

Threshold dynamics of the plant-herbivore interaction with Allee effect

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Abstract—Different species, like plants and herbivores, that are occupying similar ecosystems and living among each other interact regularly. In this interaction, understanding the conditions in which coexistence equilibrium occurs answers a key question in Ecology. In the interaction between plants and herbivores, plants serve as food for herbivores on the food chain. Then the livelihood of herbivores highly depend on the availability of food, in this case the availability of plants. With this motivation, we reformulated a mathematical model which assumes that the death rate of herbivore population is dependent on plant density. Moreover, the abundance of the plant density alone does not guarantee the non-extinction of the herbivore population as they are assumed to reproduce sexually. Therefore, we also included the concept of Allee effect for the herbivore population in the model. Using the mathematical theory of dynamical systems, threshold conditions and trapping region are obtained to ensure coexistence of the population. Moreover, it has been shown that the dynamics of the population is sensitive to the feeding rate of herbivore.

Keywords—Predator-prey, Allee effect, Equilibrium points, Threshold, Bifurcation analysis.

I. INTRODUCTION

Interaction between plants and herbivores give information about basic processes of life that are observed in ecological patterns. Plants and animal species have been co-evolving synergically ever since they

began to occupy different environments in this world. These two types of organisms have developed relationships of mutual dependence at a number of different levels, in particular those related to the feeding behavior. Relationship of this kind are based on different forms ranging from mutual benefits for both parties such as the control of plant growth by wild ruminants to the parasitic relationships observed. Any event that may cause losing of equilibrium in any herbivore activity, such as changes in the environment, species extinction or bio-invasions, may provoke significant change not only to the species involved in the interaction, but eventually also to the whole system.

Understanding the dynamics between herbivores and plants is important for land management, environmental protection and animal husbandry. The remarkable variety of dynamical behaviors exhibited by plant and herbivore species has stimulated great interest in the development of dynamical system models of populations. Classical approaches to model plant herbivore systems have same analogies with predator prey systems. There is no typical pattern resulting from plant herbivore dynamics. Instead, the evolution strongly depend upon the demographic parameters of the plant and herbivore populations and also the timing, kind and degree of density dependence that they exhibit [2]. There is strong ecological evidence showing that plant population dynamics have more important effects on plant herbivore interactions than herbivore population does [6]. Since the time of Lotka (1925) and Volterra (1926) mathematicians and biologists have been investigating the processes that affect the stability of predator-prey (plant-herbivore) systems us-

ing tools from mathematical models [1, 2, 4, 5, 6]. Terry [3] presented four predator prey models with Allee effect for predator reproduction. The analysis showed that each of the models has fixed points that represents predator extinction which is always locally stable. In this paper, we will study the plant herbivore interaction by considering logistic growth rate in the plant population and the Holling type II functional response. For birth rate of herbivore population we take the constant multiple of the feeding relation together with the Allee effect. Moreover a variable herbivore mortality rate which is dependent on plant density is formulated and analyzed.

The paper is organized as follows order. In section 2, We define and describe the mathematical model. In section 3, The formulated model is analysed using the theories of dynamical systems. The existence and stability of equilibrium points is established. In section 4 Bifurcation analysis is discussed. In the last section we give conclusion and some guidelines for future work.

II. MODEL DESCRIPTION

The basic system we consider in this paper is the classical predator prey ecological model in the version presented in [3], where the Allee effects for the herbivore are taken into account. We are studying two different populations, the plant and herbivore. Therefore, the state variables are given by: Plants, P , and herbivores H . The dynamics is represented by the following system of nonlinear ordinary differential equations:

$$\dot{P} = rP \left(1 - \frac{P}{K}\right) - HF(P) \quad (1)$$

$$\dot{H} = H \left[cF(P) \left(\frac{H}{h+H} \right) - D(F(P)) \right] \quad (2)$$

