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Approximating Solutions of Fixed Point, Variational Inequality and Hammerstein Type Equation Problems

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by

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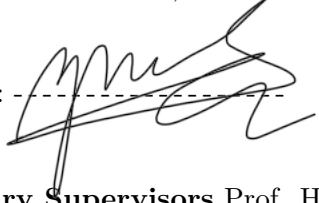
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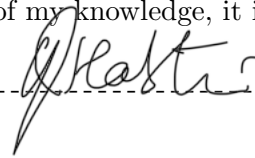
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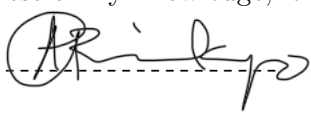
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Abstract

Many physical problems in real life can be modeled as nonlinear problems. In general, finding the exact solutions of many nonlinear problems is difficult. Thus, development of different approximation methods for solving nonlinear problems has gained significant interest once the existence of these solutions has been established.

In this thesis, we construct different iterative algorithms to approximate solutions of such nonlinear problems.

First, we introduce an iterative algorithm which solves the split equality fixed point problem involving uniformly continuous pseudocontractive mappings in the real Hilbert space settings. We also introduce a method for approximating solutions of split equality common f, g -fixed point problems involving uniformly continuous f, g -pseudocontractive mappings in reflexive real Banach spaces. We discuss their convergence analysis and give numerical examples to support each algorithm's theoretical formulation.

Next, we introduce the split equality Minty variational inequality problems and construct an iterative algorithm for approximating its solution, where the underlying mappings are Lipschitz quasimonotone in reflexive real Banach spaces and discuss its strong convergence analysis.

Then, we construct an iterative algorithm for solving bilevel variational inequality problem with fixed point constraint, where the mapping in the lower level variational inequality problem is uniformly continuous pseudomonotone and the mapping for the fixed point problem is a general demimetric mapping. Moreover, we introduce a method which approximates solutions of the split equality of variational inequality and f, g -fixed point problems in reflexive real Banach spaces and discuss its convergence analysis. We prove strong convergence theorems and provide numerical examples for each algorithm.

Finally, we introduce the split equality Hammerstein type equation problem and propose methods for approximating its solution. We construct a method for approximating its solutions, where the underlying mappings are uniformly continuous monotone in reflexive real Banach spaces and its strong convergence analysis is discussed. Moreover, we introduce an iterative algorithm which involves continuous monotone mappings for solving split equality Hammerstein type equation problems in reflexive real Banach spaces. We discuss its convergence analysis and give examples for each algorithm to reveal their effectiveness.

Dedications

This thesis is dedicated to my father Mr. Abebe Belay & my mother Mrs. Zewdu Maryo for all their sacrifices, prayers, endless supports and words of encouragement. It is also dedicated to my late mother in law Mrs. Shushay Teklegiorgis who was so excited to see this result, but unluckily passed away recently.



Publications

The following papers are our contributions to the body of scientific knowledge that constitute this thesis.

List of published papers included in this thesis

- 1) **Yirga Abebe Belay**, Habtu Zegeye and Oganeditse A. Boikanyo, “An inertial method for split equality common f, g -fixed point problems of f, g -pseudocontractive mappings in reflexive real Banach spaces”, *The Journal of Analysis*, **31(2)** (2022), 963–1000, <https://doi.org/10.1007/s41478-022-00489-9>.
- 2) **Yirga Abebe Belay**, Habtu Zegeye and Oganeditse A. Boikanyo, “Solutions of Split Equality Hammerstein Type Equation Problems in Reflexive Real Banach Spaces”, *Carpathian Journal of Mathematics*, **39(1)** (2023), 45–72, <https://doi.org/10.37193/CJM.2023.01.03>.
- 3) **Yirga Abebe Belay**, Habtu Zegeye and Oganeditse A. Boikanyo, “Approximation Methods for Solving Split Equality of Variational Inequality and f, g -Fixed Point Problems in Reflexive Banach Spaces”, *Nonlinear Functional Analysis and Applications*, **28(1)** (2023), 135–173, <https://doi.org/10.22771/nfaa.2023.28.01.09>.
- 4) **Yirga Abebe Belay**, Habtu Zegeye and Oganeditse A. Boikanyo, “Inertial Method for Solving Split Equality Hammerstein Type Equation Problems in Reflexive Banach Spaces”, *Optimization*, (2023), 1–34, <https://doi.org/10.1080/02331934.2023.2264904>.
- 5) **Yirga Abebe Belay**, Habtu Zegeye and Oganeditse A. Boikanyo, Hagos H. Gidey, Dintle Kagiso “An Inertial Method for Solving Split Equality Quasimonotone Minty Variational Inequality Problems in Reflexive Banach Spaces”, *Rendiconti del Circolo Matematico di Palermo Series 2*, (2024), <https://doi.org/10.1007/s12215-024-01025-3>.

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- 6) **Yirga Abebe Belay**, Habtu Zegeye and Oganeditse A. Boikanyo, “An Inertial method for solving split equality fixed point problems for non-Lipschitz pseudocontractive mappings”, In press, *Fixed Point Theory Journal*, (2023).
- 7) **Yirga Abebe Belay**, Habtu Zegeye and Oganeditse A. Boikanyo, Dintle Kagiso, Hagos H. Gidey “An inertial method for solving bilevel variational inequality problems with fixed point constraints”, under referee, *Annali dell’Universita’ di Ferrara*, (2024).

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Contents

Table of Contents	vii
1 Introduction	1
1.1 Background of the Study	1
1.2 Preliminary Concepts	4
1.2.1 Geometry of Hilbert and Banach Spaces	4
1.2.2 Monotonicity of Mappings	6
1.2.3 Convex Functions	8
1.2.4 Bregman Distances and their Properties	10
1.2.5 Nonlinear Mappings	12
1.3 Research Motivation and Statement of the Problem	14
1.4 Research Questions	20
1.5 Objectives	20
1.5.1 General Objective	20
1.5.2 Specific Objectives	20
1.6 Scope of the Study	21
2 Literature Review	22
2.1 Approximation of Solutions of Fixed Point Problems Involving Pseudocontractive Mappings	22
2.2 Approximating Solutions of Variational Inequality Problems	26
2.3 Solutions of Variational Inequality Problems Involving Fixed Points of Some Mappings	29
2.4 Approximating Solutions of Hammerstein Type Equation Problems	33
3 Approximating Solutions of Split Equality Fixed Point Problems	36
3.1 Preliminaries	36
3.2 An Inertial Method for Solving Split Equality Fixed Point Problems for Non-Lipschitz Pseudocontractive Mappings in Real Hilbert Spaces	39
3.2.1 Main Results	39
3.2.2 Some Theoretical Applications	50
3.2.3 Numerical Examples	51
3.3 An Inertial Method for Split Equality Common f, g -fixed Point Problems of f, g -Pseudocontractive Mappings in Reflexive Real Banach Spaces	55
3.3.1 Main Results	55
3.3.2 Some Theoretical Applications	74
3.3.3 Numerical Examples	75
4 Approximating Solutions of Split Equality Quasimonotone Minty Variational	

Inequality Problems	78
4.1 Main Results	78
4.2 Some Theoretical Applications	93
4.3 Numerical Examples	94
5 Solutions of Problems Involving Variational Inequalities and Fixed Points	97
5.1 Preliminaries	97
5.2 Inertial Methods for Solving Bilevel Variational Inequality Problems with Fixed Point Constraints	97
5.2.1 Main Results	98
5.2.2 Some Theoretical Applications	110
5.2.3 Numerical Examples	110
5.3 Approximation Methods for Solving Split Equality Variational Inequality and f, g -Fixed Point Problems in Reflexive Real Banach Spaces	113
5.3.1 Main Results	114
5.3.2 Some Theoretical Applications	130
5.3.3 Numerical Example	132
6 Approximating solutions of Split Equality Hammerstein-Type Equation Problems	135
6.1 Preliminaries	135
6.2 Solutions of Split Equality Hammerstein Type Equation Problems Involving Uniformly Continuous Monotone Mappings in Reflexive Real Banach Spaces	136
6.2.1 Main Results	136
6.2.2 Some Theoretical Applications	154
6.2.3 Numerical Example	155
6.3 Inertial Method for Solving Split Equality Hammerstein Type Equation Problems Involving Continuous Monotone Mappings in Reflexive Real Banach Spaces	157
6.3.1 Main Results	157
6.3.2 Some Theoretical Applications	171
6.3.3 Numerical Examples	173
7 Conclusion, Contribution to the Current Knowledge and Upcoming Research Works	177
7.1 Conclusion and Contribution to the Existing Knowledge	177
7.2 Future Research Works	179

1. Introduction

1.1 Background of the Study

The main focus of this thesis is the wide area of nonlinear functional analysis and its applications, an active research area in the recent years, with special emphasis given to iterative algorithms for approximating solutions of various nonlinear equations and inequalities. Several problems in the area of applied mathematics reduce to finding solutions of nonlinear equations and inequalities. However, there are no closed way methods of solving such problems, unlike those of linear problems. Consequently, various approximation techniques have been employed for solving such equations and inequalities. This thesis specifically focuses on construction of different iterative algorithms for approximating solutions of problems related to fixed point, variational inequality, and Hammerstein type equations.

One of the most interesting theories in the study of nonlinear analysis is the theory of fixed points which was introduced by Poincaré [129] in 1886. Fixed point theory is highly applicable in the study of applied sciences and engineering as many problems posed in these areas can be modeled as finding fixed points of some nonlinear mappings. In particular, fixed point theorems are applicable in differential inclusions, computer sciences, convex optimization, control theory, economics, graph theory and variational analysis. Due to these applications, the area of fixed point theory has attracted the interests of many researchers recently. Consequently, considerable research efforts have been made on the existence and approximation of solutions of fixed point problems (see, for instance, [20, 55, 92, 112, 148, 182, 188, 189, 190] and the references therein).

Specifically, several research works have been made on the concept of fixed points of mappings with different pseudocontractivity properties. The class of monotone mappings, which was originally defined by Kacurovskii [90], is closely related to the class of pseudocontractive mappings in Hilbert spaces. In this regard, if H is a Hilbert space and $T: H \rightarrow H$ is a mapping, then $F = I - T$ is monotone if and only if T is pseudocontractive, where I is the identity mapping on H . Thus, the set of fixed points of T , denoted by $F(T)$, is the same as the set of null points of F . Monotone mappings are very interesting as they are highly applicable for solving different real life problems. For instance, many physical problems can be modeled in the form of

$$z'(t) + Fz(t) = 0, \quad z(0) = 0, \quad (1.1)$$

where F is a monotone mapping in a suitable Banach space E (see, [139]). Heat, wave and Schrodinger equations are typical examples where (1.1) can be applied.

The concepts of fixed points and pseudocontractive mappings have been extended from Hilbert spaces to Banach spaces as semi-fixed points and semi-pseudocontractive mappings, respectively, by Zegeye [184]. The concepts of semi-pseudocontractive mappings and semi-fixed points have further been extended to the concepts of f -pseudocontractive mappings and f -fixed

points, respectively, by Zegeye and Wega [191]. It was pointed out by Zegeye and Wega [191] that, for a mapping $T: E \rightarrow E^*$, the mapping $F = \nabla f - T$ is monotone if and only if T is f -pseudocontractive, where E^* is the dual of E and ∇f is the gradient of a real valued function f defined on E satisfying certain properties. Likewise, the set of f -fixed points of the mapping T is the same as the set of null points of the mapping F . Thus, the interest of researchers on f -fixed points of f -pseudocontractive mappings is mainly due to their connection with the crucial class of certain monotone mappings.

The variational inequality problem is another important area in the study of nonlinear functional analysis. It was first introduced by Stampacchia [147] in 1964 as a tool for the study of partial differential equations and later studied by Hartman and Stampacchia [81] in 1966. Since then, they have been studied by many authors (see, for instance, [9, 36, 60, 73, 83, 118, 158, 184, 185] and the references therein).

Variational inequality problems are of crucial importance in the study of optimization and equilibrium problems and have been applied in solving real life problems of various fields such as transportation, engineering mechanics, circuits in electronics, network analysis, control theory, economics among others (see, for instance, [1, 14, 19, 72, 95, 104, 120, 174, 184] and references therein).

Many other problems in the real world can be formulated and studied as variational inequality problems. Financial equilibrium problems, migration equilibrium problems, environmental network and ecology problems, knowledge network problems, supply chain network equilibrium problems are some of the problems that can be formulated as variational inequality problems. Thus, the variational inequality problem has become an active area of research nowadays. Consequently, a lot of attention has been given to introducing iterative algorithms for approximating solutions of variational inequality problems (see, for instance, [9, 36, 60, 73, 83, 118, 138, 142, 143, 145, 158, 184, 185] and the references therein).

Variational inequality problems are also closely related to the fixed point problems as they can be formulated as finding fixed points of certain mappings.

Besides, several algorithms have also been introduced for finding common solutions of variational inequality and fixed point problems (see, for instance, [150, 163, 172]).

The other important area in the study of nonlinear analysis is the concept of Hammerstein type equation problems. The Hammerstein nonlinear integral equation is an equation of the form

$$u(x) + \int_{\Omega} k(x, y)f(y, u(y))dy = 0, \quad (1.2)$$

where Ω is a measure space, dy is a σ -finite measure on Ω , the real kernel k is defined on Ω , f is a nonlinear function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ and u is a real-valued function defined on Ω [192]. Interest in (1.2) stems mainly from the fact that several problems that arise from differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's functions can be transformed into the form (1.2). Consider, for instance, the following problem of a pendulum

with finite amplitude:

$$\frac{d^2\theta(t)}{dt^2} + c^2 \sin \theta(t) = z(t), \quad t \in [0, 1], \quad \theta(0) = \theta(1) = 0, \quad (1.3)$$

where θ is the amplitude (angular displacement) of the bob from the equilibrium point, z is the driving force and c is a nonzero constant which depends on the length of the pendulum and gravitational acceleration. After a series of computational steps, equation (1.3) can be transformed into the form (1.2) (see, [46]). The integral equation (1.2) can further be transformed in to a more general equation called the Hammerstein Type Equation.

Different problems that emerge from network systems, automation and optimal control can also be formulated as Hammerstein type equation problems (see, [68, 125]). Due to these and other applications, the Hammerstein type equation problems have been recently attracting the interests of researchers. Consequently, several existence and uniqueness theorems have been proved for Hammerstein type equation problems [26, 27, 28, 29]. Besides, many authors have developed different algorithms to approximate solutions of Hammerstein type equation problems (see, for instance, [46, 48, 52, 54, 56, 67, 76, 119, 165, 166, 186] and references therein).

Therefore, this thesis focuses on construction and study of different iterative algorithms for: approximation of solutions of the split equality fixed point problems involving non-Lipschitzian mappings in Hilbert spaces; solving split equality common f, g -fixed point problems in real Banach spaces; finding solutions of the split equality quasimonotone Minty variational inequality problems in the setting of reflexive Banach spaces; approximating solutions of bilevel variational inequality problems with fixed point constraints in Hilbert spaces; finding solutions of split equality variational inequality and f, g -fixed point problems in Banach spaces; approximating solutions of split equality Hammerstein type equation problems involving monotone mappings with different continuity properties in reflexive Banach spaces.

The remaining part of the thesis is organized as follows: Section 1.2 focuses on preliminary concepts on the geometry of Hilbert and Banach spaces, convex functions, Bregman distances and their properties, monotonicity and other properties of nonlinear mappings. Sections 1.3–1.6 contain the problem statement, research questions, objectives of the study, and scope of the study, respectively.

The second chapter deals with the detailed review of related literature on approximating solutions of fixed point problems both in Hilbert and Banach space settings, finding solutions of variational inequality problems, finding common solutions of the variational inequality and fixed point problems and approximating solutions of Hammerstein type equation problems.

In Chapter three, we introduce iterative algorithms which approximate solutions of split equality fixed point problems both in Hilbert and Banach space settings. In Section 3.1, we collect some preliminary concepts and definitions that will be used in the sections that follow. In Section 3.2, we construct an iterative algorithm for solving split equality fixed point problems involving non Lipschitz pseudocontractive mappings in real Hilbert spaces and we discuss its convergence analysis. Finally, numerical examples are provided to reveal the effectiveness of the method. In

Section 3.3, an algorithm for solving split equality common f, g -fixed point problems involving f, g -pseudocontractive mappings in reflexive real Banach spaces is introduced and a strong convergence analysis is discussed along with numerical illustrations.

Chapter four deals with approximation methods for solving split equality quasimonotone Minty variational inequality problems in Reflexive Banach spaces. We construct an iterative algorithm which approximates a solution of split equality Minty variational inequality problem involving quasimonotone mappings in real Banach spaces. After discussing a strong convergence analysis, a numerical example is provided to illustrate the effectiveness of the algorithm.

In Chapter five, we construct an iterative algorithm which approximates solutions of variational inequality problems involving fixed points of certain mappings. In Section 5.1, we collect some preliminary lemmas and definitions. In Section 5.2, we introduce an iterative algorithm which approximates solutions of the bilevel variational inequality problems with fixed point constraints and discuss strong convergence analysis of the algorithm introduced. Section 5.3 deals with an approximation method for solving split equality of variational inequality and f, g -fixed point problems in reflexive real Banach spaces along with a strong convergence result. Numerical illustrations are provided for each method introduced.

Chapter six focuses on introducing iterative algorithms for solving split equality Hammerstein type equation problems in reflexive real Banach spaces. Some basic concepts and preliminary lemmas are provided in Section 6.1. Section 6.2 deals with approximating solutions of split equality Hammerstein type equation problems involving uniformly continuous monotone mappings in reflexive real Banach spaces. Section 6.3 deals with construction of iterative algorithms and prove strong convergence theorems on split equality Hammerstein type equation problems involving mappings which are continuous and monotone in reflexive real Banach spaces. Proofs of strong convergence theorems along with illustrative numerical examples are also given.

The seventh chapter includes discussion and conclusions. We also point out a few unresolved issues with our findings that need further research.

1.2 Preliminary Concepts

In this section, we introduce some fundamental concepts, notations and important results that are useful in establishing our main results.

1.2.1 Geometry of Hilbert and Banach Spaces

Let H be a real Hilbert space. Then (see, e. g., [103, 151])

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \text{ for all } x, y \in H, \quad (1.4)$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \text{ for all } x, y \in H \text{ and } \lambda \in \mathbb{R}. \quad (1.5)$$

The following elementary properties also readily follow from the relation $\|x\|^2 = \langle x, x \rangle$ of Hilbert spaces.

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \text{ and} \quad (1.6)$$

$$\langle x, y \rangle = \frac{1}{2}[\|x\|^2 - \|x - y\|^2 + \|y\|^2], \text{ for all } x, y \in H. \quad (1.7)$$

Definition 1.2.1. [175] *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Then the metric projection of a point $x \in H$ onto C is the unique point, $P_C x$, of C which is characterized by*

$$\|P_C x - x\| = \inf_{y \in C} \{\|y - x\|\}.$$

The metric projection is nonexpansive and it satisfies the following properties (see, e.g., [175]):

$$z = P_C x \text{ if and only if } \langle x - z, y - z \rangle \leq 0, \text{ for all } y \in C, x \in H, \quad (1.8)$$

$$\|x - P_C x\|^2 + \|P_C x - y\|^2 \leq \|x - y\|^2, \text{ for all } y \in C \text{ and } x \in H. \quad (1.9)$$

Throughout the subsequent definitions, the notation $S_E = \{x \in E : \|x\| = 1\}$ will be used, where E is a Banach space.

Definition 1.2.2. [2] *A Banach space E is called strictly convex if $\frac{\|x + y\|}{2} < 1$, for all $x, y \in S_E$ with $x \neq y$.*

Definition 1.2.3. [2] *Let E be a Banach space. The function $\delta_E: (0, 2] \rightarrow [0, 1]$ defined by*

$$\delta_E(\varepsilon) = \inf_{x, y \in S_E} \left\{ \frac{2 - \|x + y\|}{2} : \|x - y\| \geq \varepsilon \right\},$$

is called the modulus of convexity of E .

Definition 1.2.4. [2] *A Banach space E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0, 2]$. We say that E is p -uniformly convex if there exists $d_p > 0$ such that $\delta_E(\varepsilon) \geq d_p \varepsilon^p$, for all $\varepsilon \in (0, 2]$.*

For the special case when $p = 2$, the space E is said to be 2-uniformly convex. It is noted in [173] that L_p and l_p are 2-uniformly convex if $p \in (1, 2]$ and p -uniformly convex if $p \geq 2$. We obviously observe that the class of p -uniformly convex spaces is subclass of the class of uniformly convex spaces. It can also be observed (see, [2]) that every uniformly convex Banach space is strictly convex.

Definition 1.2.5. [149] *A Banach space E is said to be smooth if*

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}, \quad (1.10)$$

exists for all $x, y \in S_E$.

Definition 1.2.6. [2] The function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ given as

$$\rho_E(t) = \sup_{x \in S_E} \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|y\| = t \right\},$$

is called modulus of smoothness of E .

Definition 1.2.7. [2] A Banach space E is said to be uniformly smooth if $\lim_{t \rightarrow 0^+} \frac{\rho_E(t)}{t} = 0$. Moreover, we say that E is q -uniformly smooth if there exists a positive constant c such that $\rho_E(t) \leq ct^q$, for all $t > 0$ and $q > 1$.

It is well known that E is a uniformly convex if and only if E^* is uniformly smooth, where E^* is the dual of E (see, for instance, [43]).

Definition 1.2.8. [2] A Banach space E is said to be reflexive if the natural embedding mapping $\psi: E \rightarrow E^{**}$ defined by $\psi(x) = f_x$ is onto, where $x \in E$, f_x is a bounded linear functional on E^* and E^{**} is the bidual or second dual of E .

All Hilbert spaces, the l_p and L_p spaces, for $p \in (1, \infty)$, are examples of reflexive Banach spaces. Every uniformly convex Banach space is also reflexive (see, [2]). But, neither of the spaces l_1 , L_1 , l_∞ and L_∞ is a reflexive Banach space (see, [2]). One of the most important properties of reflexive Banach spaces is that every bounded sequence in the space has a weakly convergent subsequence (see, [149]).

Definition 1.2.9. Let E_1 and E_2 be real Banach spaces with duals E_1^* and E_2^* , respectively, and let $A: E_1 \rightarrow E_2$ be a bounded linear mapping. A mapping $A^*: E_2^* \rightarrow E_1^*$ is called an adjoint of A if the equality $\langle x, A^*y \rangle = \langle Ax, y \rangle$ holds true for every $x \in E_1$ and $y \in E_2^*$.

For the remaining part of the thesis, we use E to denote a real Banach space, E^* to denote the dual space of E , that is, the collection of all bounded linear functionals defined on E and H to denote a real Hilbert space, unless otherwise specified. The generalized duality relation between E and E^* and the induced norm will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

1.2.2 Monotonicity of Mappings

Definition 1.2.10. A mapping $F: C \subseteq E \rightarrow E$ is said to be L -Lipschitz continuous on C if there exists a constant $L \geq 0$ such that

$$\|Fx - Fy\| \leq L\|x - y\|, \text{ for all } x, y \in C.$$

In this case, if $L < 1$, then F is called a contraction and if $L = 1$, then F is said to be nonexpansive.

Definition 1.2.11. Let C be a nonempty subset of E . A mapping $F: C \rightarrow E^*$ is said to be

(a) α -inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \alpha\|Fx - Fy\|^2, \text{ for all } x, y \in C;$$

(b) α -strongly monotone on C (see, [93]) if there exists $\alpha > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \alpha \|x - y\|^2, \text{ for all } x, y \in C;$$

(c) *monotone on C (see, [93]) if*

$$\langle Fx - Fy, x - y \rangle \geq 0, \text{ for all } x, y \in C;$$

(d) *α -strongly pseudomonotone on C (see, [93]) if there exists $\alpha > 0$ such that*

$$\langle Fx, y - x \rangle \geq 0 \text{ implies } \langle Fy, y - x \rangle \geq \alpha \|x - y\|^2, \text{ for all } x, y \in C;$$

(e) *pseudomonotone on C (see, [93]) if*

$$\langle Fx, y - x \rangle \geq 0 \text{ implies } \langle Fy, y - x \rangle \geq 0, \text{ for all } x, y \in C;$$

(f) *quasimonotone on C if*

$$\langle Fx, y - x \rangle > 0 \text{ implies } \langle Fy, y - x \rangle \geq 0, \text{ for all } x, y \in C.$$

A monotone mapping $F: C \rightarrow E^*$ is said to be *maximal monotone* if its graph, $Gph(F) = \{(x, Fx) : x \in C\}$, is not properly contained in $Gph(G)$ for any monotone mapping $G: C \rightarrow E^*$ other than F . That is, a monotone mapping F is maximal monotone if and only if $y = Fx$, whenever $(x, y) \in C \times E^*$ and $\langle x - u, y - v \rangle \geq 0$ for every $(u, v) \in Gph(F)$.

Remark 1.2.1. (i). We observe from the above definitions that $(a) \implies (c) \implies (e) \implies (f)$, but the converse relations may not be always true.

(ii). It is also easy to see that every α -strongly monotone mapping is α -strongly pseudomonotone and every α -inverse strongly monotone mapping is $\frac{1}{\alpha}$ -Lipschitz continuous.

(iii). The classes of inverse strongly monotone and strongly monotone mappings are independent.

For verification of the inclusion of monotonicity properties of mappings, we consider the following some examples.

Example 1.2.1. Let $F: [0, 1] \rightarrow \mathbb{R}$ be defined by $Fx = x^2$. Then

$$\begin{aligned} \langle Fx - Fy, x - y \rangle &= (x + y)(x - y)^2 \geq \frac{(x + y)}{2}(x - y)^2 \\ &\geq \frac{(x + y)^2}{4}(x - y)^2 = \frac{1}{4}|Fx - Fy|^2, \end{aligned} \tag{1.11}$$

showing that F is $\frac{1}{4}$ -inverse strongly monotone. However, we can not find $\alpha > 0$ such that $\langle Fx - Fy, x - y \rangle = (x + y)|x - y|^2 \geq \alpha|x - y|^2$, for all $x, y \in [0, 1]$ showing that F is not strongly monotone.

Example 1.2.2. The mapping $F: [1, \infty) \rightarrow \mathbb{R}$ defined by $Fx = x^2$ is 2-strongly monotone but not inverse strongly monotone.

Example 1.2.3. The mapping $F: \mathbb{R} \rightarrow \mathbb{R}$ defined by $F(x) = c$, for all $x \in \mathbb{R}$, where c is a real number, is monotone but not strongly monotone.

Example 1.2.4. Let $F: (0, \infty) \rightarrow (0, \infty)$ be defined by $F(x) = \frac{a}{x+a}$, $a > 0$. Let us now assume that $\langle Fx, y-x \rangle \geq 0$, for all $x, y \in (0, \infty)$, that is, $\left\langle \frac{a}{x+a}, y-x \right\rangle \geq 0$, for all $x, y \in (0, \infty)$. It then follows that $y-x \geq 0$, and thus $\langle Fy, y-x \rangle = \left\langle \frac{a}{y+a}, y-x \right\rangle \geq 0$, showing that F is pseudomonotone. To show that F is not monotone, let us take arbitrary distinct points $x, y \in (0, \infty)$. Then we have $\langle Fx - Fy, x - y \rangle = \left\langle x - y, \frac{a}{x+a} - \frac{a}{y+a} \right\rangle = \frac{-a\|x-y\|^2}{(x+a)(y+a)} < 0$, and thus F is not monotone.

Example 1.2.5. Let $C = [-1, 1]$. Then the mapping $F: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$Fx = \begin{cases} |2x-4|, & \text{if } x > 1, \\ x^2 + 1, & \text{if } x \in C, \\ -2x, & \text{if } x < -1, \end{cases}$$

is quasimonotone but not pseudomonotone [168]. In fact, if we take $x = 2$ and $y = \frac{-1}{2}$, then we get $\langle Fx, y-x \rangle = \left\langle 0, \frac{-1}{2} - 2 \right\rangle = 0$, but $\langle Fy, y-x \rangle = \left\langle \frac{5}{4}, \frac{-1}{2} - 2 \right\rangle = \left(\frac{5}{4}\right) \cdot \left(\frac{-5}{2}\right) < 0$, showing that F is not pseudomonotone.

We note from Cioranescu [57] that in a reflexive real Banach space, every continuous monotone mapping is maximal monotone.

Definition 1.2.12. The generalized duality mapping is a mapping $J_p: E \rightarrow 2^{E^*}$ defined by

$$J_p x = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p \text{ and } \|x^*\| = \|x\|^{p-1}\}, \text{ for all } x \in E.$$

The Hahn Banach Theorem guarantees that $J_p x \neq \emptyset$, for each $x \in E$. If $p = 2$, then $J_p = J_2$ is called the normalized duality mapping usually denoted by J . The normalized duality mapping is a multi-valued monotone mapping. Generally, we have that $J_p x = \|y\|^{p-2} Jx$. If, in addition, $E = H$, then J is the same as the identity mapping on H . We also note that the generalized duality mapping is multi-valued monotone.

1.2.3 Convex Functions

Let $f: E \rightarrow \mathbb{R}$ be a function. The domain of f , denoted by $\text{Dom } f$, is defined as $\text{Dom } f = \{x \in E : f(x) < +\infty\}$.

Definition 1.2.13. The right-hand derivative of the function $f: E \rightarrow (-\infty, +\infty]$ at a point $x \in \text{int}(\text{Dom } f)$ in the direction of y is defined as

$$f'(x, y) = \lim_{t \downarrow 0} \frac{f(x+ty) - f(x)}{t}. \quad (1.12)$$

Definition 1.2.14. Given a function $f: E \rightarrow \mathbb{R}$. Then the function $f^*: E^* \rightarrow \mathbb{R}$, defined by $f^*(y^*) = \sup \{\langle y^*, y \rangle - f(y) : y \in E\}$ for any $y^* \in E^*$ is called the Fenchel conjugate of f .

Definition 1.2.15. [180] Let E be a vector space and let C be a convex subset of E . Then the function $f: C \rightarrow (-\infty, +\infty]$ is said to be

i) *proper if it has nonempty domain;*

ii) *convex if for all $w, z \in C$ and $\alpha \in [0, 1]$, one has*

$$f(\alpha w + (1 - \alpha)z) \leq \alpha f(w) + (1 - \alpha)f(z);$$

iii) *uniformly convex with modulus ψ , if for each w, z in its domain and $\alpha \in [0, 1]$, one has*

$$f(\alpha w + (1 - \alpha)z) \leq \alpha f(w) + (1 - \alpha)f(z) - \alpha(1 - \alpha)\psi(\|w - z\|),$$

where ψ is an increasing function with $\psi(0) = 0$;

iv) *lower semicontinuous at x if*

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n),$$

for every sequence $\{x_n\} \subseteq \text{Dom } f$ with $\lim_{n \rightarrow \infty} x_n = x$;

v) *Gâteaux differentiable at $x \in \text{int}(\text{Dom } f)$ if for any $y \in E$ the derivative*

$$f'(x, y) = \lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t}, \quad (1.13)$$

exists. In this case, $f'(x, y)$ coincides with $\nabla f x$, where $\nabla f x$ is the value of the gradient of f at x . Moreover, the function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each point $x \in \text{int}(\text{Dom } f)$;

vi) *Fréchet differentiable at x if the limit in (1.13) is attained uniformly for every $\|y\| = 1$;*

vii) *uniformly Fréchet differentiable on a subset C of E , if the limit in (1.13) is attained uniformly for any $x \in C$ and $\|y\| = 1$;*

xiii) *strongly coercive if $\frac{f(x)}{\|x\|} \rightarrow +\infty$ as $\|x\| \rightarrow \infty$.*

If a function is proper, convex and lower semi-continuous, then it is continuous (see, [16]).

A Fréchet differentiable function is Gâteaux differentiable and continuous (see, [126], p. 142).

A function $f: E \rightarrow \mathbb{R}$ is said to be a *Legendre function* (see, [134]) if and only if the following two conditions hold:

- (1) f is Gâteaux differentiable, $\text{int}(\text{Dom } f) \neq \emptyset$ and $\text{Dom } \nabla f = \text{int}(\text{Dom } f)$;
- (2) f^* is Gâteaux differentiable, $\text{int}(\text{Dom } f^*) \neq \emptyset$ and $\text{Dom } \nabla f^* = \text{int}(\text{Dom } f^*)$.

If E is a strictly convex and smooth Banach space, then the function defined by $f(x) = \frac{1}{p}\|x\|^p$, $p \in (1, \infty)$ is a proper, lower semicontinuous Legendre function with Fenchel conjugate $f^*(x^*) = \frac{1}{q}\|x^*\|^q$, $q \in (1, \infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$ (see, [15]). In this case, we have $\nabla f = J_p$.

For a reflexive Banach space E and a Legendre function $f: E \rightarrow (-\infty, +\infty]$, we have $\nabla f^* = (\nabla f)^{-1}$ [24]. We also have that f is a Legendre function on E if and only if f^* is a Legendre

function on E^* [15].

Definition 1.2.16. [21] A Gâteaux differentiable function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be β -strongly convex if there exists a positive constant β such that

$$\langle \nabla f x - \nabla f y, x - y \rangle \geq \beta \|x - y\|^2,$$

for all $x, y \in \text{Dom } f$, or equivalently [122]

$$f(y) \geq f(x) + \langle \nabla f x, y - x \rangle + \frac{\beta}{2} \|x - y\|^2.$$

The constant β in Definition 1.2.16 is called the *strong convexity constant* of f .

If E is a smooth and 2-uniformly convex Banach space, then the function $f(x) = \frac{1}{2}\|x\|^2$ is strongly coercive, lower semi-continuous, uniformly Fréchet differentiable, bounded and strongly convex with strong convexity constant $\beta \in (0, 1]$ and conjugate $f^*(x^*) = \frac{1}{2}\|x^*\|^2$. We also have that if f is β -strongly convex, then ∇f^* is $\frac{1}{\beta}$ -Lipschitz continuous (see, Zhong [194]).

Note that the class of uniformly convex functions contains that of strongly convex functions.

The subdifferential of f at x , denoted by ∂f , is the set defined by

$$\partial f(x) = \{x^* \in E^* : \langle x^*, y - x \rangle \leq f(y) - f(x), \text{ for all } y \in E\}, \text{ (see, [85]).}$$

For a nonempty, closed and convex subset C of $E = E_1 \times E_2$ and let $f: E_1 \rightarrow (-\infty, +\infty]$, $g: E_2 \rightarrow (-\infty, +\infty]$ be convex Gâteaux differentiable functions. Now, for any $(x, y) \in E$, if $(x^*, y^*) = P_C^h(x, y)$, then

$$\langle \nabla h(x, y) - \nabla h(x^*, y^*), (u, v) - (x^*, y^*) \rangle \leq 0, \quad (1.14)$$

for all $(u, v) \in C$, where $h: E \rightarrow (-\infty, +\infty]$ is the function defined by $h = (f, g)$.

1.2.4 Bregman Distances and their Properties

Definition 1.2.17. [25, 39] The Bregman distance with respect to a Gâteaux differentiable convex function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is the function $D_f: \text{Dom } f \times \text{int}(\text{Dom } f) \rightarrow [0, +\infty)$ defined as

$$D_f(y, x) = f(y) - f(x) - \langle \nabla f x, y - x \rangle.$$

The Bregman distance is not a distance in the usual sense because it is not symmetric and does not satisfy the triangle inequality, but it satisfies the following important properties (see, e.g., [33, 133, 135]):

(i) *Three point identity:* For any $w \in \text{Dom } f$ and any $x, y \in \text{int}(\text{Dom } f)$

$$D_f(w, x) + D_f(x, y) - D_f(w, y) = \langle \nabla f x - \nabla f y, x - w \rangle. \quad (1.15)$$

(ii) *Four point identity:* For any $x, w \in \text{Dom } f$ and any $y, z \in \text{int}(\text{Dom } f)$

$$D_f(x, z) + D_f(w, y) - D_f(x, y) - D_f(w, z) = \langle \nabla f y - \nabla f z, x - w \rangle. \quad (1.16)$$

Definition 1.2.18. [35] Let $C \subseteq \text{int}(\text{Dom } f)$ be a nonempty, convex and closed subset of E , where $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is a Gâteaux differentiable and convex function. The Bregman projection of $x \in \text{int}(\text{Dom } f)$ onto C is the unique vector $P_C^f(x)$ of C with the property

$$D_f(P_C^f(x), x) = \min \{D_f(y, x) : y \in C\}. \quad (1.17)$$

The Bregman projection has the following important properties (see, [35]):

$$w = P_C^f(x) \text{ if and only if } \langle \nabla f x - \nabla f w, y - w \rangle \leq 0, \text{ for all } y \in C, \quad (1.18)$$

and

$$D_f(w, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(w, x), \text{ for all } x \in E, w \in C. \quad (1.19)$$

Remark 1.2.2. The Bregman projection has the following special cases [3]:

- (i) If, in (1.17), E is a smooth Banach space and $f(x) = \frac{1}{2}\|x\|^2$, then $P_C^f(x)$ reduces to $\Pi_C(x)$, where Π_C is the generalized projection mapping (see, e.g., [8]) characterized by

$$\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x),$$

where ϕ is the Lyapunov function defined by $\phi(w, z) = \|w\|^2 - 2\langle w, Jz \rangle + \|z\|^2$.

- (ii) If, in (1.17), $E = H$ with $f(x) = \frac{1}{2}\|x\|^2$, then $P_C^f(x)$ further reduces to $P_C x$.

The modulus of total convexity of a Gâteaux differentiable function f is the function $v_f: \text{int}(\text{Dom } f) \times [0, \infty) \rightarrow [0, \infty)$ defined by $v_f(x, t) = \inf \{D_f(y, x) : y \in \text{Dom } f, \|y - x\| = t\}$. In this case, f is said to be *totally convex* at the point $x \in \text{int}(\text{Dom } f)$ if $v_f(x, t) > 0$ whenever $t > 0$ (see, [34]) and f is said to be *totally convex* if it is totally convex at every point in the interior of its domain. On bounded subsets of E , the concepts of uniform convexity and total convexity are the same (see, [35]).

Let $f: E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and Legendre function. The non-negative real-valued function $V_f: E \times E^* \rightarrow [0, +\infty)$ associated with f defined by

$$V_f(y, y^*) = f(y) - \langle y^*, y \rangle + f^*(y^*), \text{ for all } y \in E, y^* \in E^*, \quad (1.20)$$

satisfies the following two properties (see, [141]):

$$V_f(y, y^*) = D_f(y, \nabla f^* y^*), \quad (1.21)$$

and

$$V_f(y, y^*) + \langle z^*, \nabla f^* y^* - y \rangle \leq V_f(y, y^* + z^*), \text{ for all } y \in E, y^*, z^* \in E^*. \quad (1.22)$$

1.2.5 Nonlinear Mappings

Definition 1.2.19. A mapping $T: H \rightarrow H$ is said to be

i. *firmly nonexpansive* [31] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \text{ for all } x, y \in H.$$

ii. γ -*strictly pseudocontractive* (in the sense of Browder and Petryshyn [30]) if there exists $\gamma \in (0, 1)$ such that

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \gamma \|(x - y) - (Tx - Ty)\|^2, \text{ for all } x, y \in H;$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \gamma \|(x - Tx) - (y - Ty)\|^2, \text{ for all } x, y \in H;$$

iii. *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in H,$$

or equivalently (due to Kato [94])

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \text{ for all } x, y \in H.$$

Remark 1.2.3. We observe here that if a mapping is firmly nonexpansive, then it is strictly pseudocontractive and hence it is pseudocontractive.

Definition 1.2.20. A point $p \in C \subseteq E$ is said to be a fixed point of the mapping $T: C \rightarrow E$ if $Tp = p$. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{p \in C : Tp = p\}$.

Definition 1.2.21 ([82, 117]). A mapping $T: H \rightarrow H$ with $F(T) \neq \emptyset$ is said to be ν -demicontractive if there exists a number $\nu \in [0, 1)$ with

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \nu \|Tx - x\|^2, \text{ for all } x \in H \text{ and } p \in F(T). \quad (1.23)$$

If (1.23) is satisfied for $\nu = -1$, then T is said to be firmly quasi-nonexpansive; and T is said to be quasi-nonexpansive if $\nu = 0$.

Definition 1.2.22. [13] A mapping $T: H \rightarrow H$ is said to be ν -demimetric if $F(T) \neq \emptyset$ and there exists $\nu \in (-\infty, 1)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \nu \|Tx - x\|^2, \text{ for all } x \in H \text{ and } p \in F(T),$$

or equivalently

$$\langle x - p, x - Tx \rangle \geq \frac{1 - \nu}{2} \|x - Tx\|^2, \text{ for all } x \in C \text{ and } p \in F(T).$$

We notice here that the class of ν -demictractive mappings is a larger class of mappings which contains the families of quasi-nonexpansive mappings, firmly nonexpansive mappings with nonempty set of fixed points, strictly pseudocontractive mappings with nonempty set of fixed points and firmly quasi-nonexpansive mappings (see, [59]).

It can also be observed that the class of ν -demictractive mappings is contained in the class of ν -demimetric mappings, and thus the class of ν -demimetric mappings is a larger class of mappings.

It is shown in [44] that if T is a ν -demictractive mapping, then $F(T)$ is closed and convex subset of H .

Definition 1.2.23. *A mapping $T: H \rightarrow H$ is said to be semicompact if, for any bounded sequence $\{x_n\} \subseteq H$ with $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some $y \in H$.*

Definition 1.2.24. *A mapping $T: H \rightarrow H$ is said to demicompact if, for any sequence $\{y_n\} \subseteq H$ which is bounded and $\{\|y_n - Ty_n\|\}$ converges to some y_0 , there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ converges strongly to some $y^* \in H$ and $Ty^* - y^* = y_0$.*

Definition 1.2.25. [2] *A sequence $\{x_n\}$ in E is said to converge weakly to $p \in E$ if $f(x_n) \rightarrow f(p)$ for all $f \in E^*$. The weak convergence of the sequence $\{x_n\}$ to p is denoted by $x_n \rightharpoonup p$.*

Definition 1.2.26. [100] *A mapping $T: C \subseteq E \rightarrow E$ is said to satisfy the demiclosedness property or $(I - T)$ is demiclosed at 0 if $\{x_n\}$ is any sequence in C such that $x_n \rightharpoonup p$ and $\|(I - T)x_n\| \rightarrow 0$, then $Tp = p$.*

Definition 1.2.27. *A mapping $T: C \subseteq E \rightarrow E^*$ is said to be weakly sequentially continuous on C if $Tx_n \rightharpoonup Tx$ whenever $\{x_n\}$ is a sequence in C with $x_n \rightarrow x$.*

Definition 1.2.28. [181] *Let C be a nonempty subset of a smooth Banach space E . A mapping $T: C \rightarrow E^*$ is said to be semi-pseudocontractive if for any $x, y \in C$, we have*

$$\langle Tx - Ty, x - y \rangle \leq \langle Jx - Jy, x - y \rangle.$$

Definition 1.2.29. [181] *Let $T: C \subseteq E \rightarrow E^*$ be a mapping, where E is a smooth Banach space. A point $x^* \in C$ is said to be a semi-fixed point of T if $Jx^* = Tx^*$. We denote the set of semi-fixed points of the mapping T by $F_s(T)$.*

Definition 1.2.30. [191] *Let C be a nonempty subset of E . A mapping $T: C \rightarrow E^*$ is called f -pseudocontractive if*

$$\langle Tx - Ty, x - y \rangle \leq \langle \nabla f x - \nabla f y, x - y \rangle,$$

for all $x, y \in C$, where $f: E \rightarrow (-\infty, +\infty]$ is a proper, lower-semicontinuous, convex, Fréchet differentiable function.

Definition 1.2.31. [191] *Let C be a nonempty, closed and convex subset of E and let $f: E \rightarrow (-\infty, +\infty]$ be a proper, lower semi-continuous and convex Fréchet differentiable function. Let*

$T: C \rightarrow E^*$ be any mapping. A point $p \in C$ is called

- (i) an f -fixed point of T if $Tp = \nabla f p$. We denote the set of all f -fixed points of T by $F_f(T)$.
- (ii) an f -asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C such that $x_n \rightarrow p$ and $\lim_{n \rightarrow \infty} \|\nabla f x_n - T x_n\| = 0$. The set of all f -asymptotic fixed points of T is denoted by $\widehat{F_f(T)}$.

If $f(x) = \frac{1}{2}\|x\|^2$, then $\nabla f = J$ and hence the class of f -pseudocontractive mappings reduces to the class of semi-pseudocontractive mappings. In this case, the concepts of f -fixed points and semi-fixed points coincide.

If in addition, $E = H$, where H is a real Hilbert space, then $\nabla f = J = I$, where I is the identity mapping on H . In this case, the concepts of f -pseudocontractive mappings and pseudocontractive mappings coincide. Moreover, the set of f -fixed points and fixed points of T are equal. As a result, we conclude that the class of f -pseudocontractive mappings contains that of semi-pseudocontractive mappings. Thus, the concept of f -pseudocontractivity is an extension of pseudocontractivity of mappings from Hilbert spaces to Banach spaces.

Definition 1.2.32. [172] Let $f: E \rightarrow (-\infty, +\infty]$ be a proper, lower semi-continuous, Gâteaux differentiable and convex function. A mapping $T: C \rightarrow E^*$ is called Bregman relatively f -nonexpansive if the following properties hold:

- i. $F_f(T)$ is nonempty;
- ii. $D_f(p, \nabla f^* T x) \leq D_f(p, x)$, for all $x \in C, p \in \widehat{F_f(T)}$; and
- iii. $\widehat{F_f(T)} = F_f(T)$.

The class of f -pseudocontractive mappings has a close connection with the class of monotone mappings as a mapping T is f -pseudocontractive if and only if $F := \nabla f - T$ is monotone. In this case, the set of f -fixed points of T is equal to the set of zero points, $N(F)$, of F , where $N(F) = \{x \in \text{Dom}(F) : Fx = 0\}$.

Moreover, we have from Reich and Sabach [136] that if ∇f and T are continuous, then F is maximal monotone and the set of f -fixed points of T is closed and convex.

1.3 Research Motivation and Statement of the Problem

This section covers the research problems and study motivation.

The fixed point problem is defined as finding a point $x^* \in E$ with the property

$$Tx^* = x^*,$$

where $T: E \rightarrow E$ is a nonlinear mapping.

Picard [128] was the first to construct iterative algorithms for solving boundary value problems involving integral operators, which can be modeled as fixed point problems of certain

nonlinear mappings. Since then, several iterative algorithms have been proposed and studied for approximating fixed points of nonexpansive, strictly pseudocontractive and pseudocontractive mappings both in Hilbert space and Banach space frameworks (see, for instance, [20, 30, 51, 55, 92, 112, 148, 191] and other references therein).

In connection with the fixed point problems, Moudafi and Al-Shemas [116] introduced the Split Equality Fixed Point Problem (SEFPP) defined as finding a point

$$(x^*, y^*) \in F(S_1) \times F(S_2) \text{ such that } Ax^* = By^*, \quad (1.24)$$

where H_1 and H_2 are real Hilbert spaces, $S_1: H_1 \rightarrow H_1$ and $S_2: H_2 \rightarrow H_2$ are nonlinear mappings, $A: H_1 \rightarrow H_3$ and $B: H_2 \rightarrow H_3$ are bounded linear mappings, where H_3 is another real Hilbert space.

The split equality fixed point problem is a quite general problem which contains, as special cases, the Split Equality Feasibility Problem (SEFP) and Split Feasibility Problem (SFP). Since it is highly applicable in many areas of applied mathematics, the split equality fixed point problem has been widely studied [41].

Consequently, many authors have introduced and studied different methods which approximate solutions of the SEFPP involving firmly quasi-nonexpansive, quasi-nonexpansive and quasi-pseudocontractive mappings (see, for instance, [40, 42, 114]).

Chang et al. [40] proposed and studied an iterative algorithm which approximates solutions of SEFPP involving Lipschitz quasi-asymptotically pseudocontractive mappings which are more general than quasi-nonexpansive mappings in real Hilbert spaces. They established a weak convergence result under certain conditions.

It is worth mentioning here that, according to what we currently know, there is no strong convergence result for solving SEFPP for uniformly continuous pseudocontractive mappings.

On the other hand, many authors have introduced different methods for approximating semi-fixed points, f -fixed points and common fixed points of different types of mappings in the Banach space settings (see, for instance, [50, 148, 187, 191]).

Recently, Zegeye and Wega [191] introduced an iterative algorithm which approximates a common f -fixed point of two Lipschitz f -pseudocontractive mappings in reflexive Banach spaces. However, the results obtained are restricted to a specific problem involving Lipschitz continuous mappings.

Thus, it is natural to introduce a more general problem called the Split Equality Common f, g -Fixed Point Problem (SECFPP) in reflexive Banach spaces with more general mappings.

The Split Equality Common f, g -Fixed Point Problem (SECFPP) is defined as finding a point

$$(x^*, y^*) \in (F_f(T_1) \cap F_f(T_2)) \times (F_g(S_1) \cap F_g(S_2)) \text{ such that } Ax^* = By^*, \quad (1.25)$$

where E_1 and E_2 are reflexive real Banach spaces, $f: E_1 \rightarrow (-\infty, +\infty]$ and $g: E_2 \rightarrow (-\infty, +\infty]$

are convex, proper, lower-semi-continuous and Gâteaux differentiable functions, $T_1, T_2: E_1 \rightarrow E_1^*$ are f -pseudocontractive, $S_1, S_2: E_2 \rightarrow E_2^*$ are g -pseudocontractive mappings and $A: E_1 \rightarrow E_3$ and $B: E_2 \rightarrow E_3$ are bounded linear mappings, where E_3 is another reflexive real Banach space.

The SECFPP is a more general problem which contains the Split Equality Fixed Point Problem, Split Equality f, g -Fixed Point Problem and Split Common f, g -Fixed Point Problems as its special cases.

Whether we can construct an iterative method which approximates a solution of the split equality common f, g -fixed point problem involving uniformly continuous, f, g -pseudocontractive mappings is an open problem.

The other problem in the study of nonlinear functional analysis is the Variational Inequality Problem (VIP) defined as finding a point $x^* \in C$ such that

$$\langle Fx^*, y - x^* \rangle \geq 0, \text{ for all } y \in C, \quad (1.26)$$

where C is a nonempty, convex and closed subset of E and $F: C \rightarrow E^*$ is a monotone mapping. We usually denote the solution set of the VIP (1.26) by $VI(C, F)$. The variational inequality problems are closely related to the fixed point problems. Indeed, a point p is a solution of the variational inequality problem (1.26) if and only if it is a fixed point of the mapping $P_C(x - \lambda F(x))$ for any $\lambda > 0$.

Several authors have proposed different iterative algorithms to approximate solutions of variational inequality problems involving mappings with different monotonicity properties (see, for instance, [74, 75, 78, 88, 97, 98, 101, 107, 110, 160] and the references therein). The projected gradient method, extragradient method, subgradient extragradient method and Tseng's extragradient method are some of the methods employed for solving variational inequality problems.

In connection with the variational inequality problem, we have the dual variational inequality problem, commonly called the *Minty Variational Inequality Problem* (MVIP), which is defined as finding an element $z^* \in C$ such that

$$\langle Fz, z - z^* \rangle \geq 0, \text{ for all } z \in C, \quad (1.27)$$

where C is a nonempty, convex and closed subset of E and $F: C \rightarrow E^*$ is a mapping. The set of solutions of MVIP (1.27) is commonly denoted by $MVI(C, F)$. Clearly, if the mapping F is pseudomonotone, then $MVI(C, F)$ is a closed and convex set. If F is a continuous mapping and C is convex, then $MVI(C, F) \subseteq VI(C, F)$ [193]. If F is continuous and pseudomonotone, then we have that $VI(C, F) = MVI(C, F)$ [58]. However, the inclusion $VI(C, F) \subseteq MVI(C, F)$ does not always hold if F is continuous and quasimonotone mapping (see, [179]).

There has been a growing interest of researchers on speeding up the convergence rates of different approximation methods. The inertial technique which originates from the implicit time discretization of a second-order dynamical system, is one of the methods of accelerating the convergence of algorithms and it was introduced by Polyak [130] in 1964. It is an iterative method

in which a subsequent term of a sequence is obtained from the combination of the two immediate preceding terms of the sequence. Nowadays, introducing and studying inertial algorithms for solving variational inequality problems, Hammerstein type equation problems and fixed point problems is attracting interests of many authors (see, [45, 123, 137, 155]).

Several authors have introduced different inertial algorithms for solving the Minty variational inequality problems involving quasimonotone mappings (see, for instance, [87, 168, 169]).

In 2022, Izuchukwu et al. [87] introduced an inertial extrapolation method which solves the MVIP in a real Hilbert space. They established a weak convergence of their method under the assumption that the underlying mapping is a Lipschitz continuous quasimonotone. However, the method in Izuchukwu et al. [87] is based on the metric projection. More recently, Wang et al. [169] introduced a more general inertial-like algorithm which approximates solutions of quasimonotone Minty variational inequalities in real Hilbert spaces which makes use of the Bregman projection method, which is more general than the metric projection considered in [87].

The aforementioned results deal only with MVIP in the Hilbert space settings and the convergence results obtained are weak convergence. As a result, further study is required to determine whether the existing problems can be extended to more general problems with strong convergence results in more general Banach spaces.

Thus, it is natural to introduce a new problem called the split equality Minty variational inequality problem and construct an iterative algorithm which approximates its solution in reflexive Banach spaces.

We introduce the Split Equality Minty Variational Inequality Problem (SEMVIP) as finding a point

$$(x^*, y^*) \in MVI(C, F) \times MVI(D, G) \text{ such that } Ax^* = By^*, \quad (1.28)$$

where $\emptyset \neq C$ and $\emptyset \neq D$ are convex and closed subsets of the Banach spaces E_1 and E_2 , respectively, $F: E_1 \rightarrow E_1^*$ and $G: E_2 \rightarrow E_2^*$ are nonlinear mappings, $A: E_1 \rightarrow E_3$ and $B: E_2 \rightarrow E_3$ are bounded linear mappings, where E_3 is a Banach space.

Another problem which is formulated in terms of the VIPs is the Bilevel Variational Inequality Problem (BVIP) which is defined as finding an element

$$x^* \in VI(C, F) : \langle Gx^*, x - x^* \rangle \geq 0, \text{ for all } x \in VI(C, F), \quad (1.29)$$

where C is a nonempty, convex and closed subset of H , $F: C \rightarrow H$ and $G: C \rightarrow H$ are mappings. Here, the variational inequality problem which involves the mapping F is called the lower level problem or the constraint problem. The set of solutions of (1.29) is denoted by $VI(VI(C, F), G)$.

Bilevel variational inequality problems (1.29) are special classes of quasivariational inequalities (see, for instance, [14, 72, 177]) and that of equilibrium problems with equilibrium constraints (see, [63, 64, 113]).

Several nonlinear optimization problems are covered by the BVIP. For instance, fixed point

problems, complementary problems, saddle point problems and minimum norm problems are some special cases of the BVIP.

For solving the bilevel variational inequality problems, several authors (see, [10, 11, 12, 65, 66, 91, 153, 160] and the references therein) have introduced different iterative algorithms and discussed their convergence analyses.

Many authors have also been working on the approximation of solutions of bilevel variational inequality problems that involve fixed points of certain mappings in their lower level problems (see, for instance, [77, 79]).

The *Bilevel Variational Inequality Problem with Fixed Point Constraint* (BVIPFPC) is defined as finding $z^* \in \Omega$ with the property

$$\langle x - z^*, Gz^* \rangle \geq 0, \text{ for all } x \in \Omega, \quad (1.30)$$

where $\Omega = \{y \in VI(C, F) : Ay \in F(T)\}$, C is a nonempty, convex and closed subset of the real Hilbert space H_1 ; $F, G: H_1 \rightarrow H_1$ and $T: H_2 \rightarrow H_2$ are nonlinear mappings; $A: H_1 \rightarrow H_2$ is a bounded linear mapping.

In 2021, Hai et al. [79] introduced an iterative algorithm which approximates a solution of (1.30). They proved a strong convergence theorem under the conditions that F is Lipschitz pseudomonotone and $T: H_2 \rightarrow H_2$ is a demicontractive mapping with $(I - T)$ demiclosed at 0.

However, to the best of our knowledge, there is no strong convergence result for BVIPFPC involving beyond Lipschitz continuous mappings in the lower level variational inequality problem and a mapping more general than demicontractive for the fixed point constraint.

One of the generalizations of the variational inequality problems is the Split Equality Variational Inequality Problem (SEVIP), which is defined as finding a point

$$(x^*, y^*) \in VI(C, S) \times VI(D, T) \text{ such that } Ax^* = By^*, \quad (1.31)$$

where, $\emptyset \neq C$ and $\emptyset \neq D$ are convex and closed subsets of the real Banach spaces E_1 and E_2 , respectively, $S: E_1 \rightarrow E_1^*$ and $T: E_2 \rightarrow E_2^*$ are nonlinear mappings and $A: E_1 \rightarrow E_3$ and $B: E_2 \rightarrow E_3$ are bounded linear mappings, where E_3 is another real Banach space. Some of the special cases of SEVIP are Common Solutions of Variational Inequality Problem (CSVIP), Split Variational Inequality Problem (SVIP), Split Equality Feasibility Problem (SEFP) introduced by Moudafi [115], Split Feasibility Problem (SFP) introduced by Censor and Elfving [38] and the Split Equality Null Point Problem (SENPP).

Recently, Boikanyo and Zegeye [23] introduced a new algorithm which approximates SEVIP for uniformly continuous pseudomonotone mappings which are sequentially weakly continuous in uniformly convex Banach spaces and they obtained strong convergence results.

However, their results are still confined to the classes of uniformly convex Banach spaces.

Whether the results in Boikanyo and Zegeye [23] can be extended to more general Banach spaces

with more general mappings is an open problem which needs further studies. Based on these results, it is natural to introduce a new problem called the split equality of variational inequality and f, g -fixed point problem and propose a method for approximating its solution.

We propose the split equality of variational inequality and f, g -fixed point problem as finding a point

$$(x^*, u^*) \in (VI(C, A) \cap F_f(G)) \times (VI(D, B) \cap F_g(K)) : Tx^* = Su^*, \quad (1.32)$$

where E_1 and E_2 are reflexive real Banach spaces, $A: E_1 \rightarrow E_1^*$ and $B: E_2 \rightarrow E_2^*$ are any mappings, $G: C \rightarrow E_1^*$ and $K: D \rightarrow E_2^*$ are Bregman relatively f -nonexpansive and Bregman relatively g -nonexpansive mappings, respectively, $f: E_1 \rightarrow \mathbb{R}$ and $g: E_2 \rightarrow \mathbb{R}$ are proper, convex, lower semicontinuous, uniformly Fréchet differentiable, strongly convex, strongly coercive Legendre functions which are bounded.

The other important problem in the field of nonlinear functional analysis is the Hammerstein type equation problem which is defined as finding a point u^* in E which solves the equation

$$u + KF u = 0, \quad (1.33)$$

where E is a Banach space; and $F: E \rightarrow E^*$ and $K: E^* \rightarrow E$ are monotone mappings.

With regard to approximating solutions of the Hammerstein-type equation problems, different iterative methods have been introduced (see, for instance, [32, 47, 49, 54, 56, 61, 67, 70, 76, 119, 165, 166, 186]).

In 2013, Zegeye and Malonza [183] introduced an iterative algorithm in a uniformly convex and uniformly smooth Banach space. They proved that the method they proposed converges strongly to a solution of the Hammerstein-type equation problem under the assumption that the underlying mappings are continuous monotone.

To extend the settings under consideration, Tufa et al. [164] proposed an algorithm which approximates a solution of (1.33) in reflexive real Banach spaces. They obtained a strong convergence result under the assumption that the governing mappings are Lipschitz monotone.

Based on the results obtained in Tufa et al. [164], we introduce a more general problem called the Split Equality Hammerstein Type Equation Problem (SEHTEP) which is defined as finding a point $(u, p) \in E_1 \times E_2$ such that

$$u + K_1 F_1 u = 0, \quad p + K_2 F_2 p = 0 \text{ such that } Au = Bp, \quad (1.34)$$

where E_1 and E_2 are reflexive real Banach spaces with dual spaces E_1^* and E_2^* , respectively, $F_1: E_1 \rightarrow E_1^*$, $K_1: E_1^* \rightarrow E_1$, $F_2: E_2 \rightarrow E_2^*$ and $K_2: E_2^* \rightarrow E_2$ are monotone mappings, $A: E_1 \rightarrow E_3$ and $B: E_2 \rightarrow E_3$ are bounded linear mappings and E_3 is another reflexive real Banach space.

The SEHTEP includes Hammerstein Type Equation Problems, Common Solutions of Hammerstein Type Equation Problems and Split Hammerstein Type Equation Problems as special cases.

Whether it is possible to introduce a strongly convergent algorithm for solving the split equality Hammerstein type equation problem (1.34) involving more general types of mappings in reflexive Banach spaces is not known and it needs further study.

1.4 Research Questions

Based on the aforementioned gaps, we raise the following major research questions.

1. Can we construct and study an algorithm which approximates a solution of the split equality f, g -fixed point problems for mappings which are more general than Lipschitz continuous f, g -pseudocontractive in reflexive Banach spaces?
2. Can we introduce and study a strongly convergent algorithm for solving split equality Minty variational inequality problems involving quasimonotone mappings in reflexive Banach spaces?
3. Is it possible to introduce and study a strong convergence method for bilevel variational inequality problems with fixed point constraints involving uniformly continuous mappings in their lower level variational inequality problems?
4. Is it possible to construct and study an algorithm which approximates solutions of the split equality variational inequality and f, g -fixed point problems for mappings which are more general than Lipschitz continuous in Banach spaces?
5. Can we develop an algorithm that converges strongly to a solution of split equality Hammerstein type equation problem involving mappings which are more general Lipschitz continuous in more general Banach spaces?

1.5 Objectives

1.5.1 General Objective

The general objective of the study is to construct and study different iterative methods for solving fixed point, variational inequality, and Hammerstein type equation problems involving more general class of mappings in general Banach spaces.

1.5.2 Specific Objectives

The specific objectives of the study are to

1. develop an algorithm which approximates a solution of the split equality f, g -fixed point problems involving f, g -pseudocontractive mappings with more general continuity property than Lipschitz continuous in reflexive real Banach spaces and discuss its convergence analysis;
2. prove a strong convergence theorem for solving split equality quasimonotone Minty variational inequality problems in reflexive Banach spaces;

3. introduce and study an iterative algorithm for solving bilevel variational inequality problems with fixed point constraints involving uniformly continuous mapping in the lower level problem and discuss its strong convergence theorem;
4. construct and study an algorithm for solving split equality variational inequality and f, g -fixed point problems involving mappings more general than Lipschitz continuous in Banach spaces;
5. introduce and study a strongly convergent algorithm for solving split equality Hammerstein type equation problems in reflexive Banach spaces where the underlying mappings are more general than Lipschitz continuous.

1.6 Scope of the Study

The scope of this thesis is limited to

- approximating f, g -fixed points of f, g -pseudocontractive mappings in Hilbert spaces and reflexive Banach spaces;
- solving split equality Minty variational inequality problems involving quasimonotone mappings in reflexive Banach spaces;
- approximating bilevel variational inequality problems with fixed point constraints in Hilbert spaces;
- finding a common solution of the split equality of variational inequality and f, g -fixed point problems in reflexive real Banach spaces;
- approximating solutions of split equality Hammerstein type equation problems involving uniformly continuous monotone mappings in reflexive real Banach spaces.

2. Literature Review

In this chapter, we present detailed review of the related literature which is relevant to the remaining part of the thesis.

2.1 Approximation of Solutions of Fixed Point Problems Involving Pseudocontractive Mappings

In this section, we present some research findings that have been done on the approximation of fixed points of different types of pseudocontractive mappings in Hilbert and Banach spaces.

For solving the fixed point problems, Picard [128] introduced the following iterative algorithm: Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T: C \rightarrow C$ be a contraction mapping and let $\{u_n\}$ be a sequence generated from arbitrary $u_0 \in H$ as

$$u_{n+1} = Tu_n, \quad n \geq 0. \quad (2.1)$$

It has been proved that the sequence $\{u_n\}$ converges strongly to the unique fixed point of the mapping T .

