zero of a maximal monotone mapping.

Corollary 3.2.5. Let H be a real Hilbert space. Let $A : H \rightarrow 2^H$ be a maximal monotone mapping satisfying $\Omega := \{x \in H : 0 \in Ax\} \neq \emptyset$. Let $g : H \rightarrow H$ be contraction mapping with constant α . For arbitrary $x_0 \in C$ define an iterative algorithm by

$$\begin{cases} y_n = 2J_{\gamma A}x_n - x_n, \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)y_n, \forall n \geq 0. \end{cases} \quad (3.17)$$

Then, $x_n \rightarrow x^* \in \Omega$ as $n \rightarrow \infty$.

As an application of Corollary 3.2.5 we have the following corollary for multi-valued pseudocontractive mapping.

Corollary 3.2.6. Let H be a real Hilbert space. Let $T : H \rightarrow 2^H$ be a pseudocontractive mapping with $(I - T)$ maximal monotone. Assume that $\Omega := \{x \in H : x \in Tx\} \neq \emptyset$. Let $g : H \rightarrow H$ be a contraction mapping with constant α . For arbitrary $x_0 \in H$ define an iterative algorithm by

$$\begin{cases} y_n = 2J_{\gamma(I-T)}x_n - x_n, \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)y_n, \forall n \geq 0. \end{cases} \quad (3.18)$$

Then, $\{x_n\}$ converges strongly to $z \in F(T)$ as $n \rightarrow \infty$.

3.2.1 Application to Convex Minimization Problems

In this subsection, we shall apply Theorem 3.2.1 and 3.2.2 to the convex minimization problems.

Theorem 3.2.7. Let H be a real Hilbert space. Let $h_i : H \rightarrow \mathbb{R}, i = 1, 2, \dots, N$, be a convex, lower semicontinuous function such that $\Omega = \min_{x \in H} \{\sum_{i=1}^N h_i(x)\} \neq \emptyset$.

Let $g : H \rightarrow H$ be a contraction mapping with constant α . For arbitrary $x_{i,0} \in H$, for $i \in I$, define an iterative algorithm by

$$\begin{cases} p_n = \frac{1}{N} \sum_{i=1}^N x_{i,n}, \\ d_{i,n} = J_{\gamma A_i} x_{i,n}, \forall i = 1, \dots, N, \\ q_n = \frac{1}{N} \sum_{i=1}^N d_{i,n}, \\ y_{i,n} = x_{i,n} + 2(2q_n - p_n - d_{i,n}), \forall i = 1, \dots, N, \\ x_{i,n+1} = \alpha_n g(x_{i,n}) + (1 - \alpha_n) y_{i,n}, \forall i = 1, \dots, N, \forall n \geq 0, \end{cases} \quad (3.19)$$

where $A_i = (\partial h_i)$, for $i = 1, 2, \dots, N$. Then, $\{p_n\}$ and $\{q_n\}$ converge strongly to $x \in \Omega$ as $n \rightarrow \infty$.

Theorem 3.2.8. Let H be a real Hilbert space. Let $h : H \rightarrow \mathbb{R}$ be a convex smooth function and $f : H \rightarrow \mathbb{R}$ be a convex and lower semicontinuous function such that $\Omega = \min_{x \in H} \{(h(x) + f(x))\} \neq \emptyset$. Let $g : H \rightarrow H$ be a contraction mapping with constant α . For arbitrary $x_0 \in H$ define an iterative algorithm by

$$\begin{cases} d_n = J_{\gamma \partial f} x_n, \\ y_n = 2J_{\gamma \nabla h}(2J_{\gamma \partial f} - I)x_n - 2J_{\gamma \partial f} x_n + x_n. \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) y_n, \forall n \geq 0. \end{cases} \quad (3.20)$$

Then, $x_n \rightarrow z$ as $n \rightarrow \infty$ such that $J_{\gamma \partial f} z \in \Omega$ and $d_n = J_{\gamma \partial f} x_n \rightarrow J_{\gamma \partial f} z$.

Proof. Let $A = \nabla h$ and $B = \partial f$ in Theorem 3.2.8. The scheme in (3.20) reduces to the scheme in (3.4). Thus, Theorem 3.2.1 provides the conclusion of the theorem. $\square$

3.2.2 Numerical Example

In this subsection, we present some numerical experiments to explain the conclusion of our result. The following numerical example verifies the conclusion of

Theorem 3.2.1.

Example 3.2.9. Let $H = \mathbb{R}^3$. Let $A, B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by $Ax = 2x + (1, 2, 3)$ and $Bx = 5x$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We see that A and B are maximal monotone with $R(I + \gamma A) = l_2 = (I + \gamma B)$ for all $\gamma > 0$. Now, by direct calculation we get that $Tx = \frac{x(1-5\gamma)}{1+5\gamma} \left(\frac{2}{1+2\gamma} - 1 \right) - \frac{2\gamma}{1+2\gamma}(1, 2, 3)$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Thus, if we assume $\alpha_n = \frac{1}{10n+100}$, $\gamma = 1$, for all $n \geq 0$, and $g(x) = \frac{x}{2}$ with initial point $x_0 = (2, 3, 4)$, then the iteration Scheme (3.4) provides the following numerical result (see, Table 3.1). All numerical computations were carried out using Matlab 2020(a) on a hp personal computer, 12-Gb RAM. From this we obtain that the Algorithm $d_n = J_{\gamma B}x_n$ generated by (3.4) converges strongly to $x^* = (-\frac{1}{7}, -\frac{2}{7}, -\frac{3}{7}) \in (A + B)^{-1}(0)$.

TABLE 3.1: Convergence of the sequence $\{d_n\}$

n	$d_n = J_B x_n$	$\ d_{n+1} - d_n\ _{l_2}$
1	(2.0000, 3.0000, 4.0000)	
2	(-0.0350, -0.1075, -0.1800)	5.5920
3	(-0.1180, -0.2444, -0.3708)	0.2491
4	(-0.1367, -0.2752, -0.4138)	0.0561
5	(-0.1409, -0.2823, -0.4236)	0.0128
10	(-0.1423, -0.2847, -0.4270)	0.0044
100	(-0.1428, -0.2855, -0.4283)	0.0016
200	(-0.1428, -0.2856, -0.4284)	1.4142e-04
300	(-0.1428, -0.2857, -0.4285)	1.4142e-04
400	(-0.1428, -0.2857, -0.4285)	0
500	(-0.1428, -0.2857, -0.4285)	0
	$\vdots$	$\vdots$
	$\downarrow$	$\downarrow$
	$(-\frac{1}{7}, -\frac{2}{7}, -\frac{3}{7})$	0

In addition, we have sketched the error term $\|d_n - x^*\|$ for different initial points such as $x_0 = (0, -0.2, -0.4)$, $x'_0 = (0, 1, 2)$ and $x''_0 = (2, 3, 4)$. From the sketch we observe that the convergence rate of the sequence d_n generated by (3.4) converges faster as the initial point gets closer to the solution (see, Figure 3.1, below).

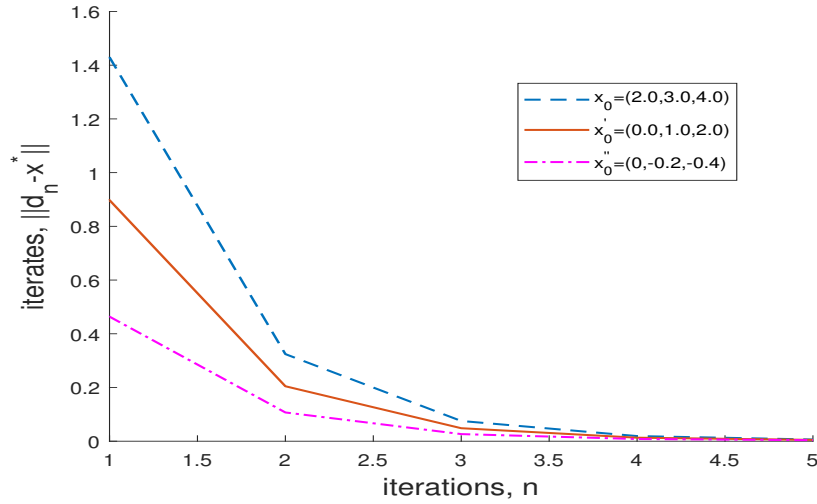


FIGURE 3.1: The graph of $\|d_n - x^*\|$ versus number of iterations with different choices of x_0

3.3 A Strong Convergence Results for Zeros of the Sum of a Finite Family of Maximal Monotone Mappings

In this section, we introduce an algorithm for finding a point in $S_e(A_1, \dots, A_m)$ which will lead us to a solution of the sum of a finite family of maximal monotone mappings in a Hilbert space and discuss its strong convergence.

For the rest of this section, let H be a real Hilbert space and $A_k : H \rightarrow 2^H, k = 1, \dots, m$ ($m \geq 2$), be a finite family of maximal monotone mappings satisfying $S_e(A_1, \dots, A_m) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, and let $\{\beta_n\}$ be a decreasing sequence in $(0, 1]$ with $\beta_0 = 1$.

We now propose the following algorithm.

Algorithm 3.3.1.

Step 0: Select initial guess $u_0 = (z_0, w_{1,0}, \dots, w_{m,0}) \in V$.

Step 1: Given n^{th} iterates ($n \geq 0$), compute $x_{k,n}$ and $y_{k,n}$ as follows:

$$\begin{aligned} x_{k,n} &= (I + \lambda_{k,n}A_k)^{-1}(z_n + \lambda_{k,n}w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} &= w_{k,n} + \frac{z_n - x_{k,n}}{\lambda_{k,n}}, & k = 1, 2, \dots, m, \end{aligned}$$

where the real sequence $\lambda_{k,n} > 0$, for each $k = 1, 2, \dots, m$ and $n \geq 0$.

Step 2: Define $\varphi_i : V \rightarrow \mathbb{R}$ by

$$\varphi_i(z, w_1, \dots, w_m) = \sum_{k=1}^m \langle z - x_{k,i}, y_{k,i} - w_k \rangle, \quad (3.21)$$

and compute

$$T_i u_n = (z_n, w_{1,n}, \dots, w_{m,n}) - \max \left\{ 0, \frac{\varphi_i(z_n, w_{1,n}, \dots, w_{m,n})}{\|\nabla \varphi_i\|^2} \right\} \nabla \varphi_i, \quad (3.22)$$

where $\nabla \varphi_i = (\sum_{k=1}^m y_{k,i}, x_{1,i} - \bar{x}_i, x_{2,i} - \bar{x}_i, \dots, x_{m,i} - \bar{x}_i)$ and $\|\nabla \varphi_i\|^2 = \sum_{k=1}^m \|x_{k,i} - \bar{x}_i\|^2 + \|\sum_{k=1}^m y_{k,i}\|^2$ with $\bar{x}_i = \frac{1}{m} \sum_{k=1}^m x_{k,i}$, which is the projection of $u_n = (z_n, w_{1,n}, \dots, w_{m,n})$ onto the half space

$$H_i = \{(z, w_1, \dots, w_m) \in V : \varphi_i(z, w_1, \dots, w_m) \leq 0\}.$$

Step 3: Compute

$$\begin{cases} v_n = \beta_n u_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i u_n, \\ u_{n+1} = \alpha_n f(u_n) + (1 - \alpha_n) v_n, \forall n \geq 0, \end{cases} \quad (3.23)$$

where $f : V \rightarrow V$ is a contraction mapping with constant α . Set $n := n + 1$ and go to step 1.

Remark 3.3.2. The following points indicate that Algorithm 3.1 is well defined.

- (i) By maximality of A_k the resolvent mapping $(I + \lambda_{k,n}A_k)^{-1}$ is single-valued

and everywhere defined for each $k = 1, 2, \dots, m$ (see, e.g., [69]). By rearranging the equation in step 1, one has the following equation:

$$x_{k,n} + \lambda_{k,n}y_{k,n} = z_n + \lambda_{k,n}w_{k,n}, \quad \forall n \geq 0, \quad (3.24)$$

where for each $k = 1, 2, \dots, m$ and $n \geq 0$, $\lambda_{k,n} > 0$ and $y_{k,n} \in A_k(x_{k,n})$. Hence, for each $k = 1, 2, \dots, m$, $(x_{k,n}, y_{k,n}) \in \text{Gph}(A_k)$ exists and is unique. Thus, the decomposable separators function φ in Lemma 3.1.7 is well defined.

- (ii) Since $\varphi_i : V \rightarrow \mathbb{R}$ is affine on V (see, Lemma 3.1.7) and the half space H_i is closed and convex subspace of V for all $i \geq 1$, the projection of u_n onto H_i given by (3.22) exists and is firmly nonexpansive (see, for example, [100]).

Lemma 3.3.3. *The sequence $\{u_n\}$ generated by Algorithm 3.3.1 is bounded.*

Proof. Let $u^* = (z^*, w_1^*, w_2^*, \dots, w_m^*) \in S_e(A_1, \dots, A_m)$, then by Lemma 3.1.7 we have $u^* \in H_i$ for all $i \geq 0$ which implies that $S_e(A_1, \dots, A_m) \subset \cap_{i=1}^{\infty} F(T_i) = \mathcal{F} \neq \emptyset$. Now, the fact that T_i is nonexpansive and (3.23) yield

$$\begin{aligned} \|v_n - u^*\| &= \|\beta_n u_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i u_n - u^*\| \\ &= \|\beta_n u_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i u_n - \sum_{i=1}^n (\beta_{i-1} - \beta_i) u^* \\ &\quad + \sum_{i=1}^n (\beta_{i-1} - \beta_i) u^* - u^*\| \\ &= \|\beta_n (u_n - u^*) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) (T_i u_n - u^*)\| \\ &\leq \beta_n \|u_n - u^*\| + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|u_n - u^*\| = \|u_n - u^*\|. \end{aligned} \quad (3.25)$$

Thus, it follows that

$$\begin{aligned}
 \|u_{n+1} - u^*\| &= \|\alpha_n f(u_n) + (1 - \alpha_n)v_n - u^*\| \\
 &\leq \alpha_n \|f(u_n) - u^*\| + (1 - \alpha_n)\|v_n - u^*\| \\
 &\leq \alpha_n \|f(u_n) - f(u^*)\| + \alpha_n \|f(u^*) - u^*\| + (1 - \alpha_n)\|v_n - u^*\| \\
 &\leq (1 - \alpha_n(1 - \alpha))\|u_n - u^*\| + \alpha_n \|f(u^*) - u^*\| \\
 &\leq \max \left\{ \|u_n - u^*\|, \frac{\|f(u^*) - u^*\|}{1 - \alpha} \right\}.
 \end{aligned}$$

Then, we have

$$\|u_n - u^*\| \leq \max \left\{ \|u_0 - u^*\|, \frac{\|f(u^*) - u^*\|}{1 - \alpha} \right\},$$

which implies that $\{u_n\}$ is bounded. Hence, we can obtain that $\{T_i u_n\}$, $\{v_n\}$ and $\{f(u_n)\}$ are bounded. $\square$

Lemma 3.3.4. *The sequence $\{u_n\}$ generated by Algorithm 3.3.1 converges strongly to $u^* = (z^*, w_1^*, \dots, w_m^*) \in \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof. We proceed with the following steps.

Step 1. First we show that $\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0$.

From (3.23), we have

$$\begin{aligned}
 \|v_n - v_{n-1}\| &= \|\beta_n u_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i u_n - \beta_{n-1} u_{n-1} \\
 &\quad - \sum_{i=1}^{n-1} (\beta_{i-1} - \beta_i) T_i u_{n-1}\| \\
 &\leq \beta_n \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| \|u_{n-1}\| \\
 &\quad + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|T_i u_n - T_i u_{n-1}\| + |\beta_n - \beta_{n-1}| \|T_i u_{n-1}\| \\
 &\leq \beta_n \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| \|u_{n-1}\| \\
 &\quad + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| \|T_i u_{n-1}\| \\
 &\leq \|u_n - u_{n-1}\| + (\beta_{n-1} - \beta_n) M,
 \end{aligned} \tag{3.26}$$

for some $M > 0$ as the sequences $\{u_n\}$ and $\{T_i u_{n-1}\}$ are bounded. Thus, (3.23) and (3.26) imply that

$$\begin{aligned}
 \|u_{n+1} - u_n\| &= \|\alpha_n f(u_n) + (1 - \alpha_n)v_n - \alpha_{n-1}f(u_{n-1}) - (1 - \alpha_{n-1})v_{n-1}\| \\
 &\leq \alpha_n \|f(u_n) - f(u_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|f(u_{n-1})\| \\
 &\quad + (1 - \alpha_n) \|v_n - v_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|v_{n-1}\| \\
 &\leq \alpha \alpha_n \|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| (\|f(u_{n-1})\| + \|v_{n-1}\|) \\
 &\quad + (1 - \alpha_n) (\|u_n - u_{n-1}\| + (\beta_{n-1} - \beta_n)M) \\
 &= (1 - \alpha_n(1 - \alpha)) \|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| (\|f(u_{n-1})\| + \|v_{n-1}\|) \\
 &\quad + (1 - \alpha_n)(\beta_{n-1} - \beta_n)M. \tag{3.27}
 \end{aligned}$$

Since $\{\beta_n\}$ is strictly decreasing, it implies that $\sum_{n=1}^{\infty} (\beta_{n-1} - \beta_n) < \infty$. Hence, from (3.27), conditions of $\{\alpha_n\}$ and Lemma 3.1.9, we immediately obtain that

$$\|u_{n+1} - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.28}$$

Step 2. We show that $\|T_i u_n - u_n\| \rightarrow 0$ as $n \rightarrow \infty$. Take $u^* = P_{\mathcal{F}}(f(u^*))$. Note that

$$\begin{aligned}
 \|u_n - u^*\|^2 &\geq \|T_i u_n - T_i u^*\|^2 \\
 &= \|T_i u_n - u_n + u_n - T_i u^*\|^2 \\
 &= \|T_i u_n - u_n\|^2 + \|u_n - u^*\|^2 + 2\langle T_i u_n - u_n, u_n - u^* \rangle, \tag{3.29}
 \end{aligned}$$

which yields

$$\frac{1}{2} \|T_i u_n - u_n\|^2 \leq \langle u_n - T_i u_n, u_n - u^* \rangle. \tag{3.30}$$

Furthermore, from (3.23) and (3.30), we immediately obtain

$$\begin{aligned}
 \frac{1}{2} \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|T_i u_n - u_n\|^2 &\leq \sum_{i=1}^n (\beta_{i-1} - \beta_i) \langle u_n - T_i u_n, u_n - u^* \rangle \\
 &= \langle (1 - \beta_n) u_n - \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i u_n, u_n - u^* \rangle \\
 &= \langle (1 - \beta_n) u_n - v_n + \beta_n u_n, u_n - u^* \rangle \\
 &= \langle u_n - v_n, u_n - u^* \rangle \\
 &= \langle u_n - u_{n+1}, u_n - u^* \rangle \\
 &\quad + \langle u_{n+1} - v_n, u_n - u^* \rangle, \tag{3.31}
 \end{aligned}$$

and from (3.23) and (3.31), we have

$$\begin{aligned}
 \frac{1}{2} \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|T_i u_n - u_n\|^2 &\leq \langle u_n - u_{n+1}, u_n - u^* \rangle \\
 &\quad + \langle \alpha_n f(u_n) + (1 - \alpha_n) v_n - v_n, u_n - u^* \rangle \\
 &\leq \|u_n - u_{n+1}\| \|u_n - u^*\| \\
 &\quad + \langle \alpha_n f(u_n) - \alpha_n v_n, u_n - u^* \rangle \\
 &\leq \|u_n - u_{n+1}\| \|u_n - u^*\| \\
 &\quad + \alpha_n \|f(u_n) - v_n\| \|u_n - u^*\|. \tag{3.32}
 \end{aligned}$$

Thus, from Lemma 3.3.3, (3.28), (3.32) and the condition on $\{\alpha_n\}$, we conclude that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|T_i u_n - u_n\|^2 = 0. \tag{3.33}$$

Since $\{\beta_n\}$ is strictly decreasing, for every $i \in \mathbb{N}$, equality (3.33) yields

$$\lim_{n \rightarrow \infty} \|T_i u_n - u_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|T_n u_n - u_n\| = 0. \tag{3.34}$$

Step 3. We show that $\limsup_{n \rightarrow \infty} \langle f(u^*) - u^*, u_n - u^* \rangle \leq 0$.

Since $\{u_n\}$ is bounded, there exist $u \in H \times H^m$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$

such that

$$\limsup_{n \rightarrow \infty} \langle f(u^*) - u^*, u_n - u^* \rangle = \lim_{k \rightarrow \infty} \langle f(u^*) - u^*, u_{n_k} - u^* \rangle,$$

and $u_{n_k} \rightharpoonup u$. From (3.34) and Lemma 3.1.4, we obtain that $u \in F(T_i)$ for each $i \geq 1$ and hence $u \in \mathcal{F}$. Since $u^* = P_{\mathcal{F}}(f(u^*))$, by Lemma 3.1.8, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle f(u^*) - u^*, u_n - u^* \rangle &= \lim_{k \rightarrow \infty} \langle f(u^*) - u^*, u_{n_k} - u^* \rangle \\ &= \langle f(u^*) - u^*, u - u^* \rangle \leq 0, \text{ as required.} \end{aligned} \quad (3.35)$$

Step 4. Finally we show that $u_n \rightarrow u^*$ as $n \rightarrow \infty$. From Lemma 3.1.3, (3.23) and (3.25), we have

$$\begin{aligned} \|u_{n+1} - u^*\|^2 &= \|\alpha_n f(u_n) + (1 - \alpha_n)u_n - u^*\|^2 \\ &\leq \left(\alpha_n \|u_n - u^*\| + (1 - \alpha_n) \|u_n - u^*\| \right)^2 \\ &\quad + 2\alpha_n \langle f(u^*) - u^*, u_{n+1} - u^* \rangle \\ &\leq (1 - (1 - \alpha)\alpha_n) \|u_n - u^*\|^2 + 2\alpha_n \langle f(u^*) - u^*, u_{n+1} - u^* \rangle \\ &\leq (1 - (1 - \alpha)\alpha_n) \|u_n - u^*\|^2 + 2\alpha_n \langle f(u^*) - u^*, u_n - u^* \rangle \\ &\quad + 2\alpha_n \|f(u^*) - u^*\| \|u_{n+1} - u_n\|. \end{aligned} \quad (3.36)$$

Therefore, from (3.28), (3.35), (3.36) and Lemma 3.1.9, we conclude that $u_n \rightarrow u^*$ as $n \rightarrow \infty$. $\square$

Lemma 3.3.5. *If in Algorithm 3.3.1 there exists $\bar{\lambda} > \underline{\lambda} > 0$ such that the sequence $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \bar{\lambda}]$, for each $k = 1, 2, \dots, m$ and $n \geq 0$, then there exists $\eta > 0$ such that*

$$\varphi_n(u_n) \geq \eta \|\nabla \varphi_n\|^2 \quad \forall n \geq 0, \quad (3.37)$$

where $\nabla \varphi_n(u_n) = (\sum_{k=1}^m y_{k,n}, x_{1,n} - \bar{x}_n, x_{2,n} - \bar{x}_n, \dots, x_{m,n} - \bar{x}_n)$ and $\|\nabla \varphi_n(u_n)\|^2 = \sum_{k=1}^m \|x_{k,n} - \bar{x}_n\|^2 + \|\sum_{k=1}^m y_{k,n}\|^2$ with $\bar{x}_n = \frac{1}{m} \sum_{k=1}^m x_{k,n}$.

Proof. From equation (3.24), we have

$$\frac{z_n - x_{k,n}}{\lambda_{k,n}} = y_{k,n} - w_{k,n}. \quad (3.38)$$

From (3.21), (3.38) and condition of $\lambda_{k,n}$, we get

$$\begin{aligned} \varphi_n(u_n) &= \sum_{k=1}^m \langle z_n - x_{k,n}, y_{k,n} - w_{k,n} \rangle \\ &= \sum_{k=1}^m \langle z_n - x_{k,n}, \frac{z_n - x_{k,n}}{\lambda_{k,n}} \rangle \\ &= \sum_{k=1}^m \frac{1}{\lambda_{k,n}} \|z_n - x_{k,n}\|^2 \\ &\geq \frac{1}{\lambda} \sum_{k=1}^m \|z_n - x_{k,n}\|^2. \end{aligned} \quad (3.39)$$

By rearranging equation (3.38), we can also get

$$y_{k,n} = \frac{z_n - x_{k,n}}{\lambda_{k,n}} + w_{k,n}, \quad (3.40)$$

which implies that

$$\sum_{k=1}^m y_{k,n} = \sum_{k=1}^m \frac{z_n - x_{k,n}}{\lambda_{k,n}} + \sum_{k=1}^m w_{k,n} = \sum_{k=1}^m \frac{z_n - x_{k,n}}{\lambda_{k,n}}, \quad (3.41)$$

and hence

$$\begin{aligned} \left\| \sum_{k=1}^m y_{k,n} \right\|^2 &= \left\| \sum_{k=1}^m \frac{z_n - x_{k,n}}{\lambda_{k,n}} \right\|^2 \\ &\leq \frac{1}{\lambda^2} \left\| \sum_{k=1}^m z_n - x_{k,n} \right\|^2 \\ &= \frac{1}{\lambda^2} \|m(z_n - \bar{x}_n)\|^2 \\ &= \frac{m^2}{\lambda^2} \|z_n - \bar{x}_n\|^2, \end{aligned} \quad (3.42)$$

which is equivalent to

$$\frac{\lambda^2}{m} \left\| \sum_{k=1}^m y_{k,n} \right\|^2 \leq m \|z_n - \bar{x}_n\|^2. \quad (3.43)$$

In addition, from the properties of inner product, we have

$$\begin{aligned} \sum_{k=1}^m \|x_{k,n} - \bar{x}_n\|^2 &= \sum_{k=1}^m \|x_{k,n} - z_n + z_n - \bar{x}_n\|^2 \\ &= \sum_{k=1}^m \|(x_{k,n} - z_n) - (\bar{x}_n - z_n)\|^2 \\ &= \sum_{k=1}^m \|x_{k,n} - z_n\|^2 - 2 \left\langle \sum_{k=1}^m (x_{k,n} - z_n), \bar{x}_n - z_n \right\rangle \\ &\quad + \sum_{k=1}^m \|\bar{x}_n - z_n\|^2 \\ &= \sum_{k=1}^m \|x_{k,n} - z_n\|^2 - 2m \langle \bar{x}_n - z_n, \bar{x}_n - z_n \rangle + m \|\bar{x}_n - z_n\|^2 \\ &= \sum_{k=1}^m \|x_{k,n} - z_n\|^2 - 2m \|\bar{x}_n - z_n\|^2 + m \|\bar{x}_n - z_n\|^2 \\ &= \sum_{k=1}^m \|x_{k,n} - z_n\|^2 - m \|\bar{x}_n - z_n\|^2, \end{aligned} \quad (3.44)$$

which implies that

$$\sum_{k=1}^m \|x_{k,n} - \bar{x}_n\|^2 + m \|\bar{x}_n - z_n\|^2 = \sum_{k=1}^m \|x_{k,n} - z_n\|^2. \quad (3.45)$$

From (3.43) and (3.45), we get

$$\sum_{k=1}^m \|x_{k,n} - \bar{x}_n\|^2 + \frac{\lambda^2}{m} \left\| \sum_{k=1}^m y_{k,n} \right\|^2 \leq \sum_{k=1}^m \|x_{k,n} - z_n\|^2, \quad (3.46)$$

and hence,

$$\frac{1}{\lambda} \left(\sum_{k=1}^m \|x_{k,n} - \bar{x}_n\|^2 + \frac{\lambda^2}{m} \left\| \sum_{k=1}^m y_{k,n} \right\|^2 \right) \leq \frac{1}{\lambda} \sum_{k=1}^m \|x_{k,n} - z_n\|^2. \quad (3.47)$$

Thus, from (3.47), (3.39) and setting $\eta = \min \left\{ \frac{1}{\lambda}, \frac{\lambda^2}{m\lambda} \right\}$, we get

$$\eta \left(\sum_{k=1}^m \|x_{k,n} - \bar{x}_n\|^2 + \left\| \sum_{k=1}^m y_{k,n} \right\|^2 \right) = \eta \|\nabla \varphi_n\|^2 \leq \varphi_n(u_n). \quad (3.48)$$

□

Next, we state and prove our main theorem.

Theorem 3.3.6. *Let $A_k : H \rightarrow 2^H$, for $k \in \{1, 2, \dots, m\}$ and $m \geq 2$, be maximal monotone mappings satisfying $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. Let $f : V \rightarrow V$ be a contraction mapping with constant α . Let the real sequence $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and for each $k = 1, 2, \dots, m$ and $n \geq 0$. Then, the sequence $\{u_n\}$ generated by Algorithm 3.3.1 converges strongly to an element $u^* = (z^*, w_1^*, \dots, w_m^*) \in S_e(A_1, A_2, \dots, A_m)$, satisfying the variational inequality*

$$\langle (f - I)u^*, u^* - x \rangle \geq 0, \forall x \in S_e(A_1, A_2, \dots, A_m), \quad (3.49)$$

where $z^* \in \Omega$.

Proof. By Lemma 3.3.5, there exists $\eta > 0$ such that

$$\varphi_n(u_n) \geq \eta \|\nabla \varphi_n\|^2 \quad \forall n \geq 0. \quad (3.50)$$

This implies that $\varphi_n(u_n)$ is always nonnegative, and from (3.22), we obtain

$$T_n u_n - u_n = - \left(\frac{\varphi_n(u_n)}{\|\nabla \varphi_n\|^2} \right) \nabla \varphi_n, \quad (3.51)$$

which implies

$$\|T_n u_n - u_n\| = \left\| - \left(\frac{\varphi_n(u_n)}{\|\nabla \varphi_n\|^2} \right) \nabla \varphi_n \right\| = \frac{\varphi_n(u_n)}{\|\nabla \varphi_n\|}, \quad (3.52)$$

for all n such that $\nabla\varphi_n \neq 0$. Thus, dividing both sides of the inequality (3.50) by $\|\nabla\varphi_n\|$, we obtain

$$\|T_n u_n - u_n\| \geq \eta \|\nabla\varphi_n\|, \quad (3.53)$$

which is also true for n having $\nabla\varphi_n = 0$. From (3.34), we have $\|T_n u_n - u_n\| \rightarrow 0$ as $n \rightarrow \infty$, so (3.53) implies $\nabla\varphi_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, it follows from (3.52) that

$$\lim_{n \rightarrow \infty} \varphi_n(u_n) = 0. \quad (3.54)$$

From the expression for $\nabla\varphi_n$, we have $\sum_{k=1}^m y_{k,n} \rightarrow 0$ and $x_{k,n} - \bar{x}_n \rightarrow 0$ for $k = 1, 2, \dots, m$ as $n \rightarrow \infty$. Moreover, subtracting $x_{k,n} + \lambda_{k,n} w_{k,n}$ from both sides of equation (3.24), we obtain

$$z_n - x_{k,n} = \lambda_{k,n}(y_{k,n} - w_{k,n}). \quad (3.55)$$

This and the definition of φ_n imply that

$$\begin{aligned} \varphi_n(u_n) &= \sum_{k=1}^m \langle z_n - x_{k,n}, y_{k,n} - w_{k,n} \rangle \\ &= \sum_{k=1}^m \langle \lambda_{k,n}(y_{k,n} - w_{k,n}), y_{k,n} - w_{k,n} \rangle \\ &= \sum_{k=1}^m \lambda_{k,n} \|y_{k,n} - w_{k,n}\|^2. \end{aligned} \quad (3.56)$$

Hence, from (3.54), (3.56) and the fact that $\lambda_{k,n} > \underline{\lambda}$, we have

$$\lim_{n \rightarrow \infty} \|y_{k,n} - w_{k,n}\| = 0, \quad \text{for all } k = 1, 2, \dots, m, \quad (3.57)$$

and from (3.55) and (3.57), we obtain

$$\lim_{n \rightarrow \infty} \|z_n - x_{k,n}\| = 0, \quad \text{for all } k = 1, 2, \dots, m. \quad (3.58)$$

Moreover, from Lemma 3.3.4, the sequence $\{u_n\} = \{(z_n, w_{1,n}, \dots, w_{m,n})\}$ converges strongly to a point $u^* = (z^*, w_1^*, \dots, w_m^*) \in \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$. In addition, equations (3.57) and (3.58) imply that $y_{k,n} \rightarrow w_k^*$ and $x_{k,n} \rightarrow z^*$, for each $k = 1, 2, \dots, m$. Since $Gph(A_k)$ is closed and $(x_{k,n}, y_{k,n}) \in Gph(A_k)$ for all $k = 1, 2, \dots, m$ and $n \geq 0$, we get $w_k^* \in A_k(z^*)$. Furthermore, since $\{u_n\} \subset V$ and V is a closed subspace, we also have $u^* \in V$ and hence, $u^* = (z^*, w_1^*, \dots, w_m^*) \in S_e(A_1, A_2, \dots, A_m)$ and by Lemma 3.1.5 we obtain $z^* \in (A_1 + A_2 + \dots + A_m)^{-1}(0)$. Moreover, since $u^* = P_{\mathcal{F}}(f(u^*))$ by Lemma 3.1.8, we obtain the variational inequality

$$\langle (f - I)u^*, u^* - x \rangle \geq 0, \forall x \in S_e(A_1, A_2, \dots, A_m), \quad (3.59)$$

as $S_e(A_1, A_2, \dots, A_m) \subset \mathcal{F}$. The proof is complete. $\square$

Remark 3.3.7. We observe that the algorithm given by 3.3.1 is equivalent to the following scheme:

$$\begin{cases} u_0 = (z_0, w_{1,0}, \dots, w_{m,0}) \in V \text{ chosen arbitrarily,} \\ x_{k,n} = (I + \lambda_{k,n}A_k)^{-1}(z_n + \lambda_{k,n}w_{k,n}), \quad k = 1, 2, \dots, m, \\ y_{k,n} = w_{k,n} + \frac{z_n - x_{k,n}}{\lambda_{k,n}}, \quad k = 1, 2, \dots, m, \\ u_{n+1} = \alpha_n f(z_n, w_{1,n}, \dots, w_{m,n}) + (1 - \alpha_n)(c_n, d_{1,n}, \dots, d_{m,n}), n \geq 0, \end{cases} \quad (3.60)$$

where $f : V \rightarrow V$ is a contraction mapping with constant α , $\lambda_{k,n} > 0$ for $k = 1, 2, \dots, m$, and $n \geq 0$, and

$$\begin{aligned} c_n &= \beta_n z_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \hat{c}_i, \\ d_{k,n} &= \beta_n w_{k,n} + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \hat{d}_{k,i}, \end{aligned}$$

$$\hat{c}_i = \begin{cases} z_n, & \text{if } \varphi_i(u_n) \leq 0, \\ z_n - \delta_i(\sum_{k=1}^m y_{k,i}), & \text{if } \varphi_i(u_n) > 0, \end{cases}$$

$$\hat{d}_{k,i} = \begin{cases} w_{k,n}, & \text{if } \varphi_i(u_n) \leq 0, \\ w_{k,n} - \delta_i(x_{k,i} - \bar{x}_i), & \text{if } \varphi_i(u_n) > 0, \end{cases}$$

with

$$\delta_i = \frac{\sum_{k=1}^m \langle z_n - x_{k,i}, y_{k,i} - w_{k,n} \rangle}{\|\sum_{k=1}^m y_{k,i}\|^2 + \sum_{k=1}^m \|x_{k,i} - \bar{x}_i\|^2}, \quad \text{for } \bar{x}_i = \frac{1}{m} \sum_{k=1}^m x_{k,i}.$$

Remark 3.3.8. At this point, we know that $S_e(A_1, A_2, \dots, A_m) \subset \mathcal{F}$ and one can show that $\mathcal{F} \subset S_e(A_1, A_2, \dots, A_m)$ and hence $S_e(A_1, A_2, \dots, A_m) = \mathcal{F}$.

If, in (3.60) we assume $\lambda_{k,n} = \lambda > 0$, for $k = 1, 2, \dots, m$ and $n \geq 0$, then we get the following corollary for the sum of a finite family of maximal monotone mappings in Hilbert spaces.

Corollary 3.3.9. Let $A_k : H \rightarrow 2^H$, for $k \in \{1, 2, \dots, m\}$ and $m \geq 2$, be maximal monotone mappings satisfying $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. Let $f : V \rightarrow V$ be a contraction mapping with constant α . For arbitrary $u_0 = (z_0, w_{1,0}, \dots, w_{m,0}) \in V$ define an iterative algorithm by

$$\begin{cases} x_{k,n} = (I + \lambda A_k)^{-1}(z_n + \lambda w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = w_{n,k} + \frac{z_n - x_{k,n}}{\lambda}, & k = 1, 2, \dots, m, \\ u_{n+1} = \alpha_n f(z_n, w_{1,n}, \dots, w_{m,n}) + (1 - \alpha_n)(c_n, d_{1,n}, \dots, d_{m,n}), & n \geq 0, \end{cases} \quad (3.61)$$

where $\{c_n\}$ and $\{d_{k,n}\}$ are as in (3.60) and $\lambda > 0$. Then, $\{u_n\}$ converges strongly to an element $u^* = (z^*, w_1^*, \dots, w_m^*)$ of $S_e(A_1, A_2, \dots, A_m)$, where $z^* \in \Omega$.

If, in Theorem 3.3.6 we replace the contraction mapping f by a constant $u \in V$, then we get the following corollary for approximating the zero of the sum of a finite family of maximal monotone mappings in Hilbert spaces.

Corollary 3.3.10. Let $A_k : H \rightarrow 2^H$, for $k \in \{1, 2, \dots, m\}$ ($m \geq 2$), be maximal monotone mappings satisfying $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$.

For arbitrary $u_0 = (z_0, w_{1,0}, \dots, w_{m,0}) \in V$ define an iterative algorithm by

$$\begin{cases} x_{k,n} = (I + \lambda_{k,n}A_k)^{-1}(z_n + \lambda_{k,n}w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = w_{k,n} + \frac{z_n - x_{k,n}}{\lambda_{k,n}}, & k = 1, 2, \dots, m, \\ u_{n+1} = \alpha_n u + (1 - \alpha_n)(c_n, d_{1,n}, \dots, d_{m,n}), & n \geq 0, \end{cases} \quad (3.62)$$

where $\{c_n\}$ and $\{d_{k,n}\}$ are as in (3.60), $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and for each $k = 1, 2, \dots, m$ and $n \geq 0$, and $u = (z, w_1, \dots, w_m) \in V \subset H^{m+1}$. Then, $\{u_n\}$ converges strongly to an element $u^* = (z^*, w_1^*, \dots, w_m^*)$ of $S_e(A_1, A_2, \dots, A_m)$, where $z^* \in \Omega$.

We note that if in Corollary 3.3.10 we assume that $u = (0, 0, \dots, 0) \in V$ we get the following result for the minimum-norm point of the extended solution set of the sum of a finite family of maximal monotone mappings in Hilbert spaces.

Theorem 3.3.11. *Let $A_k : H \rightarrow 2^H$, for $k \in \{1, 2, \dots, m\}$ and $m \geq 2$, be maximal monotone mappings satisfying $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. For arbitrary $u_0 = (z_0, w_{0,1}, \dots, w_{0,m}) \in V$ define an iterative algorithm by*

$$\begin{cases} x_{k,n} = (I + \lambda_{k,n}A_k)^{-1}(z_n + \lambda_{k,n}w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = w_{k,n} + \frac{z_n - x_{k,n}}{\lambda_{k,n}}, & k = 1, 2, \dots, m, \\ u_{n+1} = (1 - \alpha_n)(c_n, d_{1,n}, \dots, d_{m,n}), & n \geq 0, \end{cases} \quad (3.63)$$

where $\{c_n\}$ and $\{d_{k,n}\}$ are as in (3.60) and $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and for each $k = 1, 2, \dots, m$ and $n \geq 0$. Then, $\{u_n\}$ converges strongly to the minimum-norm point $u^* = (z^*, w_1^*, \dots, w_m^*)$ of $S_e(A_1, A_2, \dots, A_m)$, where $z^* \in \Omega$.

Proof. We note that since $u^* = (z^*, w_1^*, \dots, w_m^*) = P_{\mathcal{F}}(0)$, where $u^* \in S_e(A_1, A_2, \dots, A_m) \subset \mathcal{F}$, we obtain that u^* is the minimum-norm point of $S_e(A_1, A_2, \dots, A_m)$. $\square$

3.3.1 Application to Convex Minimization Problems

In this subsection, we apply Theorem 3.3.6 to study the convex minimization problems.