$P(0) \geq 0, H(0) \geq 0$, where the upper dot denotes the time derivative, r, K, c, h are positive constants, denoting the net growth rate of plant population, the carrying capacity, efficacy of giving birth to a new herbivore and Allee constant, respectively. $F(P) = \frac{aP}{b+P}$ is a function representing the value of consumption of plants by herbivores. Here a is the maximum feeding rate of each herbivore and b is the half saturation constant, namely the density of plants at which the predation rate is half of the maximum it may consume to be satisfied at a time. It is assumed that $D(F(P)) = D(F)$ is a continuously differentiable

function of the variable $F, F \geq 0$; and also, for $F \geq 0$, we have $dD/dF \leq 0$, and

$$0 < d_1 \leq D(F(P)) \leq d_2, \quad (3)$$

for some constants d_1 and d_2 representing the minimum and maximum herbivore death rate, respectively. It is common in literature to use constant mortality rate for herbivore but this is undefendable assumption, as it ignores the fact that mortality of herbivores is dependent on the availability of food source, that is, plant biomass. The life process of a herbivore is fueled by plant consumption and this includes most fundamentally staying alive [2]. If a herbivore eats little, it will be more likely for the animal to starve and die. On the other hand for all sufficiently high levels of consumption, starvation will not occur. Hence, herbivore mortality should be a decreasing function of consumption $F(P)$. Moreover, regardless of particular factors affecting herbivore mortality, no animal lives forever. So, it is sensible to assume that herbivore mortality is always bounded below by some positive constant. With this motivation we define the term $D(F(P))$ satisfying the following conditions. Let us consider the following form to describe it:

$$D(F) = d_2 - \frac{F}{g+F}, \quad (4)$$

for some parameter g . Then, substituting the values $P = 0$ and $P = K$ in (4) and solving for g gives us $g = \frac{aK(1-d_2+d_1)}{(d_2-d_1)(b+K)}$. Then, $D(F(P))$ can have the expression:

$$D(F(P)) = d_2 - \frac{\frac{aP}{b+P}}{\frac{aK(1-d_2+d_1)}{(d_2-d_1)(b+K)} + \frac{aP}{b+P}} \quad (5)$$

The system under study, therefore, takes the following form.

$$\begin{aligned} \frac{dP}{dt} &= rP \left[1 - \frac{P}{K} \right] - \frac{aP}{b+P} H \\ \frac{dH}{dt} &= H \left[c \left(\frac{aP}{b+P} \right) \left(\frac{H}{h+H} \right) - D(F(P)) \right], \end{aligned} \quad (6)$$

where $D(F(P))$ is defined in equation (5)

A. Region of Positivity

It is important to show positivity and boundedness of the model system (6) as the variables represent population densities. Positivity implies survival, and

boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources. We will show in this section that all solution for (6) satisfies positivity. That is, for this model to be mathematically and biologically well posed, we need to ensure that for all positive times the state variables $P(t)$ and $H(t)$ must be non-negative for each solution of the system starting with $P(0) \geq 0, H(0) \geq 0$. This is equivalent to saying that the positive quadrant $\mathbb{R}_2^+ = \{(P, H) \in \mathbb{R}_2 : P \geq 0, H \geq 0\}$ is positively invariant. We denote by $\mathbb{R}_2^+$ the non-negative quadrant, and by $\text{int}(\mathbb{R}_2^+)$ the positive quadrant.