It was remarked here that the convergence of the method (2.1) is not guaranteed whenever T is generally nonexpansive. If, for example, we take C to be the unit disc in $\mathbb{R}^2$ with center at the origin and T is the mapping defined by the rotation of C about the origin with an angle $\frac{\pi}{4}$. Then T is a nonexpansive mapping and 0 is the unique fixed point of T . It is not difficult to visualize that the sequence $\{x_n\}$ generated by this rotation from any $x_0 \neq 0$ on the unit disc does not converge to 0. Thus, several improvements have been made on the work of Picard [128] (see, for instance [86, 108, 124])

To resolve the drawback in Picard iteration [128], Mann [108] introduced an iterative algorithm for approximating solutions of fixed point problems involving nonexpansive mappings in the Hilbert space settings. Mann [108] proved that the sequence $\{u_n\}$ obtained from arbitrary $u_0 \in C \subseteq H$ by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n, \quad n \geq 0, \quad (2.2)$$

where $T: C \rightarrow C$ is a nonexpansive mapping and $\alpha_n \in (0, 1)$ satisfies some appropriate conditions, is a sequence of approximate fixed point, that is, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. To obtain a strong convergence result of $\{u_n\}$ to a fixed point of T (when exists), one needs some compactness type assumptions either on the set C (e.g., C is compact) or on the mapping T (e.g., T is demicompact or semicompact).

However, the algorithm introduced by Mann [108] does not necessarily converge to a fixed point of T , whenever T is a more general pseudocontractive mapping even when some compactness assumptions are imposed on C or T . Chidume and Mutangadura [51] provided an example of a

Lipschitz pseudocontractive self mapping of a convex and compact subset of a Hilbert space for which the Mann iterative scheme fails to converge.

In 1974, Ishikawa [86] introduced the following iterative algorithm in a real Hilbert space H : Let K be a convex and compact subset of H and let $T: K \rightarrow K$ be a self mapping. Then the sequence $\{u_n\}$ generated from arbitrary point $u_1 \in K$ by

$$u_{n+1} = \alpha_n T[\beta_n T u_n + (1 - \beta_n) u_n] + (1 - \alpha_n) u_n, \quad n \geq 1, \quad (2.3)$$

where the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy certain conditions, converges strongly to a fixed point of T provided that T is a Lipschitz pseudocontractive mapping. It can be observed that if $\beta_n = 0$, for all $n \geq 1$ in (2.3), then the Ishikawa iteration scheme reduces to the Mann iteration scheme (2.2). We notice here that the compactness condition imposed on the set K is a quite strong condition.

In 2013, Osilike et al. [124] introduced and studied the Modified Ishikawa Algorithm in real Hilbert space settings: Let $T: H \rightarrow H$ be a Lipschitz pseudocontractive mapping. Let $\{u_n\}$ be arbitrarily defined from $u_1 \in H$ by

$$u_{n+1} = (1 - \alpha_n)(1 - t_n)u_n + \alpha_n T[(1 - \beta_n)(1 - t_n)u_n + \beta_n [T(1 - t_n)u_n]], \quad n \geq 1, \quad (2.4)$$

where $\{t_n\}, \{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ with some suitable conditions. Then the authors proved a strong convergence of $\{u_n\}$ to a point $p = P_{F(T)}(0)$, provided that T is a Lipschitz pseudocontractive mapping with $F(T) \neq \emptyset$. It can be seen that the Modified Ishikawa Iteration method is more general than that of Ishikawa and can be reduced to the normal Ishikawa iteration [86] when $t_n = 0$. The other advantage of the result obtained in Osilike [124] over Ishikawa [86] is that the Modified Ishikawa Algorithm does not require the compactness condition on the feasible set.

Many other authors have also proposed and studied different iterative algorithms which approximate fixed points of pseudocontractive mappings (see, for instance, [4, 20, 55, 89, 92, 112, 148, 182, 188, 189, 190, 195] and the references therein).

The split equality fixed point problem (SEFPP), which is a generalization of the split feasibility problem and of the convex feasibility problem, has been extensively investigated recently due to its extraordinary utility and applicability in several areas of applied mathematics (see, for instance, [41, 156]).

In 2011, Moudafi and Al-Shemas [116] proposed the following algorithm which approximates a solution of SEFPP (1.24): Let H_1, H_2 and H_3 be real Hilbert spaces and let $T: H_1 \rightarrow H_1$ and $S: H_2 \rightarrow H_2$ be firmly quasi-nonexpansive mappings. Let $\{(x_n, y_n)\}$ be the sequence obtained by the following iteration:

$$\begin{cases} x_{n+1} = T(x_n - \beta_n A^*(Ax_n - By_n)) \\ y_{n+1} = S(y_n + \beta_n B^*(Ax_n - By_n)), \end{cases} \quad (2.5)$$

where $\{\beta_n\}$ is a real sequence satisfying some conditions, $A: H_1 \rightarrow H_3$ and $B: H_2 \rightarrow H_3$

are bounded linear mappings. Then the authors proved that the sequence $\{(x_n, y_n)\}$ converges weakly to a solution of the SEFPP (1.24) .

However, the calculation of the step size $\{\beta_n\}$ in (2.5) is dependent on the operators norms $\|A\|$ and $\|B\|$.

In 2015, Che and Li [42] proposed the following algorithm for solving SEFPP (1.24) which does not require prior information about the operator norms for calculating the step sizes: Let H_1 , H_2 and H_3 be real Hilbert spaces and let $T_1: H_1 \rightarrow H_1$ and $T_2: H_2 \rightarrow H_2$ be quasi-nonexpansive mappings. Let $\{(x_n, y_n)\}$ be the sequence defined from arbitrary $(x_0, y_0) \in H_1 \times H_2$ as

$$\begin{cases} c_n = x_n - \theta_n A^*(Ax_n - By_n), \\ x_{n+1} = \varrho_n x_n + (1 - \varrho_n) T_1 c_n, \\ d_n = y_n - \theta_n B^*(By_n - Ax_n), \\ y_{n+1} = \varrho_n y_n + (1 - \varrho_n) T_2 d_n, \end{cases}$$

where $A: H_1 \rightarrow H_3$ and $B: H_2 \rightarrow H_3$ are bounded linear mappings. They established a weak convergence result under some appropriate conditions on the control sequences $\{\varrho_n\}$ and $\{\theta_n\}$.

Though the firmly quasi-nonexpansive assumption imposed on the mappings in Moudafi and Al-Shemas [116] is extended to the relatively bigger class of quasi-nonexpansive mappings by Che and Li [42], the quasi-nonexpansiveness property is still a quite strong condition. Consequently, Chang et al. [40] proposed the following algorithm for solving SEFPP (1.24) involving more general mappings: Let H_1 , H_2 and H_3 be real Hilbert spaces and let $T: H_1 \rightarrow H_1$ and $S: H_2 \rightarrow H_2$ be Lipschitz quasi-pseudocontractive mappings. Let $A: H_1 \rightarrow H_3$ and $B: H_2 \rightarrow H_3$ are bounded linear mappings with adjoints A^* and B^* , respectively. Let $\{(x_n, y_n)\}$ be the sequence obtained from the following scheme:

$$\begin{cases} c_n = x_n - \theta_n A^*(Ax_n - By_n), \\ x_{n+1} = \varrho_n x_n + (1 - \varrho_n) [(1 - \zeta_n)I + \zeta_n T ((1 - \eta_n)I + \eta_n T)] c_n, \\ d_n = y_n - \theta_n B^*(By_n - Ax_n), \\ y_{n+1} = \varrho_n y_n + (1 - \varrho_n) [(1 - \zeta_n)I + \zeta_n S ((1 - \eta_n)I + \eta_n S)] d_n, \end{cases} \quad (2.6)$$

Then they proved a weak convergence theorem under some appropriate conditions on the sequences $\{\varrho_n\}$, $\{\theta_n\}$, $\{\eta_n\}$ and $\{\zeta_n\}$. They have also obtained a strong convergence result if in addition T and S are semi-compact.

However, no strong convergence result is obtained for the split equality fixed point problem involving mappings which are more general than Lipschitz continuous and this gap needs further studies.

Many authors have also proposed different iterative algorithms for approximating common fixed points of pseudocontractive, semi-pseudocontractive and f -pseudocontractive mappings both in Hilbert and Banach space settings (see, for instance, [131, 187, 191]).

In 2013, Zegeye and Shahzad [187] introduced the following iterative algorithm to approximate

a solution of common fixed point problem which involves two pseudocontractive mappings: Let C be a nonempty, convex and closed subset of H and let $T_1, T_2: C \rightarrow C$ be Lipschitz pseudocontractive mappings with constants L_1 and L_2 , respectively. Assume that $\Gamma = F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be defined from arbitrary $x_0, x \in C$ by

$$\begin{cases} d_n = (1 - c_n)x_n + c_n T_2 x_n, \\ e_n = (1 - \beta_n)x_n + \beta_n T_1 x_n, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)(\theta_n x_n + \delta_n T_1 e_n + \gamma_n T_2 d_n), \end{cases} \quad n \geq 0, \quad (2.7)$$

where $\{\theta_n\}, \{\delta_n\}, \{\gamma_n\} \subset (a, b) \subset (0, 1)$ and $\{\alpha_n\} \subset (0, c) \subset (0, 1)$ are real sequences which satisfy the conditions: (i) $\theta_n + \delta_n + \gamma_n = 1$; (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$; (iii) $\delta_n + \gamma_n \leq c_n, \beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$ for all $n \geq 1$, and $L := \max\{L_1, L_2\}$. Then they obtained a strong convergence of $\{x_n\}$ to an element $x^* \in \Gamma$.

In 2021, Zegeye and Wega [191] introduced the following algorithm which converges to an element of the set of common f -fixed points of two f -pseudocontractive mappings in a reflexive real Banach space E : Let $T_i: E \rightarrow E^*$, $i = 1, 2$, be Lipschitz f -pseudocontractive mappings with constants L_i and assume that $\Lambda = F_f(T_1) \cap F_f(T_2) \neq \emptyset$. Let $x_0, x \in E$ be arbitrary. Define the sequence $\{x_n\}$ by

$$\begin{cases} s_n = \nabla f^*((1 - \eta_n)\nabla f x_n + \eta_n T_1 x_n), \\ y_n = \nabla f^*((1 - \eta_n)\nabla f x_n + \eta_n T_2 x_n), \\ u_n = \nabla f^*((1 - \eta_n)\nabla f s_n + \eta_n \nabla f x_n + \eta_n (T_1 s_n - T_1 x_n)), \\ v_n = \nabla f^*((1 - \eta_n)\nabla f y_n + \eta_n \nabla f x_n + \eta_n (T_2 y_n - T_2 x_n)), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f x + (1 - \alpha_n)\nabla f z_n), \end{cases} \quad (2.8)$$

where $z_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n \nabla f u_n + \theta_n \nabla f v_n)$, $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \underline{\eta} \leq \eta_n \leq \bar{\eta} < \frac{\beta}{\beta+L}$, for $L = \max\{L_1, L_2\}$, $\{\beta_n\}, \{\delta_n\}, \{\theta_n\} \subset [c, 1) \subset (0, 1)$ such that $\beta_n + \delta_n + \theta_n = 1$, for all $n \geq 0$ and $f: E \rightarrow (-\infty, +\infty]$ is a strongly coercive, bounded, uniformly Fréchet differentiable, Legendre, and strongly convex function with constant β . Then they obtained a strong convergence result of the sequence $\{x_n\}$ to an element $x^* = P_{\Lambda}^f(x)$.

Motivated by the findings presented above, one of the goals of this thesis is to construct an inertial algorithm which approximates a solution of the split equality fixed point problem involving uniformly continuous pseudocontractive mappings. It is also our aim to construct a method which approximates solutions of the split equality common f, g -fixed point problems involving f, g -pseudocontractive mappings in reflexive Banach spaces.

2.2 Approximating Solutions of Variational Inequality Problems

This section is devoted to collecting some research findings on iterative algorithms for approximating solutions of variational inequality problems.

Many authors have been constructing and studying different iterative algorithms involving mappings with different continuity and monotonicity properties to approximate solutions of the variational inequality problems (1.26).

The simplest method which solves the variational inequality problem (1.26) is the projected gradient method introduced by Goldstein [78] which is given as

$$x_{n+1} = P_C (x_n - \lambda F x_n), \quad (2.9)$$

where C is a nonempty, convex and closed subset of H and $F: C \rightarrow H$ is a mapping. Goldstein proved a strong convergence of the sequence $\{x_n\}$ to a solution of the VIP under L -Lipschitz continuity and γ -strong monotonicity assumptions on F and $\lambda \in \left(0, \frac{2\gamma}{L^2}\right)$. Weak convergence of (2.9) can also be obtained if F is inverse strongly monotone and Lipschitz continuous. We note here that the strong monotonicity condition on the governing mapping is a quite strong condition.

It is shown in [80] that a VIP has unique solution when the underlying mapping is strongly monotone.

In 2015, Malitsky [107] proposed the following method for monotone mappings: Let $u_0, y_0 \in H$ and let $\delta \in \left(0, \frac{\sqrt{2}-1}{L}\right)$. Let $\{u_n\}$ and $\{y_n\}$ be the sequences generated by

$$\begin{cases} u_{n+1} = P_C (u_n - \delta F y_n), \\ y_{n+1} = 2u_{n+1} - u_n, n \geq 0. \end{cases} \quad (2.10)$$

They obtained weak convergence results under the assumption that the mapping $F: H \rightarrow H$ is L -Lipschitz monotone and C is a nonempty, convex and closed subset of H . One observes here that the monotonicity condition in [107] is more relaxed than that of [78].

In 2020, Thong et al. [160] proposed the following projection type algorithm to solve (1.26) in a Hilbert space setting which further relaxes the monotonicity condition on the underlying mapping. Let $l \in (0, 1), \mu > 0, \beta \in \left(0, \frac{1}{\mu}\right)$ and $x_1 \in C$ be given. Then compute

$$\begin{cases} s_n = P_C [x_n - \beta F x_n], \\ w_n = x_n - \gamma_n r_\beta(x_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_{C_n}(x_n), \end{cases} \quad (2.11)$$

where $F: H \rightarrow H$ is a uniformly continuous pseudomonotone mapping that is sequentially weakly continuous on bounded subsets of C ; $f: C \rightarrow C$ is a contraction mapping with a coefficient $\delta \in [0, 1)$; $r_\beta(x_n) = x_n - s_n$; $\{\alpha_n\}$ is a sequence of real numbers in $(0, 1)$ such that

$\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; $\gamma_n = l^{j_n}$, j_n being the smallest non-negative integer j which satisfies

$$\langle Fx_n - F(x_n - l^j r_\beta(x_n)), r_\beta(x_n) \rangle \leq \mu \|r_\beta(x_n)\|^2,$$

$C_n = \{x \in C : h_n(x) \leq 0\}$ and $h_n(x) = \langle Fw_n, x - w_n \rangle$. They proved that the sequence $\{x_n\}$ converges strongly to $x^* = P_{VI(C,F)}f(x^*)$ provided that $VI(C, F) \neq \emptyset$.

Since the monotonicity of mappings has a big impact on the convergence analysis of VIPs, there has been a huge demand to expand the monotonicity property of mappings in solving variational inequality problems (see, [6, 107, 140, 168, 170]) to the bigger class of quasimonotone mappings.

In 2020, Liu and Yang [101] proposed the following algorithm to approximate solutions of VIPs involving quasimonotone mappings in Hilbert spaces: Let C be a nonempty, convex and closed subset of H and let $\lambda_0 > 0$, $x_0 \in H$, $0 < \mu < 1$ and $\{q_n\}$ be a sequence of non-negative real numbers such that $\sum_{n=0}^{\infty} q_n < +\infty$. Let $\{x_n\}$ be the sequence obtained from

$$\begin{cases} w_n = P_C(x_n - \lambda_n Fx_n), \\ x_{n+1} = w_n + \lambda_n(Fx_n - Fw_n), \end{cases} \quad (2.12)$$

where

$$\lambda_{n+1} = \begin{cases} \min \left\{ \lambda_n + q_n, \frac{\mu \|x_n - w_n\|}{\|Fx_n - Fw_n\|} \right\}, & \text{if } Fx_n \neq Fw_n, \\ \lambda_n + q_n, & \text{otherwise,} \end{cases}$$

and F is Lipschitz continuous, quasimonotone and weakly sequentially continuous. They proved, under certain conditions, that the sequence $\{x_n\}$ obtained from (2.12) converges weakly to an element of $VI(C, F)$ provided that $VI(C, F) \neq \emptyset$.

Many authors have also introduced and studied different iterative algorithms for solving the Minty variational inequality problems involving quasimonotone mappings (see, for instance, [87, 169, 170]).

In 2022, Izuchukwu et al. [87] constructed and studied the following algorithm which approximates a solution of the variational inequality problem, where the underlying mapping is quasimonotone: Let H be a real Hilbert space and let $\rho_0, \rho_1 > 0$, $\mu \in \left(\gamma, \frac{1-2\gamma}{2}\right)$, $\gamma \in \left(0, \frac{1}{4}\right)$ and $\alpha \in [0, 1)$. Let $\{\beta_n\}$ be a non-negative sequence with the property $\sum_{n=0}^{\infty} \beta_n < \infty$. Let $x_0, x_1 \in H$ be given starting points and let $\{x_n\}$ be the sequence defined by

$$\begin{cases} t_n = x_n + \alpha(x_n - x_{n-1}), \\ x_{n+1} = P_C(t_n - ((\rho_n + \rho_{n-1})Fx_n - \rho_{n-1}Fx_{n-1})), \end{cases} \quad (2.13)$$

where

$$\rho_{n+1} = \begin{cases} \min \left\{ \beta_n + \rho_n, \frac{\mu \|x_n - x_{n+1}\|}{\|Fx_n - Fx_{n+1}\|} \right\}, & \text{if } Fx_n \neq Fx_{n+1}, \\ \beta_n + \rho_n, & \text{otherwise,} \end{cases}$$

C is a nonempty, convex and closed subset of H and $F: C \rightarrow H$ is Lipschitz continuous and quasimonotone with the property $\|Fx\| \leq \liminf_{n \rightarrow \infty} \|Fx_n\|$, for every sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup x$. They proved, under certain conditions on the control sequences, that the sequence $\{x_n\}$ obtained from (2.13) converges weakly to an element x^* of $MVI(C, F)$ provided that $MVI(C, F) \neq \emptyset$ and $Fx \neq 0$, for all $x \in C$.

Very recently, Wang et al. [169] constructed the following algorithm with Bregman distance approach for solving quasimonotone variational inequalities in real Hilbert spaces: Let C be a nonempty, convex and closed subset of H and let $F: H \rightarrow H$ be Lipschitz continuous and quasimonotone with the property $\|Fx\| \leq \liminf_{n \rightarrow \infty} \|Fx_n\|$, for every sequence $\{x_n\}$ with $x_n \rightharpoonup x$. Let $\theta \in [0, 1)$, $\lambda \in (0, 1]$, $\gamma_1 > 0$, $\mu \in (0, \beta)$. Let $\{\delta_n\}$ and $\{\zeta_n\}$ be non-negative real sequences satisfying certain conditions. Let $x_0, x_1 \in H$ be arbitrary and let $\{x_n\}$ be the sequence generated by

$$\begin{cases} w_n = \nabla f^* (\nabla f x_n + \theta_n (\nabla f x_{n-1} - \nabla f x_n)), \\ z_n = P_C^f \nabla f^* (\nabla f w_n - \gamma_n F w_n), \\ y_n = \nabla f^* (\nabla f z_n - \gamma_n (F z_n - F w_n)), \\ x_{n+1} = \nabla f^* ((1 - \lambda) \nabla f x_n + \lambda \nabla f y_n), \end{cases} \quad (2.14)$$

where $\theta_n \in [0, \bar{\theta}_n]$ with

$$\bar{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\zeta_n}{\|\nabla f x_n - \nabla f x_{n-1}\|} \right\}, & \text{if } \nabla f x_n \neq \nabla f x_{n-1}, \\ \theta, & \text{otherwise,} \end{cases}$$

$$\gamma_{n+1} = \begin{cases} \min \left\{ \gamma_n + \delta_n, \frac{\mu \|w_n - z_n\|}{\|F w_n - F z_n\|} \right\}, & \text{if } F w_n \neq F z_n, \\ \gamma_n + \delta_n, & \text{otherwise,} \end{cases}$$

$f: H \rightarrow \mathbb{R}$ is a Legendre, uniformly Fréchet differentiable and strongly convex function with strong convexity constant $\beta > 0$ which is bounded on bounded subsets of H , ∇f is the gradient of the function f , ∇f^* is the gradient of f^* and P_C^f is the Bregman projection of H onto C . They established a strong convergence result of sequence $\{x_n\}$ to an element of $MVI(C, F)$, provided that $MVI(C, F) \neq \emptyset$ and $Fx \neq 0$ for all $x \in C$.

We note here that the previously mentioned algorithms converge weakly to solutions of the MVIP. However, we find in some applied disciplines that the results obtained from strong convergence are more significant than that of weak convergences [17].

Inspired and driven by the outcomes discussed above, one of our goals in this thesis is to construct a scheme for approximating solutions of the split equality Minty variational inequality problem involving quasimonotone mappings in reflexive real Banach spaces.

2.3 Solutions of Variational Inequality Problems Involving Fixed Points of Some Mappings

Many authors have proposed and studied different methods to approximate solutions of the bilevel variational inequality problem BVIP (see, for instance, [10, 11, 12, 65, 66, 91, 153, 154, 160] and the references therein).

In 2014, Anh et al. [12] constructed and studied the following algorithm for solving the BVIP: Let C be a nonempty, convex and closed subset of H . Let F be η -inverse strongly monotone on H and let G be κ -strongly monotone and L -Lipschitz continuous on C . For arbitrary $x_0 \in C$, compute the sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda F x_n), \\ x_{n+1} = y_n - \mu \alpha_n G y_n, \end{cases} \quad (2.15)$$

where $\lambda \in (0, 2\eta]$, $\mu \in \left(0, \frac{2\kappa}{L^2}\right)$, $\tau = 1 - \sqrt{1 - \mu(2\kappa - \mu L^2)}$, $0 < \alpha_n \leq \min\left\{1, \frac{1}{\tau}\right\}$ with $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \left| \frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n} \right| = 0$. They proved that the sequences $\{x_n\}$ and $\{y_n\}$ generated by (2.15) converge strongly to an element of the set of solutions of BVIP (1.29).

In order to extend monotonicity property of the underlying mappings in BVIP, authors (see, for instance, [152]) have studied bilevel variational inequality problems which involve pseudomonotone mappings in their lower level problems.

In 2020, Thong and Hieu [159] proposed and studied the following algorithm for solving pseudomonotone BVIP in Hilbert spaces: Let C be a nonempty, convex and closed subset of H and let $F: H \rightarrow H$ be pseudomonotone, L_1 -Lipschitz continuous on H and sequentially weakly continuous on C . Let $G: H \rightarrow H$ be κ -strongly monotone and L_2 -Lipschitz continuous on H . For arbitrary $x_0 \in H$, compute the sequence $\{x_n\}$ as

$$\begin{cases} u_n = P_C(x_n - \lambda F x_n), \\ w_n = P_{T_n}(x_n - \lambda \delta \eta_n F u_n), \\ x_{n+1} = w_n - \gamma \alpha_n G w_n, \end{cases} \quad (2.16)$$

where $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lambda \in \left(0, \frac{1}{L_1}\right)$, $\delta \in (0, 2)$, $\gamma \in \left(0, \frac{2\kappa}{L_2^2}\right)$, $T_n = \{x \in H : \langle x_n - \lambda F x_n - u_n, x - u_n \rangle \leq 0\}$, $d_n = x_n - u_n - \lambda(F x_n - F u_n)$, and

$$\eta_n = \begin{cases} \frac{\langle x_n - u_n, d_n \rangle}{d_n^2}, & \text{if } d_n \neq 0, \\ 0, & \text{if } d_n = 0. \end{cases}$$

They proved that $\{x_n\}$ converges strongly to the unique solution of (1.29).

The method in (2.16) involves a projection on to the half-space T_n which is costly in computations.

In 2022, Liu and Yao [102] introduced and studied the following single projection inertial algorithm to solve the bilevel variational inequality problem in a real Hilbert space H which does not require projections on to half spaces unlike that of Thong and Hieu [159]: Let $x_0, x_1 \in H$ be arbitrarily chosen and let $\{\delta_n\} \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} \delta_n = 0$, $\sum_{n=0}^{\infty} \delta_n = \infty$. Let $0 < a \leq \lambda_n \leq b < \frac{1}{L_1}$ for some $a, b \in (0, 1)$. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}); \\ v_n = P_C(u_n - \lambda_n F u_n), \\ x_{n+1} = (I - \beta \delta_n G)((I - \lambda_n F)v_n + \lambda_n F u_n), \end{cases} \quad (2.17)$$

where C is a nonempty, convex and closed subset of H , the mapping $F: H \rightarrow H$ is L_1 -Lipschitz continuous, pseudomonotone and weakly sequentially continuous, $G: H \rightarrow H$ is L_2 -Lipschitz κ -strongly monotone, and

$$\alpha_n = \begin{cases} \Upsilon_n \|x_n - x_{n-1}\|, & \text{when } \|x_n - x_{n-1}\| \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where Υ_n is a sequence with the property $\frac{\Upsilon_n}{\delta_n} \rightarrow 0$ as $n \rightarrow \infty$, and $\beta \in \left(0, \frac{2\kappa}{L_2^2}\right)$. They proved that the sequence $\{x_n\}$ generated by (2.17) converges strongly to the unique solution of (1.29).

Many authors have also been working on the approximation of solutions of the bilevel variational inequality problems with fixed point constraints (1.30) (see, for instance, [77, 79]).

In 2021, Hai et al. [79] proposed the following algorithm for approximating solutions of (1.30). Let $\{\omega_n\}$, $\{\lambda_n\}$, and $\{\alpha_n\}$ be positive sequences satisfying some appropriate conditions and let $x_0 \in H_1$. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} u_n = A(x_n) \text{ and } y_n = x_n + \omega_n A^*(S(u_n) - u_n), \\ z_n = P_C(y_n - \lambda_n G(y_n)) \text{ and } t_n = P_{T_n}(y_n - \lambda_n G(z_n)), \\ x_{n+1} = (I - \alpha_n F)t_n, \end{cases} \quad (2.18)$$

where $T_n = \{\omega \in H_1 : \langle y_n - \lambda_n G(y_n) - z_n, \omega - z_n \rangle \leq 0\}$, $G: H_1 \rightarrow H_1$ is Lipschitz pseudomonotone mapping, $F: H_1 \rightarrow H_1$ is Lipschitz strongly monotone, $S: H_2 \rightarrow H_2$ is demicontractive $I - S$ demiclosed at 0 and $A: H_1 \rightarrow H_2$ is a bounded linear mapping. They obtained a strong convergence result under the assumption that $\limsup_{n \rightarrow \infty} \langle Gx_n, y - y_n \rangle \leq \langle Gx^*, y - y^* \rangle$ for any sequences $\{x_n\}$, $\{y_n\}$ in H_1 which converge weakly to x^* and y^* , respectively.

Inspired by the results presented above, one of the goals of this thesis is to construct a scheme for approximating solutions of (1.30) which involves a non-Lipschitz pseudomonotone mapping in its lower level variational inequality problem and a general demimetric mapping in the fixed

point constraint.

On the other hand, many authors have proposed and studied different algorithms which approximate common points of the solution sets of variational inequality and fixed point problems (see, for instance, [5, 146, 150, 163, 172]). This method became very important in optimization theory because it is applicable in different problems whose constraints can be modeled as both problems. This arises in areas like signal processing, image recovery, and network resource allocation. An instance of this is in network bandwidth allocation problem for two services in a heterogeneous wireless access networks in which the bandwidth of the services is mathematically related (see, for instance, [84, 105, 106]).

In 2003, Takahashi and Toyoda [150] introduced an iterative method which approximates a common solution of fixed point and variational inequality problems in a Hilbert space, where the underlying mapping in the fixed point problem is nonexpansive and that of the variational inequality problem is inverse strongly monotone mapping. Given a nonempty, convex and closed subset C of H . Let $\{x_n\}$ be defined from arbitrary $x_0 \in C$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) GP_D(x_n - \eta_n F x_n), \quad (2.19)$$

where n is a non-negative integer, $F: C \rightarrow H$ is β -inverse strongly monotone mapping, $G: C \rightarrow H$ is a nonexpansive mapping, $\{\alpha_n\} \subset (0, 1)$ and $\{\eta_n\} \subset (0, 2\beta)$. They proved that if $VI(C, F) \cap F(G) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (2.19) converges weakly to an element $p \in VI(C, F) \cap F(G)$.

The result obtained in Takahashi and Toyoda [150] is a weak convergence result and the setting they considered was confined to Hilbert spaces. Besides, the conditions on the mappings are quite strong conditions. In 2021, Wega and Zegeye [172] introduced and studied an iterative algorithm which approximates a common element of the set of g -fixed points of a Bregman relatively g -nonexpansive mapping and the solution set of a variational inequality problem in a reflexive real Banach space. In fact, let E be a reflexive real Banach space and let $g: E \rightarrow (-\infty, \infty]$ be a strongly coercive, bounded, λ -strongly convex on bounded subsets of E , uniformly Fréchet differentiable Legendre function. Let C be a nonempty, convex and closed subset of E . Let $F: C \rightarrow E^*$ be a Lipschitz monotone mapping with Lipschitz constant L . Let $G: C \rightarrow E^*$ be a Bregman relatively g -nonexpansive mapping with $\Gamma = VI(C, F) \cap F_g(G) \neq \emptyset$. Given $x_0, x \in C$, let the sequence $\{x_n\}$ be generated by

$$\begin{cases} s_n = P_C^g \nabla g^* (\nabla g x_n - \eta_n F x_n), \\ t_n = \nabla g^* [\xi_n \nabla g x_n + \beta_n G x_n + \tau_n \nabla g w_n], \\ x_{n+1} = P_C^g \nabla g^* (\alpha_n \nabla g(x) + (1 - \alpha_n) \nabla g t_n), \end{cases} \quad (2.20)$$

where $P_C^g(x)$ is the Bregman projection of $x \in \text{int}(\text{Dom } g)$ onto C , $w_n = P_C^g \nabla g^* (\nabla g x_n - \eta_n F s_n)$, $0 < \underline{\eta} \leq \eta_n \leq \bar{\eta} < \frac{\lambda}{L}$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\xi_n\}, \{\beta_n\}, \{\tau_n\} \subset [\delta, 1) \subset (0, 1)$ such that $\xi_n + \beta_n + \tau_n = 1$. They established a strong convergence result of $\{x_n\}$ to some element $\bar{x}$, where $\bar{x} = P_\Gamma^g(x)$.

Many authors have also introduced different iterative algorithms for solving the split equality variational inequality problem.

In 2021, Kwelegano et al. [99] introduced the following algorithm in real Hilbert spaces for pseudomonotone uniformly continuous mappings $T : H_1 \rightarrow H_1$ and $S : H_2 \rightarrow H_2$ which are weakly sequentially continuous: Let $\ell \in (0, 1)$, $\mu > 0$, $\lambda \in (0, \frac{1}{\mu})$ and let $\{(x_n, u_n)\}$ be the sequence generated from arbitrary $(x_0, u_0) \in C \times D$ by

$$\left\{ \begin{array}{l} z_n = P_C(x_n - \lambda T x_n), \\ y_n = x_n - \tau_n(x_n - z_n), \\ v_n = P_D(u_n - \lambda S u_n), \\ s_n = u_n - \delta_n(u_n - v_n), \\ d_n = P_C[x_n - \gamma_n A^*(A x_n - B v_n)], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[a_n P_{C_n} x_n + (1 - a_n) d_n], \\ t_n = P_D[u_n - \gamma_n B^*(B u_n - A x_n)], \\ u_{n+1} = \alpha_n g(u_n) + (1 - \alpha_n)[a_n P_{D_n} u_n + (1 - a_n) t_n], \end{array} \right. \quad (2.21)$$

where

$$C_n = \{x \in C : \langle T y_n, x - y_n \rangle \leq 0\}, \quad D_n = \{u \in D : \langle S s_n, v - s_n \rangle \leq 0\},$$

$\tau_n = \ell^{j_n}$ and $\delta_n = \ell^{k_n}$ and j_n and k_n are the smallest non-negative integers j and k , respectively such that

$$\begin{aligned} \langle T x_n - T[x_n - \ell^j(x_n - z_n)], x_n - z_n \rangle &\leq \mu \|x_n - z_n\|^2, \\ \langle S u_n - S[u_n - \ell^k(u_n - v_n)], u_n - v_n \rangle &\leq \mu \|u_n - v_n\|^2, \end{aligned} \quad (2.22)$$

where $\{\gamma_n\}, \{\alpha_n\}, \{a_n\}$ are positive sequences satisfying some conditions. Under suitable assumptions, they proved that the resulting sequence converges strongly to a solution of the SEVIP (1.31).

In 2021, Boikanyo and Zegeye [23] introduced an iterative algorithm for solving SEVIP (1.31) for uniformly continuous and pseudomonotone in uniformly smooth and uniformly convex Banach spaces.

The result obtained in Boikanyo and Zegeye [23] improved that of Kwelegano et al. [99] in the sense that it extended the settings from Hilbert spaces to uniformly convex and uniformly smooth Banach space. However, whether the result can be extended to more general reflexive Banach spaces still needs further research.

Motivated and inspired by the works of Boikanyo and Zegeye [23] and Wega and Zegeye [172], one of our aims in this thesis is to introduce the split equality of variational inequality and f, g -fixed point problems and construct a strongly convergent algorithm involving mappings more general than Lipschitz continuous in the variational inequality part and Bregman relatively f, g -nonexpansive mappings for the f, g -fixed point part of the problem.

2.4 Approximating Solutions of Hammerstein Type Equation Problems

In this section, we collect some findings for approximating solutions of Hammerstein type equation problems both in Hilbert and Banach space settings.

Many authors have studied and proved several existence and uniqueness results for Hammerstein type equation problem (1.33), (see, for instance, [26, 27, 28, 29, 69]). Generally, as Hammerstein type equations involve nonlinear mappings, there is no closed way method to solve such type of equations. Thus, several authors have constructed and studied different approximation methods for solving Hammerstein type equations (see, for instance, [46, 48, 52, 54, 56, 67, 70, 76, 119, 165, 166, 186]). Chidume and Zegeye [48, 54, 56] were the first to construct and study iterative algorithms which approximate solutions of the Hammerstein type equation problem (1.33).

In 2005, Chidume and Zegeye [56] introduced the following iterative algorithm in real Hilbert spaces. Let $F: D(F) \rightarrow H$ and $K: D(K) \rightarrow H$ be bounded monotone mappings with $R(F) \subset D(K)$, where $R(F)$ and $D(K)$ are closed convex subsets of H satisfying certain conditions. Let $\{u_n\}$ and $\{v_n\}$ be sequences generated from arbitrary elements $u_0 \in D(F)$ and $v_0 \in D(K)$, respectively, by

$$\begin{cases} u_{n+1} = P_{D(F)} [u_n - \gamma_n (Fu_n - v_n + \theta_n(u_n - w_1))], \\ v_{n+1} = P_{D(K)} [v_n - \gamma_n (Kv_n + u_n + \theta_n(v_n - w_2))], n \geq 0, \end{cases} \quad (2.23)$$

where (w_1, w_2) is an arbitrary fixed element of $D(F) \times D(K)$ and $\gamma_n, \theta_n \subset (0, 1)$. They proved, under some conditions, that there exists $\gamma_0 > 0$ such that if $\gamma_n \leq \gamma_0$ and $\frac{\gamma_n}{\theta_n} \leq \gamma_0^2$ for all $n \geq 0$, then the sequence $\{(u_n, v_n)\}$ converges strongly to $\{(u^*, v^*)\}$, where u^* is a solution of (1.33) with $v^* = Fu^*$.

In 2015, Tufa et al. [165] pointed out that the convergence in the work of Chidume and Zegeye [56] depends on the existence of a constant γ_0 , though it is not clear how to choose such γ_0 during implementation. They also constructed and studied an iterative algorithm which approximates a solution of the general Hammerstein type equation, $u + KF u = 0$, where K and F are Lipschitz monotone mappings. Though, the convergence of their method does not require the existence of a constant γ_0 , it only holds true in the Hilbert space settings.

In 2016, Uba et al. [166] proposed the following algorithm for solving (1.33) in a Banach space setting: Let E be a uniformly convex and uniformly smooth real Banach space and let $F: E \rightarrow E^*$, $K: E^* \rightarrow E$ be maximal monotone and bounded mappings. For $u_1 \in E$, $v_1 \in E^*$ define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^* , respectively, by

$$\begin{cases} u_{n+1} = J^{-1} [Ju_n - \lambda_n (Fu_n - v_n) - \lambda_n \theta_n (Ju_n - Ju_1)], \\ v_{n+1} = J_*^{-1} [J_* v_n - \lambda_n (Kv_n + u_n) - \lambda_n \theta_n (J_* v_n - J_* v_1)], \end{cases} \quad (2.24)$$

where $\{\lambda_n\}, \{\theta_n\} \subset (0, 1)$ satisfy the relation $\lambda_n \leq \gamma_0 \theta_n$, where $\gamma_0 > 0$. Under the assumption that the equation $u + KF u = 0$ has a solution, they proved that the sequences $\{u_n\}$ and $\{v_n\}$

converge strongly to u^* and v^* , respectively, where u^* is the solution of (1.33) with $v^* = Fu^*$.

In 2013, Zegeye and Malonza [183] introduced the following iterative algorithm in a uniformly convex and uniformly smooth Banach space: Let $F: E \rightarrow E^*$, $K: E^* \rightarrow E$ be continuous, bounded and monotone mappings. Define $A: X \rightarrow X^*$ by $A(u, v) = (Fu - v, Kv + u)$, where $X = E \times E^*$. For $r > 0$, denote

$$T_r x = \left\{ z \in X : \langle Az, y - z \rangle + \frac{1}{r} \langle Jz - Jx, y - z \rangle \geq 0, \text{ for all } y \in X \right\}, \quad (2.25)$$

for all $x \in X$, where $J = J_X$ is the normalized duality mapping on X . Define the sequence $\{x_n\}$ by

$$\begin{cases} y_0 = w \in X, \text{ chosen arbitrarily,} \\ u_n = T_{r_n} y_n, \\ y_{n+1} = J^{-1} [\beta_n Jw + (1 - \beta_n) Ju_n], \end{cases} \quad (2.26)$$

where $\{\beta_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$, $\{r_n\} \subset [c_1, +\infty)$ for some $c_1 > 0$, for all $n \in \mathbb{N}$. They established a strong convergence of the sequence $\{y_n\}$ to some point $y^* = (u^*, v^*)$, where u^* is a solution of (1.33) with $v^* = Fu^*$.

There are also several studies carried out for the approximation of solutions of Hammerstein type equation $u + KF u = 0$, where K and F are accretive mappings in Banach spaces (see, for example, [45, 48, 52, 53]).

The aforementioned results are valid only in Hilbert spaces or uniformly smooth and uniformly convex Banach spaces. Besides, the convergence processes of some of the methods depend on the existence of a constant γ_0 and the conditions on the underlying mappings are very strong.

In 2021, Tufa et al. [164] introduced the following iterative algorithm in a reflexive real Banach space E : Given the Lipschitz monotone mappings $F: E \rightarrow E^*$ and $K: E^* \rightarrow E$ with constants L_1 and L_2 , respectively, and let $L = \sqrt{2} \max \{ \sqrt{L_1^2 + 1}, \sqrt{L_2^2 + 1} \}$. Let $g: E \rightarrow (-\infty, +\infty]$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function on bounded subsets of E . Let g and g^* be strongly convex with constants μ_1 and μ_2 , respectively, and let $\mu = \min \{ \mu_1, \mu_2 \}$. Let the sequence $\{(u_n, v_n)\} \subset E \times E^*$ be generated from arbitrary points $(u_0, v_0), (\bar{u}, \bar{v}) \in E \times E^*$ by

$$\begin{aligned} u_{n+1} &= \nabla g^* [\alpha_n \nabla g \bar{u} + (1 - \alpha_n) (\nabla g u_n - \sigma_n s_n)], \\ v_{n+1} &= \nabla g [\alpha_n \nabla g^* \bar{v} + (1 - \alpha_n) (\nabla g^* v_n - \sigma_n t_n)], \end{aligned} \quad (2.27)$$

where

$$\begin{aligned} s_n &= F [\nabla g^* (\nabla g u_n - \sigma_n (Fu_n - v_n))] - \nabla g [\nabla g^* v_n - \sigma_n (Kv_n + u_n)], \\ t_n &= K [\nabla g (\nabla g^* v_n - \sigma_n (Kv_n + u_n))] - \nabla g^* [\nabla g u_n - \sigma_n (Fu_n - v_n)], \end{aligned}$$

and $\{\sigma_n\}, \{\alpha_n\}$ are sequences satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sigma_n \subseteq [a, b] \subset \left(0, \frac{\mu}{L}\right)$.

They proved that the sequences $\{u_n\}$ and $\{v_n\}$ generated by (2.27) converge strongly to u^* and v^* , respectively, where u^* is a solution of (1.33) with $v^* = Fu^*$.

Motivated by the results discussed above, one of the goals of this thesis is to construct and study inertial iterative algorithms for solving the split equality Hammerstein type equation problem (1.34) in reflexive real Banach spaces involving mappings with general continuity property than Lipschitz continuity.



3. Approximating Solutions of Split Equality Fixed Point Problems

Three sections make up this chapter. The first section deals with some preliminary lemmas that will be used in the convergence analysis of the subsequent sections. In section two, we introduce an iterative algorithm for solving split equality fixed point problems with non-Lipschitz pseudocontractive self mappings in the setting of Hilbert spaces and discuss its convergence analysis. In section three, we construct an algorithm for solving split equality common f, g -fixed point problems involving uniformly continuous f, g -pseudocontractive mappings in reflexive real Banach spaces and we discuss its convergence analysis.

3.1 Preliminaries

Lemma 3.1.1. [144] *Let α and γ be any real numbers and $x, y \in H$. Then $\|\alpha x + \gamma y\|^2 = \alpha(\alpha + \gamma)\|x\|^2 + \gamma(\alpha + \gamma)\|y\|^2 - \alpha\gamma\|x - y\|^2$.*

Lemma 3.1.2. [62] *Let C be a nonempty, convex and closed subset of H . Then the following inequalities hold for all $u \in H$ and $\alpha \geq \beta > 0$:*

$$\left\| \frac{u - P_C(u - \alpha Au)}{\alpha} \right\| \leq \left\| \frac{u - P_C(u - \beta Au)}{\beta} \right\| \text{ and } \|u - P_C(u - \beta Au)\| \leq \|u - P_C(u - \alpha Au)\|.$$

Lemma 3.1.3. [22] *If H_1 and H_2 are real Hilbert spaces, then $H = H_1 \times H_2$ is also a real Hilbert space with inner product*

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \text{ for all } (x_1, y_1), (x_2, y_2) \in H = H_1 \times H_2.$$

Thus, it can be easily shown from the definition of weak convergence that

$$(x_n, y_n) \rightharpoonup (x, y) \text{ implies } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y. \quad (3.1)$$

Lemma 3.1.4. [22] *Let H_1 and H_2 be real Hilbert spaces, $H = H_1 \times H_2$ and C be a nonempty, convex and closed subset of H . If $(u, v) \in H$ and $(u^*, v^*) = P_C(u, v)$, then*

$$\langle (u, v) - (u^*, v^*), (x, y) - (u^*, v^*) \rangle \leq 0, \text{ for all } (x, y) \in C.$$

Lemma 3.1.5. [2] *Let E_1 and E_2 be reflexive real Banach spaces. Then $E = E_1 \times E_2$ is also a reflexive Banach space with dual $E^* = E_1^* \times E_2^*$ and duality pairing*

$$\langle (x_2, y_2), (x_1, y_1) \rangle = \langle x_2, x_1 \rangle + \langle y_2, y_1 \rangle,$$

for all $(x_1, y_1) \in E$, $(x_2, y_2) \in E^*$, and $(x_n, y_n) \rightarrow (x, y)$ implies $x_n \rightarrow x$ and $y_n \rightarrow y$.

Lemma 3.1.6. [176] Let $\{b_n\}$ be a sequence of non-negative real numbers such that

$$b_{n+1} \leq (1 - \alpha_n) b_n + \alpha_n d_n,$$

where $\{\alpha_n\} \subset (0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ is a sequence of real numbers with $\limsup_{n \rightarrow \infty} d_n \leq 0$. Then $\lim_{n \rightarrow \infty} b_n = 0$.

Lemma 3.1.7. [104] Let $\{c_n\}$ be a sequence of non-negative real numbers. If $\{c_{n_i}\}$ is a subsequence of $\{c_n\}$ such that $c_{n_i} < c_{n_{i+1}}$ for all $i \in \mathbb{N}$, then there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:

$$c_{m_k} \leq c_{m_k+1} \quad \text{and} \quad c_k \leq c_{m_k+1}.$$

In fact, $m_k = \max\{n \leq k : c_n < c_{n+1}\}$.

Lemma 3.1.8. [43] Let J_E be the normalized duality mapping on E . Then for any $x, y \in E$, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle j(x + y), y \rangle, \text{ for all } j(x + y) \in J_E(x + y).$$

Lemma 3.1.9. [127] Let $f: E \rightarrow (-\infty, +\infty]$ be a proper, lower semi-continuous, convex and Gâteaux differentiable function. Then $f^*: E^* \rightarrow (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Moreover, for all $x \in E$, we have

$$D_f \left(x, \nabla f^* \left(\sum_{i=1}^N \alpha_i \nabla f(w_i) \right) \right) \leq \sum_{i=1}^N \alpha_i D_f(x, w_i),$$

where $\{w_i\} \subseteq E$ and $\{\alpha_i\} \subseteq (0, 1)$ with $\sum_{i=1}^N \alpha_i = 1$.

Lemma 3.1.10. [171] Let f be a strongly convex function with convexity constant β . Then

$$D_f(y, x) \geq \frac{\beta}{2} \|x - y\|^2,$$

for all $x \in \text{int}(\text{Dom } f)$ and $y \in \text{Dom } f$.

Lemma 3.1.11. [109] Let $f: E \rightarrow \mathbb{R}$ be a totally convex function. If the sequence of real numbers $\{D_f(x_n, x_0)\}$ is bounded for any $x_0 \in E$, then $\{x_n\}$ is bounded.

Lemma 3.1.12. [132] Let $f: E \rightarrow \mathbb{R}$ be a uniformly Fréchet differentiable function which is bounded on bounded subsets of E . Then ∇f is norm-to-norm uniformly continuous on bounded subsets of E and hence both f and ∇f are bounded on bounded subsets of E .

Lemma 3.1.13. [121] Let $f: E \rightarrow (-\infty, +\infty]$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E . Let the sequences $\{x_n\}$ and $\{y_n\}$ be bounded in E . Then $\lim_{n \rightarrow \infty} D_f(x_n, y_n) = 0$ if and only if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 3.1.14. *Let $f: E \rightarrow (-\infty, +\infty]$ be a convex and lower semicontinuous function. Then the following are equivalent (see, [180]):*

- (i) f is uniformly convex;
- (ii) for all $(x, x^*), (y, y^*) \in \text{Gph}(\partial f)$, $f(x) + \langle x^*, y - x \rangle + \psi(\|x - y\|) \leq f(y)$, where ψ is the modulus of uniform convexity of f ;
- (iii) f^* is Fréchet differentiable, ∇f^* is uniformly continuous and $\text{dom } f^* = E^*$.

In Hilbert spaces, the Lyapunov function introduced by Alber [7] takes the form

$$\phi(y, x) = \|y\|^2 + \|x\|^2 - 2\langle y, x \rangle, \text{ for all } x, y \in H, \quad (3.2)$$

and the following identities readily follow:

- (i) *Three point identity:*

$$\phi(x, y) + \phi(y, w) - \phi(x, w) = 2\langle y - w, y - x \rangle, \text{ for all } x, y, w \in H. \quad (3.3)$$

- (ii) *Four point identity:*

$$\phi(y, z) + \phi(x, w) - \phi(y, w) - \phi(x, z) = 2\langle w - z, y - x \rangle, \text{ for all } x, y, w, z \in H. \quad (3.4)$$

Moreover, the following relation holds (see, [96]):

$$\phi(x, x^*) + 2\langle y^*, x^* - x \rangle \leq \phi(x, x^* + y^*), \text{ for all } x, x^*, y^* \in H. \quad (3.5)$$

We also have that

$$\phi(y, P_C x) + \phi(P_C x, x) \leq \phi(y, x), \text{ for all } x \in H, y \in C. \quad (3.6)$$

The following lemmas are special cases of Lemma 3.1.9, Lemma 3.1.10, Lemma 3.1.11 and Lemma 3.1.13 in the order they appear.

Lemma 3.1.15. [127] *Let $\{x_i\} \subseteq H$ and $\{\alpha_i\} \subseteq (0, 1)$ be such that $\sum_{i=1}^N \alpha_i = 1$. Then*

$$\phi\left(x, \sum_{i=1}^N \alpha_i x_i\right) \leq \sum_{i=1}^N \alpha_i \phi(x, x_i), \text{ for all } x \in H.$$

Lemma 3.1.16. [171] *Let H be a real Hilbert space. Then $\phi(y, x) \geq \frac{1}{2}\|x - y\|^2$, for all $x, y \in H$.*

Lemma 3.1.17. [109] *If the sequence $\{\phi(x_n, x_0)\}$ is bounded for any $x_0 \in H$, then $\{x_n\}$ is bounded.*

Lemma 3.1.18. [121] *Let the sequences $\{x_n\}$ and $\{y_n\}$ be bounded in H . Then $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$ if and only if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

3.2 An Inertial Method for Solving Split Equality Fixed Point Problems for Non-Lipschitz Pseudocontractive Mappings in Real Hilbert Spaces

This section is devoted to construction of an inertial algorithm involving uniformly continuous pseudocontractive mappings for solving SEFPP in real Hilbert spaces.

3.2.1 Main Results

We will make use of the following conditions in the convergence analysis of our main result.

- (C1) Let C and D be nonempty, convex and closed subsets of the real Hilbert spaces H_1 and H_2 , respectively;
- (C2) Let $T_1: C \rightarrow C$ and $T_2: D \rightarrow D$ be uniformly continuous pseudocontractive self mappings;
- (C3) Let $B_1: H_1 \rightarrow H_3$ and $B_2: H_2 \rightarrow H_3$ be bounded linear mappings with adjoints B_1^* and B_2^* , respectively, where H_3 is a real Hilbert space;
- (C4) Let $\Omega = \{(\bar{x}, \bar{y}) \in F(T_1) \times F(T_2) : B_1\bar{x} = B_2\bar{y}\} \neq \emptyset$;
- (C5) Let $\{\alpha_n\} \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C6) Let $\{\zeta_n\}$ be a sequence of positive numbers such that $\zeta_n \in \left(0, \frac{1}{4}\right)$ for all $n \geq 0$, and $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$.

We now state our proposed algorithm and discuss its convergence analysis.

Algorithm 3.1.

Initialization: Let $x_0, x_1 \in H_1$; $y_0, y_1 \in H_2$; $\mu, \sigma \in \left(0, \frac{1}{2}\right)$, $l, \gamma \in (0, 1)$ and $\theta \in [0, 1)$. For arbitrary points $u \in H_1$ and $v \in H_2$, calculate $\{x_n\}$ and $\{y_n\}$ as follows:

Step 1: Given $x_{n-1}, x_n \in H_1$ and $y_{n-1}, y_n \in H_2$, choose θ_n such that $0 \leq \theta_n \leq \sigma_n$, where

$$\sigma_n = \begin{cases} \min \left\{ \theta, \frac{\zeta_n}{\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|} \right\}, & \text{if } \|x_n - x_{n-1}\| + \|y_n - y_{n-1}\| \neq 0, \\ \theta, & \text{otherwise.} \end{cases} \quad (3.7)$$

Step 2: Compute

$$\begin{aligned} a_n &= P_C(x_n + \theta_n(x_n - x_{n-1})), \\ b_n &= P_D(y_n + \theta_n(y_n - y_{n-1})). \end{aligned} \quad (3.8)$$

Step 3: Compute

$$\begin{aligned} c_n &= P_C(a_n - \gamma_n B_1^*(B_1 a_n - B_2 b_n)), \\ d_n &= P_D(b_n - \gamma_n B_2^*(B_2 b_n - B_1 a_n)), \end{aligned} \quad (3.9)$$

where $0 < \rho \leq \gamma_n \leq \rho_n$ with

$$\rho_n = \min \left\{ \rho + 1, \frac{\|B_1 a_n - B_2 b_n\|^2}{2[\|B_1^*(B_1 a_n - B_2 b_n)\|^2 + \|B_2^*(B_2 b_n - B_1 a_n)\|^2]} \right\}, \quad (3.10)$$

for $n \in \Upsilon = \{m \in \mathbb{N} : B_1 a_m - B_2 b_m \neq 0\}$, otherwise $\gamma_n = \rho$, for some $\rho > 0$.

Step 4: Compute

$$e_n = c_n - \lambda_n(I_1 - T_1)c_n, \quad (3.11)$$

$$h_n = d_n - \eta_n(I_2 - T_2)d_n,$$

where $\lambda_n = \gamma l^{j_n}$ and j_n is the smallest non-negative integer j satisfying the relation

$$\|(I_1 - T_1)(c_n - \gamma l^j(I_1 - T_1)c_n) - (I_1 - T_1)c_n\| \leq \mu \|(I_1 - T_1)c_n\|, \quad (3.12)$$

and $\eta_n = \gamma l^{m_n}$, where m_n is the smallest non-negative integer m satisfying the relation

$$\|(I_2 - T_2)(d_n - \gamma l^m(I_2 - T_2)d_n) - (I_2 - T_2)d_n\| \leq \sigma \|(I_2 - T_2)d_n\|. \quad (3.13)$$

Step 5: Compute

$$p_n = e_n - \lambda_n((I_1 - T_1)e_n - (I_1 - T_1)c_n), \quad (3.14)$$

$$q_n = h_n - \eta_n((I_2 - T_2)h_n - (I_2 - T_2)d_n).$$

Step 6: Compute

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)p_n, \quad (3.15)$$

$$y_{n+1} = \alpha_n v + (1 - \alpha_n)q_n.$$

Set $n = n + 1$ and go to **Step 1**.

Remark 3.2.1. The novelty of our method over the existing ones is that it extends the nature of the underlying mappings to the more general class of uniformly continuous mappings while the results in the literature are confined to Lipschitz continuous mappings.

Lemma 3.2.1. Assume that conditions (C1) – (C6) hold. Then the Armijo line-search rules (3.12) and (3.13) in Algorithm 3.1 are well defined.

Proof. If c_n is a fixed point of T_1 , then obviously $j = 0$ satisfies the relation (3.12). Assume now that c_n is not a fixed point of T_1 , for all $n \geq 1$. Then $\|(I_1 - T_1)c_n\| > 0$.

Since $l \in (0, 1)$ and T_1 is continuous on C , it follows that

$$\lim_{j \rightarrow \infty} \|(I_1 - T_1)(c_n - \gamma l^j(I_1 - T_1)c_n) - (I_1 - T_1)c_n\| = 0. \quad (3.16)$$

On the other hand, we have $\|(I_1 - T_1)c_n\| > 0$. This together with (3.16) guarantees the existence of a non-negative integer j which satisfies (3.12). Similarly, we can find a non-negative integer m which satisfies the relation (3.13) and hence the proof is complete. $\square$

Theorem 3.2.1. *Assume that conditions (C1) – (C6) hold. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 3.1 are bounded.*

Proof. Let $(\hat{x}, \hat{y}) \in \Omega$. From (3.15) and Lemma 3.1.15, we obtain

$$\begin{aligned}\phi(\hat{x}, x_{n+1}) &= \phi(\hat{x}, \alpha_n u + (1 - \alpha_n)p_n) \\ &\leq \alpha_n \phi(\hat{x}, u) + (1 - \alpha_n) \phi(\hat{x}, p_n).\end{aligned}\tag{3.17}$$

From (3.14) and (3.2), we get

$$\begin{aligned}\phi(\hat{x}, p_n) &= \phi(\hat{x}, e_n - \lambda_n((I_1 - T_1)e_n - (I_1 - T_1)c_n)) \\ &= \|\hat{x}\|^2 - 2\langle e_n - \lambda_n((I_1 - T_1)e_n - (I_1 - T_1)c_n), \hat{x} \rangle + \|p_n\|^2 \\ &= \|\hat{x}\|^2 + \|p_n\|^2 - 2\langle e_n, \hat{x} \rangle + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} \rangle \\ &= \|\hat{x}\|^2 + \|e_n\|^2 - \|e_n\|^2 - 2\langle e_n, \hat{x} \rangle + \|p_n\|^2 + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} \rangle \\ &= \phi(\hat{x}, e_n) - \|e_n\|^2 + \|p_n\|^2 + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} \rangle \\ &= \phi(\hat{x}, e_n) - \|e_n\|^2 - \|p_n\|^2 + 2\|p_n\|^2 - 2\langle e_n, p_n \rangle + 2\langle e_n, p_n \rangle \\ &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} \rangle \\ &= \phi(\hat{x}, e_n) - \phi(p_n, e_n) + 2\|p_n\|^2 - 2\langle e_n, p_n \rangle + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} \rangle \\ &= \phi(\hat{x}, e_n) - \phi(p_n, e_n) + 2\langle p_n, p_n \rangle - 2\langle e_n, p_n \rangle + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} \rangle \\ &= \phi(\hat{x}, e_n) - \phi(p_n, e_n) + 2\langle p_n, p_n - e_n \rangle + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} \rangle \\ &= \phi(\hat{x}, e_n) - \phi(p_n, e_n) - 2\langle p_n, \lambda_n[(I_1 - T_1)e_n - (I_1 - T_1)c_n] \rangle \\ &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} \rangle \\ &= \phi(\hat{x}, e_n) - \phi(p_n, e_n) + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} - p_n \rangle.\end{aligned}\tag{3.18}$$

By (3.4), we have

$$\phi(\hat{x}, e_n) - \phi(p_n, e_n) = \phi(\hat{x}, c_n) - \phi(p_n, c_n) + 2\langle e_n - c_n, p_n - \hat{x} \rangle.\tag{3.19}$$

Substituting (3.19) into (3.18), we obtain

$$\phi(\hat{x}, p_n) = \phi(\hat{x}, c_n) - \phi(p_n, c_n) + 2\langle e_n - c_n, p_n - \hat{x} \rangle + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} - p_n \rangle.\tag{3.20}$$

By (3.3), we have

$$\phi(p_n, c_n) = \phi(p_n, e_n) + \phi(e_n, c_n) - 2\langle e_n - c_n, e_n - p_n \rangle.\tag{3.21}$$

Substituting (3.21) into (3.20), we obtain

$$\begin{aligned}\phi(\hat{x}, p_n) &= \phi(\hat{x}, c_n) - \phi(p_n, e_n) - \phi(e_n, c_n) + 2\langle e_n - c_n, e_n - p_n \rangle \\ &\quad + 2\langle e_n - c_n, p_n - \hat{x} \rangle + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} - p_n \rangle.\end{aligned}\tag{3.22}$$

From (3.9), (3.5) and (3.6), we have

$$\begin{aligned}
 \phi(\hat{x}, c_n) &= \phi(\hat{x}, P_C(a_n - \gamma_n B_1^*(B_1 a_n - B_2 b_n))) \\
 &\leq \phi(\hat{x}, a_n - \gamma_n B_1^*(B_1 a_n - B_2 b_n)) \\
 &\leq \phi(\hat{x}, a_n) - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle,
 \end{aligned} \tag{3.23}$$

where $s_n = a_n - \gamma_n B_1^*(B_1 a_n - B_2 b_n)$. Combining (3.22) and (3.23), we obtain

$$\begin{aligned}
 \phi(\hat{x}, p_n) &\leq \phi(\hat{x}, a_n) - \phi(p_n, e_n) - \phi(e_n, c_n) + 2\langle e_n - c_n, e_n - p_n \rangle \\
 &\quad + 2\langle e_n - c_n, p_n - \hat{x} \rangle + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} - p_n \rangle \\
 &\quad - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle.
 \end{aligned} \tag{3.24}$$

From (3.8) and (3.2), we obtain

$$\begin{aligned}
 \phi(\hat{x}, a_n) &= \phi(\hat{x}, P_C[x_n + \theta_n(x_n - x_{n-1})]) \\
 &\leq \phi(\hat{x}, x_n + \theta_n(x_n - x_{n-1})) \\
 &= \|\hat{x}\|^2 - 2\langle x_n + \theta_n(x_n - x_{n-1}), \hat{x} \rangle + \|a_n\|^2 \\
 &= \|\hat{x}\|^2 - 2\langle \hat{x}, x_n \rangle - 2\langle \hat{x}, \theta_n(x_n - x_{n-1}) \rangle + \|a_n\|^2 \\
 &= \|\hat{x}\|^2 - 2\langle \hat{x}, x_n \rangle + \|x_n\|^2 - \|x_n\|^2 - 2\langle \hat{x}, \theta_n(x_n - x_{n-1}) \rangle + \|a_n\|^2 \\
 &= \phi(\hat{x}, x_n) - \|x_n\|^2 - 2\langle \hat{x}, \theta_n(x_n - x_{n-1}) \rangle + \|a_n\|^2 \\
 &= \phi(\hat{x}, x_n) - \|x_n\|^2 - 2\langle \hat{x}, \theta_n(x_n - x_{n-1}) \rangle - \|a_n\|^2 \\
 &\quad + 2\|a_n\|^2 - 2\langle x_n, a_n \rangle + 2\langle x_n, a_n \rangle \\
 &= \phi(\hat{x}, x_n) - \phi(a_n, x_n) - 2\langle \hat{x}, \theta_n(x_n - x_{n-1}) \rangle \\
 &\quad + 2\|a_n\|^2 - 2\langle x_n, a_n \rangle \\
 &= \phi(\hat{x}, x_n) - \phi(a_n, x_n) - 2\langle \hat{x}, \theta_n(x_n - x_{n-1}) \rangle \\
 &\quad + 2\langle a_n, a_n \rangle - 2\langle x_n, a_n \rangle \\
 &= \phi(\hat{x}, x_n) - \phi(a_n, x_n) - 2\langle \hat{x}, \theta_n(x_n - x_{n-1}) \rangle - 2\langle x_n - a_n, a_n \rangle \\
 &= \phi(\hat{x}, x_n) - \phi(a_n, x_n) - 2\langle \hat{x}, \theta_n(x_n - x_{n-1}) \rangle \\
 &\quad + 2\langle \theta_n(x_n - x_{n-1}), a_n \rangle \\
 &= \phi(\hat{x}, x_n) - \phi(a_n, x_n) - 2\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n \rangle.
 \end{aligned} \tag{3.25}$$

Now, substitution of (3.25) into (3.24) gives

$$\begin{aligned}
 \phi(\hat{x}, p_n) &\leq \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) \\
 &\quad + 2\langle e_n - c_n, e_n - p_n \rangle + 2\langle e_n - c_n, p_n - \hat{x} \rangle \\
 &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} - p_n \rangle \\
 &\quad - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle - 2\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n \rangle,
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \phi(\hat{x}, p_n) &\leq \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) + 2\langle e_n - c_n, e_n - \hat{x} \rangle \\
 &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} - e_n \rangle \\
 &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, e_n - p_n \rangle \\
 &\quad - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle - 2\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n \rangle \\
 &= \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) \\
 &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n - (e_n - c_n), \hat{x} - e_n \rangle \\
 &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, e_n - p_n \rangle \\
 &\quad - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x \rangle - 2\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n \rangle.
 \end{aligned} \tag{3.26}$$

Let $\tau_n = (I_1 - T_1)e_n$. Then we have from the definition of e_n that

$$e_n + \lambda_n \tau_n = c_n + \lambda_n [(I_1 - T_1)e_n - (I_1 - T_1)c_n],$$

which implies that

$$\tau_n = \frac{1}{\lambda_n} [\lambda_n ((I_1 - T_1)e_n - (I_1 - T_1)c_n) - (e_n - c_n)]. \tag{3.27}$$

Since $(I_1 - T_1)\hat{x} = 0$ and $I_1 - T_1$ is monotone, we have that

$$\langle \tau_n, e_n - \hat{x} \rangle \geq 0. \tag{3.28}$$

Thus, substituting (3.27) into (3.28), we obtain

$$\langle \lambda_n ((I_1 - T_1)e_n - (I_1 - T_1)c_n) - (e_n - c_n), e_n - \hat{x} \rangle \geq 0. \tag{3.29}$$

From (3.29) and (3.26), we get

$$\begin{aligned}
 \phi(\hat{x}, p_n) &\leq \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - D(e_n, c_n) \\
 &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, e_n - p_n \rangle \\
 &\quad - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle - 2\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n \rangle.
 \end{aligned} \tag{3.30}$$

Moreover, we obtain from the Cauchy–Schwarz Inequality and Lemma 3.1.16 that

$$\begin{aligned}
 -\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n \rangle &\leq \theta_n \|x_n - x_{n-1}\| \|\hat{x} - a_n\| = \theta_n \|x_n - x_{n-1}\| \|(\hat{x} - a_n) \times 1\| \\
 &\leq \frac{\theta_n}{2} \|x_n - x_{n-1}\| [\|\hat{x} - a_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|x_n - x_{n-1}\| [2\|\hat{x} - x_n\|^2 + 2\|x_n - a_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|x_n - x_{n-1}\| [4\phi(\hat{x}, x_n) + 4\phi(a_n, x_n) + 1] \\
 &= 2\theta_n \|x_n - x_{n-1}\| \phi(\hat{x}, x_n) + 2\theta_n \|x_n - x_{n-1}\| \phi(a_n, x_n) + \frac{\theta_n}{2} \|x_n - x_{n-1}\|.
 \end{aligned} \tag{3.31}$$

From (3.31), (3.30), (3.27) and using the fact that $\theta_n \|x_n - x_{n-1}\| \leq \zeta_n$, we obtain

$$\begin{aligned}
 \phi(\hat{x}, p_n) &\leq \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) \\
 &\quad + 4\theta_n \|x_n - x_{n-1}\| \phi(\hat{x}, x_n) + 4\theta_n \|x_n - x_{n-1}\| \phi(a_n, x_n) \\
 &\quad + 2\lambda_n \langle (I - T_1)e_n - (I - T_1)c_n, e_n - p_n \rangle + \theta_n \|x_n - x_{n-1}\| \\
 &\quad - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle \\
 &\leq \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) \\
 &\quad + 4\zeta_n \phi(\hat{x}, x_n) + 4\zeta_n \phi(a_n, x_n) + 2\mu \|e_n - c_n\| \|p_n - e_n\| + \zeta_n \\
 &\quad - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle \\
 &\leq (1 + 4\zeta_n) \phi(\hat{x}, x_n) - (1 - 4\zeta_n) \phi(a_n, x_n) - (1 - 2\mu) \phi(p_n, e_n) \\
 &\quad - (1 - 2\mu) \phi(e_n, c_n) + \zeta_n - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle.
 \end{aligned} \tag{3.32}$$

Since $\zeta_n \in \left(0, \frac{1}{4}\right)$ and $\mu \in \left(0, \frac{1}{2}\right)$, we obtain from (3.32) that

$$\phi(\hat{x}, p_n) \leq (1 + 4\zeta_n) \phi(\hat{x}, x_n) + \zeta_n - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle. \tag{3.33}$$

Substituting (3.32) into (3.17), we obtain

$$\begin{aligned}
 \phi(\hat{x}, x_{n+1}) &\leq \alpha_n \phi(\hat{x}, u) + (1 - \alpha_n) (1 + 4\zeta_n) \phi(\hat{x}, x_n) - (1 - \alpha_n) (1 - 4\zeta_n) \phi(a_n, x_n) \\
 &\quad - (1 - \alpha_n) (1 - 2\mu) \phi(p_n, e_n) - (1 - \alpha_n) (1 - 2\mu) \phi(e_n, c_n) \\
 &\quad + (1 - \alpha_n) \zeta_n - 2(1 - \alpha_n) \langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle.
 \end{aligned} \tag{3.34}$$

Similarly, we obtain

$$\begin{aligned}
 \phi(\hat{y}, y_{n+1}) &\leq \alpha_n \phi(\hat{y}, v) + (1 - \alpha_n) (1 + 4\zeta_n) \phi(\hat{y}, y_n) - (1 - \alpha_n) (1 - 4\zeta_n) \phi(b_n, y_n) \\
 &\quad - (1 - \alpha_n) (1 - 2\sigma) \phi(q_n, h_n) - (1 - \alpha_n) (1 - 2\sigma) \phi(h_n, d_n) \\
 &\quad + (1 - \alpha_n) \zeta_n - 2(1 - \alpha_n) \langle \gamma_n B_2^*(B_2 b_n - B_1 a_n), r_n - \hat{y} \rangle,
 \end{aligned} \tag{3.35}$$

where $r_n = b_n - \gamma_n B_2^*(B_2 b_n - B_1 a_n)$.