Let $g_k : H \rightarrow \mathbb{R}$, $k = 1, 2, \dots, m$ ($m \geq 2$), be a finite family of convex and lower semicontinuous functions. We consider the problem of finding $z^* \in H$ such that

$$g_1(z^*) + g_2(z^*) + \dots + g_m(z^*) = \min_{z \in H} \{g_1(z) + g_2(z) + \dots + g_m(z)\}. \quad (3.64)$$

We note that problem (3.64) is equivalent, by Fermat's rule, to the problem of finding $z^* \in H$ such that

$$0 \in \partial g_1(z^*) + \partial g_2(z^*) + \dots + \partial g_m(z^*), \quad (3.65)$$

where ∂g is a subdifferential of g , which is maximal monotone (see, e.g., [95]). So, we obtain the following theorem from Theorem 3.3.6.

Theorem 3.3.12. *Let $g_k : H \rightarrow \mathbb{R}$, for each $k = 1, 2, 3, \dots, m$ and $m \geq 2$, be a convex and lower semicontinuous function such that $\Omega = \min_{z \in H} \{g_1(z) + g_2(z) + \dots + g_m(z)\} \neq \emptyset$. Let $f : V \rightarrow V$ be a contraction mapping with constant α . For arbitrary $u_0 = (z_0, w_{0,1}, \dots, w_{0,m}) \in V$ define an iterative algorithm by*

$$\begin{cases} x_{k,n} = (I + \lambda_{k,n} \partial g_k)^{-1}(z_n + \lambda_{k,n} w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = w_{k,n} + \frac{z_n - x_{k,n}}{\lambda_{k,n}}, & k = 1, 2, \dots, m, \\ u_{n+1} = \alpha_n f(z_n, w_{1,n}, \dots, w_{m,n}) + (1 - \alpha_n)(c_n, d_{1,n}, \dots, d_{m,n}), & n \geq 0, \end{cases} \quad (3.66)$$

where for each $k = 1, 2, \dots, m$ and $n \geq 0$, $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$,

$$\begin{aligned} c_n &= \beta_n z_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \hat{c}_i, \\ d_{k,n} &= \beta_n w_{k,n} + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \hat{d}_{k,i}, \end{aligned}$$

$$\hat{c}_i = \begin{cases} z_n, & \text{if } \varphi_i(u_n) \leq 0, \\ z_n - \delta_i(\sum_{k=1}^m y_{k,i}), & \text{if } \varphi_i(u_n) > 0, \end{cases}$$

$$\hat{d}_{k,i} = \begin{cases} w_{k,n}, & \text{if } \varphi_i(u_n) \leq 0, \\ w_{k,n} - \delta_i(x_{k,i} - \bar{x}_i), & \text{if } \varphi_i(u_n) > 0, \end{cases}$$

with

$$\delta_i = \frac{\sum_{k=1}^m \langle z_n - x_{k,i}, y_{k,i} - w_{k,n} \rangle}{\|\sum_{k=1}^m y_{k,i}\|^2 + \sum_{k=1}^m \|x_{k,i} - \bar{x}_i\|^2},$$

for $\bar{x}_i = \frac{1}{m} \sum_{k=1}^m x_{k,i}$. Then, $\{u_n\}$ converges strongly to an element $u^* = (z^*, w_1^*, \dots, w_m^*)$ of $S_e(\partial g_1, \partial g_2, \dots, \partial g_m)$, where $z^* \in \Omega$.

Proof. Set $A_k = \partial g_k$, for $k = 1, 2, \dots, m$. Then, we have that A_k , $k = 1, 2, \dots, m$, is maximal monotone with $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. Hence, the conclusion follows from Theorem 3.3.6. $\square$

3.3.2 Numerical Example

In this subsection, we present a numerical experiment to illustrate our result. The following numerical example verifies the conclusion of Corollary 3.3.9.

Example 3.3.13. Let $H = l_2$, where l_2 is the space of sequences. Let $A_1, A_2, A_3 : l_2 \rightarrow l_2$ be defined by $A_1 x = x + (1, 2, 3, 0, 0, \dots)$, $A_2 x = 2x + (3, 4, 5, 0, 0, \dots)$ and $A_3 x = 3x - (2, 2, 2, 0, 0, \dots)$, where $x = (x_1, x_2, \dots) \in l_2$. We see that A_1, A_2 and A_3 are maximal monotone with $R(I + \lambda A_1) = R(I + \lambda A_2) = R(I + \lambda A_3) = l_2$ for each $\lambda > 0$. Now, by direct calculation we get that

$$\begin{aligned} x_{1,n} &= \frac{z_n + \lambda w_{1,n} - \lambda(1, 2, 3, 0, 0, \dots)}{1 + \lambda}, & y_{1,n} &= \frac{z_n + \lambda w_{1,n} + (1, 2, 3, 0, 0, \dots)}{1 + \lambda}, \\ x_{2,n} &= \frac{z_n + \lambda w_{2,n} - \lambda(3, 4, 5, 0, 0, \dots)}{1 + 2\lambda}, & y_{2,n} &= \frac{2z_n + 2\lambda w_{2,n} + (3, 4, 5, 0, 0, \dots)}{1 + 2\lambda}, \\ x_{3,n} &= \frac{z_n + \lambda w_{3,n} + \lambda(2, 2, 2, 0, 0, \dots)}{1 + 3\lambda} \text{ and } & y_{3,n} &= \frac{3z_n + 3\lambda w_{3,n} - (2, 2, 2, 0, 0, \dots)}{1 + 3\lambda}, \end{aligned}$$

for some $\lambda > 0$. Thus, if we assume $\lambda = 1$, $\alpha_n = \frac{1}{100(n+100)}$, $\beta_n = \frac{1}{n}$, for all $n \geq 1$, and $f(x) = \frac{x}{1000}$ then the Algorithm in (3.61) reduces to the following:

$$u_{n+1} = \frac{1}{10^5(n+10^2)}(z_n, w_{1,n}, w_{2,n}, w_{3,n}) + (1 - \frac{1}{10^2(n+10^2)})(c_n, d_{1,n}, d_{2,n}, d_{3,n}), \quad (3.67)$$

where $\{c_n\}$ and $\{d_{k,n}\}$, for $k = 1, 2, 3$, are as in (3.60).

Now, if we take initial point $u_1 = (z_1, w_{1,1}, w_{2,1}, w_{3,1})$, where $z_1 = (2, -1, -2, 0, 0, \dots)$, $w_{1,1} = (1, 1, 1, 0, 0, \dots)$, $w_{2,1} = (0, 0, 0, 0, 0, \dots)$ and $w_{3,1} = (-1, -1, -1, 0, 0, \dots)$, then the numerical result using MATLAB provide that the first component $\{z_n\}$ of $\{u_n\} = \{(z_n, w_{1,n}, w_{2,n}, w_{3,n})\}$ generated by (3.67) converges strongly to the solution $z^* = (-\frac{1}{3}, -\frac{2}{3}, -1, 0, 0, \dots) \in (A_1 + A_2 + A_3)^{-1}(0)$ (see, Table 3.2, below).

TABLE 3.2: Convergence of the first component $\{z_n\}$ of $\{u_n\}$ generated by (3.67)

n	z_n	$\ z_{n+1} - z_n\ _{l_2}$
1	(2.0000, -1.0000, -2.0000, 0, 0, ...)	
2	(2.1662, -1.3413, -2.5015, 0, 0, ...)	0.690
3	(1.3286, -1.4059, -2.3823, 0, 0, ...)	0.8485
4	(0.7003, -1.3819, -2.2615, 0, 0, ...)	0.6403
5	(0.4264, -1.3101, -2.1714, 0, 0, ...)	0.2971
10	(0.0554, -1.0218, -1.7395, 0, 0, ...)	0.0999
100	(-0.3081, -0.5944, -0.9294, 0, 0, ...)	9.4868e-04
200	(-0.2977, -0.6367, -0.9519, 0, 0, ...)	3.6056e-04
300	(-0.3064, -0.6500, -0.9776, 0, 0, ...)	2.4495e-04
400	(-0.3141, -0.6539, -0.9906, 0, 0, ...)	1.4142e-04
500	(-0.3196, -0.6549, -0.9969, 0, 0, ...)	1.4142e-04
1000	(-0.3300, -0.6554, -1.0006, 0, 0, ...)	0
2000	(-0.3324, -0.6597, -0.9987, 0, 0, ...)	0
3000	(-0.3330, -0.6635, -1.0001, 0, 0, ...)	0
	$\vdots$	$\vdots$
	$\downarrow$	$\downarrow$
	$(-\frac{1}{3}, -\frac{2}{3}, -1, 0, 0, \dots)$	0

Chapter 4

Convergence Theorems for a Zero of the Sum of Maximal Monotone Mappings in Banach Spaces

In this chapter, we present some preliminary results that will be useful throughout the chapter and study the forward-backward algorithm for a zero of the sum of maximal monotone mappings in the setting of Banach spaces. In addition, we prove strong convergence results for finding a zero of a finite family of maximal monotone mappings in Banach

4.1 Preliminaries

We shall need the following known results in the sequel.

In this section, let E be a real reflexive Banach space with its dual E^* .

Lemma 4.1.1. *[87] Let $f : E \rightarrow \mathbb{R}$ be a uniformly Fréchet differentiable and bounded on bounded subsets of E . Then, ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* .*

Lemma 4.1.2. [28, 122] *Let $f : E \rightarrow \mathbb{R}$ be a strongly coercive Legendre function. Then*

- i) $\nabla f : E \rightarrow E^*$ is one-to-one, onto and norm-to-weak* continuous;
- ii) $\{x \in E : D_f(x, y) \leq r\}$ is bounded for all $y \in E$ and $r > 0$;
- iii) $\text{dom} f^* = E^*$, f^* is Gâteaux differentiable and $\nabla f^* = (\nabla f)^{-1}$.

Lemma 4.1.3. [68] *If $f : E \rightarrow \mathbb{R}$ is a Gâteaux differentiable and totally convex function such that for $x_0 \in E$, the sequence $D_f(x_n, x_0)$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Proposition 4.1.4. [28] *Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable, strictly convex and lower semicontinuous function. Then the following hold:*

- i) $D_f(x, y) \geq 0 \ \forall x, y \in E$ and $D_f(x, y) = 0$ if and only if $x = y$.
- ii) $D_f(x, y) + D_f(y, x) = \langle x - y, \nabla f(x) - \nabla f(y) \rangle, \forall x, y \in E$.

Lemma 4.1.5. [25] *Let C be a nonempty, closed and convex subset of E . Let $f : E \rightarrow \mathbb{R}$ be Gâteaux differentiable function and total convex function and let $x \in E$. The Bregman projection P_C^f from E onto C is a unique vector with the following properties:*

- i) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \forall y \in C$.
- ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x)$, for all $y \in C$ and $x \in E$.

Lemma 4.1.6. [82] *Let $f : E \rightarrow (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $f^* : E^* \rightarrow (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^N t_i D_f(z, x_i), \quad (4.1)$$

where $\{x_i\} \subseteq E$ and $\{t_i\} \subseteq (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 4.1.7. (*[76], see also, [88]*) Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subset of E . Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E . Then the following assertions are equivalent

$$i) \lim_{n \rightarrow \infty} D_f(x_n, y_n) = 0;$$

$$ii) \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

Lemma 4.1.8. (*[122]*) Let $f : E \rightarrow \mathbb{R}$ be a convex and continuous function which is bounded on bounded subsets of E . Then the following are equivalent:

i) f^* is Fréchet differentiable and ∇f^* is uniformly norm to-norm continuous on bounded subsets of E^* , $\text{dom } f^* = E^*$;

ii) f is coercive and uniformly convex on bounded subset of E ;

iii) f^* is bounded on bounded subsets and uniformly smooth on bounded subsets of E^* , $\text{dom } f^* = E^*$;

Let $f : E \rightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous, convex and Gâteaux differentiable function and $\nabla f : E \times E^* \rightarrow \mathbb{R}$ be its Gâteaux derivative. We will denote the family of such function by $\mathcal{F}(E)$.

Lemma 4.1.9. (*[27]*) Let E be a Banach space and let $A : E \rightarrow 2^E$ be maximal monotone mapping. Let $f \in \mathcal{F}(E)$ such that ∇f is onto. Then given $z \in E^*$ there exists a unique $x \in E$ such that $z \in \nabla f(x) + Ax$.

The method of proof in [53] provides the following lemma.

Lemma 4.1.10. Let E and K be real Banach spaces. Take $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(K)$, for each $k = 1, 2, \dots, m$. Define $h : E \times K^m \rightarrow \mathbb{R}$ as $h(x, y_1, y_2, \dots, y_m) = f(x) + \sum_{k=1}^m g(y_k)$. Then, if f and g_k , for each $k = 1, 2, \dots, m$, are strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E and K respectively, then h is also strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $E \times K^m$.

Proposition 4.1.11. [87] Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable function. If ∇f is uniformly continuous on bounded subsets of E , then both f and ∇f are bounded on bounded subset of E .

Lemma 4.1.12. [41] Let $F : E \rightarrow 2^{E^*}$ be a maximal monotone mapping and $G : E \rightarrow E^*$ be a bounded and continuous monotone mapping. Then the mapping $F + G$ is a maximal monotone mapping.

Now, we shall prove the following lemma.

Lemma 4.1.13. Let f be a strongly convex function with constant $\mu > 0$. Then for all $y \in \text{dom } f$ and $x \in \text{int } \text{dom } f$, we have

$$D_f(y, x) \geq \frac{\mu}{2} \|x - y\|^2, \quad (4.2)$$

where, $D_f(y, x)$ is a Bregman distance with respect to f .

Proof. Let $y \in \text{dom } f$ and $x \in \text{int } \text{dom } f$. By Definition 1.2.32, we have

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|^2. \quad (4.3)$$

Hence, we obtain

$$D_f(y, x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle \geq \frac{\mu}{2} \|x - y\|^2, \quad (4.4)$$

which is the desired result. $\square$

Lemma 4.1.14. Let f be a convex and lower semi-continuous function on a Banach space E . The following assertions are equivalent (see, [122]):

- i) f is uniformly convex;
- ii) There exists modulus ϕ , $\forall (x, x^*), (y, y^*) \in \text{Gph}(\partial f)$ such that

$$f(y) \geq f(x) + \langle y - x, x^* \rangle + \phi(\|x - y\|);$$

iii) $\text{dom } f^* = E^*$, f^* is Fréchet differentiable and ∇f^* is uniformly continuous.

Lemma 4.1.15. [65] Let $\{s_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ such that $s_{k_j} < s_{k_j+1}$ for all $j \geq 0$. Define an integer sequence $\{m_k\}_{k \geq k_0}$ as

$$m_k = \max\{k_0 \leq l \leq k : s_l < s_{l+1}\}. \quad (4.5)$$

Then $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and for all $k \geq k_0$

$$\max\{s_{m_k}, s_k\} \leq s_{m_k+1}. \quad (4.6)$$

4.2 Convergence Results of Forward-Backward Method for a Zero of the Sum of Maximal Monotone Mappings in Banach Spaces

In this section, we study the forward-backward algorithm and prove strong convergence of the method to a zero of the sum of maximal monotone and Lipschitz continuous monotone mappings in real reflexive Banach spaces. For the rest of this section, let E be a real reflexive Banach space with its dual E^* .

Theorem 4.2.1. Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $F : E \rightarrow 2^{E^*}$ be a maximal monotone and $G : E \rightarrow E^*$ be a L -Lipschitz continuous monotone mappings such that $\Omega = \{x \in E : 0 \in Fx + Gx\} \neq \emptyset$. For arbitrary $u_0, u \in E$ define an iterative algorithm by

$$\begin{cases} z_n = (\nabla f + \lambda_n F)^{-1}(\nabla f u_n - \lambda_n G u_n); \\ u_{n+1} = \nabla f^*[\alpha_n \nabla f u + (1 - \alpha_n)(\nabla f z_n - \lambda_n (G z_n - G u_n))], \end{cases} \quad (4.7)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{L}$, for all $n \geq 0$. Then, $\{u_n\}$ is bounded.

Proof. Let $u^* \in \Omega$. Let $w_n = \nabla f^*(\nabla f z_n - \lambda_n(Gz_n - Gu_n))$. Then, from (4.7) and Lemma 4.1.6, we have

$$\begin{aligned} D_f(u^*, u_{n+1}) &= D_f(u^*, \nabla f^*[\alpha_n \nabla f u + (1 - \alpha_n) \nabla f w_n]) \\ &\leq \alpha_n D_f(u^*, u) + (1 - \alpha_n) D_f(u^*, w_n). \end{aligned} \quad (4.8)$$

Now, from (1.15), we obtain

$$\begin{aligned} D_f(u^*, w_n) &= D_f(u^*, \nabla f^*(\nabla f z_n - \lambda_n(Gz_n - Gu_n))) \\ &= f(u^*) - \langle u^* - w_n, \nabla f z_n - \lambda_n(Gz_n - Gu_n) \rangle - f(w_n) \\ &= f(u^*) + \langle w_n - u^*, \nabla f z_n \rangle + \langle u^* - w_n, \lambda_n(Gz_n - Gu_n) \rangle - f(w_n) \\ &= f(u^*) - \langle u^* - z_n, \nabla f z_n \rangle - f(z_n) + \langle u^* - z_n, \nabla f z_n \rangle + f(z_n) \\ &\quad + \langle w_n - u^*, \nabla f z_n \rangle + \langle u^* - w_n, \lambda_n(Gz_n - Gu_n) \rangle - f(w_n) \\ &= D_f(u^*, z_n) + \langle w_n - z_n, \nabla f z_n \rangle + f(z_n) - f(w_n) \\ &\quad - \langle w_n - u^*, \lambda_n(Gz_n - Gu_n) \rangle \\ &= D_f(u^*, z_n) - D_f(w_n, z_n) - \langle w_n - u^*, \lambda_n(Gz_n - Gu_n) \rangle, \end{aligned} \quad (4.9)$$

and from (1.17), we get

$$\begin{aligned} D_f(u^*, z_n) - D_f(w_n, z_n) &= D_f(u^*, u_n) - D_f(w_n, u_n) \\ &\quad + \langle \nabla f u_n - \nabla f z_n, u^* - w_n \rangle. \end{aligned} \quad (4.10)$$

Now, combining (4.9) and (4.10), we obtain

$$\begin{aligned} D_f(u^*, w_n) &= D_f(u^*, u_n) - D_f(w_n, u_n) + \langle \nabla f u_n - \nabla f z_n, u^* - w_n \rangle \\ &\quad - \langle w_n - u^*, \lambda_n(Gz_n - Gu_n) \rangle. \end{aligned} \quad (4.11)$$

Furthermore, from (1.16), we get

$$D_f(w_n, u_n) = D_f(w_n, z_n) + D_f(z_n, u_n) - \langle \nabla f u_n - \nabla f z_n, w_n - z_n \rangle. \quad (4.12)$$

Substituting (4.12) into (4.11), we have

$$\begin{aligned}
 D_f(u^*, w_n) &= D_f(u^*, u_n) - D_f(w_n, z_n) - D_f(z_n, u_n) + \langle \nabla f u_n - \nabla f z_n, w_n - z_n \rangle \\
 &\quad + \langle \nabla f u_n - \nabla f z_n, u^* - w_n \rangle - \langle w_n - u^*, \lambda_n(Gz_n - Gu_n) \rangle \\
 &= D_f(u^*, u_n) - D_f(w_n, z_n) - D_f(z_n, u_n) + \langle \nabla f u_n - \nabla f z_n, u^* - z_n \rangle \\
 &\quad - \langle w_n - u^*, \lambda_n(Gz_n - Gu_n) \rangle \\
 &= D_f(u^*, u_n) - D_f(w_n, z_n) - D_f(z_n, u_n) + \langle \nabla f u_n - \nabla f z_n, u^* - z_n \rangle \\
 &\quad - \langle w_n - z_n + z_n - u^*, \lambda_n(Gz_n - Gu_n) \rangle \\
 &= D_f(u^*, u_n) - D_f(w_n, z_n) - D_f(z_n, u_n) + \langle \nabla f u_n - \nabla f z_n, u^* - z_n \rangle \\
 &\quad - \langle w_n - z_n, \lambda_n(Gz_n - Gu_n) \rangle - \langle z_n - u^*, \lambda_n(Gz_n - Gu_n) \rangle \\
 &= D_f(u^*, u_n) - D_f(w_n, z_n) - D_f(z_n, u_n) - \langle w_n - z_n, \lambda_n(Gz_n - Gu_n) \rangle \\
 &\quad - \langle z_n - u^*, \lambda_n(Gz_n - Gu_n) - (\nabla f z_n - \nabla f u_n) \rangle. \tag{4.13}
 \end{aligned}$$

But $z_n = (\nabla f + \lambda_n F)^{-1}(\nabla f u_n - \lambda_n G u_n)$ implies that $\nabla f u_n - \lambda_n G u_n \in (\nabla f + \lambda_n F)z_n$ and hence there exists $t_n \in Fz_n$ such that $\nabla f u_n - \lambda_n G u_n = \nabla f z_n + \lambda_n t_n$. Therefore,

$$t_n = \frac{1}{\lambda_n}(\nabla f u_n - \lambda_n G u_n - \nabla f z_n). \tag{4.14}$$

On the other hand we know that $0 \in (Fu^* + Gu^*)$ and $t_n + Gz_n \in (F + G)z_n$. Since $F + G$ is monotone, we obtain

$$\langle t_n + Gz_n, z_n - u^* \rangle \geq 0. \tag{4.15}$$

Now, putting (4.14) into (4.15), we obtain

$$\langle z_n - u^*, \lambda_n(Gz_n - Gu_n) - (\nabla f z_n - \nabla f u_n) \rangle \geq 0. \tag{4.16}$$

Now, from (4.13), (4.16), Lemma 4.1.13 and the fact that G is Lipschitz continuous, we have

$$\begin{aligned}
 D_f(u^*, w_n) &\leq D_f(u^*, u_n) - [D_f(w_n, z_n) + D_f(z_n, u_n)] - \langle w_n - z_n, \lambda_n(Gz_n - Gu_n) \rangle \\
 &\leq D_f(u^*, u_n) - [D_f(w_n, z_n) + D_f(z_n, u_n)] + \lambda_n \|z_n - w_n\| \|Gz_n - Gu_n\| \\
 &\leq D_f(u^*, u_n) - [D_f(w_n, z_n) + D_f(z_n, u_n)] + \lambda_n L \|z_n - w_n\| \|z_n - Gu_n\| \\
 &\leq D_f(u^*, u_n) - \frac{\mu}{2} [\|w_n - z_n\|^2 + \|z_n - u_n\|^2] \\
 &\quad + \frac{\lambda_n L}{2} [\|w_n - z_n\|^2 + \|z_n - u_n\|^2] \\
 &= D_f(u^*, u_n) - \left(\frac{\mu - \lambda_n L}{2} \right) [\|w_n - z_n\|^2 + \|z_n - u_n\|^2], \tag{4.17}
 \end{aligned}$$

and hence

$$D_f(u^*, w_n) \leq D_f(u^*, u_n). \tag{4.18}$$

Finally, from (4.8) and (4.18), we get

$$\begin{aligned}
 D_f(u^*, u_{n+1}) &\leq \alpha_n D_f(u^*, u) + (1 - \alpha_n) D_f(u^*, u_n) \\
 &\leq \max\{D_f(u^*, u), D_f(u^*, u_n)\}, \tag{4.19}
 \end{aligned}$$

and by induction we obtain

$$D_f(u^*, u_n) \leq \max\{D_f(u^*, u), D_f(u^*, u_0)\}. \tag{4.20}$$

This implies $\{D_f(u^*, u_n)\}$ is bounded. Therefore, by Lemma 4.1.3, $\{u_n\}$ is bounded. $\square$

Theorem 4.2.2. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $F : E \rightarrow 2^{E^*}$ be a maximal monotone and $G : E \rightarrow E^*$ be a L -Lipschitz continuous monotone mapping such that $\Omega = \{x \in$*

$E : 0 \in Fx + Gx\} \neq \emptyset$. For arbitrary $u_0, u \in E$ define an iterative algorithm by

$$\begin{cases} z_n = (\nabla f + \lambda_n F)^{-1}(\nabla f u_n - \lambda_n G u_n); \\ u_{n+1} = \nabla f^*[\alpha_n \nabla f u + (1 - \alpha_n)(\nabla f z_n - \lambda_n(G z_n - G u_n))], \end{cases} \quad (4.21)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{L}$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to an element $u^* = P_{\Omega}^f(u)$.

Proof. From Theorem 4.2.1, we know that $\{u_n\}$ is bounded. Let $u^* = P_{\Omega}^f(u)$. We note that from Lemma 4.1.5 (i), we get

$$\langle v - u^*, \nabla f u - \nabla f u^* \rangle \leq 0, \forall u \in \Omega. \quad (4.22)$$

Now, from (1.20), (1.21), (4.7), Lemma 4.1.6 and (4.18), we obtain

$$\begin{aligned} D_f(u^*, u_{n+1}) &= D_f(u^*, \nabla f^*(\alpha_n \nabla f u + (1 - \alpha_n) \nabla f w_n)) \\ &= V_f(u^*, \alpha_n \nabla f u + (1 - \alpha_n) \nabla f w_n) \\ &\leq V_f(u^*, \alpha_n \nabla f u + (1 - \alpha_n) \nabla f w_n - \alpha_n(\nabla f u - \nabla f u^*)) \\ &\quad + \langle \alpha_n(\nabla f u - \nabla f u^*), u_{n+1} - u^* \rangle \\ &= V_f(u^*, \alpha_n \nabla f u^* + (1 - \alpha_n) \nabla f w_n) + \langle \alpha_n(\nabla f u - \nabla f u^*), u_{n+1} - u^* \rangle \\ &= D_f(u^*, \nabla f^*(\alpha_n \nabla f u^* + (1 - \alpha_n) \nabla f w_n)) \\ &\quad + \langle \alpha_n(\nabla f u - \nabla f u^*), u_{n+1} - u^* \rangle \\ &\leq \alpha_n D_f(u^*, u^*) + (1 - \alpha_n) D_f(u^*, w_n) \\ &\quad + \alpha_n \langle \nabla f u - \nabla f u^*, u_{n+1} - u^* \rangle \\ &= (1 - \alpha_n) D_f(u^*, w_n) + \alpha_n \langle \nabla f u - \nabla f u^*, u_{n+1} - u^* \rangle \\ &\leq (1 - \alpha_n) D_f(u^*, u_n) + \alpha_n \langle \nabla f u - \nabla f u^*, u_{n+1} - u^* \rangle \\ &\leq (1 - \alpha_n) D_f(u^*, u_n) + \alpha_n \|\nabla f u - \nabla f u^*\| \|u_{n+1} - u_n\| \\ &\quad + \alpha_n \langle \nabla f u - \nabla f u^*, u_n - u^* \rangle. \end{aligned} \quad (4.23)$$

Moreover, from (4.8) and (4.17), we obtain

$$\begin{aligned} D_f(u^*, u_{n+1}) &\leq \alpha_n D_f(u^*, u) + (1 - \alpha_n) D_f(u^*, u_n) \\ &\quad - (1 - \alpha_n) \left(\frac{\mu - \lambda_n L}{2} \right) \left[\|w_n - z_n\|^2 + \|z_n - u_n\|^2 \right] \end{aligned}$$

which implies that

$$\begin{aligned} &(1 - \alpha_n) \left(\frac{\mu - \lambda_n L}{2} \right) \left[\|w_n - z_n\|^2 + \|z_n - u_n\|^2 \right] \\ &\leq \alpha_n (D_f(u^*, u) - D_f(u^*, u_n)) + D_f(u^*, u_n) - D_f(u^*, u_{n+1}). \end{aligned} \quad (4.24)$$

Now, we divide the rest of the proof into two parts as follows:

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that the sequence of real numbers $\{D_f(u^*, u_n)\}$ is decreasing for all $n \geq n_0$. It then follows that $\{D_f(u^*, u_n)\}$ is convergent and hence $D_f(u^*, u_n) - D_f(u^*, u_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. Thus from (4.24) and the conditions of α_n we get

$$\lim_{n \rightarrow \infty} [\|w_n - z_n\|^2 + \|z_n - u_n\|^2] = 0, \quad (4.25)$$

which implies that

$$\lim_{n \rightarrow \infty} \|w_n - z_n\| = \lim_{n \rightarrow \infty} \|z_n - u_n\| = 0. \quad (4.26)$$

Moreover, from (4.21), we have

$$\lim_{n \rightarrow \infty} \|\nabla f u_{n+1} - \nabla f w_n\| = \alpha_n \|\nabla f u - \nabla f w_n\| = 0. \quad (4.27)$$

Since ∇f^* is uniformly continuous on bounded subset of E^* , we obtain

$$\lim_{n \rightarrow \infty} \|u_{n+1} - w_n\| = 0. \quad (4.28)$$

Consequently, from (4.26) and (4.28), we get

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|u_{n+1} - w_n\| + \|w_n - z_n\| \\ &+ \|z_n - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.29)$$

Now, since $\{u_n\}$ is bounded in E there exist $v \in E$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup v$ and

$$\limsup_{n \rightarrow \infty} \langle u_n - u^*, \nabla f u - \nabla f u^* \rangle = \lim_{k \rightarrow \infty} \langle u_{n_k} - u^*, \nabla f u - \nabla f u^* \rangle. \quad (4.30)$$

Now, we show that $v \in \Omega$, that is, $v \in (F+G)^{-1}(0)$. Suppose $(x, y) \in \text{Gph}(F+G)$. This implies that $y - Gx \in Fx$. Furthermore, we obtain from

$$z_{n_k} = (\nabla f + \lambda_{n_k} F)^{-1}(\nabla f u_{n_k} - \lambda_{n_k} G u_{n_k}) \quad \text{that} \quad (4.31)$$

$$(\nabla f u_{n_k} - \lambda_{n_k} G u_{n_k}) \in (\nabla f + \lambda_{n_k} F) z_{n_k} \quad (4.32)$$

and thus,

$$\frac{1}{\lambda_{n_k}}(\nabla f u_{n_k} - \nabla f z_{n_k} - \lambda_{n_k} G u_{n_k}) \in F z_{n_k}. \quad (4.33)$$

In addition, monotonicity of F yields that

$$\langle x - z_{n_k}, y - Gx - \frac{1}{\lambda_{n_k}}(\nabla f u_{n_k} - \nabla f z_{n_k} - \lambda_{n_k} G u_{n_k}) \rangle \geq 0. \quad (4.34)$$

Thus, we have

$$\begin{aligned} \langle x - z_{n_k}, y \rangle &\geq \langle x - z_{n_k}, Gx + \frac{1}{\lambda_{n_k}}(\nabla f u_{n_k} - \nabla f z_{n_k} - \lambda_{n_k} G u_{n_k}) \rangle \\ &= \langle x - z_{n_k}, Gx - G u_{n_k} \rangle + \langle x - z_{n_k}, \frac{1}{\lambda_{n_k}}(\nabla f u_{n_k} - \nabla f z_{n_k}) \rangle \\ &= \langle x - z_{n_k}, Gx - G z_{n_k} \rangle + \langle x - z_{n_k}, G z_{n_k} - G u_{n_k} \rangle \\ &\quad + \langle x - z_{n_k}, \frac{1}{\lambda_{n_k}}(\nabla f u_{n_k} - \nabla f z_{n_k}) \rangle \end{aligned}$$

$$\begin{aligned} \langle x - z_{n_k}, y \rangle &\geq \langle x - z_{n_k}, Gz_{n_k} - Gu_{n_k} \rangle \\ &\quad + \langle x - z_{n_k}, \frac{1}{\lambda_{n_k}}(\nabla f u_{n_k} - \nabla f z_{n_k}) \rangle, \end{aligned} \quad (4.35)$$

Furthermore, from (4.26), the fact that G and ∇f are uniformly continuous, we obtain $\lim_{n \rightarrow \infty} \|Gz_n - Gu_n\| = 0$ and $\lim_{n \rightarrow \infty} \|\nabla f u_n - \nabla f z_n\| = 0$. Consequently, inequality (4.35) provides that $\langle x - v, y \rangle \geq 0$ and hence by maximality of $F + G$ we obtain $v \in (F + G)^{-1}(0)$. Thus, by Lemma 4.1.5, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u_n - u^*, \nabla f u - \nabla f u^* \rangle &= \lim_{k \rightarrow \infty} \langle u_{n_k} - u^*, \nabla f u - \nabla f u^* \rangle \\ &= \langle v - u^*, \nabla f u - \nabla f u^* \rangle \leq 0. \end{aligned} \quad (4.36)$$

Therefore, from (4.23), (4.29), (4.36) and Lemma 3.1.9, we conclude that $D_f(u^*, u_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence, by Lemma 4.1.7 $u_n \rightarrow u^*$ as $n \rightarrow \infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_f(u^*, u_{n_i}) < D_f(u^*, u_{n_i+1}), \forall i \in \mathbb{N}. \quad (4.37)$$

Then, by Lemma 4.1.15, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and $\max\{D_f(u^*, u_{m_k}), D_f(u^*, u_k)\} \leq D_f(u^*, u_{m_k+1})$ for all $k \in \mathbb{N}$. Thus, from (4.24) and the conditions of α_n , we get

$$\lim_{k \rightarrow \infty} \|w_{m_k} - z_{m_k}\| = \lim_{k \rightarrow \infty} \|z_{m_k} - u_{m_k}\| = 0. \quad (4.38)$$

Moreover, following the methods in Case 1, we get

$$\lim_{k \rightarrow \infty} \|u_{m_k} - u_{m_k+1}\| = 0, \quad (4.39)$$

and

$$\limsup_{k \rightarrow \infty} \langle u_{m_k} - u^*, \nabla f u - \nabla f u^* \rangle \leq 0. \quad (4.40)$$

Now, from (4.23), we have

$$\begin{aligned}
 \alpha_{m_k}(D_f(u^*, u_{m_k})) &\leq D_f(u^*, u_{m_k}) - D_f(u^*, u_{m_k+1}) \\
 &\quad + \alpha_{m_k} \|u_{m_k} - u_{m_k+1}\| \|\nabla f u - \nabla f u^*\| \\
 &\quad + \alpha_{m_k} \langle u_{m_k} - u^*, \nabla f u - \nabla f u^* \rangle \\
 &\leq \alpha_{m_k} \|u_{m_k} - u_{m_k+1}\| \|\nabla f u - \nabla f u^*\| \\
 &\quad + \alpha_{m_k} \langle u_{m_k} - u^*, \nabla f u - \nabla f u^* \rangle.
 \end{aligned} \tag{4.41}$$

Consequently, since $\alpha_{m_k} > 0$, we get

$$\begin{aligned}
 D_f(u^*, u_{m_k}) &\leq \|u_{m_k} - u_{m_k+1}\| \|\nabla f u - \nabla f u^*\| \\
 &\quad + \langle u_{m_k} - u^*, \nabla f u - \nabla f u^* \rangle.
 \end{aligned} \tag{4.42}$$

Therefore, from (4.39), (4.40) and (4.42), we obtain $D_f(u^*, u_{m_k}) \rightarrow 0$ as $k \rightarrow \infty$. But from (4.23), we obtain that $D_f(u^*, u_{m_k+1}) \rightarrow 0$ as $k \rightarrow \infty$ and hence the fact $D_f(u^*, u_k) \leq D_f(u^*, u_{m_k+1})$ implies $\limsup_{k \rightarrow \infty} D_f(u^*, u_k) = 0$. Thus, $u_k \rightarrow u^*$ as $k \rightarrow \infty$. $\square$

If, in Theorem 4.2.2, we assume that $F = 0$, then we get the following corollary for a zero of monotone mapping in Banach spaces.

Corollary 4.2.3. Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $G : E \rightarrow E^*$ be a L -Lipschitz continuous monotone mapping such that $\Omega = \{x \in E : Gx = 0\} \neq \emptyset$. For arbitrary $u_0, u \in E$ define an iterative algorithm by

$$\begin{cases} z_n = \nabla f^*(\nabla f u_n - \lambda_n G u_n), \\ u_{n+1} = \nabla f^*[\alpha_n \nabla f u + (1 - \alpha_n)(\nabla f z_n - \lambda_n (G z_n - G u_n))], \end{cases} \tag{4.43}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{L}$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to an element $u^* = P_{\Omega}^f(u)$.

If, in Theorem 4.2.2, we assume that $G = 0$, then we get the following corollary for a zero of maximal monotone mapping in Banach spaces.

Corollary 4.2.4. Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $F : E \rightarrow 2^{E^*}$ be a maximal monotone mapping such that $\Omega = \{x \in E : 0 \in Fx\} \neq \emptyset$. For arbitrary $u_0, u \in E$ define an iterative algorithm by

$$\begin{cases} z_n = (\nabla f + \lambda_n F)^{-1}(\nabla f u_n), \\ u_{n+1} = \nabla f^*[\alpha_n \nabla f u + (1 - \alpha_n)(\nabla f z_n)], \end{cases} \quad (4.44)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda}$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to an element $u^* = P_{\Omega}^f(u)$.

If, in Theorem 4.2.2, E is in addition a smooth, strictly convex, then $f(x) = \frac{1}{2}\|x\|^2$ is a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with the conjugate $f^*(x^*) = \frac{1}{2}\|x^*\|^2$. This yields that $\nabla f = J_2 = J$ with $\nabla f^* = J_2^{-1} = J^{-1}$, where J is the normalized duality mapping. Thus, we get the following corollary for approximating a zero of the sum of two maximal monotone mappings in Banach spaces.

Corollary 4.2.5. Let E be a smooth, strictly convex and real reflexive Banach space with its dual E^* . Let $F : E \rightarrow 2^{E^*}$ be a maximal monotone mapping and $G : E \rightarrow E^*$ be an L -Lipschitz continuous monotone mapping such that $\Omega = \{x \in E : 0 \in Fx + Gx\} \neq \emptyset$. For arbitrary $u_0, u \in E$ define an iterative algorithm by

$$\begin{cases} z_n = (J + \lambda_n F)^{-1}(Ju_n - \lambda_n Gu_n), \\ u_{n+1} = J^{-1}[\alpha_n Ju + (1 - \alpha_n)(Jz_n - \lambda_n(Gz_n - Gu_n))], \end{cases} \quad (4.45)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{1}{L}$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to an element u^* of Ω .

Proof. Take $f(x) = \frac{1}{2}\|x\|^2$ which is strongly coercive and strongly convex Legendre functions, then the scheme in (4.7) reduces to the scheme in (4.45). Thus, the conclusion follows from Theorem 4.2.2 $\square$

If, in Theorem 4.2.2, we assume that $E = L_p$ ($1 < p < \infty$) spaces, then these spaces are smooth, strictly convex and real reflexive Banach spaces and hence we have the following corollary.

Corollary 4.2.6. Let $E = L_p$ spaces ($1 < p < \infty$). Let $F : E \rightarrow 2^{E^*}$ be a maximal monotone mapping and $G : E \rightarrow E^*$ be a L -Lipschitz continuous monotone mapping such that $\Omega = \{x \in E : 0 \in Fx + Gx\} \neq \emptyset$. For arbitrary $u_0, u \in E$ define an iterative algorithm by

$$\begin{cases} z_n = (J + \lambda_n F)^{-1}(Ju_n - \lambda_n Gu_n), \\ u_{n+1} = J^{-1}[\alpha_n Ju + (1 - \alpha_n)(Jz_n - \lambda_n(Gz_n - Gu_n))], \end{cases} \quad (4.46)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{L}$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to an element u^* of Ω .

If, in Corollary 4.2.5, we assume $E = H$, a real Hilbert space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have $\nabla f = J_2 = J_2^* = I$, identity mapping on H . Thus, we get the following corollary for approximating a zero of the sum of two monotone mappings in Hilbert spaces.

Corollary 4.2.7. Let H be a real Hilbert space. Let $F : H \rightarrow 2^H$ be a maximal monotone mapping and $G : H \rightarrow H$ be a L -Lipschitz continuous monotone mapping such that $\Omega = \{x \in H : 0 \in Fx + Gx\} \neq \emptyset$. For arbitrary $u_0, u \in H$ define an iterative algorithm by

$$\begin{cases} z_n = (I + \lambda_n F)^{-1}(u_n - \lambda_n Gu_n), \\ u_{n+1} = \alpha_n u + (1 - \alpha_n)(z_n - \lambda_n(Gz_n - Gu_n)), \end{cases} \quad (4.47)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{1}{L}$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to an element u^* of Ω .

4.2.1 Application to Convex Minimization Problems

Let $k : E \rightarrow \mathbb{R}$ be a convex, Gâteaux differentiable function with derivative ∇k which is Lipschitz continuous with constant L and $l : E \rightarrow \mathbb{R}$ be a convex, lower semicontinuous function. We consider the problem of finding $x^* \in E$ such that

$$k(x^*) + l(x^*) = \min_{x \in E} \{k(x) + l(x)\}. \quad (4.48)$$

This problem is equivalent, by Fermat's rule, to the problem of finding $x^* \in E$ such that

$$0 \in \nabla k(x^*) + \partial l(x^*), \quad (4.49)$$

where ∂l is a subdifferential of l . Note that ∇k is monotone mapping (see, Theorem 3.13,[81]) and ∂l is maximal monotone mapping (see, e.g., [95]). Thus, the following result can be obtained from Theorem 4.2.2.