Lemma 1 (Positivity) *All solutions of system (6) starting in $\text{int}(\mathbb{R}_2^+)$, remain bounded and positive.*

This is obvious from system (6). Using the Cauchy-Lipschitz theorem, as the P -axis and the H -axis are orbits, the positive quadrant $\text{int}(\mathbb{R}_2^+)$ is positively invariant. $\square$

Lemma 2 *Let $\hat{P} := \max\{P(0), K\}$ and $\hat{H} := \max\{cP(0) + H(0), \frac{\eta}{d_1}\}$ where d_1 is defined as in (3) and set $\eta = \frac{K(r+d_1)^2}{4r}$. Then, the set on which the model system (6) is defined, $\Omega = \{(P, H) \in \mathbb{R}_2 : P \geq 0, H \geq 0, P \leq \hat{P}, H \leq \hat{H}\}$ is positively invariant. i.e., all solutions of (6) starting in $\mathbb{R}_2^+$ are ultimately bounded with respect to $\mathbb{R}_2^+$ and enter the attracting set Ω for positive time ($t \geq 0$).*

Proof: Let $(P(0), H(0)) \in \Omega$, as $P(t)$ and $H(t)$ are positive for $t > 0$, the basic existence and uniqueness theorem for differential equations (Picard-Lindelof Theorem) ensures that the system has positive solutions that cannot intersect the axes. It remains to show boundedness. We first proof $P(t) \leq K$ for all $t > 0$ since $P > 0$ and $H > 0$ in $\text{int}(\mathbb{R}_2^+)$ every solution $(P(t), H(t))$ of the model system (6) which starts in $\text{int}(\mathbb{R}_2^+)$, satisfies the inequality $\frac{dP}{dt} \leq rP(1 - \frac{P}{K})$. But since we know that $P(0) \geq 0$, by a standard comparison argument we have, $P(t) \leq P_1(t)$ for $t \in [0, T]$, where $P_1(0) = P(0) \geq 0$ and for $t \in [0, T]$, P_1 is a solution of $\frac{dP_1}{dt} = rP_1(1 - \frac{P_1}{K})$. Then, $(1 - \frac{P_1}{K})$ is strictly decreasing for $P \geq \hat{P}$ where $0 \leq \hat{P} < K$, and it follows that $\frac{dP_1}{dt} = rP_1(1 - \frac{P_1}{K})$ is negative for $P_1 > K$. Using the Intermediate Value Theorem and Mean Value Theorem we must have $P_1(t) \leq \max\{P(0), K\}$. Hence, $P(t) \leq P_1(t) \leq K$ and $P(t)$ is bounded above.

Next, we consider a bound for $H(t)$ from above. We have the emergence (birth) rate of herbivores which is always less than a linear multiple of the functional response. That is, $cF(P)(\frac{H}{h+H}) < eF(P)$ for a positive constant $e > 0$ and also $D(F(P)) \geq d_1$ for positive constant d_1 . From model (6) when $P \geq 0$ and $H \geq 0$ finally we have $rP(1 - \frac{P}{K})$ is bounded above by a positive constant (say Q) for $P \geq 0$. Note that, $rP(1 - \frac{P}{K})$ is continuous on $[0, K]$ and therefore bounded, by Minmax Theorem and also $rP(1 - \frac{P}{K}) < 0$ for $P > K$. Combining these observations, the results in Lemma 2, positivity and using equations of the model system (6), we have for $t \geq 0$

$\frac{d}{dt}(eP + H) = e\frac{dP}{dt} + \frac{dH}{dt} \leq erP(1 - \frac{P}{K}) + d_1eP - d_1(eP + H)$. Here, we can see that $erP(1 - \frac{P}{K}) + d_1eP$ is equal to $-er\frac{P^2}{K} + e(r + d_1)P$ a quadratic expression in P and is easily seen to have a global maximum at $\eta = \frac{Ke(r+d_1)^2}{4r}$. From these we get $erP(1 - \frac{P}{K}) + d_1eP \leq \frac{Ke(r+d_1)^2}{4r}$. Which means, $\frac{d}{dt}(eP + H) \leq \frac{Ke(r+d_1)^2}{4r} - d_1(eP + H)$. Then, it follows from the last step that $eP(t) + H(t) \leq W(t)$ for all $t \geq 0$, where $W(0) = eP(0) + H(0) \geq 0$ and $W(t)$ is a solution of $\frac{dW}{dt} = \eta - d_1W$. Solving for W we have $W = \frac{\eta}{d_1} - \frac{K}{d_1}e^{-d_1t}$ and then $W(t) \leq \max\{W(0), \frac{\eta}{d_1}\} = \max\{eP(0) + H(0), \frac{\eta}{d_1}\}$. Hence, $eP(t) + H(t)$ is bounded by $\hat{H}$. Since P satisfies the Positivity condition and is bounded above by $\hat{P} = \max\{P(0), K\}$, then H itself must be bounded above by $\hat{H}$. $\square$