Since $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$, for any $\varepsilon \in \left(0, \frac{1}{4}\right)$ there exists $n_0 \in \mathbb{N}$ such that $\zeta_n < \varepsilon \alpha_n$, for all $n \geq n_0$. This together with the properties of α_n , μ and σ implies that

$$\begin{aligned}
 \phi(\hat{x}, x_{n+1}) &\leq \alpha_n \phi(\hat{x}, u) + (1 - \alpha_n) (1 + 4\varepsilon \alpha_n) \phi(\hat{x}, x_n) \\
 &\quad + \varepsilon \alpha_n - 2(1 - \alpha_n) \langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x} \rangle, \text{ and}
 \end{aligned} \tag{3.36}$$

$$\begin{aligned}
 \phi(\hat{y}, y_{n+1}) &\leq \alpha_n \phi(\hat{y}, v) + (1 - \alpha_n) (1 + 4\varepsilon \alpha_n) \phi(\hat{y}, y_n) \\
 &\quad + \varepsilon \alpha_n - 2(1 - \alpha_n) \langle \gamma_n B_2^*(B_2 b_n - B_1 a_n), r_n - \hat{y} \rangle, \text{ for all } n \geq n_0.
 \end{aligned} \tag{3.37}$$

Denote $\Pi_n = \phi(\hat{x}, x_n) + \phi(\hat{y}, y_n)$ and $\Sigma = \phi(\hat{x}, u) + \phi(\hat{y}, v)$. Then the combination of (3.36) and

(3.37) gives

$$\begin{aligned} \Pi_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) (1 + 4\varepsilon \alpha_n) \Pi_n + 2\varepsilon \alpha_n \\ &\quad - 2(1 - \alpha_n) [\gamma_n \langle B_1 a_n - B_2 b_n, B_1 s_n - B_2 r_n \rangle], \end{aligned} \quad (3.38)$$

for all $n \geq n_0$. Furthermore, we obtain by the Cauchy–Schwarz Inequality that

$$\begin{aligned} -\langle B_1 a_n - B_2 b_n, B_1 s_n - B_2 r_n \rangle &= -\langle B_1 a_n - B_2 b_n, B_1 a_n - B_2 b_n \rangle \\ &\quad - \langle B_1 a_n - B_2 b_n, B_1 s_n - B_1 a_n \rangle \\ &\quad - \langle B_1 a_n - B_2 b_n, B_2 b_n - B_2 r_n \rangle \\ &= -\|B_1 a_n - B_2 b_n\|^2 - \langle B_1^*(B_1 a_n - B_2 b_n), s_n - a_n \rangle \\ &\quad - \langle B_2^*(B_1 a_n - B_2 b_n), b_n - r_n \rangle \\ &\leq -\|B_1 a_n - B_2 b_n\|^2 + \|s_n - a_n\| \|B_1^*(B_1 a_n - B_2 b_n)\| \\ &\quad + \|b_n - r_n\| \|B_2^*(B_1 a_n - B_2 b_n)\|. \end{aligned} \quad (3.39)$$

Moreover,

$$\|s_n - a_n\| = \|a_n - \gamma_n B_1^*(B_1 a_n - B_2 b_n) - a_n\| = \gamma_n \|B_1^*(B_1 a_n - B_2 b_n)\|. \quad (3.40)$$

Similarly, we obtain

$$\|r_n - b_n\| = \gamma_n \|B_2^*(B_2 b_n - B_1 a_n)\|. \quad (3.41)$$

Combining (3.39), (3.40), (3.41) and (3.10), we obtain

$$\begin{aligned} -2\gamma_n \langle B_1 a_n - B_2 b_n, B_1 s_n - B_2 r_n \rangle &\leq -2\gamma_n \|B_1 a_n - B_2 b_n\|^2 + 2\gamma_n^2 \|B_1^*(B_1 a_n - B_2 b_n)\|^2 \\ &\quad + 2\gamma_n^2 \|B_2^*(B_2 b_n - B_1 a_n)\|^2 \\ &\leq -\rho \|B_1 a_n - B_2 b_n\|^2 - \gamma_n \|B_1 a_n - B_2 b_n\|^2 \\ &\quad + \gamma_n \{2\gamma_n [\|B_1^*(B_1 a_n - B_2 b_n)\|^2 + \|B_2^*(B_2 b_n - B_1 a_n)\|^2]\} \\ &\leq -\rho \|B_1 a_n - B_2 b_n\|^2. \end{aligned} \quad (3.42)$$

Substituting (3.42) into (3.38), we obtain

$$\begin{aligned} \Pi_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) (1 + 4\varepsilon \alpha_n) \Pi_n + 2\varepsilon \alpha_n - (1 - \alpha_n) \rho \|B_1 a_n - B_2 b_n\|^2 \\ &\leq \alpha_n \Sigma + (1 - \alpha_n) (1 + 4\varepsilon \alpha_n) \Pi_n + 2\varepsilon \alpha_n \\ &\leq [1 - \alpha_n (1 - 4\varepsilon)] \Pi_n + \alpha_n [\Sigma + 2\varepsilon] \\ &= [1 - \alpha_n (1 - 4\varepsilon)] \Pi_n + \alpha_n (1 - 4\varepsilon) \left[\frac{\Sigma + 2\varepsilon}{1 - 4\varepsilon} \right] \\ &\leq \max \left\{ \Pi_n, \frac{\Sigma + 2\varepsilon}{1 - 4\varepsilon} \right\} \text{ for all } n \geq n_0, \end{aligned} \quad (3.43)$$

which implies by the Principle of Mathematical Induction that $\Pi_n \leq \max \left\{ \Pi_{n_0}, \mathcal{K}, \frac{\Sigma + 2\varepsilon}{1 - 4\varepsilon} \right\}$, where $\mathcal{K} = \max \{\Pi_0, \Pi_1, \dots, \Pi_{n_0-1}\}$. Thus, we conclude that $\{\Pi_n\}$ is bounded and hence the

sequences $\{\phi(\hat{x}, x_n)\}$ and $\{\phi(\hat{y}, y_n)\}$ are bounded, which implies by Lemma 3.1.17 that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. $\square$

Theorem 3.2.2. *Assume that conditions (C1) – (C6) hold. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Omega(u, v)$.*

Proof. Let $(x^*, y^*) = P_\Omega(u, v)$. Then we have by Lemma 3.1.4 that

$$\langle (u, v) - (x^*, y^*), (s, t) - (x^*, y^*) \rangle \leq 0, \text{ for all } (s, t) \in \Omega. \quad (3.44)$$

From (3.15), (3.5) and Lemma 3.1.15 we get

$$\begin{aligned} \phi(x^*, x_{n+1}) &= \phi(x^*, \alpha_n u + (1 - \alpha_n)p_n) \\ &\leq \phi(x^*, \alpha_n u + (1 - \alpha_n)p_n - \alpha_n(u - x^*)) + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\ &= \phi(x^*, (1 - \alpha_n)p_n + \alpha_n x^*) + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\ &\leq \alpha_n \phi(x^*, x^*) + (1 - \alpha_n)\phi(x^*, p_n) + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\ &= (1 - \alpha_n)\phi(x^*, p_n) + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle. \end{aligned} \quad (3.45)$$

Combining (3.45) and (3.33) with $\hat{x} = x^*$, we obtain

$$\begin{aligned} \phi(x^*, x_{n+1}) &\leq (1 - \alpha_n) [(1 + 4\zeta_n) \phi(x^*, x_n) + \zeta_n - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^* \rangle] \\ &\quad + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n)\phi(x^*, x_n) + 4\zeta_n \phi(x^*, x_n) + (1 - \alpha_n)\zeta_n \\ &\quad - 2(1 - \alpha_n) \langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^* \rangle + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle. \end{aligned} \quad (3.46)$$

Since $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$, for every $\varepsilon \in \left(0, \frac{1}{4}\right)$, there exists $N_1 \in \mathbb{N}$ such that $\zeta_n < \varepsilon \alpha_n$, for all $n \geq N_1$. Thus, we obtain from (3.46) and the Cauchy–Schwarz Inequality that

$$\begin{aligned} \phi(x^*, x_{n+1}) &\leq (1 - \alpha_n)\phi(x^*, x_n) + 4\varepsilon \alpha_n \phi(x^*, x_n) - 2(1 - \alpha_n) \langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^* \rangle \\ &\quad + 2\alpha_n \|u - x^*\| \|x_{n+1} - x_n\| + 2\alpha_n \langle u - x^*, x_n - x^* \rangle + \zeta_n \\ &= [1 - \alpha_n(1 - 4\varepsilon)] \phi(x^*, x_n) - 2(1 - \alpha_n) \langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^* \rangle \\ &\quad + 2\alpha_n \|u - x^*\| \|x_{n+1} - x_n\| + 2\alpha_n \langle u - x^*, x_n - x^* \rangle + \frac{\alpha_n \zeta_n}{\alpha_n} \\ &= [1 - \alpha_n(1 - 4\varepsilon)] \phi(x^*, x_n) - 2(1 - \alpha_n) \langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^* \rangle \\ &\quad + \alpha_n(1 - 4\varepsilon) \left[\frac{2[\|u - x^*\| \|x_{n+1} - x_n\| + \langle u - x^*, x_n - x^* \rangle]}{1 - 4\varepsilon} + \frac{\zeta_n}{(1 - 4\varepsilon)\alpha_n} \right], \end{aligned} \quad (3.47)$$

for all $n \geq N_1$. Similarly,

$$\begin{aligned} \phi(y^*, y_{n+1}) &\leq [1 - \alpha_n (1 - 4\varepsilon)] \phi(y^*, y_n) - 2(1 - \alpha_n) \langle \gamma_n B_2^* (B_2 b_n - B_1 a_n), r_n - y^* \rangle \\ &\quad + \alpha_n (1 - 4\varepsilon) \left[\frac{2 [\|v - y^*\| \|y_{n+1} - y_n\| + \langle v - y^*, y_n - y^* \rangle]}{1 - 4\varepsilon} + \frac{\zeta_n}{(1 - 4\varepsilon)\alpha_n} \right], \end{aligned} \quad (3.48)$$

for all $n \geq N_1$. Now, denote $\Pi_n^* = \phi(x^*, x_n) + \phi(y^*, y_n)$ and $\Sigma^* = \phi(x^*, u) + \phi(y^*, v)$.

Combining (3.47) and (3.48) and using the relation (3.42), we obtain

$$\Pi_{n+1}^* \leq [1 - \alpha_n (1 - 4\varepsilon)] \Pi_n^* + \alpha_n (1 - 4\varepsilon) (\Delta_n + \Gamma_n), \quad (3.49)$$

where

$$\Delta_n = \frac{2 [\|u - x^*\| \|x_{n+1} - x_n\| + \langle u - x^*, x_n - x^* \rangle]}{1 - 4\varepsilon} + \frac{\zeta_n}{(1 - 4\varepsilon)\alpha_n}, \quad (3.50)$$

and

$$\Gamma_n = \frac{2 [\|v - y^*\| \|y_{n+1} - y_n\| + \langle v - y^*, y_n - y^* \rangle]}{1 - 4\varepsilon} + \frac{\zeta_n}{(1 - 4\varepsilon)\alpha_n}. \quad (3.51)$$

Combining (3.34) and (3.35) and using the relation (3.42) gives

$$\begin{aligned} &(1 - \alpha_n) (1 - 4\zeta_n) \phi(a_n, x_n) + (1 - \alpha_n) (1 - 2\mu) \phi(p_n, e_n) + (1 - \alpha_n) (1 - 2\mu) \phi(e_n, c_n) \\ &+ (1 - \alpha_n) (1 - 4\zeta_n) \phi(b_n, y_n) + (1 - \alpha_n) (1 - 2\sigma) \phi(q_n, h_n) + (1 - \alpha_n) (1 - 2\sigma) \phi(h_n, d_n) \\ &+ (1 - \alpha_n) \rho \|B_1 a_n - B_2 b_n\|^2 \leq \Pi_n^* - \Pi_{n+1}^* + \alpha_n \left[\Sigma^* + (4\varepsilon - 1) \Pi_n^* + \frac{2\zeta_n}{\alpha_n} \right]. \end{aligned} \quad (3.52)$$

Now, we show that the sequence of non-negative real numbers $\{\Pi_n^*\}$ converges to zero by considering two cases:

Case I. If there exists a natural number N_2 such that $\Pi_{n+1}^* \leq \Pi_n^*$ for all $n \geq N_2$, then $\{\Pi_n^*\}$ converges and hence $\lim_{n \rightarrow \infty} (\Pi_n^* - \Pi_{n+1}^*) = 0$. So, taking limit of (3.52) as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \|B_1 a_n - B_2 b_n\| = 0, \quad (3.53)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \phi(a_n, x_n) &= \lim_{n \rightarrow \infty} \phi(p_n, e_n) = \lim_{n \rightarrow \infty} \phi(e_n, c_n) = 0, \\ \lim_{n \rightarrow \infty} \phi(b_n, y_n) &= \lim_{n \rightarrow \infty} \phi(q_n, h_n) = \lim_{n \rightarrow \infty} \phi(h_n, d_n) = 0, \end{aligned} \quad (3.54)$$

which implies by Lemma 3.1.18 that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|a_n - x_n\| &= \lim_{n \rightarrow \infty} \|p_n - e_n\| = \lim_{n \rightarrow \infty} \|e_n - c_n\| = 0, \\ \lim_{n \rightarrow \infty} \|b_n - y_n\| &= \lim_{n \rightarrow \infty} \|q_n - h_n\| = \lim_{n \rightarrow \infty} \|h_n - d_n\| = 0. \end{aligned} \quad (3.55)$$

From (3.11), the nonexpansive property of the metric projection and (3.53), we obtain

$$\begin{aligned}
 \|a_n - c_n\| &= \|a_n - P_C [a_n - \gamma_n B_1^* (B_1 a_n - B_2 b_n)]\| \\
 &\leq \|a_n - [a_n - \gamma_n B_1^* (B_1 a_n - B_2 b_n)]\| \\
 &= \gamma_n \|B_1^* (B_1 a_n - B_2 b_n)\| \\
 &\leq (\rho + 1) \|B_1^* (B_1 a_n - B_2 b_n)\| \\
 &\leq (\rho + 1) \|B_1^*\| \|B_1 a_n - B_2 b_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.
 \end{aligned} \tag{3.56}$$

From (3.15) and the condition on $\{\alpha_n\}$ we obtain

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \|x_{n+1} - p_n\| &= \lim_{n \rightarrow \infty} \|\alpha_n u + (1 - \alpha_n)p_n - p_n\| \\
 &= \lim_{n \rightarrow \infty} \alpha_n \|u - p_n\| = 0.
 \end{aligned} \tag{3.57}$$

Thus, we obtain from (3.55), (3.56) and (3.57) that

$$\begin{aligned}
 \|x_{n+1} - x_n\| &\leq \|x_{n+1} - p_n\| + \|p_n - e_n\| + \|e_n - c_n\| \\
 &\quad + \|c_n - a_n\| + \|a_n - x_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.
 \end{aligned} \tag{3.58}$$

Similarly, we have

$$\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0. \tag{3.59}$$

Since the sequence $\{(x_n, y_n)\}$ is bounded in $H_1 \times H_2$, there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ and an element $(\bar{x}, \bar{y})$ of $H_1 \times H_2$ such that $(x_{n_k}, y_{n_k}) \rightharpoonup (\bar{x}, \bar{y})$ and

$$\limsup_{n \rightarrow \infty} \langle (u, v) - (x^*, y^*), (x_n, y_n) - (x^*, y^*) \rangle = \lim_{k \rightarrow \infty} \langle (u, v) - (x^*, y^*), (x_{n_k}, y_{n_k}) - (x^*, y^*) \rangle, \tag{3.60}$$

Moreover, we have by (3.1) that $x_{n_k} \rightharpoonup \bar{x}$ and $y_{n_k} \rightharpoonup \bar{y}$. From (3.55) and (3.56), we obtain $c_{n_k} \rightharpoonup \bar{x}$.

Now, we show that $(\bar{x}, \bar{y}) \in \Omega$. Put $z_{n_k} = P_C [c_{n_k} - \lambda_{n_k} l^{-1} (I_1 - T_1) c_{n_k}]$. By Lemma 3.1.2 and (3.55), we have

$$\|c_{n_k} - z_{n_k}\| \leq \frac{1}{l} \|c_{n_k} - e_{n_k}\| \rightarrow 0, \text{ as } k \rightarrow \infty. \tag{3.61}$$

Therefore, $z_{n_k} \rightharpoonup \bar{x}$. Thus, $\{z_{n_k}\}$ is bounded. Since $I_1 - T_1$ is uniformly continuous, we have

$$\|(I_1 - T_1)c_{n_k} - (I_1 - T_1)z_{n_k}\| \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{3.62}$$

By the Armijo line search rule (3.12), we have

$$\|(I_1 - T_1)(c_{n_k} - \lambda_{n_k} l^{-1} (I_1 - T_1) c_{n_k}) - (I_1 - T_1)c_{n_k}\| > \mu \|(I_1 - T_1)c_{n_k}\|,$$

which implies

$$\frac{1}{\mu} \|(I_1 - T_1)(c_{n_k} - \lambda_{n_k} l^{-1} (I_1 - T_1) c_{n_k}) - (I_1 - T_1)c_{n_k}\| > \|(I_1 - T_1)c_{n_k}\|. \tag{3.63}$$

Thus, we have from (3.62) and (3.63) that $\lim_{k \rightarrow \infty} (I_1 - T_1)c_{n_k} = 0$ which implies from the fact $c_{n_k} \rightharpoonup \bar{x}$ and demiclosedness of T_1 that $(I_1 - T_1)\bar{x} = 0$ and hence $\bar{x} \in F(T_1)$. One can show in a similar fashion that $\bar{y} \in F(T_2)$.

Next we show that $B_1\bar{x} = B_2\bar{y}$. Indeed, we have by (1.4) that

$$\begin{aligned} \|B_1\bar{x} - B_2\bar{y}\|^2 &= \|B_1a_{n_k} - B_2b_{n_k} + B_1\bar{x} - B_1a_{n_k} + B_2b_{n_k} - B_2\bar{y}\|^2 \\ &\leq \|B_1a_{n_k} - B_2b_{n_k}\|^2 + 2\langle B_1\bar{x} - B_2\bar{y}, B_1\bar{x} - B_1a_{n_k} + B_2b_{n_k} - B_2\bar{y} \rangle. \end{aligned} \quad (3.64)$$

Thus, we obtain from (3.64), (3.53) and the facts $B_1a_{n_k} \rightharpoonup B_1\bar{x}$ and $B_2b_{n_k} \rightharpoonup B_2\bar{y}$ as $k \rightarrow \infty$ that $B_1\bar{x} = B_2\bar{y}$ and hence $(\bar{x}, \bar{y}) \in \Omega$. Consequently, we obtain using Lemma 3.1.4, (3.44) and (3.60) that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (u, v) - (x^*, y^*), (x_n, y_n) - (x^*, y^*) \rangle &= \lim_{k \rightarrow \infty} \langle (u, v) - (x^*, y^*), (x_{n_k}, y_{n_k}) - (x^*, y^*) \rangle \\ &= \langle (u, v) - (x^*, y^*), (\bar{x}, \bar{y}) - (x^*, y^*) \rangle \leq 0. \end{aligned} \quad (3.65)$$

Combining (3.50), (3.51) and (3.65), we obtain

$$\limsup_{n \rightarrow \infty} (\Delta_n + \Gamma_n) \leq 0. \quad (3.66)$$

From (3.49), (3.66) and Lemma 3.1.6, we conclude that $\lim_{n \rightarrow \infty} \Pi_n^* = 0$, which implies that $\lim_{n \rightarrow \infty} \phi(x^*, x_n) = \lim_{n \rightarrow \infty} \phi(y^*, y_n) = 0$, which in turn implies by Lemma 3.1.18 that $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$ and $\lim_{n \rightarrow \infty} \|y_n - y^*\| = 0$.

Case II. Suppose now that there exists a subsequence $\{\Pi_{n_i}^*\}$ of $\{\Pi_n^*\}$ such that $\Pi_{n_i}^* < \Pi_{n_i+1}^*$, for all $i \geq 0$. Then by Lemma 3.1.7, there exists a nondecreasing sequence $\{m_k\}$ of positive integers such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\Pi_{m_k}^* \leq \Pi_{m_k+1}^* \quad \text{and} \quad \Pi_k^* \leq \Pi_{m_k+1}^*, \quad (3.67)$$

for all $k \in \mathbb{N}$. We have from (3.52) that

$$\begin{aligned} &(1 - \alpha_{m_k})(1 - 4\zeta_{m_k})\phi(a_{m_k}, x_{m_k}) + (1 - \alpha_{m_k})(1 - 2\mu)\phi(p_{m_k}, e_{m_k}) \\ &+ (1 - \alpha_{m_k})(1 - 2\mu)\phi(e_{m_k}, c_{m_k}) + (1 - \alpha_{m_k})(1 - 4\zeta_{m_k})\phi(b_{m_k}, y_{m_k}) \\ &+ (1 - \alpha_{m_k})(1 - 2\sigma)\phi(q_{m_k}, h_{m_k}) + (1 - \alpha_{m_k})(1 - 2\mu)\phi(h_{m_k}, d_{m_k}) \\ &+ (1 - \alpha_{m_k})\rho\|B_1a_{m_k} - B_2b_{m_k}\|^2 \leq \Pi_{m_k}^* - \Pi_{m_k+1}^* + \alpha_{m_k} \left[\Sigma^* + (4\varepsilon - 1)\Pi_{m_k}^* + \frac{2\zeta_{m_k}}{\alpha_{m_k}} \right]. \end{aligned} \quad (3.68)$$

Making use of (3.67) and the conditions on α_{m_k} and ζ_{m_k} and taking limit as $k \rightarrow \infty$ of (3.68) we obtain that

$$\lim_{k \rightarrow \infty} \|B_1a_{m_k} - B_2b_{m_k}\| = 0.$$

Following similar steps as in Case I, we obtain that

$$\lim_{k \rightarrow \infty} \|x_{m_k+1} - x_{m_k}\| = \lim_{k \rightarrow \infty} \|y_{m_k+1} - y_{m_k}\| = 0 \text{ and } \limsup_{k \rightarrow \infty} (\Delta_{m_k} + \Gamma_{m_k}) \leq 0.$$

From (3.49) and (3.67), we have that

$$\begin{aligned} \alpha_{m_k} (1 - 4\varepsilon) \Pi_{m_k}^* &\leq \Pi_{m_k}^* - \Pi_{m_k+1}^* + \alpha_{m_k} (1 - 4\varepsilon) (\Delta_{m_k} + \Gamma_{m_k}) \\ &\leq \alpha_{m_k} (1 - 4\varepsilon) (\Delta_{m_k} + \Gamma_{m_k}), \end{aligned}$$

and thus we have $\Pi_{m_k}^* \leq \Delta_{m_k} + \Gamma_{m_k}$. Since $\limsup_{k \rightarrow \infty} (\Delta_{m_k} + \Gamma_{m_k}) \leq 0$, we have that $\lim_{k \rightarrow \infty} \Pi_{m_k}^* = 0$, which together with (3.49) implies that $\lim_{k \rightarrow \infty} \Pi_{m_k+1}^* = 0$. Since $\Pi_k^* \leq \Pi_{m_k+1}^*$, it follows that $\lim_{k \rightarrow \infty} \Pi_k^* = 0$. Thus, we have $\lim_{k \rightarrow \infty} \phi(x^*, x_k) = \lim_{k \rightarrow \infty} \phi(y^*, y_k) = 0$, which implies by Lemma 3.1.18 that $\lim_{k \rightarrow \infty} \|x_k - x^*\| = \lim_{k \rightarrow \infty} \|y_k - y^*\| = 0$.

Thus, we conclude from Cases I and II, that the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) = P_\Omega(u, v)$, and this completes the proof. $\square$

Corollary 3.2.1. *Assume that conditions (C1), (C3) – (C6) hold. If $T_1: C \rightarrow C$ and $T_2: D \rightarrow D$ are Lipschitz pseudocontractive mappings, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Omega(u, v)$.*

Proof. As every Lipschitz continuous mapping is uniformly continuous, the result follows from Theorem 3.2.2. $\square$

If $u = 0 = v$, then Algorithm 3.1 can be used to locate an element of the solution with the minimum norm.

Corollary 3.2.2. *Assume that conditions (C1)–(C6) are satisfied. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1, with $u = v = 0$, converges strongly to the minimum norm point $(x^*, y^*) \in \Omega$, that is, $(x^*, y^*) = P_\Omega(0, 0)$.*

3.2.2 Some Theoretical Applications

In this section, we present some theoretical applications of the main result.

Split Equality Null Point Problem

If in Theorem 3.2.2, we take $C = H_1$ and $D = H_2$, then the SEFPP reduces to *split equality null point problem* which is defined as finding a point $(x^*, y^*) \in H_1 \times H_2$ such that $(x^*, y^*) \in N(A_1) \times N(A_2)$ such that $B_1 x^* = B_2 y^*$, where $A_i = I_i - T_i$, for $i = 1, 2$. Denote

$$\Phi = \{(x^*, y^*) \in N(A_1) \times N(A_2) : B_1 x^* = B_2 y^*\}.$$

Corollary 3.2.3. *Given the real Hilbert spaces H_1, H_2 and H_3 . Let $A_1: H_1 \rightarrow H_1$ and $A_2: H_2 \rightarrow H_2$ be uniformly continuous monotone mappings. Assume that the set $\Phi \neq \emptyset$. If the conditions (C3), (C5) and (C6) hold, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1, with $T_i = I_i - A_i$, for $i = 1, 2$, converges strongly to an element (x^*, y^*) , where $(x^*, y^*) = P_\Phi(u, v)$.*

Corollary 3.2.4. *Given the real Hilbert spaces H_1, H_2 and H_3 . Let $A_1: H_1 \rightarrow H_1$ and $A_2: H_2 \rightarrow H_2$ be Lipschitz continuous monotone mappings. Assume that the set $\Phi \neq \emptyset$. If the conditions (C3), (C5) and (C6) hold, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1, with $T_i = I_i - A_i$, converges strongly to an element (x^*, y^*) , where $(x^*, y^*) = P_\Phi(u, v)$.*

Split fixed Point Problems

If in Theorem 3.2.2, we consider $H_2 = H_3$ and $B_2 = I_{H_2}$, then the SEFPP reduces to *split fixed point problem* which is finding a point $(x^*, y^*) \in F(T_1) \times F(T_2)$ such that $B_1 x^* = y^*$. Denote

$$\Theta = \{(x^*, y^*) \in F(T_1) \times F(T_2) : B_1 x^* = y^*\}.$$

Corollary 3.2.5. *Assume that conditions (C1), (C2), (C5) and (C6), with $H_2 = H_3$ and $B_2 = I_{H_2}$ hold. If $\Theta \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Theta(u, v)$.*

Corollary 3.2.6. *Assume that conditions (C1), (C5) and (C6), with $H_2 = H_3$ and $B_2 = I_{H_2}$ hold. Let $T_1: C \rightarrow C$ and $T_2: D \rightarrow D$ be Lipschitz pseudocontractive mappings. If $\Theta \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Theta(u, v)$.*

Common Fixed Point Problems

If in Theorem 3.2.2, we consider $H_1 = H_2 = H_3 = H$ and $B_1 = B_2 = I_H$, then SEFPP reduces to *common fixed point problem*, which is defined as finding two points $x^*, y^* \in H$ with $(x^*, y^*) \in F(T_1) \times F(T_2)$ such that $x^* = y^*$.

Denote

$$\Theta^* = \{(x^*, y^*) \in F(T_1) \times F(T_2) : x^* = y^*\}.$$

Corollary 3.2.7. *Assume that conditions (C1), (C2), (C5) and (C6), with $H_1 = H_2 = H_3 = H$ hold. If $\Theta^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1, with $B_1 = B_2 = I_H$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Theta^*}(u, v)$.*

Corollary 3.2.8. *Assume that conditions (C1), (C5) and (C6), with $H_1 = H_2 = H_3 = H$ hold. Let $T_1: C \rightarrow C$ and $T_2: D \rightarrow D$ be Lipschitz pseudocontractive mappings. If $\Theta^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1, with $B_1 = B_2 = I_H$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Theta^*}(u, v)$.*

3.2.3 Numerical Examples

Examples of mappings which exhibit the conditions of Theorem 3.2.2 are provided hereunder. Besides, a numerical experiment is provided to demonstrate the efficiency of the method.

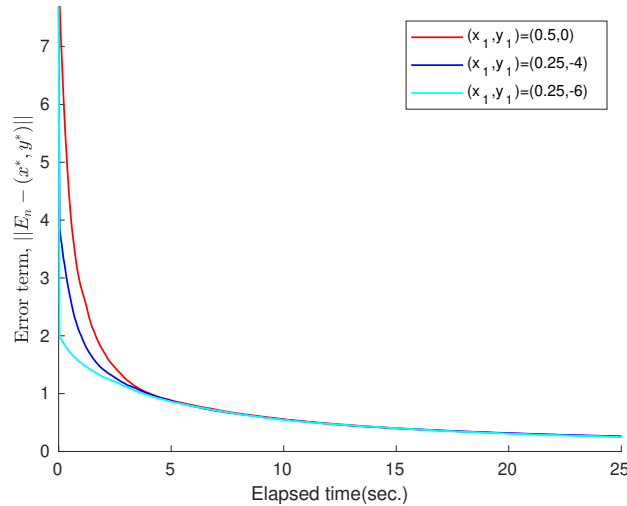
Example 3.2.1. *Let $H_1 = \mathbb{R}$ with $C = [0, \infty)$ and let $H_2 = H_3 = \mathbb{R} = D$. Let $T_1: C \rightarrow C$ and $T_2: D \rightarrow D$, be defined by*

$$T_1(x) = x - \sqrt{x} + \frac{1}{2}, \text{ and } T_2(y) = y - \sqrt[3]{y} - 2.$$

The mapping T_1 is uniformly continuous pseudocontractive on C which is not Lipschitz continuous. In fact, let $K > 0$ be given and choose $y = 0$ and $0 < x < \frac{1}{K^2}$ so that $K < \frac{1}{\sqrt{x}}$. Now,

$$\frac{|T_1(x) - T_1(y)|}{|x - y|} = \frac{|x - \sqrt{x}|}{|x|} = \left| \frac{1}{\sqrt{x}} - 1 \right| > K - 1.$$

Since K is arbitrary, one concludes that T_1 is not Lipschitz continuous. Similarly, it can be shown that T_2 is uniformly continuous pseudocontractive on D which is not Lipschitz continuous. Clearly, $x^* = \frac{1}{4} \in F(T_1)$ and $y^* = -8 \in F(T_2)$. Now, define $B_1: H_1 \rightarrow H_3$ and $B_2: H_2 \rightarrow H_3$ by $B_1(x) = 8x$ and $B_2(y) = -\frac{1}{4}y$. Then B_1 and B_2 are bounded linear mappings with adjoints $B_1^*(x) = 8x$ and $B_2^*(y) = -\frac{1}{4}y$. Moreover, we have that $B_1\left(\frac{1}{4}\right) = B_2(-8) = 2$. Thus, $(x^*, y^*) = \left(\frac{1}{4}, -8\right) \in \Omega$. For the control sequences $\zeta_n = \frac{1}{n^2 + 5}$, $\alpha_n = \frac{1}{n + 100}$ and the parameters $\sigma = 0.4$, $\gamma = 0.5$, $l = 0.5$, $\mu = 0.4$, $\theta = 0.4$, the conditions (C1) – (C6) are satisfied. A numerical experiment was carried out by taking the points $(u, v) = (-1, 0)$, $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (0.5, 0)$ and the results are depicted in Figure 3.1 and Table 3.1.



Parameters: $\gamma = l = 0.5$, $\mu = \sigma = 0.4$, $\theta = 0.4$.

Figure 3.1: Convergence of the sequence $\{(x_n, y_n)\}$ for different initial points (x_1, y_1) .

Non inertial version ($\theta_n = 0$)		Inertial version ($\theta_n \neq 0$)	
Error (E)	Time(sec.)		Time(sec.)
4.9	03568		0.3113
4.4	0.4279		0.3877
3.9	0.5432		0.4912
1.8	1.5955		1.4970
1.5	1.9915		1.9111
1.0	3.8809		3.6960

Table 3.1: Comparison of rates of convergence for the inertial and non inertial algorithms.

Remark 3.2.2. Figure 3.1 shows the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges to the solution $(x^*, y^*) = \left(\frac{1}{4}, -8\right)$ for different initial points and the convergence gets relatively faster as the initial point gets closer to the solution of the given problem. Table 3.1 shows the comparison of rates of convergence for the inertial and non inertial versions of the algorithm. The time taken for the inertial version to reach the error value $E = 1.0$ is 3.6960s, whereas the time taken by the non inertial version to reach the same error value is about 3.8809s. Thus, we conclude that the inertial version of the algorithm is faster than that of the non inertial version.

Example 3.2.2. Let $H = H_1 = H_2 = H_3 = L_2[0, 1]$ with inner product $\langle x(t), y(t) \rangle = \int_0^1 x(t)y(t)dt$ and norm $\|x\|_{L_2} = \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$.

Let $T_1: C \rightarrow C$ and $T_2: D \rightarrow D$ be the mappings defined by

$$(T_1x)(t) = x(t) - \max\{0, x(t)\}, \text{ and } (T_2y)(t) = \min\{0, y(t)\},$$

where $C = D = \{x \in H : \|x\|_{L_2} \leq 1\}$. It is proved in Tian and Xu [162] that the mapping $(Ax)(t) = \max\{0, x(t)\}$ is a monotone mapping.

Thus, we have that the mapping $(T_1x)(t) = x(t) - (Ax)(t) = x(t) - \max\{0, x(t)\}$ is a pseudocontractive mapping. It is also shown in [162] that T_1 is a uniformly continuous mapping.

Similarly, one can show that T_2 is a uniformly continuous pseudocontractive mapping.

Clearly,

$$x^*(t) = 0 \in F(T_1) \text{ and } y^*(t) = 0 \in F(T_2).$$

Let $B_1, B_2: H \rightarrow H$ be mappings defined by

$$(B_1x)(t) = 3x(t) \text{ and } (B_2y)(t) = 5y(t).$$

It can be shown that B_1 and B_2 are bounded linear mappings with adjoints $(B_1^*x)(t) = 3x(t)$ and $(B_2^*y)(t) = 5y(t)$.

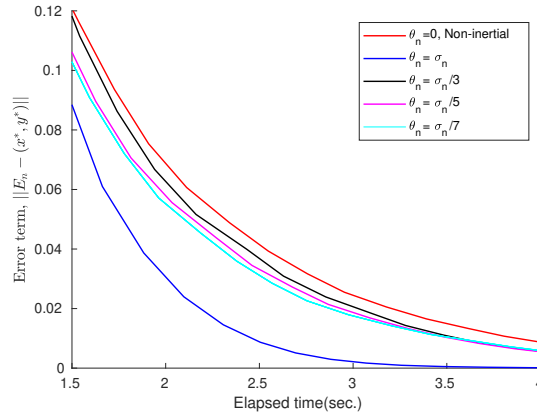
Moreover, we have $B_1(0) = B_2(0) = 0$. Therefore, $(x^*(t), y^*(t)) = (0, 0) \in \Omega$.

Now, upon choosing the parameters $\zeta_n = \frac{1}{n^2 + 5}$, $\alpha_n = \frac{1}{n + 100}$, $\sigma = 0.4$, $\gamma = 0.5$, $l = 0.5$, $\mu = 0.4$, $\theta = 0.4$ and

$$\gamma_n = \begin{cases} \min \left\{ \frac{1}{100000} + 1, \left(\frac{1}{2} \right) \frac{|3a_n - 5b_n|^2}{|9a_n - 15b_n|^2 + |25b_n - 15a_n|^2} \right\} & \text{if } n \in \Upsilon, \\ \frac{1}{100000} & \text{if } n \notin \Upsilon, \end{cases}$$

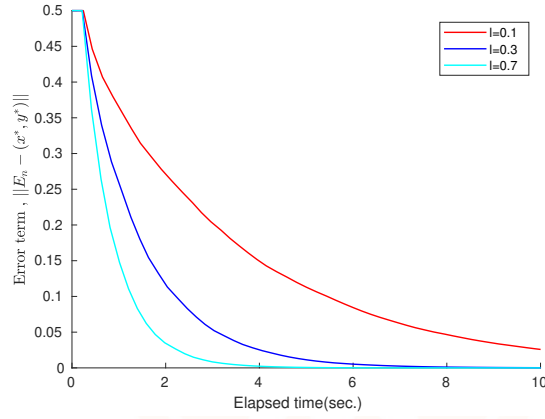
the, conditions (C1) – (C6) are satisfied.

Let us denote $E_n = (x_n, y_n)$. For the points $(u, v) = (0, 0)$, $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (0.5, 0)$ and for different choices of the inertial parameter θ_n and different values of the parameter l , the figures below reveal that the error term sequence $\{E_n - (x^*, y^*)\}$ converges to zero.



Parameters: $\gamma = l = 0.5$, $\mu = \sigma = 0.4$.

Figure 3.2: Convergence of the sequence $\{(x_n, y_n)\}$ for different values of the inertial parameter θ_n .



Parameters: $\gamma = 0.5$, $\theta = 0.4$, $\mu = \sigma = 0.4$.

Figure 3.3: Convergence of the sequence $\{(x_n, y_n)\}$ with different values of the parameter l .

Remark 3.2.3. From Figure 3.2, we observe that the method with nonzero inertial parameter has a faster rate of convergence than that of the non-inertial ($\theta_n = 0$) version. Figure 3.3 reveals that the rate of convergence of the method is different for different choices of the parameter l . In fact, it can be observed that the bigger the value of l results in a relatively faster rate of convergence of the sequence.

3.3 An Inertial Method for Split Equality Common f, g -fixed Point Problems of f, g -Pseudocontractive Mappings in Reflexive Real Banach Spaces

In this section, we propose an iterative method which approximates a solution of the split equality common f, g -fixed point problem involving uniformly continuous f, g -pseudocontractive mappings in reflexive real Banach spaces. We discuss its convergence analysis and finally provide a numerical example to verify the implementation of our algorithm.

3.3.1 Main Results

We begin by proving the following lemma before introducing our algorithm.

Lemma 3.3.1. *Let C be a nonempty subset of a reflexive real Banach space E and $A: C \rightarrow E^*$ be any mapping. Let $f: E \rightarrow \mathbb{R}$ be a strongly convex function with constant η and has Lipschitz continuous gradient with constant γ . If $\gamma \leq \eta$, then for any $x \in E$ and $\alpha \geq \beta > 0$, the following inequality holds:*

$$\frac{\|x - P_C^f \nabla f^*(\nabla f(x) - \alpha Ax)\|}{\alpha} \leq \frac{\|x - P_C^f \nabla f^*(\nabla f(x) - \beta Ax)\|}{\beta}.$$

Proof. Let $x_\alpha = P_C^f \nabla f^*(\nabla f(x) - \alpha Ax)$ and $x_\beta = P_C^f \nabla f^*(\nabla f(x) - \beta Ax)$. From (1.18) it follows that $\langle \nabla f(x) - \alpha Ax - \nabla f(x_\alpha), y - x_\alpha \rangle \leq 0$ and $\langle \nabla f(x) - \beta Ax - \nabla f(x_\beta), y - x_\beta \rangle \leq 0$, for all $y \in C$, which implies that

$$\left\langle \frac{\nabla f(x_\alpha) - \nabla f(x)}{\alpha} + Ax, x_\beta - x_\alpha \right\rangle \geq 0 \text{ and } \left\langle \frac{\nabla f(x_\beta) - \nabla f(x)}{\beta} + Ax, x_\alpha - x_\beta \right\rangle \geq 0. \quad (3.69)$$

Adding both inequalities of (3.69) and using the Cauchy–Schwarz inequality and strong convexity of f and Lipschitz continuity of ∇f , we get

$$\begin{aligned} 0 &\leq \left\langle \frac{\nabla f(x) - \nabla f(x_\alpha)}{\alpha} - \frac{\nabla f(x) - \nabla f(x_\beta)}{\beta}, x_\alpha - x_\beta \right\rangle \\ &= \left\langle \frac{\nabla f(x) - \nabla f(x_\alpha)}{\alpha} - \frac{\nabla f(x) - \nabla f(x_\beta)}{\beta}, (x - x_\beta) - (x - x_\alpha) \right\rangle \\ &= -\left\langle \frac{\nabla f(x) - \nabla f(x_\alpha)}{\alpha}, x - x_\alpha \right\rangle + \left\langle \frac{\nabla f(x) - \nabla f(x_\alpha)}{\alpha}, x - x_\beta \right\rangle \\ &\quad - \left\langle \frac{\nabla f(x) - \nabla f(x_\beta)}{\beta}, x - x_\beta \right\rangle + \left\langle \frac{\nabla f(x) - \nabla f(x_\beta)}{\beta}, x - x_\alpha \right\rangle \\ &\leq -\frac{1}{\alpha} \left\langle \nabla f(x) - \nabla f(x_\alpha), x - x_\alpha \right\rangle - \frac{1}{\beta} \left\langle \nabla f(x) - \nabla f(x_\beta), x - x_\beta \right\rangle \\ &\quad + \frac{1}{\alpha} \|\nabla f(x) - \nabla f(x_\alpha)\| \|x - x_\beta\| + \frac{1}{\beta} \|\nabla f(x) - \nabla f(x_\beta)\| \|x - x_\alpha\| \\ &\leq -\frac{\eta}{\alpha} \|x - x_\alpha\|^2 - \frac{\eta}{\beta} \|x - x_\beta\|^2 + \frac{\gamma}{\alpha} \|x - x_\alpha\| \|x - x_\beta\| + \frac{\gamma}{\beta} \|x - x_\beta\| \|x - x_\alpha\| \\ &\leq \frac{\eta}{\alpha} \|x - x_\alpha\|^2 - \frac{\eta}{\beta} \|x - x_\beta\|^2 + \frac{\eta}{\alpha} \|x - x_\alpha\| \|x - x_\beta\| + \frac{\eta}{\beta} \|x - x_\beta\| \|x - x_\alpha\|. \end{aligned} \quad (3.70)$$

Thus, we obtain from (3.70) that

$$0 \geq (\|x - x_\alpha\| - \|x - x_\beta\|) \left(\frac{\|x - x_\alpha\|}{\alpha} - \frac{\|x - x_\beta\|}{\beta} \right). \quad (3.71)$$

Now, suppose on the contrary that we have

$$\frac{\|x - x_\alpha\|}{\alpha} > \frac{\|x - x_\beta\|}{\beta}, \quad (3.72)$$

which implies that

$$\|x - x_\alpha\| > \|x - x_\beta\|. \quad (3.73)$$

Combining (3.72) and (3.73), we obtain

$$0 < (\|x - x_\alpha\| - \|x - x_\beta\|) \left(\frac{\|x - x_\alpha\|}{\alpha} - \frac{\|x - x_\beta\|}{\beta} \right),$$

which is a contradiction to (3.71) hence the proof is complete. $\square$

Before stating our algorithm, we put the following conditions that are useful in establishing the convergence analysis of the method.

- (D1) Let E_1, E_2 and E_3 be reflexive real Banach spaces with duals E_1^*, E_2^* and E_3^* , respectively;
- (D2) Let $f: E_1 \rightarrow (-\infty, +\infty]$ and $g: E_2 \rightarrow (-\infty, +\infty]$ be strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre functions on bounded subsets with strongly convex conjugates f^* and g^* , respectively. Let the strong convexity constants of f and g be β_1 and β_2 , respectively, and let $\beta = \min \{\beta_1, \beta_2\}$;
- (D3) Let $T_1, T_2: E_1 \rightarrow E_1^*$ be uniformly continuous f -pseudocontractive and $S_1, S_2: E_2 \rightarrow E_2^*$ be uniformly continuous g -pseudocontractive mappings;
- (D4) Let $A: E_1 \rightarrow E_3$ and $B: E_2 \rightarrow E_3$ be bounded linear mappings with adjoints A^* and B^* , respectively;
- (D5) Let $\Omega = \{(x^*, y^*) \in (F_f(T_1) \cap F_f(T_2)) \times (F_g(S_1) \cap F_g(S_2)) : Ax^* = By^*\} \neq \emptyset$;
- (D6) Let $\{\alpha_n\} \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (D7) Let $\{\zeta_n\}$ be a sequence of positive numbers such that $\zeta_n \in \left(0, \frac{\beta}{2}\right)$ for all $n \geq 1$ and $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$.

The following notation will also be used in the subsequent algorithm:

$$\Theta_n = \|\nabla f x_n - \nabla f x_{n-1}\| + \|\nabla g y_n - \nabla g y_{n-1}\|,$$

for all integers $n \geq 1$.

Algorithm 3.2.

Initialization: Let $x_0, x_1 \in E_1$, $y_0, y_1 \in E_2$, $\lambda, \sigma, \tau, \pi \in (0, \beta)$, $l, \gamma \in (0, 1)$ and $\theta \in [0, 1)$. For arbitrary points $x \in E_1$ and $y \in E_2$, calculate $\{x_n\}$ and $\{y_n\}$ as follows:

Step 1: Given $x_{n-1}, x_n \in E_1$ and $y_{n-1}, y_n \in E_2$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\zeta_n}{\Theta_n} \right\}, & \text{if } \Theta_n \neq 0, \\ \theta, & \text{otherwise.} \end{cases} \quad (3.74)$$

Step 2: Compute

$$a_n = \nabla f^* [\nabla f x_n + \theta_n (\nabla f x_n - \nabla f x_{n-1})], \quad (3.75)$$

$$b_n = \nabla g^* [\nabla g y_n + \theta_n (\nabla g y_n - \nabla g y_{n-1})].$$

Step 3: Compute

$$c_n = \nabla f^* [\nabla f a_n - \gamma_n A^* J_{E_3} (Aa_n - Bb_n)], \quad (3.76)$$

$$z_n = \nabla g^* [\nabla g b_n - \gamma_n B^* J_{E_3} (Bb_n - Aa_n)],$$

where $0 < \rho \leq \gamma_n \leq \rho_n$ with

$$\rho_n = \min \left\{ \rho + 1, \frac{\beta \|Aa_n - Bb_n\|^2}{2[\|A^* J_{E_3} (Aa_n - Bb_n)\|^2 + \|B^* J_{E_3} (Bb_n - Aa_n)\|^2]} \right\}, \quad (3.77)$$

for $n \in \Upsilon = \{m \in \mathbb{N} : Aa_m - Bb_m \neq 0\}$, otherwise $\gamma_n = \rho$.

Step 4: Compute

$$e_n = \nabla f^* ((1 - \lambda_n) \nabla f c_n + \lambda_n T_1 c_n), \quad (3.78)$$

$$h_n = \nabla f^* ((1 - \sigma_n) \nabla f c_n + \sigma_n T_2 c_n),$$

$$p_n = \nabla g^* ((1 - \tau_n) \nabla g z_n + \tau_n S_1 z_n), \quad (3.79)$$

$$q_n = \nabla g^* ((1 - \pi_n) \nabla g z_n + \pi_n S_2 z_n),$$

where $\lambda_n = \gamma l^{j_n}$ and $\sigma_n = \gamma l^{m_n}$ for j_n and m_n being the smallest non-negative integers j and m , respectively, satisfying the relations

$$\begin{aligned} \gamma l^j \|(\nabla f - T_1)(\nabla f^* ((1 - \gamma l^j) \nabla f c_n + \gamma l^j T_1 c_n)) - (\nabla f - T_1) c_n\| \\ \leq \lambda \|\nabla f^* ((1 - \gamma l^j) \nabla f c_n + \gamma l^j T_1 c_n) - c_n\|, \end{aligned} \quad (3.80)$$

$$\begin{aligned} \gamma l^m \|(\nabla f - T_2)(\nabla f^* ((1 - \gamma l^m) \nabla f c_n + \gamma l^m T_2 c_n)) - (\nabla f - T_2) c_n\| \\ \leq \sigma \|\nabla f^* ((1 - \gamma l^m) \nabla f c_n + \gamma l^m T_2 c_n) - c_n\|, \end{aligned} \quad (3.81)$$

and $\tau_n = \gamma l^{k_n}$, $\pi_n = \gamma l^{t_n}$, where k_n and t_n are the smallest non-negative integers k and t ,

respectively, satisfying the relations

$$\begin{aligned} \gamma l^k \|(\nabla g - S_1)(\nabla g^*((1 - \gamma l^k)\nabla g z_n + \gamma l^k S_1 z_n)) - (\nabla g - S_1)z_n\| \\ \leq \tau \|\nabla g^*((1 - \gamma l^k)\nabla g z_n + \gamma l^k S_1 z_n) - z_n\|, \end{aligned} \quad (3.82)$$

and

$$\begin{aligned} \gamma l^t \|(\nabla g - S_2)(\nabla g^*((1 - \gamma l^t)\nabla g z_n + \gamma l^t S_2 z_n)) - (\nabla g - S_2)z_n\| \\ \leq \pi \|\nabla g^*((1 - \gamma l^t)\nabla g z_n + \gamma l^t S_2 z_n) - z_n\|. \end{aligned} \quad (3.83)$$

Step 5: Compute

$$r_n = \nabla f^*((1 - \lambda_n)\nabla f e_n + \lambda_n \nabla f c_n + \lambda_n(T_1 e_n - T_1 c_n)), \quad (3.84)$$

$$s_n = \nabla f^*((1 - \sigma_n)\nabla f h_n + \sigma_n \nabla f c_n + \sigma_n(T_2 h_n - T_2 c_n)),$$

$$u_n = \nabla g^*((1 - \tau_n)\nabla g p_n + \tau_n \nabla g z_n + \tau_n(S_1 p_n - S_1 z_n)),$$

$$v_n = \nabla g^*((1 - \pi_n)\nabla g q_n + \pi_n \nabla g z_n + \pi_n(S_2 q_n - S_2 z_n)).$$

Step 6: Compute

$$w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n \nabla f r_n + \phi_n \nabla f s_n), \quad (3.85)$$

$$\bar{w}_n = \nabla g^*(\beta_n \nabla g y_n + \delta_n \nabla g u_n + \phi_n \nabla g v_n),$$

where $\beta_n + \delta_n + \phi_n = 1$.

Step 7: Compute

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f x + (1 - \alpha_n)\nabla f w_n), \quad (3.86)$$

$$y_{n+1} = \nabla g^*(\alpha_n \nabla g y + (1 - \alpha_n)\nabla g \bar{w}_n).$$

Set $n = n + 1$ and go to **Step 1**.

Remark 3.3.1. Some of the novelties of our method over the existing ones are:

- As far as our knowledge is concerned, our method is the first method of its kind proposed in the setting of reflexive Banach spaces.
- It extends the nature of the underlying mappings from Lipschitz continuous to the more class of uniformly continuous mappings in reflexive Banach spaces.

Lemma 3.3.2. Assume that conditions (D1) – (D7) hold. Then the Armijo line search rules (3.80) – (3.83) in Algorithm 3.2 are well defined.

Proof. We consider two cases concerning c_n and z_n :

Case I. Assume that c_n is an f -fixed point of T_1 . That is, $(\nabla f - T_1)c_n = 0$. Then (3.80) is satisfied for $j = 0$. Similarly, if we assume that c_n is an f -fixed point of T_2 , then (3.81) is satisfied for $m = 0$. If we assume that z_n is a g -fixed point of S_1 and S_2 , then (3.82) and (3.83) are satisfied for $k = 0$ and $t = 0$, respectively.

Case II. Assume that c_n is not an f -fixed point of T_1 and suppose on the contrary that for all natural numbers j we have

$$\|(\nabla f - T_1)k_{n_j} - (\nabla f - T_1)c_n\| > \frac{\lambda}{\gamma l^j} \|k_{n_j} - c_n\|, \quad (3.87)$$

where $k_{n_j} = \nabla f^* ((1 - \gamma l^j) \nabla f c_n + \gamma l^j T_1 c_n)$.