Theorem 4.2.8. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $k : E \rightarrow \mathbb{R}$ be a convex, Gâteaux differentiable function with derivative ∇k which is Lipschitz continuous with constant L and $l : E \rightarrow \mathbb{R}$ be a convex, lower semicontinuous function such that $\Omega = \min_{x \in E} \{k(x) + l(x)\} \neq \emptyset$. For arbitrary $u_0, u \in E$ define an iterative algorithm by*

$$\begin{cases} z_n = (\nabla f + \lambda_n \partial l)^{-1}(\nabla f u_n - \lambda_n \nabla k u_n), \\ u_{n+1} = \nabla f^*[\alpha_n \nabla f u + (1 - \alpha_n)(\nabla f z_n - \lambda_n(\nabla k z_n - \nabla k u_n))], \end{cases} \quad (4.50)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{L}$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to an element $u^* = P_{\Omega}^f(u)$.

Proof. Set $F = \partial l$ and $G = \nabla k$. Then, we have that ∂l is maximal monotone mapping, ∇k is Lipschitz continuous monotone mapping with $\Omega = \{x \in E : 0 \in Fx + Gx\} \neq \emptyset$. Hence, the conclusion follows from Theorem 4.2.2. $\square$

4.2.2 Numerical Example

In this subsection, we present a numerical result to illustrate our result. The following numerical example verifies the conclusion of Corollary 4.2.7.

Example 4.2.9. Let $H = \mathbb{R}$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$, by $f(x) = \frac{x^2}{2}$, then $J_2 = x = J_2^*$. Let $F, G : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $Fx = 3x + 1$ and $Gx = 2x - 11$, then F is maximal monotone mapping on $\mathbb{R}$ and G is 2-Lipschitz continuous monotone mapping with $2 \in (F + G)^{-1}(0)$. Thus, the Algorithm $\{u_n\}$ generated by (4.47) reduces to the following:

$$u_{n+1} = \alpha_n u + (1 - \alpha_n) \left[\frac{u_n - 2\lambda_n u_n + 10\lambda_n - 10\lambda_n^2(2 - u_n)}{1 + 3\lambda_n} \right]. \quad (4.51)$$

Therefore, if we assume $u = 1$, $\lambda_n = \frac{1}{5}$, $\alpha_n = \frac{1}{100(n+100)}$, for all $n \geq 0$, and take different initial points $u_0 = 0$, $u_0 = 1$, $u_0 = 3$ and $u_0 = 4$, then, in all cases, the numerical experiment results using MATLAB provide that the sequence $\{u_n\}$ generated by (4.51) converges strongly to the solution $u^* = 2$ (see, Figure 4.1, below).

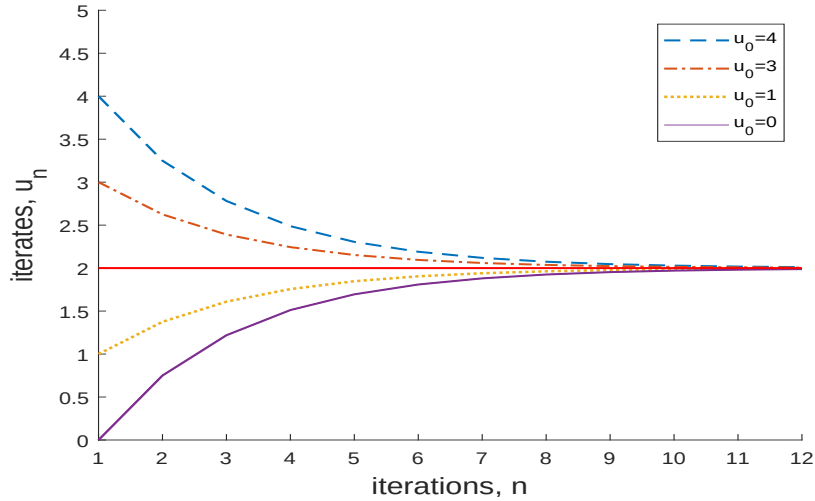


FIGURE 4.1: Convergence of the sequence $\{u_n\}$ generated by (4.51)

4.3 Convergence Results for a Zero of the Sum of a Finite Family of Maximal Monotone Mappings in Banach Spaces

In this section, we propose an algorithm for approximating a solution of the sum of a finite family of maximal monotone mappings in Banach spaces, and prove a strong convergence of the sequence generated by the proposed algorithm.

For the rest of this section, let E be a real reflexive Banach space with its dual E^* . Let $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(E^*)$, for each $k = 1, 2, \dots, m$. Let $A_k : E \rightarrow 2^{E^*}$, $k = 1, 2, \dots, m$, where $m \geq 2$, be a finite family of maximal monotone mappings satisfying $\Omega := \{z \in E : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\beta_n\}$ is a decreasing sequence in $(0, 1]$ such that $\beta_0 = 1$.

We now propose the following algorithm.

Algorithm 4.3.1.

Step 0: Select initial guess $v_0 = (z_0, w_{1,0}, w_{2,0}, \dots, w_{m,0}) \in V$, where

$$V = \{(z, w_1, w_2, \dots, w_m) \in E \times E^{*m} : \sum_{k=1}^m w_k = 0\}.$$

Step 1: Given $v_n = (z_n, w_{1,n}, w_{2,n}, \dots, w_{m,n}) \in V$, for $n \geq 0$, compute

$$\begin{aligned} x_{k,n} &= (\nabla f + \lambda_{k,n} A_k)^{-1}(\nabla f(z_n) + \lambda_{k,n} w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} &= \frac{\nabla f(z_n) - \nabla f(x_{k,n})}{\lambda_{k,n}} + w_{k,n}, & k = 1, 2, \dots, m, \end{aligned}$$

where $\lambda_{k,n} \in [\underline{\lambda}, \bar{\lambda}]$, for some $\bar{\lambda} > \underline{\lambda} > 0$.

Step 2 : Define

$$\begin{cases} \gamma_n = \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle, \\ \delta_n = \langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle x_{k,n} - \bar{x}_n, w_{k,n} \rangle \end{cases} \quad (4.52)$$

where $\bar{x}_n = \frac{1}{m} \sum_{k=1}^m x_{k,n}$. If $\gamma_n = \delta_n$ then **stop**. Otherwise, compute

$$T_i v_n = \begin{cases} v_n, & \text{if } v_n \in H_i, \\ (z', w'_1, w'_2, \dots, w'_m), & \text{if } v_n \notin H_i, \end{cases} \quad (4.53)$$

which is the Bregman projection of v_n onto the set

$$H_i = \{(z, w_1, w_2, \dots, w_m) \in V : \langle (z, w_1, w_2, \dots, w_m), (\sum_{k=1}^m y_{k,i}, x_{1,i} - \bar{x}_i, x_{2,i} - \bar{x}_i, \dots, x_{m,i} - \bar{x}_i) \rangle \leq \sigma_i\},$$

for each $i = 1, 2, \dots, n$, with

$$\begin{aligned} \sigma_i &= \langle \nabla f^*(\nabla f(z_n) + \eta_i \sum_{k=1}^m y_{k,i}), \sum_{k=1}^m y_{k,i} \rangle \\ &\quad + \sum_{k=1}^m \langle x_{k,i} - \bar{x}_i, \nabla g_k^*(\nabla g_k(w_n) + \eta_i(x_{k,i} - \bar{x}_i)) \rangle, \end{aligned}$$

where $(z', w'_1, w'_2, \dots, w'_m) \in E \times E^{*^m}$ is given by

$$\begin{cases} z' = \nabla f^*(\nabla f(z_n) + \eta_i(\sum_{k=1}^m y_{k,i})), \\ w'_k = \nabla g_k^*(\nabla g_k(w_{k,n}) + \eta_i(x_{k,i} - \bar{x}_i)), \\ \gamma_i = \langle z', \sum_{k=1}^m y_{k,i} \rangle + \sum_{k=1}^m \langle x_{k,i} - \bar{x}_i, w'_k \rangle, \end{cases} \quad (4.54)$$

for each $k = 1, 2, \dots, m$, and for some $\eta_i \in \mathbb{R}$.

Step 3: Compute

$$\begin{cases} u_n = \nabla h^*(\beta_n \nabla h(v_n) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \nabla h(T_i v_n)), \\ v_{n+1} = \nabla h^*(\alpha_n \nabla h(u) + (1 - \alpha_n) \nabla h(u_n)), \forall n \geq 0, \end{cases} \quad (4.55)$$

where $h : E \times E^{*m} \rightarrow \mathbb{R}$ is defined by $h(z, w_1, w_2, \dots, w_m) = f(z) + g_1(w_1) + g_2(w_2) + \dots + g_m(w_m)$ and $u \in E \times E^{*m}$. Set $n := n + 1$ and go to **Step 1**.

Remark 4.3.2.

We make the following observations to justify that Algorithm 4.3.1 is well defined.

1. From the maximality of A_k and the fact that ∇f is onto (see, Lemma 4.1.2) we obtain that the mapping $(\nabla f + \lambda_n A_{k,n})^{-1}$ is single-valued and hence $x_{k,n}$ and $y_{k,n}$ in Step 1 exist and are well defined for each $k = 1, 2, \dots, m$.
2. By rearranging equations in Step 1, we get

$$\nabla f(x_{k,n}) + \lambda_{k,n} y_{k,n} = \nabla f(z_n) + \lambda_{k,n} w_{k,n}, \quad (4.56)$$

as a consequence, we have $y_{k,n} \in A_k(x_{k,n})$.

3. From the definition of Bregman projection, it follows that there exists a point $(z', w'_1, w'_2, \dots, w'_m) \in E \times E^{*m}$ satisfying the system of equations in (4.54) which is the first order optimality condition for the Bregman projection of $(z_n, w_{1,n}, w_{2,n}, \dots, w_{m,n})$ onto the hyperplane H_i with respect to the auxiliary function h . Since H_i is closed and convex, the existence of the point $(z', w'_1, w'_2, \dots, w'_m)$ follows from Lemma 4.1.5.

Next, we shall prove the following propositions and lemmas which will be needed in the proof of our main theorem.

Proposition 4.3.3. If $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(E^*)$, for each $k = 1, 2, \dots, m$, are strongly coercive Legendre functions which are bounded, uniformly Fréchet

differentiable and totally convex on bounded subset of E and E^* , respectively, then $\gamma_n \leq \delta_n$, with γ_n, δ_n as defined in (4.52), which is equivalent to

$$\gamma_n \leq (1 - \epsilon)\gamma_n + \epsilon\delta_n \quad (4.57)$$

where $\epsilon \in (0, 1]$ for all $n \geq 0$.

Proof. From (4.52), (4.56) and the fact $\lambda_{k,n}$ is positive for each $k = 1, 2, \dots, m$, we get

$$\begin{aligned} \delta_n - \gamma_n &= \langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle x_{k,n} - \bar{x}_n, w_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle \\ &= \langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle x_{k,n}, w_{k,n} \rangle - \sum_{k=1}^m \langle \bar{x}_n, w_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle \\ &= \langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle x_{k,n}, w_{k,n} \rangle - \langle \bar{x}_n, \sum_{k=1}^m w_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle \end{aligned}$$

and

$$\begin{aligned} \delta_n - \gamma_n &= \langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle x_{k,n}, w_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle \\ &= \langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle x_{k,n}, w_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle - \langle z_n, \sum_{k=1}^m w_{k,n} \rangle \\ &= \sum_{k=1}^m \langle z_n - x_{k,n}, y_{k,n} - w_{k,n} \rangle \\ &= \sum_{k=1}^m \langle z_n - x_{k,n}, \frac{\nabla f(z_n) - \nabla f(x_{k,n})}{\lambda_{k,n}} \rangle \geq 0, \end{aligned} \quad (4.58)$$

where the last inequality holds by the virtue of strict convexity of f . This implies that $\gamma_n \leq \delta_n$, for all $n \geq 0$. The proof is complete. $\square$

Proposition 4.3.4. Let $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(E^*)$, for each $k = 1, 2, \dots, m$, be strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and uniformly convex on bounded subsets of E and E^* , respectively. If Algorithm 4.3.1 stops at n , then $v_n = (z_n, w_{1,n}, w_{2,n}, \dots, w_{m,n})$ belongs to $S_e(A_1, A_2, \dots, A_m)$, that is v_n solves problem (1.1).

Proof. Algorithm 4.3.1 stops at n if $\gamma_n = \delta_n$. This implies, from (4.58), that

$$\sum_{k=1}^m \left\langle z_n - x_{k,n}, \frac{\nabla f(z_n) - \nabla f(x_{k,n})}{\lambda_{k,n}} \right\rangle = 0. \quad (4.59)$$

By positivity of $\lambda_{k,n}$ and strict convexity of f , we obtain

$$z_n = x_{k,n}, \quad (4.60)$$

for each $k = 1, 2, \dots, m$. As a consequence, this and equation (4.56) yield

$$y_{k,n} = w_{k,n}, \quad (4.61)$$

for each $k = 1, 2, \dots, m$. Therefore,

$$\begin{aligned} 0 = \sum_{k=1}^m w_{k,n} = \sum_{k=1}^m y_{k,n} &\in A_1 x_{1,n} + A_2 x_{2,n} + \dots + A_m x_{m,n} \\ &= A_1 z_n + A_2 z_n + \dots + A_m z_n \end{aligned} \quad (4.62)$$

and hence $v_n = (z_n, w_{1,n}, w_{2,n}, \dots, w_{m,n}) \in S_e(A_1, A_2, \dots, A_m)$. $\square$

Lemma 4.3.5. *Let $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(E^*)$, for each $k = 1, 2, \dots, m$, be strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and uniformly convex on bounded subsets of E and E^* , respectively. Let $\{v_n\}$ be the sequence generated by Algorithm 4.3.1. Then, $S_e(A_1, A_2, \dots, A_m) \subseteq H_i$, for each $i \geq 1$.*

Proof. From (4.54) and the definition of σ_i , we have,

$$\gamma_i \leq \sigma_i, \text{ for each } i \geq 1. \quad (4.63)$$

Now, it suffices to show that for any $(z^*, w_1^*, w_2^*, \dots, w_m^*) \in S_e(A_1, A_2, \dots, A_m)$

$$\left\langle z^*, \sum_{k=1}^m y_{k,n} \right\rangle + \sum_{k=1}^m \langle x_{k,i} - \bar{x}_i, w_k^* \rangle \leq \gamma_i. \quad (4.64)$$

But, from (4.52) and (4.64), we have

$$\begin{aligned}
 & \gamma_i - [\langle z^*, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle x_{k,i} - \bar{x}_i, w_k^* \rangle] \\
 &= \sum_{k=1}^m \langle x_{k,i}, y_{k,i} \rangle - \langle z^*, \sum_{k=1}^m y_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,i} - \bar{x}_i, w_k^* \rangle \\
 &= \sum_{k=1}^m \langle x_{k,i}, y_{k,i} \rangle - \langle z^*, \sum_{k=1}^m y_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,i}, w_k^* \rangle + \sum_{k=1}^m \langle \bar{x}_i, w_k^* \rangle \\
 &= \sum_{k=1}^m \langle x_{k,i}, y_{k,i} \rangle - \langle z^*, \sum_{k=1}^m y_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,i}, w_k^* \rangle \\
 &= \sum_{k=1}^m \langle z^*, w_k^* \rangle + \sum_{k=1}^m \langle x_{k,i}, y_{k,i} \rangle - \langle z^*, \sum_{k=1}^m y_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,i}, w_k^* \rangle \\
 &= \sum_{k=1}^m \langle z^* - x_{k,i}, w_k^* - y_{k,i} \rangle \geq 0,
 \end{aligned}$$

where the last inequality follows by the virtue of the monotonicity of A_k , $w_k^* \in A_k(z^*)$ and $y_{k,i} \in A_k(x_{k,i})$, for each $k = 1, 2, \dots, m$. Therefore, inequality (4.64) holds and hence $S_e(A_1, A_2, \dots, A_m) \subseteq H_i$ for each $i \geq 1$. $\square$

Lemma 4.3.6. *Let $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(E^*)$, for each $k = 1, 2, \dots, m$, be strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and uniformly convex on bounded subsets of E and E^* , respectively. Then the sequence $\{v_n\}$ generated by Algorithm 4.3.1 is bounded in $E \times E^{*m}$.*

Proof. Since $\Omega \neq \emptyset$, from Lemma 3.1.5, we have $S_e(A_1, A_2, \dots, A_m) \neq \emptyset$. Let $v^* \in S_e(A_1, A_2, \dots, A_m)$. Then, from (4.55), Lemma 4.1.6 and the fact that T_i is Bregman strongly nonexpansive, for each $i \geq 1$, we have

$$\begin{aligned}
 D_h(v^*, u_n) &= D_h(v^*, \nabla h^*(\beta_n \nabla h(v_n) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \nabla h(T_i v_n))) \\
 &\leq \beta_n D_h(v^*, v_n) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) D_h(v^*, T_i v_n) \\
 &\leq \beta_n D_h(v^*, v_n) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) D_h(v^*, v_n) \\
 &= D_h(v^*, v_n).
 \end{aligned} \tag{4.65}$$

From Lemma 4.1.6 and (4.65), we also get

$$\begin{aligned} D_h(v^*, v_{n+1}) &= D_h(v^*, \nabla h^*(\alpha_n \nabla h(u) + (1 - \alpha_n) \nabla h(u_n))) \\ &\leq \alpha_n D_h(v^*, u) + (1 - \alpha_n) D_h(v^*, u_n) \\ &\leq \alpha_n D_h(v^*, u) + (1 - \alpha_n) D_h(v^*, v_n) \end{aligned} \quad (4.66)$$

$$\leq \max\{D_h(v^*, u), D_h(v^*, v_n)\}, \quad (4.67)$$

and hence by induction we obtain

$$D_h(v^*, v_n) \leq \max\{D_h(v^*, u), D_h(v^*, v_0)\}. \quad (4.68)$$

This, implies that $\{D_h(v^*, v_n)\}$ is bounded. Moreover, by Lemma 4.1.3, the sequence $\{v_n\}$ is bounded and so are the sequences $\{u_n\}$ and $\{T_i v_n\}$, for each $i \geq 1$. $\square$

Lemma 4.3.7. *If $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(E^*)$, for each $k = 1, 2, \dots, m$, are strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and uniformly convex on bounded subsets of E and E^* , respectively, then the sequence $\{v_n\}$ generated by Algorithm 4.3.1 converges strongly to an element v^* of $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof. Since $\Omega \neq \emptyset$, from Lemma 4.3.5 we have $\Omega \subset \mathcal{F}$ and hence $\mathcal{F} \neq \emptyset$. Let $v^* = P_{\mathcal{F}}^h(u)$. From (4.55) and the properties of V_h , we get

$$\begin{aligned} D_h(v^*, v_{n+1}) &= V_h(v^*, \nabla h(v_{n+1})) \\ &\leq V_h(v^*, \nabla h(v_{n+1}) - \alpha_n(\nabla h(u) - \nabla h(v^*))) \\ &\quad - \langle v_{n+1} - v^*, -(\alpha_n(\nabla h(u) - \nabla h(v^*))) \rangle \\ &= V_h(v^*, \alpha_n \nabla h(u) + (1 - \alpha_n) \nabla h(u_n) - \alpha_n \nabla h(u) + \alpha_n \nabla h(v^*)) \\ &\quad + \alpha_n \langle v_{n+1} - v^*, \nabla h(u) - \nabla h(v^*) \rangle \end{aligned}$$

and

$$\begin{aligned}
 D_h(v^*, v_{n+1}) &\leq V_h(v^*, \alpha_n \nabla h(v^*) + (1 - \alpha_n) \nabla h(u_n)) \\
 &\quad + \alpha_n \langle v_{n+1} - v^*, \nabla h(u) - \nabla h(v^*) \rangle \\
 &= D_h(v^*, \nabla h^*(\alpha_n \nabla h(v^*) + (1 - \alpha_n) \nabla h(u_n))) \\
 &\quad + \alpha_n \langle v_{n+1} - v^*, \nabla h(u) - \nabla h(v^*) \rangle \\
 &\leq \alpha_n D_h(v^*, v^*) + (1 - \alpha_n) D_h(v^*, u_n) \\
 &\quad + \alpha_n \langle v_{n+1} - v^*, \nabla h(u) - \nabla h(v^*) \rangle \\
 &\leq (1 - \alpha_n) D_h(v^*, v_n) + \alpha_n \langle v_{n+1} - v_n, \nabla h(u) - \nabla h(v^*) \rangle \\
 &\quad + \alpha_n \langle v_n - v^*, \nabla h(u) - \nabla h(v^*) \rangle \\
 &\leq (1 - \alpha_n) D_h(v^*, v_n) + \alpha_n \|v_{n+1} - v_n\| \|\nabla h(u) - \nabla h(v^*)\| \\
 &\quad + \alpha_n \langle v_n - v^*, \nabla h(u) - \nabla h(v^*) \rangle. \tag{4.69}
 \end{aligned}$$

Moreover, from (4.55), (4.66) and Lemma 4.1.6, we get

$$\begin{aligned}
 D_h(v^*, v_{n+1}) &\leq \alpha_n D_h(v^*, u) + (1 - \alpha_n) D_h(v^*, u_n) \\
 &= \alpha_n D_h(v^*, u) \\
 &\quad + (1 - \alpha_n) D_h(v^*, \nabla h^*(\beta_n \nabla h(v_n) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i v_n)) \\
 &\leq \alpha_n D_h(v^*, u) + (1 - \alpha_n) \beta_n D_h(v^*, v_n) \\
 &\quad + (1 - \alpha_n) \sum_{i=1}^n (\beta_{i-1} - \beta_i) D_h(v^*, T_i v_n) \\
 &= \alpha_n D_h(v^*, u) + (1 - \alpha_n) \beta_n D_h(v^*, v_n) \\
 &\quad + (1 - \alpha_n) \sum_{i=1}^n (\beta_{i-1} - \beta_i) D_h(v^*, T_i v_n) \\
 &\quad - (1 - \alpha_n) \sum_{i=1}^n (\beta_{i-1} - \beta_i) D_h(v^*, v_n) \\
 &\quad + (1 - \alpha_n) \sum_{i=1}^n (\beta_{i-1} - \beta_i) D_h(v^*, v_n) \\
 &= \alpha_n D_h(v^*, u) + (1 - \alpha_n) \beta_n D_h(v^*, v_n) \\
 &\quad + (1 - \alpha_n) \sum_{i=1}^n (\beta_{i-1} - \beta_i) (D_h(v^*, T_i v_n) - D_h(v^*, v_n)) \\
 &\quad + (1 - \alpha_n) D_h(v^*, v_n) - (1 - \alpha_n) \beta_n D_h(v^*, v_n) \\
 &= \alpha_n (D_h(v^*, u) - D_h(v^*, v_n)) + D_h(v^*, v_n) \\
 &\quad + (1 - \alpha_n) \sum_{i=1}^n (\beta_{i-1} - \beta_i) (D_h(v^*, T_i v_n) - D_h(v^*, v_n)), \tag{4.70}
 \end{aligned}$$

which implies that

$$\begin{aligned} & (1 - \alpha_n) \sum_{i=1}^n (\beta_{i-1} - \beta_i) (D_h(v^*, v_n) - D_h(v^*, T_i v_n)) \\ & \leq \alpha_n (D_h(v^*, u) - D_h(v^*, v_n)) + D_h(v^*, v_n) - D_h(v^*, v_{n+1}) \end{aligned} \quad (4.71)$$

Now, we divide the rest of the proof into two parts as follows:

Case 1. Suppose that there exist $n_0 \in \mathbb{N}$ such that $\{D_h(v^*, v_n)\}$ is decreasing. Thus, $\{D_h(v^*, v_n)\}$ converges and hence $D_h(v^*, v_n) - D_h(v^*, v_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. From (4.71) and the conditions of α_n , we get

$$\lim_{n \rightarrow \infty} (D_h(v^*, v_n) - D_h(v^*, T_i v_n)) = 0. \quad (4.72)$$

Now, since T_i is Bregman strongly nonexpansive we get

$$\lim_{n \rightarrow \infty} D_h(v_n, T_i v_n) = 0, \text{ for each } i \geq 1. \quad (4.73)$$

Thus, by Lemma 4.1.7, we obtain

$$\lim_{n \rightarrow \infty} \|T_i v_n - v_n\| = 0, \text{ for each } i \geq 1, \quad (4.74)$$

and hence

$$\lim_{n \rightarrow \infty} \|T_n v_n - v_n\| = 0. \quad (4.75)$$

Furthermore, since E is reflexive and $\{v_n\}$ is bounded, there exist $v \in E \times E^{*m}$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $\{v_{n_k}\}$ converge weakly to v and

$$\limsup_{n \rightarrow \infty} \langle v_n - v^*, \nabla h(u) - \nabla h(v^*) \rangle = \lim_{k \rightarrow \infty} \langle v_{n_k} - v^*, \nabla h(u) - \nabla h(v^*) \rangle.$$

Now, using (4.74) and the fact that T_i is Bregman strongly nonexpansive mapping we have $v \in F(T_i)$ for each $i \geq 1$ and hence $v \in \mathcal{F}$. Since $v^* = P_{\mathcal{F}}^h(u)$ by Lemma

4.1.5, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle v_n - v^*, \nabla h(u) - \nabla h(v^*) \rangle &= \lim_{k \rightarrow \infty} \langle v_{n_k} - v^*, \nabla h(u) - \nabla h(v^*) \rangle \\ &= \langle v - v^*, \nabla h(u) - \nabla h(v^*) \rangle \leq 0. \end{aligned} \quad (4.76)$$

Moreover, from (4.55) and Lemma 4.1.6, we have

$$\begin{aligned} D_h(v_n, v_{n+1}) &= D_h(v_n, \nabla h^*(\alpha_n \nabla h(u) + (1 - \alpha_n) \nabla h(u_n))) \\ &\leq \alpha_n D_h(v_n, u) + (1 - \alpha_n) D_h(v_n, u_n) \\ &\leq \alpha_n D_h(v_n, u) \\ &\quad + (1 - \alpha_n) D_h(v_n, \nabla h^*(\beta_n \nabla h(v_n) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \nabla h(T_i v_n))) \\ &\leq \alpha_n D_h(v_n, u) + (1 - \alpha_n) \beta_n D_h(v_n, v_n) \\ &\quad + (1 - \alpha_n) \sum_{i=1}^n (\beta_{i-1} - \beta_i) D_h(v_n, T_i v_n). \end{aligned} \quad (4.77)$$

Thus, from the conditions of α_n , (4.73) and (4.77), we obtain

$$\lim_{n \rightarrow \infty} D_h(v_n, v_{n+1}) = 0, \quad (4.78)$$

and hence, by Lemma 4.1.7, we get

$$\lim_{n \rightarrow \infty} \|v_n - v_{n+1}\| = 0. \quad (4.79)$$

Therefore, the relations in (4.69), (4.76), (4.79) and Lemma 3.1.9 imply that $D_h(v^*, v_n) \rightarrow 0$ as $n \rightarrow \infty$. Consequently, by Lemma 4.1.7, we obtain $\{v_n\}$ converges strongly to v^* .

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_h(v^*, v_{n_i}) < D_h(v^*, v_{n_i+1}), \forall i \in \mathbb{N}. \quad (4.80)$$

Then, by Lemma 4.1.15, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and $\max\{D_h(v^*, v_{m_k}), D_h(v^*, v_k)\} \leq D_h(v^*, v_{m_k+1})$ for

all $k \in \mathbb{N}$. Thus, from (4.71), conditions of α_n , we get

$$\lim_{k \rightarrow \infty} D_h(v^*, v_{m_k}) - D_h(v^*, T_i v_{m_k}) = 0, \quad (4.81)$$

for each $i \in \mathbb{N}$. Therefore, from the fact that T_i is Bregman strongly nonexpansive, (4.73) and Lemma 4.1.7, we obtain

$$\lim_{k \rightarrow \infty} \|v_{m_k} - T_i v_{m_k}\| = 0, \quad (4.82)$$

for each $i \in \mathbb{N}$ and hence following the methods in Case 1, we get

$$\limsup_{k \rightarrow \infty} \langle v_{m_k} - v^*, \nabla h(u) - \nabla h(v^*) \rangle \leq 0 \quad (4.83)$$

and

$$\lim_{k \rightarrow \infty} \|v_{m_k} - v_{m_k+1}\| = 0. \quad (4.84)$$

Now, from (4.69) we have

$$\begin{aligned} \alpha_{m_k} D_h(v^*, v_{m_k}) &\leq D_h(v^*, v_{m_k}) - D_h(v^*, v_{m_k+1}) \\ &\quad + \alpha_{m_k} \|v_{m_k+1} - v_{m_k}\| \|\nabla h(v) - \nabla h(v^*)\| \\ &\quad + \alpha_{m_k} \langle v_{m_k} - v^*, \nabla h(u) - \nabla h(v^*) \rangle. \end{aligned} \quad (4.85)$$

Consequently, since $\alpha_{m_k} > 0$ and $D_h(v^*, v_{m_k}) \leq D_h(v^*, v_{m_k+1})$, we get

$$\begin{aligned} D_h(v^*, v_{m_k}) &\leq \|v_{m_k+1} - v_{m_k}\| \|\nabla h(v) - \nabla h(v^*)\| \\ &\quad + \langle v_{m_k} - v^*, \nabla h(u) - \nabla h(v^*) \rangle. \end{aligned} \quad (4.86)$$

Therefore, from (4.83), (4.84) and (4.86), we obtain $\lim_{k \rightarrow \infty} D_h(v^*, v_{m_k}) = 0$. This and (4.69) imply that $\limsup_{k \rightarrow \infty} D_h(v^*, v_{m_k+1}) = 0$. But, the fact that $D_h(v^*, v_k) \leq D_h(v^*, v_{m_k+1})$ provides

$$\limsup_{k \rightarrow \infty} D_h(v^*, v_k) = 0. \quad (4.87)$$

Thus, by Lemma 4.1.7 we obtain that $v_k \rightarrow v^*$ as $k \rightarrow \infty$. $\square$

Theorem 4.3.8. *Let $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(E^*)$, for each $k = 1, 2, \dots, m$, be strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and uniformly convex on bounded subsets of E and E^* respectively. Then, the sequence $\{v_n\}$ generated by Algorithm 4.3.1 converges strongly to an element $v^* = (z^*, w_1^*, w_2^*, \dots, w_m^*)$ of $S_e(A_1, A_2, \dots, A_m)$, where $z^* \in \Omega$.*

Proof. Note that the equality in (4.75) yields

$$\lim_{n \rightarrow \infty} \|z_n - \nabla f^*(\nabla f(z_n) + \eta_n \sum_{k=1}^m y_{k,n})\| = 0, \quad (4.88)$$

and

$$\lim_{n \rightarrow \infty} \|w_{k,n} - \nabla g_k^*(\nabla g_k(w_{k,n}) + \eta_n(x_{k,n} - \overline{x_n}))\| = 0, \quad (4.89)$$

for all $k = 1, 2, \dots, m$. Moreover, using Proposition 4.3.3, we have

$$\begin{aligned} & \langle \nabla f^*(\nabla f(z_n) + \eta_n \sum_{k=1}^m y_{k,n}), \sum_{k=1}^m y_{k,n} \rangle \\ & + \sum_{k=1}^m \langle \nabla g_k^*(\nabla g_k(w_{k,n}) + \eta_n(x_{k,n} - \overline{x_n})), x_{k,n} - \overline{x_n} \rangle \\ & \leq (1 - \epsilon) \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle + \epsilon [\langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle w_{k,n}, x_{k,n} - \overline{x_n} \rangle], \end{aligned}$$

which implies that

$$\begin{aligned} & \langle -\nabla f^*(\nabla f(z_n) + \eta_n \sum_{k=1}^m y_{k,n}), \sum_{k=1}^m y_{k,n} \rangle \\ & + \sum_{k=1}^m \langle -\nabla g_k^*(\nabla g_k(w_{k,n}) + \eta_n(x_{k,n} - \overline{x_n})), x_{k,n} - \overline{x_n} \rangle \\ & \geq -(1 - \epsilon) \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle - \epsilon [\langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle w_{k,n}, x_{k,n} - \overline{x_n} \rangle]. \quad (4.90) \end{aligned}$$

By adding $\langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle w_{k,n}, x_{k,n} - \bar{x}_n \rangle$ on both sides of inequality (4.90) and using equation (4.56), we get

$$\begin{aligned}
 & \langle z_n - \nabla f^*(\nabla f(z_n) + \eta_n \sum_{k=1}^m y_{k,n}), \sum_{k=1}^m y_{k,n} \rangle \\
 & + \sum_{k=1}^m \langle w_{k,n} - \nabla g_k^*(\nabla g_k(w_{k,n}) + \eta_n(x_{k,n} - \bar{x}_n)), x_{k,n} - \bar{x}_n \rangle \\
 \geq & -(1 - \epsilon) \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle + (1 - \epsilon) [\langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle w_{k,n}, x_{k,n} - \bar{x}_n \rangle] \\
 = & (1 - \epsilon) [\langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle w_{k,n}, x_{k,n} - \bar{x}_n \rangle - \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle] \\
 = & (1 - \epsilon) [\langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle w_{k,n}, x_{k,n} \rangle - \sum_{k=1}^m \langle w_{k,n}, \bar{x}_n \rangle - \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle] \\
 = & (1 - \epsilon) [\langle z_n, \sum_{k=1}^m y_{k,n} \rangle + \sum_{k=1}^m \langle w_{k,n}, x_{k,n} \rangle - \sum_{k=1}^m \langle x_{k,n}, y_{k,n} \rangle - \sum_{k=1}^m \langle z_n, w_{k,n} \rangle] \\
 = & (1 - \epsilon) \sum_{k=1}^m \langle z_n - x_{k,n}, y_{k,n} - w_{k,n} \rangle \\
 = & (1 - \epsilon) \sum_{k=1}^m \langle z_n - x_{k,n}, \frac{\nabla f(z_n) - \nabla f(x_{k,n})}{\lambda_{k,n}} \rangle \\
 \geq & \frac{1 - \epsilon}{\bar{\lambda}} \sum_{k=1}^m \langle z_n - x_{k,n}, \nabla f(z_n) - \nabla f(x_{k,n}) \rangle,
 \end{aligned}$$

and hence

$$\begin{aligned}
 & \langle z_n - \nabla f^*(\nabla f(z_n) + \eta_n \sum_{k=1}^m y_{k,n}), \sum_{k=1}^m y_{k,n} \rangle \\
 & + \sum_{k=1}^m \langle w_{k,n} - \nabla g_k^*(\nabla g_k(w_{k,n}) + \eta_n(x_{k,n} - \bar{x}_n)), x_{k,n} - \bar{x}_n \rangle \\
 \geq & \frac{1 - \epsilon}{\bar{\lambda}} \sum_{k=1}^m D_f(z_n, x_{k,n}) + D_f(x_{k,n}, z_n),
 \end{aligned} \tag{4.91}$$

where inequality (4.91) follows from (6.2.1). Thus, using the relations in (4.88), (4.89), (4.91) and the fact that the sequences $\{x_{k,n}\}$ and $\{y_{k,n}\}$ are bounded, we get

$$\lim_{n \rightarrow \infty} D_f(z_n, x_{k,n}) = \lim_{n \rightarrow \infty} D_f(x_{k,n}, z_n) = 0. \tag{4.92}$$

Consequently, from Lemma 4.1.7, we obtain

$$\lim_{n \rightarrow \infty} \|z_n - x_{k,n}\| = 0, \quad (4.93)$$

for each $k = 1, 2, \dots, m$. In addition, since the sequences $\{x_{k,n}\}$, $\{y_{k,n}\}$ and $\{z_n\}$ are bounded and ∇f is uniformly continuous on bounded sets, from (4.93), we get

$$\lim_{n \rightarrow \infty} \|\nabla f(z_n) - \nabla f(x_{k,n})\| = 0. \quad (4.94)$$

Thus, from (4.56), we obtain

$$\lim_{n \rightarrow \infty} \|y_{k,n} - w_{k,n}\| = 0, \text{ for each } k = 1, 2, \dots, m. \quad (4.95)$$

Moreover, from Lemma 4.3.7 the sequence $\{v_n\} = \{(z_n, w_{1,n}, \dots, w_{m,n})\}$ converges strongly to a point $v^* = (z^*, w_1^*, \dots, w_m^*) \in \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$. In addition, the equations in (4.93) and (4.95) imply that $x_{k,n} \rightarrow z^*$ and $y_{k,n} \rightarrow w_k^*$, for each $k = 1, 2, \dots, m$. Since $Gph(A_k)$ is closed and $(x_{k,n}, y_{k,n}) \in Gph(A_k)$, for all $k = 1, 2, \dots, m$ and $n \geq 0$, we get $w_k^* \in A_k(z^*)$. Furthermore, since $\{v_n\} \subset V$ and V is a closed subspace, we also have $v^* = (z^*, w_1^*, \dots, w_m^*) \in V$ and hence, $v^* \in S_e(A_1, A_2, \dots, A_m)$, where $z^* \in \Omega$. $\square$

Remark 4.3.9. We observe that Algorithm 4.3.1 is equivalent to the following scheme:

$$\begin{cases} v_0 = (z_0, w_{1,0}, \dots, w_{m,0}) \in V \text{ chosen arbitrarily,} \\ x_{k,n} = (\nabla f + \lambda_{k,n} A_k)^{-1}(\nabla f(z_n) + \lambda_{k,n} w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = \frac{\nabla f(z_n) - \nabla f(x_{k,n})}{\lambda_{k,n}} + w_{k,n}, & k = 1, 2, \dots, m, \\ v_{n+1} = \nabla h^*(\alpha_n \nabla h(z, w_1, w_2, \dots, w_m) + (1 - \alpha_n) \nabla h(a_n, b_{1,n}, \dots, b_{m,n})), \end{cases} \quad (4.96)$$

where $(z, w_1, w_2, \dots, w_m) \in V$,

$$\begin{aligned} a_n &= \nabla f^*(\beta_n \nabla f(z_n) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \nabla f(\widehat{a}_i)), \\ b_{k,n} &= \nabla g_k^*(\beta_n \nabla g_k(w_{k,n}) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \nabla g_k(\widehat{b}_{k,i})) \end{aligned}$$

and $\lambda_{k,n} \in [\underline{\lambda}, \bar{\lambda}]$, for some $\bar{\lambda} > \underline{\lambda} > 0$, for each $k = 1, 2, \dots, m$, with

$$\widehat{a}_i = \begin{cases} z_n, & \text{if } v_n \in H_i, \\ \nabla f^*(\nabla f(z_n) + \eta_i(\sum_{k=1}^m y_{k,i})), & \text{if } v_n \notin H_i \end{cases}$$

$$\widehat{b}_{k,i} = \begin{cases} w_{k,n}, & \text{if } v_n \in H_i, \\ \nabla g_k^*(\nabla g_k(w_{k,n}) + \eta_i(x_{k,i} - \bar{x}_i)), & \text{if } v_n \notin H_i, \end{cases}$$

such that

$$\begin{aligned} \gamma_i &= \langle \nabla f^*(\nabla f(z_n) + \eta_i(\sum_{k=1}^m y_{k,i})), \sum_{k=1}^m y_{k,i} \rangle \\ &\quad + \sum_{k=1}^m \langle x_{k,i} - \bar{x}_i, \nabla g_k^*(\nabla g_k(w_{k,n}) + \eta_i(x_{k,i} - \bar{x}_i)) \rangle \end{aligned}$$

for $y_{k,i} \in A_k(x_{k,i})$, $\eta_i \in \mathbb{R}$ and H_i is as Algorithm 4.3.1.

We note that if in (4.96) we assume that $(z, w_1, w_2, \dots, w_m) = (0, 0, 0, \dots, 0) \in V$ we get the following theorem for approximating the minimum-norm point of the extended solution set of the sum of a finite family of maximal monotone mappings in Banach spaces.