III. EQUILIBRIUM ANALYSIS

A. Existence of equilibria and stability

Equilibrium points of the system (6) are obtained by putting the right hand side equal to zero. When there is no herbivore, that is, when $H = 0$, P needs to satisfy $Pr(1 - \frac{P}{K}) = 0$ which implies that existence of two equilibria is possible. One is the trivial equilibrium $(0, 0)$ which is unstable and the other is herbivore free equilibrium $(K, 0)$, which is locally asymptotically stable. When the herbivore population $H > 0$, from equation (2) the term $c(\frac{aP}{b+P})(\frac{H}{h+H})$ represents the emergence of herbivore and is an increasing function of P . Similarly the term $D(F(P))$ represents the rate of removal of herbivores from the system and is a decreasing function of P . Moreover from the assumption that $P \leq K$, we have, $\frac{aP}{b+P} \leq$

$\frac{aK}{b+K}$, and $D(F(K)) \geq d_1$. Now, consider the case $\frac{caK}{(b+K)} \left(\frac{H}{h+H} \right) < d_1$. In this case herbivore mortality would always be higher than herbivore birth rate, the herbivore population would decay until extinction. Therefore, for the non extinction of herbivores we must always have $\frac{caK}{(b+K)} \left(\frac{H}{h+H} \right) > d_1$.

Define $\mathcal{R}_0 = \frac{caK}{d_1(b+K)}$ to be a threshold parameter value for the system, and we shall call it the reproduction number of the herbivore population. Then we have the following theorem

Theorem 3 Suppose the model system (6) is given with the initial plant population is positive $P(0) > 0$. If $\mathcal{R}_0 \leq 1$ then the equilibrium $(K, 0)$ is locally asymptotically stable.

Proof: Let us bound $\frac{dH}{dt}$ from above. To do this, we bound the herbivore death rate from below and the herbivore birth rate from above. In view of positivity (Lemma 1), the herbivore death rate function $D(F(P))$ is bounded below by the parameter d_1 . Since the herbivore birth rate is $c \left(\frac{aP}{b+P} \right) \left(\frac{H}{h+H} \right)$, to bound this from above, first, by positivity where $\hat{H}$ is the maximum value for herbivore population, $H \leq \hat{H}$ and we have $\frac{H}{h+H} \leq \frac{\hat{H}}{h+\hat{H}}$ for $t \geq 0$, where $\hat{H}$ is a positive constant defined in Lemma 2. Second, note that by positivity for $t \geq 0$ in the proof of Lemma 2 we saw that when $P(0) > 0$, $P(t) \rightarrow K$ as $t \rightarrow \infty$. Hence we can have the relation $P(t) < K$ for $t \geq T_1$ for some $T_1 > 0$. Using this, Lemma 1, and Standard Comparison argument [3] we have for $t \geq T_1$, $P \leq K$ and which implies that $\frac{caP}{b+P} \leq \frac{caK}{b+K}$. Combining these observations we get,

$$\frac{dH}{dt} \leq \hat{H} \left[\frac{caK}{b+K} \left(\frac{\hat{H}}{h+\hat{H}} \right) - d_1 \right]. \quad (7)$$