From the continuity of ∇f^* , we get

$$\lim_{j \rightarrow \infty} \|k_{n_j} - c_n\| = \lim_{j \rightarrow \infty} \|\nabla f^* ((1 - \gamma l^j) \nabla f c_n + \gamma l^j T_1 c_n) - c_n\| = 0. \quad (3.88)$$

Since $\nabla f - T_1$ is uniformly continuous, we obtain from (3.88) that

$$\lim_{j \rightarrow \infty} \|(\nabla f - T_1)k_{n_j} - (\nabla f - T_1)c_n\| = 0. \quad (3.89)$$

Combining (3.87) and (3.89), we obtain

$$\lim_{j \rightarrow \infty} \frac{\lambda}{\gamma l^j} \|k_{n_j} - c_n\| = 0. \quad (3.90)$$

The Lipschitz continuity of ∇f together with (3.90) gives

$$\lim_{j \rightarrow \infty} \left(\frac{\nabla f k_{n_j} - \nabla f c_n}{\gamma l^j} \right) = 0. \quad (3.91)$$

Since $k_{n_j} \in E_1$, one can write $k_{n_j} = P_{E_1}^f \nabla f^* ((1 - \gamma l^j) \nabla f c_n + \gamma l^j T_1 c_n)$. Thus, we have by (1.18) that

$$\langle (1 - \gamma l^j) \nabla f c_n + \gamma l^j T_1 c_n - \nabla f k_{n_j}, s - k_{n_j} \rangle \leq 0, \text{ for all } s \in E_1,$$

which implies that

$$\left\langle \frac{\nabla f c_n - \nabla f k_{n_j}}{\gamma l^j}, s - k_{n_j} \right\rangle - \langle (\nabla f - T_1)c_n, s - k_{n_j} \rangle \leq 0, \text{ for all } s \in E_1. \quad (3.92)$$

Taking limit as $j \rightarrow \infty$ in (3.92) and using (3.91), we obtain

$$- \lim_{j \rightarrow \infty} \langle (\nabla f - T_1)c_n, s - k_{n_j} \rangle \leq 0, \text{ for all } s \in E_1. \quad (3.93)$$

Putting $s = -J^{-1}[(\nabla f - T_1)c_n] + k_{n_j}$ into (3.93), we obtain

$$- \lim_{j \rightarrow \infty} \langle (\nabla f - T_1)c_n, -J^{-1}[(\nabla f - T_1)c_n] + k_{n_j} - k_{n_j} \rangle = \|(\nabla f - T_1)c_n\|^2 \leq 0, \quad (3.94)$$

which implies that

$$(\nabla f - T_1)c_n = 0, \quad (3.95)$$

that is, c_n is an f -fixed point of T_1 , which is a contradiction. Thus, (3.80) is well defined. Following similar steps, we can easily show that (3.81) is

well defined. Considering similar case on z_n , it can be shown that (3.82) and (3.83) are well defined and hence the proof is complete. $\square$

Theorem 3.3.1. *Assume that conditions (D1) – (D7) hold. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 3.2 are bounded.*

Proof. Let $(x^*, y^*) \in \Omega$. We have from (3.86) and Lemma 3.1.9 that

$$\begin{aligned} D_f(x^*, x_{n+1}) &= D_f(x^*, \nabla f^*((1 - \alpha_n)\nabla f w_n + \alpha_n \nabla f x)) \\ &\leq \alpha_n D_f(x^*, x) + (1 - \alpha_n) D_f(x^*, w_n). \end{aligned} \quad (3.96)$$

Similarly, we have from (3.85) and Lemma 3.1.9 that

$$D_f(x^*, w_n) \leq \beta_n D_f(x^*, x_n) + \delta_n D_f(x^*, r_n) + \phi_n D_f(x^*, s_n). \quad (3.97)$$

From Definition 1.2.17 and (3.84), we obtain

$$\begin{aligned} D_f(x^*, r_n) &= D_f\left(x^*, \nabla f^*((1 - \lambda_n)\nabla f e_n + \lambda_n \nabla f c_n + \lambda_n (T_1 e_n - T_1 c_n))\right) \\ &= f(x^*) - \langle (1 - \lambda_n)\nabla f e_n + \lambda_n \nabla f c_n + \lambda_n (T_1 e_n - T_1 c_n), x^* - r_n \rangle - f(r_n) \\ &= f(x^*) - \langle \nabla f e_n, x^* - r_n \rangle \\ &\quad - \langle \lambda_n (T_1 e_n - T_1 c_n), x^* - r_n \rangle - \langle \lambda_n (\nabla f c_n - \nabla f e_n), x^* - r_n \rangle - f(r_n) \\ &= f(x^*) - \langle \nabla f e_n, x^* - e_n \rangle - f(e_n) + \langle \nabla f e_n, x^* - e_n \rangle + f(e_n) - \langle \nabla f e_n, x^* - r_n \rangle \\ &\quad - \langle \lambda_n (T_1 e_n - T_1 c_n), x^* - r_n \rangle - \langle \lambda_n (\nabla f c_n - \nabla f e_n), x^* - r_n \rangle - f(r_n) \\ &= D_f(x^*, e_n) + \langle \nabla f e_n, r_n - e_n \rangle + f(e_n) - f(r_n) \\ &\quad - \langle \lambda_n ((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), x^* - r_n \rangle \\ &= D_f(x^*, e_n) - D_f(r_n, e_n) - \langle \lambda_n ((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), x^* - r_n \rangle. \end{aligned} \quad (3.98)$$

From (1.16), we have

$$D_f(x^*, e_n) - D_f(r_n, e_n) = D_f(x^*, c_n) - D_f(r_n, c_n) + \langle \nabla f c_n - \nabla f e_n, x^* - r_n \rangle. \quad (3.99)$$

Combining (3.98) and (3.99), we obtain

$$\begin{aligned} D_f(x^*, r_n) &= D_f(x^*, c_n) - D_f(r_n, c_n) + \langle \nabla f c_n - \nabla f e_n, x^* - r_n \rangle \\ &\quad - \langle \lambda_n ((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), x^* - r_n \rangle. \end{aligned} \quad (3.100)$$

From (1.15), we obtain

$$D_f(r_n, c_n) = D_f(r_n, e_n) + D_f(e_n, c_n) - \langle \nabla f e_n - \nabla f c_n, e_n - r_n \rangle. \quad (3.101)$$

Combining (3.100) and (3.101), we get

$$\begin{aligned}
 D_f(x^*, r_n) &= D_f(x^*, c_n) - D_f(r_n, e_n) - D_f(e_n, c_n) + \langle \nabla f e_n - \nabla f c_n, e_n - r_n \rangle \\
 &\quad + \langle \nabla f c_n - \nabla f e_n, x^* - r_n \rangle - \langle \lambda_n((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), x^* - r_n \rangle \\
 &= D_f(x^*, c_n) - D_f(r_n, e_n) - D_f(e_n, c_n) + \langle \nabla f c_n - \nabla f e_n, x^* - e_n \rangle \\
 &\quad - \langle \lambda_n((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), x^* - e_n \rangle - \langle \lambda_n((\nabla f - T_1)c_n \\
 &\quad - (\nabla f - T_1)e_n), e_n - r_n \rangle,
 \end{aligned}$$

that is,

$$\begin{aligned}
 D_f(x^*, r_n) &= D_f(x^*, c_n) - D_f(r_n, e_n) - D_f(e_n, c_n) \\
 &\quad + \langle \lambda_n((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), r_n - e_n \rangle \\
 &\quad - \langle \lambda_n((\nabla f - T_1)e_n - (\nabla f - T_1)c_n) - (\nabla f e_n - \nabla f c_n), e_n - x^* \rangle.
 \end{aligned} \tag{3.102}$$

Since $e_n = \nabla f^*((1 - \lambda_n)\nabla f c_n + \lambda_n T_1 c_n)$ we have that $\nabla f e_n = (1 - \lambda_n)\nabla f c_n + \lambda_n T_1 c_n$. Let $\psi_n = (\nabla f - T_1)e_n$. Then we get $\nabla f c_n + \lambda_n((\nabla f - T_1)e_n - (\nabla f - T_1)c_n) = \nabla f e_n + \lambda_n \psi_n$, which implies that

$$\psi_n = \frac{1}{\lambda_n} [\lambda_n((\nabla f - T_1)e_n - (\nabla f - T_1)c_n) - (\nabla f e_n - \nabla f c_n)]. \tag{3.103}$$

Since $(\nabla f - T_1)x^* = 0$ and $\nabla f - T_1$ is monotone, we obtain that

$$\langle \psi_n, e_n - x^* \rangle \geq 0. \tag{3.104}$$

Thus, substituting (3.103) into (3.104), we obtain

$$\langle \lambda_n((\nabla f - T_1)e_n - (\nabla f - T_1)c_n) - (\nabla f e_n - \nabla f c_n), e_n - x^* \rangle \geq 0. \tag{3.105}$$

From (3.102) and (3.105), we get

$$\begin{aligned}
 D_f(x^*, r_n) &\leq D_f(x^*, c_n) - D_f(r_n, e_n) - D_f(e_n, c_n) \\
 &\quad + \langle \lambda_n((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), r_n - e_n \rangle.
 \end{aligned} \tag{3.106}$$

From (3.76), (1.21) and (1.22), we have

$$\begin{aligned}
 D_f(x^*, c_n) &= D_f\left(x^*, \nabla f^*(\nabla f a_n - \gamma_n A^* J_{E_3}(Aa_n - Bb_n))\right) \\
 &= V_f(x^*, \nabla f a_n - \gamma_n A^* J_{E_3}(Aa_n - Bb_n)) \\
 &\leq V_f(x^*, \nabla f a_n) - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle \\
 &= D_f(x^*, a_n) - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle.
 \end{aligned} \tag{3.107}$$

Combining (3.106) and (3.107), we obtain

$$\begin{aligned}
 D_f(x^*, r_n) &\leq D_f(x^*, a_n) - D_f(r_n, e_n) - D_f(e_n, c_n) \\
 &\quad + \langle \lambda_n((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), r_n - e_n \rangle \\
 &\quad - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle.
 \end{aligned} \tag{3.108}$$

From (3.75) and Definition 1.2.17, we have

$$\begin{aligned}
 D_f(x^*, a_n) &= D_f(x^*, \nabla f^*(\nabla f x_n + \theta_n(\nabla f x_n - \nabla f x_{n-1}))) \\
 &= f(x^*) - \langle \nabla f x_n + \theta_n(\nabla f x_n - \nabla f x_{n-1}), x^* - a_n \rangle - f(a_n) \\
 &= f(x^*) - \langle \nabla f x_n, x^* - a_n \rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), x^* - a_n \rangle - f(a_n) \\
 &= f(x^*) - \langle \nabla f x_n, x^* - x_n \rangle - f(x_n) + \langle \nabla f x_n, x^* - x_n \rangle + f(x_n) \\
 &\quad - \langle \nabla f x_n, x^* - a_n \rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), x^* - a_n \rangle - f(a_n) \\
 &= D_f(x^*, x_n) + \langle \nabla f x_n, a_n - x_n \rangle + f(x_n) - f(a_n) - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), x^* - a_n \rangle \\
 &= D_f(x^*, x_n) - D_f(a_n, x_n) - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), x^* - a_n \rangle.
 \end{aligned} \tag{3.109}$$

Combining (3.108) and (3.109), we obtain

$$\begin{aligned}
 D_f(x^*, r_n) &\leq D_f(x^*, x_n) - D_f(r_n, e_n) - D_f(e_n, c_n) - D_f(a_n, x_n) \\
 &\quad + \langle \lambda_n((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), r_n - e_n \rangle \\
 &\quad - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), x^* - a_n \rangle.
 \end{aligned} \tag{3.110}$$

Moreover, we obtain from the Cauchy–Schwarz Inequality and Lemma 3.1.10 that

$$\begin{aligned}
 -\langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), x^* - a_n \rangle &\leq \theta_n \|\nabla f x_n - \nabla f x_{n-1}\| \|(x^* - a_n) \times 1\| \\
 &\leq \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| [\|x^* - a_n\|^2 + 1] \\
 &= \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| [\|x^* - x_n + x_n - a_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| [2\|x^* - x_n\|^2 + 2\|x_n - a_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| \left[\frac{4}{\beta} D_f(x^*, x_n) + \frac{4}{\beta} D_f(a_n, x_n) + 1 \right] \\
 &= \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(x^*, x_n) \\
 &\quad + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(a_n, x_n) \\
 &\quad + \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\|.
 \end{aligned} \tag{3.111}$$

From (3.110) and (3.111), we obtain that

$$\begin{aligned}
 D_f(x^*, r_n) &\leq D_f(x^*, x_n) - D_f(r_n, e_n) - D_f(e_n, c_n) - D_f(a_n, x_n) \\
 &\quad + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(x^*, x_n) + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(a_n, x_n) \\
 &\quad + \langle \lambda_n((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), r_n - e_n \rangle + \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| \\
 &\quad - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle,
 \end{aligned}$$

which implies by the relation $\theta_n \|\nabla f x_n - \nabla f x_{n-1}\| \leq \zeta_n$ that

$$\begin{aligned} D_f(x^*, r_n) &\leq D_f(x^*, x_n) - D_f(r_n, e_n) - D_f(e_n, c_n) - D_f(a_n, x_n) + \frac{2\zeta_n}{\beta} D_f(x^*, x_n) \\ &\quad + \frac{2\zeta_n}{\beta} D_f(a_n, x_n) + \langle \lambda_n((\nabla f - T_1)c_n - (\nabla f - T_1)e_n), r_n - e_n \rangle + \frac{\zeta_n}{2} \\ &\quad - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle. \end{aligned} \quad (3.112)$$

From (3.112), Lemma 3.1.10 and the Cauchy–Schwarz Inequality, we obtain

$$\begin{aligned} D_f(x^*, r_n) &\leq D_f(x^*, x_n) - D_f(r_n, e_n) - D_f(e_n, c_n) - D_f(a_n, x_n) + \frac{2\zeta_n}{\beta} D_f(x^*, x_n) \\ &\quad + \frac{2\zeta_n}{\beta} D_f(a_n, x_n) + \lambda_n \|(\nabla f - T_1)c_n - (\nabla f - T_1)e_n\| \|r_n - e_n\| + \frac{\zeta_n}{2} \\ &\quad - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle \\ &\leq D_f(x^*, x_n) - D_f(r_n, e_n) - D_f(e_n, c_n) - D_f(a_n, x_n) + \frac{2\zeta_n}{\beta} D_f(x^*, x_n) \\ &\quad + \frac{2\zeta_n}{\beta} D_f(a_n, x_n) + \lambda \|c_n - e_n\| \|r_n - e_n\| + \frac{\zeta_n}{2} \\ &\quad - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle \\ &\leq D_f(x^*, x_n) - D_f(r_n, e_n) - D_f(e_n, c_n) - D_f(a_n, x_n) + \frac{2\zeta_n}{\beta} D_f(x^*, x_n) \\ &\quad + \frac{2\zeta_n}{\beta} D_f(a_n, x_n) + \frac{\lambda}{2} [\|c_n - e_n\|^2 + \|r_n - e_n\|^2] \\ &\quad + \frac{\zeta_n}{2} - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle \\ &\leq D_f(x^*, x_n) - D_f(r_n, e_n) - D_f(e_n, c_n) - D_f(a_n, x_n) + \frac{2\zeta_n}{\beta} D_f(x^*, x_n) \\ &\quad + \frac{2\zeta_n}{\beta} D_f(a_n, x_n) + \frac{\lambda}{\beta} D_f(e_n, c_n) + \frac{\lambda}{\beta} D_f(r_n, e_n) \\ &\quad + \frac{\zeta_n}{2} - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle. \\ &= \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(x^*, x_n) - \left(1 - \frac{2\zeta_n}{\beta}\right) D_f(a_n, x_n) - \left(1 - \frac{\lambda}{\beta}\right) D_f(r_n, e_n) \\ &\quad - \left(1 - \frac{\lambda}{\beta}\right) D_f(e_n, c_n) + \frac{\zeta_n}{2} - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle. \end{aligned} \quad (3.113)$$

Since $\zeta_n \in \left(0, \frac{\beta}{2}\right)$ and $\lambda < \beta$, we obtain from (3.113) that

$$D_f(x^*, r_n) \leq \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(x^*, x_n) + \frac{\zeta_n}{2} - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle. \quad (3.114)$$

Similarly,

$$\begin{aligned} D_f(x^*, s_n) &\leq \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(x^*, x_n) - \left(1 - \frac{2\zeta_n}{\beta}\right) D_f(a_n, x_n) - \left(1 - \frac{\sigma}{\beta}\right) D_f(s_n, h_n) \\ &\quad - \left(1 - \frac{\sigma}{\beta}\right) D_f(h_n, c_n) + \frac{\zeta_n}{2} - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle, \end{aligned} \quad (3.115)$$

which together with the conditions on $\{\zeta_n\}$ and σ implies that

$$D_f(x^*, s_n) \leq \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(x^*, x_n) + \frac{\zeta_n}{2} - \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle. \quad (3.116)$$

From (3.96), (3.97), (3.113) and (3.115), we get

$$\begin{aligned} D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, x) + (1 - \alpha_n) \beta_n D_f(x^*, x_n) + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) \delta_n D_f(x^*, x_n) \\ &\quad - (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) \delta_n D_f(a_n, x_n) - (1 - \alpha_n) \left(1 - \frac{\lambda}{\beta}\right) \delta_n D_f(r_n, e_n) \\ &\quad - (1 - \alpha_n) \left(1 - \frac{\lambda}{\beta}\right) \delta_n D_f(e_n, c_n) + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) \phi_n D_f(x^*, x_n) \\ &\quad - (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) \phi_n D_f(a_n, x_n) - (1 - \alpha_n) \left(1 - \frac{\sigma}{\beta}\right) \phi_n D_f(s_n, h_n) \\ &\quad - (1 - \alpha_n) \left(1 - \frac{\sigma}{\beta}\right) \phi_n D_f(h_n, c_n) + (1 - \alpha_n) \frac{\phi_n \zeta_n}{2} + (1 - \alpha_n) \frac{\delta_n \zeta_n}{2} \\ &\quad - (1 - \alpha_n) \gamma_n (\delta_n + \phi_n) \langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle. \end{aligned} \quad (3.117)$$

From (3.117) and the conditions on $\alpha_n, \beta_n, \delta_n$ and ϕ_n , we obtain

$$\begin{aligned} D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, x) + (1 - \alpha_n) \beta_n D_f(x^*, x_n) + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) \delta_n D_f(x^*, x_n) \\ &\quad + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) \phi_n D_f(x^*, x_n) + (1 - \alpha_n) \frac{\phi_n \zeta_n}{2} + (1 - \alpha_n) \frac{\delta_n \zeta_n}{2} \\ &\quad - (1 - \alpha_n) \gamma_n (\delta_n + \phi_n) \langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle. \\ &= \alpha_n D_f(x^*, x) + (1 - \alpha_n) D_f(x^*, x_n) + (1 - \alpha_n) \frac{2\zeta_n}{\beta} \delta_n D_f(x^*, x_n) \\ &\quad + (1 - \alpha_n) \frac{2\zeta_n}{\beta} \phi_n D_f(x^*, x_n) + (1 - \alpha_n) \frac{\phi_n \zeta_n}{2} + (1 - \alpha_n) \frac{\delta_n \zeta_n}{2} \\ &\quad - (1 - \alpha_n) \gamma_n (\delta_n + \phi_n) \langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle \\ &\leq \alpha_n D_f(x^*, x) + (1 - \alpha_n) D_f(x^*, x_n) + (1 - \alpha_n) \frac{2\zeta_n}{\beta} D_f(x^*, x_n) + \zeta_n \\ &\quad + (1 - \alpha_n) \frac{2\zeta_n}{\beta} D_f(x^*, x_n) - (1 - \alpha_n) (\delta_n + \phi_n) [\gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle] \\ &= \alpha_n D_f(x^*, x) + (1 - \alpha_n) D_f(x^*, x_n) + (1 - \alpha_n) \frac{4\zeta_n}{\beta} D_f(x^*, x_n) + \zeta_n \\ &\quad - (1 - \alpha_n) (\delta_n + \phi_n) [\gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle], \end{aligned}$$

which implies that

$$\begin{aligned} D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, x) + (1 - \alpha_n) \left(1 + \frac{4\zeta_n}{\beta}\right) D_f(x^*, x_n) + \zeta_n \\ &\quad - (1 - \alpha_n) (\delta_n + \phi_n) [\gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle]. \end{aligned} \quad (3.118)$$

Since $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$, for every $\varepsilon \in \left(0, \frac{\beta}{4}\right)$ there exists $N \in \mathbb{N}$ such that $\zeta_n < \varepsilon\alpha_n$, for all $n \geq N$. Thus, we obtain from (3.118) that

$$\begin{aligned} D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, x) + (1 - \alpha_n) \left(1 + \frac{4\varepsilon\alpha_n}{\beta}\right) D_f(x^*, x_n) + \varepsilon\alpha_n \\ &\quad - (1 - \alpha_n)(\delta_n + \phi_n) [\gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle], \end{aligned} \quad (3.119)$$

for all $n \geq N$. Similarly, we have

$$\begin{aligned} D_g(y^*, y_{n+1}) &\leq \alpha_n D_g(y^*, y) + (1 - \alpha_n)\beta_n D_g(y^*, y_n) + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) \delta_n D_g(y^*, y_n) \\ &\quad - (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) \delta_n D_g(b_n, y_n) - (1 - \alpha_n) \left(1 - \frac{\tau}{\beta}\right) \delta_n D_g(u_n, p_n) \\ &\quad - (1 - \alpha_n) \left(1 - \frac{\tau}{\beta}\right) \delta_n D_g(p_n, z_n) + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) \phi_n D_g(y^*, y_n) \\ &\quad - (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) \phi_n D_g(b_n, y_n) - (1 - \alpha_n) \left(1 - \frac{\pi}{\beta}\right) \phi_n D_g(v_n, q_n) \\ &\quad - (1 - \alpha_n) \left(1 - \frac{\pi}{\beta}\right) \phi_n D_g(q_n, z_n) + (1 - \alpha_n) \frac{\phi_n \zeta_n}{2} + (1 - \alpha_n) \frac{\delta_n \zeta_n}{2} \\ &\quad - (1 - \alpha_n) \gamma_n (\delta_n + \phi_n) \langle B^* J_{E_3}(Bb_n - Aa_n), z_n - y^* \rangle, \end{aligned} \quad (3.120)$$

which implies that

$$\begin{aligned} D_g(y^*, y_{n+1}) &\leq \alpha_n D_g(y^*, y) + (1 - \alpha_n) \left(1 + \frac{4\varepsilon\alpha_n}{\beta}\right) D_g(y^*, y_n) + \varepsilon\alpha_n \\ &\quad - (1 - \alpha_n)(\delta_n + \phi_n) [\gamma_n \langle B^* J_{E_3}(Bb_n - Aa_n), z_n - y^* \rangle]. \end{aligned} \quad (3.121)$$

Denote $\Omega_n = D_f(x^*, x_n) + D_g(y^*, y_n)$ and $\Sigma = D_f(x^*, x) + D_g(y^*, y)$. Then combination of (3.119) and (3.121) gives

$$\begin{aligned} \Omega_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{4\varepsilon\alpha_n}{\beta}\right) \Omega_n + 2\varepsilon\alpha_n \\ &\quad - (1 - \alpha_n)(\delta_n + \phi_n) [\gamma_n \langle J_{E_3}(Aa_n - Bb_n), Ac_n - Bz_n \rangle]. \end{aligned} \quad (3.122)$$

Furthermore, we have

$$\begin{aligned} -\langle J_{E_3}(Aa_n - Bb_n), Ac_n - Bz_n \rangle &= -\langle J_{E_3}(Aa_n - Bb_n), Aa_n - Bb_n \rangle \\ &\quad - \langle J_{E_3}(Aa_n - Bb_n), Ac_n - Aa_n \rangle \\ &\quad - \langle J_{E_3}(Aa_n - Bb_n), Bb_n - Bz_n \rangle \\ &= -\|Aa_n - Bb_n\|^2 - \langle A^* J_{E_3}(Aa_n - Bb_n), c_n - a_n \rangle \\ &\quad - \langle B^* J_{E_3}(Aa_n - Bb_n), b_n - z_n \rangle \\ &\leq -\|Aa_n - Bb_n\|^2 + \|c_n - a_n\| \|A^* J_{E_3}(Aa_n - Bb_n)\| \\ &\quad + \|b_n - z_n\| \|B^* J_{E_3}(Aa_n - Bb_n)\|. \end{aligned} \quad (3.123)$$

The Lipschitz continuity of ∇f^* gives that

$$\begin{aligned} \|c_n - a_n\| &= \|\nabla f^*(\nabla f(a_n) - \gamma_n A^* J_{E_3}(Aa_n - Bb_n)) - \nabla f^*(\nabla f(a_n))\| \\ &\leq \frac{\gamma_n}{\beta} \|A^* J_{E_3}(Aa_n - Bb_n)\|. \end{aligned} \quad (3.124)$$

Similarly, we obtain from the Lipschitz continuity of ∇g^* that

$$\begin{aligned} \|d_n - b_n\| &= \|\nabla g^*(\nabla g(b_n) - \gamma_n B^* J_{E_3}(Bb_n - Aa_n)) - \nabla g^*(\nabla g(b_n))\| \\ &\leq \frac{\gamma_n}{\beta} \|B^* J_{E_3}(Bb_n - Aa_n)\|. \end{aligned} \quad (3.125)$$

Combining (3.123), (3.124) and (3.125) and using (3.77), we obtain

$$\begin{aligned} &-\gamma_n \langle J_{E_3}(Aa_n - Bb_n), Ac_n - Bz_n \rangle \\ &\leq -\gamma_n \|Aa_n - Bb_n\|^2 + \frac{\gamma_n^2}{\beta} \|A^* J_{E_3}(Aa_n - Bb_n)\|^2 \\ &\quad + \frac{\gamma_n^2}{\beta} \|B^* J_{E_3}(Bb_n - Aa_n)\|^2 \\ &\leq -\frac{\rho}{2} \|Aa_n - Bb_n\|^2 - \frac{\gamma_n}{2} \|Aa_n - Bb_n\|^2 \\ &\quad + \frac{\gamma_n}{2} \left\{ \frac{2\gamma_n}{\beta} [\|A^* J_{E_3}(Aa_n - Bb_n)\|^2 + \|B^* J_{E_3}(Bb_n - Aa_n)\|^2] \right\} \\ &\leq -\frac{\rho}{2} \|Aa_n - Bb_n\|^2. \end{aligned} \quad (3.126)$$

Since $(1 - \alpha_n)(\delta_n + \phi_n) > 0$, we conclude from (3.126) that

$$-(1 - \alpha_n)(\delta_n + \phi_n) [\gamma_n \langle J_{E_3}(Aa_n - Bb_n), Ac_n - Bz_n \rangle] \leq -(1 - \alpha_n)(\delta_n + \phi_n) \frac{\rho}{2} \|Aa_n - Bb_n\|^2 \leq 0. \quad (3.127)$$

Thus, we obtain from (3.127) and (3.122) that

$$\begin{aligned} \Omega_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{4\varepsilon\alpha_n}{\beta} \right) \Omega_n + 2\varepsilon\alpha_n \\ &\quad - (1 - \alpha_n)(\delta_n + \phi_n) \frac{\rho}{2} \|Aa_n - Bb_n\|^2 \\ &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{4\varepsilon\alpha_n}{\beta} \right) \Omega_n + 2\varepsilon\alpha_n \\ &\leq \left[1 - \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) \right] \Omega_n + \alpha_n [\Sigma + 2\varepsilon] \\ &= \left[1 - \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) \right] \Omega_n + \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) \left[\frac{\beta(\Sigma + 2\varepsilon)}{\beta - 4\varepsilon} \right] \\ &\leq \max \left\{ \Omega_n, \frac{\beta(\Sigma + 2\varepsilon)}{\beta - 4\varepsilon} \right\}, \end{aligned}$$

for all $n \geq N$. Thus, we have by induction that $\Omega_n \leq \max \left\{ \Omega_N, \frac{\beta(\Sigma + 2\varepsilon)}{\beta - 4\varepsilon} \right\}$ for all $n \geq 1$, where $\mathcal{K} = \max \{\Omega_1, \Omega_2, \dots, \Omega_{N-1}\}$. Thus, the sequence $\{\Omega_n\}$ is bounded and hence the sequences $\{D_f(x^*, x_n)\}$ and $\{D_g(y^*, y_n)\}$ are bounded. Thus, it follows by Lemma 3.1.11 that

the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. $\square$

Theorem 3.3.2. *Assume that conditions (D1) – (D7) hold. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Omega^h(x, y)$ and $h = (f, g)$.*

Proof. Let $(x^*, y^*) = P_\Omega^h(x, y)$. Then we have by (1.18) that

$$\langle (\nabla f x, \nabla g y) - (\nabla f x^*, \nabla g y^*), (u, v) - (x^*, y^*) \rangle \leq 0, \text{ for all } (u, v) \in \Omega. \quad (3.128)$$

From Definition 1.2.17, (1.21), (1.22) and Lemma 3.1.9, we get

$$\begin{aligned} D_f(x^*, x_{n+1}) &= D_f(x^*, \nabla f^*(\alpha_n \nabla f x + (1 - \alpha_n) \nabla f w_n)) \\ &= V_f(x^*, \alpha_n \nabla f x + (1 - \alpha_n) \nabla f w_n) \\ &\leq V_f(x^*, \alpha_n \nabla f x + (1 - \alpha_n) \nabla f w_n - \alpha_n (\nabla f x - \nabla f x^*)) \\ &\quad + \langle \alpha_n (\nabla f x - \nabla f x^*), x_{n+1} - x^* \rangle \\ &= V_f(x^*, \alpha_n \nabla f x^* + (1 - \alpha_n) \nabla f w_n) + \langle \alpha_n (\nabla f x - \nabla f x^*), x_{n+1} - x^* \rangle \quad (3.129) \\ &= D_f(x^*, \nabla f^*(\alpha_n \nabla f x^* + (1 - \alpha_n) \nabla f w_n)) \\ &\quad + \langle \alpha_n (\nabla f x - \nabla f x^*), x_{n+1} - x^* \rangle \\ &\leq \alpha_n D_f(x^*, x^*) + (1 - \alpha_n) D_f(x^*, w_n) + \langle \alpha_n (\nabla f x - \nabla f x^*), x_{n+1} - x^* \rangle \\ &= (1 - \alpha_n) D_f(x^*, w_n) + \langle \alpha_n (\nabla f x - \nabla f x^*), x_{n+1} - x^* \rangle. \end{aligned}$$

From (3.85), Lemma 3.1.9, (3.114) and (3.116), we get

$$\begin{aligned} D_f(x^*, w_n) &= D_f(x^*, \nabla f^*(\beta_n \nabla f x_n + \delta_n \nabla f r_n + \phi_n \nabla f s_n)) \\ &\leq \beta_n D_f(x^*, x_n) + \delta_n D_f(x^*, r_n) + \phi_n D_f(x^*, s_n) \\ &\leq \beta_n D_f(x^*, x_n) + \left(1 + \frac{2\zeta_n}{\beta}\right) \delta_n D_f(x^*, x_n) + \delta_n \frac{\zeta_n}{2} \quad (3.130) \\ &\quad - \delta_n \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n) c_n - x^* \rangle + \phi_n \frac{\zeta_n}{2} \\ &\quad + \left(1 + \frac{2\zeta_n}{\beta}\right) \phi_n D_f(x^*, x_n) - \phi_n \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle. \end{aligned}$$

From (3.130) and the conditions on β_n , δ_n and ϕ_n , we obtain

$$\begin{aligned} D_f(x^*, w_n) &\leq D_f(x^*, x_n) + \delta_n \frac{2\zeta_n}{\beta} D_f(x^*, x_n) + \phi_n \frac{2\zeta_n}{\beta} D_f(x^*, x_n) + \delta_n \frac{\zeta_n}{2} + \phi_n \frac{\zeta_n}{2} \\ &\quad - \delta_n \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle - \phi_n \langle \gamma_n A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle \\ &\leq D_f(x^*, x_n) + \frac{2\zeta_n}{\beta} D_f(x^*, x_n) + \frac{2\zeta_n}{\beta} D_f(x^*, x_n) + \zeta_n \\ &\quad - (\delta_n + \phi_n) \gamma_n [\langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle] \\ &= D_f(x^*, x_n) + \frac{4\zeta_n}{\beta} D_f(x^*, x_n) + \zeta_n - (\delta_n + \phi_n) \gamma_n [\langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle]. \quad (3.131) \end{aligned}$$

Substituting (3.131) into (3.129) and using Cauchy–Schwarz Inequality, we obtain

$$\begin{aligned}
 D_f(x^*, x_{n+1}) &\leq (1 - \alpha_n) \left[D_f(x^*, x_n) + \frac{4\zeta_n}{\beta} D_f(x^*, x_n) + \zeta_n \right] + \langle \alpha_n (\nabla f x - \nabla f x^*), x_{n+1} - x^* \rangle \\
 &\quad - (1 - \alpha_n)(\delta_n + \phi_n)\gamma_n [\langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle] \\
 &\leq (1 - \alpha_n) \left(1 + \frac{4\zeta_n}{\beta} \right) D_f(x^*, x_n) + (1 - \alpha_n)\zeta_n \\
 &\quad + \alpha_n \|\nabla f x - \nabla f x^*\| \|x_{n+1} - x_n\| + \alpha_n \langle \nabla f x - \nabla f x^*, x_n - x^* \rangle \\
 &\quad - (1 - \alpha_n)(\delta_n + \phi_n)\gamma_n [\langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle].
 \end{aligned} \tag{3.132}$$

Since $\zeta_n < \varepsilon\alpha_n$, we obtain from (3.132) that

$$\begin{aligned}
 D_f(x^*, x_{n+1}) &\leq (1 - \alpha_n) \left(1 + \frac{4\varepsilon\alpha_n}{\beta} \right) D_f(x^*, x_n) + \alpha_n \frac{\zeta_n}{\alpha_n} \\
 &\quad + \alpha_n \|\nabla f x - \nabla f x^*\| \|x_{n+1} - x_n\| + \alpha_n \langle \nabla f x - \nabla f x^*, x_n - x^* \rangle \\
 &\quad - (1 - \alpha_n)(\delta_n + \phi_n)\gamma_n [\langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle] \\
 &\leq \left[1 - \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) \right] D_f(x^*, x_n) \\
 &\quad + \alpha_n \left[\frac{\zeta_n}{\alpha_n} + \|\nabla f x - \nabla f x^*\| \|x_{n+1} - x_n\| + \langle \nabla f x - \nabla f x^*, x_n - x^* \rangle \right] \\
 &\quad - (1 - \alpha_n)(\delta_n + \phi_n)\gamma_n [\langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle] \\
 &= \left[1 - \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) \right] D_f(x^*, x_n) \\
 &\quad + \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) \left[\frac{\frac{\zeta_n}{\alpha_n} + \|\nabla f x - \nabla f x^*\| \|x_{n+1} - x_n\| + \langle \nabla f x - \nabla f x^*, x_n - x^* \rangle}{\left(1 - \frac{4\varepsilon}{\beta} \right)} \right] \\
 &\quad - (1 - \alpha_n)(\delta_n + \phi_n)\gamma_n [\langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle] \\
 &= \left[1 - \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) \right] D_f(x^*, x_n) + \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) K_n \\
 &\quad - (1 - \alpha_n)(\delta_n + \phi_n)\gamma_n [\langle A^* J_{E_3}(Aa_n - Bb_n), c_n - x^* \rangle],
 \end{aligned} \tag{3.133}$$

where

$$K_n = \left[\frac{\frac{\zeta_n}{\alpha_n} + \|\nabla f x - \nabla f x^*\| \|x_{n+1} - x_n\| + \langle \nabla f x - \nabla f x^*, x_n - x^* \rangle}{\left(1 - \frac{4\varepsilon}{\beta} \right)} \right].$$

Similarly,

$$\begin{aligned}
 D_g(y^*, y_{n+1}) &\leq \left[1 - \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) \right] D_g(y^*, y_n) + \alpha_n \left(1 - \frac{4\varepsilon}{\beta} \right) M_n \\
 &\quad - (1 - \alpha_n)(\delta_n + \phi_n) \langle \gamma_n B^* J_{E_3}(Bb_n - Aa_n), z_n - y^* \rangle,
 \end{aligned} \tag{3.134}$$

where

$$M_n = \left[\frac{\frac{\zeta_n}{\alpha_n} + \|\nabla gy - \nabla gy^*\| \|y_{n+1} - y_n\| + \langle \nabla gy - \nabla gy^*, y_n - y^* \rangle}{\left(1 - \frac{4\varepsilon}{\beta}\right)} \right].$$

Denote $\Omega_n^* = D_f(x^*, x_n) + D_g(y^*, y_n)$ and $\Sigma^* = D_f(x^*, x) + D_g(y^*, y)$. Then combining (3.133) and (3.134) and using the relation (3.126), we obtain

$$\begin{aligned} \Omega_{n+1}^* &\leq \left[1 - \alpha_n \left(1 - \frac{4\varepsilon}{\beta}\right)\right] \Omega_n^* + \alpha_n \left(1 - \frac{4\varepsilon}{\beta}\right) (K_n + M_n) - (1 - \alpha_n)(\delta_n + \phi_n) \frac{\rho}{2} \|Aa_n - Bb_n\|^2 \\ &\leq \left[1 - \alpha_n \left(1 - \frac{4\varepsilon}{\beta}\right)\right] \Omega_n^* + \alpha_n \left(1 - \frac{4\varepsilon}{\beta}\right) (K_n + M_n). \end{aligned} \quad (3.135)$$

Combining (3.117) and (3.120), we obtain

$$\begin{aligned} &(1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) \delta_n D_f(a_n, x_n) + (1 - \alpha_n) \left(1 - \frac{\lambda}{\beta}\right) \delta_n D_f(r_n, e_n) \\ &+ (1 - \alpha_n) \left(1 - \frac{\lambda}{\beta}\right) \delta_n D_f(e_n, c_n) + (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) \phi_n D_f(a_n, x_n) \\ &+ (1 - \alpha_n) \left(1 - \frac{\sigma}{\beta}\right) \phi_n D_f(s_n, h_n) + (1 - \alpha_n) \left(1 - \frac{\sigma}{\beta}\right) \phi_n D_f(h_n, c_n) \\ &+ (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) \delta_n D_g(b_n, y_n) + (1 - \alpha_n) \left(1 - \frac{\tau}{\beta}\right) \delta_n D_g(u_n, p_n) \\ &+ (1 - \alpha_n) \left(1 - \frac{\tau}{\beta}\right) \delta_n D_g(p_n, z_n) + (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) \phi_n D_g(b_n, y_n) \\ &+ (1 - \alpha_n) \left(1 - \frac{\pi}{\beta}\right) \phi_n D_g(v_n, q_n) + (1 - \alpha_n) \left(1 - \frac{\pi}{\beta}\right) \phi_n D_g(q_n, z_n) \\ &+ (1 - \alpha_n)(\delta_n + \phi_n) \frac{\rho}{2} \|Aa_n - Bb_n\|^2 \\ &\leq \Omega_n^* - \Omega_{n+1}^* + \alpha_n \left[\Sigma^* + \left(\frac{4\varepsilon}{\beta} - 1\right) \Omega_n^* + 2\frac{\zeta_n}{\alpha_n} \right]. \end{aligned} \quad (3.136)$$

For the rest of the proof, we consider two cases on Ω_n^* .

Case I. Suppose that the sequence $\{\Omega_n^*\}$ is nonincreasing. It then follows from the Monotone Convergence Theorem that $\{\Omega_n^*\}$ is convergent. Taking limit on both sides of (3.136) as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \|Aa_n - Bb_n\| = 0. \quad (3.137)$$

Taking limit on both sides of (3.136) as $n \rightarrow \infty$ and making use of Lemma 3.1.13, we obtain

$$\lim_{n \rightarrow \infty} \|a_n - x_n\| = \lim_{n \rightarrow \infty} \|r_n - e_n\| = \lim_{n \rightarrow \infty} \|e_n - c_n\| = 0, \quad (3.138)$$

$$\lim_{n \rightarrow \infty} \|s_n - h_n\| = \lim_{n \rightarrow \infty} \|h_n - c_n\| = \lim_{n \rightarrow \infty} \|b_n - y_n\| = 0, \text{ and} \quad (3.139)$$

$$\lim_{n \rightarrow \infty} \|u_n - p_n\| = \lim_{n \rightarrow \infty} \|p_n - z_n\| = \lim_{n \rightarrow \infty} \|v_n - q_n\| = \lim_{n \rightarrow \infty} \|q_n - z_n\| = 0. \quad (3.140)$$

From (3.76) and (3.137), we obtain

$$\begin{aligned}\|\nabla f c_n - \nabla f a_n\| &= \gamma_n \|A^* J_{E_3}(Aa_n - Bb_n)\| \\ &\leq (\rho + 1) \|A^* J_{E_3}(Aa_n - Bb_n)\| \\ &\leq (\rho + 1) \|A^*\| \|J_{E_3}(Aa_n - Bb_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty,\end{aligned}$$

and this implies by the uniform continuity of ∇f^* that

$$\lim_{n \rightarrow \infty} \|c_n - a_n\| = 0. \quad (3.141)$$

From (3.86) and the condition on α_n , we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \|\nabla f x_{n+1} - \nabla f w_n\| &= \lim_{n \rightarrow \infty} \|\alpha_n \nabla f x + (1 - \alpha_n) \nabla f w_n - \nabla f w_n\| \\ &= \lim_{n \rightarrow \infty} \alpha_n \|\nabla f x - \nabla f w_n\| = 0.\end{aligned} \quad (3.142)$$

Since ∇f^* is uniformly continuous, we obtain from (3.142) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - w_n\| = 0. \quad (3.143)$$

From (3.138) and (3.141), we obtain

$$\|r_n - x_n\| \leq \|r_n - e_n\| + \|e_n - c_n\| + \|c_n - a_n\| + \|a_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.144)$$

Similarly,

$$\lim_{n \rightarrow \infty} \|s_n - x_n\| = 0. \quad (3.145)$$

From (3.85) and Lemma 3.1.9 we obtain

$$\begin{aligned}D_f(x_n, w_n) &= D_f(x_n, \nabla f^*(\beta_n \nabla f x_n + \delta_n \nabla f r_n + \phi_n \nabla f s_n)) \\ &\leq \beta_n D_f(x_n, x_n) + \delta_n D_f(x_n, r_n) + \phi_n D_f(x_n, s_n).\end{aligned} \quad (3.146)$$

Combining (3.144), (3.145), (3.146) and using Lemma 3.1.13 we obtain

$$\lim_{n \rightarrow \infty} \|x_n - w_n\| = 0. \quad (3.147)$$

Consequently, we obtain from (3.143) and (3.147) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - w_n\| + \lim_{n \rightarrow \infty} \|w_n - x_n\| = 0. \quad (3.148)$$

Similarly,

$$\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0. \quad (3.149)$$

Since the sequence $\{(x_n, y_n)\}$ is bounded in $E_1 \times E_2$, where $E_1 \times E_2$ is reflexive (Lemma 3.1.5), there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ and an element $(\tilde{x}, \tilde{y})$ of $E_1 \times E_2$ such that $(x_{n_k}, y_{n_k}) \rightharpoonup (\tilde{x}, \tilde{y})$ and

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle (\nabla f x, \nabla g y) - (\nabla f x^*, \nabla g y^*), (x_n, y_n) - (x^*, y^*) \rangle \\ &= \lim_{k \rightarrow \infty} \langle (\nabla f x, \nabla g y) - (\nabla f x^*, \nabla g y^*), (x_{n_k}, y_{n_k}) - (x^*, y^*) \rangle. \end{aligned}$$

Moreover, we have $x_{n_k} \rightharpoonup \tilde{x}$ and $y_{n_k} \rightharpoonup \tilde{y}$. Now, we show that $(\tilde{x}, \tilde{y}) \in \Omega$. From the definition of e_n and the projection property we have

$$\frac{1}{\lambda_{n_k}} \langle \nabla f c_{n_k} - \nabla f e_{n_k}, x - e_{n_k} \rangle \leq \langle (\nabla f - T_1) c_{n_k}, x - e_{n_k} \rangle \text{ for all } x \in E_1,$$

or equivalently,

$$\begin{aligned} & \frac{1}{\lambda_{n_k}} \langle \nabla f c_{n_k} - \nabla f e_{n_k}, x - e_{n_k} \rangle + \langle (\nabla f - T_1) c_{n_k}, e_{n_k} - c_{n_k} \rangle \\ & \leq \langle (\nabla f - T_1) c_{n_k}, x - c_{n_k} \rangle \text{ for all } x \in E_1. \end{aligned} \quad (3.150)$$

Now, we show that $\liminf_{k \rightarrow \infty} \langle (\nabla f - T_1) c_{n_k}, x - c_{n_k} \rangle \geq 0$. To show this, we consider two possible cases. Suppose first that $\liminf_{k \rightarrow \infty} \lambda_{n_k} > 0$. Since ∇f is uniformly continuous, we have from (3.138) that $\nabla f c_{n_k} - \nabla f e_{n_k} \rightarrow 0$ as $k \rightarrow \infty$. Since $\nabla f - T_1$ is uniformly continuous and $\{c_{n_k}\}$ is bounded, we have that the sequence $\{(\nabla f - T_1) c_{n_k}\}$ is bounded. Thus, taking limit of (3.150) as $k \rightarrow \infty$ and taking (3.138) into account, we obtain $\liminf_{k \rightarrow \infty} \langle (\nabla f - T_1) c_{n_k}, x - c_{n_k} \rangle \geq 0$. We now assume the case $\liminf_{k \rightarrow \infty} \lambda_{n_k} = 0$. Put $\omega_{n_k} = \nabla f^* [\nabla f c_{n_k} - \lambda_{n_k} l^{-1} (\nabla f - T_1) c_{n_k}]$. Note that $\lambda_{n_k} l^{-1} > \lambda_{n_k}$. Thus, we obtain by Lemma 3.3.1 and (3.138) that

$$\|c_{n_k} - \omega_{n_k}\| \leq \frac{1}{l} \|c_{n_k} - e_{n_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (3.151)$$

This implies by the uniform continuity of $\nabla f - T_1$ that,

$$\lim_{k \rightarrow \infty} \|(\nabla f - T_1) c_{n_k} - (\nabla f - T_1) \omega_{n_k}\| = 0. \quad (3.152)$$

By the Armijo line-search rule (3.80), we have

$$\begin{aligned} & \lambda_{n_k} l^{-1} \|(\nabla f - T_1) c_{n_k} - (\nabla f - T_1) (\nabla f^* (\nabla f c_{n_k} - \lambda_{n_k} l^{-1} (\nabla f - T_1) c_{n_k}))\| \\ & > \mu \|c_{n_k} - \nabla f^* (\nabla f c_{n_k} - \lambda_{n_k} l^{-1} (\nabla f - T_1) c_{n_k})\|. \end{aligned}$$

That is,

$$\frac{1}{\mu} \|(\nabla f - T_1) c_{n_k} - (\nabla f - T_1) \omega_{n_k}\| > \frac{\|c_{n_k} - \omega_{n_k}\|}{\lambda_{n_k} l^{-1}}. \quad (3.153)$$

Combining (3.152) and (3.153), we obtain that $\lim_{k \rightarrow \infty} \frac{\|c_{n_k} - \omega_{n_k}\|}{\lambda_{n_k} l^{-1}} = 0$, which implies by the Lipschitz continuity of ∇f that

$$\lim_{k \rightarrow \infty} \frac{\|\nabla f c_{n_k} - \nabla f \omega_{n_k}\|}{\lambda_{n_k} l^{-1}} = 0. \quad (3.154)$$

Furthermore, we have

$$\langle \nabla f c_{n_k} - \lambda_{n_k} l^{-1} (\nabla f - T_1) c_{n_k} - \nabla f \omega_{n_k}, x - \omega_{n_k} \rangle \leq 0, \text{ for all } x \in E_1,$$

which together with the monotonicity of $\nabla f - T_1$ implies that

$$\begin{aligned} & \frac{1}{\lambda_{n_k} l^{-1}} \langle \nabla f c_{n_k} - \nabla f \omega_{n_k}, x - \omega_{n_k} \rangle + \langle (\nabla f - T_1) c_{n_k}, \omega_{n_k} - c_{n_k} \rangle \\ & \leq \langle (\nabla f - T_1) c_{n_k}, x - c_{n_k} \rangle \leq \langle (\nabla f - T_1) x, x - c_{n_k} \rangle, \text{ for all } x \in E_1. \end{aligned} \quad (3.155)$$

Taking limit inferior of (3.155) as $k \rightarrow \infty$ and using (3.154) and (3.151), we get $\liminf_{k \rightarrow \infty} \langle (\nabla f - T_1) c_{n_k}, x - c_{n_k} \rangle \geq 0$. Consequently, we obtain from the monotonicity of $\nabla f - T_1$ that $\liminf_{k \rightarrow \infty} \langle (\nabla f - T_1) x, x - c_{n_k} \rangle = \langle (\nabla f - T_1) x, x - \tilde{x} \rangle \geq 0$, for all $x \in E_1$, or equivalently $\langle (\nabla f - T_1) \tilde{x}, x - \tilde{x} \rangle \geq 0$, for all $x \in E_1$ (by the pseudomonotonicity of $\nabla f - T_1$). Thus, we obtain that $\tilde{x} \in (\nabla f - T_1)^{-1}(0)$ and hence $\tilde{x} \in F_f(T_1)$. By the same token, we can show that $\tilde{x} \in F_f(T_2)$. Thus, $\tilde{x} \in F_f(T_1) \cap F_f(T_2)$. Similarly, we have that $\tilde{y} \in F_g(S_1) \cap F_g(S_2)$.

Moreover, Lemma 3.1 gives

$$\begin{aligned} \|\tilde{x} - \tilde{y}\|^2 &= \|Aa_{n_k} - Bb_{n_k} + A\tilde{x} - Aa_{n_k} + Bb_{n_k} - B\tilde{y}\|^2 \\ &\leq \|Aa_{n_k} - Bb_{n_k}\|^2 + 2\langle J_{E_3}(A\tilde{x} - B\tilde{y}), A\tilde{x} - Aa_{n_k} + Bb_{n_k} - B\tilde{y} \rangle. \end{aligned} \quad (3.156)$$

From (3.138) and (3.139), we have that $a_{n_k} \rightharpoonup \tilde{x}$ and $b_{n_k} \rightharpoonup \tilde{y}$. Thus, from the sequentially weakly continuity of A and B , we have $Aa_{n_k} \rightharpoonup A\tilde{x}$ and $Bb_{n_k} \rightharpoonup B\tilde{y}$. Thus, from (3.137) and (3.156), we obtain that $A\tilde{x} = B\tilde{y}$. Therefore, $(\tilde{x}, \tilde{y}) \in \Omega$. By (1.14), we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle (\nabla f x, \nabla g y) - (\nabla f x^*, \nabla g y^*), (x_n, y_n) - (x^*, y^*) \rangle \\ &= \lim_{k \rightarrow \infty} \langle (\nabla f x, \nabla g y) - (\nabla f x^*, \nabla g y^*), (x_{n_k}, y_{n_k}) - (x^*, y^*) \rangle \\ &= \langle (\nabla f x, \nabla g y) - (\nabla f x^*, \nabla g y^*), (\tilde{x}, \tilde{y}) - (x^*, y^*) \rangle \leq 0. \end{aligned} \quad (3.157)$$

From (3.148), (3.149) and (3.157), we obtain

$$\limsup_{n \rightarrow \infty} (K_n + M_n) \leq 0. \quad (3.158)$$

Consequently, we obtain from (3.135), (3.158) and Lemma 3.1.6 that

$$\lim_{n \rightarrow \infty} \Omega_n^* = 0,$$

which implies that $\lim_{n \rightarrow \infty} D_f(x^*, x_n) = \lim_{n \rightarrow \infty} D_g(y^*, y_n) = 0$ which in turn implies by Lemma 3.1.13 that $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$ and $\lim_{n \rightarrow \infty} \|y_n - y^*\| = 0$ and hence

$$\lim_{n \rightarrow \infty} x_n = x^* \text{ and } \lim_{n \rightarrow \infty} y_n = y^*.$$

Case II. Suppose now that there exists a subsequence $\{\Omega_{n_i}^*\}$ of $\{\Omega_n^*\}$ such that $\Omega_{n_i}^* < \Omega_{n_i+1}^*$, for all $i \geq 0$. Then by Lemma 3.1.7, there exists a nondecreasing sequence $\{m_k\}$ of positive

integers such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\Omega_{m_k}^* \leq \Omega_{m_k+1}^* \quad \text{and} \quad \Omega_k^* \leq \Omega_{m_k+1}^*. \quad (3.159)$$

Thus, we have from (3.136) that

$$\begin{aligned} & (1 - \alpha_{m_k}) \left(1 - \frac{2\zeta_{m_k}}{\beta}\right) \delta_{m_k} D_f(a_{m_k}, x_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{\lambda}{\beta}\right) \delta_{m_k} D_f(r_{m_k}, e_{m_k}) \\ & + (1 - \alpha_{m_k}) \left(1 - \frac{\lambda}{\beta}\right) \delta_{m_k} D_f(e_{m_k}, c_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{2\zeta_{m_k}}{\beta}\right) \phi_{m_k} D_f(a_{m_k}, x_{m_k}) \\ & + (1 - \alpha_{m_k}) \left(1 - \frac{\sigma}{\beta}\right) \phi_{m_k} D_f(s_{m_k}, h_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{\sigma}{\beta}\right) \phi_{m_k} D_f(h_{m_k}, c_{m_k}) \\ & + (1 - \alpha_{m_k}) \left(1 - \frac{2\zeta_{m_k}}{\beta}\right) \delta_{m_k} D_g(b_{m_k}, y_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{\tau}{\beta}\right) \delta_{m_k} D_g(u_{m_k}, p_{m_k}) \\ & + (1 - \alpha_{m_k}) \left(1 - \frac{\tau}{\beta}\right) \delta_{m_k} D_g(p_{m_k}, z_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{2\zeta_n}{\beta}\right) \phi_{m_k} D_g(b_{m_k}, y_{m_k}) \\ & + (1 - \alpha_{m_k}) \left(1 - \frac{\pi}{\beta}\right) \phi_{m_k} D_g(v_{m_k}, q_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{\pi}{\beta}\right) \phi_{m_k} D_g(q_{m_k}, z_{m_k}) \\ & + (1 - \alpha_{m_k}) (\delta_{m_k} + \phi_{m_k}) \frac{\rho}{2} \|Aa_{m_k} - Bb_{m_k}\|^2 \\ & \leq \Omega_{m_k}^* - \Omega_{m_k+1}^* + \alpha_{m_k} \left[\Sigma^* + \left(\frac{4\varepsilon}{\beta} - 1\right) \Omega_{m_k}^* + 2\frac{\zeta_{m_k}}{\alpha_{m_k}} \right]. \end{aligned} \quad (3.160)$$

Taking limit as $k \rightarrow \infty$ of (3.160) and making use of (3.159) and the conditions on the sequences $\{\alpha_{m_k}\}$, $\{\sigma_{m_k}\}$ and $\{\phi_{m_k}\}$, we obtain that

$$\lim_{k \rightarrow \infty} \|Aa_{m_k} - Bb_{m_k}\| = 0. \quad (3.161)$$

Following similar steps as in Case I, we obtain that

$$\lim_{k \rightarrow \infty} \|x_{m_k+1} - x_{m_k}\| = 0 = \lim_{k \rightarrow \infty} \|y_{m_k+1} - y_{m_k}\| \quad \text{and} \quad \limsup_{k \rightarrow \infty} (K_{m_k} + M_{m_k}) \leq 0.$$

From (3.135) and (3.159), we obtain that

$$\begin{aligned} \alpha_{m_k} \left(1 - \frac{4\varepsilon}{\beta}\right) \Omega_{m_k}^* & \leq \Omega_{m_k}^* - \Omega_{m_k+1}^* + \alpha_{m_k} \left(1 - \frac{4\varepsilon}{\beta}\right) (K_{m_k} + M_{m_k}) \\ & \leq \alpha_{m_k} \left(1 - \frac{4\varepsilon}{\beta}\right) (K_{m_k} + M_{m_k}), \end{aligned}$$

and thus we have that $\Omega_{m_k}^* \leq K_{m_k} + M_{m_k}$. Since $\limsup_{k \rightarrow \infty} (K_{m_k} + M_{m_k}) \leq 0$, it follows that $\lim_{k \rightarrow \infty} \Omega_{m_k}^* = 0$, which together with (3.135) implies that $\lim_{k \rightarrow \infty} \Omega_{m_k+1}^* = 0$. This in turn implies by (3.159) that $\lim_{k \rightarrow \infty} \Omega_k^* = 0$ and thus $\lim_{k \rightarrow \infty} D_f(x^*, x_n) = 0$ and $\lim_{k \rightarrow \infty} D_g(y^*, y_n) = 0$. Thus, we have by Lemma 3.1.13 that $x_k \rightarrow x^*$ and $y_k \rightarrow y^*$ as $k \rightarrow \infty$ and hence the proof is complete. $\square$

If we assume, in Theorem 3.3.2, that T_1 and T_2 are Lipschitz continuous f -pseudocontractive and S_1 and S_2 are Lipschitz continuous g -pseudocontractive mappings, then we obtain the following corollary.

Corollary 3.3.1. *Assume that the conditions (D1), (D2), (D4)–(D7) hold. If $T_1, T_2 : E_1 \rightarrow E_1^*$ are Lipschitz continuous f –pseudocontractive and $S_1, S_2 : E_2 \rightarrow E_2^*$ are Lipschitz continuous g –pseudocontractive mappings, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Omega^h(x, y)$.*

Proof. As every Lipschitz continuous mapping is uniformly continuous, the result follows from Theorem 3.3.2. $\square$

If we assume $f(\cdot) = \frac{1}{2}\|\cdot\|^2$ and $g(\cdot) = \frac{1}{2}\|\cdot\|^2$, in Theorem 3.3.2, then $\nabla f = J_{E_1}$, $\nabla g = J_{E_2}$, $\nabla f^* = J_{E_1^*}^{-1}$ and $\nabla g^* = J_{E_2^*}^{-1}$. Thus, we get the following corollary.

Corollary 3.3.2. *Assume that the conditions (D1), (D3)–(D7) are satisfied. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2, with $\nabla f = J_{E_1}$, $\nabla g = J_{E_2}$, $\nabla f^* = J_{E_1^*}^{-1}$ and $\nabla g^* = J_{E_2^*}^{-1}$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = \Pi_\Omega(x, y)$, where $\Pi_\Omega(x, y)$ is the generalized projection mapping of (x, y) onto Ω .*

If we assume, in Theorem 3.3.2, that E_1, E_2 and E_3 are real Hilbert spaces, $f(\cdot) = \frac{1}{2}\|\cdot\|^2$ and $g(\cdot) = \frac{1}{2}\|\cdot\|^2$, then $\nabla f = J_{E_1} = I_{E_1}$, $\nabla g = J_{E_2} = I_{E_2}$, $\nabla f^* = J_{E_1^*}^{-1} = I_{E_1^*}^{-1}$ and $\nabla g^* = J_{E_2^*}^{-1} = I_{E_2^*}^{-1}$ and hence we obtain the following corollary.

Corollary 3.3.3. *Let E_1, E_2 and E_3 be real Hilbert spaces and assume that the conditions (D3)–(D7) are satisfied. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2, with $\nabla f = J_{E_1} = I_{E_1}$, $\nabla g = J_{E_2} = I_{E_2}$, $\nabla f^* = J_{E_1^*}^{-1} = I_{E_1^*}^{-1}$ and $\nabla g^* = J_{E_2^*}^{-1} = I_{E_2^*}^{-1}$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Omega(x, y)$.*

3.3.2 Some Theoretical Applications

In this section, we present some theoretical applications of the main result.

The Split Common f, g –fixed Point Problem

If we consider $E_2 = E_3$ and $B = I_{E_2}$ in Algorithm 3.2, then the SECFPP reduces to *split common f, g –fixed point problem* which is finding a point $(x^*, y^*) \in E_1 \times E_2$ such that $(x^*, y^*) \in (F_f(T_1) \cap F_f(T_2)) \times (F_g(S_1) \cap F_g(S_2))$ such that $Ax^* = y^*$. Denote

$$\Gamma^* = \{(x^*, y^*) \in (F_f(T_1) \cap F_f(T_2)) \times (F_g(S_1) \cap F_g(S_2)) : Ax^* = y^*\}.$$

Corollary 3.3.4. *Assume that conditions (D1)–(D4), (D6), (D7), with $E_2 = E_3$ and $B = I_{E_2}$ hold. If $\Gamma^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Gamma^*}^h(x, y)$.*

Corollary 3.3.5. *Assume that conditions (D1), (D2), (D4), (D6), (D7), with $E_2 = E_3$ $B = I_{E_2}$ hold. Let $T_1, T_2 : E_1 \rightarrow E_1^*$ be Lipschitz f –pseudocontractive and $S_1, S_2 : E_2 \rightarrow E_2^*$ be g –pseudocontractive mappings. If $\Gamma^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Gamma^*}^h(x, y)$.*

Common Solutions of Common f, g -Fixed Point Problems

If we take $E_1 = E_2 = E_3 = E$ and $A = B = I_E$ in Algorithm 3.2, then SECFPP reduces to *common solutions of common f, g -fixed point problem*, which is defined as finding two points $x^*, y^* \in E$ with $(x^*, y^*) \in (F_f(T_1) \cap F_f(T_2)) \times (F_g(S_1) \cap F_g(S_2))$ such that $x^* = y^*$.

Denote

$$\Omega^* = \{(x^*, y^*) \in (F_f(T_1) \cap F_f(T_2)) \times (F_g(S_1) \cap F_g(S_2)) : x^* = y^*\}.$$

Corollary 3.3.6. *Assume that conditions (D1) – (D3) and (D6) and (D7), with $E_1 = E_2 = E_3 = E$ hold. If $\Omega^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2, with $A = B = I_E$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Omega^*}^h(x, y)$.*

Corollary 3.3.7. *Assume that conditions (D1), (D2), (D6) and (D7), with $E_1 = E_2 = E_3 = E$ hold. Let $T_1, T_2: E \rightarrow E^*$ be Lipschitz f -pseudocontractive and $S_1, S_2: E \rightarrow E^*$ be g -pseudocontractive mappings. If $\Omega^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2, with $A = B = I_E$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Omega^*}^h(x, y)$.*

Split equality f, g - Fixed Point Problems

If we take $T_2x = \nabla fx$ for all $x \in E_1$ and $S_2y = \nabla gy$ for all $y \in E_2$ in Algorithm 3.2, then the SECFPP reduces to *split equality f, g -fixed point problem* which seeks to find a point $(x^*, y^*) \in F_f(T_1) \times F_g(S_1)$ such that $Ax^* = By^*$. Denote

$$\Sigma^* = \{(x^*, y^*) \in F_f(T_1) \times F_g(S_1) : Ax^* = By^*\}.$$

Corollary 3.3.8. *Assume that conditions (D1)–(D4), (D6) and (D7), with $T_2x = \nabla fx$, for all $x \in E_1$ and $S_2y = \nabla gy$, for all $y \in E_2$ hold. If $\Sigma^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Sigma^*}^h(x, y)$.*

3.3.3 Numerical Examples

Example 3.3.1. Let $E = E_1 = E_2 = E_3 = L_p^{\mathbb{R}}([0, 1])$, where $1 < p < \infty$, with the norm

$$\|x\|_{L_p} = \left(\int_0^1 |x(t)|^p dt \right)^{\frac{1}{p}} \text{ and } f, g: E \rightarrow (-\infty, +\infty] \text{ be defined by } f(x) = g(x) = \frac{1}{p} \|x\|^p. \text{ Then}$$

$E^* = E_1^* = E_2^* = E_3^* = L_q^{\mathbb{R}}([0, 1])$, where $\frac{1}{p} + \frac{1}{q} = 1$. Define the mappings T_1, T_2, S_1, S_2 :

$$L_p^{\mathbb{R}}([0, 1]) \rightarrow L_q^{\mathbb{R}}([0, 1]) \text{ by } (T_1x)(t) = \frac{1}{2} \nabla fx(t) - 2, (T_2x)(t) = -\frac{2}{9} \nabla fx(t) - \frac{44}{9}, (S_1y)(t) = -\frac{1}{5} \nabla gy(t) + 6 \text{ and } (S_2y)(t) = \frac{2}{3} \nabla gy(t) + \frac{5}{3}. \text{ Clearly, } T_1 \text{ and } T_2 \text{ are uniformly continuous } f\text{-pseudocontractive and } S_1 \text{ and } S_2 \text{ are uniformly continuous } g\text{-pseudocontractive mappings.}$$

If we take $p = 2$ for simplicity, then $E = E_1 = E_2 = E_3 = L_2^{\mathbb{R}}([0, 1])$, $f(x) = g(x) = \frac{1}{2} \|x\|^2$,

$$\nabla f = \nabla g = I_E \text{ and the mappings } T_1, T_2, S_1, S_2 \text{ will be reduced to } (T_1x)(t) = \frac{1}{2} x(t) - 2, (T_2x)(t) = -\frac{2}{9} x(t) - \frac{44}{9}, (S_1y)(t) = -\frac{1}{5} y(t) + 6 \text{ and } (S_2y)(t) = \frac{2}{3} y(t) + \frac{5}{3}.$$

In this case, one can easily show that $x^*(t) = -4 \in F_f(T_1) \cap F_f(T_2)$ and $y^*(t) = 5 \in F_g(S_1) \cap F_g(S_2)$. Now, define $A, B: E \rightarrow E$ by $(Ax)(t) = \frac{2}{3} x(t)$ and $(By)(t) = -\frac{8}{15} y(t)$. Clearly,

A and B are bounded linear mappings with adjoints $(A^*x)(t) = \frac{2}{3}x(t)$ and $(B^*y)(t) = \frac{-8}{15}y(t)$. Moreover, $A(-4) = B(5) = \frac{-8}{3}$. Therefore, $(-4, 5) \in \Omega$. If we take $\zeta_n = \frac{1}{n^3 + 1000000}$, $\alpha_n = \frac{1}{n + 10000000}$, $\lambda = 0.4$, $\sigma = 0.1$, $\tau = 0.1$, $\pi = 0.1$, $\gamma = 0.4$, $\theta = 0.001$, $\beta_n = \delta_n = \phi_n = \frac{1}{3}$ and

$$\gamma_n = \begin{cases} \left(\frac{\beta}{2}\right) \frac{\left\|\frac{2}{3}a_n + \frac{8}{15}b_n\right\|^2}{\left\|\frac{4}{9}a_n + \frac{16}{45}b_n\right\|^2 + \left\|\frac{64}{225}b_n + \frac{16}{45}a_n\right\|^2} & \text{if } n \in \Upsilon, \\ \frac{1}{100000000} & \text{if } n \notin \Upsilon, \end{cases}$$

then the conditions (D1) – (D7) are satisfied and the numerical MATLAB experiment shows that the sequence $\{d_n\} = \{(x_n(t), y_n(t))\}$ generated by Algorithm 3.2 converges strongly to the solution $(x^*(t), y^*(t)) = (-4, 5)$ of the SECFPP for $(x(t), y(t)) = (1, 1)$, $(x_0, y_0) = (0, 0)$ and different values of the parameter l and strong convexity constant, β , of f .

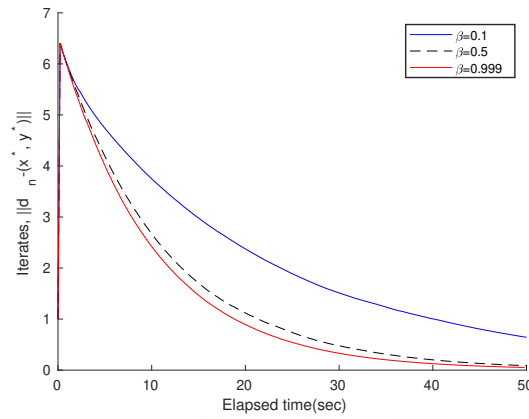


Figure 3.4: Different values of the strong convexity constant β .

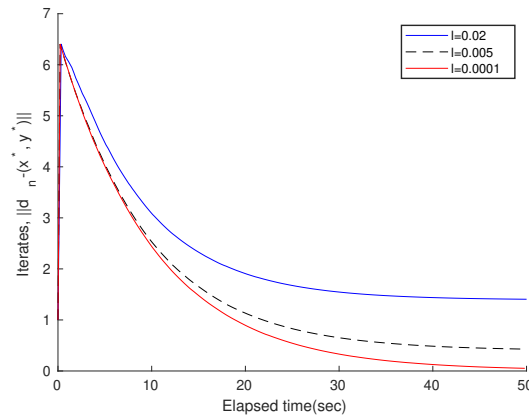


Figure 3.5: Different values of the parameter l .

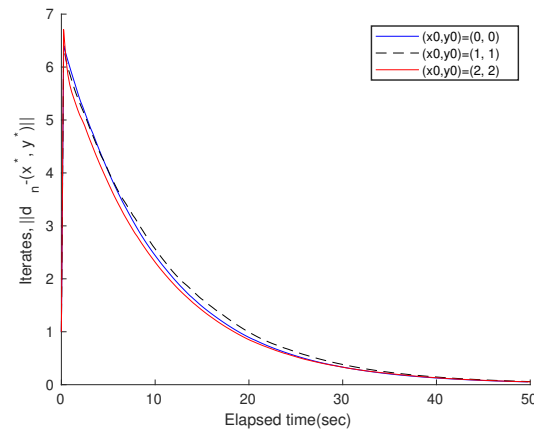


Figure 3.6: Different values of the initial point (x_0, y_0) .

Remark 3.3.2. We can observe from Figure 3.4 that the convergence of the method gets faster as the strong convexity constant β gets closer to 1 keeping the initial points and all other parameters fixed. Figure 3.5 depicts that the convergence gets faster as the parameter l gets closer to zero. On the other hand, Figure 3.6 reveals that the sequence $\{d_n\}$ has almost the same rate of convergence for different choices of initial points.

4. Approximating Solutions of Split Equality Quasimonotone Minty Variational Inequality Problems

In this chapter, we construct an algorithm for approximating solutions of the split equality quasimonotone Minty variational inequality problem in reflexive real Banach spaces and discuss its convergence analysis and provide numerical examples.

4.1 Main Results

The following conditions that are useful in the convergence analysis of the algorithm will be employed throughout the section.