Theorem 4.3.10. *Let $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(E^*)$, for each $k = 1, 2, \dots, m$, be strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and uniformly convex on bounded subsets of E and E^* , respectively. Let $A_k : E \rightarrow 2^{E^*}$, for $k \in \{1, 2, \dots, m\}$ and $m \geq 2$, be maximal monotone mappings satisfying $\Omega := \{z \in E : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. For arbitrary*

$v_0 = (z_0, w_{0,1}, \dots, w_{0,m}) \in V$ define an iterative algorithm by

$$\begin{cases} x_{k,n} = (\nabla f + \lambda_{k,n} A_k)^{-1}(\nabla f(z_n) + \lambda_{k,n} w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = \frac{\nabla f(z_n) - \nabla f(x_{k,n})}{\lambda_{k,n}} + w_{k,n}, & k = 1, 2, \dots, m, \\ v_{n+1} = \nabla h^*((1 - \alpha_n) \nabla h(a_n, b_{1,n}, \dots, b_{m,n})), \end{cases}$$

where $\{a_n\}$ and $\{b_{k,n}\}$ are as in (4.96), $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and for each $k = 1, 2, \dots, m$ and $n \geq 0$. Then, $\{v_n\}$ converges strongly to the minimum-norm point $v^* = (z^*, w_1^*, \dots, w_m^*)$ of $S_e(A_1, A_2, \dots, A_m)$ with respect to the Bregman distance and the auxiliary function h , where $z^* \in \Omega$.

If, in (4.96), E is smooth, strictly convex and real reflexive Banach space with its dual E^* and $f : E \rightarrow \mathbb{R}$ and $g_k : E^* \rightarrow \mathbb{R}$ are defined by $f(x) = \frac{1}{p} \|x\|^p$, $g_k(y_k) = \frac{1}{q} \|y_k\|^q$, for each $k = 1, 2, \dots, m$, and $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ with $h(x, y_1, y_2, \dots, y_m) = \frac{1}{p} \|x\|^p + \frac{1}{q} \sum_{k=1}^m \|y_k\|^q$. Then, f , g_k and h are proper, lower semicontinuous, strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and uniformly convex on bounded subsets of E , E^* and $E \times E^{*m}$, respectively. Moreover, the corresponding conjugates are given by $f^*(x^*) = \frac{1}{q} \|x^*\|^q$, $g_k^*(y_k^*) = \frac{1}{p} \|y_k^*\|^p$, and $h^*(x^*, y_1^*, y_2^*, \dots, y_m^*) = \frac{1}{q} \|x^*\|^q + \frac{1}{p} \sum_{k=1}^m \|y_k^*\|^p$ and the corresponding gradients are given by $\nabla f = J_p$, $\nabla g_k = J_q$, $\nabla h = (J_p, J_q, \dots, J_q)$ with $\nabla f^* = J_q$, $\nabla g_k^* = J_p$, for each $k = 1, 2, \dots, m$ and $\nabla h^* = (J_q, J_p, \dots, J_p)$. Thus, we get the following corollary for approximating a zero of the sum of a finite family of maximal monotone mappings in Banach spaces.

Corollary 4.3.11. Let E be a smooth, strictly convex and reflexive real Banach space with its dual E^* . Let $f(x) = \frac{1}{p} \|x\|^p$ and $g_k(y_k) = \frac{1}{q} \|y_k\|^q$, for $k = 1, 2, \dots, m$ and $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ be functions on E and E^* respectively with $h(x, y_1, y_2, \dots, y_m) = f(x) + \sum_{k=1}^m g_k(y_k)$. Let $A_k : E \rightarrow 2^{E^*}$, for $k \in \{1, 2, \dots, m\}$ and $m \geq 2$, be maximal monotone mappings satisfying $\Omega := \{z \in E : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. For arbitrary $v_0 = (z_0, w_{0,1}, \dots, w_{0,m}) \in V$ define an

iterative algorithm by

$$\begin{cases} x_{k,n} = (J_p + \lambda_{k,n}A_k)^{-1}(J_p(z_n) + \lambda_{k,n}w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = \frac{J_p(z_n) - J_p(x_{k,n})}{\lambda_{k,n}} + w_{k,n}, & k = 1, 2, \dots, m, \\ v_{n+1} = \nabla h^*(\alpha_n \nabla h(z, w_1, w_2, \dots, w_m) + (1 - \alpha_n) \nabla h(a_n, b_{1,n}, \dots, b_{m,n})), \end{cases}$$

where $(z, w_1, w_2, \dots, w_m) \in V$, $\nabla h = (J_p, J_q, \dots, J_q)$, $\nabla h^* = (J_q, J_p, \dots, J_p)$

$$\begin{aligned} a_n &= J_q(\beta_n J_p(z_n) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) J_p(\widehat{a}_i)) \\ b_{k,n} &= J_p(\beta_n J_q(w_{k,n}) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) J_q(\widehat{b}_{k,i})) \end{aligned}$$

and $\lambda_{k,n} \in [\underline{\lambda}, \bar{\lambda}]$, for some $\bar{\lambda} > \underline{\lambda} > 0$, for each $k = 1, 2, \dots, m$, with

$$\widehat{a}_i = \begin{cases} z_n & \text{if } v_n \in H_i, \\ J_q(J_p z_n) + \eta_i(\sum_{k=1}^m y_{k,i}) & \text{if } v_n \notin H_i \end{cases}$$

$$\widehat{b}_{k,i} = \begin{cases} w_{k,n} & \text{if } v_n \in H_i, \\ J_p(J_q(w_{k,n}) + \eta_i(x_{k,i} - \bar{x}_i)) & \text{if } v_n \notin H_i, \end{cases}$$

such that

$$\begin{aligned} \gamma_i &= \langle J_q(J_p(z_n) + \eta_i(\sum_{k=1}^m y_{k,i})), \sum_{k=1}^m y_{k,i} \rangle \\ &\quad + \sum_{k=1}^m \langle x_{k,i} - \bar{x}_i, J_p(J_q(w_{k,n}) + \eta_i(x_{k,i} - \bar{x}_i)) \rangle \end{aligned}$$

for $y_{k,i} \in A_k(x_{k,i})$ and $\eta_i \in \mathbb{R}$. Then, $\{v_n\}$ converges strongly to an element $v^* = (z^*, w_1^*, w_2^*, \dots, w_m^*)$ of $S_e(A_1, A_2, \dots, A_m)$, where $z^* \in \Omega$.

If, in (4.96), E is smooth and strictly convex with its dual E^* and $p = 2$, then $\nabla h = (J, J^{-1}, \dots, J^{-1})$ and $\nabla h^* = (J^{-1}, J, \dots, J)$. Thus, we get the following

corollary for a zero of the sum of a finite family of maximal monotone mappings in Banach spaces.

Corollary 4.3.12. Let E be a smooth, strictly convex and reflexive real Banach space with its dual E^* . Let $f(x) = \frac{1}{2}\|x\|^2$ and $g_k(y_k) = \frac{1}{2}\|y_k\|^2$, for $k = 1, 2, \dots, m$, be functions on E and E^* respectively with $h(x, y_1, y_2, \dots, y_m) = f(x) + \sum_{k=1}^m g_k(y_k)$. Let $A_k : E \rightarrow 2^{E^*}$, for $k \in \{1, 2, \dots, m\}$ and $m \geq 2$, be maximal monotone mappings satisfying $\Omega := \{z \in E : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. For arbitrary $v_0 = (z_0, w_{0,1}, \dots, w_{0,m}) \in V$ define an iterative algorithm by

$$\begin{cases} x_{k,n} = (J + \lambda_{k,n} A_k)^{-1}(J(z_n) + \lambda_{k,n} w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = \frac{J(z_n) - J(x_{k,n})}{\lambda_{k,n}} + w_{k,n}, & k = 1, 2, \dots, m, \\ v_{n+1} = \nabla h^*(\alpha_n \nabla h(z, w_1, w_2, \dots, w_m) + (1 - \alpha_n) \nabla h(a_n, b_{1,n}, \dots, b_{m,n})) \end{cases}$$

where $(z, w_1, w_2, \dots, w_m) \in V$, $\nabla h = (J, J^{-1}, \dots, J^{-1})$, $\nabla h^* = (J^{-1}, J, \dots, J)$

$$\begin{aligned} a_n &= J^{-1}(\beta_n J(z_n) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) J(\widehat{a}_i)) \\ b_{k,n} &= J(\beta_n J^{-1}(w_{k,n}) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) J^{-1}(\widehat{b}_{k,i})) \end{aligned}$$

and $\lambda_{k,n} \in [\underline{\lambda}, \bar{\lambda}]$, for some $\bar{\lambda} > \underline{\lambda} > 0$, for each $k = 1, 2, \dots, m$, with

$$\begin{aligned} \widehat{a}_i &= \begin{cases} z_n & \text{if } v_n \in H_i, \\ J^{-1}(J(z_n) + \eta_i(\sum_{k=1}^m y_{k,i})) & \text{if } v_n \notin H_i \end{cases} \\ \widehat{b}_{k,i} &= \begin{cases} w_{k,n} & \text{if } v_n \in H_i, \\ J(J^{-1}(w_{k,n}) + \eta_i(x_{k,i} - \bar{x}_i)) & \text{if } v_n \notin H_i, \end{cases} \end{aligned}$$

such that

$$\begin{aligned} \gamma_i &= \langle J^{-1}(J(z_n) + \eta_i(\sum_{k=1}^m y_{k,i})), \sum_{k=1}^m y_{k,i} \rangle \\ &\quad + \sum_{k=1}^m \langle x_{k,i} - \bar{x}_i, J(J^{-1}(w_{k,n}) + \eta_i(x_{k,i} - \bar{x}_i)) \rangle \end{aligned}$$

for $y_{k,i} \in A_k(x_{k,i})$ and $\eta_i \in \mathbb{R}$. Then, $\{v_n\}$ converges strongly to an element $v^* = (z^*, w_1^*, w_2^*, \dots, w_m^*)$ of $S_e(A_1, A_2, \dots, A_m)$, where $z^* \in \Omega$.

If, in Corollary 4.3.12, we assume $E = H$, a real Hilbert space, then $\nabla h = \nabla h^* = I$, where I is the identity mapping on H^{m+1} . Thus, we get the following corollary for approximating a zero of the sum of maximal monotone mappings in Hilbert spaces.

Corollary 4.3.13. Let H be a real Hilbert space. Let $f(x) = \frac{1}{2}||x||^2$, $g_k(y_k) = \frac{1}{2}||y_k||^2$, for $k = 1, 2, \dots, m$, be functions on H with $h(x, y_1, y_2, \dots, y_m) = f(x) + \sum_{k=1}^m g_k(y_k)$. Let $A_k : H \rightarrow 2^{H^*}$, for $k \in \{1, 2, \dots, m\}$ and $m \geq 2$, be maximal monotone mappings satisfying $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. For arbitrary $v_0 = (z_0, w_{0,1}, \dots, w_{0,m}) \in V$ define an iterative algorithm by

$$\begin{cases} x_{k,n} = (I + \lambda_{k,n} A_k)^{-1}(z_n + \lambda_{k,n} w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = \frac{z_n - x_{k,n}}{\lambda_{k,n}} + w_{k,n}, & k = 1, 2, \dots, m, \\ v_{n+1} = \alpha_n(z, w_1, w_2, \dots, w_m) + (1 - \alpha_n)(a_n, b_{1,n}, \dots, b_{m,n}) \end{cases} \quad (4.97)$$

where $(z, w_1, w_2, \dots, w_m) \in V$,

$$\begin{aligned} a_n &= \beta_n z_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \widehat{a}_i \\ b_{k,n} &= \beta_n w_{k,n} + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \widehat{b}_{k,i} \end{aligned}$$

and $\lambda_{k,n} \in [\underline{\lambda}, \bar{\lambda}]$, for some $\bar{\lambda} > \underline{\lambda} > 0$, for each $k = 1, 2, \dots, m$, with

$$\widehat{a}_i = \begin{cases} z_n & \text{if } v_n \in H_i, \\ z_n + \eta_i \sum_{k=1}^m y_{k,i} & \text{if } v_n \notin H_i \end{cases}$$

$$\widehat{b}_{k,i} = \begin{cases} w_{k,n} & \text{if } v_n \in H_i, \\ w_{k,n} + \eta_i(x_{k,i} - \bar{x}_i) & \text{if } v_n \notin H_i, \end{cases}$$

such that

$$\begin{aligned} \gamma_i &= \langle z_n + \eta_i \sum_{k=1}^m y_{k,i}, \sum_{k=1}^m y_{k,i} \rangle \\ &\quad + \sum_{k=1}^m \langle x_{k,i} - \bar{x}_i, w_{k,n} + \eta_i(x_{k,i} - \bar{x}_i) \rangle \end{aligned}$$

for $y_{k,i} \in A_k(x_{k,i})$ and $\eta_i \in \mathbb{R}$. Then, $\{v_n\}$ converges strongly to an element $v^* = (z^*, w_1^*, w_2^*, \dots, w_m^*)$ of $S_e(A_1, A_2, \dots, A_m)$, where $z^* \in \Omega$.

4.3.1 Application to Convex Minimization Problems

Let $l_k : E \rightarrow \mathbb{R}$, $k = 1, 2, \dots, m$, where $m \geq 2$, be a finite family of convex and lower semicontinuous functions. We consider the problem of finding $z^* \in E$ such that

$$l_1(z^*) + l_2(z^*) + \dots + l_m(z^*) = \min_{z \in E} \{l_1(z) + l_2(z) + \dots + l_m(z)\}. \quad (4.98)$$

We note that problem (4.98) is equivalent, by Fermat's rule, to the problem of finding $z^* \in E$ such that

$$0 \in \partial l_1(z^*) + \partial l_2(z^*) + \dots + \partial l_m(z^*), \quad (4.99)$$

where ∂l is a subdifferential of l , which is maximal monotone (see, e.g., [95]). So, we obtain the following theorem from Theorem 4.3.8.

Theorem 4.3.14. *Let $f \in \mathcal{F}(E)$ and $g_k \in \mathcal{F}(E^*)$, for each $k = 1, 2, \dots, m$, be strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and uniformly convex on bounded subsets of E and E^* , respectively. Let $l_k : E \rightarrow \mathbb{R}$, for each $k = 1, 2, 3, \dots, m$ and $m \geq 2$, be convex and lower semi-continuous functions such that $\Omega = \min_{z \in E} \{l_1(z) + l_2(z) + \dots + l_m(z)\} \neq \emptyset$. For arbitrary $v_0 = (z_0, w_{0,1}, \dots, w_{0,m}) \in V$ define an iterative algorithm by*

$$\begin{cases} x_{k,n} = (\nabla f + \lambda_{k,n} \partial l_k)^{-1}(\nabla f(z_n) + \lambda_{k,n} w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = \frac{\nabla f(z_n) - \nabla f(x_{k,n})}{\lambda_{k,n}} + w_{k,n}, & k = 1, 2, \dots, m, \\ v_{n+1} = \nabla h^*(\alpha_n \nabla h(z, w_1, w_2, \dots, w_m) + (1 - \alpha_n) \nabla h(a_n, b_{1,n}, \dots, b_{m,n})), \end{cases}$$

where $(z, w_1, w_2, \dots, w_m) \in V$, $\{a_n\}$ and $\{b_{k,n}\}$ are as in (4.96) for $y_{k,n} \in \partial l_k(x_{k,n})$ for each $k = 1, 2, 3, \dots, m$. Then, $\{v_n\}$ converges strongly to an element $v^* = (z^*, w_1^*, w_2^*, \dots, w_m^*)$ of $S_e(\partial l_1, \partial l_2, \dots, \partial l_m)$, where $z^* \in \Omega$.

4.3.2 Numerical Example

In this subsection, we present a numerical example to explain the conclusion of our result. The following example verifies the conclusion of Corollary 4.3.13.

Example 4.3.15. Let $H = \mathbb{R}$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$, by $f(x) = \frac{x^2}{2}$ and $g_1(x) = \frac{x^2}{2} = g_2(x)$, then $\nabla f(x) = x = \nabla g_1(x) = \nabla g_2(x)$, $\nabla f(x^*) = x^* = \nabla g_1(x^*) = \nabla g_2(x^*)$ and $\nabla h = (x, x, x) = \nabla h^*$. Let $A_1, A_2 : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $A_1 x = 2x - 3$ and $A_2 x = x^3$, then A_1 and A_2 are maximal monotone mapping on $\mathbb{R}$ and $1 \in (A_1 + A_2)^{-1}(0)$. Now, by direct calculation we have

$$x_{1,n} = \frac{z_n + \lambda_{1,n} w_{1,n} + 3\lambda_{1,n}}{1 + 2\lambda_{1,n}}, \quad y_{1,n} = \frac{z_n + 2\lambda_{1,n} w_{1,n} - 3}{1 + 3\lambda_{2,n}},$$

$$x_{2,n} = \sqrt[3]{\frac{z_n + \lambda_{2,n}w_{2,n}}{2\lambda_{2,n}} + \sqrt{\left(\frac{z_n + \lambda_{2,n}w_{2,n}}{2\lambda_{2,n}}\right)^2 + \left(\frac{1}{3\lambda_{2,n}}\right)^3}} + \sqrt[3]{\frac{z_n + \lambda_{2,n}w_{2,n}}{2\lambda_{2,n}} - \sqrt{\left(\frac{z_n + \lambda_{2,n}w_{2,n}}{2\lambda_{2,n}}\right)^2 + \left(\frac{1}{3\lambda_{2,n}}\right)^3}},$$

and $y_{2,n} = (x_{2,n})^3$, for some $\lambda_{1,n}, \lambda_{2,n} > 0$. Thus, if we assume $u = (z, w_1, w_2) = (0.001, -0.01, 0.01)$, $\lambda_{1,n} = \lambda_{2,n} = 1$, $\alpha_n = \frac{1}{100(n+10^4)}$, $\beta_n = \frac{1}{n+1}$, for all $n \geq 0$, and take different initial points $v_0 = (0, 4, -4)$, $v_0 = (0.5, 6, -6)$, $v_0 = (1.5, -3, 3)$ and $v_0 = (2, -4, 4)$, then, in all cases, the numerical experiment results using MATLAB provide that the sequence $\{v_n\} = \{(z_n, w_{1,n}, w_{2,n})\}$ generated by (4.97) converges strongly to $v^* = (1, -1, 1)$, where $z^* = 1 \in (A_1 + A_2)^{-1}(0)$. The Figure 4.2 shown below indicates that the convergence of the first component $\{z_n\}$ of $\{v_n\} = \{(z_n, w_{1,n}, w_{2,n})\}$ to the solution $z^* = 1$, for all different initial points.

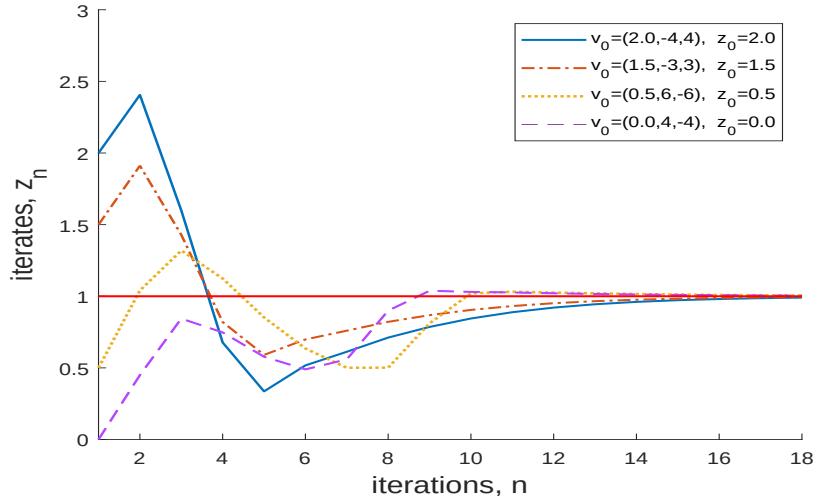


FIGURE 4.2: Convergence of the first component $\{z_n\}$ of $\{v_n\}$ generated by (4.97)

Chapter 5

Split Equality Methods for Solutions of Monotone Inclusion Problems

We divide this chapter into two sections. In the first section, we collect some preliminary results that will be useful throughout the study of the chapter. In the second section, we introduce our algorithm for finding a point which is a solution of the split equality inclusion problem for the sum of maximally monotone mappings in Hilbert spaces and discuss its convergence

5.1 Preliminaries

Lemma 5.1.1. *[47] Finding a point in $S_e(A, B)$ is equivalent to solving (1.24) in the sense that*

$$z \in (A + B)^{-1}(0) \Leftrightarrow \exists w \in H : (z, w) \in S_e(A, B). \quad (5.1)$$

Proof. Given $z \in H$, one has $z \in (A + B)^{-1}(0) \Leftrightarrow 0 \in A(z) + B(z) \Leftrightarrow \exists w \in H : w \in B(z), -w \in A(z) \Leftrightarrow \exists w \in H : (z, w) \in S_e(A, B)$. $\square$

Lemma 5.1.2. [47] *Given any two maximal monotone mappings $A, B : H \rightarrow 2^H$, the corresponding extended solution set $S_e(A, B)$ is closed and convex in $H \times H$.*

Lemma 5.1.3. [47] *Given any two points $(x, b) \in \text{Gph}(B)$ and $(y, a) \in \text{Gph}(A)$, define the function $\varphi : H \times H \rightarrow \mathbb{R}$ via*

$$\varphi(z, w) = \langle z - x, b - w \rangle + \langle z - y, a + w \rangle.$$

Then, for any $(z, w) \in S_e(A, B)$ we have $\varphi(z, w) \leq 0$, that is

$$S_e(A, B) \subseteq \{(z, w) \in H \times H \mid \varphi(z, w) \leq 0\}.$$

Additionally, φ is both continuous and affine, $\nabla \varphi = (a + b, x - y)$, and

$$\nabla \varphi = 0 \Leftrightarrow (x, b) \in S_e(A, B), x = y, a = -b.$$

Furthermore, $\nabla \varphi = 0$ implies $\varphi(z, w) = 0$ for all $z, w \in H$.

The function φ in Lemma 5.1.3 is called decomposable separators.

Definition 5.1.4. Let H be a real Hilbert space and $A : H \rightarrow H$ be a bounded linear mapping. We define a mapping $A^* : H \rightarrow H$ by the relation

$$\langle Ax, y \rangle = \langle x, A^*y \rangle,$$

for all $x, y \in H$. The mapping A is called adjoint/dual of A .

5.2 Iterative Algorithm for Solving Monotone Inclusion Problems

For the rest of this section, let H_1, H_2 and H_3 be real Hilbert spaces and $F_1, G_1 : H_1 \rightarrow 2^{H_1}$ and $F_2, G_2 : H_2 \rightarrow 2^{H_2}$ be maximally monotone mappings satisfying $\Gamma := \{(x^*, y^*) \in H_1 \times H_2 : x^* \in (F_1 + G_1)^{-1}(0), y^* \in (F_2 + G_2)^{-1}(0) \text{ and } Ax^* =$

$By^*\} \neq \emptyset$, where $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ are bounded linear mappings with adjoints A^* and B^* , respectively. Let $\mathcal{A} : H_1 \times H_1 \rightarrow H_3 \times H_3$ and $\mathcal{B} : H_2 \times H_2 \rightarrow H_3 \times H_3$ be given by

$$\mathcal{A}(z, w) = (Az, 0) \quad \text{and} \quad \mathcal{B}(\hat{z}, \hat{w}) = (B\hat{z}, 0). \quad (5.2)$$

Then we observe that

$$\mathcal{A}^*(z, w) = (A^*z, 0) \quad \text{and} \quad \mathcal{B}^*(\hat{z}, \hat{w}) = (B^*\hat{z}, 0). \quad (5.3)$$

Let $f_1 : H_1 \times H_1 \rightarrow H_1 \times H_1$ and $f_2 : H_2 \times H_2 \rightarrow H_2 \times H_2$ be contraction mappings with common constant $\alpha \in [0, \frac{1}{\sqrt{2}})$. Let $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a decreasing sequence in $(0, 1]$ such that $\beta_0 = 1$ and $\{\omega_n\}$ be a real sequence satisfying $0 < \liminf_{n \rightarrow \infty} \omega_n \leq \limsup_{n \rightarrow \infty} \omega_n < 1$.

We now propose the following algorithm.

Algorithm 5.2.1.

Step 0: Select initial guess $p_0 = (z_0, w_0) \in H_1 \times H_1$ and $v_0 = (\hat{z}_0, \hat{w}_0) \in H_2 \times H_2$.

Step 1: Given (z_n, w_n) and $(\hat{z}_n, \hat{w}_n)$, for $n \geq 0$, compute

$$\begin{aligned} x_n &= (I + \lambda_n G_1)^{-1}(z_n + \lambda_n w_n), \\ y_n &= (I + \mu_n F_1)^{-1}((1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n), \\ \hat{x}_n &= (I + \lambda_n G_2)^{-1}(\hat{z}_n + \lambda_n \hat{w}_n), \\ \hat{y}_n &= (I + \mu_n F_2)^{-1}((1 - \theta_n)\hat{z}_n + \theta_n \hat{x}_n - \mu_n \hat{w}_n), \end{aligned}$$

where $\lambda_n, \mu_n > 0$ and $\theta_n \in \mathbb{R}$.

Step 2: Compute

$$\begin{cases} d_n = \frac{1}{\mu_n}((1 - \theta_n)z_n + \theta_n x_n - y_n) - w_n, \\ e_n = w_n + \frac{1}{\lambda_n}(z_n - x_n), \\ \widehat{d}_n = \frac{1}{\mu_n}((1 - \theta_n)\widehat{z}_n + \theta_n \widehat{x}_n - \widehat{y}_n) - \widehat{w}_n, \\ \widehat{e}_n = \widehat{w}_n + \frac{1}{\lambda_n}(\widehat{z}_n - \widehat{x}_n). \end{cases}$$

Step 3 : Define $\varphi_i : H_1 \times H_1 \rightarrow \mathbb{R}$ by

$$\varphi_i(z, w) := \langle z - x_i, e_i - w \rangle + \langle z - y_i, d_i + w \rangle,$$

and $\widehat{\varphi}_i : H_2 \times H_2 \rightarrow \mathbb{R}$ by

$$\widehat{\varphi}_i(\widehat{z}, \widehat{w}) := \langle \widehat{z} - \widehat{x}_i, \widehat{e}_i - \widehat{w} \rangle + \langle \widehat{z} - \widehat{y}_i, \widehat{d}_i + \widehat{w} \rangle,$$

and compute

$$S_i p_n = \begin{cases} (z_n, w_n), & \text{if } \varphi_i(z_n, w_n) \leq 0, \\ (z_n, w_n) - \left(\frac{\varphi_i(z_n, w_n)}{\|\nabla \varphi_i\|^2} \right) \nabla \varphi_i, & \text{if } \varphi_i(z_n, w_n) > 0, \end{cases} \quad (5.4)$$

and

$$T_i v_n = \begin{cases} (\widehat{z}_n, \widehat{w}_n), & \text{if } \widehat{\varphi}_i(\widehat{z}_n, \widehat{w}_n) \leq 0, \\ (\widehat{z}_n, \widehat{w}_n) - \left(\frac{\widehat{\varphi}_i(\widehat{z}_n, \widehat{w}_n)}{\|\nabla \widehat{\varphi}_i\|^2} \right) \nabla \widehat{\varphi}_i, & \text{if } \widehat{\varphi}_i(\widehat{z}_n, \widehat{w}_n) > 0, \end{cases} \quad (5.5)$$

where $S_i p_n$ is the projection of $p_n = (z_n, w_n)$ onto the half space

$$H_i = \{(z, w) \in H_1 \times H_1 : \varphi_i(z, w) \leq 0\}, \text{ for each } i = 1, 2, \dots, n$$

and $T_i v_n$ is the projection of $v_n = (\widehat{z}_n, \widehat{w}_n)$ onto the half space

$$\widehat{H}_i = \{(\widehat{z}, \widehat{w}) \in H_2 \times H_2 : \widehat{\varphi}_i(\widehat{z}, \widehat{w}) \leq 0\}, \text{ for each } i = 1, 2, \dots, n.$$

Step 4: Compute

$$\begin{cases} q_n = p_n - \gamma_n \mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n), \\ p_{n+1} = \alpha_n f_1(p_n) + (1 - \alpha_n)[(1 - \omega_n)q_n \\ \quad + \omega_n(\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n)], \\ u_n = v_n + \gamma_n \mathcal{B}^*(\mathcal{A}p_n - \mathcal{B}v_n), \\ v_{n+1} = \alpha_n f_2(v_n) + (1 - \alpha_n)[(1 - \omega_n)u_n \\ \quad + \omega_n(\beta_n u_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)T_i u_n)], \end{cases} \quad (5.6)$$

where $\mathcal{A} : H_1 \times H_1 \rightarrow H_3 \times H_3$ and $\mathcal{B} : H_2 \times H_2 \rightarrow H_3 \times H_3$ are defined as in (5.2) with adjoints $\mathcal{A}^*$ and $\mathcal{B}^*$ respectively and for some $\epsilon > 0$,

$$\gamma_n \in \left(\epsilon, \frac{2\|\mathcal{A}p_n - \mathcal{B}v_n\|^2}{\|\mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2 + \|\mathcal{B}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2} - \epsilon \right), \text{ for } n \in \Omega$$

otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), such that the indexes $\Omega = \{n \in \mathbb{N} : \mathcal{A}p_n - \mathcal{B}v_n \neq 0\}$. Set $n := n + 1$ and go to **Step 1**.

Remark 5.2.2. By maximality of G_1 the resolvent mapping $(I + \lambda_n G_1)^{-1}$ is single-valued and well-defined (see, e.g., [69]) and hence

$$(x_n, e_n) = ((I + \lambda_n G_1)^{-1}(z_n + \lambda_n w_n), w_n + \frac{1}{\lambda_n}(z_n - x_n)),$$

exists and is unique. Also by maximality of F_1 the resolvent mapping $(I + \mu_n F_1)^{-1}$ is single-valued and well-defined, which implies that

$$(y_n, d_n) = ((I + \mu_n F_1)^{-1}((1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n), \frac{1}{\mu_n}((1 - \theta_n)z_n + \theta_n x_n - y_n) - w_n),$$

exists and is unique. Similarly, by maximality of F_2 and G_2 we have $(\widehat{y}_n, \widehat{d}_n)$ and $(\widehat{x}_n, \widehat{e}_n)$ exist. By rearranging equations in Step 1, one has the following equations:

$$\begin{cases} x_n + \lambda_n e_n = z_n + \lambda_n w_n, \\ y_n + \mu_n d_n = (1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n, \end{cases} \quad (5.7)$$

$$\begin{cases} \widehat{x}_n + \lambda_n \widehat{e}_n = \widehat{z}_n + \lambda_n \widehat{w}_n, \\ \widehat{y}_n + \mu_n \widehat{d}_n = (1 - \theta_n) \widehat{z}_n + \theta_n \widehat{x}_n - \mu_n \widehat{w}_n, \end{cases} \quad (5.8)$$

where $d_n \in F_1(y_n)$, $e_n \in G_1(x_n)$, $\widehat{d}_n \in F_2(\widehat{y}_n)$ and $\widehat{e}_n \in G_2(\widehat{x}_n)$. Hence decomposable separators function φ_i and $\widehat{\varphi}_i$ in Step 3 are well defined.

We proceed with the following lemmas which will be used to prove our main theorem.

Lemma 5.2.3. *The sequence $\{p_n\}$ and $\{v_n\}$ generated by Algorithm 5.2.1 are bounded in $H_1 \times H_1$ and $H_2 \times H_2$, respectively.*

Proof. Let $(z^*, \widehat{z}^*) \in \Gamma$. Then by Lemmas 5.1.1 and 5.1.3 there exist $w^* \in H_1$ and $\widehat{w}^* \in H_2$ such that $p^* = (z^*, w^*) \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i)$, $v^* = (\widehat{z}^*, \widehat{w}^*) \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$ and hence $(p^*, v^*) \in \mathcal{F} := \{(p, v) : p \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i), v \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \text{ and } \mathcal{A}p = \mathcal{B}v\}$. Thus, from (5.6) we get

$$\begin{aligned} \|q_n - p^*\|^2 &= \|p_n - \gamma \mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n) - p^*\|^2 \\ &= \|p_n - p^* - \gamma \mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2 \\ &= \|p_n - p^*\|^2 - 2\gamma \langle p_n - p^*, \mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n) \rangle \\ &\quad + \gamma^2 \|\mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2. \end{aligned} \quad (5.9)$$

Observe that

$$\begin{aligned} -2\langle p_n - p^*, \mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n) \rangle &= -2\langle \mathcal{A}p_n - \mathcal{A}p^*, \mathcal{A}p_n - \mathcal{B}v_n \rangle \\ &= \|\mathcal{B}v_n - \mathcal{A}p^*\|^2 - \|\mathcal{A}p_n - \mathcal{A}p^*\|^2 \\ &\quad - \|\mathcal{A}p_n - \mathcal{B}v_n\|^2. \end{aligned} \quad (5.10)$$

Thus, from (5.9) and (5.10), we obtain

$$\begin{aligned} \|q_n - p^*\|^2 &= \|p_n - p^*\|^2 - \gamma_n \|\mathcal{A}p_n - \mathcal{A}p^*\|^2 - \gamma_n \|\mathcal{A}p_n - \mathcal{B}v_n\|^2 \\ &\quad + \gamma_n \|\mathcal{B}v_n - \mathcal{A}p^*\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2. \end{aligned} \quad (5.11)$$

Similarly, we have

$$\begin{aligned} \|u_n - v^*\|^2 &= \|v_n - v^*\|^2 + \gamma_n \|\mathcal{A}p_n - \mathcal{B}v^*\|^2 - \gamma_n \|\mathcal{B}v_n - \mathcal{B}v^*\|^2 \\ &\quad - \gamma_n \|\mathcal{A}p_n - \mathcal{B}v_n\|^2 + \gamma_n^2 \|\mathcal{B}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2. \end{aligned} \quad (5.12)$$

By adding the equations (5.11) and (5.12) and using the fact that $\mathcal{A}p^* = \mathcal{B}v^*$, we obtain

$$\begin{aligned} \|q_n - p^*\|^2 + \|u_n - v^*\|^2 &= \|p_n - p^*\|^2 + \|v_n - v^*\|^2 - \gamma_n [2\|\mathcal{A}p_n - \mathcal{B}v_n\|^2 \\ &\quad - \gamma_n (\|\mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2 + \|\mathcal{B}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2)] \\ &\leq \|p_n - p^*\|^2 + \|v_n - v^*\|^2. \end{aligned} \quad (5.13)$$

Thus, from (5.6), firmly nonexpansiveness property of S_i and T_i for all $i \in \mathbb{N}$ and Lemma 3.1.3, we have

$$\begin{aligned} \|p_{n+1} - p^*\|^2 &= \|\alpha_n f_1(p_n) + (1 - \alpha_n)[(1 - \omega_n)q_n \\ &\quad + \omega_n(\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n)] - p^*\|^2 \\ &\leq \alpha_n \|f_1(p_n) - p^*\|^2 + (1 - \alpha_n) \|(1 - \omega_n)(q_n - p^*) \\ &\quad + \omega_n(\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n - p^*)\|^2 \\ &\leq \alpha_n (\|f_1(p_n) - f_1(p^*)\| + \|f_1(p^*) - p^*\|)^2 \\ &\quad + (1 - \alpha_n) [(1 - \omega_n)\|(q_n - p^*)\|^2 \\ &\quad + \omega_n \|(\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n) - p^*\|^2] \\ &\leq 2\alpha_n (\alpha^2 \|p_n - p^*\|^2 + \|f_1(p^*) - p^*\|^2) \\ &\quad + (1 - \alpha_n) \|q_n - p^*\|^2. \end{aligned} \quad (5.14)$$

Similarly, we have

$$\|v_{n+1} - v^*\|^2 \leq 2\alpha_n (\alpha^2 \|v_n - v^*\|^2 + \|f_2(v^*) - v^*\|^2) + (1 - \alpha_n) \|u_n - v^*\|^2. \quad (5.15)$$

Adding up the inequalities (5.14) and (5.15), setting $R_n(p^*, v^*) = \|p_n - p^*\|^2 + \|v_n - v^*\|^2$ and using inequality (5.13), we obtain

$$\begin{aligned} R_{n+1}(p^*, v^*) &\leq (1 - \alpha_n(1 - 2\alpha^2))R_n(p^*, v^*) \\ &\quad + 2\alpha_n(\|f_1(p) - p^*\|^2 + \|f_2(v) - v^*\|^2) \\ &\leq \max \left\{ R_n(p^*, v^*), \frac{2}{1 - 2\alpha^2}(\|f_1(p^*) - p^*\|^2 + \|f_2(v^*) - v^*\|^2) \right\}, \end{aligned}$$

and hence by induction

$$R_n(p^*, v^*) \leq \max \left\{ R_0(p^*, v^*), \frac{2(\|f_1(p^*) - p^*\|^2 + \|f_2(v^*) - v^*\|^2)}{1 - 2\alpha^2} \right\},$$

which implies that $\{p_n\}$, $\{v_n\}$ and hence $\{q_n\}$, $\{u_n\}$, $\{S_i q_n\}$ and $\{T_i u_n\}$ are bounded. $\square$

Lemma 5.2.4. *The sequence $\{(p_n, v_n)\}$ generated by Algorithm 5.2.1 converges strongly to an element (p^*, v^*) of*

$$\mathcal{F} := \{(p, v) : p \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i), v \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \text{ and } \mathcal{A}p = \mathcal{B}v\}.$$

Proof. Let $(p^*, v^*) = P_{\mathcal{F}}(f_1(p^*), f_2(v^*))$, where $p^* = (z^*, w^*)$ and $v^* = (\widehat{z}^*, \widehat{w}^*)$. It follows that $S_i p^* = p^*$ and $T_i v^* = v^*$, for each $i \in \mathbb{N}$ and $\mathcal{A}p^* = \mathcal{B}v^*$. From (5.6) and Lemma 3.1.3 and following the arguments in the proof of Lemma 5.2.3 we

have

$$\begin{aligned}
 \|p_{n+1} - p^*\|^2 &= \|\alpha_n f_1(p_n) + (1 - \alpha_n)[(1 - \omega_n)q_n \\
 &\quad + \omega_n(\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n] - p^*\|^2 \\
 &\leq \|\alpha_n(f_1(p_n) - f_1(p^*)) + (1 - \alpha_n)[(1 - \omega_n)q_n \\
 &\quad + \omega_n(\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n - p^*]\|^2 \\
 &\quad + 2\alpha_n \langle f_1(p^*) - p^*, p_{n+1} - p^* \rangle \\
 &\leq \alpha_n \|f_1(p_n) - f_1(p^*)\|^2 + (1 - \alpha_n) \|[(1 - \omega_n)q_n \\
 &\quad + \omega_n(\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n] - p^*\|^2 \\
 &\quad + 2\alpha_n \langle f_1(p^*) - p^*, p_{n+1} - p^* \rangle \\
 &\leq \alpha_n \alpha^2 \|p_n - p^*\|^2 + (1 - \alpha_n) \|q_n - p^*\|^2 \\
 &\quad - \omega_n(1 - \omega_n)(1 - \alpha_n) \|q_n - (\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n)\|^2 \\
 &\quad + 2\alpha_n \langle f_1(p^*) - p^*, p_{n+1} - p^* \rangle. \tag{5.16}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \|v_{n+1} - v^*\|^2 &\leq \alpha_n \alpha^2 \|v_n - v^*\|^2 + (1 - \alpha_n) \|u_n - v^*\|^2 \\
 &\quad - \omega_n(1 - \omega_n)(1 - \alpha_n) \|u_n - (\beta_n u_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)T_i u_n)\|^2 \\
 &\quad + 2\alpha_n \langle f_2(v^*) - v^*, v_{n+1} - v^* \rangle. \tag{5.17}
 \end{aligned}$$

By adding (5.16) with (5.17) and using (5.13), we obtain

$$\begin{aligned}
 R_{n+1}(p^*, v^*) &\leq (1 - \alpha_n(1 - \alpha^2))R_n(p^*, v^*) + 2\alpha_n [\langle f_1(p^*) - p^*, p_{n+1} - p^* \rangle \\
 &\quad + \langle f_2(v^*) - v^*, v_{n+1} - v^* \rangle] \\
 &\quad - \omega_n(1 - \omega_n)(1 - \alpha_n) \|q_n - (\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n)\|^2 \\
 &\quad - \omega_n(1 - \omega_n)(1 - \alpha_n) \|u_n - (\beta_n u_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)T_i u_n)\|^2 \\
 &\quad - (1 - \alpha_n)\gamma_n [2\|\mathcal{A}p_n - \mathcal{B}v_n\|^2 - \gamma_n(\|\mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2 \\
 &\quad + \|\mathcal{B}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2)]. \tag{5.18}
 \end{aligned}$$

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{R_n(p^*, v^*)\}$ is decreasing, then $\{R_n(p^*, v^*)\}$ converges and $R_n(p^*, v^*) - R_{n+1}(p^*, v^*) \rightarrow 0$ as $n \rightarrow \infty$. Now from (5.18), we have

$$\|q_n - (\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) S_i q_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (5.19)$$

$$\|u_n - (\beta_n u_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i u_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (5.20)$$

and

$$\lim_{n \rightarrow \infty} \left[\|\mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2 + \|\mathcal{B}^*(\mathcal{A}p_n - \mathcal{B}v_n)\|^2 \right] = 0. \quad (5.21)$$