Now, for $\mathcal{R}_0 \leq 1$, $\frac{caK}{b+K} \leq d_1$, and $0 < \left(\frac{\hat{H}}{h+\hat{H}} \right) < 1$, then the term in the square bracket of equation (7) is negative constant say $-\alpha$ (where $\alpha > 0$). Then, we have $H(t) \leq H_1(t)$ for $t \geq T_1$, where $H_1(t) = H(t)$ for $t \leq T_1$ and H_1 is a solution of $\frac{dH_1}{dt} = -\alpha H_1$. Since H satisfies positivity and is bounded above (by Lemma 2) then $H_1(t_1) = H(t_1)$ is non negative and finite. Hence $H_1(t) \rightarrow 0$ as $t \rightarrow \infty$, which implies $H(t) \rightarrow 0$ as $t \rightarrow \infty$, since $0 \leq H(t) \leq H_1(t)$ for $t \geq T_1$.

Finally, we can show that $P(t) \rightarrow K$ as $t \rightarrow \infty$. We choose any $\varepsilon_1 > 0$. Then there exists $T_2 > 0$ such that $P(t) < K + \varepsilon_1$ for $t \geq T_2$. Now, choose $\varepsilon_2 > 0$ such

that $\varepsilon_2 < \max \left\{ \frac{r}{a}, \frac{r\varepsilon_1}{2aK} \right\}$, since $H(t) \rightarrow 0$ as $t \rightarrow \infty$, there exists $T_3 > 0$ such that $0 < H(t) < \varepsilon_2$ for $t \geq T_3$. Using this and also positivity, we have $\frac{aPH}{b+P} < \frac{a\varepsilon_2 P}{b+P} < a\varepsilon_2 P$, for $t \geq T_3$. Hence $\frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right) - \frac{aPH}{b+P}$. Thus, $\frac{dP}{dt} \geq rP \left(1 - \frac{P}{K} \right) - a\varepsilon_2 P$ or $\frac{dP}{dt} \geq rP \left(1 - \frac{a\varepsilon_2}{r} - \frac{P}{K} \right)$. We also know that $P(T_3)$ is bounded and positive. Then $P(t) \geq P_2(t)$ for $t > T_3$ where $P_2(T_3) = P(T_3) > 0$ and P_2 is a solution for $\frac{dP_2}{dt} = rP_2 \left(1 - \frac{a\varepsilon_2}{r} - \frac{P_2}{K} \right)$. Since by definition $\varepsilon_2 < \frac{r}{a}$, then $1 - \frac{a\varepsilon_2}{r} > 0$, so that by solving explicitly for $P_2(t)$ we have $\frac{dP_2}{dt} = rP_2 \left(1 - \frac{a\varepsilon_2}{r} - \frac{P_2}{K} \right)$ which solves to $\frac{P_2}{\left(1 - \frac{a\varepsilon_2}{r} - \frac{P_2}{K} \right)} = Ae^{(1 - \frac{a\varepsilon_2}{r})rt}$ for some constant A . Now, as $t \rightarrow \infty$ since $1 - \frac{a\varepsilon_2}{r} > 0$ we have $\left(1 - \frac{a\varepsilon_2}{r} - \frac{P_2}{K} \right) \rightarrow 0$ or $\lim_{t \rightarrow \infty} P_2 = K \left(1 - \frac{a\varepsilon_2}{r} \right)$.

In particular, $P_2 > K \left(1 - \frac{2a\varepsilon_2}{r} \right)$ for $t \geq T_4$ for some $T_4 \geq T_3$. But by our choice for ε_2 , we have $\varepsilon_2 < \frac{r\varepsilon_1}{2aK}$. So that $1 - \frac{2a\varepsilon_2}{r} > 1 - \frac{\varepsilon_1}{K}$. Hence $P(t) \geq P_2(t) > K \left(1 - \frac{\varepsilon_1}{K} \right) = K - \varepsilon_1$ for $t \geq T_4$. Combining our observations, and letting $T_5 = \max \{T_2, T_4\}$, we see that $K - \varepsilon_1 < P(t) < K + \varepsilon_1$ for $t \geq T_5$. Thus $P(t) \rightarrow K$ as $t \rightarrow \infty$, as required. $\square$

This theorem describes that the herbivore birth rate is less than the mortality rate then more herbivores are dying compared to the new borns. Therefore, in the long run it results in the growth of plant population up to the carrying capacity of the environment while the herbivore population goes to extinction.