- (Q1) Let E_1 and E_2 be reflexive real Banach spaces with duals E_1^* and E_2^* , respectively, and let C and D be nonempty, closed and convex subsets of E_1 and E_2 , respectively;
- (Q2) Let $f: E_1 \rightarrow (-\infty, +\infty]$ and $g: E_2 \rightarrow (-\infty, +\infty]$ be strongly coercive, strongly convex, bounded and uniformly Fréchet differentiable Legendre functions which are bounded on bounded subsets of E_1 and E_2 , respectively, with conjugates f^* and g^* . Let the strong convexity constants of f and g be β_1 and β_2 , respectively, and let $\beta = \min \{\beta_1, \beta_2\}$.
- (Q3) Let $F: E_1 \rightarrow E_1^*$ and $G: E_2 \rightarrow E_2^*$ be Lipschitz continuous and quasimonotone mappings with Lipschitz constants L_1 and L_2 , respectively, and satisfying the conditions

$$\|Fx^*\| \leq \liminf_{n \rightarrow \infty} \|Fx_n\| \text{ and } \|Gy^*\| \leq \liminf_{n \rightarrow \infty} \|Gy_n\|, \quad (4.1)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in E_1 and E_2 , respectively, such that $x_n \rightharpoonup x^*$ and $y_n \rightharpoonup y^*$;

- (Q4) Let $A: E_1 \rightarrow E_3$ and $B: E_2 \rightarrow E_3$ be bounded linear mappings with adjoints A^* and B^* , respectively, where E_3 is another reflexive real Banach space;
- (Q5) Let the set $\Omega = \{(x^*, y^*) \in MVI(C, F) \times MVI(D, G) : Ax^* = By^*\}$ be nonempty;
- (Q6) Let $\{\alpha_n\} \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (Q7) Let $\{\zeta_n\}$ be a sequence of positive numbers such that $\zeta_n \in \left(0, \frac{\beta}{2}\right)$ for all $n \geq 0$ and $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$.

The notation $\Theta_n = \|\nabla f x_n - \nabla f x_{n-1}\| + \|\nabla g y_n - \nabla g y_{n-1}\|$ for all integers $n \geq 1$ will be used in the proofs the of the subsequent results of this chapter.

Algorithm 4.1.

Initialization: Let $x_0, x_1 \in E_1$, $y_0, y_1 \in E_2$, $\theta \in [0, 1)$. Let $\mu, \delta \in (0, \beta)$, $\lambda_1 > 0$, $\eta_1 > 0$. Let $\{\phi_n\} \subset [0, \infty)$ and $\{\varphi_n\} \subset [0, \infty)$ be such that $\sum_{n=1}^{\infty} \phi_n < \infty$ and $\sum_{n=1}^{\infty} \varphi_n < \infty$. Set $n := 1$. For arbitrary points $\hat{x} \in E_1$ and $\hat{y} \in E_2$, calculate $\{x_n\}$ and $\{y_n\}$ as follows:

Step 1. Given the iterates $x_{n-1}, x_n \in E_1$ and $y_{n-1}, y_n \in E_2$, choose θ_n such that $0 \leq \theta_n \leq \sigma_n$, where

$$\sigma_n = \begin{cases} \min \left\{ \theta, \frac{\zeta_n}{\Theta_n} \right\}, & \text{if } \Theta_n \neq 0, \\ \theta, & \text{otherwise.} \end{cases} \quad (4.2)$$

Step 2. Compute

$$\begin{aligned} a_n &= \nabla f^* (\nabla f x_n + \theta_n (\nabla f x_n - \nabla f x_{n-1})), \\ b_n &= \nabla g^* (\nabla g y_n + \theta_n (\nabla g y_n - \nabla g y_{n-1})). \end{aligned} \quad (4.3)$$

Step 3. Compute

$$\begin{aligned} p_n &= \nabla f^* (\nabla f a_n - \gamma_n A^* J_{E_3} (Aa_n - Bb_n)), \\ q_n &= \nabla g^* (\nabla g b_n - \gamma_n B^* J_{E_3} (Bb_n - Aa_n)), \end{aligned} \quad (4.4)$$

where $0 < \rho \leq \gamma_n \leq \rho_n$ with

$$\rho_n = \min \left\{ \rho + 1, \frac{\beta \|Aa_n - Bb_n\|^2}{2[\|A^* J_{E_3} (Aa_n - Bb_n)\|^2 + \|B^* J_{E_3} (Bb_n - Aa_n)\|^2]} \right\}, \quad (4.5)$$

for $n \in \Upsilon = \{m \in \mathbb{N} : Aa_m - Bb_m \neq 0\}$, otherwise $\gamma_n = \rho$.

Step 4. Compute

$$\begin{aligned} u_n &= P_C^f \nabla f^* (\nabla f p_n - \lambda_n F p_n), \\ v_n &= P_D^g \nabla g^* (\nabla g q_n - \eta_n G q_n). \end{aligned} \quad (4.6)$$

Step 5. Compute

$$\begin{aligned} w_n &= \nabla f^* (\nabla f u_n - \lambda_n (F u_n - F p_n)), \\ z_n &= \nabla g^* (\nabla g v_n - \eta_n (G v_n - G q_n)). \end{aligned} \quad (4.7)$$

Step 6. Compute

$$\begin{aligned} x_{n+1} &= \nabla f^* (\alpha_n \nabla f \hat{x} + (1 - \alpha_n) \nabla f w_n), \\ y_{n+1} &= \nabla g^* (\alpha_n \nabla g \hat{y} + (1 - \alpha_n) \nabla g z_n). \end{aligned} \quad (4.8)$$

Update λ_n and η_n as follows

$$\lambda_{n+1} = \begin{cases} \min \left\{ \lambda_n + \phi_n, \frac{\mu \|u_n - p_n\|}{\|F u_n - F p_n\|} \right\}, & \text{if } F u_n \neq F p_n, \\ \lambda_n + \phi_n, & \text{otherwise,} \end{cases} \quad (4.9)$$

and

$$\eta_{n+1} = \begin{cases} \min \left\{ \eta_n + \varphi_n, \frac{\delta \|v_n - q_n\|}{\|Gv_n - Gq_n\|} \right\}, & \text{if } Gv_n \neq Gq_n, \\ \eta_n + \varphi_n, & \text{otherwise.} \end{cases} \quad (4.10)$$

and then go to **Step 1**.

Remark 4.1.1. Some of the advantages of our method over the existing ones are:

- The method we introduced is a newly proposed inertial single projection method in the setting of reflexive Banach spaces.
- It is a more general problem which encompasses all the Minty variational inequality problems in the literature
- It does not require prior knowledge of the Lipschitz constants of the underlying mappings.

Remark 4.1.2. Condition (4.1) is weaker condition than the weak sequential continuity condition imposed on F (see, [101, 140, 168]). The mapping $F(x) = \|x\|x$ is an example which satisfies condition (4.1) but is not sequentially weakly continuous (see, [161]).

Lemma 4.1.1. Let $\{\lambda_n\}$ and $\{\eta_n\}$ be as in (4.9) and (4.10), respectively. Then

- (i) $\lim_{n \rightarrow \infty} \lambda_n = \lambda$ and $\lambda \in \left[\min \left\{ \frac{\mu}{L_1}, \lambda_1 \right\}, \lambda_1 + \mathbb{K} \right]$, where $\mathbb{K} = \sum_{n=1}^{\infty} \phi_n$;
- (ii) $\lim_{n \rightarrow \infty} \eta_n = \eta$ and $\eta \in \left[\min \left\{ \frac{\delta}{L_2}, \eta_1 \right\}, \eta_1 + \mathbb{M} \right]$, where $\mathbb{M} = \sum_{n=1}^{\infty} \varphi_n$.

Proof. The proof follows the same method as in ([101], Lemma 3.1). □

Theorem 4.1.1. Assume that conditions (Q1) – (Q7) hold. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 4.1 are bounded.

Proof. Let $(\bar{x}, \bar{y}) \in \Omega$. Then, $\bar{x} \in C$ and $\bar{y} \in D$. From (4.8) and Lemma 3.1.9, we obtain that

$$\begin{aligned} D_f(\bar{x}, x_{n+1}) &= D_f(\bar{x}, \nabla f^* (\alpha_n \nabla f \hat{x} + (1 - \alpha_n) \nabla f w_n)) \\ &\leq \alpha_n D_f(\bar{x}, \hat{x}) + (1 - \alpha_n) D_f(\bar{x}, w_n). \end{aligned} \quad (4.11)$$

From (4.7) and (1.2.17), we obtain that

$$\begin{aligned} D_f(\bar{x}, w_n) &= D_f(\bar{x}, \nabla f^* (\nabla f u_n - \lambda_n (F u_n - F p_n))) \\ &= f(\bar{x}) - \langle \nabla f u_n - \lambda_n (F u_n - F p_n), \bar{x} - w_n \rangle - f(w_n) \\ &= f(\bar{x}) + \langle \nabla f u_n, w_n - \bar{x} \rangle + \lambda_n \langle F u_n - F p_n, \bar{x} - w_n \rangle - f(w_n) \\ &= f(\bar{x}) - \langle \nabla f u_n, \bar{x} - u_n \rangle - f(u_n) + \langle \nabla f u_n, \bar{x} - u_n \rangle + f(u_n) \\ &\quad + \langle \nabla f u_n, w_n - \bar{x} \rangle + \lambda_n \langle F u_n - F p_n, \bar{x} - w_n \rangle - f(w_n) \\ &= D_f(\bar{x}, u_n) + \langle \nabla f u_n, w_n - u_n \rangle + f(u_n) - f(w_n) + \lambda_n \langle F u_n - F p_n, \bar{x} - w_n \rangle \\ &= D_f(\bar{x}, u_n) - D_f(w_n, u_n) + \lambda_n \langle F u_n - F p_n, \bar{x} - w_n \rangle. \end{aligned} \quad (4.12)$$

From (1.15) we obtain that

$$D_f(\bar{x}, u_n) = D_f(\bar{x}, p_n) - D_f(u_n, p_n) + \langle \nabla f p_n - \nabla f u_n, \bar{x} - u_n \rangle. \quad (4.13)$$

Substituting (4.13) into (4.12), we obtain

$$\begin{aligned} D_f(\bar{x}, w_n) &= D_f(\bar{x}, p_n) - D_f(u_n, p_n) - D_f(w_n, u_n) \\ &\quad + \lambda_n \langle F u_n - F p_n, \bar{x} - w_n \rangle + \langle \nabla f p_n - \nabla f u_n, \bar{x} - u_n \rangle. \end{aligned} \quad (4.14)$$

We also obtain from (4.4), (1.21) and (1.22) that

$$\begin{aligned} D_f(\bar{x}, p_n) &= D_f(\bar{x}, \nabla f^* [\nabla f a_n - \gamma_n A^* J_{E_3}(A a_n - B b_n)]) \\ &= V_f(\bar{x}, \nabla f a_n - \gamma_n A^* J_{E_3}(A a_n - B b_n)) \\ &\leq V_f(\bar{x}, \nabla f a_n) - \gamma_n \langle A^* J_{E_3}(A a_n - B b_n), p_n - \bar{x} \rangle \\ &= D_f(\bar{x}, a_n) - \gamma_n \langle A^* J_{E_3}(A a_n - B b_n), p_n - \bar{x} \rangle. \end{aligned} \quad (4.15)$$

Substituting (4.15) into (4.14), we obtain

$$\begin{aligned} D_f(\bar{x}, w_n) &\leq D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) + \lambda_n \langle F u_n - F p_n, \bar{x} - w_n \rangle \\ &\quad + \langle \nabla f p_n - \nabla f u_n, \bar{x} - u_n \rangle - \gamma_n \langle A^* J_{E_3}(A a_n - B b_n), p_n - \bar{x} \rangle. \end{aligned} \quad (4.16)$$

Since $u_n = P_C^f \nabla f^*(\nabla f p_n - \lambda_n F p_n)$ and $\bar{x} \in C$, we have by (1.18) that

$$\langle \nabla f p_n - \lambda_n F p_n - \nabla f u_n, \bar{x} - u_n \rangle \leq 0,$$

which implies that

$$\langle \nabla f p_n - \nabla f u_n, \bar{x} - u_n \rangle \leq \lambda_n \langle F p_n, \bar{x} - u_n \rangle. \quad (4.17)$$

From (4.16) and (4.17) we obtain that

$$\begin{aligned} D_f(\bar{x}, w_n) &\leq D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) + \lambda_n \langle F u_n - F p_n, \bar{x} - w_n \rangle \\ &\quad + \lambda_n \langle F p_n, \bar{x} - u_n \rangle - \gamma_n \langle A^* J_{E_3}(A a_n - B b_n), p_n - \bar{x} \rangle \\ &= D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) + \lambda_n \langle F u_n, \bar{x} - w_n \rangle - \lambda_n \langle F p_n, \bar{x} - w_n \rangle \\ &\quad + \lambda_n \langle F p_n, \bar{x} - u_n \rangle - \gamma_n \langle A^* J_{E_3}(A a_n - B b_n), p_n - \bar{x} \rangle \\ &= D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) + \lambda_n \langle F u_n, \bar{x} - w_n \rangle + \lambda_n \langle F p_n, w_n - u_n \rangle \\ &\quad - \gamma_n \langle A^* J_{E_3}(A a_n - B b_n), p_n - \bar{x} \rangle \\ &= D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) - \lambda_n \langle F u_n, u_n - \bar{x} \rangle \\ &\quad + \lambda_n \langle F u_n, u_n - w_n \rangle + \lambda_n \langle F p_n, w_n - u_n \rangle - \gamma_n \langle A^* J_{E_3}(A a_n - B b_n), p_n - \bar{x} \rangle \\ &= D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) - \lambda_n \langle F u_n, u_n - \bar{x} \rangle \\ &\quad + \lambda_n \langle F p_n - F u_n, w_n - u_n \rangle - \gamma_n \langle A^* J_{E_3}(A a_n - B b_n), p_n - \bar{x} \rangle. \end{aligned} \quad (4.18)$$

Since $\bar{x} \in MVI(C, F)$ and $u_n \in C$, we have by the Minty VIP that $\langle F u_n, u_n - \bar{x} \rangle \geq 0$, which

together with (4.18) gives that

$$\begin{aligned} D_f(\bar{x}, w_n) &\leq D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) \\ &\quad + \lambda_n \langle Fp_n - Fu_n, w_n - u_n \rangle - \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - \bar{x} \rangle. \end{aligned} \quad (4.19)$$

From (4.19) and the Cauchy–Schwarz Inequality, we obtain

$$\begin{aligned} D_f(\bar{x}, w_n) &\leq D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) + \lambda_n \|Fp_n - Fu_n\| \|w_n - u_n\| \\ &\quad - \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - \bar{x} \rangle, \end{aligned}$$

which implies by (4.9) and Lemma 3.1.10 that

$$\begin{aligned} D_f(\bar{x}, w_n) &\leq D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) + \mu \frac{\lambda_n}{\lambda_{n+1}} \|p_n - u_n\| \|w_n - u_n\| \\ &\quad - \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - \bar{x} \rangle \\ &\leq D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) + \frac{\mu}{2} \frac{\lambda_n}{\lambda_{n+1}} [\|p_n - u_n\|^2 + \|w_n - u_n\|^2] \\ &\quad - \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - \bar{x} \rangle \\ &\leq D_f(\bar{x}, a_n) - D_f(u_n, p_n) - D_f(w_n, u_n) + \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}} D_f(u_n, p_n) + \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}} D_f(w_n, u_n) \\ &\quad - \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - \bar{x} \rangle \\ &= D_f(\bar{x}, a_n) - \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(u_n, p_n) - \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(w_n, u_n) \\ &\quad - \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - \bar{x} \rangle. \end{aligned} \quad (4.20)$$

From (4.3) and (1.2.17) we obtain that

$$\begin{aligned} D_f(\bar{x}, a_n) &= D_f(\bar{x}, \nabla f^*(\nabla f x_n + \theta_n(\nabla f x_n - \nabla f x_{n-1}))) \\ &= f(\bar{x}) - \langle \nabla f x_n + \theta_n(\nabla f x_n - \nabla f x_{n-1}), \bar{x} - a_n \rangle - f(a_n) \\ &= f(\bar{x}) - \langle \nabla f x_n, \bar{x} - x_n \rangle - f(x_n) + \langle \nabla f x_n, \bar{x} - x_n \rangle + f(x_n) \\ &\quad - \langle \nabla f x_n, \bar{x} - a_n \rangle - \theta_n \langle \nabla f x_n - \nabla f x_{n-1}, \bar{x} - a_n \rangle - f(a_n) \\ &= D_f(\bar{x}, x_n) + \langle \nabla f x_n, a_n - x_n \rangle + f(x_n) - f(a_n) - \theta_n \langle \nabla f x_n - \nabla f x_{n-1}, \bar{x} - a_n \rangle \\ &= D_f(\bar{x}, x_n) - D_f(a_n, x_n) - \theta_n \langle \nabla f x_n - \nabla f x_{n-1}, \bar{x} - a_n \rangle. \end{aligned} \quad (4.21)$$

Substituting (4.21) into (4.20) we obtain

$$\begin{aligned} D_f(\bar{x}, w_n) &\leq D_f(\bar{x}, x_n) - D_f(a_n, x_n) - \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(u_n, p_n) - \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(w_n, u_n) \\ &\quad - \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - \bar{x} \rangle - \theta_n \langle \nabla f x_n - \nabla f x_{n-1}, \bar{x} - a_n \rangle. \end{aligned} \quad (4.22)$$

Moreover, we have from the property of ζ_n and Lemma 3.1.10 that

$$\begin{aligned}
 -\theta_n \langle \nabla f x_n - \nabla f x_{n-1}, \bar{x} - a_n \rangle &\leq \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| [\|\bar{x} - a_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| [2\|\bar{x} - x_n\|^2 + 2\|x_n - a_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| \left[\frac{4}{\beta} D_f(\bar{x}, x_n) + \frac{4}{\beta} D_f(a_n, x_n) + 1 \right] \\
 &= \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(\bar{x}, x_n) + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(a_n, x_n) \\
 &\quad + \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| \leq \frac{2\zeta_n}{\beta} D_f(\bar{x}, x_n) + \frac{2\zeta_n}{\beta} D_f(a_n, x_n) + \frac{\zeta_n}{2}.
 \end{aligned} \tag{4.23}$$

Combining (4.22) and (4.23), we obtain

$$\begin{aligned}
 D_f(\bar{x}, w_n) &\leq \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(\bar{x}, x_n) - \left(1 - \frac{2\zeta_n}{\beta}\right) D_f(a_n, x_n) - \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(u_n, p_n) \\
 &\quad - \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(w_n, u_n) - \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - \bar{x} \rangle + \frac{\zeta_n}{2}.
 \end{aligned} \tag{4.24}$$

From (4.11) and (4.24), we obtain

$$\begin{aligned}
 D_f(\bar{x}, x_{n+1}) &\leq \alpha_n D_f(\bar{x}, \hat{x}) + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(\bar{x}, x_n) - (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) D_f(a_n, x_n) \\
 &\quad - (1 - \alpha_n) \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(u_n, p_n) - (1 - \alpha_n) \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(w_n, u_n) \\
 &\quad - (1 - \alpha_n) \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - \bar{x} \rangle + (1 - \alpha_n) \frac{\zeta_n}{2}.
 \end{aligned} \tag{4.25}$$

Since $\lim_{n \rightarrow \infty} \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \frac{\mu}{\beta} > 0$ and $\lim_{n \rightarrow \infty} \left(1 - \frac{\delta}{\beta} \frac{\eta_n}{\eta_{n+1}}\right) = 1 - \frac{\delta}{\beta} > 0$, there exists $N_0 \in \mathbb{N}$ such that $1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}} > 0$ and $1 - \frac{\delta}{\beta} \frac{\eta_n}{\eta_{n+1}} > 0$ for all $n \geq N_0$. Thus, we have from (4.25) and the conditions on μ , β and ζ_n that

$$\begin{aligned}
 D_f(\bar{x}, x_{n+1}) &\leq \alpha_n D_f(\bar{x}, \hat{x}) + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(\bar{x}, x_n) \\
 &\quad - (1 - \alpha_n) \gamma_n \langle J_{E_3}(Aa_n - Bb_n), Ap_n - A\bar{x} \rangle + (1 - \alpha_n) \frac{\zeta_n}{2}, \text{ for all } n \geq N_0.
 \end{aligned} \tag{4.26}$$

Similarly, we have

$$\begin{aligned}
 D_g(\bar{y}, y_{n+1}) &\leq \alpha_n D_g(\bar{y}, \hat{y}) + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) D_g(\bar{y}, y_n) - (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) D_g(b_n, y_n) \\
 &\quad - (1 - \alpha_n) \left(1 - \frac{\delta}{\beta} \frac{\eta_n}{\eta_{n+1}}\right) D_g(v_n, q_n) - (1 - \alpha_n) \left(1 - \frac{\delta}{\beta} \frac{\eta_n}{\eta_{n+1}}\right) D_g(z_n, v_n) \\
 &\quad - (1 - \alpha_n) \gamma_n \langle B^* J_{E_3}(Bb_n - Aa_n), q_n - \bar{y} \rangle + (1 - \alpha_n) \frac{\zeta_n}{2},
 \end{aligned} \tag{4.27}$$

and this implies that

$$D_g(\bar{y}, y_{n+1}) \leq \alpha_n D_g(\bar{y}, \hat{y}) + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) D_g(\bar{y}, y_n) - (1 - \alpha_n) \gamma_n \langle J_{E_3}(Bb_n - Aa_n), Bq_n - B\bar{y} \rangle + (1 - \alpha_n) \frac{\zeta_n}{2}, \text{ for all } n \geq N_0. \quad (4.28)$$

Denote $\Omega_n = D_f(\bar{x}, x_n) + D_g(\bar{y}, y_n)$ and $\Sigma = D_f(\bar{x}, \hat{x}) + D_g(\bar{y}, \hat{y})$. Since $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$, for every $\varepsilon \in \left(0, \frac{\beta}{2}\right)$, there exists a natural number $N_1 \geq N_0$ such that $\zeta_n < \alpha_n \varepsilon$, for all $n \geq N_1$. Now, combining (4.26) and (4.28) and using the fact that $A\bar{x} = B\bar{y}$, we obtain

$$\Omega_{n+1} \leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{2\varepsilon\alpha_n}{\beta}\right) \Omega_n + \varepsilon\alpha_n - (1 - \alpha_n) \gamma_n \langle J_{E_3}(Aa_n - Bb_n), Ap_n - Bq_n \rangle, \text{ for all } n \geq N_1. \quad (4.29)$$

Furthermore, we have

$$\begin{aligned} & -\langle J_{E_3}(Aa_n - Bb_n), Ap_n - Bq_n \rangle \\ &= -\langle J_{E_3}(Aa_n - Bb_n), Aa_n - Bb_n \rangle - \langle J_{E_3}(Aa_n - Bb_n), Ap_n - Aa_n \rangle \\ &= -\langle J_{E_3}(Aa_n - Bb_n), Bb_n - Bq_n \rangle \\ &= -\|Aa_n - Bb_n\|^2 - \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - a_n \rangle \\ &= -\langle B^* J_{E_3}(Aa_n - Bb_n), b_n - q_n \rangle \\ &\leq -\|Aa_n - Bb_n\|^2 + \|p_n - a_n\| \|A^* J_{E_3}(Aa_n - Bb_n)\| \\ &\quad + \|b_n - q_n\| \|B^* J_{E_3}(Aa_n - Bb_n)\|. \end{aligned} \quad (4.30)$$

The Lipschitz continuity of ∇f^* gives that

$$\begin{aligned} \|p_n - a_n\| &= \|\nabla f^*(\nabla f a_n - \gamma_n A^* J_{E_3}(Aa_n - Bb_n)) - \nabla f^*(\nabla f a_n)\| \\ &\leq \frac{\gamma_n}{\beta} \|A^* J_{E_3}(Aa_n - Bb_n)\|. \end{aligned} \quad (4.31)$$

Similarly, we obtain from the Lipschitz continuity of ∇g^* that

$$\|q_n - b_n\| \leq \frac{\gamma_n}{\beta} \|B^* J_{E_3}(Bb_n - Aa_n)\|. \quad (4.32)$$

Combining (4.30), (4.31) and (4.32) and making use of (4.5), we obtain

$$\begin{aligned} & -\gamma_n \langle J_{E_3}(Aa_n - Bb_n), Ap_n - Bq_n \rangle \\ &\leq -\gamma_n \|Aa_n - Bb_n\|^2 + \frac{\gamma_n^2}{\beta} \|A^* J_{E_3}(Aa_n - Bb_n)\|^2 \\ &\quad + \frac{\gamma_n^2}{\beta} \|B^* J_{E_3}(Bb_n - Aa_n)\|^2 \\ &\leq -\frac{\rho}{2} \|Aa_n - Bb_n\|^2 - \frac{\gamma_n}{2} \|Aa_n - Bb_n\|^2 \\ &\quad + \frac{\gamma_n}{2} \left\{ \frac{2\gamma_n}{\beta} [\|A^* J_{E_3}(Aa_n - Bb_n)\|^2 + \|B^* J_{E_3}(Bb_n - Aa_n)\|^2] \right\}, \end{aligned}$$

which implies that

$$-\gamma_n \langle J_{E_3}(Aa_n - Bb_n), Ap_n - Bq_n \rangle \leq -\frac{\rho}{2} \|Aa_n - Bb_n\|^2. \quad (4.33)$$

Since $1 - \alpha_n > 0$ for all $n \geq 1$, it follows from (4.33) that

$$-(1 - \alpha_n) [\gamma_n \langle J_{E_3}(Aa_n - Bb_n), Ap_n - Bq_n \rangle] \leq -(1 - \alpha_n) \frac{\rho}{2} \|Aa_n - Bb_n\|^2. \quad (4.34)$$

Thus, we obtain from (4.29) and (4.34) that

$$\begin{aligned} \Omega_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{2\varepsilon\alpha_n}{\beta}\right) \Omega_n + \varepsilon\alpha_n - (1 - \alpha_n) \frac{\rho}{2} \|Aa_n - Bb_n\|^2 \\ &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{2\varepsilon\alpha_n}{\beta}\right) \Omega_n + \varepsilon\alpha_n \\ &\leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] \Omega_n + \alpha_n [\Sigma + \varepsilon] \\ &= \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] \Omega_n + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) \left[\frac{\beta(\Sigma + \varepsilon)}{\beta - 2\varepsilon}\right] \\ &\leq \max \left\{ \Omega_n, \frac{\beta(\Sigma + \varepsilon)}{\beta - 2\varepsilon} \right\}, \text{ for all } n \geq N_1, \end{aligned}$$

which implies by induction that

$$\Omega_n \leq \max \left\{ \mathcal{M}, \Omega_{N_1}, \frac{\beta(\Sigma + \varepsilon)}{\beta - 2\varepsilon} \right\}, \text{ for all } n \geq 1,$$

where $\mathcal{M} = \max \{\Omega_1, \Omega_2, \dots, \Omega_{N_1-1}\}$. Thus, the sequence $\{\Omega_n\}$ is bounded, and hence the sequences $\{D_f(\bar{x}, x_n)\}$ and $\{D_g(\bar{y}, y_n)\}$ are bounded. Therefore, by Lemma 3.1.11, we have that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. $\square$

Lemma 4.1.2. *Assume that conditions (Q1) – (Q7) hold and let $\{u_n\}, \{p_n\}, \{q_n\}$ and $\{v_n\}$ be as defined in Algorithm 4.1. Assume that $Fx \neq 0$, for all $x \in C$ and $Gy \neq 0$, for all $y \in D$.*

- (a) *If there exist subsequences $\{p_{n_k}\}$ and $\{u_{n_k}\}$ of $\{p_n\}$ and $\{u_n\}$, respectively, with $p_{n_k} \rightharpoonup p$ and $\|p_{n_k} - u_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$, then $p \in MVI(C, F)$.*
- (b) *If there exist subsequences $\{q_{n_k}\}$ and $\{v_{n_k}\}$ of $\{q_n\}$ and $\{v_n\}$, respectively, with $q_{n_k} \rightharpoonup q$ and $\|q_{n_k} - v_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$, then $q \in MVI(D, G)$.*

Proof. (a) Let $\{p_{n_k}\}$ and $\{u_{n_k}\}$ be subsequences of $\{p_n\}$ and $\{u_n\}$, respectively, with $p_{n_k} \rightharpoonup p$ and $\lim_{k \rightarrow \infty} \|p_{n_k} - u_{n_k}\| = 0$. It immediately follows that $u_{n_k} \rightharpoonup p$. Since $\{u_{n_k}\} \subset C$ and C is weakly closed, we have that $p \in C$.

Now, if $\limsup_{k \rightarrow \infty} \|Fp_{n_k}\| = 0$, then $\lim_{k \rightarrow \infty} \|Fp_{n_k}\| = \liminf_{k \rightarrow \infty} \|Fp_{n_k}\| = 0$. Thus, we have by condition (4.1) that $0 \leq \|Fp\| \leq \liminf_{k \rightarrow \infty} \|Fp_{n_k}\| = 0$, which implies that $Fp = 0$, which is a contradiction to the hypothesis that $Fx \neq 0$ for all $x \in C$. Thus, this case that $\limsup_{k \rightarrow \infty} \|Fp_{n_k}\| = 0$ does not hold.

Thus, $\limsup_{k \rightarrow \infty} \|Fp_{n_k}\| > 0$.

From the definition of u_{n_k} and (1.18), we obtain that

$$\langle \nabla f p_{n_k} - \lambda_{n_k} F p_{n_k} - \nabla f u_{n_k}, y - u_{n_k} \rangle \leq 0, \text{ for all } y \in C,$$

or equivalently

$$\frac{1}{\lambda_{n_k}} \langle \nabla f p_{n_k} - \nabla f u_{n_k}, y - u_{n_k} \rangle + \langle F p_{n_k}, u_{n_k} - p_{n_k} \rangle \leq \langle F p_{n_k}, y - p_{n_k} \rangle, \text{ for all } y \in C. \quad (4.35)$$

Taking limit inferior on both sides of (4.35) as $k \rightarrow \infty$ and using our assumption, uniform continuity of F , uniform continuity of ∇f and the boundedness of the sequences $\{p_{n_k}\}$ and $\{u_{n_k}\}$, we obtain

$$0 \leq \liminf_{k \rightarrow \infty} \langle F p_{n_k}, y - p_{n_k} \rangle \leq \limsup_{k \rightarrow \infty} \langle F p_{n_k}, y - p_{n_k} \rangle, \text{ for all } y \in C. \quad (4.36)$$

We now consider two possible cases on $\limsup_{k \rightarrow \infty} \langle F p_{n_k}, y - p_{n_k} \rangle$.

Case A. Suppose $\limsup_{k \rightarrow \infty} \langle F p_{n_k}, y - p_{n_k} \rangle > 0$. Then we can find a subsequence $\{p_{n_{k_j}}\}$ of $\{p_{n_k}\}$ with $\lim_{j \rightarrow \infty} \langle F p_{n_{k_j}}, y - p_{n_{k_j}} \rangle > 0$. So, there exists a positive integer N_2 such that $\langle F p_{n_{k_j}}, y - p_{n_{k_j}} \rangle > 0$ for all $j \geq N_2$.

Since F is quasimonotone, it follows that

$$\langle F y, y - p_{n_{k_j}} \rangle \geq 0, \text{ for all } j \geq N_2. \quad (4.37)$$

Taking limit in (4.37) as $j \rightarrow \infty$, we obtain that $\langle F y, y - p \rangle \geq 0$ and hence $p \in MVI(C, F)$.

Case B. Suppose $\limsup_{k \rightarrow \infty} \langle F p_{n_k}, y - p_{n_k} \rangle = 0$. Then, we obtain from (4.36) that

$$\lim_{k \rightarrow \infty} \langle F p_{n_k}, y - p_{n_k} \rangle = 0. \quad (4.38)$$

Let us now choose $\tau_k = |\langle F p_{n_k}, y - p_{n_k} \rangle| + \frac{1}{k}$, $k = 1, 2, 3, \dots$. Then, we have from (4.38) that

$$\langle F p_{n_k}, y - p_{n_k} \rangle + \tau_k > 0, \text{ for all } k = 1, 2, 3, \dots. \quad (4.39)$$

Since $\limsup_{k \rightarrow \infty} \|F p_{n_k}\| > 0$, there exists a subsequence of $\{F p_{n_k}\}$, which is still denoted by $\{F p_{n_k}\}$, such that $\lim_{k \rightarrow \infty} \|F p_{n_k}\| > 0$. Thus, there exists a positive integer N_3 such that $\|F p_{n_k}\| > 0$ for

all $k \geq N_3$. Let us now set $h_{n_k} = \frac{J_{E_1}^{-1} F p_{n_k}}{\|F p_{n_k}\|^2}$. Thus, we have $\langle F p_{n_k}, h_{n_k} \rangle = 1$. It follows from (4.39) that

$$\langle F p_{n_k}, y - p_{n_k} \rangle + \tau_k \langle F p_{n_k}, h_{n_k} \rangle > 0,$$

that is,

$$\langle F p_{n_k}, y + \tau_k h_{n_k} - p_{n_k} \rangle > 0,$$

which implies by the quasimonotonicity of F that

$$\langle F(y + \tau_k h_{n_k}), y + \tau_k h_{n_k} - p_{n_k} \rangle \geq 0,$$

and thus we obtain from the Lipschitz continuity of F that

$$\begin{aligned} \langle Fy, y - p_{n_k} \rangle &\geq -\langle F(y + \tau_k h_{n_k}) - Fy, y - p_{n_k} \rangle - \langle F(y + \tau_k h_{n_k}), \tau_k h_{n_k} \rangle \\ &\geq -\|F(y + \tau_k h_{n_k}) - Fy\| \|y - p_{n_k}\| - \|F(y + \tau_k h_{n_k})\| \|\tau_k h_{n_k}\| \\ &\geq -L_1 \|\tau_k h_{n_k}\| \|y - p_{n_k}\| - \|F(y + \tau_k h_{n_k})\| \|\tau_k h_{n_k}\|. \end{aligned} \quad (4.40)$$

We now show that $\lim_{k \rightarrow \infty} (\tau_k h_{n_k}) = 0$. Since $Fp \neq 0$, we have by condition (4.1) that $0 < \|Fp\| \leq \liminf_{k \rightarrow \infty} \|Fp_{n_k}\|$. Hence, we have

$$0 \leq \limsup_{k \rightarrow \infty} \|\tau_k h_{n_k}\| = \limsup_{k \rightarrow \infty} \left(\frac{\tau_k}{\|Fp_{n_k}\|} \right) \leq \frac{\limsup_{k \rightarrow \infty} \tau_k}{\liminf_{k \rightarrow \infty} \|Fp_{n_k}\|} \leq \frac{0}{\|Fp\|} = 0.$$

Thus, $\limsup_{k \rightarrow \infty} \|\tau_k h_{n_k}\| = 0$ and hence $\lim_{k \rightarrow \infty} \|\tau_k h_{n_k}\| = 0$. So, taking limit on both sides of (4.40) as $k \rightarrow \infty$, we get $\langle Fy, y - p \rangle \geq 0$, for all $y \in C$ and hence $p \in MVI(C, F)$.

Part (b) of the lemma can be proved in a similar way as that of part (a) and hence the proof is complete. $\square$

Theorem 4.1.2. *Assume that conditions (Q1) – (Q7) hold. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1, with $F(x) \neq 0$ for all $x \in C$ and $G(y) \neq 0$ for all $y \in D$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Omega^h(\hat{x}, \hat{y})$ and $h = (f, g)$.*

Proof. Let $(x^*, y^*) = P_\Omega^h(\hat{x}, \hat{y})$. Then, we have by (1.14) that

$$\langle (\nabla f \hat{x}, \nabla g \hat{y}) - (\nabla f x^*, \nabla g y^*), (s, t) - (x^*, y^*) \rangle \leq 0, \quad (4.41)$$

for all $(s, t) \in \Omega$. From (4.8), (1.21), (1.22) and Lemma 3.1.9, we get

$$\begin{aligned} D_f(x^*, x_{n+1}) &= D_f(x^*, \nabla f^*(\alpha_n \nabla f \hat{x} + (1 - \alpha_n) \nabla f w_n)) \\ &= V_f(x^*, \alpha_n \nabla f \hat{x} + (1 - \alpha_n) \nabla f w_n) \\ &\leq V_f(x^*, \alpha_n \nabla f \hat{x} + (1 - \alpha_n) \nabla f w_n - \alpha_n (\nabla f \hat{x} - \nabla f x^*)) \\ &\quad + \alpha_n \langle \nabla f \hat{x} - \nabla f x^*, x_{n+1} - x^* \rangle \\ &= V_f(x^*, \alpha_n \nabla f x^* + (1 - \alpha_n) \nabla f w_n) + \alpha_n \langle \nabla f \hat{x} - \nabla f x^*, x_{n+1} - x^* \rangle \\ &= D_f(x^*, \nabla f^*(\alpha_n \nabla f x^* + (1 - \alpha_n) \nabla f w_n)) \\ &\quad + \alpha_n \langle \nabla f \hat{x} - \nabla f x^*, x_{n+1} - x^* \rangle \\ &\leq \alpha_n D_f(x^*, x^*) + (1 - \alpha_n) D_f(x^*, w_n) + \alpha_n \langle \nabla f \hat{x} - \nabla f x^*, x_{n+1} - x^* \rangle \\ &= (1 - \alpha_n) D_f(x^*, w_n) + \alpha_n \langle \nabla f \hat{x} - \nabla f x^*, x_{n+1} - x^* \rangle. \end{aligned} \quad (4.42)$$

Substituting (4.24) into (4.42) with $\bar{x} = x^*$ and taking the conditions on μ and ζ_n into account,

we obtain

$$\begin{aligned}
D_f(x^*, x_{n+1}) &\leq (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(x^*, x_n) - (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) D_f(a_n, x_n) \\
&\quad - (1 - \alpha_n) \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(u_n, p_n) - (1 - \alpha_n) \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(w_n, u_n) \\
&\quad - (1 - \alpha_n) \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - x^* \rangle + (1 - \alpha_n) \frac{\zeta_n}{2} \\
&\quad + \alpha_n \langle \nabla f \hat{x} - \nabla f x^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(x^*, x_n) + \alpha_n \langle \nabla f \hat{x} - \nabla f x^*, x_{n+1} - x^* \rangle \\
&\quad - (1 - \alpha_n) \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - x^* \rangle + (1 - \alpha_n) \frac{\zeta_n}{2} \\
&\leq (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(x^*, x_n) + \alpha_n \|\nabla f \hat{x} - \nabla f x^*\| \|x_{n+1} - x_n\| + \frac{\zeta_n}{2} \\
&\quad + \alpha_n \langle \nabla f \hat{x} - \nabla f x^*, x_n - x^* \rangle - (1 - \alpha_n) \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - x^* \rangle \\
&= (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(x^*, x_n) - (1 - \alpha_n) \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - x^* \rangle \\
&\quad + \alpha_n \left[\|\nabla f \hat{x} - \nabla f x^*\| \|x_{n+1} - x_n\| + \langle \nabla f \hat{x} - \nabla f x^*, x_n - x^* \rangle + \frac{\zeta_n}{2\alpha_n} \right].
\end{aligned} \tag{4.43}$$

Since $\lim_{n \rightarrow \infty} \frac{\zeta_n}{\alpha_n} = 0$, for every $\varepsilon \in \left(0, \frac{\beta}{2}\right)$ there exists $n_0 \in \mathbb{N}$ such that $\frac{\zeta_n}{\alpha_n} < \varepsilon$, that is, $\zeta_n < \varepsilon \alpha_n$ for all $n \geq n_0$. Thus, it follows from (4.43) that

$$\begin{aligned}
D_f(x^*, x_{n+1}) &\leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(x^*, x_n) - (1 - \alpha_n) \gamma_n \langle A^* J_{E_3}(Aa_n - Bb_n), p_n - x^* \rangle \\
&\quad + \alpha_n \left[\frac{\zeta_n}{2\alpha_n} + \|\nabla f \hat{x} - \nabla f x^*\| \|x_{n+1} - x_n\| + \langle \nabla f \hat{x} - \nabla f x^*, x_n - x^* \rangle \right] \\
&= \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(x^*, x_n) - (1 - \alpha_n) \gamma_n [\langle A^* J_{E_3}(Aa_n - Bb_n), p_n - x^* \rangle] \\
&\quad + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) \left[\frac{\frac{\zeta_n}{2\alpha_n} + \|\nabla f \hat{x} - \nabla f x^*\| \|x_{n+1} - x_n\| + \langle \nabla f \hat{x} - \nabla f x^*, x_n - x^* \rangle}{\left(1 - \frac{2\varepsilon}{\beta}\right)} \right] \\
&= \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(x^*, x_n) + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) \Delta_n \\
&\quad - (1 - \alpha_n) \gamma_n \langle J_{E_3}(Aa_n - Bb_n), Ap_n - Ax^* \rangle, \text{ where}
\end{aligned} \tag{4.44}$$

$$\Delta_n = \left[\frac{\frac{\zeta_n}{2\alpha_n} + \|\nabla f \hat{x} - \nabla f x^*\| \|x_{n+1} - x_n\| + \langle \nabla f \hat{x} - \nabla f x^*, x_n - x^* \rangle}{\left(1 - \frac{2\varepsilon}{\beta}\right)} \right], \text{ for all } n \geq n_0.$$

Similarly,

$$D_g(y^*, y_{n+1}) \leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_g(y^*, y_n) + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) \Gamma_n - (1 - \alpha_n) \gamma_n \langle J_{E_3}(Bb_n - Aa_n), Bq_n - By^* \rangle, \quad (4.45)$$

where

$$\Gamma_n = \left[\frac{\frac{\zeta_n}{2\alpha_n} + \|\nabla g\hat{y} - \nabla gy^*\| \|y_{n+1} - y_n\| + \langle \nabla g\hat{y} - \nabla gy^*, y_n - y^* \rangle}{\left(1 - \frac{2\varepsilon}{\beta}\right)} \right], \text{ for all } n \geq n_0.$$

Denote $\Omega_n^* = D_f(x^*, x_n) + D_g(y^*, y_n)$ and $\Sigma^* = D_f(x^*, \hat{x}) + D_g(y^*, \hat{y})$. Then combining (4.44) and (4.45) and using the relation (4.33), we obtain

$$\begin{aligned} \Omega_{n+1}^* &\leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] \Omega_n^* + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) (\Delta_n + \Gamma_n) - (1 - \alpha_n) \frac{\rho}{2} \|Aa_n - Bb_n\|^2 \\ &\leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] \Omega_n^* + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) (\Delta_n + \Gamma_n). \end{aligned} \quad (4.46)$$

Combining (4.25) and (4.27) with $\bar{x} = x^*$ and rearranging terms, we obtain

$$\begin{aligned} &(1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) D_f(a_n, x_n) + (1 - \alpha_n) \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(u_n, p_n) \\ &+ (1 - \alpha_n) \left(1 - \frac{\mu}{\beta} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(w_n, u_n) + (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) D_g(b_n, y_n) \\ &+ (1 - \alpha_n) \left(1 - \frac{\delta}{\beta} \frac{\eta_n}{\eta_{n+1}}\right) D_g(v_n, q_n) + (1 - \alpha_n) \left(1 - \frac{\delta}{\beta} \frac{\eta_n}{\eta_{n+1}}\right) D_g(z_n, v_n) \\ &+ (1 - \alpha_n) \frac{\rho}{2} \|Aa_n - Bb_n\|^2 \leq \Omega_n^* - \Omega_{n+1}^* + \alpha_n \left[\Sigma^* + \left(\frac{2\varepsilon}{\beta} - 1\right) \Omega_n^* + \frac{\zeta_n}{\alpha_n} \right]. \end{aligned} \quad (4.47)$$

We now consider two cases on the sequence $\{\Omega_n^*\}$.

Case I. If there exists a natural number N such that $\Omega_{n+1}^* \leq \Omega_n^*$ for all $n \geq N$, then we have by the Monotone Convergence Theorem that the sequence $\{\Omega_n^*\}$ converges. Taking limit as $n \rightarrow \infty$ in (4.47), we obtain

$$\lim_{n \rightarrow \infty} \|Aa_n - Bb_n\| = 0, \quad (4.48)$$

$$\lim_{n \rightarrow \infty} D_f(a_n, x_n) = \lim_{n \rightarrow \infty} D_f(u_n, p_n) = \lim_{n \rightarrow \infty} D_f(w_n, u_n) = 0,$$

and

$$\lim_{n \rightarrow \infty} D_g(b_n, y_n) = \lim_{n \rightarrow \infty} D_g(v_n, q_n) = \lim_{n \rightarrow \infty} D_g(z_n, v_n) = 0,$$

from which we obtain by Lemma 3.1.13 that

$$\lim_{n \rightarrow \infty} \|a_n - x_n\| = \lim_{n \rightarrow \infty} \|p_n - u_n\| = \lim_{n \rightarrow \infty} \|w_n - u_n\| = 0, \quad (4.49)$$

and

$$\lim_{n \rightarrow \infty} \|b_n - y_n\| = \lim_{n \rightarrow \infty} \|v_n - q_n\| = \lim_{n \rightarrow \infty} \|z_n - v_n\| = 0. \quad (4.50)$$

We also obtain from the property of γ_n , (4.48) and Lipschitz continuity of ∇f^* that

$$\begin{aligned} \|p_n - a_n\| &= \|\nabla f^*[\nabla f a_n - \gamma_n A^* J_{E_3}(Aa_n - Bb_n)] - \nabla f^*(\nabla f a_n)\| \\ &\leq \frac{1}{\beta} \gamma_n \|A^* J_{E_3}(Aa_n - Bb_n)\| \leq \frac{1}{\beta} (\rho + 1) \|A^* J_{E_3}(Aa_n - Bb_n)\| \\ &\leq \frac{1}{\beta} (\rho + 1) \|A^*\| \|J_{E_3}(Aa_n - Bb_n)\| \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.51)$$

From (4.8), uniform continuity of ∇f and the condition on α_n , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\nabla f x_{n+1} - \nabla f w_n\| &= \lim_{n \rightarrow \infty} \|\alpha_n \nabla f \hat{x} + (1 - \alpha_n) \nabla f w_n - \nabla f w_n\| \\ &= \lim_{n \rightarrow \infty} \alpha_n \|\nabla f \hat{x} - \nabla f w_n\| = 0. \end{aligned} \quad (4.52)$$

Since ∇f^* is uniformly continuous, we obtain from (4.52) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - w_n\| = 0. \quad (4.53)$$

Consequently, we obtain from (4.49), (4.51) and (4.53) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 \quad (4.54)$$

Similarly, we obtain

$$\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0. \quad (4.55)$$

Since the sequence $\{(x_n, y_n)\}$ is bounded in the reflexive Banach space $E_1 \times E_2$, there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ such that $(x_{n_k}, y_{n_k}) \rightharpoonup (\tilde{x}, \tilde{y}) \in E_1 \times E_2$ and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (\nabla f \hat{x}, \nabla g \hat{y}) - (\nabla f x^*, \nabla g y^*), (x_n, y_n) - (x^*, y^*) \rangle \\ = \lim_{k \rightarrow \infty} \langle (\nabla f \hat{x}, \nabla g \hat{y}) - (\nabla f x^*, \nabla g y^*), (x_{n_k}, y_{n_k}) - (x^*, y^*) \rangle. \end{aligned} \quad (4.56)$$

Moreover, we have $x_{n_k} \rightharpoonup \tilde{x}$ and $y_{n_k} \rightharpoonup \tilde{y}$. Now, we show that $(\tilde{x}, \tilde{y}) \in \Omega$.

Since $\|x_{n_k} - u_{n_k}\| \rightarrow 0$ and $x_{n_k} \rightharpoonup \tilde{x}$, we have that $u_{n_k} \rightharpoonup \tilde{x}$. Since C is weakly closed, we have that $\tilde{x} \in C$, and thus we have by assumption that $F\tilde{x} \neq 0$. This implies by Lemma 4.1.2 and (4.49) that $\tilde{x} \in MVI(C, F)$. One can similarly show that $\tilde{y} \in MVI(D, G)$. Now, it remains to show that $A\tilde{x} = B\tilde{y}$.

From Lemma 3.1, one obtains that

$$\begin{aligned} \|A\tilde{x} - B\tilde{y}\|^2 &= \|Aa_{n_k} - Bb_{n_k} + A\tilde{x} - Aa_{n_k} + Bb_{n_k} - B\tilde{y}\|^2 \\ &\leq \|Aa_{n_k} - Bb_{n_k}\|^2 + 2\langle J_{E_3}(A\tilde{x} - B\tilde{y}), A\tilde{x} - Aa_{n_k} + Bb_{n_k} - B\tilde{y} \rangle. \end{aligned} \quad (4.57)$$

From (4.49) and (4.50) we obtain $a_{n_k} \rightharpoonup \tilde{x}$ and $b_{n_k} \rightharpoonup \tilde{y}$, respectively. Thus, we obtain from the sequential weakly continuity of A and B that $Aa_{n_k} \rightharpoonup A\tilde{x}$ and $Bb_{n_k} \rightharpoonup B\tilde{y}$, as $k \rightarrow \infty$. Thus, from (4.48) and (4.57), we obtain that $A\tilde{x} = B\tilde{y}$. Therefore, $(\tilde{x}, \tilde{y}) \in \Omega$. Thus, we obtain by

(4.41) and (4.56) that

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \langle (\nabla f \hat{x}, \nabla g \hat{y}) - (\nabla f x^*, \nabla g y^*), (x_n, y_n) - (x^*, y^*) \rangle \\
 &= \lim_{k \rightarrow \infty} \langle (\nabla f \hat{x}, \nabla g \hat{y}) - (\nabla f x^*, \nabla g y^*), (x_{n_k}, y_{n_k}) - (x^*, y^*) \rangle \\
 &= \langle (\nabla f \hat{x}, \nabla g \hat{y}) - (\nabla f x^*, \nabla g y^*), (\tilde{x}, \tilde{y}) - (x^*, y^*) \rangle \leq 0.
 \end{aligned} \tag{4.58}$$

From (4.54), (4.55) and (4.58), we obtain

$$\limsup_{n \rightarrow \infty} (\Delta_n + \Gamma_n) \leq 0. \tag{4.59}$$

Consequently, we obtain from (4.46), (4.59) and Lemma 3.1.6 that $\lim_{n \rightarrow \infty} \Omega_n^* = 0$, which implies that $\lim_{n \rightarrow \infty} D_f(x^*, x_n) = \lim_{n \rightarrow \infty} D_g(y^*, y_n) = 0$ which in turn implies by Lemma 3.1.13 that $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$ and $\lim_{n \rightarrow \infty} \|y_n - y^*\| = 0$, that is, $\lim_{n \rightarrow \infty} x_n = x^*$ and $\lim_{n \rightarrow \infty} y_n = y^*$.

Case II. Suppose there exists a subsequence $\{\Omega_{n_i}^*\}$ of $\{\Omega_n^*\}$ with $\Omega_{n_i}^* < \Omega_{n_i+1}^*$ for all $i \geq 0$. Then, by Lemma 3.1.7, there exists a non-decreasing sequence $\{m_k\}$ of positive integers such that $\lim_{k \rightarrow \infty} m_k = \infty$ and

$$\Omega_{m_k}^* \leq \Omega_{m_k+1}^* \text{ and } \Omega_k^* \leq \Omega_{m_k+1}^*, \tag{4.60}$$

for all positive integers k . Thus, we have that $\Omega_{m_k}^* - \Omega_{m_k+1}^* \leq 0$. With this $\{m_k\}$, (4.47) becomes

$$\begin{aligned}
 & (1 - \alpha_{m_k}) \left(1 - \frac{2\zeta_{m_k}}{\beta}\right) D_f(a_{m_k}, x_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{\mu}{\beta} \frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right) D_f(u_{m_k}, p_{m_k}) \\
 &+ (1 - \alpha_{m_k}) \left(1 - \frac{\mu}{\beta} \frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right) D_f(w_{m_k}, u_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{2\zeta_{m_k}}{\beta}\right) D_g(b_{m_k}, y_{m_k}) \\
 &+ (1 - \alpha_{m_k}) \left(1 - \frac{\delta}{\beta} \frac{\eta_{m_k}}{\eta_{m_k+1}}\right) D_g(v_{m_k}, q_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{\delta}{\beta} \frac{\eta_{m_k}}{\eta_{m_k+1}}\right) D_g(z_{m_k}, v_{m_k}) \\
 &+ (1 - \alpha_{m_k}) \frac{\rho}{2} \|Aa_{m_k} - Bb_{m_k}\|^2 \leq \Omega_{m_k}^* - \Omega_{m_k+1}^* + \alpha_{m_k} \left[\Sigma^* + \left(\frac{2\varepsilon}{\beta} - 1\right) \Omega_{m_k}^* + \frac{\zeta_{m_k}}{\alpha_{m_k}} \right].
 \end{aligned} \tag{4.61}$$

Taking limit as $k \rightarrow \infty$ of (4.61) and taking (4.60) and the property of α_{m_k} into account, we obtain that $\lim_{k \rightarrow \infty} \|Aa_{m_k} - Bb_{m_k}\| = 0$. Moreover, we have that

$$\begin{aligned}
 & \lim_{k \rightarrow \infty} \|a_{m_k} - x_{m_k}\| = \lim_{k \rightarrow \infty} \|p_{m_k} - u_{m_k}\| = \lim_{k \rightarrow \infty} \|w_{m_k} - u_{m_k}\| = 0, \\
 & \lim_{k \rightarrow \infty} \|x_{m_k} - a_{m_k}\| = \lim_{k \rightarrow \infty} \|p_{m_k} - a_{m_k}\| = 0,
 \end{aligned}$$

and

$$\begin{aligned}
 & \lim_{k \rightarrow \infty} \|b_{m_k} - y_{m_k}\| = \lim_{k \rightarrow \infty} \|v_{m_k} - q_{m_k}\| = \lim_{k \rightarrow \infty} \|z_{m_k} - v_{m_k}\| = 0, \\
 & \lim_{k \rightarrow \infty} \|y_{m_k} - b_{m_k}\| = \lim_{k \rightarrow \infty} \|q_{m_k} - b_{m_k}\| = 0.
 \end{aligned}$$

Following similar methods used in Case I, we obtain

$$\lim_{k \rightarrow \infty} \|x_{m_k+1} - x_{m_k}\| = \lim_{k \rightarrow \infty} \|y_{m_k+1} - y_{m_k}\| = 0 \text{ and } \limsup_{k \rightarrow \infty} (\Delta_{m_k} + \Gamma_{m_k}) \leq 0. \tag{4.62}$$

We obtain from (4.46) and (4.60) that

$$\alpha_{m_k} \left(1 - \frac{2\varepsilon}{\beta}\right) \Omega_{m_k}^* \leq \Omega_{m_k}^* - \Omega_{m_k+1}^* + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) (\Delta_{m_k} + \Gamma_{m_k}) \leq \alpha_{m_k} \left(1 - \frac{2\varepsilon}{\beta}\right) (\Delta_{m_k} + \Gamma_{m_k}),$$

which implies that

$$\Omega_{m_k}^* \leq \Delta_{m_k} + \Gamma_{m_k}. \quad (4.63)$$

Taking limit as $k \rightarrow \infty$ on both sides of (4.63) and using (4.62), we obtain that $\lim_{k \rightarrow \infty} \Omega_{m_k}^* = 0$. This together with (4.46) implies that $\lim_{k \rightarrow \infty} \Omega_{m_k+1}^* = 0$, which implies by (4.60) that $\lim_{k \rightarrow \infty} \Omega_k^* = 0$. Thus, we have $\lim_{k \rightarrow \infty} D_f(x^*, x_k) = \lim_{k \rightarrow \infty} D_g(y^*, y_k) = 0$ and hence we obtain by Lemma 3.1.13 that $\lim_{k \rightarrow \infty} x_k = x^*$ and $\lim_{k \rightarrow \infty} y_k = y^*$.

Thus, we conclude from Cases I and II that the sequence (x_n, y_n) generated by Algorithm 4.1 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Omega}^h(\hat{x}, \hat{y})$ and hence the proof is complete. $\square$

We notice here that if the mappings F and G in Theorem 4.1.2 are assumed to be Lipschitz continuous and pseudomonotone, then we deduce the following corollary.

Corollary 4.1.1. *Assume that conditions (Q1) – (Q2) and (Q4) – (Q7) hold. If $F: E_1 \rightarrow E_1^*$ and $G: E_2 \rightarrow E_2^*$ are Lipschitz continuous and pseudomonotone mappings satisfying property (4.1). Then, the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Omega}^h(\hat{x}, \hat{y})$.*

Proof. As every pseudomonotone mapping is quasimonotone, the result follows from Theorem 4.1.2. $\square$

If we assume that E_1 and E_2 are smooth and 2–uniformly convex Banach spaces and if we take $f(\cdot) = \frac{1}{2} \|\cdot\|^2$ and $g(\cdot) = \frac{1}{2} \|\cdot\|^2$, in Theorem 4.1.2, then $\nabla f = J_{E_1}$, $\nabla g = J_{E_2}$, $\nabla f^* = J_{E_1^*}^{-1}$ and $\nabla g^* = J_{E_2^*}^{-1}$. Thus, we get the following corollary.

Corollary 4.1.2. *Let C and D be nonempty, closed and convex subsets of smooth and 2–uniformly convex Banach spaces E_1 and E_2 and assume that the conditions (Q3) – (Q7) are satisfied. Assume that $Fx \neq 0$ for all $x \in C$ and $Gy \neq 0$ for all $y \in D$. Then, the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1, with $\nabla f = J_{E_1}$, $\nabla g = J_{E_2}$, $\nabla f^* = J_{E_1^*}^{-1}$ and $\nabla g^* = J_{E_2^*}^{-1}$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = \Pi_{\Omega}(\hat{x}, \hat{y})$, where $\Pi_{\Omega}(\hat{x}, \hat{y})$ is the generalized projection mapping of $(\hat{x}, \hat{y})$ onto Ω .*

If we assume, in Theorem 4.1.2, that E_1, E_2 and E_3 are real Hilbert spaces, $f(\cdot) = \frac{1}{2} \|\cdot\|^2$ and $g(\cdot) = \frac{1}{2} \|\cdot\|^2$, then $\nabla f = I_{E_1}$, $\nabla g = I_{E_2}$, $\nabla f^* = I_{E_1^*}^{-1}$ and $\nabla g^* = I_{E_2^*}^{-1}$ and hence we obtain the following corollaries.

Corollary 4.1.3. *Assume that E_1, E_2 and E_3 are real Hilbert spaces and let C and D be nonempty, convex and closed subsets of E_1 and E_2 , respectively. Assume that the conditions (Q3) – (Q7) are satisfied. Assume that $Fx \neq 0$, for all $x \in C$ and $Gy \neq 0$, for all $y \in D$. Then, the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1, with $\nabla f = I_{E_1}$, $\nabla g = I_{E_2}$, $\nabla f^* = I_{E_1^*}^{-1}$*

and $\nabla g^* = I_{E_2^*}^{-1}$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Omega(\hat{x}, \hat{y})$, where $P_\Omega(\hat{x}, \hat{y})$ is the metric projection of $(\hat{x}, \hat{y})$ onto Ω .

Corollary 4.1.4. Assume that E_1 , E_2 and E_3 are real Hilbert spaces and let C and D be nonempty, convex and closed subsets of E_1 and E_2 , respectively. Let $F: E_1 \rightarrow E_1^*$ and $G: E_2 \rightarrow E_2^*$ be Lipschitz continuous and pseudomonotone mappings with the property (4.1). Assume that the conditions (Q4) – (Q7) are satisfied. Then, the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1, with $\nabla f = I_{E_1}$, $\nabla g = I_{E_2}$, $\nabla f^* = I_{E_1^*}^{-1}$ and $\nabla g^* = I_{E_2^*}^{-1}$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_\Omega(\hat{x}, \hat{y})$.

4.2 Some Theoretical Applications

In this section, we apply our main result to solve some specific problems.

Common Solutions of Minty Variational Inequality Problems

If we take $E_1 = E_2 = E_3 = E$, $C = D$ and $A = B = I_E$ in Algorithm 4.1, then our main result reduces to *common solution of Minty variational inequality problems*, which is defined as finding a point $x^* \in C$ with $(x^*, x^*) \in MVI(C, F) \times MVI(C, G)$.

Denote $\Omega^* = \{(x^*, x^*) \in MVI(C, F) \times MVI(C, G)\}$. We now deduce the following corollaries.

Corollary 4.2.1. Assume that conditions (Q1) – (Q3), (Q6), (Q7), with $E_1 = E_2 = E_3 = E$ and $C = D$ hold. Assume that $Fx \neq 0$ for all $x \in C$ and $Gy \neq 0$ for all $y \in C$. If $\Omega^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1, with $A = B = I_E$, converges strongly to (x^*, x^*) , where $(x^*, x^*) = P_{\Omega^*}^h(\hat{x}, \hat{y})$.

Corollary 4.2.2. Assume that conditions (Q1), (Q2), (Q6), (Q7), with $E_1 = E_2 = E_3 = E$ and $C = D$ hold. Let $F: C \rightarrow E^*$ and $G: C \rightarrow E^*$ be Lipschitz continuous pseudomonotone mappings satisfying condition (4.1). If $\Omega^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1, with $A = B = I_E$, converges strongly to (x^*, x^*) , where $(x^*, x^*) = P_{\Omega^*}^h(\hat{x}, \hat{y})$.

Split Minty Variational Inequality Problems

If we consider $E_2 = E_3$ and $B = I_{E_2}$ in Algorithm 4.1, then our main result reduces to *split Minty variational inequality problem* which is to find a point $(x^*, y^*) \in C \times D$ such that $(x^*, y^*) \in MVI(C, F) \times MVI(D, G) : Ax^* = y^*$.

Denote $\Gamma^* = \{(x^*, y^*) \in MVI(C, F) \times MVI(D, G) : Ax^* = y^*\}$. Then we have the following corollaries.

Corollary 4.2.3. Assume that conditions (Q1) – (Q4), (Q6), (Q7), with $E_2 = E_3$ and $B = I_{E_2}$ hold. Assume that $Fx \neq 0$ for all $x \in C$ and $Gy \neq 0$ for all $y \in D$. If $\Gamma^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1 converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Gamma^*}^h(\hat{x}, \hat{y})$.

Corollary 4.2.4. Assume that conditions (Q1), (Q2), (Q4), (Q6), (Q7), with $E_2 = E_3$ hold. Let $F: C \rightarrow E_1^*$ and $G: D \rightarrow E_2^*$ be Lipschitz continuous and pseudomonotone mappings satisfying condition (4.1). If $\Gamma^* \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1, with $B = I_{E_2}$, converges strongly to (x^*, y^*) , where $(x^*, y^*) = P_{\Gamma^*}^h(\hat{x}, \hat{y})$.

4.3 Numerical Examples

Example 4.3.1. Let $E = E_1 = E_2 = E_3 = \mathbb{R}^2$ with the standard norm.

Let $C = \{(x, y) \in E : (x - 2)^2 + y^2 \leq 1\}$ and $D = \{(x, y) \in E : x^2 + (y - 2)^2 \leq 1\}$. Clearly, C and D are closed and convex subsets of E . Define the mappings $F, G: E \rightarrow E$ by $F(x, y) = \left(\frac{x^2}{1+x^2}, 0\right)$ and $G(x, y) = \left(0, \frac{y^2}{1+y^2}\right)$. Clearly, the mappings F and G are Lipschitz continuous on E . It remains to show that the mappings F and G are quasimonotone on E . Indeed, suppose we have for all $(x_1, x_2), (y_1, y_2) \in E$ that $\langle F(x_1, x_2), (y_1, y_2) - (x_1, x_2) \rangle > 0$, that is,

$$\left\langle \left(\frac{x_1^2}{1+x_1^2}, 0\right), (y_1, y_2) - (x_1, x_2) \right\rangle = \frac{x_1^2}{1+x_1^2}(y_1 - x_1) > 0,$$

which implies that $y_1 - x_1 > 0$. Now,

$$\langle F(y_1, y_2), (y_1, y_2) - (x_1, x_2) \rangle = \left\langle \left(\frac{y_1^2}{1+y_1^2}, 0\right), (y_1, y_2) - (x_1, x_2) \right\rangle = \frac{y_1^2}{1+y_1^2}(y_1 - x_1) \geq 0,$$

showing that F is quasimonotone. If we take $(x_1, x_2) = (0, 0)$ and $(y_1, y_2) = \left(\frac{-1}{2}, 0\right)$, then $\langle F(x_1, x_2), (y_1, y_2) - (x_1, x_2) \rangle = 0$, but $\langle F(y_1, y_2), (y_1, y_2) - (x_1, x_2) \rangle < 0$, that is, F is not pseudomonotone. Similarly, one can show that G is quasimonotone but is not pseudomonotone. Clearly, $F(x, y) \neq 0$ for all $(x, y) \in C$ and $G(x, y) \neq 0$, for all $(x, y) \in D$. Moreover, $MVI(C, F) = \{(1, 0)\}$ and $MVI(D, G) = \{(0, 1)\}$. Let $A: E \rightarrow E$ and $B: E \rightarrow E$ be defined by $A(x_1, x_2) = \left(0, \frac{x_2}{2}\right)$ and $B(x_1, x_2) = (3x_1, 0)$. Clearly, A and B are bounded linear mappings with adjoints $A^*(x_1, x_2) = \left(0, \frac{x_2}{2}\right)$ and $B^*(x_1, x_2) = (3x_1, 0)$. Moreover, $A(1, 0) = (0, 0) = B(0, 1)$. Therefore, we have that

$$(x^*, y^*) = ((1, 0), (0, 1)) \in \Omega = \{(x, y) \in MVI(C, F) \times MVI(D, G) : Ax = By\}.$$

Define $f: E \rightarrow (-\infty, +\infty]$ and $g: E \rightarrow (-\infty, +\infty]$ by $f(x) = \frac{1}{2}\|x\|^2$ and $g(y) = \frac{1}{2}\|y\|^2$. Then, $\nabla f x = \nabla f^* x = I_E x = x$ and $\nabla g y = \nabla g^* y = I_E y = y$.