Note that $\mathcal{A}p_n - \mathcal{B}v_n = 0$ if $n \notin \Omega$. So, in general we get

$$\lim_{n \rightarrow \infty} \|\mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}v_n)\| = \lim_{n \rightarrow \infty} \|\mathcal{B}^*(\mathcal{A}p_n - \mathcal{B}v_n)\| = 0. \quad (5.22)$$

In addition, from (5.18) and (5.21) we obtain that

$$\lim_{n \rightarrow \infty} \|\mathcal{A}p_n - \mathcal{B}v_n\| = 0. \quad (5.23)$$

Since S_i , for each $i \in \mathbb{N}$, is firmly nonexpansive, we have

$$\begin{aligned} \|q_n - p^*\|^2 &\geq \|S_i q_n - p^*\|^2 \\ &= \|S_i q_n - q_n + q_n - p^*\|^2 \end{aligned}$$

which implies that

$$\frac{1}{2} \|S_i q_n - q_n\|^2 \leq \langle q_n - S_i q_n, q_n - p^* \rangle, \quad (5.24)$$

and hence,

$$\begin{aligned}
 \frac{1}{2} \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|S_i q_n - q_n\|^2 &\leq \sum_{i=1}^n (\beta_{i-1} - \beta_i) \langle q_n - S_i q_n, q_n - p^* \rangle \\
 &= \left\langle \sum_{i=1}^n (\beta_{i-1} - \beta_i) (q_n - S_i q_n), q_n - p^* \right\rangle \\
 &= \left\langle q_n - (\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) S_i q_n), q_n - p^* \right\rangle \\
 &\leq \|q_n - (\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) S_i q_n)\| \|q_n - p^*\|.
 \end{aligned}$$

Thus, from (5.19) we get

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|S_i q_n - q_n\| = 0. \quad (5.25)$$

Since $\{\beta_n\}$ strictly decreasing the above equation yields,

$$\lim_{n \rightarrow \infty} \|S_i q_n - q_n\| = 0, \quad (5.26)$$

for each $i \geq 1$ and hence

$$\lim_{n \rightarrow \infty} \|S_n q_n - q_n\| = 0. \quad (5.27)$$

Similarly, we obtain that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|T_i u_n - q_n\| = 0,$$

and hence

$$\lim_{n \rightarrow \infty} \|T_i u_n - u_n\| = 0, \quad (5.28)$$

for each $i \geq 1$ and

$$\lim_{n \rightarrow \infty} \|T_n u_n - u_n\| = 0. \quad (5.29)$$

Moreover, from (5.6) and (5.22), we get

$$\|q_n - p_n\|^2 = \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}p_n - \mathcal{B}u_n)\|^2 \quad (5.30)$$

and hence

$$\lim_{n \rightarrow \infty} \|q_n - p_n\| = 0. \quad (5.31)$$

Similarly,

$$\lim_{n \rightarrow \infty} \|u_n - v_n\| = 0. \quad (5.32)$$

From equations (5.29), (5.31), (5.32) and the fact that S_i and T_i for each $i \in \mathbb{N}$ are nonexpansive, we have

$$\begin{aligned} \|p_n - S_n p_n\| &= \|p_n - q_n + q_n - S_n q_n + S_n q_n - S_n p_n\| \\ &\leq 2\|p_n - q_n\| + \|q_n - S_n q_n\| \end{aligned} \quad (5.33)$$

and

$$\begin{aligned} \|v_n - T_n v_n\| &= \|v_n - u_n + u_n - T_n u_n + T_n u_n - T_n v_n\| \\ &\leq 2\|v_n - u_n\| + \|u_n - T_n u_n\|. \end{aligned} \quad (5.34)$$

Therefore,

$$\lim_{n \rightarrow \infty} \|p_n - S_n p_n\| = 0 = \lim_{n \rightarrow \infty} \|v_n - T_n v_n\|. \quad (5.35)$$

Now, since $\{(p_n, v_n)\}$ is bounded in $H_1 \times H_2$, there exists a subsequence $\{(p_{n_k}, v_{n_k})\}$ of $\{(p_n, v_n)\}$ such that $(p_{n_k}, v_{n_k}) \rightharpoonup (\bar{p}, \bar{v}) \in H_1 \times H_2$ and

$$\begin{aligned} \limsup_{n \rightarrow \infty} [\langle f_1(p^*) - p^*, p_n - p^* \rangle + \langle f_2(v^*) - v^*, v_n - v^* \rangle] &= \\ \lim_{k \rightarrow \infty} \langle f_1(p^*) - p^*, p_{n_k} - p^* \rangle + \lim_{k \rightarrow \infty} \langle f_2(v^*) - v^*, v_{n_k} - v^* \rangle. \end{aligned} \quad (5.36)$$

Note that $(p_{n_k}, v_{n_k}) \rightharpoonup (\bar{p}, \bar{v})$ implies $p_{n_k} \rightharpoonup \bar{p}$ and $v_{n_k} \rightharpoonup \bar{v}$. Now, using (5.26), (5.28) and Lemma 3.1.4, we obtain $\bar{p} \in \bigcap_{i=1}^{\infty} F(S_i)$ and $\bar{v} \in \bigcap_{i=1}^{\infty} F(T_i)$. Next, we show that $\mathcal{A}\bar{p} = \mathcal{B}\bar{v}$. But, observe that from (5.23) we get

$$\begin{aligned} \|\mathcal{A}\bar{p} - \mathcal{B}\bar{v}\|^2 &= \|\mathcal{A}\bar{p} - \mathcal{A}p_{n_k} + \mathcal{B}v_{n_k} - \mathcal{B}\bar{v} + \mathcal{A}p_{n_k} - \mathcal{B}v_{n_k}\|^2 \\ &\leq \|\mathcal{A}p_{n_k} - \mathcal{B}v_{n_k}\|^2 + 2\langle \mathcal{A}\bar{p} - \mathcal{B}\bar{v}, \mathcal{A}\bar{p} - \mathcal{A}p_{n_k} + \mathcal{B}v_{n_k} - \mathcal{B}\bar{v} \rangle, \\ &\rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned} \quad (5.37)$$

and this implies $\mathcal{A}\bar{p} = \mathcal{B}\bar{v}$, that is, $(\bar{p}, \bar{v}) \in \mathcal{F}$. Since $(p^*, v^*) = P_{\mathcal{F}}(f_1(p^*), f_2(v^*))$, from (5.36) and Lemma 3.1.8, we obtain

$$\begin{aligned} &\limsup_{n \rightarrow \infty} [\langle f_1(p^*) - p^*, p_n - p^* \rangle + \langle f_2(v^*) - v^*, v_n - v^* \rangle] \\ &= \lim_{k \rightarrow \infty} \langle f_1(p^*) - p^*, p_{n_k} - p^* \rangle + \lim_{k \rightarrow \infty} \langle f_2(v^*) - v^*, v_{n_k} - v^* \rangle \\ &= \langle f_1(p^*) - p^*, \bar{p} - p^* \rangle + \langle f_2(v^*) - v^*, \bar{v} - v^* \rangle \leq 0. \end{aligned} \quad (5.38)$$

Moreover, from (5.6), we have

$$\begin{aligned} \|p_{n+1} - p_n\| &= \|\alpha_n f_1(p_n) + (1 - \alpha_n)[(1 - \omega_n)q_n + \omega_n(\beta_n q_n \\ &\quad + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n] - p_n\| \\ &\leq \alpha_n \|f_1(p_n) - p_n\| + (1 - \alpha_n)[(1 - \omega_n)\|q_n - p_n\| \\ &\quad + \omega_n\|(\beta_n q_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)S_i q_n) - p_n\|] \\ &\leq \alpha_n(\alpha\|p_n - p^*\| + \|f_1(p^*) - p_n\|) + (1 - \alpha_n)\|q_n - p_n\| \\ &\quad + \omega_n(1 - \alpha_n) \sum_{i=1}^n (\beta_{i-1} - \beta_i)\|S_i q_n - q_n\|. \end{aligned} \quad (5.39)$$

Hence, from (5.27), (5.31), (5.39) and the fact that $\alpha_n \rightarrow 0$, as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \|p_{n+1} - p_n\| = 0. \quad (5.40)$$

Similarly, we obtain that

$$\lim_{n \rightarrow \infty} \|v_{n+1} - v_n\| = 0. \quad (5.41)$$

Finally, from (5.18), we have

$$\begin{aligned}
 R_{n+1}(p^*, v^*) &\leq (1 - \alpha_n(1 - \alpha^2))R_n(p^*, v^*) + 2\alpha_n \left[\langle f_1(p^*) - p^*, p_{n+1} - p^* \rangle \right. \\
 &\quad \left. + \langle f_2(v^*) - v^*, v_{n+1} - v^* \rangle \right] \\
 &\leq (1 - \alpha_n(1 - \alpha^2))R_n(p^*, v^*) + 2\alpha_n \left[\|f_1(p^*) - p^*\| \|p_{n+1} - p_n\| \right. \\
 &\quad \left. + \|f_2(v^*) - v^*\| \|v_{n+1} - v_n\| + \langle f_1(p^*) - p^*, p_n - p^* \rangle \right. \\
 &\quad \left. + \langle f_2(v^*) - v^*, v_n - v^* \rangle \right]. \tag{5.42}
 \end{aligned}$$

Thus, from (5.38)-(5.42) and Lemma 3.1.9, $R_n(p^*, v^*)$ converges strongly to $(p^*, v^*) = P_{\mathcal{F}}(f_1(p^*), f_2(v^*))$ as $n \rightarrow \infty$. That is, $\{p_n\}$ converges strongly to p^* and $\{v_n\}$ converges strongly to v^* as $n \rightarrow \infty$.

Case 2. Assume that there exists a subsequence $R_{k_j}(p^*, v^*)$ of $R_k(p^*, v^*) := \|p_k - p^*\|^2 + \|v_k - v^*\|^2$ such that $R_{k_j}(p^*, v^*) < R_{k_j+1}(p^*, v^*)$. Then by Lemma 4.1.15, we can define nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and $\max\{R_{m_k}(p^*, v^*), R_k(p^*, v^*)\} \leq R_{m_k+1}(p^*, v^*)$ for all $k \in \mathbb{N}$. From (5.18) and the fact that $\alpha_{m_k} \rightarrow 0$ as $k \rightarrow \infty$, we have

$$\lim_{k \rightarrow \infty} \|q_{m_k} - (\beta_{m_k} q_{m_k} + \sum_{i=1}^k (\beta_{i-1} - \beta_i) S_i q_{m_k})\| = 0, \tag{5.43}$$

$$\lim_{k \rightarrow \infty} \|u_{m_k} - (\beta_{m_k} u_{m_k} + \sum_{i=1}^k (\beta_{i-1} - \beta_i) T_i u_{m_k})\| = 0, \tag{5.44}$$

and

$$\lim_{k \rightarrow \infty} \|\mathcal{A}^*(\mathcal{A}p_{m_k} - \mathcal{B}v_{m_k})\| = \lim_{k \rightarrow \infty} \|\mathcal{B}^*(\mathcal{A}p_{m_k} - \mathcal{B}v_{m_k})\| = 0. \tag{5.45}$$

Moreover, following the methods in Case 1, we can also drive the limits

$$\lim_{k \rightarrow \infty} \|S_i q_{m_k} - q_{m_k}\| = \lim_{k \rightarrow \infty} \|T_i u_{m_k} - u_{m_k}\| = 0, \tag{5.46}$$

for each $i \geq 1$ and we drive

$$\lim_{k \rightarrow \infty} \|q_{m_k} - p_{m_k}\| = 0 = \lim_{k \rightarrow \infty} \|u_{m_k} - v_{m_k}\|, \quad (5.47)$$

and

$$\lim_{n \rightarrow \infty} \|S_{m_k} p_{m_k} - p_{m_k}\| = 0 = \lim_{n \rightarrow \infty} \|T_{m_k} v_{m_k} - v_{m_k}\| = 0.$$

Consequently, following similar steps as in Case 1, we obtain

$$\lim_{k \rightarrow \infty} \|p_{m_k+1} - p_{m_k}\| = 0 = \lim_{k \rightarrow \infty} \|v_{m_k+1} - v_{m_k}\|. \quad (5.48)$$

Now, we may choose a subsequence $\{(p_{n_{k_l}}, v_{n_{k_l}})\}$ of $\{(p_{n_k}, v_{n_k})\}$ such that $(p_{n_{k_l}}, v_{n_{k_l}}) \rightharpoonup (\bar{p}, \bar{v}) \in H_1 \times H_2$ and

$$\begin{aligned} & \limsup_{k \rightarrow \infty} [\langle f_1(p^*) - p^*, p_{m_k} - p^* \rangle + \langle f_2(v^*) - v^*, v_{m_k} - v^* \rangle] \\ &= \lim_{l \rightarrow \infty} \langle f_1(p^*) - p^*, p_{m_{k_l}} - p^* \rangle + \lim_{l \rightarrow \infty} \langle f_2(v^*) - v^*, v_{m_{k_l}} - v^* \rangle. \end{aligned} \quad (5.49)$$

Then, as in Case 1, we drive that $(\bar{p}, \bar{v}) \in \mathcal{F}$. So, by Lemma 3.1.8, we obtain

$$\begin{aligned} & \limsup_{k \rightarrow \infty} [\langle f_1(p^*) - p^*, p_{m_k} - p^* \rangle + \langle f_2(v^*) - v^*, v_{m_k} - v^* \rangle] \\ &= \langle f_1(p^*) - p^*, \bar{p} - p^* \rangle + \langle f_2(v^*) - v^*, \bar{v} - v^* \rangle \leq 0. \end{aligned} \quad (5.50)$$

Now, from (5.42) we have

$$\begin{aligned} R_{m_k+1}(p^*, v^*) &\leq (1 - \alpha_{m_k}(1 - \alpha^2))R_{m_k}(p^*, v^*) + 2\alpha_{m_k}[\langle f_1(p^*) - p^*, p_{m_k+1} - p^* \rangle \\ &\quad + \langle f_2(v^*) - v^*, v_{m_k+1} - v^* \rangle] \\ &\leq (1 - \alpha_{m_k}(1 - \alpha^2))R_{m_k}(p^*, v^*) + 2\alpha_{m_k}[\|f_1(p^*) - p^*\| \|p_{m_k+1} - p_{m_k}\| \\ &\quad + \|f_2(v^*) - v^*\| \|v_{m_k+1} - v_{m_k}\|] + 2\alpha_{m_k}[\langle f_1(p^*) - p^*, p_{m_k} - p^* \rangle \\ &\quad + \langle f_2(v^*) - v^*, v_{m_k} - v^* \rangle]. \end{aligned} \quad (5.51)$$

Since $R_{m_k}(p^*, v^*) \leq R_{m_k+1}(p^*, v^*)$, it follows that

$$\begin{aligned} R_{m_k}(p^*, v^*) \leq & \frac{2}{1-\alpha^2} \left[\langle f_1(p^*) - p^*, p_{m_k} - p^* \rangle + \langle f_2(v^*) - v^*, v_{m_k} - v^* \rangle \right. \\ & + \|f_1(p^*) - p^*\| \|p_{m_k+1} - p_{m_k}\| \\ & \left. + \|f_2(v^*) - v^*\| \|v_{m_k+1} - v_{m_k}\| \right]. \end{aligned} \quad (5.52)$$

Thus, from (5.50), (5.48) and (5.52), we get

$$\lim_{k \rightarrow \infty} R_{m_k}(p^*, v^*) = 0 \text{ and hence } \lim_{k \rightarrow \infty} R_{m_k+1}(p^*, v^*) = 0.$$

The fact that $R_k(p^*, v^*) \leq R_{m_k+1}(p^*, v^*)$, provides that $R_k(p^*, v^*) \rightarrow 0$ as $k \rightarrow \infty$, that is,

$$\lim_{k \rightarrow \infty} \|p_k - p^*\| = 0 = \lim_{k \rightarrow \infty} \|v_k - v^*\| = 0. \quad (5.53)$$

Hence, $\{p_k\}$ converges strongly to p^* and $\{v_k\}$ converges strongly to v^* as $k \rightarrow \infty$. We have shown in both cases that the sequence generated by Algorithm 5.2.1 converges strongly to $(p^*, v^*) \in \mathcal{F}$. $\square$

Following the method of proof of [47] we obtain the following lemma.

Lemma 5.2.5. *If in Algorithm 5.2.1, the sequences $\{\lambda_n\}, \{\mu_n\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and $\{\theta_n\} \subset \mathbb{R}$ satisfy the condition*

$$\inf_{n \geq 0} \left\{ \frac{\mu_n}{\lambda_n} - \left(\frac{\theta_n}{2} \right)^2 \right\} > 0, \quad (5.54)$$

then, there exist some scalars $\eta > 0$ and $\hat{\eta} > 0$ such that $\varphi_n(p_n) \geq \eta \|\nabla \varphi_n\|^2$ and $\widehat{\varphi}_n(v_n) \geq \hat{\eta} \|\nabla \widehat{\varphi}_n\|^2$ for all $n \geq 0$.

Next, we state and prove our main theorem.

Theorem 5.2.6. *Let H_1, H_2 and H_3 be real Hilbert spaces. Let $F_1, G_1 : H_1 \rightarrow 2^{H_1}$ and $F_2, G_2 : H_2 \rightarrow 2^{H_2}$ be maximally monotone mappings and $\Gamma \neq \emptyset$. Let $\mathcal{A} : H_1 \times H_1 \rightarrow H_3 \times H_3$ and $\mathcal{B} : H_2 \times H_2 \rightarrow H_3 \times H_3$ are as in (5.2) with adjoints*

$\mathcal{A}^*$ and $\mathcal{B}^*$, respectively. If the sequences $\{\lambda_n\}, \{\mu_n\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and $\{\theta_n\} \subset \mathbb{R}$ satisfy the condition

$$\inf_{n \geq 0} \left\{ \frac{\mu_n}{\lambda_n} - \left(\frac{\theta_n}{2} \right)^2 \right\} > 0,$$

then, the sequence $\{((z_n, w_n), (\widehat{z}_n, \widehat{w}_n))\}$ generated by Algorithm 5.2.1 converges strongly to an element $(p^*, v^*) = ((z^*, w^*), (\widehat{z}^*, \widehat{w}^*))$ of $\mathcal{F}$, where $(z^*, \widehat{z}^*) \in \Gamma$.

Proof. By Lemma 5.2.5, there exist $\eta > 0$ and $\widehat{\eta} > 0$ such that

$$\varphi_n(p_n) \geq \eta \|\nabla \varphi_n\|^2, \quad (5.55)$$

and

$$\widehat{\varphi}_n(v_n) \geq \widehat{\eta} \|\nabla \widehat{\varphi}_n\|^2. \quad (5.56)$$

This implies that $\varphi_n(p_n)$ and $\widehat{\varphi}_n(v_n)$ are always nonnegative, so from (5.4), we have

$$\|S_n p_n - p_n\| = \left\| - \left(\frac{\varphi_n(z_n, w_n)}{\|\nabla \varphi_n\|^2} \right) \nabla \varphi_n \right\| = \frac{\varphi_n(z_n, w_n)}{\|\nabla \varphi_n\|}, \quad (5.57)$$

for all n for which $\nabla \varphi_n \neq 0$. Thus, dividing both sides of the inequality (5.55) by $\|\nabla \varphi_n\|$, we obtain

$$\|S_n p_n - p_n\| \geq \eta \|\nabla \varphi_n\|, \quad (5.58)$$

which is also true for n with $\nabla \varphi_n = 0$. Similarly, from (5.5), we have

$$\|T_n v_n - v_n\| \geq \widehat{\eta} \|\nabla \widehat{\varphi}_n\|. \quad (5.59)$$

Now, (5.35), (5.58) and (5.59) imply that $\nabla \varphi_n \rightarrow 0$ and $\nabla \widehat{\varphi}_n \rightarrow 0$ as $n \rightarrow \infty$.

Now, from the expression for $\nabla \varphi_n = (d_n + e_n, x_n - y_n)$ and $\nabla \widehat{\varphi}_n = (\widehat{d}_n + \widehat{e}_n, \widehat{x}_n - \widehat{y}_n)$,

we have

$$x_n - y_n \rightarrow 0, \text{ and } d_n + e_n \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (5.60)$$

and

$$\widehat{x}_n - \widehat{y}_n \rightarrow 0 \text{ and } \widehat{d}_n + \widehat{e}_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.61)$$

Moreover, subtracting y_n from both sides of the first equation and adding $\mu_n e_n - \theta_n y_n$ to both sides of the second equation in (5.7) and rearranging we obtain

$$\begin{cases} (z_n - y_n) + \lambda_n(w_n - e_n) = (x_n - y_n), \\ (\theta_n - 1)(z_n - y_n) + \mu_n(w_n - e_n) = \theta_n(x_n - y_n) - \mu_n(d_n + e_n), n \geq 0. \end{cases}$$

Thus, solving for $(w_n - e_n)$ from the above equations, we obtain that

$$(\mu_n + (1 - \theta_n)\lambda_n)(w_n - e_n) = (x_n - y_n) - \mu_n(d_n + e_n). \quad (5.62)$$

Similarly, subtracting $\widehat{y}_n$ from both sides of the first equation and adding $\mu_n \widehat{e}_n - \theta_n \widehat{y}_n$ to both sides of the second equation in (5.8) and rearranging we get

$$\begin{cases} (\widehat{z}_n - \widehat{y}_n) + \lambda_n(\widehat{w}_n - \widehat{e}_n) = (\widehat{x}_n - \widehat{y}_n), \\ (\theta_n - 1)(\widehat{z}_n - \widehat{y}_n) + \mu_n(\widehat{w}_n - \widehat{e}_n) = \theta_n(\widehat{x}_n - \widehat{y}_n) - \mu_n(\widehat{d}_n + \widehat{e}_n), n \geq 0. \end{cases}$$

Now, solving $(\widehat{w}_n - \widehat{e}_n)$ from this equations, we get

$$(\mu_n + (1 - \theta_n)\lambda_n)(\widehat{w}_n - \widehat{e}_n) = (\widehat{x}_n - \widehat{y}_n) - \mu_n(\widehat{d}_n + \widehat{e}_n). \quad (5.63)$$

Therefore, since $(\mu_n + (1 - \theta_n)\lambda_n) > 0$, from (5.60)-(5.63), we have $w_n - e_n \rightarrow 0$ and hence this with (5.60) yields $z_n - y_n \rightarrow 0$, as $n \rightarrow \infty$. Similarly, $\widehat{w}_n - \widehat{e}_n \rightarrow 0$ as $n \rightarrow \infty$ and hence this with (5.61) yields $\widehat{z}_n - \widehat{y}_n \rightarrow 0$, as $n \rightarrow \infty$. As a consequence, we obtain

$$z_n - x_n \rightarrow 0, z_n - y_n \rightarrow 0, w_n - e_n \rightarrow 0, w_n + d_n \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (5.64)$$

and

$$\widehat{z}_n - \widehat{x}_n \rightarrow 0, \widehat{z}_n - \widehat{y}_n \rightarrow 0, \widehat{w}_n - \widehat{e}_n \rightarrow 0, \widehat{w}_n + \widehat{d}_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.65)$$

In addition, from Lemma 5.2.4 we know that the sequence

$\{(p_n, v_n)\} = \{(z_n, w_n), (\widehat{z}_n, \widehat{w}_n)\}$ converges strongly to a point $(p^*, v^*) = ((z^*, w^*), (\widehat{z}^*, \widehat{w}^*)) \in \mathcal{F}$. Now, we show that $p^* \in S_e(F_1, G_1)$ and $v^* \in S_e(F_2, G_2)$. But $z_n - x_n \rightarrow 0$ implies that $x_n \rightarrow z^*$. Furthermore, since $w_n - e_n \rightarrow 0$, we have $e_n \rightarrow w^*$. Similarly, $\widehat{z}_n - \widehat{x}_n \rightarrow 0$ implies that $\widehat{x}_n \rightarrow \widehat{z}^*$. Since $\widehat{w}_n - \widehat{e}_n \rightarrow 0$, we have $\widehat{e}_n \rightarrow \widehat{w}^*$. As a consequence, since the $Gph(G_1)$ is closed and $(x_n, e_n) \in Gph(G_1)$ for all n , we conclude that $(z^*, w^*) \in Gph(G_1)$. Similarly, $(z^*, -w^*) \in Gph(F_1)$, so $p^* = (z^*, w^*) \in S_e(F_1, G_1)$. Moreover, since the $Gph(F_2)$ and $Gph(G_2)$ are closed by the same argument, we have $v^* = (\widehat{z}^*, \widehat{w}^*) \in S_e(F_2, G_2)$. Therefore, by Lemma 5.1.1, we obtain $z^* \in (F_1 + G_1)^{-1}(0)$, $\widehat{z}^* \in (F_2 + G_2)^{-1}(0)$ and hence $(z^*, \widehat{z}^*) \in \Gamma$. The proof is complete. $\square$

Remark 5.2.7. We observe that Algorithm 5.2.1 is equivalent to the following scheme:

$$\left\{ \begin{array}{l} (z_0, w_0) \in H_1 \times H_1 \text{ and } (\widehat{z}_0, \widehat{w}_0) \in H_2 \times H_2 \text{ chosen arbitrary,} \\ x_n = (I + \lambda_n G_1)^{-1}(z_n + \lambda_n w_n), \\ y_n = (I + \mu_n F_1)^{-1}((1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n), \\ \widehat{x}_n = (I + \lambda_n G_2)^{-1}(\widehat{z}_n + \lambda_n \widehat{w}_n), \\ \widehat{y}_n = (I + \mu_n F_2)^{-1}((1 - \theta_n)\widehat{z}_n + \theta_n \widehat{x}_n - \mu_n \widehat{w}_n), \\ p_{n+1} = \alpha_n f_1(z_n, w_n) + (1 - \alpha_n)[(1 - \omega_n)(a_n, w_n) + \omega_n(h_n, g_n)], \\ v_{n+1} = \alpha_n f_2(\widehat{z}_n, \widehat{w}_n) + (1 - \alpha_n)[(1 - \omega_n)(\widehat{a}_n, \widehat{w}_n) + \omega_n(\widehat{h}_n, \widehat{g}_n)], \end{array} \right. \quad (5.66)$$

where

$$a_n = z_n - \gamma_n A^*(Az_n - B\widehat{z}_n),$$

$$\widehat{a}_n = \widehat{z}_n + \gamma_n A^*(Az_n - B\widehat{z}_n),$$

$$\begin{aligned}
 h_n &= \beta_n a_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \varsigma_i \\
 g_n &= \beta_n b_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \tau_i \\
 \widehat{h}_n &= \beta_n \widehat{a}_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \widehat{\varsigma}_i \\
 \widehat{g}_n &= \beta_n \widehat{b}_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \widehat{\tau}_i,
 \end{aligned}$$

such that

$$\varsigma_i = \begin{cases} a_n & \text{if } \varphi_i(z_n, w_n) \leq 0, \\ a_n - \delta_i(d_i + e_i) & \text{if } \varphi_i(z_n, w_n) > 0, \end{cases}$$

$$\tau_i = \begin{cases} b_n & \text{if } \varphi_i(z_n, w_n) \leq 0, \\ b_n - \delta_i(x_i - y_i) & \text{if } \varphi_i(z_n, w_n) > 0, \end{cases}$$

$$\widehat{\varsigma}_i = \begin{cases} \widehat{a}_n & \text{if } \widehat{\varphi}_i(\widehat{z}_n, \widehat{w}_n) \leq 0, \\ \widehat{a}_n - \widehat{\delta}_i(\widehat{d}_i + \widehat{e}_i) & \text{if } \widehat{\varphi}_i(\widehat{z}_n, \widehat{w}_n) > 0, \end{cases}$$

$$\widehat{\tau}_i = \begin{cases} \widehat{b}_n & \text{if } \widehat{\varphi}_i(\widehat{z}_n, \widehat{w}_n) \leq 0, \\ \widehat{b}_n - \widehat{\delta}_i(\widehat{x}_i - \widehat{y}_i) & \text{if } \widehat{\varphi}_i(\widehat{z}_n, \widehat{w}_n) > 0, \end{cases}$$

$\lambda_n, \mu_n \in [\underline{\lambda}, \overline{\lambda}]$ for some $\overline{\lambda} > \underline{\lambda} > 0$, $\theta_n \in \mathbb{R}$, with

$$\delta_i = \frac{\langle a_n - x_i, e_i - w_n \rangle + \langle a_n - y_i, d_i + w_n \rangle}{\|d_i + e_i\|^2 + \|x_i - y_i\|^2}$$

and

$$\widehat{\delta}_i = \frac{\langle \widehat{a}_n - \widehat{x}_i, \widehat{e}_i - \widehat{w}_n \rangle + \langle \widehat{a}_n - \widehat{y}_i, \widehat{d}_i + \widehat{w}_n \rangle}{\|\widehat{d}_i + \widehat{e}_i\|^2 + \|\widehat{x}_i - \widehat{y}_i\|^2}$$

for $e_i \in G_1(x_i)$, $d_i \in F_1(y_i)$, $\widehat{e}_i \in G_2(\widehat{x}_i)$ and $\widehat{d}_i \in F_2(\widehat{y}_i)$.

If, in Theorem 5.2.6, we assume $f_1 = (z, w) \in H_1 \times H_1$ and $f_2 = (\widehat{z}, \widehat{w}) \in H_2 \times H_2$, then we get the following corollary for approximating a solution of the split equality monotone inclusion problem (2.13).

Corollary 5.2.8. Let H_1, H_2 and H_3 be real Hilbert spaces. Let $F_1, G_1 : H_1 \rightarrow 2^{H_1}$ and $F_2, G_2 : H_2 \rightarrow 2^{H_2}$ be maximally monotone mappings and $\Gamma := \{(x, y) \in H_1 \times H_2 : x \in (F_1 + G_1)^{-1}(0), y \in (F_2 + G_2)^{-1}(0) \text{ and } Ax = By\} \neq \emptyset$. Let $\mathcal{A} : H_1 \times H_1 \rightarrow H_3 \times H_3$ and $\mathcal{B} : H_2 \times H_2 \rightarrow H_3 \times H_3$ are as in (5.2) with adjoints $\mathcal{A}^*$ and $\mathcal{B}^*$, respectively. Let the sequences $\{\lambda_n\}, \{\mu_n\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and $\{\theta_n\} \subset \mathbb{R}$ satisfy the condition

$$\inf_{n \geq 0} \left\{ \frac{\mu_n}{\lambda_n} - \left(\frac{\theta_n}{2} \right)^2 \right\} > 0.$$

For arbitrary $(z_0, w_0), (z, w) \in H_1 \times H_1$ and $(\widehat{z}_0, \widehat{w}_0), (\widehat{z}, \widehat{w}) \in H_2 \times H_2$ define an iterative algorithm by

$$\begin{cases} x_n = (I + \lambda_n G_1)^{-1}(z_n + \lambda_n w_n), \\ y_n = (I + \mu_n F_1)^{-1}((1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n), \\ \widehat{x}_n = (I + \lambda_n G_2)^{-1}(\widehat{z}_n + \lambda_n \widehat{w}_n), \\ \widehat{y}_n = (I + \mu_n F_2)^{-1}((1 - \theta_n)\widehat{z}_n + \theta_n \widehat{x}_n - \mu_n \widehat{w}_n), \\ p_{n+1} = \alpha_n(z, w) + (1 - \alpha_n)[(1 - \omega_n)(a_n, w_n) + \omega_n(h_n, g_n)], \\ v_{n+1} = \alpha_n(\widehat{z}, \widehat{w}) + (1 - \alpha_n)[(1 - \omega_n)(\widehat{a}_n, \widehat{w}_n) + \omega_n(\widehat{h}_n, \widehat{g}_n)], \end{cases} \quad (5.67)$$

where $a_n, \widehat{a}_n, h_n, g_n, \widehat{h}_n$ and $\widehat{g}_n$ are as in (5.66). Then, $\{(z_n, w_n), (\widehat{z}_n, \widehat{w}_n)\}$ converges strongly to an element $(p^*, v^*) = ((z^*, w^*), (\widehat{z}^*, \widehat{w}^*))$ of $\mathcal{F}$, where $(z^*, \widehat{z}^*) \in \Gamma$.

If, in Theorem 5.2.6, we assume $\theta_n = 0$ for all $n \geq 0$ and then we get the following method for approximating the split equality monotone inclusion problem 2.13.

Corollary 5.2.9. Let H_1, H_2 and H_3 be real Hilbert spaces. Let $F_1, G_1 : H_1 \rightarrow 2^{H_1}$ and $F_2, G_2 : H_2 \rightarrow 2^{H_2}$ be maximally monotone mappings and $\Gamma := \{(x, y) \in$

$H_1 \times H_2 : x \in (F_1 + G_1)^{-1}(0), y \in (F_2 + G_2)^{-1}(0) \text{ and } Ax = By\} \neq \emptyset$. Let $\mathcal{A} : H_1 \times H_1 \rightarrow H_3 \times H_3$ and $\mathcal{B} : H_2 \times H_2 \rightarrow H_3 \times H_3$ are as in (5.2) with adjoints $\mathcal{A}^*$ and $\mathcal{B}^*$, respectively. Let $\lambda_n, \mu_n \in [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$. For arbitrary $(z_0, w_0) \in H_1 \times H_1$ and $(\widehat{z}_0, \widehat{w}_0) \in H_2 \times H_2$ define an iterative algorithm by

$$\begin{cases} x_n = (I + \lambda_n G_1)^{-1}(z_n + \lambda_n w_n), \\ y_n = (I + \mu_n F_1)^{-1}(z_n - \mu_n w_n), \\ \widehat{x}_n = (I + \lambda_n G_2)^{-1}(\widehat{z}_n + \lambda_n \widehat{w}_n), \\ \widehat{y}_n = (I + \mu_n F_2)^{-1}(\widehat{z}_n - \mu_n \widehat{w}_n), \\ p_{n+1} = \alpha_n f_1(z_n, w_n) + (1 - \alpha_n)[(1 - \omega_n)(a_n, w_n) + \omega_n(h_n, g_n)], \\ v_{n+1} = \alpha_n f_2(\widehat{z}_n, \widehat{w}_n) + (1 - \alpha_n)[(1 - \omega_n)(\widehat{a}_n, \widehat{w}_n) + \omega_n(\widehat{h}_n, \widehat{g}_n)], \end{cases} \quad (5.68)$$

where $a_n, \widehat{a}_n, h_n, g_n, \widehat{h}_n$ and $\widehat{g}_n$ are as in (5.66). Then, $\{(z_n, w_n), (\widehat{z}_n, \widehat{w}_n)\}$ converges strongly to an element $(p^*, v^*) = ((z^*, w^*), (\widehat{z}^*, \widehat{w}^*))$ of $\mathcal{F}$, where $(z^*, \widehat{z}^*) \in \Gamma$.

If, in Theorem 5.2.6, we take $H_2 = H_3$ and $B = I$, then we obtain a convergence result for a solution of the split monotone inclusion problem (SMIP) given in (2.10).

Theorem 5.2.10. *Let H_1 and H_2 be real Hilbert spaces. Let $F_1, G_1 : H_1 \rightarrow 2^{H_1}$ and $F_2, G_2 : H_2 \rightarrow 2^{H_2}$ be maximally monotone mappings and $\bar{\Gamma} := \{(x, y) \in H_1 \times H_2 : x \in (F_1 + G_1)^{-1}(0) \text{ such that } y = Ax \in (F_2 + G_2)^{-1}(0)\} \neq \emptyset$. Let $\mathcal{A} : H_1 \rightarrow H_2$. If the sequences $\{\lambda_n\}, \{\mu_n\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and $\{\theta_n\} \subset \mathbb{R}$ satisfy the condition*

$$\inf_{n \geq 0} \left\{ \frac{\mu_n}{\lambda_n} - \left(\frac{\theta_n}{2} \right)^2 \right\} > 0,$$

then, the sequence $\{((z_n, w_n), (\widehat{z}_n, \widehat{w}_n))\}$ generated by Algorithm 5.2.1, with $B = I$, identity mapping on H_2 , converges strongly to an element $(p^*, v^*) = ((z^*, w^*), (\widehat{z}^*, \widehat{w}^*))$ of $\mathcal{F}$, where $(z^*, \widehat{z}^*) \in \bar{\Gamma}$.

5.2.1 Application to Convex Minimization Problem

In this subsection, we shall apply Theorem 5.2.6 to study the convex minimization problem.

Theorem 5.2.11. *Let H_1, H_2 and H_3 be real Hilbert spaces. Let $h_1 : H_1 \rightarrow \mathbb{R}$ and $h_2 : H_2 \rightarrow \mathbb{R}$ be convex smooth functions and $k_1 : H_1 \rightarrow \mathbb{R}$ and $k_2 : H_2 \rightarrow \mathbb{R}$ be convex, lower semicontinuous functions such that $\Gamma = \{(x, y) \in H_1 \times H_2 : h_1(x) + k_1(x) = \min_{z \in H} \{h_1(z) + k_1(z)\}, h_2(y) + k_2(y) = \min_{\widehat{z} \in H} \{h_2(\widehat{z}) + k_2(\widehat{z})\} \text{ and } Ax = By\} \neq \emptyset$. Let $\mathcal{A} : H_1 \times H_1 \rightarrow H_3 \times H_3$ and $\mathcal{B} : H_2 \times H_2 \rightarrow H_3 \times H_3$ are as in (5.2) with adjoints $\mathcal{A}^*$ and $\mathcal{B}^*$ respectively. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be real sequences such that $\{\lambda_n\}, \{\mu_n\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and $\{\theta_n\} \subset \mathbb{R}$ satisfy the condition*

$$\inf_{n \geq 0} \left\{ \frac{\mu_n}{\lambda_n} - \left(\frac{\theta_n}{2} \right)^2 \right\} > 0.$$

For arbitrary $(z_0, w_0) \in H_1 \times H_1$ and $(\widehat{z}_0, \widehat{w}_0) \in H_2 \times H_2$ define an iterative algorithm by

$$\begin{cases} x_n = (I + \lambda_n \partial k_1)^{-1}(z_n + \lambda_n w_n), \\ y_n = (I + \mu_n \nabla h_1)^{-1}((1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n), \\ \widehat{x}_n = (I + \lambda_n \partial k_2)^{-1}(\widehat{z}_n + \lambda_n \widehat{w}_n), \\ \widehat{y}_n = (I + \mu_n \nabla h_2)^{-1}((1 - \theta_n)\widehat{z}_n + \theta_n \widehat{x}_n - \mu_n \widehat{w}_n), \\ p_{n+1} = \alpha_n f_1(z_n, w_n) + (1 - \alpha_n)[(1 - \omega_n)(a_n, w_n) + \omega_n(h_n, g_n)], \\ v_{n+1} = \alpha_n f_2(\widehat{z}_n, \widehat{w}_n) + (1 - \alpha_n)[(1 - \omega_n)(\widehat{a}_n, \widehat{w}_n) + \omega_n(\widehat{h}_n, \widehat{g}_n)], \end{cases} \quad (5.69)$$

where $a_n, \widehat{a}_n, h_n, g_n, \widehat{h}_n$ and $\widehat{g}_n$ are as in (5.66) for $e_i \in \partial k_1(x_i)$, $d_i \in \nabla h_1(y_i)$, $\widehat{e}_i \in \partial k_2(\widehat{x}_i)$ and $\widehat{d}_i \in \nabla h_2(\widehat{y}_i)$. Then, $\{(z_n, w_n), (\widehat{z}_n, \widehat{w}_n)\}$ converges strongly to an element $(p^*, v^*) = ((z^*, w^*), (\widehat{z}^*, \widehat{w}^*))$ of $\mathcal{F}$, where $(z^*, \widehat{z}^*) \in \Gamma$.

Proof. Let $F_1 = \nabla h_1$, $G_1 = \partial k_1$, $F_2 = \nabla h_2$ and $G_2 = \partial k_2$ be as in Theorem 5.2.11. Then, Theorem 5.2.6 provides the conclusion of the theorem. $\square$

5.2.2 Numerical Example

In this subsection, we present a numerical experiment result to explain the conclusion of our result. The following numerical example verifies the conclusion of Corollary 5.2.9.