Theorem 4 Suppose that $\mathcal{R}_0 > 1$ and define $H_m = \frac{(d_1)h(b+K)}{caK - d_1(b+K)}$. Then the following hold.

- If $H(0) < H_m$ and $P(0) > 0$, then $(P(t), H(t)) \rightarrow (K, 0)$ as $t \rightarrow \infty$.
- If $H(0) \geq H_m$ and $0 < P(0) < \frac{1}{(ca\beta e^{r\beta})}$, where $\beta = \left(\frac{2}{d_1} \right) \left[1 + \ln \left(\frac{H(0)}{H_m} \right) \right]$ is a positive constant; then $(P(t), H(t)) \rightarrow (K, 0)$ as $t \rightarrow \infty$.

Proof: From Lemma 2 we have $P(t) \leq \max \{P(0), K\} = \hat{P}$ for $t \geq 0$. Let $\sigma(H) = \left(\frac{caK}{b+K} \right) \left(\frac{H}{h+H} \right) - d_1$. Then clearly $\sigma(0) = -d_1 < 0$. Also $\sigma(H)$ is continuous in $H \geq 0$. Moreover $\sigma(H) > 0$ for all H sufficiently large because $\mathcal{R}_0 > 1$. Therefore, by Intermediate Value Theorem there is a least positive solution H_m to $\sigma(H) = 0$, such that we have $\sigma(H) < 0$ for $0 \leq H < H_m$.

(i) For the first part of the theorem; from the proof of Theorem 3 we find that for $t \geq 0$ that $\frac{dH}{dt} \leq H \left[\frac{caK}{b+K} \left(\frac{H}{h+H} \right) - d_1 \right] = H\sigma(H)$. Then by the Standard Comparison argument there exists H_1 such that $H(t) \leq H_1(t)$ for $t > 0$ where $H_1(0) = H(0) \geq 0$, and H_1 is a solution of $\frac{dH_1}{dt} = H_1\sigma(H_1)$. From the hypothesis we have $H(0) < H_m$ and we also saw that $\sigma(H) < 0$ for $0 \leq H < H_m$. Therefore, we have $H_1(t) \rightarrow 0$ as $t \rightarrow \infty$, which in turn implies $H(t) \rightarrow 0$ as $t \rightarrow \infty$ since we have $0 < H(t) \leq H_1(t)$ for $t > 0$. In similar manner as in the proof of Theorem(3) one can easily show that $P \rightarrow K$ as $t \rightarrow \infty$.

(ii) To prove the second part, we suppose $H(0) \geq H_m$. Then since $H_m > 0$, let us define $\beta = \left(\frac{2}{d_1} \right) \left[1 + \ln \left(\frac{H(0)}{H_m} \right) \right]$. Hence, $ca\beta e^{r\beta} > 0$ as $\frac{H(0)}{H_m} \geq 1$. Choose $P(0)$ so that $0 < P(0) < \frac{1}{ca\beta e^{r\beta}}$. Then using the equation for $\frac{dP}{dt}$ and positivity of P we have $\frac{dP}{dt} < rP$ for $t \geq 0$. Moreover, from $P > 0$, it follows that $P(t) < P(0)e^{rt}$ for $t \geq 0$. Using our choice of $P(0) < \frac{1}{ca\beta e^{r\beta}}$, we get

$$P(t) < \frac{1}{ca\beta} \text{ for } 0 \leq t \leq \beta. \quad (8)$$

Now, using the equation for $\frac{dH}{dt}$ and positivity of H we have $\frac{dH}{dt} = H \left(\frac{caP}{b+P} \left(\frac{H}{h+H} \right) - D(F(P)) \right) < H(caP - d_1)$ which means,