If we take $\zeta_n = \frac{1}{n^2 + 2}$, $\alpha_n = \frac{1}{n + 100}$, $\beta = 1$, $\mu = 0.4$, and $\delta = 0.3$, then the conditions (Q1) – (Q7) are satisfied. The figures below reveal convergence of the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1 for different values of the inertial parameter θ_n and different initial values (x_1, y_1) for the points $(\hat{x}, \hat{y}) = ((1.5, 0), (0, 1.5))$ and $(x_0, y_0) = ((2, 0), (0, 2))$. Figure 4.1 indicates the convergence of the sequence with the aforementioned parameters for different values of θ_n and initial value $(x_1, y_1) = ((2, 0), (0, 2))$. Figure 4.2 shows the convergence of the sequence for the inertial parameter $\theta_n = \frac{\sigma_n}{7}$ and different initial values $(x_1, y_1) = ((2, 0), (0, 2))$, $(x_1, y_1) = ((1.5, 0.3), (0.3, 1.5))$ and $(x_1, y_1) = ((1.9, -0.1), (-0.1, 1.9))$.

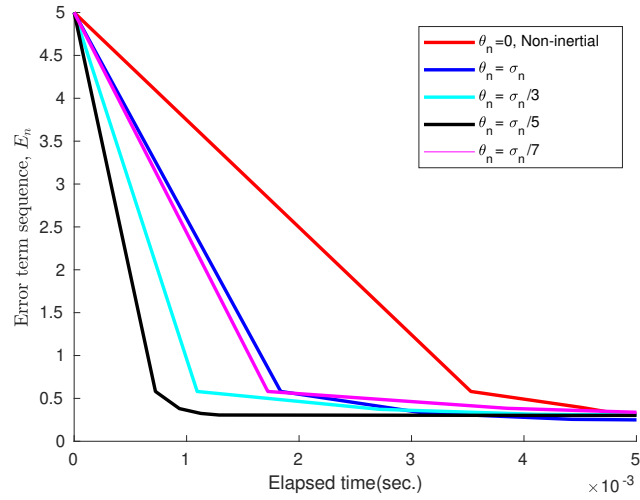


Figure 4.1: Rate of convergence for different values of the inertial term θ_n and initial point $(x_1, y_1) = ((2, 0), (0, 2))$.

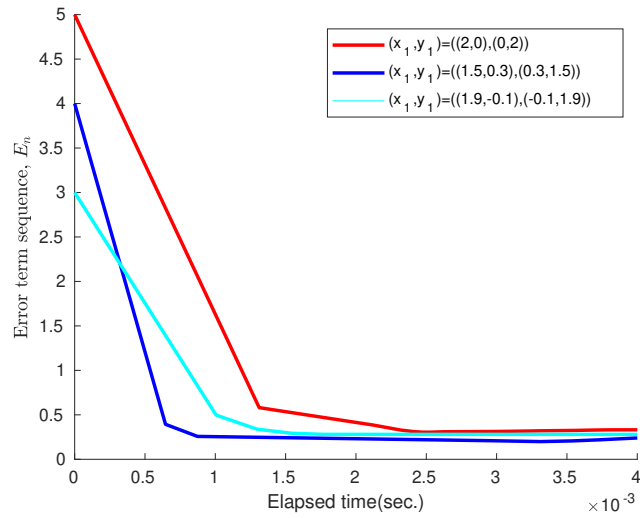


Figure 4.2: Rate of convergence for different choices of initial points (x_1, y_1) .

Number of Iterations	$E_n = \ (x_n, y_n) - (x^*, y^*)\ $	Elapsed time in seconds
100	0.3323	0.0216
200	0.3305	0.0291
300	0.3296	0.0447
500	0.3288	0.0621
700	0.3283	0.0775
900	0.3280	0.0886

Table 4.1: Approximate rates of convergence of (x_n, y_n) .

Remark 4.3.1. The result of the numerical experiment shows that the absolute error term sequence $E_n = \|(x_n, y_n) - (x^*, y^*)\|$ converges strongly to zero, that is, the sequence $\{(x_n, y_n)\}$ converges strongly to the point $(x^*, y^*) = ((1, 0), (0, 1))$ of Ω (See , Figure 4.1 and Figure 4.2).

Figure 4.1 shows that the convergence rate of the method with inertial extrapolation ($\theta_n \neq 0$) is faster than that of the non-inertial version ($\theta_n = 0$). It can be observed from Figure 4.2 that the convergence of the method is independent of our choice of the initial points, that is, the sequence (x_n, y_n) generated by Algorithm 4.1 converges strongly to the solution $((1, 0), (0, 1))$ of the given problem for any initial point (x_1, y_1) . Table 4.1 also illustrates $\{(x_n, y_n)\}$ converges strongly to the point $(x^*, y^*) = ((1, 0), (0, 1))$ of Ω .



5. Solutions of Problems Involving Variational Inequalities and Fixed Points

This chapter is divided into three sections. The first section deals with some preliminary concepts that are important to prove the main results. In section two, we introduce an algorithm for approximating solutions of the bilevel variational inequality problems with fixed point constraints in Hilbert space settings. We also discuss the strong convergence analysis of the proposed method and provide illustrative examples. The third section deals with a method of approximation of a solution of split equality common solution of variational inequality and f, g -fixed point problems involving uniformly continuous pseudomonotone mappings in the variational inequality problems and Bregman relatively f, g -nonexpansive mappings in the fixed point problems in the setting of reflexive real Banach space. We prove strong convergence theorem under some mild conditions on the control sequences and provide numerical examples to demonstrate the effectiveness of our method.

5.1 Preliminaries

Lemma 5.1.1. [58] *Let C be a nonempty, closed and convex subset of E . If $F: C \rightarrow E^*$ is a continuous and pseudomonotone mapping, then $VI(C, F)$ is closed and convex. Moreover, $p \in VI(C, F)$ if and only if $p \in MVI(C, F)$.*

Lemma 5.1.2. [178] *Let $G: H \rightarrow H$ be an l -Lipschitz continuous and κ -strongly monotone mapping. Let $\beta \in \left(0, \frac{2\kappa}{l^2}\right)$ and define the mapping F by $F(x) = x - \alpha\beta G(x)$ for any $\alpha \in (0, 1)$. Then*

$$\|F(x) - F(y)\| \leq (1 - \alpha\tau)\|x - y\|,$$

holds for all $x, y \in H$, where $\tau = 1 - \sqrt{1 - \beta(2\kappa - \beta l^2)} \in (0, 1)$.

Lemma 5.1.3. [102] *Let $G: H \rightarrow H$ be an l -Lipschitz continuous and κ -strongly monotone mapping. Assume that $\beta \in \left(0, \frac{2\kappa}{l^2}\right)$. Then $I - \beta G$ is a contraction mapping with contraction constant $1 - \omega$, where $\omega = \frac{\beta}{2}(2\kappa - \beta l^2)$ and I is the identity mapping on H .*

5.2 Inertial Methods for Solving Bilevel Variational Inequality Problems with Fixed Point Constraints

This section is devoted to introducing an iterative algorithm for solving bilevel variational inequality problems involving fixed point constraints in their lower level problems. Then strong convergence analysis will be discussed, and finally some numerical examples will be provided to support the theoretical findings.

5.2.1 Main Results

Before introducing our algorithm, we put the following conditions that are useful in the convergence analysis of the method.

- (P1) Let C be a nonempty, closed and convex subset of a real Hilbert space H_1 and let H_2 be another real Hilbert space;
- (P2) Let $F: H_1 \rightarrow H_1$ be a uniformly continuous, pseudomonotone mapping on C that satisfies the condition: $\|Fx^*\| \leq \liminf_{n \rightarrow \infty} \|Fx_n\|$ for every sequence $\{x_n\} \subseteq C$ with $x_n \rightharpoonup x^*$;
- (P3) Let $G: H_1 \rightarrow H_1$ be a κ -strongly monotone and L -Lipschitz continuous mapping and let $\beta \in \left(0, \frac{2\kappa}{L^2}\right)$;
- (P4) Let $A: H_1 \rightarrow H_2$ be a nonzero bounded linear mapping with adjoint A^* ;
- (P5) Let $T: H_2 \rightarrow H_2$ be a ν -demimetric with $I_2 - T$ being demiclosed at 0;
- (P6) Let $\Omega = \{x^* \in VI(C, F) : Ax^* \in F(T)\} \neq \emptyset$;
- (P7) Let $\{\delta_n\} \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (P8) Let $\{\zeta_n\}$ be a positive sequence such that $\frac{\zeta_n}{\delta_n} \rightarrow 0$ as $n \rightarrow \infty$.

We now state our proposed algorithm.

Algorithm 5.1.

Initialization: Let $x_0, x_1 \in H_1$, $\alpha \in [0, 1)$, $\eta > 0$, $\rho, \gamma, l \in (0, 1)$. Set $n = 1$. Then compute the sequence $\{x_n\}$ as follows:

Step 1: Given $x_{n-1}, x_n \in H_1$, choose α_n such that $0 \leq \alpha_n \leq \sigma_n$, where

$$\sigma_n = \begin{cases} \min \left\{ \alpha, \frac{\zeta_n}{\|x_n - x_{n-1}\|} \right\}, & \text{if } \|x_n - x_{n-1}\| \neq 0, \\ \alpha, & \text{otherwise.} \end{cases} \quad (5.1)$$

Step 2: Compute

$$w_n = P_C(x_n + \alpha_n(x_n - x_{n-1})). \quad (5.2)$$

Step 3: Compute

$$z_n = P_C(w_n - \eta_n A^*(I_2 - T)Aw_n); \quad (5.3)$$

where η_n is chosen to be a bounded sequence with the property

$$0 < \epsilon \leq \eta_n \leq \frac{(1 - \nu)\|Aw_n - TAw_n\|^2}{\|A^*(Aw_n - TAw_n)\|^2} - \epsilon, \text{ if } Aw_n \neq TAw_n, \quad (5.4)$$

otherwise set $\eta_n = \eta > 0$.

Step 4: Compute

$$y_n = P_C(z_n - \lambda_n F z_n), \quad (5.5)$$

where $\lambda_n = \gamma l^{j_m}$ and j_m is the smallest non-negative integer j satisfying

$$\gamma l^j \|F(P_C(z_n - \gamma l^j F z_n)) - F z_n\| \leq \rho \|P_C(z_n - \gamma l^j F z_n) - z_n\|. \quad (5.6)$$

Step 5: Compute

$$s_n = y_n - \lambda_n (F y_n - F z_n). \quad (5.7)$$

Step 6: Compute

$$x_{n+1} = P_C(s_n - \beta \delta_n G s_n). \quad (5.8)$$

Remark 5.2.1. Some of the advantages of our method over the existing ones are:

- The mapping considered in the lower level VIP is a uniformly continuous not just Lipschitz continuous.
- It generalizes some results since the mappings of the lower level problem is the more general class of pseudomonotone mappings.
- It is novel in the sense that the mapping for the fixed point constraint is a demimetric mapping which is more general than demicontractive mappings.

Remark 5.2.2. We deduce from (5.1) and condition (P8) that

$$\lim_{n \rightarrow \infty} \frac{\alpha_n}{\delta_n} \|x_n - x_{n-1}\| = 0, \quad (5.9)$$

and this together with the condition on δ_n gives that

$$\lim_{n \rightarrow \infty} \alpha_n \|x_n - x_{n-1}\| = 0. \quad (5.10)$$

Lemma 5.2.1. Assume that the conditions (P1) – (P8) are satisfied. Then the line search rule (5.6) is well defined.

Proof. Denote $h_j = P_C(z_n - \gamma l^j F z_n)$. If we consider the case when $z_n \in VI(C, F)$, then $z_n = P_C(z_n - \gamma l^j F z_n)$. Thus, we have $z_n = h_j$ and hence (5.6) is satisfied for $j = 0$. Now, assume the case when $z_n \notin VI(C, F)$ and assume on the contrary that

$$\gamma l^j \|F(P_C(z_n - \gamma l^j F z_n)) - F z_n\| > \rho \|P_C(z_n - \gamma l^j F z_n) - z_n\|, \quad (5.11)$$

for all $j \geq 0$. That is,

$$\frac{1}{\rho} \|F P_C(z_n - \gamma l^j F z_n) - F z_n\| > \frac{1}{\gamma l^j} \|P_C(z_n - \gamma l^j F z_n) - z_n\|, \text{ for all } j \geq 0. \quad (5.12)$$

Since the mapping P_C is continuous, we have

$$\lim_{j \rightarrow \infty} \|P_C(z_n - \gamma l^j F z_n) - z_n\| = 0, \quad (5.13)$$

which implies by the uniform continuity of F on C that

$$\lim_{j \rightarrow \infty} \|F P_C(z_n - \gamma l^j F z_n) - F z_n\| = 0. \quad (5.14)$$

Thus, we have from (5.12) and (5.14) that

$$\lim_{j \rightarrow \infty} \frac{1}{\gamma l^j} \|P_C(z_n - \gamma l^j F z_n) - z_n\| = 0. \quad (5.15)$$

From the definition of h_j and (1.18) we have that

$$\langle z_n - \gamma l^j F z_n - h_j, z - h_j \rangle \leq 0, \quad \text{for all } z \in C,$$

which implies that

$$\left\langle \frac{z_n - h_j}{\gamma l^j}, z - h_j \right\rangle + \langle F z_n, h_j - z_n \rangle \leq \langle F z_n, z - z_n \rangle, \quad \text{for all } z \in C. \quad (5.16)$$

Taking limit as $j \rightarrow \infty$ of (5.16) and making use of (5.15) and (5.13), we obtain that $\langle F z_n, z - z_n \rangle \geq 0$, for all $z \in C$, which is a contradiction to our assumption that $z_n \notin VI(C, F)$. Thus, there exists a non-negative integer j that satisfies (5.6). Thus, (5.6) is well defined and hence the proof is complete. $\square$

We now show that the sequence $\{x_n\}$ is bounded.

Theorem 5.2.1. *Assume that conditions (P1) – (P8) hold. Then the sequence $\{x_n\}$ generated by Algorithm 5.1 is bounded.*

Proof. Let $p \in \Omega$. Now, we have from (1.6), (5.6) and the Cauchy–Schwarz Inequality that

$$\begin{aligned} \|s_n - p\|^2 &= \|y_n - p - \lambda_n (F y_n - F z_n)\|^2 \\ &= \|y_n - p\|^2 - 2\lambda_n \langle F y_n - F z_n, y_n - p \rangle + \lambda_n^2 \|F y_n - F z_n\|^2 \\ &= \|y_n - z_n + z_n - p\|^2 - 2\lambda_n \langle F y_n - F z_n, y_n - p \rangle + \lambda_n^2 \|F y_n - F z_n\|^2 \\ &= \|y_n - z_n\|^2 + \|z_n - p\|^2 + 2\langle y_n - z_n, z_n - p \rangle \\ &\quad - 2\lambda_n \langle F y_n - F z_n, y_n - p \rangle + \lambda_n^2 \|F y_n - F z_n\|^2 \\ &\leq \|z_n - p\|^2 - \|z_n - y_n\|^2 + 2\langle y_n - z_n, y_n - p \rangle \\ &\quad - 2\lambda_n \langle F y_n - F z_n, y_n - p \rangle + \rho^2 \|y_n - z_n\|^2 \\ &= \|z_n - p\|^2 - (1 - \rho^2) \|z_n - y_n\|^2 + 2\langle y_n - z_n, y_n - p \rangle \\ &\quad - 2\lambda_n \langle F y_n - F z_n, y_n - p \rangle. \end{aligned} \quad (5.17)$$

Since $y_n = P_C(z_n - \lambda_n F z_n)$ and $p \in C$, we have by (1.18) that $\langle z_n - \lambda_n F z_n - y_n, p - y_n \rangle \leq 0$,

that is

$$\langle y_n - z_n, y_n - p \rangle \leq -\lambda_n \langle Fz_n, y_n - p \rangle. \quad (5.18)$$

Combining (5.17) and (5.18), we obtain

$$\begin{aligned} \|s_n - p\|^2 &\leq \|z_n - p\|^2 - (1 - \rho^2)\|z_n - y_n\|^2 - 2\lambda_n \langle Fz_n, y_n - p \rangle - 2\lambda_n \langle Fy_n - Fz_n, y_n - p \rangle \\ &= \|z_n - p\|^2 - (1 - \rho^2)\|z_n - y_n\|^2 - 2\lambda_n \langle Fy_n, y_n - p \rangle. \end{aligned} \quad (5.19)$$

Since $p \in VI(C, F)$ and $y_n \in C$, we have by Lemma 5.1.1 that $\lambda_n \langle Fy_n, y_n - p \rangle \geq 0$. Thus, (5.19) becomes

$$\|s_n - p\|^2 \leq \|z_n - p\|^2 - (1 - \rho^2)\|z_n - y_n\|^2, \quad (5.20)$$

which implies by the condition on ρ that

$$\|s_n - p\| \leq \|z_n - p\|. \quad (5.21)$$

From (1.6), (5.3) and demimetric property of T , we have

$$\begin{aligned} \|z_n - p\|^2 &= \|P_C(w_n - \eta_n A^*(I_2 - T)Aw_n) - p\|^2 \\ &\leq \|w_n - p - \eta_n A^*(I_2 - T)Aw_n\|^2 \\ &= \|w_n - p\|^2 + \eta_n^2 \|A^*(I_2 - T)Aw_n\|^2 - 2\eta_n \langle A^*(I_2 - T)Aw_n, w_n - p \rangle \\ &\leq \|w_n - p\|^2 + \eta_n^2 \|A^*(I_2 - T)Aw_n\|^2 - (1 - \nu)\eta_n \|(I_2 - T)Aw_n\|^2 \\ &= \|w_n - p\|^2 + \eta_n [\eta_n \|A^*(I_2 - T)Aw_n\|^2 - (1 - \nu)\|(I_2 - T)Aw_n\|^2]. \end{aligned} \quad (5.22)$$

But, we have from (5.4) that

$$\epsilon + \eta_n \leq \frac{(1 - \nu)\|Aw_n - TAw_n\|^2}{\|A^*(Aw_n - TAw_n)\|^2},$$

which implies that

$$\eta_n [\eta_n \|A^*(I_2 - T)Aw_n\|^2 - (1 - \nu)\|(I_2 - T)Aw_n\|^2] \leq -\epsilon \eta_n \|A^*(Aw_n - TAw_n)\|^2. \quad (5.23)$$

Since $0 < \epsilon \leq \eta_n$, we have that $-\epsilon^2 \geq -\epsilon \eta_n$ and hence we obtain

$$-\epsilon^2 \|A^*(I_2 - T)Aw_n\|^2 \geq -\epsilon \eta_n \|A^*(I_2 - T)Aw_n\|^2. \quad (5.24)$$

Combining (5.22), (5.23) and (5.24), we obtain

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \epsilon^2 \|A^*(I_2 - T)Aw_n\|^2, \quad (5.25)$$

which implies that

$$\|z_n - p\| \leq \|w_n - p\|. \quad (5.26)$$

From (1.4), (5.2) and non expansiveness of the metric projection, we have

$$\begin{aligned}
 \|w_n - p\|^2 &\leq \|x_n - p + \alpha_n(x_n - x_{n-1})\|^2 \\
 &\leq \|x_n - p\|^2 + 2\alpha_n \langle x_n - x_{n-1}, x_n - p + \alpha_n(x_n - x_{n-1}) \rangle \\
 &\leq \|x_n - p\|^2 + 2\alpha_n^2 \|x_n - x_{n-1}\|^2 + 2\alpha_n \|x_n - x_{n-1}\| \|x_n - p\|.
 \end{aligned} \tag{5.27}$$

From (5.2) and non expansiveness of P_C , we obtain

$$\|w_n - p\| \leq \|x_n - p\| + \alpha_n \|x_n - x_{n-1}\|. \tag{5.28}$$

From (5.21), (5.26) and (5.28), we obtain that

$$\|s_n - p\| \leq \|x_n - p\| + \alpha_n \|x_n - x_{n-1}\|. \tag{5.29}$$

From (5.20), (5.25), (5.27) and the conditions on ϵ and ρ , we obtain

$$\|s_n - p\|^2 \leq \|x_n - p\|^2 + 2\alpha_n \|x_n - x_{n-1}\| \|x_n - p\| + 2\alpha_n^2 \|x_n - x_{n-1}\|^2. \tag{5.30}$$

From (5.8), nonexpansive property of P_C , Lemma 5.1.2 and (5.29), one obtains

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq \|(I_1 - \beta\delta_n G)s_n - p\| \\
 &= \|(I_1 - \beta\delta_n G)s_n - (I_1 - \beta\delta_n G)p - \beta\delta_n Gp\| \\
 &\leq \|(I_1 - \beta\delta_n G)s_n - (I_1 - \beta\delta_n G)p\| + \|\beta\delta_n Gp\| \\
 &\leq (1 - \tau\delta_n) \|s_n - p\| + \beta\delta_n \|Gp\| \\
 &\leq (1 - \tau\delta_n) \|x_n - p\| + \delta_n \frac{\alpha_n}{\delta_n} \|x_n - x_{n-1}\| + \delta_n M_1,
 \end{aligned} \tag{5.31}$$

where $\tau = 1 - \sqrt{1 - \beta(2\kappa - \beta L^2)}$ and $M_1 = \beta\|Gp\|$.

Since $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\delta_n} \|x_n - x_{n-1}\| = 0$, there exists $N \in \mathbb{N}$ and a constant $M_2 > 0$ such that $\frac{\alpha_n}{\delta_n} \|x_n - x_{n-1}\| < M_2$, for all $n \geq N$. Thus, we have from (5.31) that

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq (1 - \tau\delta_n) \|x_n - p\| + \tau\delta_n M \\
 &\leq \max \{ \|x_n - p\|, M \}, \text{ for all } n \geq N,
 \end{aligned}$$

where $M = \frac{M_1 + M_2}{\tau}$. Thus, we obtain by induction that

$$\|x_n - p\| \leq \max \{ K, \|x_N - p\|, M \}, \text{ for all } n \geq 1,$$

where $K = \max \{ \|x_0 - p\|, \|x_1 - p\|, \dots, \|x_{N-1} - p\| \}$. Thus, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{w_n\}, \{z_n\}, \{y_n\}$ and $\{Gs_n\}$ are bounded and hence the proof is complete. $\square$

Lemma 5.2.2. Assume that conditions (P1) – (P8) hold. Let $\{z_n\}$ and $\{y_n\}$ be as defined in Algorithm 5.1. If there are subsequences $\{z_{n_k}\}$ and $\{y_{n_k}\}$ of $\{z_n\}$ and $\{y_n\}$, respectively, such

that $z_{n_k} \rightharpoonup p^* \in C$ and $\|z_{n_k} - y_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$, then $p^* \in VI(C, F)$.

Proof. Let $e_{n_k} = P_C(z_{n_k} - \lambda_{n_k} l^{-1} F z_{n_k})$. Then we have by Lemma 3.1.2 that

$$\|z_{n_k} - e_{n_k}\| \leq \frac{1}{l} \|z_{n_k} - y_{n_k}\| \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (5.32)$$

Thus, we get that $e_{n_k} \rightharpoonup p^*$ and hence the sequence $\{e_{n_k}\}$ is bounded. Since F is uniformly continuous, we have from (5.32) that

$$\lim_{k \rightarrow \infty} \|F z_{n_k} - F e_{n_k}\| = 0. \quad (5.33)$$

We also have by (5.6) that

$$\lambda_{n_k} l^{-1} \|F e_{n_k} - F z_{n_k}\| > \rho \|z_{n_k} - e_{n_k}\|,$$

that is,

$$\frac{1}{\rho} \|F e_{n_k} - F z_{n_k}\| > \frac{1}{\lambda_{n_k} l^{-1}} \|z_{n_k} - e_{n_k}\|. \quad (5.34)$$

From (5.33) and (5.34), we obtain that

$$\lim_{k \rightarrow \infty} \frac{1}{\lambda_{n_k} l^{-1}} \|z_{n_k} - e_{n_k}\| = 0. \quad (5.35)$$

By the definition of e_{n_k} and (1.18), we have

$$\langle z_{n_k} - \lambda_{n_k} l^{-1} F z_{n_k} - e_{n_k}, u - e_{n_k} \rangle \leq 0, \text{ for all } u \in C,$$

or equivalently,

$$\frac{1}{\lambda_{n_k} l^{-1}} \langle z_{n_k} - e_{n_k}, u - e_{n_k} \rangle \leq \langle F z_{n_k}, u - z_{n_k} + z_{n_k} - e_{n_k} \rangle, \text{ for all } u \in C,$$

and thus

$$\frac{1}{\lambda_{n_k} l^{-1}} \langle z_{n_k} - e_{n_k}, u - e_{n_k} \rangle + \langle F z_{n_k}, e_{n_k} - z_{n_k} \rangle \leq \langle F z_{n_k}, u - z_{n_k} \rangle, \text{ for all } u \in C. \quad (5.36)$$

Taking limit inferior as $k \rightarrow \infty$ in (5.36) and making use of (5.32), (5.35) and the boundedness of $\{e_{n_k}\}$, we obtain

$$\liminf_{k \rightarrow \infty} \langle F z_{n_k}, u - z_{n_k} \rangle \geq 0, \text{ for all } u \in C. \quad (5.37)$$

Now, let $\{\varepsilon_k\}$ be a sequence of decreasing non negative numbers such that $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. For each ε_k , we choose N_k to be the smallest positive integer such that

$$\langle F z_{n_k}, u - z_{n_k} \rangle + \varepsilon_k \geq 0, \text{ for all } k \geq N_k, \quad (5.38)$$

where the existence of N_k follows from (5.37). Since $\{\varepsilon_k\}$ is decreasing, $\{N_k\}$ is increasing.

Now, if there exists $N \in \mathbb{N}$ such that $Fz_{N_k} = 0$, for all $k \geq N$, then $\langle Fz_{N_k}, u - z_{N_k} \rangle \geq 0$, for all $k \geq N$ and all $u \in C$.

Since F is pseudomonotone on C , we have that

$$\langle Fu, u - z_{N_k} \rangle \geq 0, \text{ for all } k \geq N \text{ and } u \in C. \quad (5.39)$$

Taking limit on both sides of (5.39) as $k \rightarrow \infty$, we obtain $\langle Fu, u - p^* \rangle \geq 0$, for all $u \in C$. By Lemma 5.1.1, we conclude that $p^* \in VI(C, F)$.

If there exists a subsequence $\{N_{k_i}\}$ of $\{N_k\}$, again denoted by $\{N_k\}$, such that $Fz_{N_k} \neq 0$ for all $k \in \mathbb{N}$, then setting $t_{N_k} = \frac{Fz_{N_k}}{\|Fz_{N_k}\|^2}$, we get $\langle Fz_{N_k}, t_{N_k} \rangle = 1$ for each k . Thus, we have from (5.38) that

$$\langle Fz_{N_k}, u + \varepsilon_k t_{N_k} - z_{N_k} \rangle \geq 0.$$

Since F is pseudomonotone, we have that

$$\langle F(u + \varepsilon_k t_{N_k}), u + \varepsilon_k t_{N_k} - z_{N_k} \rangle \geq 0. \quad (5.40)$$

Since $z_{n_k} \rightarrow p^*$ as $k \rightarrow \infty$, we deduce by the condition under (P2) that

$$0 < \|Fp^*\| \leq \liminf_{k \rightarrow \infty} \|Fz_{n_k}\|. \quad (5.41)$$

The first inequality of (5.41) can be taken without loss of generality since the case when $Fp^* = 0$ also implies that $p^* \in VI(C, F)$. Since $\{z_{N_k}\} \subset \{z_{n_k}\}$ and $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, we have that

$$0 \leq \limsup_{k \rightarrow \infty} \|\varepsilon_k t_{N_k}\| = \limsup_{k \rightarrow \infty} \left(\frac{\varepsilon_k}{\|Fz_{N_k}\|} \right) \leq \frac{\limsup_{k \rightarrow \infty} \varepsilon_k}{\liminf_{k \rightarrow \infty} \|Fz_{N_k}\|} \leq \frac{0}{\|Fp^*\|} = 0.$$

Hence, $\limsup_{k \rightarrow \infty} \|\varepsilon_k t_{N_k}\| = 0$. So, taking limit on both sides of (5.40) as $k \rightarrow \infty$, we get

$$\langle Fu, u - p^* \rangle \geq 0, \text{ for all } u \in C.$$

Therefore, by Lemma 5.1.1, we obtain that $p^* \in VI(C, F)$. □

Theorem 5.2.2. *Assume that the conditions (P1)–(P8) are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 5.1 converges strongly to the unique solution of (1.30).*

Proof. We begin the proof by showing that the mapping $P_\Omega(I_1 - \beta G)$ is a contraction mapping. Indeed, we have from Lemma 5.1.3 and the nonexpansive property of P_Ω that for any $x, y \in H_1$

$$\begin{aligned} \|P_\Omega(I_1 - \beta G)x - P_\Omega(I_1 - \beta G)y\| &\leq \|(I_1 - \beta G)x - (I_1 - \beta G)y\| \\ &\leq (1 - \omega)\|x - y\|, \end{aligned}$$

where $\omega = \frac{1}{2}\beta(2\kappa - \beta L^2) \in (0, 1)$. Thus, the mapping $P_\Omega(I_1 - \beta G)$ is a contraction mapping with constant $1 - \omega$. By the Banach Contraction Principle, $P_\Omega(I_1 - \beta G)$ has a unique fixed point, say $q \in H_1$, that is, $q = P_\Omega(I_1 - \beta G)q$. By the projection property, this unique fixed point solves the variational inequality $\langle Gx, s - x \rangle \geq 0$, for all $s \in \Omega$. We now show that $x_n \rightarrow q$ as $n \rightarrow \infty$. From (5.8), (1.5) Lemma 3.1.1, (5.20) (5.25), (5.27), we obtain

$$\begin{aligned}
 \|x_{n+1} - q\|^2 &= \|P_C((I_1 - \beta\delta_n G)s_n) - P_C q\|^2 \leq \|(I_1 - \beta\delta_n G)s_n - q\|^2 \\
 &= \|\delta_n((I_1 - \beta G)s_n - q) + (1 - \delta_n)(s_n - q)\|^2 \\
 &\leq \delta_n\|(I_1 - \beta G)s_n - q\|^2 + (1 - \delta_n)\|s_n - q\|^2 \\
 &\leq \delta_n\|(I_1 - \beta G)s_n - q\|^2 + \|z_n - q\|^2 - (1 - \delta_n)(1 - \rho^2)\|z_n - y_n\|^2 \\
 &\leq \delta_n\|(I_1 - \beta G)s_n - q\|^2 + \|w_n - q\|^2 - \epsilon^2\|A^*(I_2 - T)Aw_n\|^2 \\
 &\quad - (1 - \delta_n)(1 - \rho^2)\|z_n - y_n\|^2 \\
 &\leq \delta_n\|(I_1 - \beta G)s_n - q\|^2 + \|x_n - q\|^2 + 2\alpha_n^2\|x_n - x_{n-1}\|^2 \\
 &\quad + 2\alpha_n\|x_n - x_{n-1}\|\|x_n - q\| - \epsilon^2\|A^*(I_2 - T)Aw_n\|^2 \\
 &\quad - (1 - \delta_n)(1 - \rho^2)\|z_n - y_n\|^2.
 \end{aligned} \tag{5.42}$$

Now, denote $c_n = s_n - \beta\delta_n Gs_n$. Then we have

$$\|c_n - q\| \leq \|s_n - q\| + \beta\delta_n\|Gs_n\|. \tag{5.43}$$

From (5.8), Lemma 5.1.3, (1.4) (5.42), (5.43) and the Cauchy–Schwarz Inequality, we obtain

$$\begin{aligned}
 \|x_{n+1} - q\|^2 &\leq \|\delta_n((I_1 - \beta G)s_n - q) + (1 - \delta_n)(s_n - q)\|^2 \\
 &\leq (1 - \delta_n)^2\|s_n - q\|^2 + 2\delta_n\langle(I_1 - \beta G)s_n - q, c_n - q\rangle \\
 &= (1 - \delta_n)^2\|s_n - q\|^2 + 2\delta_n\langle(I_1 - \beta G)s_n - (I_1 - \beta G)q + (I_1 - \beta G)q - q, c_n - q\rangle \\
 &\leq (1 - \delta_n)^2\|s_n - q\|^2 + 2\delta_n\|(I_1 - \beta G)s_n - (I_1 - \beta G)q\|\|c_n - q\| \\
 &\quad + 2\delta_n\langle(I_1 - \beta G)q - q, c_n - q\rangle \\
 &\leq (1 - \delta_n)^2\|s_n - q\|^2 + 2(1 - \omega)\delta_n\|s_n - q\|\|c_n - q\| \\
 &\quad + 2\delta_n\langle(I_1 - \beta G)q - q, c_n - q\rangle \\
 &\leq (1 - \delta_n)^2\|s_n - q\|^2 + 2(1 - \omega)\delta_n\|s_n - q\|(\|s_n - q\| + \beta\delta_n\|Gs_n\|) \\
 &\quad + 2\delta_n\langle(I_1 - \beta G)q - q, c_n - q\rangle \\
 &\leq (1 - \delta_n)^2\|s_n - q\|^2 + 2(1 - \omega)\delta_n\|s_n - q\|^2 + 2\beta\delta_n^2\|s_n - q\|\|Gs_n\| \\
 &\quad + 2\delta_n\langle(I_1 - \beta G)q - q, c_n - q\rangle \\
 &\leq (1 - \omega\delta_n)\|s_n - q\|^2 + \delta_n^2\|s_n - q\|^2 + 2\beta\delta_n^2\|s_n - q\|\|Gs_n\| \\
 &\quad + 2\delta_n\langle(I_1 - \beta G)q - q, c_n - q\rangle,
 \end{aligned} \tag{5.44}$$

where $\omega = \frac{\beta}{2} (2\kappa - \beta L^2) \in (0, 1)$. From (5.30) and (5.44), one obtains that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1 - \omega\delta_n) \left(\|x_n - q\|^2 + 2\alpha_n^2 \|x_n - x_{n-1}\|^2 + 2\alpha_n \|x_n - x_{n-1}\| \|x_n - q\| \right) \\ &\quad + \delta_n^2 \|s_n - q\|^2 + 2\beta\delta_n^2 \|s_n - q\| \|Gs_n\| + 2\delta_n \langle (I_1 - \beta G)q - q, c_n - q \rangle \\ &\leq (1 - \omega\delta_n) \|x_n - q\|^2 + 2\alpha_n^2 \|x_n - x_{n-1}\|^2 + 2\alpha_n \|x_n - x_{n-1}\| \|x_n - q\| \\ &\quad + \delta_n^2 \|s_n - q\|^2 + 2\beta\delta_n^2 \|s_n - q\| \|Gs_n\| + 2\delta_n \langle (I_1 - \beta G)q - q, c_n - q \rangle. \end{aligned} \quad (5.45)$$

Moreover, we have by Cauchy–Schwarz Inequality that

$$\begin{aligned} \langle (I_1 - \beta G)q - q, c_n - q \rangle &= \langle (I_1 - \beta G)q - q, s_n - q - \beta\delta_n Gs_n \rangle \\ &= \langle (I_1 - \beta G)q - q, s_n - q \rangle - \langle (I_1 - \beta G)q - q, \beta\delta_n Gs_n \rangle \\ &\leq \langle (I_1 - \beta G)q - q, s_n - q \rangle + \beta\delta_n \|(I_1 - \beta G)q - q\| \|Gs_n\| \\ &\leq \langle (I_1 - \beta G)q - q, x_n - q \rangle + \|(I_1 - \beta G)q - q\| \|s_n - x_n\| \\ &\quad + \beta\delta_n \|(I_1 - \beta G)q - q\| \|Gs_n\|. \end{aligned} \quad (5.46)$$

Substituting (5.46) into (5.45), we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1 - \omega\delta_n) \|x_n - q\|^2 + 2\alpha_n^2 \|x_n - x_{n-1}\|^2 + 2\alpha_n \|x_n - x_{n-1}\| \|x_n - q\| \\ &\quad + \delta_n^2 \|s_n - q\|^2 + 2\beta\delta_n^2 \|s_n - q\| \|Gs_n\| \\ &\quad + 2\delta_n \left(\|(I_1 - \beta G)q - q\| \|s_n - x_n\| + \beta\delta_n \|(I_1 - \beta G)q - q\| \|Gs_n\| \right) \\ &\quad + 2\delta_n \langle (I_1 - \beta G)q - q, x_n - q \rangle. \end{aligned} \quad (5.47)$$

Now, we consider two cases on $\{\|x_n - q\|^2\}$.

Case I. Suppose that the sequence $\{\|x_n - q\|^2\}$ is decreasing. Then it follows from the Monotone Convergence Theorem that $\{\|x_n - q\|^2\}$ is convergent and hence $\|x_n - q\|^2 - \|x_{n+1} - q\|^2 \rightarrow 0$, as $n \rightarrow \infty$. From (5.42), we have

$$\begin{aligned} \epsilon^2 \|A^*(I_2 - T)Aw_n\|^2 + (1 - \delta_n)(1 - \rho^2) \|z_n - y_n\|^2 \\ \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \delta_n \|(I_1 - \beta G)s_n - q\|^2 \\ + 2\alpha_n \|x_n - x_{n-1}\| \|x_n - q\| + 2\alpha_n^2 \|x_n - x_{n-1}\|^2 \end{aligned} \quad (5.48)$$

Taking limit of both sides of (5.48) and using the fact in (5.10) and the conditions on δ_n and ρ , we obtain that

$$\lim_{n \rightarrow \infty} \|A^*(I_2 - T)Aw_n\| = 0, \quad (5.49)$$

and

$$\lim_{n \rightarrow \infty} \|y_n - z_n\| = 0. \quad (5.50)$$

From (5.3), boundedness of η_n and (5.49), we obtain

$$\|z_n - w_n\| \leq \|w_n - \eta_n A^*(I_2 - T)Aw_n - w_n\| = \eta_n \|A^*(I_2 - T)Aw_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (5.51)$$

We also have from (5.2) and (5.10) that

$$\lim_{n \rightarrow \infty} \|w_n - x_n\| \leq \lim_{n \rightarrow \infty} \alpha_n \|x_n - x_{n-1}\| = 0. \quad (5.52)$$

From (5.51) and (5.52), we obtain

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| \leq \lim_{n \rightarrow \infty} \|z_n - w_n\| + \lim_{n \rightarrow \infty} \|w_n - x_n\| = 0. \quad (5.53)$$

Moreover, we have from (5.50) and (5.53)

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| \leq \lim_{n \rightarrow \infty} \|y_n - z_n\| + \lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \quad (5.54)$$

Since the sequence $\{x_n\}$ is a bounded sequence in C , we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a point $\bar{x} \in H_1$ and

$$\limsup_{n \rightarrow \infty} \langle (I_1 - \beta G)q - q, x_n - q \rangle = \lim_{k \rightarrow \infty} \langle (I_1 - \beta G)q - q, x_{n_k} - q \rangle. \quad (5.55)$$

Since C is weakly closed, we have that $\bar{x} \in C$. We now claim that $\bar{x} \in \Omega$. We have from (5.54) that $y_{n_k} \rightharpoonup \bar{x}$, as $k \rightarrow \infty$. Thus, we conclude from (5.50) and Lemma 5.2.2 that $\bar{x} \in VI(C, F)$. It remains to show that $A\bar{x} \in F(T)$.

Indeed, we have from the ν -demimetricity of T and (5.49) that

$$\begin{aligned} \frac{(1 - \nu)}{2} \|Aw_{n_k} - TAw_{n_k}\|^2 &\leq \langle Aw_{n_k} - TAw_{n_k}, Aw_{n_k} - Aq \rangle \\ &\leq \|A^*(I_2 - T)Aw_{n_k}\| \|w_{n_k} - q\| \rightarrow 0, \text{ as } k \rightarrow \infty, \end{aligned}$$

and thus, we have

$$\lim_{k \rightarrow \infty} \|(I_2 - T)Aw_{n_k}\| = 0. \quad (5.56)$$

From (5.52), we have that $w_{n_k} \rightharpoonup \bar{x}$, and thus $Aw_{n_k} \rightharpoonup A\bar{x}$. Thus, it follows from the demiclosedness of $I_2 - T$ that $(I_2 - T)A\bar{x} = 0$ and hence $A\bar{x} = TA\bar{x}$. Thus, we conclude that $\bar{x} \in \Omega$.

From the fact $q = P_\Omega(I_1 - \beta G)q$, we have by (1.18) that

$$\langle (I_1 - \beta G)q - q, \bar{x} - q \rangle \leq 0. \quad (5.57)$$

Thus, we obtain from (5.55) and (5.57) that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (I_1 - \beta G)q - q, x_n - q \rangle &= \lim_{k \rightarrow \infty} \langle (I_1 - \beta G)q - q, x_{n_k} - q \rangle \\ &= \langle (I_1 - \beta G)q - q, \bar{x} - q \rangle \leq 0. \end{aligned} \quad (5.58)$$

From (5.7) and (5.50), we obtain

$$\|s_n - y_n\| = \|y_n - \lambda_n(Fy_n - Fz_n) - y_n\| \leq \rho \|y_n - z_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (5.59)$$

From (5.50), (5.51), (5.52) and (5.59), we obtain

$$\begin{aligned} \|s_n - x_n\| &= \|s_n - y_n + y_n - z_n + z_n - w_n + w_n - x_n\| \\ &\leq \|s_n - y_n\| + \|y_n - z_n\| + \|w_n - x_n\| + \|z_n - w_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (5.60)$$

Now, we have from (5.47) that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1 - \omega\delta_n)\|x_n - q\|^2 + 2\alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - x_{n-1}\|\|x_n - q\| \\ &\quad + \delta_n^2\|s_n - q\|^2 + 2\beta\delta_n^2\|s_n - q\|\|Gs_n\| \\ &\quad + 2\delta_n(\|(I_1 - \beta G)q - q\|\|s_n - x_n\| + \beta\delta_n\|(I_1 - \beta G)q - q\|\|Gs_n\|) \\ &\quad + 2\delta_n\langle(I_1 - \beta G)q - q, x_n - q\rangle \\ &= (1 - \omega\delta_n)\|x_n - q\|^2 + \omega\delta_n K_n, \end{aligned} \quad (5.61)$$

where

$$\begin{aligned} K_n &= \frac{2\delta_n}{\omega} \frac{\alpha_n^2}{\delta_n^2} \|x_n - x_{n-1}\|^2 + \frac{2}{\omega} \frac{\alpha_n}{\delta_n} \|x_n - x_{n-1}\|\|x_n - q\| \\ &\quad + \frac{\delta_n}{\omega} \|s_n - q\|^2 + \frac{2\beta}{\omega} \delta_n \|s_n - q\|\|Gs_n\| \\ &\quad + \frac{2}{\omega} (\|(I_1 - \beta G)q - q\|\|s_n - x_n\| + \beta\delta_n\|(I_1 - \beta G)q - q\|\|Gs_n\|) \\ &\quad + \frac{2}{\omega} \langle(I_1 - \beta G)q - q, x_n - q\rangle. \end{aligned} \quad (5.62)$$

From (5.9), (5.58), (5.60), (5.62) and the condition on δ_n we obtain that

$$\limsup_{n \rightarrow \infty} K_n \leq 0. \quad (5.63)$$

From (5.61), (5.63) and Lemma 3.1.6, we obtain that $\|x_n - q\| \rightarrow 0$, as $n \rightarrow \infty$ and hence $x_n \rightarrow q$, as $n \rightarrow \infty$.

Case II. Suppose now that there is a subsequence $\{\|x_{n_i} - q\|^2\}$ such that $\|x_{n_i} - q\|^2 < \|x_{n_i+1} - q\|^2$, for all $i \geq 0$. Then by Lemma 3.1.7, there exists an increasing sequence $\{m_k\}$ of positive integers such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\|x_{m_k} - q\| \leq \|x_{m_k+1} - q\| \text{ and } \|x_k - q\| \leq \|x_{m_k+1} - q\|, \quad (5.64)$$

for all $k \in \mathbb{N}$. From (5.42), we have

$$\begin{aligned} &\epsilon^2 \|A^*(I_2 - T)Aw_{m_k}\|^2 + (1 - \delta_{m_k})(1 - \rho^2)\|z_{m_k} - y_{m_k}\|^2 \\ &\leq \|x_{m_k} - q\|^2 - \|x_{m_k+1} - q\|^2 + \delta_{m_k}\|(I_1 - \beta G)s_{m_k} - q\|^2 \\ &\quad + 2\alpha_{m_k}^2\|x_{m_k} - x_{m_k-1}\|^2 + 2\alpha_{m_k}\|x_{m_k} - x_{m_k-1}\|\|x_{m_k} - q\|. \end{aligned} \quad (5.65)$$

Using the relations (5.10), (5.63) and the condition on δ_{m_k} , taking limit of both sides of (5.65)

as $k \rightarrow \infty$ and following similar argument as in Case I, we obtain

$$\lim_{k \rightarrow \infty} \|y_{m_k} - z_{m_k}\| = 0, \text{ and } \lim_{k \rightarrow \infty} \|A^*(I_2 - T)Aw_{m_k}\| = 0.$$

Furthermore, following similar steps as in Case I, we obtain

$$\lim_{k \rightarrow \infty} \|s_{m_k} - x_{m_k}\| = 0, \text{ and} \quad (5.66)$$

$$\limsup_{k \rightarrow \infty} \langle (I_1 - \beta G)q - q, x_{m_k} - q \rangle \leq 0. \quad (5.67)$$

We also obtain from (5.47) and (5.64) that

$$\begin{aligned} \|x_{m_k+1} - q\|^2 &\leq (1 - \omega\delta_{m_k})\|x_{m_k+1} - q\|^2 + 2\alpha_{m_k}^2\|x_{m_k} - x_{m_k-1}\|^2 + 2\alpha_{m_k}\|x_{m_k} - x_{m_k-1}\|\|x_{m_k} - q\| \\ &\quad + \delta_{m_k}^2\|s_{m_k} - q\|^2 + 2\beta\delta_{m_k}^2\|s_{m_k} - q\|\|Gs_{m_k}\| \\ &\quad + 2\delta_{m_k}(\|(I_1 - \beta G)q - q\|\|s_{m_k} - x_{m_k}\| + \beta\delta_{m_k}\|(I_1 - \beta G)q - q\|\|Gs_{m_k}\|) \\ &\quad + 2\delta_{m_k}\langle (I_1 - \beta G)q - q, x_{m_k} - q \rangle, \end{aligned}$$

which, up on rearrangement, gives

$$\begin{aligned} \|x_{m_k+1} - q\|^2 &\leq \frac{2}{\omega}\delta_{m_k}\frac{\alpha_{m_k}^2}{\delta_{m_k}^2}\|x_{m_k} - x_{m_k-1}\|^2 + \frac{2}{\omega}\frac{\alpha_{m_k}}{\delta_{m_k}}\|x_{m_k} - x_{m_k-1}\|\|x_{m_k} - q\| \\ &\quad + \frac{1}{\omega}\delta_{m_k}\|s_{m_k} - q\|^2 + \frac{2\beta}{\omega}\delta_{m_k}\|s_{m_k} - q\|\|Gs_{m_k}\| \\ &\quad + \frac{2}{\omega}(\|(I_1 - \beta G)q - q\|\|s_{m_k} - x_{m_k}\| + \beta\delta_{m_k}\|(I_1 - \beta G)q - q\|\|Gs_{m_k}\|) \\ &\quad + \frac{2}{\omega}\langle (I_1 - \beta G)q - q, x_{m_k} - q \rangle. \end{aligned} \quad (5.68)$$

Taking limit on both sides of (5.68) and making use of (5.66), (5.67) and (5.9), one obtains that $\lim_{k \rightarrow \infty} \|x_{m_k+1} - q\| = 0$, which implies by (5.64) that $\lim_{k \rightarrow \infty} \|x_k - q\| = 0$ and hence the proof is complete. $\square$

If, in Algorithm 5.1, the mapping F is a Lipschitz continuous, pseudomonotone and $\|Fx^*\| \leq \liminf_{n \rightarrow \infty} \|Fx_n\|$ whenever $\{x_n\} \subset C$ with $x_n \rightharpoonup x^*$, then we get the following corollary:

Corollary 5.2.1. *Suppose that conditions (P1), (P3)–(P8) are satisfied and let F be a Lipschitz continuous, pseudomonotone mapping with $\|Fx^*\| \leq \liminf_{n \rightarrow \infty} \|Fx_n\|$ whenever $x_n \rightharpoonup x^*$. Then the sequence $\{x_n\}$ converges strongly to the unique solution of (1.30).*

Proof. As every Lipschitz continuous mapping is uniformly continuous, the result follows from Theorem 5.2.2. $\square$

Corollary 5.2.2. *Suppose that conditions (P1), (P3)–(P8) are satisfied and let $F: H_1 \rightarrow H_1$ be a uniformly continuous, pseudomonotone and weakly sequentially continuous mapping. Then the sequence $\{x_n\}$ generated by Algorithm 5.1 converges strongly to the unique solution of (1.30).*

If the mapping F in (P2) is monotone, then the sequential weakly continuity requirement is dispensed and we get the following corollary:

Corollary 5.2.3. *Suppose that conditions (P1), (P3) – (P8) are satisfied and let $F: H_1 \rightarrow H_1$ be a uniformly continuous monotone mapping. Then the sequence $\{x_n\}$ generated by Algorithm 5.1 converges strongly to the unique solution of (1.30).*

If the mapping T in (P5) is a demicontractive mapping on H_2 with $I_2 - T$ is demiclosed at 0, then we have the following corollary:

Corollary 5.2.4. *Suppose the conditions (P1) – (P4) and (P6) – (P8) hold. Then the sequence $\{x_n\}$ generated by Algorithm 5.1, with T a demicontractive mapping on H_2 and $I_2 - T$ is demiclosed at 0, converges strongly to the unique solution of (1.30).*

Proof. As every demicontractive mapping is demimetric, the result follows from Theorem 5.2.2. $\square$

5.2.2 Some Theoretical Applications

Bilevel Variational Inequality Problems

If the mapping T in (P5) is the identity mapping on H_2 , then (1.30) reduces to the Bilevel Variational Inequality Problems (BVIP) of finding

$$x^* \in VI(C, F) \text{ such that } \langle Gx^*, x - x^* \rangle \geq 0, \text{ for all } x \in VI(C, F).$$

Thus, we obtain the following corollary:

Corollary 5.2.5. *Suppose the conditions (P1) – (P4) and (P6) – (P8) hold. Then the sequence $\{x_n\}$ generated by Algorithm 5.1, with $T = I_2$, converges strongly to the unique solution of (1.29).*

5.2.3 Numerical Examples

Example 5.2.1. *Let us consider the Hilbert space $H = H_1 = H_2 = L_2^{\mathbb{R}}([0, 1])$, with inner product $\langle x(t), y(t) \rangle = \int_0^1 x(t)y(t)dt$ and norm $\|x(t)\|_{L_2} = \left(\int_0^1 |x(t)|^2 dt \right)^{\frac{1}{2}}$.*

Let r and R be positive real numbers such that $\frac{R}{k+1} < \frac{r}{k} < r < R$, for some $k > 1$. Let $C = \{x(t) \in H : \|x(t)\| \leq r\}$.

Define the mapping $F: H \rightarrow H$ by $(Fx)(t) = (R - \|x\|)x(t)$, where $t \in [0, 1]$. Let $G: H \rightarrow H$ be defined by $(Gx)(t) = \frac{5}{3}x(t)$. It can be shown that the mapping F is pseudomonotone and uniformly continuous (see for reference, [154]) and G is $\frac{5}{3}$ -strongly monotone and $\frac{5}{3}$ -Lipschitz continuous.

If we specifically take $r = 2$ and $R = 3$, then we have $(Fx)(t) = (3 - \|x\|)x(t)$. Taking $x = 1.250$ and $y = 1.875$, we obtain that $\langle Fx - Fy, x - y \rangle < 0$. Thus, F is not monotone. It can be easily

shown that $x^*(t) = 0 \in VI(C, F)$.

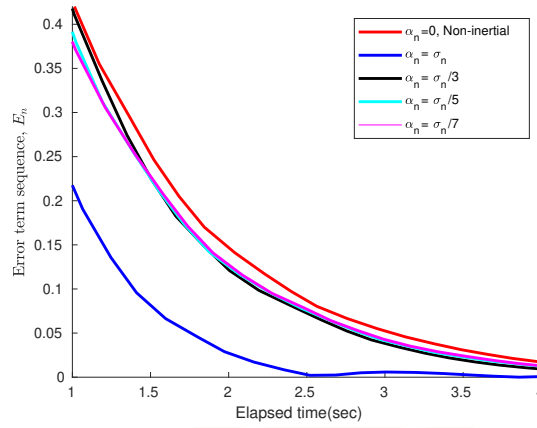
We now define the mapping $A: H \rightarrow H$ by

$$(Ax)(t) = \frac{2}{3}x(t).$$

Clearly, A is a bounded linear mapping with adjoint $(A^*x)(t) = \frac{2}{3}x(t)$. Let $T: H \rightarrow H$ be defined by $(Tx)(t) = \frac{3}{4}x(t)$. It can be easily shown that T is a demimetric mapping.

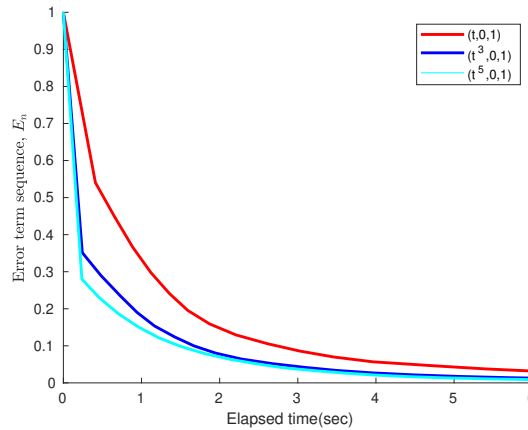
Moreover, $(TAx^*)(t) = (Ax^*)(t) = 0$. Thus, $Ax^*(t) \in F(T)$ and hence $x^*(t) = 0 \in \Omega$. Let $\zeta_n = \frac{1}{n^2 + 1}$, $\delta_n = \frac{1}{n + 100}$, $n \geq 1$, $\beta = 1$, $\gamma = 0.4$, $l = 0.1$, $\eta = \frac{1}{1000}$, $\alpha = 0.8$, $\rho = 0.3$, $x_0(t) = 0$ and $x_1(t) = 1$.

Then conditions (P1) – (P8) are satisfied. We have conducted the numerical experiments to demonstrate that the error term sequence $E_n = \{x_n - x^*\}$, $n \geq 1$, converges strongly to zero for different values of the inertial parameter α_n and different choices of initial points.



$$\alpha = 0.8, \quad l = 0.1, \quad \gamma = 0.4, \quad \rho = 0.3, \quad \beta = 1, \quad x_0(t) = 0, \quad x_1(t) = 1.$$

Figure 5.1: Convergence rate for different values of the inertial parameter α_n .



$$\alpha = 0.8, \quad l = 0.1, \quad \gamma = 0.4, \quad \rho = 0.3, \quad \beta = 1, \quad \alpha_n = \frac{\sigma_n}{7}, \quad x_0(t) = 0.$$

Figure 5.2: Convergence rate for different choices of the initial point $x_1(t)$.

Remark 5.2.3. One can observe from Figure 5.1 that the inertial version ($\alpha_n \neq 0$) of the algorithm has faster rate of convergence than that of the non-inertial version ($\alpha_n = 0$) and Figure 5.2 reveals that the rate of convergence gets faster as the degree of t in the initial point increases.

Example 5.2.2. Let $H = H_1 = H_2 = \mathbb{R}^2$ with norm $\|x\|_{\mathbb{R}^2} = \left(\sum_{i=1}^2 |x_i|^2\right)^{\frac{1}{2}}$ and inner product

$$\langle x, y \rangle = \sum_{i=1}^2 x_i y_i, \text{ for all } x = (x_1, x_2) \text{ and } y = (y_1, y_2) \in H.$$

Let $C = \{x = (x_1, x_2) \in H : |x_i| \leq 1, i = 1, 2\}$. Define the mapping $F: H \rightarrow H$ by $F(x) = \left(\|x\| + \frac{1}{\|x\| + b}\right)x$, for some $b > 0$. Now, we show that F is pseudomonotone and uniformly continuous on C . Indeed, let $x, y \in C$ be such that $\langle Fx, y - x \rangle \geq 0$. This implies that $\langle x, y - x \rangle \geq 0$. Now, we have

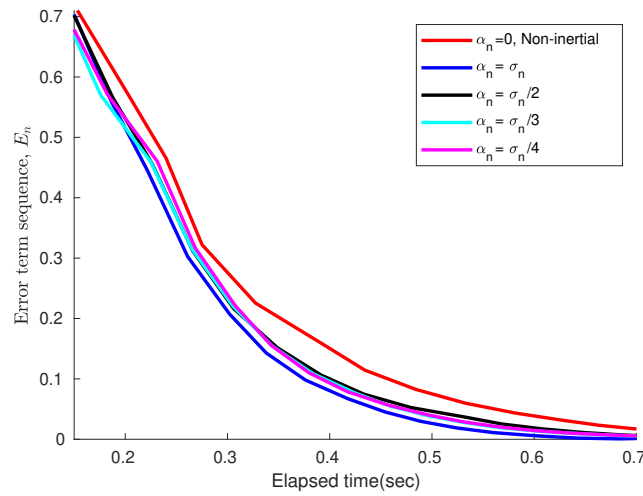
$$\begin{aligned} \langle Fy, y - x \rangle &= \left(\|y\| + \frac{1}{\|y\| + b}\right) \langle y, y - x \rangle \\ &\geq \left(\|y\| + \frac{1}{\|y\| + b}\right) \langle y, y - x \rangle - \langle x, y - x \rangle \\ &= \left(\|y\| + \frac{1}{\|y\| + b}\right) \|y - x\|^2 \geq 0, \end{aligned}$$

which shows that F is pseudomonotone on C . Since C is compact, the mapping F is uniformly continuous and sequentially weakly continuous on C (see, [157]) and hence condition (P2) is satisfied.

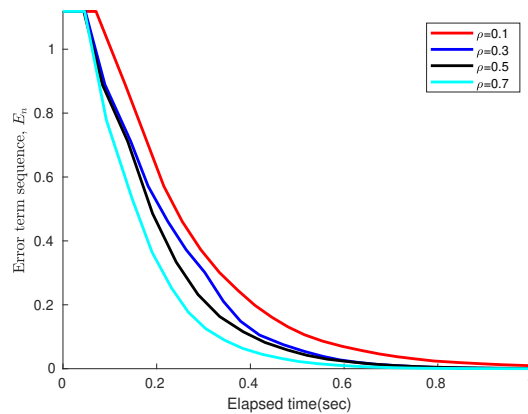
Let $G: H \rightarrow H$ be defined by $G(x) = 3x$. Then G is 3-strongly monotone and 3-Lipschitz continuous. Let $T: H \rightarrow H$ be defined by $T(x) = \frac{1}{3}x$ and $A: H \rightarrow H$ by $A(x) = 5x$. Clearly, A is a bounded linear mapping with adjoint $A^*(x) = 5x$. If we take $x^* = 0$, then we have for $\nu = 0.5$ that

$$\begin{aligned} \langle x - T(x), x - x^* \rangle &= \left\langle x - \frac{1}{3}x, x - 0 \right\rangle = \frac{2}{3} \langle x, x \rangle \\ &= \frac{2}{3} \sum_{i=1}^2 |x_i|^2 \geq \frac{1}{9} \sum_{i=1}^2 |x_i|^2 = \frac{1}{4} \left\| \frac{2}{3}x \right\|^2 = \frac{1}{4} \left\| x - \frac{1}{3}x \right\|^2 = \frac{(1-\nu)}{2} \|x - T(x)\|^2. \end{aligned}$$

Thus, T is 0.5-demimetric. Moreover, $x^* \in VI(C, F)$ and $TA(x^*) = A(x^*) = 0$. Thus, $x^* = 0 \in \Omega$. Let $\zeta_n = \frac{1}{n^2 + 100}$, $\delta_n = \frac{1}{n + 100000}$, $n \geq 1$, $b = 3$, $\beta = \frac{1}{6}$, $\alpha = 0.8$, $\gamma = 0.4$, $l = 0.1$, $\eta = \frac{1}{10000000}$, $\rho = 0.4$ and the initial points $x_0 = (0, 0)$ and $x_1 = (1, 1)$. Then conditions (P1) – (P8) are satisfied and the numerical experiments reveal that the error term sequence $E_n = \{x_n - x^*\}$, for each $n \geq 1$, converges strongly to zero.



$$\alpha = 0.8, \quad l = 0.1, \quad \gamma = 0.4, \quad \rho = 0.4, \quad \beta = \frac{1}{6}, \quad x_0 = (0, 0), \quad x_1 = (1, 1).$$

 Figure 5.3: Convergence rate for different values of the inertial parameter α_n .


$$\alpha = 0.8, \quad l = 0.1, \quad \gamma = 0.4, \quad \beta = \frac{1}{6}, \quad \alpha_n = \frac{\sigma_n}{4}, \quad x_0 = (0, 0), \quad x_1 = (1, 1).$$

 Figure 5.4: Convergence of the sequence $\{x_n\}$ for different choices of the parameter ρ .

Remark 5.2.4. Figure 5.3 shows the comparison between inertial and non-inertial version of the Algorithm 5.1. It reveals that the rate of convergence of the algorithm is relatively faster for the inertial version ($\alpha_n \neq 0$) than that of the non-inertial ($\alpha_n = 0$) version. From Figure 5.4, we observe that the rate of convergence gets faster as the parameter ρ increases and approaches 1.

5.3 Approximation Methods for Solving Split Equality Variational Inequality and f, g -Fixed Point Problems in Reflexive Real Banach Spaces

This section is devoted to approximating solutions of split equality of variational inequality and f, g -fixed point problems. First, we construct an iterative algorithm involving uni-

formly continuous pseudomonotone mappings in the variational inequality problem and Bregman relatively f, g -nonexpansive mappings in the f, g -fixed point problem part. Then we prove strong convergence of the algorithm introduced and provide a numerical example to illustrate the effectiveness of our method.

5.3.1 Main Results

The following conditions will be used in establishing the convergence results of the method.