Example 5.2.12. Let $H_1 = H_2 = H_3 = l_2$, where l_2 is the space of sequences. Let $\mathcal{A}, \mathcal{B} : l_2 \times l_2 \rightarrow l_2 \times l_2$ be defined by $\mathcal{A}(z, w) = (\frac{z}{2}, 0)$ and $\mathcal{B}(\widehat{z}, \widehat{w}) = (\widehat{z}, 0)$, where $z = (z_1, z_2, \dots), w = (w_1, w_2, \dots), \widehat{z} = (\widehat{z}_1, \widehat{z}_2, \dots), \widehat{w} = (\widehat{w}_1, \widehat{w}_2, \dots) \in l_2$. Let $F_1, G_1, F_2, G_2 : l_2 \rightarrow l_2$ be defined by $F_1x = x + (1, 1.25, 1, 0, 0, \dots)$, $G_1x = 2x + (5, 1.75, 0.5, 0, 0, \dots)$, $F_2x = x + (-1, 1, 1.25, 0, 0, \dots)$ and $G_2x = 2x + (3, 0, -0.75, 0, 0, \dots)$, where $x = (x_1, x_2, \dots) \in l_2$. We see that F_1, G_1, F_2 and G_2 are maximally monotone with $R(I + \gamma F_1) = R(I + \gamma G_1) = l_2 = R(I + \gamma F_2) = R(I + \gamma G_2)$ for each $\gamma > 0$. Now, by direct calculation we get that

$$\begin{aligned} x_n &= \frac{z_n + \lambda w_n - \lambda(5, 1.75, 0.5, 0, 0, \dots)}{1 + 2\lambda}, & y_n &= \frac{z_n - \lambda w_n - \lambda(1, 1.25, 1, 0, 0, \dots)}{1 + \lambda}, \\ d_n &= \frac{z_n - \lambda w_n + (1, 1.25, 1, 0, 0, \dots)}{1 + \lambda}, & e_n &= \frac{2z_n - 2\lambda w_n + (5, 1.75, 0.5, 0, 0, \dots)}{1 + 2\lambda}, \\ \widehat{x}_n &= \frac{\widehat{z}_n + \lambda \widehat{w}_n - \lambda(3, 0, -0.75, 0, 0, \dots)}{1 + \lambda}, & \widehat{y}_n &= \frac{\widehat{z}_n - \lambda \widehat{w}_n - \lambda(-1, 1, 1.25, 0, 0, \dots)}{1 + \lambda}, \\ \widehat{d}_n &= \frac{\widehat{z}_n - \lambda \widehat{w}_n - (-1, 1, 1.25, 0, 0, \dots)}{1 + \lambda} \text{ and } & \widehat{e}_n &= \frac{\widehat{z}_n + \lambda \widehat{w}_n + (3, 0, -0.75, 0, 0, \dots)}{1 + \lambda}, \end{aligned}$$

for some $\lambda > 0$. Thus, if we assume $\lambda_n = \mu_n = 1$, $\alpha_n = \frac{1}{10^5(n+100)}$, $\beta_n = \frac{1}{n}$, $\omega_n = \frac{9n+100}{10n+100}$ for all $n \geq 1$, and $f_1(x) = f_2(x) = \frac{x}{10}$ with initial points $z_1 = (-1, 1, -2, 0, 0, \dots)$, $w_1 = (1, 0, -2, 0, 0, \dots)$, $\widehat{z}_1 = (-1, -1, 1, 0, 0, \dots)$ and $\widehat{w}_1 = (-1, 2, -1, 0, 0, \dots)$, then the iteration Scheme (5.68) provides the following numerical experiment result using MATLAB (see, Table 5.1 and Table 5.2) below. From this we obtain that the Algorithms z_n and $\widehat{z}_n$ converges strongly to $z^* = (-2, -1, -\frac{1}{2}, 0, 0, \dots) \in (F_1 + G_1)^{-1}(0)$ and $\widehat{z}^* = (-1, -\frac{1}{2}, -\frac{1}{4}, 0, 0, \dots) \in (F_2 + G_2)^{-1}(0)$ respectively, where $A(z^*) = B(\widehat{z}^*)$.

TABLE 5.1: Convergence of the sequence $\{z_n\}$

n	z_n	$\ z_{n+1} - z_n\ _{l_2}$
1	$(-1.0000, 1.0000, -2.0000, 0, 0, \dots)$	
5	$(-1.6546, -0.3399, -0.9301, 0, 0, \dots)$	0.5854
10	$(-1.7874, -0.6085, -0.7857, 0, 0, \dots)$	0.0389
100	$(-1.9605, -0.9364, -0.5666, 0, 0, \dots)$	9.0000e-04
500	$(-1.9900, -0.9876, -0.5125, 0, 0, \dots)$	1.0000e-04
600	$(-1.9916, -0.9903, -0.5095, 0, 0, \dots)$	1.0000e-04
700	$(-1.9928, -0.9922, -0.5072, 0, 0, \dots)$	1.0000e-04
800	$(-1.9937, -0.9936, -0.5055, 0, 0, \dots)$	0
900	$(-1.9944, -0.9947, -0.5042, 0, 0, \dots)$	0
1000	$(-1.9950, -0.9957, -0.5031, 0, 0, \dots)$	0
	$\vdots$	$\vdots$
	$\downarrow$	$\downarrow$
	$(-2, -1, -\frac{1}{2}, 0, 0, \dots)$	0

TABLE 5.2: Convergence of the sequence $\{\widehat{z}_n\}$

n	$\widehat{z}_n$	$\ \widehat{z}_{n+1} - \widehat{z}_n\ _{l_2}$
1	$(-1.0000, -1.0000, -1.0000, 0, 0, \dots)$	
5	$(-0.9072, -0.4382, -0.7684, 0, 0, \dots)$	0.0604
10	$(-0.9085, -0.3628, -0.6164, 0, 0, \dots)$	0.0258
100	$(-0.9799, -0.4674, -0.2866, 0, 0, \dots)$	5.3852e-04
500	$(-0.9950, -0.4938, -0.2564, 0, 0, \dots)$	1.4142e-04
600	$(-0.9958, -0.4951, -0.2549, 0, 0, \dots)$	0
700	$(-0.9964, -0.4960, -0.2537, 0, 0, \dots)$	0
800	$(-0.9968, -0.4968, -0.2528, 0, 0, \dots)$	0
900	$(-0.9972, -0.4974, -0.2521, 0, 0, \dots)$	0
1000	$(-0.9975, -0.4978, -0.2516, 0, 0, \dots)$	0
	$\vdots$	$\vdots$
	$\downarrow$	$\downarrow$
	$(-1, -\frac{1}{2}, -\frac{1}{4}, 0, 0, \dots)$	0

Chapter 6

Approximation of Common f -Fixed Points of f -Pseudocontractive Mappings in Banach Spaces

In this chapter, we introduce and study a new class of mappings called f -pseudocontractive mappings with the notion of f -fixed points and construct an algorithm which converges strongly to a common f -fixed point of such mappings in Banach space.

6.1 Iterative Schemes for a Common f -fixed Point of f -Pseudocontractive Mappings in Banach Spaces

In this section, let E be a real reflexive Banach space with its dual E^* and let $f : E \rightarrow (-\infty, +\infty]$ be a proper, lower-semi-continuous and Gâteaux differentiable convex function on E with $f^* : E^* \rightarrow (-\infty, +\infty]$ as the Fenchel conjugate of f .

We introduce the following definition.

Definition 6.1.1. Let C be a nonempty subset of E . A mapping $T : C \rightarrow E^*$ is said to be *f*-pseudocontractive if for each $x, y \in C$ the following inequality holds:

$$\langle Tx - Ty, x - y \rangle \leq \langle \nabla fx - \nabla fy, x - y \rangle.$$

It is called strictly *f*-pseudocontractive if there exists $\gamma > 0$ such that for each $x, y \in C$

$$\langle Tx - Ty, x - y \rangle \leq \langle \nabla fx - \nabla fy, x - y \rangle - \gamma \|(\nabla fx - \nabla fy) - (Tx - Ty)\|^2.$$

Example 6.1.2. Let $1 < q < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Define $f : \ell_p \rightarrow [0, \infty)$ by $f(x) = \frac{\|x\|^p}{p}$ and $T : \ell_p \rightarrow \ell_q$ by $Tx = -\nabla fx$, then T is *f*-pseudocontractive mapping.

Definition 6.1.3. Let C be a nonempty, closed and convex subset of E . Let $T : C \rightarrow E^*$ be a map. A point $p \in C$ is called *f*-fixed point of T if $Tp = \nabla fp$. We denote the set of *f*-fixed points of T by $F_f(T)$, that is, $F_f(T) := \{p \in C : Tp = \nabla fp\}$.

Example 6.1.4. Let $1 < q < p < \infty$ $\frac{1}{p} + \frac{1}{q} = 1$ and let $\alpha_1, \alpha_2 \in \{x \in \mathbb{R} : x \geq 0\}$ be arbitrary. Define $T : \ell_p \rightarrow \ell_q$ by $T(x_1, x_2, x_3, \dots) = (|x_1|^{p-1}, |x_2|^{p-1}, 0, 0, \dots)$ and $f : \ell_p \rightarrow [0, \infty)$ by $f(x) = \frac{\|x\|^p}{p}$. Then, $\alpha_x := (\alpha_1, \alpha_2, 0, 0, \dots)$ is *f*-fixed point of T .

We observe that if $f(x) = \frac{1}{2}\|x\|^2$, then $\nabla f = J$, the normalized duality mapping from E into 2^{E^*} , and hence *f*-pseudocontractive mappings become semi-pseudocontractive mappings and *f*-fixed points of T reduces to semi-fixed points of T . If, in addition, $E = H$, a real Hilbert space, then *f*-pseudocontractive mapping reduces to pseudocontractive mapping. Therefore, we obtain that the class of *f*-pseudocontractive mappings includes the class of semi-pseudocontractive mappings and hence it is an extension of pseudocontractive mappings to Banach spaces.

We remark that this class of f -pseudocontractive mappings in Banach spaces is closely related with the important class of nonlinear monotone mappings. Observe that T is f -pseudocontractive if and only if $A := \nabla f - T$ is monotone and if T is strictly f -pseudocontractive, then $A := \nabla f - T$ is γ -inverse strongly monotone. Furthermore, we observe that zeros of A correspond to f -fixed points of T . In fact, if T and ∇f are continuous then A is maximal monotone and the set of zeros of A and hence the set of f -fixed points of an f -pseudocontractive mapping T is closed and convex (see, [93]).

Theorem 6.1.5. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $T_i : E \rightarrow E^*$ be a Lipschitz and f -pseudocontractive mappings with Lipschitz constants L_i , for $i = 1, 2$ such that $\Omega = F_f(T_1) \cap F_f(T_2) \neq \emptyset$. For arbitrary $x_0, v \in E$ define an iterative algorithm by*

$$\begin{cases} y_n = \nabla f^*((1 - \lambda_n)\nabla f x_n + \lambda_n T_1 x_n), \\ z_n = \nabla f^*((1 - \lambda_n)\nabla f x_n + \lambda_n T_2 x_n), \\ v_n = \nabla f^*((1 - \lambda_n)\nabla f y_n + \lambda_n \nabla f x_n + \lambda_n (T_1 y_n - T_1 x_n)), \\ u_n = \nabla f^*((1 - \lambda_n)\nabla f z_n + \lambda_n \nabla f x_n + \lambda_n (T_2 z_n - T_2 x_n)), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f v + (1 - \alpha_n) \nabla f w_n), \end{cases} \quad (6.1)$$

where $w_n = \nabla f^*(\beta_n \nabla f x_n + \eta_n \nabla f v_n + \theta_n \nabla f u_n)$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{\mu+L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ is bounded.

Proof. Let $x^* \in \Omega$. Then, from (6.1) and Lemma 4.1.6, we get

$$\begin{aligned} D_f(x^*, x_{n+1}) &= D_f(x^*, \nabla f^*[\alpha_n \nabla f v + (1 - \alpha_n) \nabla f w_n]) \\ &\leq \alpha_n D_f(x^*, v) + (1 - \alpha_n)[\beta_n D_f(x^*, x_n) + \eta_n D_f(x^*, v_n) \\ &\quad + \theta_n D_f(x^*, u_n)]. \end{aligned} \quad (6.2)$$

Now, from (1.15), we obtain

$$\begin{aligned}
 D_f(x^*, v_n) &= D_f(x^*, \nabla f^*((1 - \lambda_n)\nabla f y_n + \lambda_n \nabla f x_n + \lambda_n(T_1 y_n - T_1 x_n))) \\
 &= f(x^*) - \langle x^* - v_n, (1 - \lambda_n)\nabla f y_n + \lambda_n \nabla f x_n + \lambda_n(T_1 y_n - T_1 x_n) \rangle \\
 &\quad - f(v_n) \\
 &= f(x^*) - \langle x^* - v_n, (1 - \lambda_n)\nabla f y_n \rangle - \langle x^* - v_n, \lambda_n \nabla f x_n \rangle \\
 &\quad - \langle x^* - v_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle - f(v_n) \\
 &= f(x^*) - \langle x^* - v_n, \nabla f y_n \rangle - \langle x^* - v_n, \lambda_n(\nabla f x_n - \nabla f y_n) \rangle \\
 &\quad - \langle x^* - v_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle - f(v_n) \\
 &= D_f(x^*, y_n) - [f(v_n) - \langle v_n - y_n, \nabla f y_n \rangle - f(y_n)] \\
 &\quad - \langle x^* - v_n, \lambda_n(\nabla f x_n - \nabla f y_n) \rangle - \langle x^* - v_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle \\
 &= D_f(x^*, y_n) - D_f(v_n, y_n) - \langle x^* - v_n, \lambda_n(\nabla f x_n - \nabla f y_n) \rangle \\
 &\quad - \langle x^* - v_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle, \tag{6.3}
 \end{aligned}$$

and from (1.17), we obtain

$$\begin{aligned}
 D_f(x^*, y_n) - D_f(v_n, y_n) &= D_f(x^*, x_n) - D_f(v_n, x_n) \\
 &\quad + \langle \nabla f x_n - \nabla f y_n, x^* - v_n \rangle. \tag{6.4}
 \end{aligned}$$

Now, combining (6.3) and (6.4), we obtain

$$\begin{aligned}
 D_f(x^*, v_n) &= D_f(x^*, x_n) - D_f(v_n, x_n) + \langle \nabla f x_n - \nabla f y_n, x^* - v_n \rangle \\
 &\quad - \langle x^* - v_n, \lambda_n(\nabla f x_n - \nabla f y_n) \rangle \\
 &\quad - \langle x^* - v_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle. \tag{6.5}
 \end{aligned}$$

From (1.16), we get

$$D_f(v_n, x_n) = D_f(v_n, y_n) + D_f(y_n, x_n) - \langle \nabla f x_n - \nabla f y_n, v_n - y_n \rangle. \tag{6.6}$$

Substituting (6.6) into (6.5), we have

$$\begin{aligned}
 D_f(x^*, v_n) &= D_f(x^*, x_n) - D_f(v_n, y_n) - D_f(y_n, x_n) \\
 &\quad + \langle \nabla f x_n - \nabla f y_n, v_n - y_n \rangle + \langle \nabla f x_n - \nabla f y_n, x^* - v_n \rangle \\
 &\quad - \langle x^* - v_n, \lambda_n(\nabla f x_n - \nabla f y_n) \rangle - \langle x^* - v_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle \\
 &= D_f(x^*, x_n) - D_f(v_n, y_n) - D_f(y_n, x_n) + \langle \nabla f x_n - \nabla f y_n, x^* - y_n \rangle \\
 &\quad - \langle x^* - v_n, \lambda_n(\nabla f x_n - \nabla f y_n) \rangle - \langle x^* - v_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle \\
 &= D_f(x^*, x_n) - D_f(v_n, y_n) - D_f(y_n, x_n) + \langle \nabla f x_n - \nabla f y_n, x^* - y_n \rangle \\
 &\quad - \langle x^* - y_n + y_n - v_n, \lambda_n(\nabla f x_n - \nabla f y_n) \rangle \\
 &\quad - \langle x^* - y_n + y_n - v_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle \\
 &= D_f(x^*, x_n) - D_f(v_n, y_n) - D_f(y_n, x_n) + \langle \nabla f x_n - \nabla f y_n, x^* - y_n \rangle \\
 &\quad - \langle x^* - y_n, \lambda_n(\nabla f x_n - \nabla f y_n) \rangle - \langle y_n - v_n, \lambda_n(\nabla f x_n - \nabla f y_n) \rangle \\
 &\quad - \langle x^* - y_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle - \langle y_n - v_n, \lambda_n(T_1 y_n - T_1 x_n) \rangle \\
 &= D_f(x^*, x_n) - D_f(v_n, y_n) - D_f(y_n, x_n) \\
 &\quad - \langle y_n - x^*, \lambda_n((\nabla f - T_1)y_n - (\nabla f - T_1)x_n) - (\nabla f y_n - \nabla f x_n) \rangle \\
 &\quad + \lambda_n \langle v_n - y_n, \nabla f x_n - \nabla f y_n \rangle + \lambda_n \langle v_n - y_n, T_1 y_n - T_1 x_n \rangle. \quad (6.7)
 \end{aligned}$$

Furthermore, from (1.16), we obtain

$$\begin{aligned}
 \langle v_n - y_n, \nabla f x_n - \nabla f y_n \rangle &= D_f(v_n, y_n) + D_f(y_n, x_n) - D_f(v_n, x_n) \\
 &\leq D_f(v_n, y_n) + D_f(y_n, x_n). \quad (6.8)
 \end{aligned}$$

From (6.7) and (6.8), we get

$$\begin{aligned}
 D_f(x^*, v_n) &\leq D_f(x^*, x_n) - (1 - \lambda_n)[D_f(v_n, y_n) + D_f(y_n, x_n)] \\
 &\quad - \langle y_n - x^*, \lambda_n((\nabla f - T_1)y_n - (\nabla f - T_1)x_n) - (\nabla f y_n - \nabla f x_n) \rangle \\
 &\quad + \lambda_n \langle v_n - y_n, T_1 y_n - T_1 x_n \rangle \quad (6.9)
 \end{aligned}$$

But $y_n = \nabla f^*((1 - \lambda_n)\nabla f x_n + \lambda_n T_1 x_n)$ implies that $\nabla f x_n - \lambda_n(\nabla f - T_1)x_n = \nabla f y_n$.

Now, let $s_n = (\nabla f - T_1)y_n$ so that we have $\nabla f x_n + \lambda_n(\nabla f - T_1)y_n - \lambda_n(\nabla f - T_1)x_n =$

$\nabla f y_n + \lambda_n s_n$. This implies that

$$s_n = \frac{1}{\lambda_n} \left(\lambda_n (\nabla f - T_1) y_n - \lambda_n (\nabla f - T_1) x_n - (\nabla f y_n - \nabla f x_n) \right). \quad (6.10)$$

On the other hand we know that $0 = (\nabla f - T_1) x^*$. Since $\nabla f - T_1$ is monotone, we obtain

$$\langle y_n - x^*, s_n \rangle \geq 0. \quad (6.11)$$

Now, putting (6.10) into (6.11), we obtain

$$\langle y_n - x^*, \lambda_n (\nabla f - T_1) y_n - \lambda_n (\nabla f - T_1) x_n - (\nabla f y_n - \nabla f x_n) \rangle \geq 0. \quad (6.12)$$

From (6.9), (6.12), Lemma 4.1.13 and the fact that T_1 is Lipschitz continuous, we have

$$\begin{aligned} D_f(x^*, v_n) &\leq D_f(x^*, x_n) - (1 - \lambda_n)[D_f(v_n, y_n) + D_f(y_n, x_n)] \\ &\quad + \lambda_n \langle v_n - y_n, T_1 y_n - T_1 x_n \rangle \\ &\leq D_f(x^*, x_n) - (1 - \lambda_n)[D_f(v_n, y_n) + D_f(y_n, x_n)] \\ &\quad + \lambda_n \|v_n - y_n\| \|T_1 y_n - T_1 x_n\| \\ &\leq D_f(x^*, x_n) - (1 - \lambda_n)[D_f(v_n, y_n) + D_f(y_n, x_n)] \\ &\quad + \lambda_n L \|v_n - y_n\| \|y_n - x_n\| \\ &\leq D_f(x^*, x_n) - (1 - \lambda_n)[D_f(v_n, y_n) + D_f(y_n, x_n)] \\ &\quad + \frac{\lambda_n L}{2} [\|v_n - y_n\|^2 + \|y_n - x_n\|^2] \\ &\leq D_f(x^*, x_n) - (1 - \lambda_n)[D_f(v_n, y_n) + D_f(y_n, x_n)] \\ &\quad + \frac{\lambda_n L}{\mu} [D_f(v_n, y_n) + D_f(y_n, x_n)] \\ &= D_f(x^*, x_n) - \left(\frac{\mu - \lambda_n(\mu + L)}{\mu} \right) D_f(v_n, y_n) \\ &\quad - \left(\frac{\mu - \lambda_n(\mu + L)}{\mu} \right) D_f(y_n, x_n). \end{aligned} \quad (6.13)$$

Similarly, we obtain that

$$\begin{aligned} D_f(x^*, u_n) &\leq D_f(x^*, x_n) - \left(\frac{\mu - \lambda_n(\mu + L)}{\mu} \right) D_f(u_n, z_n) \\ &\quad - \left(\frac{\mu - \lambda_n(\mu + L)}{\mu} \right) D_f(z_n, x_n), \end{aligned} \quad (6.14)$$

and hence since by hypothesis $\lambda_n < \frac{\mu}{\mu+L}$ we obtain

$$D_f(x^*, v_n) \leq D_f(x^*, x_n), \quad (6.15)$$

and

$$D_f(x^*, u_n) \leq D_f(x^*, x_n). \quad (6.16)$$

Finally, from (6.2), (6.15) and (6.16), we get

$$\begin{aligned} D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, v) + (1 - \alpha_n) D_f(x^*, x_n) \\ &\leq \max\{D_f(x^*, v), D_f(x^*, x_n)\}, \end{aligned} \quad (6.17)$$

and by induction, we obtain

$$D_f(x^*, x_n) \leq \max\{D_f(x^*, v), D_f(x^*, x_0)\}. \quad (6.18)$$

This implies that $\{D_f(x^*, x_n)\}$ is bounded. Therefore, by Lemma 4.1.3 we have $\{x_n\}$ is bounded and so is the sequence $\{v_n\}$. $\square$

Next, we prove our main theorem.

Theorem 6.1.6. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $T_i : E \rightarrow E^*$ be Lipschitz and f -pseudocontractive mappings with Lipschitz constants L_i , for $i = 1, 2$, such that $\Omega = F_f(T_1) \cap F_f(T_2) \neq \emptyset$.*

0. For arbitrary $x_0, v \in E$, define an iterative algorithm by

$$\begin{cases} y_n = \nabla f^*((1 - \lambda_n)\nabla f x_n + \lambda_n T_1 x_n), \\ z_n = \nabla f^*((1 - \lambda_n)\nabla f x_n + \lambda_n T_2 x_n), \\ v_n = \nabla f^*((1 - \lambda_n)\nabla f y_n + \lambda_n \nabla f x_n + \lambda_n (T_1 y_n - T_1 x_n)), \\ u_n = \nabla f^*((1 - \lambda_n)\nabla f z_n + \lambda_n \nabla f x_n + \lambda_n (T_2 z_n - T_2 x_n)), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f v + (1 - \alpha_n) \nabla f w_n), \end{cases} \quad (6.19)$$

where $w_n = \nabla f^*(\beta_n \nabla f x_n + \eta_n \nabla f v_n + \theta_n \nabla f u_n)$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{\mu+L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* = P_{\Omega}^f(v)$.

Proof. From Theorem 7.2.5, we know that $\{x_n\}$ is bounded. Let $x^* = P_{\Omega}^f(v)$. We note that from Lemma 4.1.5 (i), we get

$$\langle u - x^*, \nabla f v - \nabla f x^* \rangle \leq 0, \forall u \in \Omega. \quad (6.20)$$

From (6.19), Lemma 4.1.6, (6.15) and (6.16), we have

$$\begin{aligned} D_f(x^*, w_n) &= D_f(x^*, \nabla f^*(\beta_n \nabla f x_n + \eta_n \nabla f v_n + \theta_n \nabla f u_n)) \\ &\leq \beta_n D_f(x^*, x_n) + \eta_n D_f(x^*, v_n) + \theta_n D_f(x^*, u_n) \\ &\leq D_f(x^*, x_n). \end{aligned} \quad (6.21)$$

Now, from (6.19), Lemma 4.1.6, (1.20) and (6.21), we obtain

$$\begin{aligned}
 D_f(x^*, x_{n+1}) &= D_f(x^*, \nabla f^*(\alpha_n \nabla f v + (1 - \alpha_n) \nabla f w_n)) \\
 &= V_f(x^*, \alpha_n \nabla f v + (1 - \alpha_n) \nabla f w_n) \\
 &\leq V_f(x^*, \alpha_n \nabla f v + (1 - \alpha_n) \nabla f w_n - \alpha_n (\nabla f v - \nabla f x^*)) \\
 &\quad + \langle \alpha_n (\nabla f v - \nabla f x^*), x_{n+1} - x^* \rangle \\
 &= V_f(x^*, \alpha_n \nabla f x^* + (1 - \alpha_n) \nabla f w_n) \\
 &\quad + \alpha_n \langle \nabla f v - \nabla f x^*, x_{n+1} - x^* \rangle \\
 &= D_f(x^*, \nabla f^*(\alpha_n \nabla f x^* + (1 - \alpha_n) \nabla f w_n)) \\
 &\quad + \alpha_n \langle \nabla f v - \nabla f x^*, x_{n+1} - x^* \rangle \\
 &\leq \alpha_n D_f(x^*, x^*) + (1 - \alpha_n) D_f(x^*, w_n) \\
 &\quad + \alpha_n \langle \nabla f v - \nabla f x^*, x_{n+1} - x^* \rangle \\
 &\leq (1 - \alpha_n) D_f(x^*, x_n) + \alpha_n \langle \nabla f v - \nabla f x^*, x_{n+1} - x^* \rangle \\
 &\leq (1 - \alpha_n) D_f(x^*, x_n) + \alpha_n \|\nabla f v - \nabla f x^*\| \|x_{n+1} - x_n\| \\
 &\quad + \alpha_n \langle \nabla f v - \nabla f x^*, x_n - x^* \rangle. \tag{6.22}
 \end{aligned}$$

Moreover, from (6.2), (6.13) and (6.14), we have

$$\begin{aligned}
 D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, v) + (1 - \alpha_n) D_f(x^*, x_n) \\
 &\quad - \left(\frac{\mu - \lambda_n(\mu + L)}{\mu} \right) (1 - \alpha_n) \left[\eta_n (D_f(v_n, y_n) + D_f(y_n, x_n)) \right. \\
 &\quad \left. + \theta_n (D_f(u_n, z_n) + D_f(z_n, x_n)) \right], \tag{6.23}
 \end{aligned}$$

which implies that

$$\begin{aligned}
 &\left(\frac{\mu - \lambda_n(\mu + L)}{\mu} \right) (1 - \alpha_n) \eta_n [D_f(v_n, y_n) + D_f(y_n, x_n)] \\
 &+ \left(\frac{\mu - \lambda_n(\mu + L)}{\mu} \right) (1 - \alpha_n) \theta_n [D_f(u_n, z_n) + D_f(z_n, x_n)] \\
 &\leq \alpha_n (D_f(x^*, v) - D_f(x^*, x_n)) + D_f(x^*, x_n) - D_f(x^*, x_{n+1}). \tag{6.24}
 \end{aligned}$$

Now, we divide the rest of the proof into two parts as follows:

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that the sequence of real numbers $\{D_f(x^*, x_n)\}$ is decreasing for all $n \geq n_0$. It then follows that $\{D_f(x^*, x_n)\}$ is convergent and hence $D_f(x^*, x_n) - D_f(x^*, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. Thus from (6.24) and the conditions on α_n, η_n and θ_n , we get

$$\lim_{n \rightarrow \infty} [D_f(v_n, y_n) + D_f(y_n, x_n)] = 0, \quad (6.25)$$

and

$$\lim_{n \rightarrow \infty} [D_f(u_n, z_n) + D_f(z_n, x_n)] = 0, \quad (6.26)$$

which imply that

$$\lim_{n \rightarrow \infty} \|v_n - y_n\| = \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0, \quad (6.27)$$

and

$$\lim_{n \rightarrow \infty} \|u_n - z_n\| = \lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \quad (6.28)$$

Consequently, from (6.27) and (6.28), we get

$$\|x_n - v_n\| \leq \|x_n - y_n\| + \|y_n - v_n\| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (6.29)$$

and

$$\|x_n - u_n\| \leq \|x_n - z_n\| + \|z_n - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (6.30)$$

Moreover, from (6.19), we have

$$\lim_{n \rightarrow \infty} \|\nabla f x_{n+1} - \nabla f w_n\| = \lim_{n \rightarrow \infty} \alpha_n \|\nabla f v - \nabla f w_n\| = 0, \quad (6.31)$$

and the fact that ∇f^* is uniformly continuous on bounded subset of E^* yields

$$\lim_{n \rightarrow \infty} \|x_{n+1} - w_n\| = 0. \quad (6.32)$$

Next, from (6.19), we have

$$\begin{aligned} D_f(x_n, w_n) &= D_f(x_n, \nabla f^*(\beta_n \nabla f x_n + \eta_n \nabla f v_n + \theta_n \nabla f u_n)) \\ &\leq \beta_n D_f(x_n, x_n) + \eta_n D_f(x_n, v_n) + \theta_n D_f(x_n, u_n), \end{aligned} \quad (6.33)$$

and from Lemma 4.1.7, (6.29), (6.30) and (6.33), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - w_n\| = 0. \quad (6.34)$$

Consequently, from (6.32) and (6.34), we get

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (6.35)$$

Since $\{x_n\}$ is bounded in E , there exists $p \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ and

$$\limsup_{n \rightarrow \infty} \langle x_n - x^*, \nabla f v - \nabla f x^* \rangle = \lim_{k \rightarrow \infty} \langle x_{n_k} - x^*, \nabla f v - \nabla f x^* \rangle. \quad (6.36)$$

Now, we show that $p \in \Omega$, that is $T_1 p = \nabla f p = T_2 p$. Suppose $z = (\nabla f - T_1)w$. Note that from (6.19), we get

$$\nabla f x_{n_k} - \lambda_{n_k}(\nabla f - T_1)x_{n_k} = \nabla f y_{n_k}, \quad (6.37)$$

and hence

$$\frac{1}{\lambda_{n_k}}(\nabla f x_{n_k} - \nabla f y_{n_k} + \lambda_{n_k}((\nabla f - T_1)y_{n_k} - (\nabla f - T_1)x_{n_k})) = (\nabla f - T_1)y_{n_k}.$$

In addition, monotonicity of A yields that

$$\langle w - y_{n_k}, z - \frac{1}{\lambda_{n_k}}(\nabla f x_{n_k} - \nabla f y_{n_k} + \lambda_{n_k}((\nabla f - T_1)y_{n_k} - (\nabla f - T_1)x_{n_k})) \rangle \geq 0.$$

Thus, we have

$$\begin{aligned}
 \langle w - y_{n_k}, z \rangle &\geq \langle w - y_{n_k}, \frac{1}{\lambda_{n_k}}(\nabla f x_{n_k} - \nabla f y_{n_k}) \\
 &\quad + \langle w - y_{n_k}, (\nabla f - T_1)y_{n_k} - (\nabla f - T_1)x_{n_k} \rangle \\
 &= \langle x - y_{n_k}, \frac{1}{\lambda_{n_k}}(\nabla f x_{n_k} - \nabla f y_{n_k}) \\
 &\quad + \langle w - y_{n_k}, \nabla f y_{n_k} - \nabla f x_{n_k} \rangle \\
 &\quad + \langle w - y_{n_k}, T_1 x_{n_k} - T_1 y_{n_k} \rangle,
 \end{aligned} \tag{6.38}$$

Furthermore, from (6.27), the fact that T_1 and ∇f are uniformly continuous, we obtain $\lim_{n \rightarrow \infty} \|T_1 x_{n_k} - T_1 y_{n_k}\| = 0$ and $\lim_{n \rightarrow \infty} \|\nabla f x_{n_k} - \nabla f y_{n_k}\| = 0$. Consequently, inequality (6.38) provides that $\langle w - p, z \rangle \geq 0$ and hence by the maximality of $\nabla f - T_1$, we obtain $0 = (\nabla f - T_1)p$, that is, $p \in F_f(T_1)$. Similarly, we get that $p \in F_f(T_2)$, and hence $p \in F_f(T_1) \cap F_f(T_2)$. Thus, by Lemma 4.1.5, we obtain

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \langle x_n - x^*, \nabla f v - \nabla f x^* \rangle &= \lim_{k \rightarrow \infty} \langle x_{n_k} - x^*, \nabla f v - \nabla f x^* \rangle \\
 &= \langle p - x^*, \nabla f v - \nabla f x^* \rangle \leq 0.
 \end{aligned} \tag{6.39}$$

Therefore, from (6.22), (6.35), (6.39) and Lemma 3.1.9, we conclude that $D_f(x^*, x_n) \rightarrow 0$ as $n \rightarrow \infty$ which implies that $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_f(x^*, x_{n_i}) < D_f(x^*, x_{n_i+1}), \forall i \in \mathbb{N}. \tag{6.40}$$

Then, by Lemma 4.1.15, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and $\max\{D_f(x^*, x_{m_k}), D_f(x^*, x_k)\} \leq D_f(x^*, x_{m_k+1})$ for all $k \in \mathbb{N}$. Thus, from (6.24) and the conditions on α_n, η_n and θ_n , we get

$$\lim_{k \rightarrow \infty} \|v_{m_k} - y_{m_k}\| = \lim_{k \rightarrow \infty} \|y_{m_k} - x_{m_k}\| = 0, \tag{6.41}$$

and

$$\lim_{k \rightarrow \infty} \|u_{m_k} - z_{m_k}\| = \lim_{k \rightarrow \infty} \|z_{m_k} - x_{m_k}\| = 0. \quad (6.42)$$

Moreover, following the methods in Case 1, we have

$$\lim_{k \rightarrow \infty} \|x_{m_k} - x_{m_k+1}\| = 0, \quad (6.43)$$

and

$$\limsup_{k \rightarrow \infty} \langle x_{m_k} - x^*, \nabla f v - \nabla f x^* \rangle \leq 0. \quad (6.44)$$

Now, the inequality in (6.22) provides

$$\begin{aligned} \alpha_{m_k} D_f(x^*, x_{m_k}) &\leq D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1}) \\ &\quad + \alpha_{m_k} \|x_{m_k} - x_{m_k+1}\| \|\nabla f v - \nabla f x^*\| \\ &\quad + \alpha_{m_k} \langle x_{m_k} - x^*, \nabla f v - \nabla f x^* \rangle \\ &\leq \alpha_{m_k} \|x_{m_k} - x_{m_k+1}\| \|\nabla f v - \nabla f x^*\| \\ &\quad + \alpha_{m_k} \langle x_{m_k} - x^*, \nabla f v - \nabla f x^* \rangle. \end{aligned} \quad (6.45)$$

Consequently, since $\alpha_{m_k} > 0$, we get

$$\begin{aligned} D_f(x^*, x_{m_k}) &\leq \|x_{m_k} - x_{m_k+1}\| \|\nabla f v - \nabla f x^*\| \\ &\quad + \langle x_{m_k} - x^*, \nabla f v - \nabla f x^* \rangle. \end{aligned} \quad (6.46)$$

Therefore, from (6.43), (6.44) and (6.46), we obtain $D_f(x^*, x_{m_k}) \rightarrow 0$ as $k \rightarrow \infty$ and hence from (6.23) we have $D_f(x^*, x_{m_k+1}) \rightarrow 0$ as $k \rightarrow \infty$. But the fact $D_f(x^*, x_k) \leq D_f(x^*, x_{m_k+1})$ implies that $\lim_{k \rightarrow \infty} D_f(x^*, x_k) = 0$. Therefore, $x_k \rightarrow x^*$ as $k \rightarrow \infty$. The proof is complete. $\square$

We remark that the method of proof of Theorem 6.1.6 provides the following theorem for approximating common *f*-fixed points of a finite family of Lipschitz and *f*-pseudocontractive mappings in Banach spaces.

Theorem 6.1.7. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $T_i : E \rightarrow E^*$ be Lipschitz and f -pseudocontractive mappings with Lipschitz constants L_i , for $i = 1, 2, \dots, N$ such that $\Omega = \cap_{i=1}^N F_f(T_i) \neq \emptyset$. For arbitrary $x_0, v \in E$ define an iterative algorithm by*

$$\begin{cases} y_{n,i} = \nabla f^*((1 - \lambda_n)\nabla f x_n + \lambda_n T_i x_n), \\ v_{n,i} = \nabla f^*((1 - \lambda_n)\nabla f y_{n,i} + \lambda_n \nabla f x_n + \lambda_n (T_i y_{n,i} - T_i x_n)), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f v + (1 - \alpha_n)\nabla f w_n), \end{cases} \quad (6.47)$$

where $w_n = \nabla f^*(\beta_n \nabla f x_n + \sum_{i=1}^N \eta_{n,i} \nabla f v_{n,i})$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{\mu+L}$, for $L := \max\{L_i : i = 1, 2, \dots, N\}$ and $\{\beta_n\}, \{\eta_{n,i}\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \sum_{i=1}^N \eta_{n,i} = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* = P_{\Omega}^f(v)$.

The following theorem follows from Theorem 6.1.6, when $T_2 = \nabla f$.

Theorem 6.1.8. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $T : E \rightarrow E^*$ be a Lipschitz and f -pseudocontractive mapping with Lipschitz constant L such that $F_f(T) \neq \emptyset$. For arbitrary $x_0, v \in E$, define an iterative algorithm by*

$$\begin{cases} y_n = \nabla f^*((1 - \lambda_n)\nabla f x_n + \lambda_n T x_n), \\ v_n = \nabla f^*((1 - \lambda_n)\nabla f y_n + \lambda_n \nabla f x_n + \lambda_n (T y_n - T x_n)), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f v + (1 - \alpha_n)[\beta_n \nabla f x_n + (1 - \beta_n)\nabla f v_n]), \end{cases} \quad (6.48)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{\mu+L}$ and $\{\beta_n\} \subset [\epsilon, 1) \subset (0, 1)$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* = P_{F_f(T)}(v)$.

Next, we state a theorem for a common zero of Lipschitz monotone mappings.

Theorem 6.1.9. *Let $f \in \mathcal{F}(E)$ be a strongly coercive bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $A_i : E \rightarrow E^*$ be monotone mappings such that $(\nabla f - A_i)$ are Lipschitz with Lipschitz constant L_i , for $i = 1, 2$, and $\Omega = \{x \in E : x \in \cap_{i=1}^2 A_i^{-1}(0)\} \neq \emptyset$. For arbitrary $x_0, v \in E$, define an iterative algorithm by*

$$\begin{cases} v_n = \nabla f^*(\nabla f y_n - \lambda_n(A_1 y_n - A_1 x_n)), \\ u_n = \nabla f^*(\nabla f z_n - \lambda_n(A_2 z_n - A_2 x_n)), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f v + (1 - \alpha_n) \nabla f w_n), \end{cases} \quad (6.49)$$

where $y_n = \nabla f^*(\nabla f x_n - \lambda_n A_1 x_n)$, $z_n = \nabla f^*(\nabla f x_n - \lambda_n A_2 x_n)$, $w_n = \nabla f^*(\beta_n \nabla f x_n + \eta_n \nabla f v_n + \theta_n \nabla f u_n)$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{\mu + L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* = P_{\Omega}^f(v)$.

Proof. Consider $T_i = \nabla f - A_i$, for $i = 1, 2$. Then, we have that T_i is Lipschitz f -pseudocontractive mapping and f -fixed point of T_i is the same as the zero of A_i , for $i = 1, 2$. Moreover, the scheme in (6.49) reduces to the scheme in (6.19). Therefore, Theorem 6.1.6 provides the conclusion of the theorem. $\square$

The following corollary is for a common f -fixed point of strictly f -pseudocontractive mappings which follows from Theorem 6.1.6.

Corollary 6.1.10. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $T_i : E \rightarrow E^*$ be Lipschitz and strictly f -pseudocontractive mappings with Lipschitz constant L_i , for $i = 1, 2$, such that $\Omega = F_f(T_1) \cap F_f(T_2) \neq \emptyset$. For arbitrary $x_0, v \in E$, define an iterative algorithm*

by

$$\begin{cases} y_n = \nabla f^*((1 - \lambda_n)\nabla f x_n + \lambda_n T_1 x_n), \\ z_n = \nabla f^*((1 - \lambda_n)\nabla f x_n + \lambda_n T_2 x_n), \\ v_n = \nabla f^*((1 - \lambda_n)\nabla f y_n + \lambda_n \nabla f x_n + \lambda_n (T_1 y_n - T_1 x_n)), \\ u_n = \nabla f^*((1 - \lambda_n)\nabla f z_n + \lambda_n \nabla f x_n + \lambda_n (T_2 z_n - T_2 x_n)), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f v + (1 - \alpha_n)\nabla f w_n), \end{cases} \quad (6.50)$$

where $w_n = \nabla f^*(\beta_n \nabla f x_n + \eta_n \nabla f v_n + \theta_n \nabla f u_n)$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{\mu+L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* = P_{\Omega}^f(v)$.