$$\frac{dH}{dt} \leq (caP - d_1)H \text{ for } t \geq 0. \quad (9)$$

From (8) and (9) and using positivity of H $\frac{dH}{dt} \leq \left(\frac{1}{\beta} - d_1 \right) H$ for $0 \leq t \leq \beta$, after some calculations we arrive at $H(t) \leq H(0)e^{\left(\frac{1}{\beta} - d_1 \right)t}$ for $0 \leq t \leq \beta$. Hence, $H(\beta) \leq H(0)e^{(1-\beta d_1)}$. But then from our definition of β we have $\beta = \left(\frac{2}{d_1} \right) \left[1 + \ln \left(\frac{H(0)}{H_m} \right) \right]$, and clearly we can see that $\beta > \left(\frac{1}{d_1} \right) \left[1 + \ln \left(\frac{H(0)}{H_m} \right) \right]$. After simplification of this inequality we arrive at: $H(0)e^{1-\beta d_1} < H_m$. Hence $H(\beta) < H_m$. If we now apply the argument used to prove the first part of the theorem, with the time origin shifted from $t = 0$ to $t = \beta$ we deduce that $(P(t), H(t)) \rightarrow (K, 0)$ as $t \rightarrow \infty$. $\square$

The above theorem states that when $\mathcal{R}_0 > 1$ then there is a least minimum value of H which is the corresponding solution of $\left(\frac{caK}{b+K} \right) \left(\frac{H}{h+H} \right) = d_1$. Solving for H we obtain the necessary least possible number of herbivore for coexistence. Which is given by $H_m = \frac{d_1 h(b+K)}{caK - d_1(b+K)}$. Then, the first part states that if

the initial herbivore population is less than the least number of herbivore population needed to keep the population existing in the system, $H(0) < H_m$, and the initial plant population is positive $P(0) > 0$, then herbivore extinction will always occur and the plant population grows to its carrying capacity. The second part means if the initial herbivore population is greater than the least number of herbivore population needed to keep the population existing in the system then the herbivore population would still go to extinction if there is not enough plant population to feed on and reproduce. In addition it also gives us threshold values or conditions where herbivore extinction can occur. Let us establish conditions for coexistence of plant and herbivore in the ecosystem using the following theorem.

Theorem 5 Let L_P, U_P, L_H, U_H be constants satisfying the following conditions $0 < L_P < U_P$ and $0 < L_H < U_H$

$$L_H F(U_P) > r U_P \left(1 - \frac{U_P}{K} \right) \quad (10)$$

$$U_H F(L_P) < r L_P \left(1 - \frac{L_P}{K} \right) \quad (11)$$

$$c F(L_P) \left(\frac{L_H}{h+L_H} \right) > D(F(L_P)) \quad (12)$$

$$c F(U_P) \left(\frac{U_H}{h+U_H} \right) < D(F(U_P)) \quad (13)$$

Then in model (6) if $H(0) \in [L_H, U_H]$ and $P(0) \in [L_P, U_P]$ we have $H(t) \in [L_H, U_H]$ and $P(t) \in [L_P, U_P]$.

Proof: Assuming the hypothesis of the theorem hold, we need to define $\Sigma = \{(P, H) : P \in [L_P, U_P], H \in [L_H, U_H]\}$, as a trapping region. This trapping region exists if the flow in the phase plane points inwards across all the boundaries of Σ . which is true if:

$$\frac{dP}{dt} \Big|_{P=U_P} < 0 \text{ and } \frac{dP}{dt} \Big|_{P=L_P} > 0 \text{ for } H \in [L_H, U_H] \quad (14)$$

$$\frac{dH}{dt} \Big|_{H=U_H} < 0 \text{ and } \frac{dH}{dt} \Big|_{H=L_H} > 0 \text{ for } P \in [L_P, U_P] \quad (15)$$

Using (10)-(13) one can easily show that (14)-(15) are satisfied for $P \in [L_P, U_P]$ and $H \in [L_H, U_H]$ Then we have the existence of a trapping region Σ . $\square$

The phase diagram also agrees with this theorem as it shows (figure not shown) that there is a stable limit cycle as time goes to infinity. This means that there is no specific coexistence equilibrium as it keeps appearing and disappearing but the dynamics is always attracted to this stable limit cycle. Coexistence will occur as long as P and H are in Σ .