- (A1) Let C and D be nonempty, closed and convex subsets of the smooth, strictly convex and reflexive real Banach spaces E_1 and E_2 , respectively.
- (A2) Let $f: E_1 \rightarrow \mathbb{R}$ and $g: E_2 \rightarrow \mathbb{R}$ be proper, lower semi-continuous, strongly coercive, uniformly Fréchet differentiable, strongly convex Legendre functions which are bounded on bounded subsets of E_1 and E_2 , respectively. Let f and g have Lipschitz continuous gradients with the strong convexity constant of f (respectively, g) greater than or equal to the Lipschitz constant of ∇f (respectively, ∇g).
- (B1) Let $A: C \rightarrow E_1^*$ and $B: D \rightarrow E_2^*$ be uniformly continuous, pseudomonotone and sequentially weakly continuous mappings;
- (B2) Let $G: E_1 \rightarrow E_1^*$ and $K: E_2 \rightarrow E_2^*$ be Bregman relatively f -nonexpansive and Bregman relatively g -nonexpansive mappings, respectively;
- (B3) Let $T: E_1 \rightarrow E_3$ and $S: E_2 \rightarrow E_3$ be bounded linear mappings with adjoints $T^*: E_3^* \rightarrow E_1^*$ and $S^*: E_3^* \rightarrow E_2^*$, respectively, where E_3 is another smooth, strictly convex real reflexive Banach space;
- (B4) Let $\Upsilon = \{(x, u) \in (VI(C, A) \cap F_f(G)) \times (VI(D, B) \cap F_g(K)) : Tx = Su\} \neq \emptyset$.
- (R1) Let $\beta = \min \{\beta_1, \beta_2\}$, where β_1 and β_2 are the strong convexity constants of f and g , respectively;
- (R2) Let $\{\alpha_n\} \subseteq (0, 1)$ be such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Algorithm 5.2.

Initialization: Choose $(x_1, u_1) \in E_1 \times E_2$, $\mu \in (0, \beta)$, $l, \gamma, \tau \in (0, 1)$. For $x \in C, u \in D$, compute the sequences $\{x_n\}$ and $\{u_n\}$ as follows:

Step 1: Given the current iterates x_n and u_n , compute

$$\begin{aligned} z_n &= P_C^f [\nabla f^* (\nabla f x_n - \gamma_n T^* J_{E_3} (Tx_n - Su_n))], \\ w_n &= P_D^g [\nabla g^* (\nabla g u_n - \gamma_n S^* J_{E_3} (Su_n - Tx_n))], \end{aligned} \quad (5.69)$$

where $0 < \rho \leq \gamma_n \leq \rho_n$ with

$$\rho_n = \min \left\{ \rho + 1, \frac{\beta \|Tx_n - Su_n\|^2}{2[\|T^*J_{E_3}(Tx_n - Su_n)\|^2 + \|S^*J_{E_3}(Su_n - Tx_n)\|^2]} \right\}$$

for $n \in \Omega$, otherwise $\gamma_n = \rho > 0$, where $\Omega = \{m \in \mathbb{N} : Tx_m - Su_m \neq 0\}$ and J_{E_3} is the normalized duality mapping on E_3 .

Step 2: Compute

$$y_n = P_C^f(\nabla f^*(\nabla f z_n - \lambda_n A z_n)), \quad (5.70)$$

$$v_n = P_D^g(\nabla g^*(\nabla g w_n - \eta_n B w_n)),$$

where $\lambda_n = \gamma l^{j_n}$, where j_n is the smallest non-negative integer j satisfying

$$\gamma l^j \|A(P_C^f(\nabla f^*(\nabla f z_n - \gamma l^j A z_n))) - A z_n\| \leq \mu \|P_C^f(\nabla f^*(\nabla f z_n - \gamma l^j A z_n)) - z_n\|, \quad (5.71)$$

and $\eta_n = \gamma l^{k_n}$, where k_n is the smallest non-negative integer k satisfying

$$\gamma l^k \|B(P_D^g(\nabla g^*(\nabla g w_n - \gamma l^k B w_n))) - B w_n\| \leq \mu \|P_D^g(\nabla g^*(\nabla g w_n - \gamma l^k B w_n)) - w_n\|. \quad (5.72)$$

Step 3: Compute

$$a_n = \nabla f^*(\nabla f y_n - \lambda_n (A y_n - A z_n)), \quad (5.73)$$

$$b_n = \nabla g^*(\nabla g v_n - \eta_n (B v_n - B w_n)).$$

Step 4: Compute

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f x + (1 - \alpha_n) [\tau \nabla f a_n + (1 - \tau) G(a_n)]), \quad (5.74)$$

$$u_{n+1} = \nabla g^*(\alpha_n \nabla g u + (1 - \alpha_n) [\tau \nabla g b_n + (1 - \tau) K(b_n)]).$$

Set $n := n + 1$ and go to **Step 1**.

Remark 5.3.1. Some of our contributions to the results in the literature are:

- The proposed scheme extends the existing SEVIP to the more general split equality of VIP and f, g -fixed point problem.
- It extends the all the results in the literature from Hilbert spaces and uniformly convex (smooth) spaces to the more general reflexive Banach spaces.

We first prove some results that are fundamental to the convergence analysis of the sequences generated by Algorithm 5.2. We begin by proving that the proposed algorithm is well-defined.

Lemma 5.3.1. Assume that conditions (A1) – (A2), (B1) – (B4) and (R1) – (R2) hold. Then the Armijo line-search rules (5.71) and (5.72) are well-defined.

Proof. Denote $k_j = P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j A z_n)$. If $z_n \in VI(C, A)$, then $z_n = P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j A z_n)$. In this case, we have $z_n = k_j$ and hence (5.71) holds for $j = 0$. Let us now consider

the case when $z_n \notin VI(C, A)$. Assume on the contrary that for all $j \geq 0$ we have

$$\gamma l^j \|A(P_C^f(\nabla f^*(\nabla f(z_n) - \gamma l^j Az_n)) - Az_n)\| > \mu \|P_C^f(\nabla f^*(\nabla f(z_n) - \gamma l^j Az_n)) - z_n\|.$$

That is,

$$\|AP_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n) - Az_n\| > \frac{\mu}{\gamma l^j} \|P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n) - z_n\|. \quad (5.75)$$

Since P_C^f and ∇f^* are continuous, we have that

$$\lim_{j \rightarrow \infty} \|z_n - P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n)\| = 0. \quad (5.76)$$

By the uniform continuity of the mapping A on bounded subsets of C , we obtain

$$\lim_{j \rightarrow \infty} \|AP_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n) - Az_n\| = 0. \quad (5.77)$$

Thus, we obtain from (5.75) and (5.77) that

$$\lim_{j \rightarrow \infty} \frac{\|z_n - P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n)\|}{\gamma l^j} = 0. \quad (5.78)$$

Since ∇f is Lipschitz continuous, there exists a real number $R > 0$ such that

$$\begin{aligned} 0 &\leq \lim_{j \rightarrow \infty} \frac{\|\nabla f(z_n) - \nabla f(P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n))\|}{\gamma l^j} \\ &\leq R \lim_{j \rightarrow \infty} \frac{\|z_n - P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n)\|}{\gamma l^j}, \end{aligned}$$

which implies by (5.78) that

$$\lim_{j \rightarrow \infty} \frac{\|\nabla f(z_n) - \nabla f(P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n))\|}{\gamma l^j} = 0. \quad (5.79)$$

From the definition of k_j and (1.18), we get

$$\langle \nabla f(k_j) - \nabla f(z_n) + \gamma l^j Az_n, y - k_j \rangle \geq 0, \text{ for all } y \in C,$$

which implies that

$$\left\langle \frac{\nabla f(k_j) - \nabla f(z_n)}{\gamma l^j}, y - k_j \right\rangle + \langle Az_n, z_n - k_j \rangle + \langle Az_n, y - z_n \rangle \geq 0, \text{ for all } y \in C. \quad (5.80)$$

Taking limit as $j \rightarrow \infty$ in (5.80) and using (5.76) and (5.79), we obtain

$$\langle Az_n, y - z_n \rangle \geq 0, \text{ for all } y \in C,$$

and this implies that $z_n \in VI(C, A)$, which is a contradiction. Thus, there exists a non-negative integer j for which (5.71) holds and hence it is well-defined. Similarly, one can show that there

exists a non-negative integer k for which (5.72) holds true and hence the proof is complete. $\square$

Theorem 5.3.1. *Assume that conditions (A1)–(A2), (B1)–(B4) and (R1)–(R2) hold. Then the sequences $\{x_n\}$ and $\{u_n\}$ generated by Algorithm 5.2 are bounded.*

Proof. Denote $q_n = \nabla f^*(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n))$ and $t_n = \nabla g^*(\nabla g(u_n) - \gamma_n S^* J_{E_3}(Su_n - Tx_n))$. Let $(\bar{x}, \bar{u}) \in \Upsilon$. Then by (5.74), Lemma 3.1.9 and Bregman relatively f -nonexpansiveness of G , we have

$$\begin{aligned}
 D_f(\bar{x}, x_{n+1}) &= D_f(\bar{x}, \nabla f^*(\alpha_n \nabla f(x) + (1 - \alpha_n)[\tau \nabla f(a_n) + (1 - \tau)G(a_n)])) \\
 &\leq \alpha_n D_f(\bar{x}, x) + (1 - \alpha_n) D_f(\bar{x}, \nabla f^*(\tau \nabla f(a_n) + (1 - \tau)G(a_n))) \\
 &\leq \alpha_n D_f(\bar{x}, x) + (1 - \alpha_n) \tau D_f(\bar{x}, a_n) \\
 &\quad + (1 - \alpha_n)(1 - \tau) D_f(\bar{x}, \nabla f^*(G(a_n))) \\
 &\leq \alpha_n D_f(\bar{x}, x) + (1 - \alpha_n) \tau D_f(\bar{x}, a_n) + (1 - \alpha_n)(1 - \tau) D_f(\bar{x}, a_n) \\
 &= \alpha_n D_f(\bar{x}, x) + (1 - \alpha_n) D_f(\bar{x}, a_n).
 \end{aligned} \tag{5.81}$$

From (5.73) and Definition 1.2.17, we have

$$\begin{aligned}
 D_f(\bar{x}, a_n) &= D_f(\bar{x}, \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Az_n))) \\
 &= f(\bar{x}) - \langle \nabla f(y_n) - \lambda_n(Ay_n - Az_n), \bar{x} - a_n \rangle - f(a_n) \\
 &= f(\bar{x}) + \langle \nabla f(y_n), a_n - \bar{x} \rangle + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n \rangle - f(a_n) \\
 &= f(\bar{x}) - \langle \nabla f(y_n), \bar{x} - y_n \rangle - f(y_n) + \langle \nabla f(y_n), \bar{x} - y_n \rangle + f(y_n) \\
 &\quad + \langle \nabla f(y_n), a_n - \bar{x} \rangle + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n \rangle - f(a_n) \\
 &= D_f(\bar{x}, y_n) + \langle \nabla f(y_n), a_n - y_n \rangle \\
 &\quad + f(y_n) - f(a_n) + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n \rangle \\
 &= D_f(\bar{x}, y_n) - D_f(a_n, y_n) + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n \rangle.
 \end{aligned} \tag{5.82}$$

Using (1.16), we get

$$D_f(\bar{x}, y_n) - D_f(a_n, y_n) = D_f(\bar{x}, z_n) - D_f(a_n, z_n) + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - a_n \rangle. \tag{5.83}$$

Substituting (5.83) into (5.82), we obtain

$$\begin{aligned}
 D_f(\bar{x}, a_n) &= D_f(\bar{x}, z_n) - D_f(a_n, z_n) + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - a_n \rangle \\
 &\quad + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n \rangle.
 \end{aligned} \tag{5.84}$$

Furthermore, from (1.15), we obtain

$$D_f(a_n, z_n) = D_f(a_n, y_n) + D_f(y_n, z_n) - \langle \nabla f(y_n) - \nabla f(z_n), y_n - a_n \rangle. \tag{5.85}$$

Therefore, from (5.84) and (5.85) we obtain

$$\begin{aligned}
 D_f(\bar{x}, a_n) &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - a_n \rangle \\
 &\quad + \langle \nabla f(y_n) - \nabla f(z_n), y_n - a_n \rangle + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n \rangle \\
 &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - y_n \rangle \\
 &\quad + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n \rangle \\
 &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\
 &\quad + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - y_n \rangle + \langle \lambda_n(Ay_n - Az_n), \bar{x} - y_n + y_n - a_n \rangle \\
 &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - y_n \rangle \\
 &\quad + \langle \lambda_n(Ay_n - Az_n), \bar{x} - y_n \rangle + \langle \lambda_n(Ay_n - Az_n), y_n - a_n \rangle \\
 &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \langle \lambda_n(Ay_n - Az_n) \\
 &\quad + \nabla f(z_n) - \nabla f(y_n), \bar{x} - y_n \rangle + \langle \lambda_n(Ay_n - Az_n), y_n - a_n \rangle \\
 &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - \langle \lambda_n(Ay_n - Az_n) - \\
 &\quad (\nabla f(y_n) - \nabla f(z_n)), y_n - \bar{x} \rangle + \langle \lambda_n(Ay_n - Az_n), y_n - a_n \rangle.
 \end{aligned} \tag{5.86}$$

Since $y_n = P_C^f[\nabla f^*(\nabla f(z_n) - \lambda_n Az_n)]$, by (1.18) we get

$$\langle \nabla f(z_n) - \lambda_n Az_n - \nabla f(y_n), y_n - \bar{x} \rangle \geq 0. \tag{5.87}$$

Since $y_n \in C$ and $\bar{x} \in VI(C, A)$, we obtain that $\langle A\bar{x}, y_n - \bar{x} \rangle \geq 0$. Moreover, the pseudomonotonicity of A implies that

$\langle Ay_n, y_n - \bar{x} \rangle \geq 0$, and thus

$$\langle \lambda_n Ay_n, y_n - \bar{x} \rangle \geq 0. \tag{5.88}$$

Combining (5.87) and (5.88), we get

$$\langle \lambda_n(Ay_n - Az_n) - (\nabla f(y_n) - \nabla f(z_n)), y_n - \bar{x} \rangle \geq 0. \tag{5.89}$$

Thus, from (5.86) and (5.89), we obtain that

$$\begin{aligned}
 D_f(\bar{x}, a_n) &\leq D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\
 &\quad + \langle \lambda_n(Ay_n - Az_n), y_n - a_n \rangle.
 \end{aligned} \tag{5.90}$$

Furthermore, from (5.90), Cauchy Schwarz Inequality, (5.71) and (3.1.10), we get

$$\begin{aligned}
 D_f(\bar{x}, a_n) &\leq D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \lambda_n \|y_n - a_n\| \|Ay_n - Az_n\| \\
 &\leq D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \mu \|y_n - a_n\| \|y_n - z_n\| \\
 &\leq D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \frac{\mu}{2} (\|y_n - a_n\|^2 + \|y_n - z_n\|^2) \\
 &\leq D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \frac{\mu}{\beta} (D_f(a_n, y_n) + D_f(y_n, z_n)) \\
 &= D_f(\bar{x}, z_n) - \left(1 - \frac{\mu}{\beta}\right) (D_f(a_n, y_n) + D_f(y_n, z_n)).
 \end{aligned} \tag{5.91}$$

Using (1.19), (1.21) and (1.22), we get

$$\begin{aligned}
 D_f(\bar{x}, z_n) &\leq D_f(\bar{x}, \nabla f^*(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n))) - D_f(z_n, q_n) \\
 &\leq D_f(\bar{x}, \nabla f^*(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n))) \\
 &= V_f(\bar{x}, \nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)) \\
 &\leq V_f(\bar{x}, \nabla f(x_n)) \\
 &\quad - \left\langle \gamma_n T^* J_{E_3}(Tx_n - Su_n), \right. \\
 &\quad \left. \nabla f^*(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)) - \bar{x} \right\rangle \\
 &= D_f(\bar{x}, x_n) - \gamma_n \langle T^* J_{E_3}(Tx_n - Su_n), q_n - \bar{x} \rangle \\
 &= D_f(\bar{x}, x_n) - \gamma_n \langle J_{E_3}(Tx_n - Su_n), Tq_n - T\bar{x} \rangle.
 \end{aligned} \tag{5.92}$$

Substituting (5.92) into (5.91), we obtain

$$\begin{aligned}
 D_f(\bar{x}, a_n) &\leq D_f(\bar{x}, x_n) - \left(1 - \frac{\mu}{\beta}\right) \left(D_f(a_n, y_n) + D_f(y_n, z_n)\right) \\
 &\quad - \gamma_n \langle J_{E_3}(Tx_n - Su_n), Tq_n - T\bar{x} \rangle.
 \end{aligned} \tag{5.93}$$

Thus, from (5.93) and (5.81), we get

$$\begin{aligned}
 D_f(\bar{x}, x_{n+1}) &\leq \alpha_n D_f(\bar{x}, x) + (1 - \alpha_n) D_f(\bar{x}, x_n) \\
 &\quad - (1 - \alpha_n) \left(1 - \frac{\mu}{\beta}\right) \left(D_f(a_n, y_n) + D_f(y_n, z_n)\right) \\
 &\quad - (1 - \alpha_n) \gamma_n \langle J_{E_3}(Tx_n - Su_n), Tq_n - T\bar{x} \rangle.
 \end{aligned} \tag{5.94}$$

Similarly, we have

$$\begin{aligned}
 D_g(\bar{u}, u_{n+1}) &\leq \alpha_n D_g(\bar{u}, u) + (1 - \alpha_n) D_g(\bar{u}, u_n) \\
 &\quad - (1 - \alpha_n) \left(1 - \frac{\mu}{\beta}\right) \left(D_g(b_n, v_n) + D_g(v_n, w_n)\right) \\
 &\quad - (1 - \alpha_n) \gamma_n \langle J_{E_3}(Su_n - Tx_n), St_n - S\bar{u} \rangle.
 \end{aligned} \tag{5.95}$$

Now, denote $\Omega_n = D_f(\bar{x}, x_n) + D_g(\bar{u}, u_n)$ and $\Sigma = D_f(\bar{x}, x) + D_g(\bar{u}, u)$. Since $\mu \in (0, \beta)$ and $\beta > 0$, we have that $1 > \frac{\mu}{\beta} > 0$.

Thus, $1 - \frac{\mu}{\beta} > 0$. Then, combining (5.94) and (5.95) and using the fact that $T\bar{x} = S\bar{u}$, we get

$$\Omega_{n+1} \leq \alpha_n \Sigma + (1 - \alpha_n) \Omega_n - (1 - \alpha_n) \gamma_n \langle Tq_n - St_n, J_{E_3}(Tx_n - Su_n) \rangle. \tag{5.96}$$

But, we have by the Cauchy–Schwarz Inequality that

$$\begin{aligned}
 & -\langle Tq_n - St_n, J_{E_3}(Tx_n - Su_n) \rangle \\
 & = -\langle Tx_n - Su_n, J_{E_3}(Tx_n - Su_n) \rangle - \langle Tq_n - Tx_n, J_{E_3}(Tx_n - Su_n) \rangle \\
 & \quad - \langle Su_n - St_n, J_{E_3}(Tx_n - Su_n) \rangle \\
 & = -\|Tx_n - Su_n\|^2 - \langle q_n - x_n, T^* J_{E_3}(Tx_n - Su_n) \rangle \\
 & \quad - \langle u_n - t_n, S^* J_{E_3}(Tx_n - Su_n) \rangle \\
 & \leq -\|Tx_n - Su_n\|^2 + \|q_n - x_n\| \|T^* J_{E_3}(Tx_n - Su_n)\| \\
 & \quad + \|u_n - t_n\| \|S^* J_{E_3}(Tx_n - Su_n)\|.
 \end{aligned} \tag{5.97}$$

From the strong convexity of f and the definition of q_n , we have

$$\begin{aligned}
 \|q_n - x_n\| & = \|\nabla f^*(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)) - \nabla f^*(\nabla f(x_n))\| \\
 & \leq \frac{1}{\beta_1} \|\gamma_n T^* J_{E_3}(Tx_n - Su_n)\| \\
 & \leq \frac{\gamma_n}{\beta} \|T^* J_{E_3}(Tx_n - Su_n)\|.
 \end{aligned} \tag{5.98}$$

Similarly, the strong convexity of g and the definition of t_n gives

$$\|t_n - u_n\| \leq \frac{\gamma_n}{\beta} \|S^* J_{E_3}(Su_n - Tx_n)\|. \tag{5.99}$$

Substituting (5.99) and (5.98) into (5.97) and applying the property of γ_n , we get

$$\begin{aligned}
 -\gamma_n \langle Tq_n - St_n, J_{E_3}(Tx_n - Su_n) \rangle & \leq -\gamma_n \|Tx_n - Su_n\|^2 + \frac{\gamma_n^2}{\beta} \|T^* J_{E_3}(Tx_n - Su_n)\|^2 \\
 & \quad + \frac{\gamma_n^2}{\beta} \|S^* J_{E_3}(Su_n - Tx_n)\|^2 \\
 & \leq -\frac{\rho}{2} \|Tx_n - Su_n\|^2 - \frac{\gamma_n}{2} \|Tx_n - Su_n\|^2 \\
 & \quad + \frac{\gamma_n^2}{\beta} \|T^* J_{E_3}(Tx_n - Su_n)\|^2 \\
 & \quad + \frac{\gamma_n^2}{\beta} \|S^* J_{E_3}(Su_n - Tx_n)\|^2 \\
 & \leq -\frac{\rho}{2} \|Tx_n - Su_n\|^2,
 \end{aligned} \tag{5.100}$$

and substituting (5.100) into (5.96), one obtains

$$\Omega_{n+1} \leq \alpha_n \Sigma + (1 - \alpha_n) \Omega_n - (1 - \alpha_n) \frac{\rho}{2} \|Tx_n - Su_n\|^2 \leq \alpha_n \Sigma + (1 - \alpha_n) \Omega_n,$$

which implies by the Principle of Mathematical Induction that $\Omega_n \leq \max\{\Omega_1, \Sigma\}$, for all $n \geq 1$. Thus, $\{D_f(\bar{x}, x_n) + D_g(\bar{u}, u_n)\}$ is bounded which implies that the sequences $\{D_f(\bar{x}, x_n)\}$ and $\{D_g(\bar{u}, u_n)\}$ are bounded. By Lemma 3.1.11, we have that $\{x_n\}$ and $\{u_n\}$ are bounded. $\square$

Lemma 5.3.2. *Assume that conditions (A1) – (A2), (B1) – (B4) and (R1) – (R2) hold. Let $\{z_n\}$, $\{y_n\}$, $\{w_n\}$ and $\{v_n\}$ be as defined in Algorithm 5.2.*

(a) If there exist subsequences $\{z_{n_k}\}$ and $\{y_{n_k}\}$ of $\{z_n\}$ and $\{y_n\}$, respectively, such that $z_{n_k} \rightharpoonup p \in C$ and $\|z_{n_k} - y_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$, then

i. $0 \leq \liminf_{k \rightarrow \infty} \langle Az_{n_k}, z - z_{n_k} \rangle$ for all $z \in C$;

ii. $p \in VI(C, A)$.

(b) If there exist subsequences $\{w_{n_k}\}$ and $\{v_{n_k}\}$ of $\{w_n\}$ and $\{v_n\}$, respectively, such that $w_{n_k} \rightharpoonup q \in D$ and $\|w_{n_k} - v_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$, then

i. $0 \leq \liminf_{k \rightarrow \infty} \langle Bw_{n_k}, w - w_{n_k} \rangle$ for all $w \in D$;

ii. $q \in VI(D, B)$.

Proof. (a) Let the hypotheses be satisfied.

i. Put $s_{n_k} = P_C^f \nabla f^*(\nabla f z_{n_k} - \lambda_{n_k} l^{-1} A z_{n_k})$. By Lemma 3.3.1 and (5.76) we have

$$\|z_{n_k} - s_{n_k}\| \leq \frac{1}{l} \|z_{n_k} - y_{n_k}\| \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (5.101)$$

Therefore, $s_{n_k} \rightharpoonup p \in C$ and thus we have that $\{s_{n_k}\}$ is bounded. Since A is uniformly continuous on bounded subsets of E_1 , we have

$$\|Az_{n_k} - As_{n_k}\| \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (5.102)$$

From the Armijo line-search rule (5.71), we have

$$\begin{aligned} \lambda_{n_k} l^{-1} \|AP_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k}) - A z_{n_k}\| \\ > \mu \|z_{n_k} - P_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k})\|, \end{aligned}$$

which implies

$$\begin{aligned} \frac{1}{\mu} \|AP_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k}) - A z_{n_k}\| \\ > \frac{\|z_{n_k} - P_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k})\|}{\lambda_{n_k} l^{-1}}. \end{aligned} \quad (5.103)$$

From (5.102) and (5.103), we have

$$\lim_{k \rightarrow \infty} \frac{\|z_{n_k} - P_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k})\|}{\lambda_{n_k} l^{-1}} = 0. \quad (5.104)$$

Since ∇f is Lipschitz continuous, we have

$$\begin{aligned} 0 &\leq \lim_{k \rightarrow \infty} \frac{\|\nabla f z_{n_k} - \nabla f (P_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k}))\|}{\lambda_{n_k} l^{-1}} \\ &\leq L \lim_{k \rightarrow \infty} \frac{\|z_{n_k} - P_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k})\|}{\lambda_{n_k} l^{-1}}, \end{aligned} \quad (5.105)$$

for some $L \geq 0$. Thus, we obtain from (5.104) and (5.105) that

$$\lim_{k \rightarrow \infty} \frac{\|\nabla f z_{n_k} - \nabla f (P_C^f \nabla f^* (\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k}))\|}{\lambda_{n_k} l^{-1}} = 0. \quad (5.106)$$

From the definition of s_{n_k} and (1.18), we obtain

$$\langle \nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k} - \nabla f(s_{n_k}), z - s_{n_k} \rangle \leq 0, \text{ for all } z \in C,$$

which implies

$$\begin{aligned} \left\langle \frac{\nabla f(z_{n_k}) - \nabla f(s_{n_k})}{\lambda_{n_k} l^{-1}}, z - s_{n_k} \right\rangle + \langle A z_{n_k}, s_{n_k} - z_{n_k} \rangle \\ \leq \langle A z_{n_k}, z - z_{n_k} \rangle \text{ for all } z \in C. \end{aligned} \quad (5.107)$$

Taking limit infimum on both sides of (5.107) as $k \rightarrow \infty$ and using (5.101), (5.106), uniform continuity of A and the boundedness of the sequences $\{z_{n_k}\}$ and $\{s_{n_k}\}$, we obtain

$$\liminf_{k \rightarrow \infty} \langle A z_{n_k}, z - z_{n_k} \rangle \geq 0, \text{ for all } z \in C. \quad (5.108)$$

- ii. Let $\{\varepsilon_k\}$ be a sequence of decreasing non-negative numbers such that $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. For each ε_k , we choose N_k to be the smallest positive integer such that

$$\langle A z_{n_k}, z - z_{n_k} \rangle + \varepsilon_k \geq 0, \text{ for all } k \geq N_k \quad (5.109)$$

where the existence of N_k follows from (5.108). Since $\{\varepsilon_k\}$ is decreasing, $\{N_k\}$ is increasing. If there exists $N > 0$ such that $A z_{N_k} = 0$ for all $k \geq N$, then $\langle A z_{N_k}, z - z_{N_k} \rangle \geq 0$ for all $k \geq N$ and $z \in C$.

Since A is pseudomonotone, we have that

$$\langle A z, z - z_{N_k} \rangle \geq 0 \text{ for all } k \geq N \text{ and } z \in C. \quad (5.110)$$

Taking limit on both sides of (5.110) as $k \rightarrow \infty$, we obtain $\langle A z, z - p \rangle \geq 0$ for all $k \geq N$ and all $z \in C$. By Lemma 5.1.1, we conclude that $p \in VI(C, A)$.

If there exists a subsequence $\{N_{k_i}\}$ of $\{N_k\}$, again denoted by $\{N_k\}$, such that $A z_{N_k} \neq 0$ for all $k \in \mathbb{N}$, then setting $\omega_{N_k} = \frac{J_{E_1}^{-1} A z_{N_k}}{\|A z_{N_k}\|^2}$, we get $\langle A z_{N_k}, \omega_{N_k} \rangle = 1$ for each k . Therefore, from (5.109)

$$\langle A z_{N_k}, z + \varepsilon_k \omega_{N_k} - z_{N_k} \rangle \geq 0,$$

which implies by the pseudomonotonicity of A that

$$\langle A(z + \varepsilon_k \omega_{N_k}), z + \varepsilon_k \omega_{N_k} - z_{N_k} \rangle \geq 0. \quad (5.111)$$

Since $\{z_{n_k}\}$ converges weakly to p as $k \rightarrow \infty$ and A is sequentially weakly continuous, we have that $\{A z_{n_k}\}$ converges weakly to $A p$. Suppose $A p \neq 0$ (otherwise, $p \in VI(C, A)$).

Then by the sequential weakly lower semi-continuity of the norm, we get

$$0 < \|Ap\| \leq \liminf_{k \rightarrow \infty} \|Az_{n_k}\|.$$

Since $\{z_{N_k}\} \subset \{z_{n_k}\}$ and $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, we get that

$$\begin{aligned} 0 &\leq \limsup_{k \rightarrow \infty} \|\varepsilon_k \omega_{N_k}\| = \limsup_{k \rightarrow \infty} \left(\frac{\varepsilon_k}{\|Az_{n_k}\|} \right) \\ &\leq \frac{\limsup_{k \rightarrow \infty} \varepsilon_k}{\liminf_{k \rightarrow \infty} \|Az_{n_k}\|} \leq \frac{0}{\|Ap\|} = 0 \end{aligned}$$

Hence, $\limsup_{k \rightarrow \infty} \|\varepsilon_k \omega_{N_k}\| = 0$. So, taking limit on both sides of (5.111) as $k \rightarrow \infty$, we get

$$\langle Az, z - p \rangle \geq 0, \quad \text{for all } z \in C.$$

Therefore, by Lemma 5.1.1, we have $p \in VI(C, A)$.

Part (b) of the lemma can be proved in a similar way. $\square$

Theorem 5.3.2. *Suppose that conditions (A1) – (A2), (B1) – (B4) and (R1) – (R2) hold. Then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2 converges strongly to $(\hat{x}, \hat{u}) \in \Upsilon$, where $(\hat{x}, \hat{u}) = P_{\Upsilon}^h(x, u)$, where $h = (f, g)$.*

Proof. Let $(\hat{x}, \hat{u}) = P_{\Upsilon}^h(x, u)$. Denote $C_n = \tau \nabla f(a_n) + (1 - \tau)G(a_n)$ and

$$\begin{aligned} \Delta_n &= \langle \nabla f(x) - \nabla f(\hat{x}), x_n - \hat{x} \rangle + \langle \nabla g(u) - \nabla g(\hat{u}), u_n - \hat{u} \rangle \\ &= \langle (\nabla f(x), \nabla g(u)) - (\nabla f(\hat{x}), \nabla g(\hat{u})), (x_n, u_n) - (\hat{x}, \hat{u}) \rangle. \end{aligned}$$

Then, by the Cauchy–Schwarz Inequality, we have

$$\begin{aligned} \Delta_{n+1} &= \langle \nabla f(x) - \nabla f(\hat{x}), x_n - \hat{x} \rangle + \langle \nabla g(u) - \nabla g(\hat{u}), u_n - \hat{u} \rangle \\ &\quad + \langle \nabla f(x) - \nabla f(\hat{x}), x_{n+1} - x_n \rangle \\ &\quad + \langle \nabla g(u) - \nabla g(\hat{u}), u_{n+1} - u_n \rangle \\ &\leq \Delta_n + \Lambda [\|x_{n+1} - x_n\| + \|u_{n+1} - u_n\|], \end{aligned}$$

where $\Lambda = \max \{\|\nabla f(x) - \nabla f(\hat{x})\|, \|\nabla g(u) - \nabla g(\hat{u})\|\}$.

From (5.74), (1.21), (1.22) and Lemma 3.1.9, we obtain

$$\begin{aligned}
 D_f(\hat{x}, x_{n+1}) &= D_f\left(\hat{x}, \nabla f^*((1 - \alpha_n)C_n + \alpha_n \nabla f(x))\right) \\
 &= V_f\left(\hat{x}, (1 - \alpha_n)\nabla f(\nabla f^*(C_n)) + \alpha_n \nabla f(x)\right) \\
 &\leq V_f\left(\hat{x}, \alpha_n \nabla f(x) + (1 - \alpha_n)\nabla f(\nabla f^*(C_n)) - \alpha_n(\nabla f(x) - \nabla f(\hat{x}))\right) \\
 &\quad - \left\langle -\alpha_n(\nabla f(x) - \nabla f(\hat{x})), \right. \\
 &\quad \left. \nabla f^*(\alpha_n \nabla f(x) + (1 - \alpha_n)\nabla f(\nabla f^*(C_n))) - \hat{x} \right\rangle \\
 &= V_f\left(\hat{x}, \alpha_n \nabla f(\hat{x}) + (1 - \alpha_n)\nabla f(\nabla f^*(C_n))\right) \\
 &\quad + \alpha_n \langle \nabla f(x) - \nabla f(\hat{x}), x_{n+1} - \hat{x} \rangle \\
 &= D_f(\hat{x}, \nabla f^*(\alpha_n \nabla f(\hat{x}) + (1 - \alpha_n)\nabla f(\nabla f^*(C_n)))) \\
 &\quad + \alpha_n \langle \nabla f(x) - \nabla f(\hat{x}), x_{n+1} - \hat{x} \rangle \\
 &\leq \alpha_n D_f(\hat{x}, \hat{x}) + (1 - \alpha_n) D_f(\hat{x}, \nabla f^*(C_n)) \\
 &\quad + \alpha_n \langle \nabla f(x) - \nabla f(\hat{x}), x_{n+1} - \hat{x} \rangle,
 \end{aligned}$$

that is,

$$D_f(\hat{x}, x_{n+1}) \leq (1 - \alpha_n) D_f(\hat{x}, \nabla f^*(C_n)) + \alpha_n \langle \nabla f(x) - \nabla f(\hat{x}), x_{n+1} - \hat{x} \rangle. \quad (5.112)$$

Furthermore, from (1.20), (1.21), part (iii) of Lemma 3.1.13 and Bregman relatively f -nonexpansiveness of G , we have

$$\begin{aligned}
 D_f(\hat{x}, \nabla f^*(C_n)) &= D_f(\hat{x}, \nabla f^*(\tau \nabla f(a_n) + (1 - \tau)G(a_n))) \\
 &= V_f(\hat{x}, \tau \nabla f(a_n) + (1 - \tau)G(a_n)) \\
 &= f(\hat{x}) + f^*(\tau \nabla f(a_n) + (1 - \tau)G(a_n)) \\
 &\quad - \langle \tau \nabla f(a_n) + (1 - \tau)G(a_n), \hat{x} \rangle \\
 &\leq f(\hat{x}) + \tau f^*(\nabla f(a_n)) + (1 - \tau) f^*(G(a_n)) \\
 &\quad - \tau(1 - \tau) \phi_1(\|\nabla f(a_n) - G(a_n)\|) \\
 &\quad - \tau \langle \nabla f(a_n), \hat{x} \rangle - (1 - \tau) \langle G(a_n), \hat{x} \rangle \\
 &= \tau V_f(\hat{x}, \nabla f(a_n)) + (1 - \tau) V_f(\hat{x}, G(a_n)) \\
 &\quad - \tau(1 - \tau) \phi_1(\|\nabla f(a_n) - G(a_n)\|) \\
 &= \tau D_f(\hat{x}, a_n) + (1 - \tau) D_f(\hat{x}, \nabla f^*G(a_n)) \\
 &\quad - \tau(1 - \tau) \phi_1(\|\nabla f(a_n) - G(a_n)\|) \\
 &\leq \tau D_f(\hat{x}, a_n) + (1 - \tau) D_f(\hat{x}, a_n) \\
 &\quad - \tau(1 - \tau) \phi_1(\|\nabla f(a_n) - G(a_n)\|) \\
 &= D_f(\hat{x}, a_n) - \tau(1 - \tau) \phi_1(\|\nabla f(a_n) - G(a_n)\|),
 \end{aligned} \quad (5.113)$$

where ϕ_1 is the modulus of uniform convexity of f^* . Substituting (5.113) into (5.112), we get

$$\begin{aligned} D_f(\hat{x}, x_{n+1}) &\leq (1 - \alpha_n)D_f(\hat{x}, a_n) - (1 - \alpha_n)\tau(1 - \tau)\phi_1(\|\nabla f(a_n) - G(a_n)\|) \\ &\quad + \alpha_n\langle \nabla f(x) - \nabla f(\hat{x}), x_{n+1} - \hat{x} \rangle. \end{aligned} \quad (5.114)$$

Again using (5.93), we obtain

$$\begin{aligned} D_f(\hat{x}, x_{n+1}) &\leq (1 - \alpha_n)D_f(\hat{x}, x_n) - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)\left(D_f(a_n, y_n) + D_f(y_n, z_n)\right) \\ &\quad - (1 - \alpha_n)\tau(1 - \tau)\phi_1(\|\nabla f(a_n) - G(a_n)\|) \\ &\quad - (1 - \alpha_n)\gamma_n\langle J_{E_3}(Tx_n - Su_n), Tq_n - T\hat{x} \rangle \\ &\quad + \alpha_n\langle \nabla f(x) - \nabla f(\hat{x}), x_{n+1} - \hat{x} \rangle. \end{aligned} \quad (5.115)$$

Similarly, we have

$$\begin{aligned} D_g(\hat{u}, u_{n+1}) &\leq (1 - \alpha_n)D_g(\hat{u}, u_n) - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)\left(D_g(b_n, v_n) + D_g(v_n, w_n)\right) \\ &\quad - (1 - \alpha_n)\tau(1 - \tau)\phi_2(\|\nabla g(b_n) - K(b_n)\|) \\ &\quad - (1 - \alpha_n)\gamma_n\langle J_{E_3}(Su_n - Tx_n), St_n - S\hat{u} \rangle \\ &\quad + \alpha_n\langle \nabla g(u) - \nabla g(\hat{u}), u_{n+1} - \hat{u} \rangle. \end{aligned} \quad (5.116)$$

where ϕ_2 is the modulus of uniform convexity of g^* .

Let $\Theta_n = D_f(\hat{x}, x_n) + D_g(\hat{u}, u_n)$. Then, combining (5.115) and (5.116) and using the relation in (5.100), we obtain

$$\begin{aligned} \Theta_{n+1} &\leq (1 - \alpha_n)\Theta_n - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)D_f(a_n, y_n) - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)D_f(y_n, z_n) \\ &\quad - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)D_g(b_n, v_n) - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)D_g(v_n, w_n) \\ &\quad - (1 - \alpha_n)\tau(1 - \tau)\phi_1(\|\nabla f(a_n) - G(a_n)\|) \\ &\quad - (1 - \alpha_n)\tau(1 - \tau)\phi_2(\|\nabla g(b_n) - K(b_n)\|) \\ &\quad - (1 - \alpha_n)\gamma_n\langle J_{E_3}(Tx_n - Su_n), Tq_n - T\hat{x} \rangle \\ &\quad - (1 - \alpha_n)\gamma_n\langle J_{E_3}(Su_n - Tx_n), St_n - S\hat{u} \rangle \\ &\quad + \alpha_n\langle \nabla f(x) - \nabla f(\hat{x}), x_{n+1} - \hat{x} \rangle + \alpha_n\langle \nabla g(u) - \nabla g(\hat{u}), u_{n+1} - \hat{u} \rangle \\ &\leq (1 - \alpha_n)\Theta_n - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)D_f(a_n, y_n) \\ &\quad - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)D_f(y_n, z_n) - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)D_g(b_n, v_n) \\ &\quad - (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right)D_g(v_n, w_n) - \tau(1 - \tau)(1 - \alpha_n)\phi_1(\|\nabla f(a_n) - G(a_n)\|) \\ &\quad - \tau(1 - \tau)(1 - \alpha_n)\phi_2(\|\nabla g(b_n) - K(b_n)\|) - (1 - \alpha_n)\frac{\rho}{2}\|Tx_n - Su_n\|^2 \\ &\quad + \alpha_n\langle \nabla f(x) - \nabla f(\hat{x}), x_{n+1} - \hat{x} \rangle + \alpha_n\langle \nabla g(u) - \nabla g(\hat{u}), u_{n+1} - \hat{u} \rangle, \end{aligned} \quad (5.117)$$

which implies that

$$\begin{aligned}
 & \left(1 - \frac{\mu}{\beta}\right) \left[D_f(a_n, y_n) + D_f(y_n, z_n) + D_g(b_n, v_n) + D_g(v_n, w_n) \right] \\
 & + \frac{\rho}{2} \|Tx_n - Su_n\|^2 + \tau(1 - \tau)\phi_1(\|\nabla f(a_n) - G(a_n)\|) \\
 & + \tau(1 - \tau)\phi_2(\|\nabla g(b_n) - K(b_n)\|) \leq \Theta_n - \Theta_{n+1} + \alpha_n M,
 \end{aligned} \tag{5.118}$$

for some $M > 0$, where the existence of such M is guaranteed by the boundedness of $\{x_n\}$ and $\{u_n\}$. Now, we show that the sequence of real numbers $\{\Theta_n\}$ converges strongly to zero by considering two cases.

Case I. If there exists a natural number n_0 such that $\Theta_{n+1} \leq \Theta_n$ for all $n \geq n_0$, then $\{\Theta_n\}$ converges. Taking limit as $n \rightarrow \infty$ in (5.118), we get

$$\lim_{n \rightarrow \infty} \|Tx_n - Su_n\|^2 = 0, \tag{5.119}$$

and

$$\lim_{n \rightarrow \infty} \phi_1(\|\nabla f(a_n) - G(a_n)\|) = \lim_{n \rightarrow \infty} \phi_2(\|\nabla g(b_n) - K(b_n)\|) = 0;$$

which implies that

$$\lim_{n \rightarrow \infty} \|\nabla f(a_n) - G(a_n)\| = 0 = \lim_{n \rightarrow \infty} \|\nabla g(b_n) - K(b_n)\|. \tag{5.120}$$

From (5.69), (1.21), (1.22), property of the Bregman projection, and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
 D_f(x_n, z_n) & \leq D_f\left(x_n, \nabla f^*(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n))\right) \\
 & = V_f(x_n, \nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)) \\
 & \leq V_f(x_n, \nabla f(x_n)) \\
 & \quad - \langle \gamma_n T^* J_{E_3}(Tx_n - Su_n), \nabla f^*(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)) - x_n \rangle \\
 & = D_f(x_n, x_n) - \gamma_n \langle T^* J_{E_3}(Tx_n - Su_n), q_n - x_n \rangle \\
 & \leq \gamma_n \|T^* J_{E_3}(Tx_n - Su_n)\| \|q_n - x_n\|.
 \end{aligned} \tag{5.121}$$

Substituting (5.98) into (5.121), then taking limit on both sides and using (5.119), we obtain

$$\begin{aligned}
 0 & \leq \lim_{n \rightarrow \infty} D_f(x_n, z_n) \leq \lim_{n \rightarrow \infty} \left(\frac{\gamma_n^2}{\beta} \|T\|^2 \|J_{E_3}(Tx_n - Su_n)\|^2 \right) \\
 & = \lim_{n \rightarrow \infty} \left(\frac{\gamma_n^2}{\beta} \|T\|^2 \|Tx_n - Su_n\|^2 \right) = 0,
 \end{aligned}$$

and hence $\lim_{n \rightarrow \infty} D_f(x_n, z_n) = 0$ and this implies by Lemma 3.1.13 that

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \tag{5.122}$$

Similarly, one can show that

$$\lim_{n \rightarrow \infty} \|u_n - w_n\| = 0. \quad (5.123)$$

Moreover, by taking limit as $n \rightarrow \infty$ on both sides of (5.118), we obtain

$$\lim_{n \rightarrow \infty} D_f(a_n, y_n) = 0, \quad \lim_{n \rightarrow \infty} D_f(y_n, z_n) = 0,$$

$$\lim_{n \rightarrow \infty} D_g(b_n, v_n) = 0, \quad \lim_{n \rightarrow \infty} D_g(v_n, w_n) = 0,$$

which implies that

$$\lim_{n \rightarrow \infty} \|a_n - y_n\| = 0, \quad \lim_{n \rightarrow \infty} \|y_n - z_n\| = 0, \quad (5.124)$$

and

$$\lim_{n \rightarrow \infty} \|b_n - v_n\| = 0, \quad \lim_{n \rightarrow \infty} \|v_n - w_n\| = 0. \quad (5.125)$$

From (5.122) and (5.124), we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| \leq \lim_{n \rightarrow \infty} \|x_n - z_n\| + \lim_{n \rightarrow \infty} \|z_n - y_n\| = 0, \quad (5.126)$$

and from (5.123) and (5.125), we obtain

$$\lim_{n \rightarrow \infty} \|u_n - v_n\| \leq \lim_{n \rightarrow \infty} \|u_n - w_n\| + \lim_{n \rightarrow \infty} \|w_n - v_n\| = 0. \quad (5.127)$$

From (5.124) and (5.126), we have

$$\lim_{n \rightarrow \infty} \|a_n - x_n\| \leq \lim_{n \rightarrow \infty} \|a_n - y_n\| + \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (5.128)$$

From (5.74) and (5.120), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\nabla f(x_{n+1}) - \nabla f(a_n)\| &\leq \lim_{n \rightarrow \infty} (\alpha_n \|\nabla f(x) - \tau \nabla f(a_n)\|) \\ &\quad + (1 - \tau) \lim_{n \rightarrow \infty} ((1 - \alpha_n) \|G(a_n) - \nabla f(a_n)\|) = 0. \end{aligned} \quad (5.129)$$

From (5.129) and part (iii) of Lemma 3.1.13, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - a_n\| = 0. \quad (5.130)$$

Thus, we obtain from (5.128) and (5.130) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| \leq \lim_{n \rightarrow \infty} \|x_{n+1} - a_n\| + \lim_{n \rightarrow \infty} \|a_n - x_n\| = 0. \quad (5.131)$$

Similarly, we obtain

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0. \quad (5.132)$$

Since $\{(x_n, u_n)\}$ is bounded and $E_1 \times E_2$ is reflexive (Lemma 3.1.5), there exists a subsequence $\{(x_{n_k}, u_{n_k})\}$ of $\{(x_n, u_n)\}$ which converges weakly to some $(x^*, u^*) \in E_1 \times E_2$ and

$$\limsup_{n \rightarrow \infty} \Delta_n = \lim_{k \rightarrow \infty} \Delta_{n_k}. \quad (5.133)$$

Consequently, we have $x_{n_k} \rightharpoonup x^*$ and $u_{n_k} \rightharpoonup u^*$. From (5.122) and (5.123), we have $z_{n_k} \rightharpoonup x^*$ and $w_{n_k} \rightharpoonup u^*$, respectively.

So, from (5.124), the fact that $z_{n_k} \rightharpoonup x^*$ and Lemma 5.3.2, we obtain $x^* \in VI(C, A)$. Similarly, one can show that $u^* \in VI(D, B)$. From (5.128), we have $a_n \rightharpoonup x^*$. Thus, with the help of (5.120) and the definition of f -asymptotic fixed points, we conclude that $x^* \in \widehat{F_f(G)}$. From the Bregman relatively f -nonexpansivity of G , we have $x^* \in F_f(G)$. So, $x^* \in VI(C, A) \cap F_f(G)$. Similarly, we can show that $u^* \in VI(D, B) \cap F_g(K)$. Moreover, by Lemma 3.1 we have

$$\begin{aligned} \|Tx^* - Su^*\|^2 &= \|Tx_{n_k} - Su_{n_k} + Tx^* - Tx_{n_k} + Su_{n_k} - Su^*\|^2 \\ &\leq \|Tx_{n_k} - Su_{n_k}\|^2 + 2\langle J_{E_3}(Tx^* - Su^*), Tx^* - Tx_{n_k} + Su_{n_k} - Su^* \rangle. \end{aligned} \quad (5.134)$$

Since T and S are sequentially weakly continuous, we have that $x_{n_k} \rightharpoonup x^*$ implies $Tx_{n_k} \rightharpoonup Tx^*$ and $u_{n_k} \rightharpoonup u^*$ implies $Su_{n_k} \rightharpoonup Su^*$. Thus, we obtain using (5.119) that $Tx^* = Su^*$. Therefore, $(x^*, u^*) \in \Upsilon$.

Furthermore, from the definition of Δ_{n+1} , (5.131), (5.132), (5.133) and (1.14) we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \Delta_{n+1} &\leq \limsup_{n \rightarrow \infty} \Delta_n + \Lambda \limsup_{n \rightarrow \infty} [\|x_{n+1} - x_n\| + \|u_{n+1} - u_n\|] \\ &= \lim_{k \rightarrow \infty} \Delta_{n_k} + \Lambda \lim_{k \rightarrow \infty} [\|x_{n_k+1} - x_{n_k}\| + \|u_{n_k+1} - u_{n_k}\|] \\ &= \lim_{k \rightarrow \infty} \langle (\nabla f(x), \nabla g(u)) - (\nabla f(\hat{x}), \nabla g(\hat{u})), (x_{n_k}, u_{n_k}) - (\hat{x}, \hat{u}) \rangle \\ &= \langle (\nabla f(x), \nabla g(u)) - (\nabla f(\hat{x}), \nabla g(\hat{u})), (x^*, u^*) - (\hat{x}, \hat{u}) \rangle \\ &\leq 0. \end{aligned} \quad (5.135)$$

From (5.117) we have

$$\Theta_{n+1} \leq (1 - \alpha_n)\Theta_n + \alpha_n\Delta_{n+1}. \quad (5.136)$$

So, (5.135), (5.136), Lemma 3.1.6 and the condition on α_n give that $\lim_{n \rightarrow \infty} \Theta_n = 0$, which implies that $\lim_{n \rightarrow \infty} D_f(\hat{x}, x_n) = 0$ and $\lim_{n \rightarrow \infty} D_g(\hat{u}, u_n) = 0$. It then follows by Lemma 3.1.13 that $\lim_{n \rightarrow \infty} x_n = \hat{x}$ and $\lim_{n \rightarrow \infty} u_n = \hat{u}$.

Case II. If there exists a subsequence $\{\Theta_{n_i}\}$ of $\{\Theta_n\}$ with $\Theta_{n_i} < \Theta_{n_i+1}$ for all $i \geq 0$, then by Lemma 3.1.7, we can find a nondecreasing sequence $\{m_k\}$ of positive integers such that $\lim_{k \rightarrow \infty} m_k = \infty$ and

$$\Theta_{m_k} \leq \Theta_{m_k+1} \quad \text{and} \quad \Theta_k \leq \Theta_{m_k+1}, \quad (5.137)$$

for all positive integers k . Thus, (5.118) becomes

$$\begin{aligned} &\left(1 - \frac{\mu}{\beta}\right) \left[D_f(a_{m_k}, y_{m_k}) + D_f(y_{m_k}, z_{m_k}) + D_g(b_{m_k}, v_{m_k}) + D_g(v_{m_k}, w_{m_k}) \right] \\ &+ \tau(1 - \tau) \left[\phi_1(\|\nabla f(a_{m_k}) - G(a_{m_k})\|) + \phi_2(\|\nabla g(b_{m_k}) - K(b_{m_k})\|) \right] \\ &+ \frac{\rho}{2} \|Tx_{m_k} - Su_{m_k}\|^2 \leq \Theta_{m_k} - \Theta_{m_k+1} + \alpha_{m_k}M. \end{aligned} \quad (5.138)$$

Taking limit as $k \rightarrow \infty$ to both sides of (5.138), we derive

$$\begin{aligned} \lim_{k \rightarrow \infty} \|Tx_{m_k} - Su_{m_k}\|^2 &= 0, \\ \lim_{k \rightarrow \infty} \|\nabla f(a_{m_k}) - G(a_{m_k})\| &= \lim_{k \rightarrow \infty} \|\nabla g(b_{m_k}) - K(b_{m_k})\| = 0. \end{aligned}$$

Following the method used in Case I, we get

$$\begin{aligned} \lim_{k \rightarrow \infty} \|x_{m_k} - z_{m_k}\| &= \lim_{k \rightarrow \infty} \|u_{m_k} - w_{m_k}\| = 0, \\ \lim_{k \rightarrow \infty} \|y_{m_k} - z_{m_k}\| &= \lim_{k \rightarrow \infty} \|v_{m_k} - w_{m_k}\| = 0, \\ \lim_{k \rightarrow \infty} \|x_{m_k} - y_{m_k}\| &= \lim_{k \rightarrow \infty} \|u_{m_k} - v_{m_k}\| = 0. \end{aligned}$$

Furthermore, following similar steps as in Case I, we obtain

$$\lim_{k \rightarrow \infty} \|x_{m_k+1} - x_{m_k}\| = 0, \quad \lim_{k \rightarrow \infty} \|u_{m_k+1} - u_{m_k}\| = 0 \quad \text{and} \quad \limsup_{k \rightarrow \infty} \Delta_{m_k+1} \leq 0.$$

Thus, from (5.117) and (5.137), we have

$$\alpha_{m_k} \Theta_{m_k} \leq \Theta_{m_k} - \Theta_{m_k+1} + \alpha_{m_k} \Delta_{m_k+1} \leq \alpha_{m_k} \Delta_{m_k+1},$$

which implies that

$$\Theta_{m_k} \leq \Delta_{m_k+1}. \quad (5.139)$$

Taking limit on both sides of (5.139) as $k \rightarrow \infty$ and using the fact that $\limsup_{k \rightarrow \infty} \Delta_{m_k+1} \leq 0$, we get that the sequence $\Theta_{m_k} \rightarrow 0$ as $k \rightarrow \infty$. It follows from (5.136) that $\Theta_{m_k+1} \rightarrow 0$ as $k \rightarrow \infty$. Since $\Theta_k \leq \Theta_{m_k+1}$ for all $k \geq 0$, we have that $\Theta_k \rightarrow 0$ as $k \rightarrow \infty$. Thus, we have $\lim_{k \rightarrow \infty} D_f(\hat{x}, x_k) = 0$ and $\lim_{k \rightarrow \infty} D_g(\hat{u}, u_k) = 0$, which implies by Lemma 3.1.13 that $\lim_{k \rightarrow \infty} x_k = \hat{x}$ and $\lim_{k \rightarrow \infty} u_k = \hat{u}$.

Thus, we conclude from Cases I and II, that the sequence $\{(x_n, u_n)\}$ converges strongly to $(\hat{x}, \hat{u}) = P_Y^h(x, u)$, and this completes the proof. $\square$

If A and B are uniformly continuous and monotone mappings, then the assumption that A and B are sequentially weakly continuous is not required and hence the following corollary follows.

Corollary 5.3.1. *Assume that $A: C \rightarrow E_1^*$ and $B: D \rightarrow E_2^*$ are uniformly continuous and monotone mappings. If conditions (A1) – (A2), (B2) – (B4) and (R1) – (R2) are satisfied, then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2 converges strongly to $(\hat{x}, \hat{u}) \in \Upsilon$, where $(\hat{x}, \hat{u}) = P_Y^h(x, u)$.*

Proof. As every monotone mapping is pseudomonotone, the result follows from Theorem 5.3.2. $\square$

If $x = 0 = u$, then Algorithm 5.2 can be used to locate an element of the solution with the minimum norm and hence the following corollary follows.

Corollary 5.3.2. *Suppose that the conditions (A1) – (A2), (B1) – (B4) and (R1) – (R2) are satisfied. Then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2 with $x = 0$ and $u = 0$ converges strongly to $(\hat{x}, \hat{u}) \in \Upsilon$, where $(\hat{x}, \hat{u}) = P_{\Upsilon}^h(0, 0)$.*

5.3.2 Some Theoretical Applications

Hereunder, we present some specific cases of our main problem. The following are the assumptions that will be used in these cases.

- (A3) Let C and D be nonempty, convex and closed subsets of a smooth, strictly convex, reflexive real Banach space E with dual E^* ;
- (A4) Let $f, g: E \rightarrow \mathbb{R}$ be proper, lower semi-continuous, uniformly Fréchet differentiable, strongly convex, strongly coercive, Legendre functions which are bounded on bounded subsets. Let f and g have Lipschitz continuous gradients with the strong convexity constant of f (respectively, g) greater than or equal to the Lipschitz constant of ∇f (respectively, ∇g);
- (B5) Let $A, B: E \rightarrow E^*$ be uniformly continuous, pseudomonotone and sequentially weakly continuous on bounded subsets of E ;
- (B6) Let $G, K: E \rightarrow E^*$ be Bregman relatively f -nonexpansive and Bregman relatively g -nonexpansive mappings, respectively.

Common Solutions of Variational Inequality and f, g -Fixed Point Problems

Let $E = E_1 = E_2 = E_3$, $T = S = I$ and let f and g be as in (A4). Then the split equality of variational inequality and f, g -fixed point problems reduces to finding a common solution of two variational inequality and f, g -fixed point problems. This problem can be expressed as

$$\text{find } \hat{x} \in (VI(C, A) \cap F_f(G)) \text{ and } \hat{u} \in (VI(D, B) \cap F_g(K)) \text{ such that } \hat{x} = \hat{u}.$$

Denote $\Sigma = \{(\hat{x}, \hat{u}) \in (VI(C, A) \cap F_f(G)) \times (VI(D, B) \cap F_g(K)) : \hat{x} = \hat{u}\}$. In this case, we have the following corollaries:

Corollary 5.3.3. *Assume that $\Sigma \neq \emptyset$. If conditions (A3) – (A4), (B5) – (B6), (R1) – (R2) are satisfied, then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2, with $E = E_1 = E_2 = E_3$ and $T = S = I$, converges strongly to $(\hat{x}, \hat{u}) \in \Sigma$, where $(\hat{x}, \hat{u}) = P_{\Sigma}^h(x, u)$.*

Corollary 5.3.4. *Assume that $\Sigma \neq \emptyset$ and let $A: E \rightarrow E^*$ and $B: E \rightarrow E^*$ be uniformly continuous monotone mappings. If conditions (A3) – (A4), (B6), (R1) – (R2) are satisfied, then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2, with $E = E_1 = E_2 = E_3$ and $T = S = I$, converges strongly to $(\hat{x}, \hat{u}) \in \Sigma$, where $(\hat{x}, \hat{u}) = P_{\Sigma}^h(x, u)$.*

Split Equality of Null Point and f, g -Fixed Point Problems

Let f and g be as in (A2). If $C = E_1$ and $D = E_2$, then the split equality of variational inequality and f, g -fixed point problem reduces to the split equality of null point and f, g -fixed point problem which can be described as finding a point $(\bar{x}, \bar{u})$ with the property

$$(\hat{x}, \hat{u}) \in (A^{-1}(0) \cap F_f(G)) \times (B^{-1}(0) \cap F_g(K)) \text{ such that } T\hat{x} = S\hat{u},$$

where $A^{-1}(0) = \{x \in E_1 : 0 \in Ax\}$ and $B^{-1}(0) = \{u \in E_2 : 0 \in Bu\}$. Denote

$$\Upsilon^* = \{(\hat{x}, \hat{u}) \in (A^{-1}(0) \cap F_f(G)) \times (B^{-1}(0) \cap F_g(K)) : T\hat{x} = S\hat{u}\}.$$

In this case, we have the following results:

Corollary 5.3.5. *Assume that $\Upsilon^* \neq \emptyset$. If conditions (A1) – (A2), (B1) – (B3), with $C = E_1, D = E_2$, (R1) – (R2) are satisfied, then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2 converges strongly to $(\hat{x}, \hat{u}) \in \Upsilon^*$, where $(\hat{x}, \hat{u}) = P_{\Upsilon^*}^h(x, u)$.*

Corollary 5.3.6. *Let $A: E_1 \rightarrow E_1^*$ and $B: E_2 \rightarrow E_2^*$ be uniformly continuous monotone mappings and assume that $\Upsilon^* \neq \emptyset$. If conditions (A1) – (A2), (B2) – (B3) with $C = E_1$ and $D = E_2$, (R1) – (R2) are satisfied, then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2 converges strongly to $(\hat{x}, \hat{u}) \in \Upsilon^*$, where $(\hat{x}, \hat{u}) = P_{\Upsilon^*}^h(x, u)$.*

Split Equality Variational Inequality Problem

Let f and g be as in (A2). If $Gx = \nabla f(x)$ for all $x \in C$ and $Ku = \nabla g(u)$ for all $u \in D$, then the split equality of variational inequality and f, g -fixed point problems reduces to the split equality variational inequality problem which seeks to

$$\text{find } \hat{x} \in VI(C, A) \text{ and } \hat{u} \in VI(D, B) \text{ such that } T\hat{x} = S\hat{u}.$$

Denote $\Gamma = \{(\hat{x}, \hat{u}) \in VI(C, A) \times VI(D, B) : T\hat{x} = S\hat{u}\}$. One can easily show that ∇f and ∇g are Bregman f -relatively nonexpansive and Bregman g -relatively nonexpansive, respectively, and hence we have the following results.

Corollary 5.3.7. *Assume that $\Gamma \neq \emptyset$. If conditions (A1) – (A2), (B1) – (B3), (R1) – (R2) are satisfied with $Gx = \nabla f(x)$ for all $x \in C$ and $Ku = \nabla g(u)$ for all $u \in D$, then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2 converges strongly to $(\hat{x}, \hat{u}) \in \Gamma$, where $(\hat{x}, \hat{u}) = P_{\Gamma}^h(x, u)$.*

Corollary 5.3.8. *Let $A: E_1 \rightarrow E_1^*$ and $B: E_2 \rightarrow E_2^*$ be uniformly continuous monotone mappings. Assume that $\Gamma \neq \emptyset$. If conditions (A1) – (A2), (B2) – (B3), (R1) – (R2) are satisfied with $Gx = \nabla f(x)$ for all $x \in C$ and $Ku = \nabla g(u)$ for all $u \in D$, then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2 converges strongly to $(\hat{x}, \hat{u}) \in \Gamma$, where $(\hat{x}, \hat{u}) = P_{\Gamma}^h(x, u)$.*

Split Equality f, g -Fixed Point Problem

If we have $A = 0$ and $B = 0$, in Algorithm 5.2, then the split equality of variational inequality and f, g -fixed point problem reduces to the split equality f, g -fixed point problem which seeks to

$$\text{find } \hat{x} \in F_f(G) \text{ and } \hat{u} \in F_g(K) \text{ such that } T\hat{x} = S\hat{u}.$$

Denote $\Gamma^* = \{(\hat{x}, \hat{u}) \in F_f(G) \times F_g(K) : T\hat{x} = S\hat{u}\}$.

Corollary 5.3.9. *Assume that $\Gamma^* \neq \emptyset$. If conditions (A1) – (A2), (B2) – (B3), (R1) – (R2) are satisfied. then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 5.2, with $A = B = 0$, converges strongly to $(\hat{x}, \hat{u}) \in \Gamma^*$, where $(\hat{x}, \hat{u}) = P_{\Gamma^*}(x, u)$.*

5.3.3 Numerical Example

Example 5.3.1. Given $E_1 = E_2 = E_3 = \mathbb{R}^2$. Let the norm and inner product on $\mathbb{R}^2$ be, respectively, given by $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^2 |x_i|^2}$ and $\langle x, u \rangle = \sum_{i=1}^2 x_i u_i$ for all $x, u \in \mathbb{R}^2$.

Define the mappings $A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $B: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$Ax = A(x_1, x_2) = \left(\frac{5}{2} + \sqrt{x_1^2 + x_2^2} \right) (x_1, x_2 - 1) \text{ and } Bu = B(u_1, u_2) = \sqrt{u_1^2 + u_2^2} (u_1, u_2)$$

and let C and D be given by $C = \{x \in \mathbb{R}^2 : \|x\| \leq 2\}$ and $D = \{u \in \mathbb{R}^2 : \|u\| \leq 2\}$.