If, in Theorem 6.1.6, E is in addition a smooth and strictly convex, then $f(x) = \frac{1}{2}\|x\|^2$ is a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu = 1$ and conjugate $f^*(x^*) = \frac{1}{2}\|x^*\|^2$ which imply that $\nabla f = J$ and $\nabla f^* = J^{-1}$. Thus, we have the following corollaries from Theorems 6.1.6 and 6.1.9, respectively.

Corollary 6.1.11. Let E be a smooth, strictly convex and real reflexive Banach space with its dual E^* . Let $T_i : E \rightarrow E^*$ be a Lipschitz and J -pseudocontractive mappings with Lipschitz constants L_i , for $i = 1, 2$, such that $\Omega = F_s(T_1) \cap F_s(T_2) \neq \emptyset$. For arbitrary $x_0, v \in E$, define an iterative algorithm by

$$\begin{cases} v_n = J^{-1}((1 - \lambda_n)Jy_n + \lambda_n Jx_n + \lambda_n (T_1 y_n - T_1 x_n)), \\ u_n = J^{-1}((1 - \lambda_n)Jz_n + \lambda_n Jx_n + \lambda_n (T_2 z_n - T_2 x_n)), \\ x_{n+1} = J^{-1}(\alpha_n Jv + (1 - \alpha_n)Jw_n), \end{cases} \quad (6.51)$$

where $y_n = J^{-1}((1 - \lambda_n)Jx_n + \lambda_n T_1 x_n)$, $z_n = J^{-1}((1 - \lambda_n)Jx_n + \lambda_n T_2 x_n)$, $w_n = J^{-1}(\beta_n Jx_n + \eta_n Jv_n + \theta_n Ju_n)$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{1}{1+L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$

such that $\beta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element x^* in Ω .

Corollary 6.1.12. Let E be a smooth, strictly convex and real reflexive Banach space with its dual E^* . Let $A_i : E \rightarrow E^*$ be monotone mappings such that $(J - A_i)$ are Lipschitz with constant L_i , for $i = 1, 2$, and $\Omega = \{x \in E : x \in \cap_{i=1}^2 A_i^{-1}(0)\} \neq \emptyset$. For arbitrary $x_0, v \in E$ define an iterative algorithm by

$$\begin{cases} v_n = J^{-1}(Jy_n - \lambda_n(A_1y_n - A_1x_n)), \\ u_n = J^{-1}(Jz_n - \lambda_n(A_2z_n - A_2x_n)), \\ x_{n+1} = J^{-1}(\alpha_n Jv + (1 - \alpha_n)Jw_n), \end{cases} \quad (6.52)$$

where $y_n = J^{-1}(Jx_n - \lambda_n A_1 x_n)$, $z_n = J^{-1}(Jx_n - \lambda_n A_2 x_n)$, $w_n = J^{-1}(\beta_n Jx_n + \eta_n Jv_n + \theta_n Ju_n)$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{1}{1+L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element x^* of Ω .

If, in Corollary 6.1.11, we assume $E = H$ a real Hilbert space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have $\nabla f = J = I$ and $\nabla f^* = J^{-1} = I$, where I is identity mapping on H . Thus, we have the following corollary for a common fixed point of Lipschitz and pseudocontractive mappings.

Corollary 6.1.13. Let H be a real Hilbert space. Let $T_i : H \rightarrow H$ be Lipschitz and pseudocontractive mappings with Lipschitz constant L_i , for $i = 1, 2$ such that $\Omega = F(T_1) \cap F(T_2) \neq \emptyset$. For arbitrary $x_0, v \in E$ define an iterative algorithm by

$$\begin{cases} v_n = (1 - \lambda_n)y_n + \lambda_n x_n + \lambda_n(T_1 y_n - T_1 x_n), \\ u_n = (1 - \lambda_n)z_n + \lambda_n x_n + \lambda_n(T_2 z_n - T_2 x_n), \\ x_{n+1} = \alpha_n v + (1 - \alpha_n)w_n, \end{cases} \quad (6.53)$$

where $y_n = (1 - \lambda_n)x_n + \lambda_n T_1 x_n$, $z_n = (1 - \lambda_n)x_n + \lambda_n T_2 x_n$, $w_n = \beta_n x_n + \eta_n v_n + \theta_n u_n$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{1}{1+L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element x^* in Ω .

6.2 Application to Convex Minimization Problems in Banach Spaces

Let $h : E \rightarrow \mathbb{R}$ be a convex, Gâteaux differentiable function with derivative ∇h which is Lipschitz continuous with constant L . We consider the problem of finding $x^* \in E$ such that

$$h(x^*) = \min_{x \in E} \{h(x)\}, \quad (6.54)$$

This problem is equivalent, by Fermat's rule, to the problem of finding $x^* \in E$ such that

$$0 = \nabla h(x^*). \quad (6.55)$$

Note that ∇h is monotone mapping (see, Theorem 3.13, [81]). One way of solving the equation (6.55) is by finding the f -fixed points of an f -pseudocontractive mapping $T := \nabla f - \nabla h$ since an f -fixed point of T is a zero of ∇h . Thus, we get the following result from Theorem 6.1.8.

Theorem 6.2.1. *Let $f \in \mathcal{F}(E)$ be a strongly coercive Legendre function which is bounded and strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $h : E \rightarrow \mathbb{R}$ be a convex, Gâteaux differentiable function such that $\nabla f - \nabla h$ is Lipschitz continuous with constant L and $\Omega = \min_{x \in E} \{h(x)\} \neq \emptyset$. For arbitrary $u_0, u \in E$, define an iterative algorithm by*

$$\begin{cases} z_n = \nabla f^*(\nabla f u_n - \lambda_n \nabla h u_n), \\ u_{n+1} = \nabla f^*[\alpha_n \nabla f u + (1 - \alpha_n)(\nabla f z_n - \lambda_n(\nabla h z_n - \nabla h u_n))], \end{cases} \quad (6.56)$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \frac{\mu}{\mu+L}$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to an element $x^* = P_{\Omega}^f(v)$.

Proof. Consider $T = \nabla f - \nabla h$. Then, we have that T is a Lipschitz f -pseudocontractive mapping, and an f -fixed point of T is the same as the zero of ∇h . Moreover, the scheme in (6.56) reduces to the scheme in (6.48). Therefore, Theorem 6.1.8 provides the conclusion of the theorem. $\square$

6.2.1 Numerical Example

In this subsection, we present a numerical example to explain the conclusion of our result. The following numerical example verifies the conclusion of Corollary 6.1.13.

Example 6.2.2. Let $H = \mathbb{R}^3$. Define $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, by $f(x) = \frac{1}{2}\|x\|^2$, then $\nabla f = J = I$ and $\nabla f^* = J^{-1} = I$, where I is identity mapping on $\mathbb{R}^3$. Let $T_1, T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by $T_1 x = (3, 1, 10) - (2x_1, x_2, 4x_3)$ and $T_2 x = (4, 2, 8) - 3(x_1, x_2, x_3)$ where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We see that T_1 and T_2 are Lipschitz and pseudocontractive mappings with $F(T_1) \cap F(T_2) = \{(1, \frac{1}{2}, 2)\}$.

Now, taking $v = (1, 1, 1)$, $\lambda_n = \frac{1}{6}$, $\alpha_n = \frac{1}{100(n+100)}$, $\beta_n = \frac{1}{n+1000} + 0.001 = \eta_n$ and $\theta_n = 1 - \frac{2}{n+1000} - 0.002$ for all $n \geq 0$, we observe that the conditions of Corollary 6.1.13 are satisfied. Thus, if we take different initial points $x_0 = (2, 1, 3)$, $x'_0 = (3, 2, 4)$ and $x''_0 = (4, 3, 5)$ then in all cases, the numerical experiment results using MATLAB provide that $\{x_n\}$ generated by (6.53) converges strongly to the solution $x^* = (1, \frac{1}{2}, 2) \in F(T_1) \cap F(T_2)$ (see, Table 6.1, below).

In addition, we have sketched the error term $\|x_n - x^*\|$ for each initial point. From the sketch we observe that the convergence rate of the sequence $\{x_n\}$ generated by (6.53) converges faster as the initial point gets closer to the solution (see, Figure 6.1, below).

TABLE 6.1: Convergence of the sequence $\{x_n\}$ generated by (6.53)

n	x_n	x_n	x_n
0	(2.0000, 1.0000, 3.0000)	(3.0000, 2.0000, 4.0000)	(4.0000, 3.0000, 5.0000)
1	(1.6951, 0.8891, 2.8612)	(2.3902, 1.6673, 3.7225)	(3.0853, 2.4454, 4.5838)
2	(1.4832, 0.8028, 2.7417)	(1.9663, 1.4084, 3.4835)	(2.4495, 2.0139, 4.2253)
3	(1.3358, 0.7357, 2.6387)	(1.6717, 1.2069, 3.2777)	(2.0075, 1.6781, 3.9166)
4	(1.2334, 0.6835, 2.5500)	(1.4669, 1.0501, 3.1004)	(1.7003, 1.4167, 3.6507)
5	(1.1623, 0.6428, 2.4736)	(1.3245, 0.9281, 2.9476)	(1.4868, 1.2134, 3.4217)
10	(1.0263, 0.5409, 2.2242)	(1.0527, 0.6223, 2.4488)	(1.0790, 0.7037, 2.6735)
50	(1.0000, 0.5002, 2.0001)	(1.0000, 0.5002, 2.0006)	(1.0000, 0.5002, 2.0012)
100	(1.0000, 0.5001, 1.9996)	(1.0000, 0.5001, 1.9996)	(1.0000, 0.5001, 1.9996)
150	(1.0000, 0.5001, 1.9997)	(1.0000, 0.5001, 1.9997)	(1.0000, 0.5001, 1.9997)
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
	$(1, \frac{1}{2}, 2)$	$(1, \frac{1}{2}, 2)$	$(1, \frac{1}{2}, 2)$

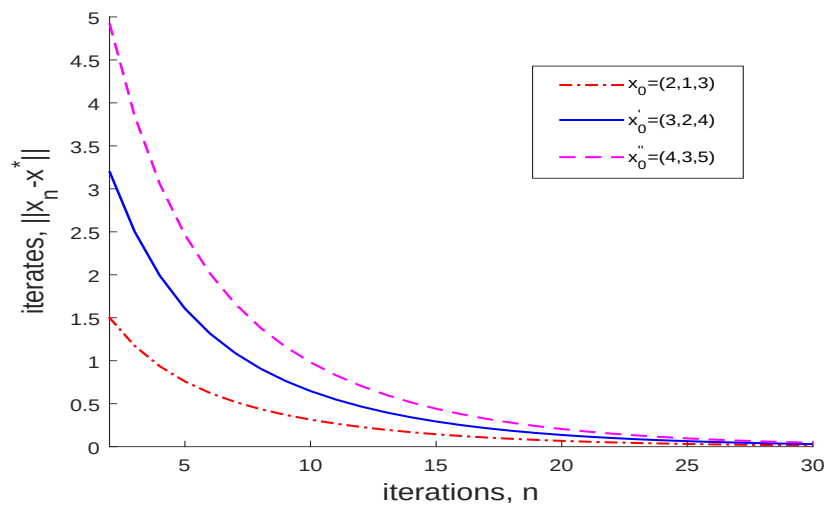


FIGURE 6.1: The graph of $\|x_n - x^*\|$ versus number of iterations with different choices of x_0

Chapter 7

Convergence Theorem for Approximating a Common Point of Solutions of Variational Inequality and f -Fixed Point Problems

In this chapter, we present some preliminary results that will be useful to prove our main theorems of the chapter. In addition, we introduce the concept of a Bregman relatively f -nonexpansive mapping and the method of approximating a common element of the set of solutions of variational inequality problems for Lipschitz monotone mappings and the set of f -fixed points of Bregman relatively f -nonexpansive mapping in the setting of Banach spaces.

7.1 Preliminaries

In the sequel, we shall need the following lemmas.

Lemma 7.1.1. [76] *Let E be a Banach space and $r > 0$ be a constant. Let $f : E \rightarrow \mathbb{R}$ be a continuous and convex function which is uniformly convex on bounded subset of E . Then*

$$f\left(\sum_{k=0}^n \alpha_k x_k\right) \leq \sum_{k=0}^n \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - x_j\|), \quad (7.1)$$

for all $0 \leq i, j \leq n$, $x_k \in B_r$, $\alpha_k \in (0, 1)$ with $\sum_{k=0}^n \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f .

Lemma 7.1.2. [122] *Let $f : E \rightarrow \mathbb{R}$ be a continuous convex function on a reflexive Banach space E which is strongly coercive. Then, the following statements are equivalent:*

- i) f is bounded on bounded subset and uniformly smooth on bounded subset of E .
- ii) f^* is Fréchet differentiable and f^* is uniformly norm-to-norm continuous on bounded subset of E^* .
- iii) $\text{dom } f^* = E^*$, f^* is strongly coercive and uniformly convex on bounded subset of E^* .

7.2 Solutions of Variational Inequality and f -Fixed Point Problems

For the rest of this section, let E be a real reflexive Banach space with its dual E^* and C be a nonempty, closed and convex subset of E . Let $f : E \rightarrow (-\infty, +\infty]$ be a proper, continuous and convex function on a real Banach space E and $f^* : E^* \rightarrow (-\infty, +\infty]$ be the Fenchel conjugate of f . We denote the family of such functions by $\mathcal{F}(E)$. Let $\{\alpha_n\} \subset (0, 1)$ be a real sequence satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Definition 7.2.1. Let $T : C \rightarrow E^*$ be a map. A point $p \in C$ is called an f -asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n \rightarrow \infty} \|\nabla f x_n - T x_n\| = 0$. We denote the set of f -asymptotic fixed points of T by $\widehat{F}_f(T)$.

Definition 7.2.2. A mapping $T : C \rightarrow E^*$ is called Bregman relatively f -nonexpansive if

- i) $F_f(T) \neq \emptyset$;
- ii) $D_f(p, \nabla f^* T y) \leq D_f(p, y), \forall y \in C, p \in \widehat{F}_f(T)$;
- iii) $\widehat{F}_f(T) = F_f(T)$.

Example 7.2.3. Let ℓ_p be the sequence spaces for $1 < q < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{x_n\} \in \ell_p$ be given by $x_n = (0, 0, 0, \dots, 0, 1, 0, \dots)$ with 1 in the n^{th} position and zero in every other position, then $x_n \rightharpoonup 0$ as $n \rightarrow \infty$. Define $f : \ell_p \rightarrow [0, \infty)$ by $f(x) = \frac{\|x\|^p}{p}$ and $T : \ell_p \rightarrow \ell_q$ by

$$Tx = \begin{cases} \nabla f \frac{x}{2} & \text{if } x \neq x_n \\ \frac{2n}{2n+3} \nabla f x_n, & \text{if } x = x_n, \end{cases} \quad (7.2)$$

then we observe that $F_f(T) = \{0\}$. Moreover, since $\|Tx_n - \nabla f x_n\| = \|\frac{2n}{2n+3} \nabla f x_n - \nabla f x_n\| = \frac{3}{2n+3} \|\nabla f x_n\| \rightarrow 0$ as $n \rightarrow \infty$, we obtain that $\widehat{F}_f(T) = \{0\} = F_f(T)$. Furthermore, the fact that

$$D_f(0, \nabla f^* Tx) = D_f(0, \frac{x}{2}) = \frac{\|x\|^p}{2^{pp}}(p-1) \leq \frac{\|x\|^p}{p}(p-1) = D_f(0, x) \quad (7.3)$$

for $x \neq x_n$ and

$$D_f(0, \nabla f^* Tx_n) = \left(\frac{2n}{2n+3}\right)^{qp-1} \frac{p-1}{p} \leq \frac{p-1}{p} = D_f(0, x_n), \quad (7.4)$$

for $x = x_n$, we obtain $D_f(0, \nabla f^* Tx) \leq D_f(0, x)$ for all $x \in \ell_p$. Thus, T is Bregman relatively f -nonexpansive.

First, we prove the following lemmas.

Lemma 7.2.4. *If $T : C \rightarrow E^*$ is a Bregman relatively f -nonexpansive mapping, then $F_f(T)$ is closed and convex.*

Proof. We first prove that $F_f(T)$ is closed. Let $\{x_n\}$ be a sequence in $F_f(T)$ such that $x_n \rightarrow p \in C$. From the definition of T , we have

$$D_f(x_n, \nabla f^*Tp) \leq D_f(x_n, p), \quad \text{for each } n \geq 0. \quad (7.5)$$

Thus,

$$\begin{aligned} D_f(p, \nabla f^*Tp) &= \lim_{n \rightarrow \infty} D_f(x_n, \nabla f^*Tp) \leq \lim_{n \rightarrow \infty} D_f(x_n, p) \\ &= D_f(p, p) = 0, \end{aligned} \quad (7.6)$$

which implies that $p = \nabla f^*Tp$ and hence, $\nabla f p = Tp$. So, we have $p \in F_f(T)$, that is, $F_f(T)$ is closed. Next, we show that $F_f(T)$ is convex. Let $p, q \in F_f(T)$ and $w = tp + (1 - t)q$, for $t \in (0, 1)$. Then, from (1.15) and the definition of T , we have

$$\begin{aligned} D_f(w, \nabla f^*Tw) &= f(w) - \langle w - \nabla f^*Tw, Tw \rangle - f(\nabla f^*Tw) \\ &= f(w) - \langle tp + (1 - t)q - \nabla f^*Tw, Tw \rangle - f(\nabla f^*Tw) \\ &= f(w) - t\langle p - \nabla f^*Tw, Tw \rangle \\ &\quad - (1 - t)\langle q - \nabla f^*Tw, Tw \rangle - f(\nabla f^*Tw) \\ &= f(w) + tf(p) - tf(p) - t\langle p - \nabla f^*Tw, Tw \rangle \\ &\quad + (1 - t)f(q) - (1 - t)f(q) \\ &\quad - (1 - t)\langle q - \nabla f^*Tw, Tw \rangle - f(\nabla f^*Tw) \\ &= f(w) + tD_f(p, \nabla f^*Tw) + (1 - t)D_f(q, \nabla f^*Tw) \\ &\quad - tf(p) - (1 - t)f(q) \\ &\leq f(w) + tD_f(p, w) + (1 - t)D_f(q, w) - tf(p) - (1 - t)f(q) \\ &= f(w) + t[f(p) - \langle p - w, \nabla fw \rangle - f(w)] \\ &\quad + (1 - t)[f(q) - \langle q - w, \nabla fw \rangle - f(w)] - tf(p) - (1 - t)f(q) \\ &= f(w) - \langle w - w, \nabla fw \rangle - f(w) = 0. \end{aligned} \quad (7.7)$$

This impels $w = \nabla f^*Tw$ and hence $\nabla fw = Tw$. Therefore, $w \in F_f(T)$, that is, $F_f(T)$ is convex. $\square$

Theorem 7.2.5. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $A_i : C \rightarrow E^*$ be L_i -Lipschitz monotone mappings with constants L_i , for $i = 1, 2$ and $T : C \rightarrow E^*$ be a Bregman relatively f -nonexpansive mapping such that $\Omega = VI(C, A_1) \cap VI(C, A_2) \cap F_f(T) \neq \emptyset$. For arbitrary $x_0, w \in C$ define an iterative algorithm by*

$$\begin{cases} y_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_1 x_n), \\ z_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_2 x_n), \\ w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n + \eta_n \nabla f u_n + \theta_n \nabla f v_n), \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n), \end{cases} \quad (7.8)$$

where $u_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)$, $v_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \bar{\gamma} < \frac{\mu}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ is bounded.

Proof. Let $q \in \Omega$. Now, from (7.8), Lemma 4.1.5 (ii), (1.15), (2.15) and monotonicity of A_1 , we have

$$\begin{aligned} D_f(q, u_n) &= D_f(q, P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)) \\ &\leq D_f(q, \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)) - D_f(u_n, \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)) \\ &= f(q) - \langle q - \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n), \nabla f x_n - \gamma_n A_1 y_n \rangle \\ &\quad - f(\nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)) - f(u_n) \\ &\quad + \langle u_n - \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n), \nabla f x_n - \gamma_n A_1 y_n \rangle \\ &\quad + f(\nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)) \\ &= f(q) + \langle u_n - q, \nabla f x_n - \gamma_n A_1 y_n \rangle - f(u_n) \end{aligned}$$

and

$$\begin{aligned}
D_f(q, u_n) &\leq f(q) + \langle u_n - q, \nabla f x_n \rangle - \langle u_n - q, \gamma_n A_1 y_n \rangle - f(u_n) \\
&= f(q) - \langle q - x_n, \nabla f x_n \rangle - f(x_n) + \langle q - x_n, \nabla f x_n \rangle + f(x_n) \\
&\quad + \langle u_n - q, \nabla f x_n \rangle - \langle u_n - q, \gamma_n A_1 y_n \rangle - f(u_n) \\
&= D_f(q, x_n) + \langle u_n - x_n, \nabla f x_n \rangle + f(x_n) - f(u_n) - \langle u_n - q, \gamma_n A_1 y_n \rangle \\
&= D_f(q, x_n) - D_f(u_n, x_n) - \langle u_n - q, \gamma_n A_1 y_n \rangle \\
&= D_f(q, x_n) - D_f(u_n, x_n) + \langle q - y_n, \gamma_n A_1 y_n \rangle + \langle y_n - u_n, \gamma_n A_1 y_n \rangle \\
&= D_f(q, x_n) - D_f(u_n, x_n) + \gamma_n \langle q - y_n, A_1 y_n - A_1 q \rangle \\
&\quad + \gamma_n \langle q - y_n, A_1 q \rangle + \langle y_n - u_n, \gamma_n A_1 y_n \rangle \\
&\leq D_f(q, x_n) - D_f(u_n, x_n) + \langle y_n - u_n, \gamma_n A_1 y_n \rangle. \tag{7.9}
\end{aligned}$$

In addition, from (1.16), we get

$$D_f(u_n, x_n) = D_f(u_n, y_n) + D_f(y_n, x_n) + \langle \nabla f x_n - \nabla f y_n, y_n - u_n \rangle. \tag{7.10}$$

Substituting (7.10) in (7.9), we obtain

$$\begin{aligned}
D_f(q, u_n) &\leq D_f(q, x_n) - D_f(u_n, y_n) - D_f(y_n, x_n) \\
&\quad - \langle \nabla f x_n - \nabla f y_n, y_n - u_n \rangle + \langle y_n - u_n, \gamma_n A_1 y_n \rangle \\
&= D_f(q, x_n) - D_f(u_n, y_n) - D_f(y_n, x_n) \\
&\quad + \langle y_n - u_n, \gamma_n A_1 y_n + \nabla f y_n - \nabla f x_n \rangle, \tag{7.11}
\end{aligned}$$

and from Lemma 4.1.5 (i), we have

$$\begin{aligned}
\langle y_n - u_n, \gamma_n A_1 y_n + \nabla f y_n - \nabla f x_n \rangle &= \langle u_n - y_n, \nabla f x_n - \gamma_n A_1 y_n - \nabla f y_n \rangle \\
&= \langle u_n - y_n, \gamma_n A_1 x_n - \gamma_n A_1 y_n \rangle \\
&\quad + \langle u_n - y_n, \nabla f x_n - \gamma_n A_1 x_n - \nabla f y_n \rangle \\
&\leq \langle u_n - y_n, \gamma_n A_1 x_n - \gamma_n A_1 y_n \rangle. \tag{7.12}
\end{aligned}$$

Furthermore, (7.11), (7.12), the fact that A_1 is L_1 -Lipschitz and Lemma 4.1.13, we obtain

$$\begin{aligned}
 D_f(q, u_n) &\leq D_f(q, x_n) - D_f(u_n, y_n) - D_f(y_n, x_n) + \langle u_n - y_n, \gamma_n A_1 x_n - \gamma_n A_1 y_n \rangle \\
 &\leq D_f(q, x_n) - D_f(u_n, y_n) - D_f(y_n, x_n) + \gamma_n \|u_n - y_n\| \|A_1 x_n - A_1 y_n\| \\
 &\leq D_f(q, x_n) - D_f(u_n, y_n) - D_f(y_n, x_n) + \gamma_n L \|u_n - y_n\| \|x_n - y_n\| \\
 &\leq D_f(q, x_n) - D_f(u_n, y_n) - D_f(y_n, x_n) + \frac{\gamma_n L}{2} [\|u_n - y_n\|^2 + \|x_n - y_n\|^2] \\
 &\leq D_f(q, x_n) - \frac{\mu}{2} \|u_n - y_n\|^2 - \frac{\mu}{2} \|y_n - x_n\|^2 \\
 &\quad + \frac{\gamma_n L}{2} [\|u_n - y_n\|^2 + \|x_n - y_n\|^2] \\
 &= D_f(q, x_n) - \left(\frac{\mu - \gamma_n L}{2} \right) [\|u_n - y_n\|^2 + \|x_n - y_n\|^2]. \tag{7.13}
 \end{aligned}$$

Similarly, we obtain that

$$D_f(q, v_n) \leq D_f(q, x_n) - \left(\frac{\mu - \gamma_n L}{2} \right) [\|v_n - z_n\|^2 + \|x_n - z_n\|^2]. \tag{7.14}$$

Again from (7.8), (1.19), (1.20) and Lemma 7.1.1, we get

$$\begin{aligned}
 D_f(q, w_n) &= D_f(q, \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n + \eta_n \nabla f u_n + \theta_n \nabla f v_n)) \\
 &= V_f(q, \beta_n \nabla f x_n + \delta_n T x_n + \eta_n \nabla f u_n + \theta_n \nabla f v_n) \\
 &= f(q) - \langle q, \beta_n \nabla f x_n + \delta_n T x_n + \eta_n \nabla f u_n + \theta_n \nabla f v_n \rangle \\
 &\quad + f^*(\beta_n \nabla f x_n + \delta_n T x_n + \eta_n \nabla f u_n + \theta_n \nabla f v_n) \\
 &\leq \beta_n f(q) + \delta_n f(q) + \eta_n f(q) + \theta_n f(q) \\
 &\quad - \langle q, \beta_n \nabla f x_n \rangle - \langle q, \delta_n T x_n \rangle - \langle q, \eta_n \nabla f u_n \rangle - \langle q, \theta_n \nabla f v_n \rangle \\
 &\quad + \beta_n f^*(\nabla f x_n) + \delta_n f^*(T x_n) + \eta_n f^*(\nabla f u_n) + \theta_n f^*(\nabla f v_n) \\
 &\quad - \beta_n \delta_n \rho_r^*(\|\nabla f x_n - T x_n\|) \\
 &= \beta_n V_f(q, \nabla f x_n) + \delta_n V_f(q, T x_n) + \eta_n V_f(q, \nabla f u_n) \\
 &\quad + \theta_n V_f(q, \nabla f v_n) - \beta_n \delta_n \rho_r^*(\|\nabla f x_n - T x_n\|) \\
 &= \beta_n D_f(q, x_n) + \delta_n D_f(q, \nabla f^* T x_n) + \eta_n D_f(q, u_n) + \theta_n D_f(q, v_n) \\
 &\quad - \beta_n \delta_n \rho_r^*(\|\nabla f x_n - T x_n\|).
 \end{aligned}$$

Using the assumption on T , inequalities (7.13) and (7.14), we obtain

$$\begin{aligned}
 D_f(q, w_n) &\leq \beta_n D_f(q, x_n) + \delta_n D_f(q, x_n) + \eta_n D_f(q, u_n) + \theta_n D_f(q, v_n) \\
 &\quad - \beta_n \delta_n \rho_r^*(\|\nabla f x_n - T x_n\|) \\
 &\leq \beta_n D_f(q, x_n) + \delta_n D_f(q, x_n) \\
 &\quad + \eta_n \left[D_f(q, x_n) - \left(\frac{\mu - \gamma_n L}{2} \right) (\|u_n - y_n\|^2 + \|x_n - y_n\|^2) \right] \\
 &\quad + \theta_n \left[D_f(q, x_n) - \left(\frac{\mu - \gamma_n L}{2} \right) (\|v_n - z_n\|^2 + \|x_n - z_n\|^2) \right] \\
 &\quad - \beta_n \delta_n \rho_r^*(\|\nabla f x_n - T x_n\|) \\
 &= D_f(q, x_n) - \eta_n \left[\left(\frac{\mu - \gamma_n L}{2} \right) (\|u_n - y_n\|^2 + \|x_n - y_n\|^2) \right] \\
 &\quad - \theta_n \left[\left(\frac{\mu - \gamma_n L}{2} \right) (\|v_n - z_n\|^2 + \|x_n - z_n\|^2) \right] \\
 &\quad - \beta_n \delta_n \rho_r^*(\|\nabla f x_n - T x_n\|) \tag{7.15}
 \end{aligned}$$

$$\leq D_f(q, x_n). \tag{7.16}$$

Now, from (7.8), Lemma 4.1.5 (ii), Lemma 4.1.6 and (7.16), we obtain

$$\begin{aligned}
 D_f(q, x_{n+1}) &= D_f(q, P_C^f \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n)) \\
 &= D_f(q, \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n)) \\
 &\leq \alpha_n D_f(q, w) + (1 - \alpha_n) D_f(q, w_n) \\
 &\leq \alpha_n D_f(q, w) + (1 - \alpha_n) D_f(q, x_n) \\
 &\leq \max\{D_f(q, w), D_f(q, x_n)\}, \tag{7.17}
 \end{aligned}$$

and by induction we get

$$D_f(q, x_n) \leq \max\{D_f(q, w), D_f(q, x_0)\}. \tag{7.18}$$

Thus, $\{D_f(q, x_n)\}$ is bounded. Therefore, by Lemma 4.1.3, $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{z_n\}$, $\{u_n\}$, $\{w_n\}$ and $\{v_n\}$. $\square$

Next, we prove the main theorem of the section.

Theorem 7.2.6. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $A_i : C \rightarrow E^*$ be L_i -Lipschitz monotone mappings with constants L_i , for $i = 1, 2$ and $T : C \rightarrow E^*$ be a Bregman relatively f -nonexpansive mapping such that $\Omega = VI(C, A_1) \cap VI(C, A_2) \cap F_f(T) \neq \emptyset$. For arbitrary $x_0, w \in C$ define an iterative algorithm by*

$$\begin{cases} y_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_1 x_n), \\ z_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_2 x_n), \\ w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n + \eta_n \nabla f u_n + \theta_n \nabla f v_n), \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n), \end{cases} \quad (7.19)$$

where $u_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)$, $v_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \bar{\gamma} < \frac{\mu}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* = P_\Omega^f(w)$.

Proof. From Theorem 7.2.5, we know that $\{x_n\}$ is bounded. Let $q = P_\Omega^f(w)$ and $r_n = \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n)$. We note that from Lemma 4.1.5 (i), we get

$$\langle u - q, \nabla f w - \nabla f q \rangle \leq 0, \forall u \in \Omega. \quad (7.20)$$

Now, from (7.19), Lemma 4.1.5 and (1.20), we get

$$\begin{aligned} D_f(q, x_{n+1}) &= D_f(q, P_C^f \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n)) \\ &\leq D_f(q, \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n)) \\ &= V_f(q, \alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n) \\ &\leq V_f(q, \alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n - \alpha_n (\nabla f w - \nabla f q)) \\ &\quad + \langle \alpha_n (\nabla f w - \nabla f q), r_n - q \rangle \end{aligned}$$

and from Lemma 4.1.6 and (7.16),

$$\begin{aligned}
D_f(q, x_{n+1}) &\leq V_f(q, \alpha_n \nabla f q + (1 - \alpha_n) \nabla f w_n) \\
&\quad + \alpha_n \langle \nabla f w - \nabla f q, r_n - q \rangle \\
&= D_f(q, \nabla f^*(\alpha_n \nabla f q + (1 - \alpha_n) \nabla f w_n)) \\
&\quad + \alpha_n \langle \nabla f w - \nabla f q, r_n - q \rangle, \\
&\leq \alpha_n D_f(q, q) + (1 - \alpha_n) D_f(q, w_n) \\
&\quad + \alpha_n \langle \nabla f w - \nabla f q, r_n - q \rangle \\
&= (1 - \alpha_n) D_f(q, w_n) + \alpha_n \langle \nabla f w - \nabla f q, r_n - q \rangle \\
&\leq (1 - \alpha_n) D_f(q, x_n) + \alpha_n \langle \nabla f w - \nabla f q, r_n - q \rangle \\
&\leq (1 - \alpha_n) D_f(q, x_n) + \alpha_n \|\nabla f w - \nabla f q\| \|r_n - x_n\| \\
&\quad + \alpha_n \langle \nabla f w - \nabla f q, x_n - q \rangle. \tag{7.21}
\end{aligned}$$

On the other hand, from (7.19), Lemma 4.1.5 and Lemma 4.1.6, we have

$$\begin{aligned}
D_f(q, x_{n+1}) &= D_f(q, P_C^f \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n)) \\
&= D_f(q, \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n)) \\
&\leq \alpha_n D_f(q, w) + (1 - \alpha_n) D_f(q, w_n). \tag{7.22}
\end{aligned}$$

Moreover, from (7.22) and (7.15), we obtain

$$\begin{aligned}
D_f(q, x_{n+1}) &\leq \alpha_n D_f(q, w) + (1 - \alpha_n) D_f(q, x_n) \\
&\quad - (1 - \alpha_n) \eta_n \left[\left(\frac{\mu - \gamma_n L}{2} \right) (\|u_n - y_n\|^2 + \|x_n - y_n\|^2) \right] \\
&\quad - (1 - \alpha_n) \theta_n \left[\left(\frac{\mu - \gamma_n L}{2} \right) (\|v_n - z_n\|^2 + \|x_n - z_n\|^2) \right] \\
&\quad - \beta_n \delta_n \rho_r^* (\|\nabla f x_n - T x_n\|), \tag{7.23}
\end{aligned}$$

which implies that

$$\begin{aligned}
 & (1 - \alpha_n)\eta_n \left[\left(\frac{\mu - \gamma_n L}{2} \right) (\|u_n - y_n\|^2 + \|x_n - y_n\|^2) \right] \\
 & + (1 - \alpha_n)\theta_n \left[\left(\frac{\mu - \gamma_n L}{2} \right) (\|v_n - z_n\|^2 + \|x_n - z_n\|^2) \right] \\
 & + \beta_n \delta_n \rho_r^* (\|\nabla f x_n - T x_n\|) \\
 \leq & \alpha_n (D_f(q, w) - D_f(q, x_n)) + D_f(q, x_n) - D_f(q, x_{n+1}). \tag{7.24}
 \end{aligned}$$

Now, we divide the rest of the proof into two parts as follows:

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that the sequence of real numbers $D_f(q, x_n)$ is decreasing for all $n \geq n_0$. It then follows that $D_f(q, x_n)$ is convergent and hence $D_f(q, x_n) - D_f(q, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. Thus, from (7.24) and the conditions on $\alpha_n, \beta_n, \delta_n, \eta_n$ and θ_n we get,

$$\lim_{n \rightarrow \infty} [\|u_n - y_n\|^2 + \|x_n - y_n\|^2] = \lim_{n \rightarrow \infty} [\|v_n - z_n\|^2 + \|x_n - z_n\|^2] = 0,$$

and

$$\lim_{n \rightarrow \infty} \rho_r^* (\|\nabla f x_n - T x_n\|) = 0, \tag{7.25}$$

which implies that

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0, \tag{7.26}$$

$$\lim_{n \rightarrow \infty} \|v_n - z_n\| = \lim_{n \rightarrow \infty} \|x_n - z_n\| = 0, \tag{7.27}$$

and

$$\lim_{n \rightarrow \infty} \|\nabla f x_n - T x_n\| = 0. \tag{7.28}$$

From (7.28) and the fact that ∇f^* is uniformly continuous on bounded subsets of E^* , we get

$$\lim_{n \rightarrow \infty} \|x_n - \nabla f^* T x_n\| = 0. \quad (7.29)$$

Moreover, from (7.19) and Lemma 4.1.6, we obtain

$$\begin{aligned} D_f(x_n, r_n) &= D_f(x_n, \nabla f^*[\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n]) \\ &\leq \alpha_n D_f(x_n, w) + (1 - \alpha_n) D_f(q, w_n) \\ &= \alpha_n D_f(x_n, w) + (1 - \alpha_n) [\beta_n D_f(x_n, x_n) + \delta_n D_f(x_n, \nabla f^* T x_n)] \\ &\quad + (1 - \alpha_n) [\eta_n D_f(x_n, u_n) + \theta_n D_f(x_n, v_n)] \end{aligned} \quad (7.30)$$

Thus, from Lemma 4.1.7, (7.26), (7.27), (7.29) and (7.30), we get

$$\lim_{n \rightarrow \infty} D_f(x_n, r_n) = 0, \quad (7.31)$$

which implies that

$$\lim_{n \rightarrow \infty} \|x_n - r_n\| = 0. \quad (7.32)$$

Now, since $\{x_n\}$ is bounded in E there exists $u \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that since $\{x_n\}$ is bounded in E there exists $u \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup u$ and

$$\limsup_{n \rightarrow \infty} \langle x_n - q, \nabla f w - \nabla f q \rangle = \lim_{k \rightarrow \infty} \langle x_{n_k} - q, \nabla f w - \nabla f q \rangle. \quad (7.33)$$

From (7.28) and the fact that T is relatively f -nonexpansive, we have $u \in F_f(T)$.

Next, we show that $u \in VI(C, A_1) \cap VI(C, A_2)$. Let

$$F_1 x = \begin{cases} A_1 x + N_C x, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases} \quad (7.34)$$

where $N_C(x) = \{v \in E^* : \langle x - z, v \rangle \geq 0 \text{ for all } z \in C\}$ is the normal cone

to C . Then, F_1 is a maximal monotone mapping and $0 \in F_1x$ if and only if $x \in VI(C, A_1)$ (see, [95]). Let $(x, y) \in Gph(F_1)$. Then $y \in A_1x + N_C(x)$ and hence $y - A_1x \in N_C(x)$. Thus we obtain

$$\langle x - v, y - A_1x \rangle \geq 0, \text{ for all } v \in C. \quad (7.35)$$

Since x and u_{n_k} are in C by Lemma 4.1.5 (i), we have $\langle \nabla f x_{n_k} - \gamma_{n_k} A_1 y_{n_k} - \nabla f u_{n_k}, u_{n_k} - x \rangle \geq 0$ which implies that $\langle x - u_{n_k}, \frac{\nabla f u_{n_k} - \nabla f x_{n_k}}{\gamma_{n_k}} + A_1 y_{n_k} \rangle \geq 0$. This and the inequality in (7.35) imply that

$$\begin{aligned} \langle x - u_{n_k}, y \rangle &\geq \langle x - u_{n_k}, A_1 x \rangle \\ &\geq \langle x - u_{n_k}, A_1 x \rangle - \langle x - u_{n_k}, \frac{\nabla f u_{n_k} - \nabla f x_{n_k}}{\gamma_{n_k}} + A_1 y_{n_k} \rangle \\ &= \langle x - u_{n_k}, A_1 x - A_1 u_{n_k} \rangle + \langle x - u_{n_k}, A_1 u_{n_k} - A_1 y_{n_k} \rangle \\ &\quad - \frac{1}{\gamma_{n_k}} \langle x - u_{n_k}, \nabla f u_{n_k} - \nabla f x_{n_k} \rangle \\ &\geq \langle x - u_{n_k}, A_1 u_{n_k} - A_1 y_{n_k} \rangle \\ &\quad - \frac{1}{\gamma_{n_k}} \langle x - u_{n_k}, \nabla f u_{n_k} - \nabla f x_{n_k} \rangle. \end{aligned} \quad (7.36)$$

Furthermore, from (7.26), the fact that A_1 and ∇f are uniformly continuous, we obtain $\lim_{n \rightarrow \infty} \|A_1 u_n - A_1 y_n\| = 0$ and $\lim_{n \rightarrow \infty} \|\nabla f u_n - \nabla f x_n\| = 0$. Consequently, inequality (7.36) provides that $\langle x - u, y \rangle \geq 0$ and hence by the maximality of F_1 we obtain $u \in F_1^{-1}(0)$. Therefore, $u \in VI(C, A_1)$. Similarly, we get that $u \in VI(C, A_2)$. Thus $u \in VI(C, A_1) \cap VI(C, A_2) \cap F_f(T)$. Hence, by Lemma 4.1.5 (i), we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle x_n - q, \nabla f w - \nabla f q \rangle &= \lim_{k \rightarrow \infty} \langle x_{n_k} - q, \nabla f w - \nabla f q \rangle \\ &= \langle u - q, \nabla f w - \nabla f q \rangle \leq 0. \end{aligned} \quad (7.37)$$

Therefore, from (7.21), (7.32), (7.37) and Lemma 3.1.9, we conclude that $D_f(q, x_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence, by Lemma 4.1.7, $x_n \rightarrow q$ as $n \rightarrow \infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_f(q, x_{n_i}) < D_f(q, x_{n_i+1}), \forall i \in \mathbb{N}. \quad (7.38)$$

Then, by Lemma 4.1.15, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and $\max\{D_f(q, x_{m_k}), D_f(q, x_k)\} \leq D_f(q, x_{m_k+1})$, for all $k \in \mathbb{N}$. Thus, from (7.24) and the conditions on $\alpha_n, \beta_n, \delta_n, \eta_n$ and θ_n , we get

$$\lim_{k \rightarrow \infty} \|u_{m_k} - y_{m_k}\| = \lim_{k \rightarrow \infty} \|x_{m_k} - y_{m_k}\| = 0, \quad (7.39)$$

$$\lim_{k \rightarrow \infty} \|v_{m_k} - z_{m_k}\| = \lim_{k \rightarrow \infty} \|x_{m_k} - z_{m_k}\| = 0, \quad (7.40)$$

and

$$\lim_{k \rightarrow \infty} \|\nabla f x_{m_k} - T x_{m_k}\| = 0. \quad (7.41)$$

Moreover, following the methods in Case 1, we get

$$\lim_{k \rightarrow \infty} \|x_{m_k} - r_{m_k}\| = 0, \quad (7.42)$$

and

$$\limsup_{k \rightarrow \infty} \langle x_{m_k} - q, \nabla f w - \nabla f q \rangle \leq 0. \quad (7.43)$$

Now, from (7.21), we obtain

$$\begin{aligned} \alpha_{m_k}(D_f(q, x_{m_k})) &\leq D_f(q, x_{m_k}) - D_f(q, x_{m_k+1}) \\ &\quad + \alpha_{m_k} \|x_{m_k} - r_{m_k}\| \|\nabla f w - \nabla f q\| \\ &\quad + \alpha_{m_k} \langle x_{m_k} - q, \nabla f w - \nabla f q \rangle \\ &\leq \alpha_{m_k} \|x_{m_k} - r_{m_k}\| \|\nabla f w - \nabla f q\| \\ &\quad + \alpha_{m_k} \langle x_{m_k} - q, \nabla f w - \nabla f q \rangle. \end{aligned} \quad (7.44)$$

Consequently, since $\alpha_{m_k} > 0$, we get

$$\begin{aligned} D_f(q, x_{m_k}) &\leq \|x_{m_k} - r_{m_k}\| \|\nabla f w - \nabla f q\| \\ &\quad + \langle x_{m_k} - q, \nabla f w - \nabla f q \rangle. \end{aligned} \quad (7.45)$$

Therefore, from (7.42), (7.43) and (7.45), we obtain $D_f(q, x_{m_k}) \rightarrow 0$ as $k \rightarrow \infty$. But from (7.23), we obtain that $D_f(q, x_{m_k+1}) \rightarrow 0$ as $k \rightarrow \infty$ and hence the fact that $D_f(q, x_k) \leq D_f(q, x_{m_k+1})$ implies $\lim_{k \rightarrow \infty} D_f(q, x_k) = 0$. Thus, by Lemma 4.1.7 $x_k \rightarrow q$ as $k \rightarrow \infty$. $\square$

We remark that the method of proof of Theorem 7.2.6 provides the following result for a common point of solutions of f -fixed points of a Bregman relatively f -nonexpansive mapping and variational inequality problems of a finite family of Lipschitz monotone mappings in Banach spaces.