B. Bifurcation analysis

The sensitivity of the threshold $\mathcal{R}_0 = \frac{caK}{d_1(b+K)}$ with respect to the maximum feeding rate parameter a was found to be a transcritical bifurcation. The herbivore population remains zero until the feeding rate reaches a certain point when the threshold value $\mathcal{R}_0 = \frac{caK}{d_1(b+K)} \leq 1$, the number of herbivores cannot reach H_m hence it will remain at zero. But once this $\mathcal{R}_0$ is slightly greater than one it means the number of herbivore population is greater than H_m so that they can reproduce and increase the population size.

IV. CONCLUSION

In this paper a plant-herbivore model with logistic growth rate for the plant population and Holling type II functional response was assumed for the interaction. For herbivore emergence rate a linear multiple of the functional response and the Allee effect were considered. Under the assumption that the mortality of herbivores depends on the availability of food source, instead of constant death rate we used density dependent variable mortality rate which is a decreasing function of the functional response. The mortality rate is bounded below since no animal can leave for ever. Then existence of two herbivore free equilibrium $(0,0)$, which is unstable and $(K,0)$, which is locally asymptotically stable, were found. Furthermore for the reproduction number $\mathcal{R}_0$, if $\mathcal{R}_0 > 1$, the plant population grows to its carrying capacity and herbivore extinction will occur either if the initial herbivore population is less than the least number of herbivore population needed to keep the population existing in the system and the initial plant population is positive or if the initial herbivore population is greater than the least number of herbivore population needed to keep the population existing in the system but there is not enough plant population to feed on and reproduce. In addition, threshold values or conditions where herbivore extinction can occur and when coexistence state can be assured were established. We have found a stable limit cycle for coexistence state inside the trapping region Σ . However, the plant growth rate is dependent on rainfall and temperature, the feeding rate of the herbivores is also dependent on the availability of food and weather conditions. Hence, these parameters require to be reconsidered

for the model to be more realistic. Therefore, considering periodic and non-periodic functions of time for the plant growth rate $r(t)$ and feeding rate $a(t)$ is to be done in the future.

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REFERENCES

- [1] A. Lebon, L. Mailleret, Y. Dumont and F. Grognaud, "Direct and apparent compensation in plant-herbivore interactions", *Ecol. Model.*, vol. 290, pp. 192–203, 2014.
- [2] A.J. Terry; "Dynamics of a structured population on two patches", *J. Math. Anal. Appl.*, vol. 378, pp. 1–15, 2011.
- [3] A.J. Terry, "Predator-prey models with component Allee effect for predator reproduction", *J. Math. Biol.*, vol. 71, pp. 1325–1352, 2015.
- [4] E.P. Holland, R.P. Pech, W.A. Ruscoe, J.P. Parks, G. Nugent and R.P. Duncan, "Thresholds in plant-herbivore interactions: Predicting plant mortality due to herbivore browse damage", *Oecologia*, vol. 172, pp. 751–766, 2013.
- [5] J.H. Reynolds, J.A. Sherratt, A. White and X. Lambin, "A comparison of the dynamical impact of seasonal mechanisms in a herbivore-plant system", *Theor. Ecol.* vol. 6, pp. 225-239, 2013.
- [6] Y. Kang ,D. Armbruster and Y. Kuang, "Dynamics of plant-herbivore model", *J. Biol. Dyn.*, vol. 2, pp. 89–101, 2008.