Theorem 7.2.7. *Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $A_i : C \rightarrow E^*$ be L_i -Lipschitz monotone mappings with constants L_i , for $i = 1, 2, \dots, N$ and $T : C \rightarrow E^*$ be a Bregman relatively f -nonexpansive mapping such that $\Omega = \cap_{i=1}^N VI(C, A_i) \cap F_f(T) \neq \emptyset$. For arbitrary $x_0, w \in C$ define an iterative algorithm by*

$$\begin{cases} y_{n,i} = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_i x_n), \\ u_{n,i} = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_i y_{n,i}), \\ w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n + \sum_{i=1}^n \eta_{n,i} \nabla f u_{n,i}), \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n), \end{cases} \quad (7.46)$$

where $0 < \underline{\gamma} \leq \gamma_n \leq \bar{\gamma} < \frac{\mu}{L}$, for $L := \max\{L_i : i = 1, 2, \dots, N\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_{n,i}\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \delta_n + \sum_{i=1}^N \eta_{n,i} = 1$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to an element $x^* = P_\Omega^f(w)$.

If, in Theorem 7.2.6, we assume that $C = E$, then the Bregman projection P_C^f reduces to the identity mapping on E and hence $VI(C, A_1) = A_1^{-1}(0)$,

$VI(C, A_2) = A_2^{-1}(0)$. Thus, we obtain the following corollary for approximating a common point in the set of solutions of f -fixed points of Bregman relatively f -nonexpansive mapping and the set of zeros of Lipschitz monotone mappings in Banach spaces.

Corollary 7.2.8. Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $A_i : E \rightarrow E^*$ be L_i -Lipschitz monotone mappings with constants L_i , for $i = 1, 2$ and $T : E \rightarrow E^*$ be a Bregman relatively f -nonexpansive mapping such that $\Omega = F_f(T) \cap A_1^{-1}(0) \cap A_2^{-1}(0) \neq \emptyset$. For arbitrary $x_0, w \in E$ define an iterative algorithm by

$$\begin{cases} y_n = \nabla f^*(\nabla f x_n - \gamma_n A_1 x_n), \\ z_n = \nabla f^*(\nabla f x_n - \gamma_n A_2 x_n), \\ w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n + \eta_n \nabla f u_n + \theta_n \nabla f v_n), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n), \end{cases} \quad (7.47)$$

where $u_n = \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)$, $v_n = \nabla f^*(\nabla f x_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \bar{\gamma} < \frac{\mu}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* = P_\Omega^f(w)$.

If, in Theorem 7.2.6, we assume that $A_i = 0$, for $i = 1, 2$, then we get a method of approximation for f -fixed points of a Bregman relatively f -nonexpansive mapping in Banach spaces.

Corollary 7.2.9. Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $T : C \rightarrow E^*$ be a Bregman relatively f -nonexpansive mapping such that $F_f(T) \neq \emptyset$. For arbitrary $x_0, w \in C$ define an

iterative algorithm by

$$\begin{cases} w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n), \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n), \end{cases} \quad (7.48)$$

where $\{\beta_n\}, \{\delta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \delta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* = P_{F_f(T)}(w)$.

If, in Theorem 7.2.7, we assume that $C = E$ and $T = \nabla f$, then we get the following result for a common point of zeros of a finite family of Lipschitz monotone mappings in Banach spaces.

Corollary 7.2.10. Let $f \in \mathcal{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E . Let $A_i : E \rightarrow E^*$ be L_i -Lipschitz monotone mappings with Lipschitz constants L_i , for $i = 1, 2, \dots, N$ such that $\Omega = \cap_{i=1}^N A_i^{-1}(0) \neq \emptyset$. For arbitrary $x_0, w \in E$, define an iterative algorithm by

$$\begin{cases} y_{n,i} = \nabla f^*(\nabla f x_n - \gamma_n A_i x_n), \\ u_{n,i} = \nabla f^*(\nabla f x_n - \gamma_n A_i y_{n,i}), \\ w_n = \nabla f^*(\beta_{n,0} \nabla f x_n + \sum_{i=1}^n \beta_{n,i} \nabla f u_{n,i}), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f w + (1 - \alpha_n) \nabla f w_n), \end{cases} \quad (7.49)$$

where $0 < \underline{\gamma} \leq \gamma_n \leq \bar{\gamma} < \frac{\mu}{L}$, for $L := \max\{L_i : i = 1, 2, \dots, N\}$ and $\{\beta_{n,i}\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\sum_{i=0}^N \beta_{n,i} = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to a point $x^* = P_\Omega^f(w)$.

If, in Corollary 7.2.8, E is a smooth, strictly convex and real reflexive Banach space then $f(x) = \frac{1}{2} \|x\|^2$ is a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu = 2$ and conjugate $f^*(x^*) = \frac{1}{2} \|x^*\|^2$. This yields that $\nabla f = J_2 = J$ with $\nabla f^* = J_2^{-1} = J^{-1}$. Thus, we get the following corollaries for approximating a common point of

solutions of semi-fixed points of relatively J -nonexpansive mapping and variational inequality problems in Banach spaces.

Corollary 7.2.11. Let E be a smooth, strictly convex and real reflexive Banach space with its dual E^* . Let $A_i : C \rightarrow E^*$ be L_i -Lipschitz monotone mappings with Lipschitz constants L_i , for $i = 1, 2$ and $T : C \rightarrow E^*$ be a relatively J -nonexpansive mapping such that $\Omega = F_f(T) \cap A_1^{-1}(0) \cap A_2^{-1}(0) \neq \emptyset$. For arbitrary $x_0, w \in C$, define an iterative algorithm by

$$\begin{cases} y_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 x_n), \\ z_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 x_n), \\ w_n = J^{-1}(\beta_n Jx_n + \delta_n T x_n + \eta_n J u_n + \theta_n J v_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J w + (1 - \alpha_n) J w_n), \end{cases} \quad (7.50)$$

where $u_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 y_n)$, $v_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \bar{\gamma} < \frac{1}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \Omega$.

Corollary 7.2.12. Let E be a smooth, strictly convex and real reflexive Banach space with its dual E^* . Let $A_i : C \rightarrow E^*$ be L_i -Lipschitz monotone mappings with Lipschitz constants L_i , for $i = 1, 2$ and $T : C \rightarrow E$ be a relatively nonexpansive mapping such that $\Omega = VI(C, A_1) \cap VI(C, A_2) \cap F(T) \neq \emptyset$. For arbitrary $x_0, w \in C$ define an iterative algorithm by

$$\begin{cases} y_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 x_n), \\ z_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 x_n), \\ w_n = J^{-1}(\beta_n Jx_n + \delta_n J T x_n + \eta_n J u_n + \theta_n J v_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J w + (1 - \alpha_n) J w_n), \end{cases} \quad (7.51)$$

where $u_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 y_n)$, $v_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \bar{\gamma} < \frac{1}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that

$\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \Omega$.

If, in Corollary 7.2.11, we assume $E = H$, a real Hilbert space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have $\nabla f = J_2 = I$ and $\nabla f^* = J_2^{-1} = I$, where I is identity mapping on H . Thus, we get the following corollary for approximating a common point of solutions of variational inequality and fixed point problems in Hilbert spaces.

Corollary 7.2.13. Let H be a real Hilbert space. Let $A_i : C \rightarrow H$ be L_i -Lipschitz monotone mappings with Lipschitz constants L_i , for $i = 1, 2$ and $T : C \rightarrow E$ be a quasi-nonexpansive mapping such that $(I - T)$ is demiclosed at 0 and $\Omega = A_1^{-1}(0) \cap A_2^{-1}(0) \cap F(T) \neq \emptyset$. For arbitrary $x_0, w \in C$, define an iterative algorithm by

$$\begin{cases} y_n = P_C(x_n - \gamma_n A_1 x_n), \\ z_n = P_C(x_n - \gamma_n A_2 x_n), \\ w_n = \beta_n x_n + \delta_n T x_n + \eta_n u_n + \theta_n v_n, \\ x_{n+1} = P_C(\alpha_n w + (1 - \alpha_n) w_n), \end{cases} \quad (7.52)$$

where $u_n = P_C(x_n - \gamma_n A_1 y_n)$, $v_n = P_C(x_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \bar{\gamma} < \frac{1}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\epsilon, 1) \subset (0, 1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* \in \Omega$.

Chapter 8

Discussion, Conclusion and Recommendation

8.1 Discussion and Conclusion

The works presented in this thesis contains different methods studied for approximating solutions of several problems in the area of Nonlinear Functional Analysis. Most of the results obtained in this thesis were carried out in Hilbert and real reflexive Banach spaces where the notion of Bregman distance was utilized. In addition, we briefly introduced new classes of nonlinear mappings and studied their properties and methods of approximating their fixed points. In particular, we summarize our contributions and provide our conclusions as follows.

- In Chapter 3, we successfully formulated the method of approximation for a zero of the sum of maximal monotone mappings in Hilbert spaces. In Section 3.2, Viscosity type Douglas-Rachford splitting algorithm was constructed and strong convergence of the scheme to a zero of the sum of a finite family of maximal monotone mappings was obtained. Moreover, in Section 3.3, we introduced an algorithm by reformulating (1.1) as the problem of locating a point in a certain extended solution set $S_e(A_1, A_2, \dots, A_m) \subset H \times H^m$ and proved its strong converges to

a zero of the sum of a finite family of maximal monotone mappings in Hilbert spaces. We also gave some applications to the minimization problems. Moreover, numerical examples which support our main result were presented. Our results in this chapter extend the results of [100], Lorenz and Pock [62], Combettes [42], Takahashi *et al.* [105] and the references therein in the sense that our convergence is strong or A and/or B is not α -inverse strongly monotone. Furthermore, our results extend the results of Bot *et al.* [22] to the case of the zero of the sum of a finite family of maximal monotone mappings. The results obtained in Section 3.2 have been published in the journal of “*Advances in Operator Theory*, volume 5, Number 2, (2020), Pages 359-370” while the results obtained in Section 3.3 have been published in the journal of “*Arab Journal of Mathematical Sciences* 2019, Pages 1-15” and the journal of “*Demonstratio Mathematica* volume 53, Number 1, (2020), Pages 152-166”.

- In Chapter 4, we proved convergence theorems for approximating a zero of the sum of maximal monotone mappings in Banach spaces. In Section 4.2, we studied the forward-backward algorithm and proved strong convergence of the method to a zero of the sum of maximal monotone and Lipschitz continuous mappings in real reflexive Banach spaces. In Section 4.3, we proposed splitting algorithm method associated with a certain extended solution set $S_e(A_1, A_2, \dots, A_m) \subset E \times E^{*^m}$, which converges strongly to a zero of the sum of a finite family of maximal monotone mappings $A_1, A_2, \dots, A_m$ in Banach spaces. We have given some applications to the minimization problems. Furthermore, we have provided numerical examples to support our main results. Our results in this chapter extend the results of [19, 40, 45, 80, 107, 108] from real Hilbert spaces to real reflexive Banach spaces without the assumption that one of the mappings is α -inverse strongly monotone. In particular, Theorem 4.3.8 extends Theorem 6 of Iusem [53] and the results in [100] in the sense that our theorems provide strong convergence in arbitrary real reflexive Banach spaces and Corollary 4.2.5 extends Theorem 3.12 of Shehu [97] from uniformly smooth real Banach spaces and 2-uniformly convex to smooth,

strictly convex and real reflexive Banach spaces. In addition, Theorem 4.3.8 extends Theorem 3 of [115] and Theorem 3.6 of [114] from Hilbert spaces to real reflexive Banach spaces. The results obtained in Section 4.2 have been published in the “*Computational and Applied Mathematics Journal*, Volume 39, Number 3, (2020), Pages 1-16” and the results obtained in Section 4.3 have been published in the journal of “*Journal of Fixed Point Theory and Applications*, Volume 22, Number 57, (2020), Pages 1-24” and accepted by the journal of “*Optimization*”.

- In Chapter 5, we constructed an algorithm for approximating a solution of the split equality monotone inclusion problem for the sum of two maximal monotone mappings in Hilbert spaces and discussed its strong convergence. The assumptions that one of the mappings is single-valued and α -inverse strongly monotone are dispensed with. We have given an application to the convex minimization problem. Moreover, a numerical example which supports our main result is presented. Our results improve and unify most of the results that have been proved in this direction. In particular Theorem 5.2.6 generalize the results in [72–74]. The results obtained in this chapter has been published in the journal of “*Linear and Nonlinear Analysis*, Volume 5, Number 3, (2019), Pages 495-516”.

- In Chapter 6, we introduced a new class of mappings called f -pseudocontractive mappings with the notion of f -fixed points in Banach spaces which are the extensions of the class of pseudocontractive mappings and the notion of fixed points in Hilbert spaces, respectively. This class of mappings also includes the class of semi-pseudocontractive in Banach spaces. In addition, an algorithm for approximating a common point of f -fixed points of Lipschitz and f -pseudocontractive mappings is established. Furthermore, we applied our main results to obtain a common zero of Lipschitz monotone mappings. Our theorems in this chapter improve and unify most of the results that have been proved for this important class of nonlinear mappings. In particular, Theorem 6.1.6 extends Theorem 3.6 of Zegeye [123], Theorem 2.5 of Su and Xu [99] and Theorem 2.4 of Liu [64] to the case of real reflexive Banach spaces. Our results also complements the results of Chidume

and Idu [37]. The results obtained in this chapter have been published in the journal of “*Rendiconti del Circolo Matematico di Palermo Series 2* (2020), Pages 1-24”.

- In Chapter 7, we introduced the concepts of Bregman relatively f -nonexpansive mappings and studied their properties and the method of approximating a common element of the set of f -fixed points of Bregman relatively f -nonexpansive mapping and the sets of solutions of variational inequality problems of Lipschitz monotone mappings in real reflexive Banach spaces. Our results in this chapter improve and extend most of the results in the literature. In particular, Theorem 7.2.6 extend the results of [52, 77, 102, 107, 121, 127] from real Hilbert spaces to real reflexive Banach spaces. Furthermore, Corollary 7.2.12 extends Theorem 3.1 of Tufa and Zegeye [113] from uniformly smooth real Banach spaces and 2-uniformly convex to smooth, strictly convex and real reflexive Banach spaces. The results obtained in this chapter have been accepted by the journal of “*Applied Set-Valued Analysis and Optimization*”.

8.2 Recommendation

To the best of our knowledge the following problems/questions are still open for investigations.

- In Chapter 3 and 4, we have obtained strong convergence results for approximating a zero of the sum of maximal monotone mappings in Hilbert spaces and Banach spaces more general than Hilbert spaces. Can we relax the assumptions indicated and/or extend the results to a more general Banach spaces?
- In Chapter 4, we have proved convergence theorems for approximating a zero of the sum of maximal monotone mappings in Banach spaces using the concept of extended solution set of the sum of maximal monotone mappings. Can we introduce a simpler iteration scheme which is efficient and easy to implement?

- In Chapter 5, we have developed an algorithm which converges strongly to a solution of the split equality monotone inclusion problem for the sum of two maximal monotone mappings in Hilbert spaces. Can we extended this result to a finite family of maximal monotone mappings in Banach spaces more general than Hilbert spaces?
- In Chapter 6, we have introduced and studied an algorithm for approximating a common point of a f -fixed point of f -pseudocontractive mappings in real reflexive Banach space. Can we obtain f -fixed point results for a class of mappings more general than Lipschitz and f -pesudocontractive mappings in spaces more general than reflexive Banach spaces?
- In Chapter 7, we have obtained strong convergence results for approximating a common element of the set of f -fixed points of Bregman relatively f -nonexpansive mapping and the sets of solutions of variational inequality problems of Lipschitz monotone mappings in real reflexive Banach spaces. Can the Lipschitz monotone mapping assumption on our mappings be relaxed in order to accommodate more class of mappings?

References

- [1] R.P. Agarwal, D. O'Regan, D.R. Sahu. Fixed Point Theory for Lipschitzian-type Mappings with Applications. Springer, New York, 2009.
- [2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda. On a strongly nonexpansive sequence in Hilbert spaces. *J. Nonlinear Convex Anal.*, 89:471-489, 2007.
- [3] A.D. Arvanitakis. A proof of generalized Banach contraction conjecture. *Proc. Amer. Math. Soc.*, 131(12):3647-3656, 2002.
- [4] H. Attouch, J. Bolte, P. Redont and A. Soubeyran. Alternating proximal algorithms for weakly coupled minimization problems. *Applications to dynamical games and PDEs, J, Convex Anal.*, 15:485-506, 2008.
- [5] H. Attouch, A. Cabot, F. Frankel and J. Peypouquet. Alternating proximal algorithms for constrained variational inequalities, Application to domain decomposition for PDEs. *Nonlinear Anal.*, 74: 7455-7473, 2011.
- [6] K. Avetisyan, O. Djordjević, and M. Pavlović. Littlewood-Paley inequality in uniformly convex and uniformly smooth Banach spaces. *J. Math. Anal. Appl.*, 336(1):31-43, 2007.
- [7] J.B. Baillon and G.Haddad. Quelques proprietes des operateurs angle-bornes et cycliquement monotones. *Israel J. Math.*, 26(2):137-150, 1977.
- [8] S. Banach. Sur les operations dans les ensembles et leur applications aux équations entégrals. *Fund. Math.*, 3(1):133-181, 1922.

- [9] H. H. Bauschke and J. M. Borwein. Legendre functions and the method of random Bregman projections. *J. Convex Anal.*, 4:27-67, 1997.
- [10] H. H. Bauschke, J. M. Borwein and P.L. Combettes . Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. *Comm. Contemp. Math.*, 3:615-647, 2001.
- [11] H.H. Bauschke and P.L. Combettes. A weak-to-strong convergence principle for Fejer-monotone methods in Hilbert spaces. *Math. Oper. Res.*, 26:248-264, 2001.
- [12] H. H. Bauschke, P. L. Combettes and S. Reich. The asymptotic behavior of the composition of two resolvents. *Nonlinear Anal.*, 60:283-301, 2005.
- [13] H.H. Bauschke and P. L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. CMS Books in Mathematics, New York, NY: Springer, 2011.
- [14] H.H. Bauschke and P.L. Combettes. A weak to strong convergence principle for Fejermonotone methods in Hilbert spaces. *Math. Oper. Res.*, 26:248-264, 2001.
- [15] J.Y. Bello Cruz and A.N. Iusem. A strongly convergent direct method for monotone variational inequalities in Hilbert space. *Numer. Funct. Anal. Optim.*, 30:23-36, 2009.
- [16] V. Berinde. Approximating fixed points of weak contractions using Picard iteration *Fixed Point Theory*. 9(1):43-53, 2004.
- [17] V. Berinde. General constructive fixed point theorems fo Ciri-type almost contractions in metric spaces. *Carpathian J. Math.*, 24(2):10-19 2008.
- [18] D. Bertsekas. *Convex Analysis and Optimization Contributors*. Angelia Nedic and Asuman E. Ozdaglar. Athena Scientific. pp. 72, 2003, ISBN 9781886529458.
- [19] L.M. Briceño-Arias. Forward-partial inverse-forward splitting for solving monotone inclusions. *J. Optim. Theory Appl.*, 166(2):391-413, 2015.

- [20] R.E. Bruck and S. Reich. Nonexpansive projections and resolvents of accretive operators in Banach spaces. *Houston J. Math.*, 3:459-470, 1977.
- [21] J. F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problem*. Springer, New York, 2000.
- [22] R.I. Bot, E.R. Csetnek and D. Meier. Inducing strong convergence into the asymptotic behaviour of proximal splitting algorithms in Hilbert spaces. *Optim. Methods Softw.*, 2018 <https://doi.org/10.1080/10556788.2018.1457151>.
- [23] D.W. Boy, J.S. and W. Wong, On Nonlinear Contractions. *Proc. Amer. Math. Soc.*, 20:458-464, 1969.
- [24] F.E. Browder. Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces. *Bull. Amer. Math. Soc.*, Vol. 73:875-882, 1967.
- [25] D. Butnariu and E. Resmerita. Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. *Abstr. Appl. Anal.*, 2006:1–39, 2006.
- [26] D. Butnariu and A. N. Iusem. *Totally convex functions for fixed points computation and infinite dimensional optimization*. Vol. 40, Klumer Academic, Dodrecht, The Netherlands, 2000.
- [27] R.S. Burachik and S. Scheimberg. A proximal point algorithm for the variational inequality problem in Banach spaces. *SIAM J. Control Optim.*, 39:1615-1632, 2001.
- [28] D. Butnariu and A.N. Iusem. *Totally convex functions for fixed points computation and infinite dimensional optimization*. Vol. 40, Klumer Academic, Dodrecht, The Netherlands, 2000.
- [29] G. Cai and S. Bu. An iterative algorithm for a general system of variational inequalities and fixed point problems in q-uniformly smooth Banach spaces. *Optim. Lett.*, 7:267-287, 2013.
- [30] Y. Censor and A. Lent. An iterative row-action method for interval convex programming. *J. Optim. Theory Appl.*, 34(3):321–353, 1981.

- [31] Y. Censor, A. Gibali and S. Reich. *The split variational inequality problem*. The Technion-Israel Institute of Technology, Haifa arXiv:1009.3780, 2010.
- [32] Y. Censor, T. Bortfeld, B. Martin and A. Trofimov. A unified approach for inversion problems intensitymodulated radiation therapy. *Phys. Med. Biol.*, 51:2353–2365, 2006.
- [33] Y. Censor and T. Elfving. A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithms*, 8(2):221–239, 1994.
- [34] S.-S. Chang, L. Wang and L.J. Qin. Split equality fixed point problem for quasi-pseudo-contractive mappings with applications. *Fixed Point Theory Appl.*, 2015(208), 2015.
- [35] Q. Cheng, Y. Su, J. Zhang, *Duality fixed point and zero point theorems and applications*, Abstr. Appl. Anal. 2012, Article ID 391301 (2012). <https://doi.org/10.1155/2012/391301>.
- [36] C. Chidume. *Geometric properties of Banach spaces and nonlinear iterations*. Springer-Verlag London, 2009.
- [37] C.E. Chidume and K.O. Idu. Approximation of zeros of bounded maximal monotone mappings, solutions of Hammerstein integral equations and convex minimization problems. *Fixed Point Theory Appl.*, (2016). <https://doi.org/10.1186/s13663-016-0582-8>.
- [38] S.-S. Chang. On Chidume’s open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces. *J. Math. Anal. Appl.*, 216(1):94–111, 1997.
- [39] C. E. Chidume and C.G. Chinedu. New algorithms for approximating zeros of inverse strongly monotone maps and J -fixed points. *Fixed Point Theory Appl.*, 2020(1), 2020.
- [40] S.Y. Cho, X. Qin, L. Wang. Strong convergence of a splitting algorithm for treating monotone operators. *Fixed Point Theory Appl.*, 2014(94), 2014.

- [41] I. Cioranescu. *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*. Kluwer Academic Publishers, Dordrecht, 1990.
- [42] P.L. Combettes. Iterative construction of the resolvent of a sum of maximal monotone operators. *J. Convex Anal.*, 16(4):727-748, 2009.
- [43] V. Dadashi and H. Khatibzadeh. On the weak and strong convergence of the proximal point algorithm in reflexive Banach spaces. *Optimization*, 66(9):1487-1494, 2017.
- [44] V. Dadashi and M. Postolache. Hybrid proximal point algorithm and applications to equilibrium problems and convex programming. *J. Optim. Theory Appl.*, 174(2):518-529, 2017.
- [45] V. Dadashi and M. Postolache. Forward backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators. *Arab. J. Math.*, 2019 <https://doi.org/10.1007/s40065-018-0236-2>.
- [46] O.A. Daman and H. Zegeye. Strong convergence theorems for a common fixed point of a finite family of pseudocontractive mappings. *Int. J. Math. Math. Sci.*, 2012, Article ID 405315 (2012) <https://doi.10.1155/2012/405315>.
- [47] J. Eckstein and B.F. Svaiter. A family of projective splitting methods of the sum of two maximal monotone operators. *Math. Program., Ser. B*, 111:173-199, 2008.
- [48] M. Edelstein. An extension of Banach's contraction principle. *Proc. Amer. Math. Soc.*, 12(1):7-10, 1961.
- [49] M. Edelstein. On fixed and periodic points under contractive mappings. *J. Lond. Math. Soc.*, 37:74-76, 1962.
- [50] B. Halpern. Fixed points of nonexpanding maps. *Bull. Amer. Math. Soc.*, 73:957-961, 1967.
- [51] P. Hartman and G. Stampacchia. On some non-linear elliptic differential-functional equations. *Acta Math.*, 115:271-310, 1966.

- [52] H. Iiduka and W. Takahashi. Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings. *Nonlinear Anal.*, 61(3):341-350, 2005.
- [53] N. Iusem and B. F. Svaiter. Splitting methods for finding zeroes of sums of maximal monotone operators in Banach spaces. *J. Nonlinear Convex Anal.*, 15(2):379-397, 2014.
- [54] S. Kamimura and W. Takahashi. Approximating solutions of maximal monotone operators in Hilbert spaces. *J. Approx. Theory*, 106(2):226-240, 2000.
- [55] S. H. Khan, M. Abbas and S. Ali. Fixed points of multivalued quasi-nonexpansive mappings using a faster iterative process. *Acta Math. Sin. (Engl. Ser.)*, 30:1231–1241, 2014.
- [56] Y. Kimura and K. Nakajo. Strong convergence for a modified forward-backward splitting method in Banach spaces. *J. Nonlinear Var. Anal.*, 3(1):5-18, 2019.
- [57] G.K. Kirk. *Topics in Metric Fixed Point Theory*. Cambridge University press, Cambridge, 1990.
- [58] P. Kumam. A hybrid approximation method for equilibrium and fixed point problems for a monotone mapping and a nonexpansive. *Nonlinear Anal. Hybrid syst.*, 2008 doi:10.1016/j.nahs.2008.09.017
- [59] W.M. Kozłowski. *Modular function spaces*. Series of Monographs and Textbook in Pure and Applied Mathematics, vol. 122 Dekker, New York, 1988.
- [60] E. Kreyszig. *Introduction to functional analysis with applications*. Canada, 1978.
- [61] G.L'opez, V. Mart'ín-M'arquez, F. Wang, and H.-K. Xu. Forward-backward splitting methods for accretive operators in Banach spaces. *Abstr. Appl. Anal.*, vol. 2012, Article ID109236, 25 pages, 2012.
- [62] D. Lorenz, and T. Pock. An inertial forward-backward algorithm for monotone inclusions. *J. Math. Imaging Vision*, 51:311-325, 2015.

- [63] P.L. Loons and B. Mercier. Splitting Algorithms for the Sum of Two Nonlinear Operators. *SIAM J. Numer. Anal.*, 16(6):964-979, 1979.
- [64] B. Liu. Fixed point of strong duality pseudocontractive mappings and applications. *Abstr. Appl. Anal.*, Article ID 623625 (2012), 2012.
- [65] P.E. Maingé. Strong convergence of projected subgradient method for non-smooth and nonstrictly convex minimization. *Set-Valued Anal.*, 16:899-912, 2008.
- [66] S. Matsushita and W. Takahashi. Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces. *Fixed Point Theory Appl.*, 2004(1):37-47, 2004.
- [67] B. Mercier. *Inequations Variationnelles de la Mecanique*. Publications Mathematiques d'Orsay 80.01, Universite Paris-Sud, Orsay, France, 1980.
- [68] V. Martin-Marquez, S. Reich, S and S. Sabach. Right Bregman nonexpansive operators in Banach spaces. *Nonlinear Anal.*, 75(14):5448-5465, 2012.
- [69] G.J. Minty. Monotone (nonlinear) operators in Hilbert space. *Duke Math. J.*, 29(3):341-346, 1962.
- [70] A. Moudafi and M. Théra. Finding a zero of the sum of two maximal monotone operators. *J. Optim. Theory Appl.*, 94(2):425-448, 1997.
- [71] A. Moudafi and M. Oliny. Convergence of a splitting inertial proximal method for monotone operators. *J. Comput. Appl. Math.*, 155:447-454, 2003.
- [72] A. Moudafi. A relaxed alternating CQ algorithm for convex feasibility problems. *Nonlinear Anal-Theor.*, 79:117-121, 2013.
- [73] A. Moudafi. Alternating CQ-algorithm for convex feasibility and split fixed-point problems. *J. Nonlinear Convex Anal.*, 15:809-818, 2014.
- [74] A. Moudafi. Split monotone variational inclusions. *J. Optim. Theory Appl.*, 150:275-283, 2011.

- [75] A. Moudafi and M. Théra. Finding a zero of the sum of two maximal monotone operators. *J. Optim. Theory Appl.*, 94(2):425-448, 1997.
- [76] E. Naraghirad and J.C. Yao. Bregman weak relatively nanexpansive mappings in Banach spaces. *Fixed Point Theory Appl.*, 2013(141), 2013.
- [77] N. Nadezhkina and W. Takahashi. Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. *J. Optim. Theory Appl.*, 128:191-201, 2006.
- [78] Y. Nesterov. *Introductory Lectures on Convex Optimization*. A Basic Course. Kluwer Academic Publishers. pp. 63–64, 2004, ISBN 9781402075537.
- [79] M.A. Noor. A class of new iterative methods for solving mixed variational inequalities. *Math. Comput. Modell.*, 31:11-19, 2000.
- [80] G.B. Passty. Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. *J. Math. Anal. Appl.*, 72(20):383-390, 1979.
- [81] J. Peypouquet. *Convex Optimization in Normed Spaces, Theory, Methods and Examples With a foreword by Hedy Attouch*. Springer Briefs in Optimization, Springer, Cham, 2015.
- [82] R.P. Phelps. *Convex functions, monotone operators, and differentiability*. Lecture Notes in Mathematics, vol. 1364, 2nd edn. Springer Verlag, Berlin, 1993.
- [83] N. Pholasaa, P. Cholamjiaka and Y.J. Chob. Modified forward-backward splitting methods for accretive operators in Banach spaces. *J. Nonlinear Sci. Appl.*, 9:2766-2778, 2016.
- [84] X. Qin, S.Y. Cho and L. Wang. A regularization method for treating zero points of the sum of two monotone operators. *Fixed Point Theory Appl.*, 2014(75), 2014.
- [85] X. Qin, S.Y. Cho and L. Wang. Convergence of splitting algorithms for the sum of two accretive operators with applications. *Fixed Point Theory Appl.*, 2014.

- [86] S. Reich. *Constructive techniques for accretive and monotone operators in Applied Nonlinear Analysis*. Academic Press, New York, 335-345, 1979.
- [87] S. Reich and S. Sabach. A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. *J. Nonlinear Convex Anal.*, 10:471-485, 2009.
- [88] S. Reich and S. Sabach. Three strong convergence methods regarding iterative methods for solving equilibrium problems in reflexive Banach spaces. *Contemporary Math.*, 568:225-240, 2012.
- [89] S. Reich and S. Sabach. Three strong convergence theorems regarding iterative methods for solving equilibrium problems in reflexive Banach spaces. *Contemp. Math.*, 568:225-240, 2012.
- [90] S. Reich and S. Sabach. Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. *Nonlinear Anal. TMA*, 73:122-135, 2010.
- [91] S. Reich and S. Sabach. *Existence and approximation of fixed-points of Bregman firmly nonexpansive mappings in reflexive Banach spaces in: Fixed-Point Algorithms for Inverse Problems in Science and Engineering*. Springer, New York ,301-316, 2011.
- [92] S. Reich and S. Sabach. A projection method for solving nonlinear problems in reflexive Banach spaces. *Journal of Fixed Point Theory Appl.*, 101-116, 2011.
- [93] S. Reich. Product formulas, nonlinear semigroups, and accretive operators. *J. Functional Anal.*, 36, 147-168 1980.
- [94] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, Marcel Dekker, New York, pp. 313-318, (1996).
- [95] R.T. Rockafellar. On the maximal monotonicity of subdifferential mappings. *Pac. J. Math.*, 33:209-216, 1970.

- [96] P. Senakka and P. Cholamjiak. Approximation method for solving fixed point problem of Bregman strongly nonexpansive mappings in reflexive Banach spaces. *Ric. Mat.*, 65(1):209-220, 2016.
- [97] Y. Shehu. Convergence Results of Forward-Backward Algorithms for Sum of Monotone Operators in Banach Spaces. *Results Math.*, 74(138), 2019.
- [98] S. Suantai, P. Cholamjiak and P. Sunthrayuth. Iterative methods with perturbations for the sum of two accretive operators in q -uniformly smooth Banach spaces. *RACSAM*, 113(2), 2017.
- [99] Y. Su and H.K. Xu. A duality fixed point theorem and applications. *Fixed Point Theory* 13(1):259-265, 2012.
- [100] B.F. Svaiter. General projective splitting methods for sums of maximal monotone operators. *SIAM J. Control Optim.*, 48(2):787-811, 2009.
- [101] B.f. Svaiter. A simplified proof of weak convergence in Douglas-Rechord method to a solution of under line inclusion problem. ar.Xiv:1809.00967V2 [math.OC], 7 Sep 2018.
- [102] A. Tada, W. Takahashi. Weak and strong convergence theorems for nonexpansive mappings and equilibrium problems. *J. Optim. Theory Appl.*, 133:359-370, 2007.
- [103] W. Takahashi, Nonlinear Functional Analysis. Fixed Point Theory and Its Applications. Yokohama Puplicher, Yokohama, Japan, 2000.
- [104] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
- [105] W. Takahashi, N.-C. Wong, and J.-C. Yao. Two generalized strong convergence theorems of Halpern's type in Hilbert spaces and applications *Taiwanese J. Math.*, 16(30):1151-1172, 2012.
- [106] W. Takahashi, N.C. Wong and J.C. Yao. Two generalized strong convergence theorems of Halpern's type in Hilbert spaces and applications. *Taiwanese J. Math.*, 16:1151-1172, 2012.

- [107] W. Takahashi. Strong Convergence Theorems for Maximal and Inverse-Strongly Monotone Mappings in Hilbert Spaces and Applications. *J. Optim. Theory Appl.*, 157(3), 2013.
- [108] W. Takahashi and M. Toyoda. Strong Convergence Theorems for Maximal Monotone Operators with Nonlinear Mappings in Hilbert Spaces. *J. Optim. Theory Appl.*, 147:27-41, 2010.
- [109] W. Takahashi and M. Toyoda. Weak Convergence Theorems for Nonexpansive Mappings and Monotone Mappings. *J. Optim. Theory Appl.*, 118:417-428, 2003.
- [110] W. Takahashi and K. Zembayashi. Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. *Nonlinear Anal.*, 70:45-57, 2009.
- [111] D.V. Thong and D. V. Hieu. Weak and strong convergence theorems for variational inequality problems. *Numer. Algorithms*, 2017.
- [112] P. Tseng. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control Optim.*, 38:431-446, 2000.
- [113] A.R. Tufa and H. Zegeye. An algorithm for finding a common point of the solutions of fixed point and variational inequality problems in Banach spaces. *Arab. J. Math.*, 4:199-213, 2015.
- [114] G.B. Wega and H. Zegeye. A Method of approximation for a zero of the sum of maximally monotone mappings in Hilbert spaces, *Arab J. Math. Sci.* 2019, <https://doi.org/10.1016/j.ajmsc.2019.05.004>.
- [115] G.B. Wega, H. Zegeye and O.A. Boikanyo. Approximating solutions of the Sum of a finite family of maximally monotone mappings in Hilbert spaces. *Adv. Oper. Theory*, 2020, <https://doi.org/10.1007/s43036-019-00026-9>.
- [116] G.B. Wega and H. Zegeye. Convergence results of Forward-Backward method for a zero of the sum of maximally monotone mappings in Banach spaces. *Comput. Appl. Math.*, (submitted).

- [117] H. Wu, C. Cheng, and D. Qu. Strong Convergence Theorems for Maximal Monotone Operators, Fixed-Point Problems, and Equilibrium Problems. *ISRN Applied Math.*, 19 June (2013).
- [118] M.G. Xu. Weak and strong convergence theorems for strict pseudo-contraction in Hilbert space, *J. Math. Anal. Appl.*, 329:336-346, 2007.
- [119] H.-K. Xu. Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, 298(1):279–291, 2004.
- [120] Y. Yao, Y. J. Cho and Y.-C. Liou. Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems. *Eur. J. Oper. Res.*, 212(2):242-250, 2011.
- [121] Y. Yao, M. Postolache and Y-C. Liou. Extragradient-type method for monotone variational inequalities. *Fixed Point Theory Appl.*, 2013(185), 2013.
- [122] C. Zalinescu. *Convex Analysis in General Vector Spaces*. World Scientific, River Edge, 2002.
- [123] Zegeye. H. Strong convergence theorems for maximal monotone mappings in Banach spaces. *J. Math. Anal. Appl.*, 343:663–671, 2008.
- [124] H. Zegeye and N. Shahzad. A hybrid scheme for finite families of equilibrium, variational inequality and fixed point problems. *Nonlinear Anal.*, 74:263-272, 2011.
- [125] H. Zegeye and N. Shahzad. Strong convergence for monotone mappings and relatively weak nonexpansive mappings. *Nonlinear Anal.*, 70:2707-2716, 2009.
- [126] H. Zegeye and N. Shahzad. Construction of a common solution of a finite family of variational inequality problems for monotone mappings. *J. Nonlinear Sci. Appl.*, 2016(9):1645-1657, 2016.
- [127] Y. Zhanga and Q. Yuanb. Iterative common solutions of fixed point and variational inequality problems. *J. Nonlinear Sci. Appl.*, 2016(6):1882-1890, 2016.