Clearly, C and D are nonempty, closed and convex subsets of $\mathbb{R}^2$. Clearly, the mappings A and B are uniformly continuous and sequentially weakly continuous on C and D , respectively. It can be easily shown that B is pseudomonotone on subsets of D . We only show here that A is pseudomonotone on $\mathbb{R}^2$. To this end, let

$$\begin{aligned} \langle Ax, y - x \rangle \geq 0 &\iff \left\langle \left(\frac{5}{2} + \sqrt{x_1^2 + x_2^2} \right) (x_1, x_2 - 1), (y_1 - x_1, y_2 - x_2) \right\rangle \geq 0 \\ &\iff \left(\frac{5}{2} + \sqrt{x_1^2 + x_2^2} \right) \langle (x_1, x_2 - 1), (y_1 - x_1, y_2 - x_2) \rangle \geq 0. \end{aligned}$$

Since $\frac{5}{2} + \sqrt{x_1^2 + x_2^2} > 0$, we conclude that $\langle (x_1, x_2 - 1), (y_1 - x_1, y_2 - x_2) \rangle \geq 0$. Now, for $(y_1, y_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} \langle Ay, y - x \rangle &= \left(\frac{5}{2} + \sqrt{y_1^2 + y_2^2} \right) \langle (y_1, y_2 - 1), (y_1 - x_1, y_2 - x_2) \rangle \\ &\geq \left(\frac{5}{2} + \sqrt{y_1^2 + y_2^2} \right) \left[\langle (y_1, y_2 - 1), (y_1 - x_1, y_2 - x_2) \rangle \right] \\ &\quad - \left(\frac{5}{2} + \sqrt{y_1^2 + y_2^2} \right) \left[\langle (x_1, x_2 - 1), (y_1 - x_1, y_2 - x_2) \rangle \right] \\ &= \left(\frac{5}{2} + \sqrt{y_1^2 + y_2^2} \right) (|y_1 - x_1|^2 + |y_2 - x_2|^2) \geq \frac{5}{2} \|y - x\|^2 \geq 0. \end{aligned}$$

Therefore, A is pseudomonotone on $\mathbb{R}^2$. Let us define $T, S: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $Tx = T(x_1, x_2) = (5x_1, 0)$ and $Su = S(u_1, u_2) = (2u_1, 3u_2)$, where $(x_1, x_2), (u_1, u_2) \in \mathbb{R}^2$. Clearly, T and S are bounded linear maps on $\mathbb{R}^2$ with adjoints $T^*x = (5x_1, 0)$ and $S^*u = (2u_1, 3u_2)$, respectively. Now, we have $\langle A(0, 1), (x_1, x_2) - (0, 1) \rangle \geq 0$, for all $(x_1, x_2) \in C$, $\langle B(0, 0), (u_1, u_2) - (0, 0) \rangle \geq 0$ for all $(u_1, u_2) \in D$. So, $\hat{x} \in VI(C, A)$ and $\hat{u} \in VI(D, B)$, where $\hat{x} = (0, 1)$ and $\hat{u} = (0, 0)$. Let us define $G: C \rightarrow \mathbb{R}^2$ and $K: D \rightarrow \mathbb{R}^2$ by

$$Gx = G(x_1, x_2) = (0, x_2) \text{ and } Ku = K(u_1, u_2) = (u_2, u_1),$$

and $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x) = \frac{1}{2} \|x\|^2$ and $g(u) = \frac{1}{2} \|u\|^2$. Then, we have $\nabla f(x) = x$, $\nabla g(u) = u$ and $J_E = I$. One can easily show that G is Bregman relatively f -nonexpansive and K is Bregman relatively g -nonexpansive. Moreover,

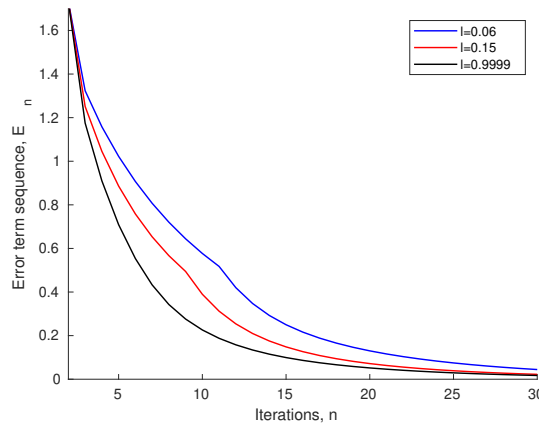
$$G\hat{x} = G(0, 1) = (0, 1) = \nabla f(0, 1) \text{ and } K\hat{u} = K(0, 0) = (0, 0) = \nabla g(0, 0).$$

From this, we conclude that $\hat{x} \in F_f(G)$ and $\hat{u} \in F_g(K)$. Moreover, $T(0, 1) = (0, 0) = S(0, 0)$. Thus we have $(\hat{x}, \hat{u}) \in (VI(C, A) \cap F_f(G)) \times (VI(D, B) \cap F_g(K))$ with $T\hat{x} = S\hat{u}$. Now, taking

$\alpha_n = \frac{1}{n + 100000}$ for $n \geq 1$, $(x_{11}, x_{12}) = (0.5, 0.5)$, $(u_{11}, u_{12}) = (0.5, 0.5)$ and

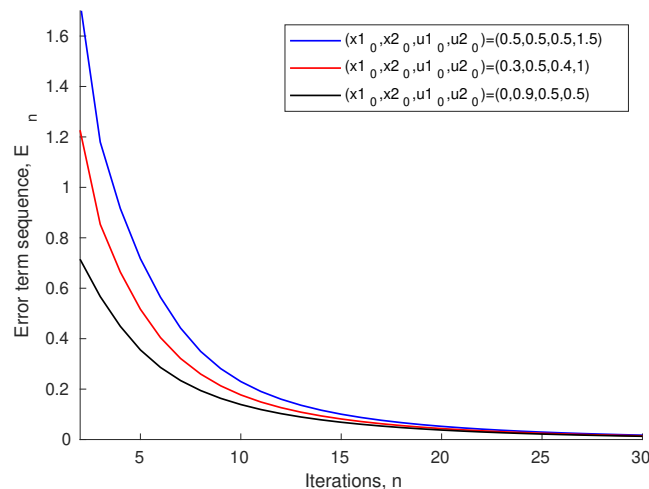
$$\gamma_n = \begin{cases} \left(\frac{2}{5}\right) \frac{(5x_{1n} - 2u_{1n})^2 + (3u_{2n})^2}{(25x_{1n} - 10u_{1n})^2 + (4u_{1n} - 10x_{1n})^2 + (9u_{2n})^2} & \text{if } n \in \Omega, \\ \frac{1}{1000000} & \text{if } n \notin \Omega. \end{cases}$$

then the Conditions (A1) – (A2), (B1) – (B4), and (R1) – (R2) are satisfied. The numerical experiments were carried out using MATLAB version R2020a and all programs were run on a 64-bit OS PC with Intel(R) Core(TM) i7-8550U CPU@1.80GHz 1.99GHz and 16GB RAM. The figures below show that the error term sequence $\{E_n\} = \{(x_n, u_n) - (\bar{x}, \bar{u})\}$ converges strongly to zero for different choices of the parameter l and different initial points.



$$((x_{1_0}, x_{2_0}), (u_{1_0}, u_{2_0})) = ((0.5, 0.5), (0.5, 1.5)), \quad \beta = 0.5, \quad \mu = 0.4, \quad \gamma = 0.5$$

Figure 5.5: Illustration of convergence rate of the sequence for different values of the parameter l .



$$\text{Parameters } l = 0.9, \quad \beta = 0.5, \quad \mu = 0.4, \quad \gamma = 0.5$$

Figure 5.6: Illustration of convergence rate of the sequence for different initial values.

Remark 5.3.2. *Figure 5.5 shows that the convergence rate of the method gets faster as l gets closer to 1. From Figure 5.6, we observe that for any choice of initial point, the sequence converges to a solution of the split equality of variational inequality and f, g –fixed point problem. That is, the ultimate convergence of the sequence does not depend on the choice of initial points.*



6. Approximating solutions of Split Equality Hammerstein-Type Equation Problems

This chapter consists of three sections. Some preliminary results are presented in the first section. In section two, we propose an iterative algorithm for solving split equality Hammerstein type equation problems which involves uniformly continuous monotone mappings in reflexive real Banach spaces and discuss its strong convergence analysis along with numerical illustrations. Section three deals with construction of an iterative algorithm for solving split equality Hammerstein type equation problems involving continuous monotone mappings in reflexive Banach spaces. A strong convergence theorem is proved under some appropriate conditions and numerical examples are provided to demonstrate the effectiveness of the proposed method.

6.1 Preliminaries

Lemma 6.1.1. [164] *Let E be a reflexive real Banach space and let $g: E \rightarrow (-\infty, +\infty]$ be a strongly coercive and strongly convex function with strongly convex conjugate g^* and let $x = (u, v) \in E \times E^*$. Then, the function $f: E \times E^* \rightarrow (-\infty, +\infty]$ defined by*

$$f(x) = g(u) + g^*(v),$$

is strongly coercive and strongly convex.

A mapping A from a Banach space E_1 into another Banach space E_2 is said to be *hemicontinuous* at $x_0 \in E_1$ if for any sequence $\{x_n\}$ converging to x_0 along a line implies $Ax_n \rightarrow Ax_0$, i.e., $Ax_n = A(x_0 + t_n x) \rightarrow Ax_0$ as $t_n \rightarrow 0$ for all $x \in E_1$.

Lemma 6.1.2. [111] *Let E be a reflexive real Banach space and let $A: E \rightarrow E^*$ be a hemicontinuous monotone mapping. Then, A is maximal monotone.*

Note that for any continuous monotone mapping A , the set $N(A) = \{x \in E : Ax = 0\}$ is closed and convex (see, e.g., [186]).

Lemma 6.1.3. [111] *Let X be a reflexive real Banach space and let $E = X \times X^*$ with norm $\|x\|_E^2 = \|u\|_X^2 + \|v\|_{X^*}^2$, where $x = (u, v) \in E$. If $F: X \rightarrow X^*$ and $K: X^* \rightarrow X$ are hemicontinuous monotone mappings, then the mapping $A: E \rightarrow E^*$ defined by $Ax = (Fu - v, Kv + u)$ is maximal monotone.*

Let $g: X \rightarrow (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. If g is strongly coercive, bounded and Legendre on bounded subsets of X , then its Fenchel conjugate $g^*: X^* \rightarrow (-\infty, +\infty]$ is also strongly coercive, bounded and Legendre on bounded subsets of X^* [164]. If we define $f: X \times X^* \rightarrow (-\infty, +\infty]$ by

$$f(x) = g(u) + g^*(v), \quad x = (u, v) \in X \times X^*,$$

then it can be easily verified that f is a bounded Legendre function on bounded subsets of $X \times X^*$. Moreover, if g is uniformly convex and uniformly Fréchet differentiable function, then f is Fréchet differentiable and $\nabla f x = (\nabla g u, \nabla g^* v)$ [164]. By Lemma 3.1.14 and Lemma 3.1.12, we have that ∇g and ∇g^* are uniformly continuous on bounded subsets of their domains and hence ∇f and ∇f^* are uniformly continuous. Inline with the argument in [133] (Lemma 1 and Lemma 2, page 130 – 131), we state the following lemma.

Lemma 6.1.4. *Let E be a reflexive real Banach space and $g: E \rightarrow (-\infty, +\infty]$ be a Legendre function. Let $A: C \rightarrow E^*$ be a continuous monotone mapping, where C is a nonempty, convex and closed subset of E . For $r > 0$ and $x \in E$, the following hold:*

1. *There exists $z \in C$ such that*

$$\langle Az, y - z \rangle + \frac{1}{r} \langle \nabla g z - \nabla g x, y - z \rangle \geq 0, \text{ for all } y \in C.$$

2. *Define the mapping $F_r^A: E \rightarrow C$ by*

$$F_r^A x = \left\{ z \in C : \langle Az, y - z \rangle + \frac{1}{r} \langle \nabla g z - \nabla g x, y - z \rangle \geq 0, \text{ for all } y \in C \right\},$$

for all $x \in E$. Let $F(F_r^A)$ be the set of fixed points of the mapping F_r^A and $VI(C, A)$ be the solution set of the variational inequality problem. Then the following hold:

- (i) F_r^A is single-valued;
- (ii) $F(F_r^A) = VI(C, A)$;
- (iii) $VI(C, A)$ is closed and convex;
- (iv) $D_g(p, F_r^A x) + D_g(F_r^A x, x) \leq D_g(p, x)$, for all $p \in F(F_r^A x)$, $x \in E$, provided that $F(F_r^A x) \neq \emptyset$.

6.2 Solutions of Split Equality Hammerstein Type Equation Problems Involving Uniformly Continuous Monotone Mappings in Reflexive Real Banach Spaces

In this section, we introduce an inertial iterative algorithm to approximate solutions of the split equality Hammerstein type equation problems involving uniformly continuous mappings in the reflexive real Banach space settings. We discuss its convergence analysis and validate its appropriateness by providing a numerical example.

6.2.1 Main Results

The following conditions will be used in establishing the convergence results of the method:

- (L1) Let X and Y be reflexive real Banach spaces with dual spaces X^* and Y^* , respectively;
- (L2) Let $F_1: X \rightarrow X^*$, $K_1: X^* \rightarrow X$, $F_2: Y \rightarrow Y^*$ and $K_2: Y^* \rightarrow Y$ be uniformly continuous monotone mappings;

- (L3) Let $T_1: X \rightarrow Z$ and $S_1: Y \rightarrow Z$ be bounded linear mappings with adjoints T_1^* and S_1^* , respectively, where Z is another reflexive real Banach space;
- (L4) Let $\Lambda = \{(u^*, p^*) \in X \times Y : u^* + K_1 F_1 u^* = 0, p^* + K_2 F_2 p^* = 0 \text{ and } T_1 u^* = S_1 p^*\} \neq \emptyset$;
- (L5) Let $g: X \rightarrow (-\infty, +\infty]$ and $h: Y \rightarrow (-\infty, +\infty]$ be strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre functions on bounded subsets with strongly convex conjugates g^* and h^* . Let the strong convexity constants of g, g^*, h and h^* be β_1, β_2, μ_1 and μ_2 , respectively, and let $\beta = \min\{\mu_1, \mu_2, \beta_1, \beta_2\}$;
- (L6) Let $\{\alpha_n\} \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (L7) Let $\{\zeta_n\}$ be a positive sequence such that $\zeta_n \in \left(0, \frac{\beta}{2}\right)$ and $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 6.2.1. *Let X be a real normed space with dual X^* . Let $F: X \rightarrow X^*$ and $K: X^* \rightarrow X$ be uniformly continuous mappings. Let $E = X \times X^*$ be with the norm $\|x\|_E^2 = \|u\|_X^2 + \|v\|_{X^*}^2$, where $x = (u, v) \in E$. Then, the mapping $A: E \rightarrow E^*$ defined by $Ax = A(u, v) = (Fu - v, Kv + u)$ is uniformly continuous.*

Proof. Let $x_n = (u_n, v_n), y_n = (w_n, z_n) \in E$ be such that $x_n - y_n \rightarrow 0$ as $n \rightarrow \infty$. That is, $u_n - w_n \rightarrow 0$ and $v_n - z_n \rightarrow 0$ as $n \rightarrow \infty$. Now,

$$\begin{aligned}
 \|Ax_n - Ay_n\| &= \|(Fu_n - v_n, Kv_n + u_n) - (Fw_n - z_n, Kz_n + w_n)\| \\
 &= \|(Fu_n - Fw_n + z_n - v_n, Kv_n - Kz_n + u_n - w_n)\| \\
 &= \left\{ \|Fu_n - Fw_n + z_n - v_n\|^2 + \|Kv_n - Kz_n + u_n - w_n\|^2 \right\}^{\frac{1}{2}} \\
 &\leq \left\{ [\|Fu_n - Fw_n\| + \|z_n - v_n\|]^2 + [\|Kv_n - Kz_n\| + \|u_n - w_n\|]^2 \right\}^{\frac{1}{2}}.
 \end{aligned} \tag{6.1}$$

Since F and K are uniformly continuous and the norm is a continuous function, taking limit as $n \rightarrow \infty$ of both sides of (6.1) yields that $\|Fu_n - Fw_n\| \rightarrow 0$ and $\|Kv_n - Kz_n\| \rightarrow 0$. Thus, we have that $\|Ax_n - Ay_n\| \rightarrow 0$ as $n \rightarrow \infty$ and hence A is uniformly continuous. $\square$

The following notations will be used in the understated algorithm.

$$\begin{aligned}
 \Theta_n &= \left\{ \|\nabla g u_n - \nabla g u_{n-1}\|^2 + \|\nabla g^* v_n - \nabla g^* v_{n-1}\|^2 \right\}^{\frac{1}{2}}, \\
 \Phi_n &= \left\{ \|\nabla h p_n - \nabla h p_{n-1}\|^2 + \|\nabla h^* q_n - \nabla h^* q_{n-1}\|^2 \right\}^{\frac{1}{2}}.
 \end{aligned}$$

Algorithm 6.1.

Initialization: Let $u_0, u_1, \in X, v_0, v_1, \in X^*, p_0, p_1, \in Y, q_0, q_1, \in Y^*, \theta \in [0, 1), \mu \in (0, \beta), l, \gamma, \in (0, 1)$. For $u \in X, v \in X^*, p \in Y$ and $q \in Y^*$, calculate $\{u_n\}, \{v_n\}, \{p_n\}$ and $\{q_n\}$ as follows:

Step 1: Let the iterates $u_{n-1}, u_n \in X$, $v_{n-1}, v_n \in X^*$, $p_{n-1}, p_n \in Y$ and $q_{n-1}, q_n \in Y^*$ be given. Choose θ_n such that $0 \leq \theta_n \leq \sigma_n$, where

$$\sigma_n = \begin{cases} \min \left\{ \theta, \frac{\zeta_n}{\Theta_n + \Phi_n} \right\}, & \text{if } \Theta_n + \Phi_n \neq 0 \\ \theta & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$\begin{aligned} e_{1n} &= \nabla g^* [\nabla g u_n + \theta_n (\nabla g u_n - \nabla g u_{n-1})], \\ e_{2n} &= \nabla g [\nabla g^* v_n + \theta_n (\nabla g^* v_n - \nabla g^* v_{n-1})], \\ h_{1n} &= \nabla h^* [\nabla h p_n + \theta_n (\nabla h p_n - \nabla h p_{n-1})], \\ h_{2n} &= \nabla h [\nabla h^* q_n + \theta_n (\nabla h^* q_n - \nabla h^* q_{n-1})]. \end{aligned}$$

Step 3: Compute

$$\begin{aligned} z_{1n} &= \nabla g^* [\nabla g e_{1n} - \gamma_n T_1^* J_Z (T_1 e_{1n} - S_1 h_{1n})], \\ r_{1n} &= \nabla h^* [\nabla h h_{1n} - \gamma_n S_1^* J_Z (S_1 h_{1n} - T_1 e_{1n})], \end{aligned}$$

where $0 < \rho \leq \gamma_n \leq \rho_1$ with

$$\rho_1 = \min \left\{ \rho + 1, \frac{\beta \|T_1 e_{1n} - S_1 h_{1n}\|^2}{2[\|T_1^* J_Z (T_1 e_{1n} - S_1 h_{1n})\|^2 + \|S_1^* J_Z (S_1 h_{1n} - T_1 e_{1n})\|^2]} \right\}$$

for $n \in \Omega = \{m \in \mathbb{N} : T_1 e_{1m} - S_1 h_{1m} \neq 0\}$, otherwise $\gamma_n = \rho$.

Step 4: Compute

$$y_{1n} = \nabla g^* [\nabla g z_{1n} - \lambda_n (F_1 z_{1n} - e_{2n})], \quad (6.2)$$

$$y_{2n} = \nabla g [\nabla g^* e_{2n} - \lambda_n (K_1 e_{2n} + z_{1n})], \quad (6.3)$$

$$t_{1n} = \nabla h^* [\nabla h r_{1n} - \eta_n (F_2 r_{1n} - h_{2n})],$$

$$t_{2n} = \nabla h [\nabla h^* h_{2n} - \eta_n (K_2 h_{2n} + r_{1n})],$$

$\lambda_n = \gamma l^{j_n}$ and $\eta_n = \gamma l^{m_n}$, where j_n is the smallest non-negative integer j satisfying

$$\begin{aligned} \gamma l^j [\|F_1 s_{n_j} - F_1 z_{1n} + e_{2n} - b_{n_j}\|^2 + \|K_1 b_{n_j} - K_1 e_{2n} + s_{n_j} - z_{1n}\|^2] \\ \leq \mu [\|s_{n_j} - z_{1n}\|^2 + \|b_{n_j} - e_{2n}\|^2], \end{aligned} \quad (6.4)$$

where $s_{n_j} = \nabla g^* [\nabla g z_{1n} - \gamma l^j (F_1 z_{1n} - e_{2n})]$ and $b_{n_j} = \nabla g [\nabla g^* e_{2n} - \gamma l^j (K_1 e_{2n} + z_{1n})]$ and m_n is the smallest non-negative integer m satisfying

$$\begin{aligned} \gamma l^m [\|F_2 c_{n_m} - F_2 r_{1n} + h_{2n} - d_{n_m}\|^2 + \|K_2 d_{n_m} - K_2 h_{2n} + c_{n_m} - r_{1n}\|^2] \\ \leq \mu [\|c_{n_m} - r_{1n}\|^2 + \|d_{n_m} - h_{2n}\|^2], \end{aligned} \quad (6.5)$$

where $c_{n_m} = \nabla h^* [\nabla h r_{1n} - \gamma l^m (F_2 r_{1n} - h_{2n})]$ and $d_{n_m} = \nabla h [\nabla h^* h_{2n} - \gamma l^m (K_2 h_{2n} + r_{1n})]$.

Step 5: Compute

$$\begin{aligned} a_{1n} &= \nabla g^* [\nabla g y_{1n} - \lambda_n (F_1 y_{1n} - F_1 z_{1n} + e_{2n} - y_{2n})], \\ a_{2n} &= \nabla g [\nabla g^* y_{2n} - \lambda_n (K_1 y_{2n} - K_1 e_{2n} + y_{1n} - z_{1n})], \\ k_{1n} &= \nabla h^* [\nabla h t_{1n} - \eta_n (F_2 t_{1n} - F_2 r_{1n} + h_{2n} - t_{2n})], \\ k_{2n} &= \nabla h [\nabla h^* t_{2n} - \eta_n (K_2 t_{2n} - K_2 h_{2n} + t_{1n} - r_{1n})]. \end{aligned}$$

Step 6: Compute

$$\begin{aligned} u_{n+1} &= \nabla g^* [\alpha_n \nabla g u + (1 - \alpha_n) \nabla g a_{1n}], \\ v_{n+1} &= \nabla g [\alpha_n \nabla g^* v + (1 - \alpha_n) \nabla g^* a_{2n}], \\ p_{n+1} &= \nabla h^* [\alpha_n \nabla h p + (1 - \alpha_n) \nabla h k_{1n}], \\ q_{n+1} &= \nabla h [\alpha_n \nabla h^* q + (1 - \alpha_n) \nabla h^* k_{2n}]. \end{aligned}$$

Remark 6.2.1. The proposed problem is a new problem which extends the existing HTEPs to the more general SEHTEP which encompasses the HTEPs as special cases.

Lemma 6.2.2. Assume that the conditions (L1)–(L7) hold. Then the Armijo line search rules (6.4) and (6.5) are well defined.

Proof. We consider two cases on z_{1n} and r_{1n} :

Case I. Assume that z_{1n} is a solution of the Hammerstein type equation $u + K_1 F_1 u = 0$. That is, $z_{1n} + K_1 F_1 z_{1n} = 0$ with $e_{2n} = F_1 z_{1n}$. Then, we have

$$(F_1 z_{1n} - e_{2n}, K_1 e_{2n} + z_{1n}) = (0, 0),$$

which implies that

$$F_1 z_{1n} - e_{2n} = 0, \tag{6.6}$$

and

$$K_1 e_{2n} + z_{1n} = 0. \tag{6.7}$$

From (6.6) and (6.7), it follows that (6.4) holds for $j = 0$.

Case II. Assume that z_{1n} is not a solution of the Hammerstein type equation $u + K_1 F_1 u = 0$ and assume on the contrary that for all j we have

$$\begin{aligned} & [\|F_1 s_{n_j} - F_1 z_{1n} + e_{2n} - b_{n_j}\|^2 + \|K_1 b_{n_j} - K_1 e_{2n} + s_{n_j} - z_{1n}\|^2] \\ & > \frac{\mu}{\gamma l^j} [\|s_{n_j} - z_{1n}\|^2 + \|b_{n_j} - e_{2n}\|^2], \end{aligned} \tag{6.8}$$

Since ∇g^* and ∇g are continuous, we have

$$\lim_{j \rightarrow \infty} \|s_{n_j} - z_{1n}\| = \lim_{j \rightarrow \infty} \|\nabla g^* [\nabla g z_{1n} - \gamma l^j (F_1 z_{1n} - e_{2n})] - z_{1n}\| = 0. \quad (6.9)$$

Similarly,

$$\lim_{j \rightarrow \infty} \|b_{n_j} - e_{2n}\| = 0. \quad (6.10)$$

Since F_1 and K_1 are uniformly continuous, we obtain from (6.9) and (6.10) that

$$\lim_{j \rightarrow \infty} (\|F_1 s_{n_j} - F_1 z_{1n} + e_{2n} - b_{n_j}\|^2 + \|K_1 b_{n_j} - K_1 e_{2n} + s_{n_j} - z_{1n}\|^2) = 0. \quad (6.11)$$

Combining (6.8) and (6.11), we obtain

$$\lim_{j \rightarrow \infty} \left(\frac{\mu}{\gamma l^j} [\|s_{n_j} - z_{1n}\|^2 + \|b_{n_j} - e_{2n}\|^2] \right) = 0,$$

from which we obtain

$$\lim_{j \rightarrow \infty} \left(\frac{s_{n_j} - z_{1n}}{\gamma l^j} \right) = \lim_{j \rightarrow \infty} \left(\frac{b_{n_j} - e_{2n}}{\gamma l^j} \right) = 0. \quad (6.12)$$

The Lipschitz continuity of ∇g together with (6.12) gives

$$\lim_{j \rightarrow \infty} \left(\frac{\nabla g s_{n_j} - \nabla g z_{1n}}{\gamma l^j} \right) = 0. \quad (6.13)$$

Since $s_{n_j} \in X$, one can write $s_{n_j} = P_X^g \nabla g^* [\nabla g z_{1n} - \gamma l^j (F_1 z_{1n} - e_{2n})]$ and thus we have by (1.18) that

$$\langle \nabla g z_{1n} - \gamma l^j (F_1 z_{1n} - e_{2n}) - \nabla g s_{n_j}, y - s_{n_j} \rangle \leq 0, \text{ for all } y \in X, \quad (6.14)$$

which implies that

$$\left\langle \frac{\nabla g z_{1n} - \nabla g s_{n_j}}{\gamma l^j}, y - s_{n_j} \right\rangle - \langle F_1 z_{1n} - e_{2n}, y - s_{n_j} \rangle \leq 0, \text{ for all } y \in X. \quad (6.15)$$

Taking limit as $j \rightarrow \infty$ in (6.15) and using (6.13), we obtain

$$- \lim_{j \rightarrow \infty} \langle F_1 z_{1n} - e_{2n}, y - s_{n_j} \rangle \leq 0, \text{ for all } y \in X. \quad (6.16)$$

Taking $y = -J^{-1}(F_1 z_{1n} - e_{2n}) + s_{n_j}$ in (6.16), we obtain $\langle J^{-1}(F_1 z_{1n} - e_{2n}), F_1 z_{1n} - e_{2n} \rangle = \|F_1 z_{1n} - e_{2n}\|^2 \leq 0$ which implies that

$$F_1 z_{1n} - e_{2n} = 0. \quad (6.17)$$

Similarly, we can show that

$$K_1 e_{2n} + z_{1n} = 0. \quad (6.18)$$

Combining (6.17) and (6.18), we get

$$(F_1 z_{1n} - e_{2n}, K_1 e_{2n} + z_{1n}) = (0, 0), \quad (6.19)$$

which means that z_{1n} is a solution of the Hammerstein type equation $u + K_1 F_1 u = 0$ which is a contradiction. Thus, (6.4) holds. Considering similar cases on r_{1n} , for each $n \geq 1$, it can be shown that (6.5) holds and hence the proof is complete. $\square$

Theorem 6.2.1. *Assume that the conditions (L1) – (L7) hold. Then the sequences $\{u_n\} \subseteq X$, $\{v_n\} \subseteq X^*$, $\{p_n\} \subseteq Y$ and $\{q_n\} \subseteq Y^*$ generated by Algorithm 6.1 are bounded.*

Proof. We have from Lemma 6.1.2 that F_1 and K_1 are maximal monotone mappings. If we define a norm on $E_1 = X \times X^*$ by $\|x\|_{E_1}^2 = \|u\|_X^2 + \|v\|_{X^*}^2$, then we have by Lemma 6.1.3 that the mapping $A_1: E_1 \rightarrow E_1^*$ defined by $A_1 x = (F_1 u - v, K_1 v + u)$, where $x = (u, v) \in E_1$, is a maximal monotone. Similarly, the mapping $A_2: E_2 \rightarrow E_2^*$ given by $A_2 w = (F_2 p - q, K_2 q + p)$, where $w = (p, q) \in E_2 = Y \times Y^*$, is a maximal monotone mapping. If we let $E_3 = Z \times Z^*$ and let $T: E_1 \rightarrow E_3$, $S: E_2 \rightarrow E_3$ be mappings defined by $T = (T_1, 0)$ and $S = (S_1, 0)$, then Algorithm 6.1 can be rewritten as follows:

Initialization: Let $x_0, x_1 \in E_1$, $w_0, w_1 \in E_2$, $\theta \in [0, 1)$, $\mu \in (0, \beta)$, $l, \gamma \in (0, 1)$. For $x = (u, v) \in E_1$ and $w = (p, q) \in E_2$, calculate (x_n, w_n) as follows:

Step 1: Given the current iterates $x_{n-1}, x_n \in E_1$ and $w_{n-1}, w_n \in E_2$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\zeta_n}{\Theta_n + \Phi_n} \right\}, & \text{if } \Theta_n + \Phi_n \neq 0 \\ \theta & \text{otherwise,} \end{cases} \quad (6.20)$$

Step 2: Compute

$$\begin{aligned} e_n &= \nabla f^* [\nabla f x_n + \theta_n (\nabla f x_n - \nabla f x_{n-1})], \\ h_n &= \nabla s^* [\nabla s w_n + \theta_n (\nabla s w_n - \nabla s w_{n-1})]. \end{aligned} \quad (6.21)$$

Step 3: Compute

$$\begin{aligned} z_n &= \nabla f^* [\nabla f e_n - \gamma_n T^* J_{E_3} (T e_n - S h_n)], \\ r_n &= \nabla s^* [\nabla s h_n - \gamma_n S^* J_{E_3} (S h_n - T e_n)]. \end{aligned} \quad (6.22)$$

Step 4: Compute

$$\begin{aligned} y_n &= \nabla f^* [\nabla f z_n - \lambda_n A_1 z_n], \\ t_n &= \nabla s^* [\nabla s r_n - \eta_n A_2 r_n], \end{aligned} \quad (6.23)$$

for $\lambda_n = \gamma l^{j_n}$ and $\eta_n = \gamma l^{m_n}$, where j_n and m_n are the smallest non negative integers j and m satisfying the relations

$$\begin{aligned} \gamma l^j \|A_1 y_n - A_1 z_n\| &\leq \mu \|y_n - z_n\| \text{ and} \\ \gamma l^m \|A_2 t_n - A_2 r_n\| &\leq \mu \|t_n - r_n\|, \end{aligned} \quad (6.24)$$

respectively.

Step 5: Compute

$$\begin{aligned} a_n &= \nabla f^* \left[\nabla f y_n - \lambda_n (A_1 y_n - A_1 z_n) \right], \\ k_n &= \nabla s^* \left[\nabla s t_n - \eta_n (A_2 t_n - A_2 r_n) \right]. \end{aligned} \quad (6.25)$$

Step 6: Compute

$$\begin{aligned} x_{n+1} &= \nabla f^* \left[(1 - \alpha_n) \nabla f a_n + \alpha_n \nabla f x \right], \\ w_{n+1} &= \nabla s^* \left[(1 - \alpha_n) \nabla s k_n + \alpha_n \nabla s w \right]. \end{aligned} \quad (6.26)$$

Set $n = n + 1$ and go to **Step 1**.

where $f: E_1 \rightarrow (-\infty, +\infty]$ and $s: E_2 \rightarrow (-\infty, +\infty]$ are defined by $f(u, v) = g(u) + g^*(v)$ and $s(p, q) = h(p) + h^*(q)$. The functions f and s are uniformly Fréchet differentiable Legendre functions which are bounded on bounded subsets of E_1 and E_2 . By Lemma 6.1.1, we have that f and s are strongly coercive and strongly convex with constant β .

Let us denote $x_n = (u_n, v_n)$ and $w_n = (p_n, q_n)$. Now, let $(u^*, p^*) \in \Lambda$. We recall that $x^* = (u^*, v^*)$ solves $A_1 x = 0$ if and only if u^* is a solution of the Hammerstein equation $u + K_1 F_1 u = 0$, where $v^* = F_1 u^*$. So, the set of null points of A_1 is nonempty. Similarly, it can be shown that the set of null points of A_2 is nonempty. In addition to this, we have $T_1 u^* = S_1 p^*$. Thus, $\Pi = \{(x^*, w^*) \in N(A_1) \times N(A_2) : Tx^* = Sw^*\}$ is nonempty.

Now, let $(\hat{x}, \hat{w}) \in \Pi$. From Lemma 3.1.9 and (6.26), we obtain that

$$\begin{aligned} D_f(\hat{x}, x_{n+1}) &= D_f(\hat{x}, \nabla f^* (\alpha_n \nabla f x + (1 - \alpha_n) \nabla f a_n)) \\ &\leq \alpha_n D_f(\hat{x}, x) + (1 - \alpha_n) D_f(\hat{x}, a_n). \end{aligned} \quad (6.27)$$

From (6.25) and Definition 1.2.17, we obtain

$$\begin{aligned} D_f(\hat{x}, a_n) &= D_f(\hat{x}, \nabla f^* (\nabla f y_n - \lambda_n (A_1 y_n - A_1 z_n))) \\ &= f(\hat{x}) - \langle \nabla f y_n - \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle - f(a_n) \\ &= f(\hat{x}) + \langle \nabla f y_n, a_n - \hat{x} \rangle + \langle \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle - f(a_n) \\ &= f(\hat{x}) - \langle \nabla f y_n, \hat{x} - y_n \rangle - f(y_n) + \langle \nabla f y_n, \hat{x} - y_n \rangle + f(y_n) \\ &\quad + \langle \nabla f y_n, a_n - \hat{x} \rangle + \langle \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle - f(a_n) \\ &= D_f(\hat{x}, y_n) + \langle \nabla f y_n, a_n - y_n \rangle + f(y_n) - f(a_n) + \langle \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle \\ &= D_f(\hat{x}, y_n) - D_f(a_n, y_n) + \langle \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle. \end{aligned} \quad (6.28)$$

From (1.16), we have that

$$D_f(\hat{x}, y_n) - D_f(a_n, y_n) = D_f(\hat{x}, z_n) - D_f(a_n, z_n) + \langle \nabla f z_n - \nabla f y_n, \hat{x} - a_n \rangle. \quad (6.29)$$

Combining (6.28) and (6.29), we obtain

$$D_f(\hat{x}, a_n) = D_f(\hat{x}, z_n) - D_f(a_n, z_n) + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle + \langle \nabla f z_n - \nabla f y_n, \hat{x} - a_n \rangle. \quad (6.30)$$

From (6.22), (1.21) and (1.22), we have

$$\begin{aligned} D_f(\hat{x}, z_n) &= D_f(\hat{x}, \nabla f^*(\nabla f e_n - \gamma_n T^* J_{E_3}(T e_n - S h_n))) \\ &\leq V_f(\hat{x}, \nabla f e_n) - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle \\ &= D_f(\hat{x}, e_n) - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle. \end{aligned} \quad (6.31)$$

Substituting (6.31) into (6.30), we get

$$\begin{aligned} D_f(\hat{x}, a_n) &\leq D_f(\hat{x}, e_n) - D_f(a_n, z_n) + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle \\ &\quad + \langle \nabla f z_n - \nabla f y_n, \hat{x} - a_n \rangle - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle. \end{aligned} \quad (6.32)$$

From (1.15), we obtain

$$D_f(a_n, z_n) = D_f(a_n, y_n) + D_f(y_n, z_n) - \langle \nabla f y_n - \nabla f z_n, y_n - a_n \rangle. \quad (6.33)$$

Combining (6.32) and (6.33), we obtain

$$\begin{aligned} D_f(\hat{x}, a_n) &\leq D_f(\hat{x}, e_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\ &\quad + \langle \nabla f y_n - \nabla f z_n, y_n - a_n \rangle + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle \\ &\quad + \langle \nabla f z_n - \nabla f y_n, \hat{x} - a_n \rangle - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle \\ &= D_f(\hat{x}, e_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \langle \nabla f y_n - \nabla f z_n, y_n - \hat{x} \rangle \\ &\quad + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle. \end{aligned} \quad (6.34)$$

From (6.21) and Definition 1.2.17, we have

$$\begin{aligned} D_f(\hat{x}, e_n) &= D_f(\hat{x}, \nabla f^*(\nabla f x_n + \theta_n(\nabla f x_n - \nabla f x_{n-1}))) \\ &= f(\hat{x}) - \langle \nabla f x_n + \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle - f(e_n) \\ &= f(\hat{x}) - \langle \nabla f x_n, \hat{x} - e_n \rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle - f(e_n) \\ &= f(\hat{x}) - \langle \nabla f x_n, \hat{x} - x_n \rangle - f(x_n) + \langle \nabla f x_n, \hat{x} - x_n \rangle + f(x_n) \\ &\quad - \langle \nabla f x_n, \hat{x} - e_n \rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle - f(e_n) \\ &= D_f(\hat{x}, x_n) + \langle \nabla f x_n, e_n - x_n \rangle + f(x_n) - f(e_n) \\ &\quad - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle \\ &= D_f(\hat{x}, x_n) - D_f(e_n, x_n) - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle. \end{aligned} \quad (6.35)$$

Substituting (6.35) into (6.34), we obtain

$$\begin{aligned}
 D_f(\hat{x}, a_n) &\leq D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n) \\
 &\quad + \langle \nabla f y_n - \nabla f z_n, y_n - \hat{x} \rangle + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle \\
 &\quad - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle \\
 &= D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n) \\
 &\quad + \langle \nabla f y_n - \nabla f z_n, y_n - \hat{x} \rangle + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - y_n + y_n - a_n \rangle \\
 &\quad - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle \\
 &= D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n) \\
 &\quad + \langle \nabla f y_n - \nabla f z_n, y_n - \hat{x} \rangle + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - y_n \rangle \\
 &\quad + \langle \lambda_n(A_1 y_n - A_1 z_n), y_n - a_n \rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle \\
 &\quad - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle \\
 &= D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n) \\
 &\quad + \langle \lambda_n(A_1 y_n - A_1 z_n), y_n - a_n \rangle - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle \\
 &\quad - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle \\
 &\quad - \langle \lambda_n(A_1 y_n - A_1 z_n) - (\nabla f y_n - \nabla f z_n), y_n - \hat{x} \rangle.
 \end{aligned} \tag{6.36}$$

Since $\hat{x} \in N(A_1)$, we have from (6.23) that

$$\begin{aligned}
 &\langle \lambda_n(A_1 y_n - A_1 z_n) - (\nabla f y_n - \nabla f z_n), y_n - \hat{x} \rangle \\
 &= \langle \lambda_n A_1 y_n - (\nabla f z_n - \nabla f y_n) - (\nabla f y_n - \nabla f z_n), y_n - \hat{x} \rangle \\
 &= \langle \lambda_n A_1 y_n, y_n - \hat{x} \rangle \geq 0,
 \end{aligned}$$

where the last inequality follows by the virtue of monotonicity of A_1 . Thus, the relation (6.36) can be simplified to

$$\begin{aligned}
 D_f(\hat{x}, a_n) &\leq D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n) \\
 &\quad + \langle \lambda_n(A_1 y_n - A_1 z_n), y_n - a_n \rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle \\
 &\quad - \langle \gamma_n T^* J_{E_3}(T e_n - S h_n), z_n - \hat{x} \rangle.
 \end{aligned} \tag{6.37}$$

Moreover, by the Cauchy–Schwarz inequality and Lemma 3.1.10 we have

$$\begin{aligned}
 \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle &\leq \theta_n \|\nabla f x_n - \nabla f x_{n-1}\| \|(\hat{x} - e_n) \times 1\| \\
 &\leq \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| [\|\hat{x} - e_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| [2\|\hat{x} - x_n\|^2 + 2\|x_n - e_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| \left[\frac{4}{\beta} D_f(\hat{x}, x_n) + \frac{4}{\beta} D_f(e_n, x_n) + 1 \right] \\
 &= \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(\hat{x}, x_n) \\
 &\quad + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(e_n, x_n) + \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\|.
 \end{aligned} \tag{6.38}$$

Substituting (6.38) into (6.37) and using (6.20) and (6.24) and the Cauchy–Schwarz Inequality, we obtain

$$\begin{aligned}
 D_f(\hat{x}, a_n) &\leq D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n) + \lambda_n \|A_1 y_n - A_1 z_n\| \|y_n - a_n\| \\
 &\quad + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(\hat{x}, x_n) + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(e_n, x_n) \\
 &\quad + \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle \\
 &\leq D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n) + \mu \|y_n - z_n\| \|y_n - a_n\| \\
 &\quad + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(\hat{x}, x_n) + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(e_n, x_n) \\
 &\quad + \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle \\
 &\leq D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n) + \frac{\mu}{\beta} D_f(y_n, z_n) + \frac{\mu}{\beta} D_f(a_n, y_n) \\
 &\quad + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(\hat{x}, x_n) + \frac{2\theta_n}{\beta} \|\nabla f x_n - \nabla f x_{n-1}\| D_f(e_n, x_n) \\
 &\quad + \frac{\theta_n}{2} \|\nabla f x_n - \nabla f x_{n-1}\| - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle \\
 &\leq D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n) + \frac{\mu}{\beta} D_f(y_n, z_n) + \frac{\mu}{\beta} D_f(a_n, y_n) \\
 &\quad + \frac{2\zeta_n}{\beta} D_f(\hat{x}, x_n) + \frac{2\zeta_n}{\beta} D_f(e_n, x_n) + \frac{\zeta_n}{2} - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle,
 \end{aligned}$$

that is,

$$\begin{aligned}
 D_f(\hat{x}, a_n) &\leq \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(\hat{x}, x_n) - \left(1 - \frac{2\zeta_n}{\beta}\right) D_f(e_n, x_n) - \left(1 - \frac{\mu}{\beta}\right) D_f(a_n, y_n) \\
 &\quad - \left(1 - \frac{\mu}{\beta}\right) D_f(y_n, z_n) + \frac{\zeta_n}{2} - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle.
 \end{aligned} \tag{6.39}$$

Since $\zeta_n \in \left(0, \frac{\beta}{2}\right)$ and $\mu \in (0, \beta)$, it follows from (6.39) that

$$D_f(\hat{x}, a_n) \leq \left(1 + \frac{2\zeta_n}{\beta}\right) D_f(\hat{x}, x_n) + \frac{\zeta_n}{2} - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle. \tag{6.40}$$

Substituting (6.39) into (6.27), we obtain that

$$\begin{aligned}
 D_f(\hat{x}, x_{n+1}) &\leq \alpha_n D_f(\hat{x}, x) + (1 - \alpha_n) \left[\left(1 + \frac{2\zeta_n}{\beta}\right) D_f(\hat{x}, x_n) - \left(1 - \frac{2\zeta_n}{\beta}\right) D_f(e_n, x_n) \right] \\
 &\quad - (1 - \alpha_n) \left[\left(1 - \frac{\mu}{\beta}\right) D_f(a_n, y_n) + \left(1 - \frac{\mu}{\beta}\right) D_f(y_n, z_n) - \frac{\zeta_n}{2} \right] \\
 &\quad - (1 - \alpha_n) \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle.
 \end{aligned} \tag{6.41}$$

Similarly, we get

$$\begin{aligned}
 D_s(\hat{w}, w_{n+1}) &\leq \alpha_n D_s(\hat{w}, w) + (1 - \alpha_n) \left[\left(1 + \frac{2\zeta_n}{\beta}\right) D_s(\hat{w}, w_n) - \left(1 - \frac{2\zeta_n}{\beta}\right) D_s(h_n, w_n) \right] \\
 &\quad - (1 - \alpha_n) \left[\left(1 - \frac{\mu}{\beta}\right) D_s(k_n, t_n) + \left(1 - \frac{\mu}{\beta}\right) D_s(t_n, r_n) - \frac{\zeta_n}{2} \right] \\
 &\quad - (1 - \alpha_n) \langle \gamma_n S^* J_{E_3}(Sh_n - Te_n), r_n - \hat{w} \rangle.
 \end{aligned} \tag{6.42}$$

Let $\Omega_n = D_f(\hat{x}, x_n) + D_s(\hat{w}, w_n)$ and $\Sigma = D_f(\hat{x}, x) + D_s(\hat{w}, w)$. Then, combining (6.41) and (6.42), we obtain

$$\begin{aligned}
 \Omega_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) \Omega_n + (1 - \alpha_n) \zeta_n \\
 &\quad - (1 - \alpha_n) \gamma_n \langle J_{E_3}(Te_n - Sh_n), Tz_n - Sr_n \rangle.
 \end{aligned} \tag{6.43}$$

But,

$$\begin{aligned}
 -\langle J_{E_3}(Te_n - Sh_n), Tz_n - Sr_n \rangle &= -\langle J_{E_3}(Te_n - Sh_n), Te_n - Sh_n \rangle \\
 &\quad - \langle J_{E_3}(Te_n - Sh_n), Tz_n - Te_n \rangle \\
 &\quad - \langle J_{E_3}(Te_n - Sh_n), Sh_n - Sr_n \rangle \\
 &= -\|Te_n - Sh_n\|^2 - \langle T^* J_{E_3}(Te_n - Sh_n), z_n - e_n \rangle \\
 &\quad - \langle S^* J_{E_3}(Te_n - Sh_n), h_n - r_n \rangle \\
 &\leq -\|Te_n - Sh_n\|^2 + \|z_n - e_n\| \|T^* J_{E_3}(Te_n - Sh_n)\| \\
 &\quad + \|h_n - r_n\| \|S^* J_{E_3}(Te_n - Sh_n)\|.
 \end{aligned} \tag{6.44}$$

From the strong convexity of f and the definition of z_n , we have

$$\begin{aligned}
 \|z_n - e_n\| &= \|\nabla f^*(\nabla f(e_n) - \gamma_n T^* J_{E_3}(Te_n - Sh_n)) - \nabla f^*(\nabla f(e_n))\| \\
 &\leq \frac{\gamma_n}{\beta} \|T^* J_{E_3}(Te_n - Sh_n)\|.
 \end{aligned} \tag{6.45}$$

Similarly, the strong convexity of s and the definition of r_n imply that

$$\begin{aligned}
 \|r_n - h_n\| &= \|\nabla s^*(\nabla sh_n - \gamma_n S^* J_{E_3}(Sh_n - Te_n)) - \nabla s^*(\nabla sh_n)\| \\
 &\leq \frac{\gamma_n}{\beta} \|S^* J_{E_3}(Sh_n - Te_n)\|.
 \end{aligned} \tag{6.46}$$

Substituting (6.45) and (6.46) into (6.44) and applying the condition on γ_n , we get

$$\begin{aligned}
 -\gamma_n \langle J_{E_3}(Te_n - Sh_n), Tz_n - Sr_n \rangle &\leq -\gamma_n \|Te_n - Sh_n\|^2 + \frac{\gamma_n^2}{\beta} \|T^* J_{E_3}(Te_n - Sh_n)\|^2 \\
 &\quad + \frac{\gamma_n^2}{\beta} \|S^* J_{E_3}(Sh_n - Te_n)\|^2 \\
 &\leq -\frac{\rho}{2} \|Te_n - Sh_n\|^2 - \frac{\gamma_n}{2} \|Te_n - Sh_n\|^2 \\
 &\quad + \frac{\gamma_n}{2} \left\{ \frac{2\gamma_n}{\beta} \|T^* J_{E_3}(Te_n - Sh_n)\|^2 \right\} \\
 &\quad + \frac{\gamma_n}{2} \left\{ \frac{2\gamma_n}{\beta} \|S^* J_{E_3}(Sh_n - Te_n)\|^2 \right\} \\
 &\leq -\frac{\rho}{2} \|Te_n - Sh_n\|^2.
 \end{aligned} \tag{6.47}$$

Let $\varepsilon \in \left(0, \frac{\beta}{2}\right)$. Since $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$, there exists a natural number $\mathcal{N}$ such that $\zeta_n < \alpha_n \varepsilon$ for all $n \geq \mathcal{N}$. So, combining (6.47) and (6.43), we obtain

$$\begin{aligned}
 \Omega_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta}\right) \Omega_n + (1 - \alpha_n) \zeta_n - (1 - \alpha_n) \frac{\rho}{2} \|Te_n - Sh_n\|^2 \\
 &\leq \alpha_n \Sigma + (1 - \alpha_n) \Omega_n + \frac{2\varepsilon \alpha_n}{\beta} \Omega_n + \varepsilon \alpha_n \\
 &= \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] \Omega_n + \alpha_n [\Sigma + \varepsilon] \\
 &= \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] \Omega_n + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) \left[\frac{\beta \Sigma}{\beta - 2\varepsilon} + \frac{\beta \varepsilon}{\beta - 2\varepsilon}\right] \\
 &\leq \max \left\{ \Omega_n, \frac{\beta \Sigma}{\beta - 2\varepsilon} + \frac{\beta \varepsilon}{\beta - 2\varepsilon} \right\},
 \end{aligned}$$

for all $n \geq \mathcal{N}$ from which we obtain by the Principle of Mathematical Induction

$$\Omega_n \leq \max \left\{ \Omega_{\mathcal{N}}, \frac{\beta \Sigma}{\beta - 2\varepsilon} + \frac{\beta \varepsilon}{\beta - 2\varepsilon} \right\}, \text{ for all } n \geq \mathcal{N},$$

which in turn implies that

$$\Omega_n \leq \max \left\{ \mathcal{M}, \Omega_{\mathcal{N}}, \frac{\beta \Sigma}{\beta - 2\varepsilon} + \frac{\beta \varepsilon}{\beta - 2\varepsilon} \right\}, \text{ for all } n \geq 1,$$

where $\mathcal{M} = \max \{\Omega_1, \Omega_2, \dots, \Omega_{\mathcal{N}-1}\}$. Thus, $\{\Omega_n\}$ is a bounded sequence. This in turn implies that the sequences $\{D_f(\hat{x}, x_n)\}$ and $\{D_f(\hat{w}, w_n)\}$ are bounded. So, by Lemma 3.1.11 we have that $\{x_n\}$ and $\{w_n\}$ are bounded and hence $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ are bounded sequences. $\square$

Theorem 6.2.2. *Assume that the conditions (L1) – (L7) hold. Then the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.1 converge strongly to $\hat{u}$, $\hat{v}$, $\hat{p}$ and $\hat{q}$, respectively, where $(\hat{u}, \hat{p}) \in \Lambda$, $\hat{v} = F_1 \hat{u}$ and $\hat{q} = F_2 \hat{p}$.*

Proof. Let $\Pi = \{(x^*, w^*) \in N(A_1) \times N(A_2) : Tx^* = Sw^*\}$, where A_1 , A_2 , T and S are defined

as in the proof of Theorem 6.2.1. Now, let $(\dot{x}, \dot{w}) = P_{\Pi}^{\bar{f}}(x, w)$, where $\bar{f} = (f, s)$, $x = (u, v)$, $w = (p, q)$, $\dot{x} = (\dot{u}, \dot{v})$ and $\dot{w} = (\dot{p}, \dot{q})$. Then, by (1.18), we have

$$\langle (\nabla f x, \nabla s w) - (\nabla f \dot{x}, \nabla s \dot{w}), (z, r) - (\dot{x}, \dot{w}) \rangle \leq 0, \quad (6.48)$$

for all $(z, r) \in \Pi$. From (6.26), (1.21), (1.22) and Lemma 3.1.9, we get

$$\begin{aligned} D_f(\dot{x}, x_{n+1}) &= D_f(\dot{x}, \nabla f^*(\alpha_n \nabla f x + (1 - \alpha_n) \nabla f a_n)) \\ &\leq V_f(\dot{x}, \alpha_n \nabla f x + (1 - \alpha_n) \nabla f a_n - \alpha_n (\nabla f x - \nabla f \dot{x})) \\ &\quad + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_{n+1} - \dot{x} \rangle \\ &= V_f(\dot{x}, \alpha_n \nabla f \dot{x} + (1 - \alpha_n) \nabla f a_n) + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_{n+1} - \dot{x} \rangle \\ &= D_f(\dot{x}, \nabla f^*(\alpha_n \nabla f \dot{x} + (1 - \alpha_n) \nabla f a_n)) + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_{n+1} - \dot{x} \rangle. \end{aligned} \quad (6.49)$$

We obtain from Lemma 3.1.9, (6.49), (6.40) and the fact $D_f(\dot{x}, \dot{x}) = 0$ that

$$\begin{aligned} D_f(\dot{x}, x_{n+1}) &\leq (1 - \alpha_n) D_f(\dot{x}, a_n) + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_{n+1} - \dot{x} \rangle \\ &\leq (1 - \alpha_n) \left[\left(1 + \frac{2\zeta_n}{\beta} \right) D_f(\dot{x}, x_n) + \frac{\zeta_n}{2} - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \dot{x} \rangle \right] \\ &\quad + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_{n+1} - \dot{x} \rangle \\ &= (1 - \alpha_n) \left(1 + \frac{2\zeta_n}{\beta} \right) D_f(\dot{x}, x_n) + (1 - \alpha_n) \frac{\zeta_n}{2} \\ &\quad - (1 - \alpha_n) \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \dot{x} \rangle + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_{n+1} - \dot{x} \rangle \\ &\leq (1 - \alpha_n) D_f(\dot{x}, x_n) + \frac{2\zeta_n}{\beta} D_f(\dot{x}, x_n) + \frac{\zeta_n}{2} \\ &\quad - (1 - \alpha_n) \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \dot{x} \rangle + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_{n+1} - \dot{x} \rangle. \end{aligned} \quad (6.50)$$

Since $\frac{\zeta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$, for every $\varepsilon \in \left(0, \frac{\beta}{2}\right)$ there exists $n_0 \in \mathbb{N}$ such that $\zeta_n \leq \varepsilon \alpha_n$, for all $n \geq n_0$. Thus, we obtain from (6.50) for all $n \geq n_0$ that

$$\begin{aligned} D_f(\dot{x}, x_{n+1}) &\leq (1 - \alpha_n) D_f(\dot{x}, x_n) + \frac{2\varepsilon \alpha_n}{\beta} D_f(\dot{x}, x_n) + \frac{\zeta_n}{2} \\ &\quad - (1 - \alpha_n) \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \dot{x} \rangle + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_{n+1} - \dot{x} \rangle \\ &= \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta} \right) \right] D_f(\dot{x}, x_n) + \frac{\zeta_n}{2} + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_{n+1} - \dot{x} \rangle \\ &\quad - (1 - \alpha_n) \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \dot{x} \rangle, \\ &\leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta} \right) \right] D_f(\dot{x}, x_n) + \alpha_n \|\nabla f x - \nabla f \dot{x}\| \|x_{n+1} - x_n\| + \frac{\alpha_n \zeta_n}{2\alpha_n} \\ &\quad + \alpha_n \langle \nabla f x - \nabla f \dot{x}, x_n - \dot{x} \rangle - (1 - \alpha_n) \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \dot{x} \rangle, \end{aligned}$$

and thus

$$\begin{aligned}
 D_f(\hat{x}, x_{n+1}) &\leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(\hat{x}, x_n) - (1 - \alpha_n) \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle \\
 &\quad + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) \left[\frac{\beta \|\nabla f x - \nabla f \hat{x}\| \|x_{n+1} - x_n\|}{\beta - 2\varepsilon} \right] \\
 &\quad + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) \left[\frac{\beta \langle \nabla f x - \nabla f \hat{x}, x_n - \hat{x} \rangle}{\beta - 2\varepsilon} + \frac{\beta \zeta_n}{(2\beta - 4\varepsilon) \alpha_n} \right] \\
 &= \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(\hat{x}, x_n) + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) \Delta_n \\
 &\quad - (1 - \alpha_n) \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle,
 \end{aligned} \tag{6.51}$$

where

$$\Delta_n = \left[\frac{\beta \|\nabla f x - \nabla f \hat{x}\| \|x_{n+1} - x_n\|}{\beta - 2\varepsilon} + \frac{\beta \langle \nabla f x - \nabla f \hat{x}, x_n - \hat{x} \rangle}{\beta - 2\varepsilon} + \frac{\beta \zeta_n}{(2\beta - 4\varepsilon) \alpha_n} \right].$$

Similarly, we obtain

$$\begin{aligned}
 D_s(\hat{w}, w_{n+1}) &\leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_s(\hat{w}, w_n) + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) \Upsilon_n \\
 &\quad - (1 - \alpha_n) \langle \gamma_n S^* J_{E_3}(Sh_n - Te_n), r_n - \hat{w} \rangle,
 \end{aligned} \tag{6.52}$$

where

$$\Upsilon_n = \left[\frac{\beta \|\nabla s w - \nabla s \hat{w}\| \|w_{n+1} - w_n\|}{\beta - 2\varepsilon} + \frac{\beta \langle \nabla s w - \nabla s \hat{w}, w_n - \hat{w} \rangle}{\beta - 2\varepsilon} + \frac{\beta \zeta_n}{(2\beta - 4\varepsilon) \alpha_n} \right].$$

Denote $\Sigma^* = D_f(\hat{x}, x) + D_s(\hat{w}, w)$ and $\Omega_n^* = D_f(\hat{x}, x_n) + D_s(\hat{w}, w_n)$. Now, combining (6.51) and (6.52) and using the relation (6.47), we obtain

$$\begin{aligned}
 \Omega_{n+1}^* &\leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] \Omega_n^* + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) (\Delta_n + \Upsilon_n) - (1 - \alpha_n) \frac{\rho}{2} \|Te_n - Sh_n\|^2 \\
 &\leq \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right)\right] \Omega_n^* + \alpha_n \left(1 - \frac{2\varepsilon}{\beta}\right) (\Delta_n + \Upsilon_n).
 \end{aligned} \tag{6.53}$$

Adding the relations (6.41) and (6.42) with $\hat{x} = \hat{x}$ and $\hat{w} = \hat{w}$ and using (6.47), we obtain after rearranging terms that

$$\begin{aligned}
 &(1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) D_f(e_n, x_n) + (1 - \alpha_n) \left(1 - \frac{\mu}{\beta}\right) D_f(a_n, y_n) \\
 &+ (1 - \alpha_n) \left(1 - \frac{\mu}{\beta}\right) D_f(y_n, z_n) + (1 - \alpha_n) \left(1 - \frac{2\zeta_n}{\beta}\right) D_s(h_n, w_n) \\
 &+ (1 - \alpha_n) \left(1 - \frac{\mu}{\beta}\right) D_s(k_n, t_n) + (1 - \alpha_n) \left(1 - \frac{\mu}{\beta}\right) D_s(t_n, r_n) \\
 &+ \frac{\rho}{2} \|Te_n - Sh_n\|^2 \leq \Omega_n^* - \Omega_{n+1}^* + \alpha_n \left[\Sigma^* + \left(\frac{2\varepsilon}{\beta} - 1\right) \Omega_n^* + \frac{\zeta_n}{\alpha_n} \right].
 \end{aligned} \tag{6.54}$$

For the rest of the proof, we consider two cases on the sequence $\{\Omega_n^*\}$.

Case I. Suppose that the sequence $\{\Omega_n^*\}$ is nonincreasing. It then follows from the Monotone Convergence Theorem that $\{\Omega_n^*\}$ is convergent and thus $\lim_{n \rightarrow \infty} (\Omega_n^* - \Omega_{n+1}^*) = 0$. Taking limit on both sides of (6.54) we obtain

$$\lim_{n \rightarrow \infty} \|Te_n - Sh_n\| = 0. \quad (6.55)$$

We also obtain using Lemma 3.1.13 and the condition on α_n that

$$\lim_{n \rightarrow \infty} \|e_n - x_n\| = \lim_{n \rightarrow \infty} \|a_n - y_n\| = \lim_{n \rightarrow \infty} \|y_n - z_n\| = 0, \quad (6.56)$$

and

$$\lim_{n \rightarrow \infty} \|h_n - w_n\| = \lim_{n \rightarrow \infty} \|k_n - t_n\| = \lim_{n \rightarrow \infty} \|t_n - r_n\| = 0. \quad (6.57)$$

From (6.22) and (6.55), we obtain

$$\begin{aligned} \|\nabla f z_n - \nabla f e_n\| &= \gamma_n \|T^* J_{E_3}(Te_n - Sh_n)\| \\ &\leq (\rho + 1) \|T^* J_{E_3}(Te_n - Sh_n)\| \\ &\leq (\rho + 1) \|T^*\| \|J_{E_3}(Te_n - Sh_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

which together with the uniform continuity of ∇f^* implies that

$$\lim_{n \rightarrow \infty} \|z_n - e_n\| = 0. \quad (6.58)$$

From (6.26) and the condition on α_n , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\nabla f x_{n+1} - \nabla f a_n\| &= \lim_{n \rightarrow \infty} \|\alpha_n \nabla f x + (1 - \alpha_n) \nabla f a_n - \nabla f a_n\| \\ &= \lim_{n \rightarrow \infty} \alpha_n \|\nabla f x - \nabla f a_n\| = 0. \end{aligned} \quad (6.59)$$

From (6.59) and the uniform continuity of ∇f^* , we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - a_n\| = 0. \quad (6.60)$$

Consequently, from (6.56), (6.58) and (6.60), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - a_n\| + \|a_n - y_n\| + \|y_n - z_n\| \\ &\quad + \|z_n - e_n\| + \|e_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (6.61)$$

Similarly,

$$\lim_{n \rightarrow \infty} \|w_{n+1} - w_n\| = 0. \quad (6.62)$$

Since $\{(x_n, w_n)\}$ is bounded in $E_1 \times E_2$ and $E_1 \times E_2$ is reflexive, there exists a subsequence $\{(x_{n_k}, w_{n_k})\}$ of $\{(x_n, w_n)\}$ and an element $(\tilde{x}, \tilde{w})$ of $E_1 \times E_2$ such that $(x_{n_k}, w_{n_k}) \rightharpoonup (\tilde{x}, \tilde{w})$ and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (\nabla f x, \nabla s w) - (\nabla f \tilde{x}, \nabla s \tilde{w}), (x_n, w_n) - (\tilde{x}, \tilde{w}) \rangle \\ = \lim_{k \rightarrow \infty} \langle (\nabla f x, \nabla s w) - (\nabla f \tilde{x}, \nabla s \tilde{w}), (x_{n_k}, w_{n_k}) - (\tilde{x}, \tilde{w}) \rangle. \end{aligned} \quad (6.63)$$

Moreover, we have $x_{n_k} \rightharpoonup \tilde{x}$ and $w_{n_k} \rightharpoonup \tilde{w}$. Now, we show that $(\tilde{x}, \tilde{w}) \in \Pi$. From the definition of y_n , we have $y_{n_k} = \nabla f^*(\nabla f z_{n_k} - \lambda_{n_k} A_1 z_{n_k})$, and hence

$$\frac{1}{\lambda_{n_k}} \langle \nabla f z_{n_k} - \nabla f y_{n_k}, x - y_{n_k} \rangle \leq \langle A_1 z_{n_k}, x - y_{n_k} \rangle \text{ for all } x \in E_1,$$

which implies that

$$\frac{1}{\lambda_{n_k}} \langle \nabla f z_{n_k} - \nabla f y_{n_k}, x - y_{n_k} \rangle + \langle A_1 z_{n_k}, y_{n_k} - z_{n_k} \rangle \leq \langle A_1 z_{n_k}, x - z_{n_k} \rangle \text{ for all } x \in E_1. \quad (6.64)$$

Now, we show that $\liminf_{k \rightarrow \infty} \langle A_1 z_{n_k}, x - z_{n_k} \rangle \geq 0$. To show this, we consider two possible cases. Suppose first that $\liminf_{k \rightarrow \infty} \lambda_{n_k} > 0$. Since A_1 is uniformly continuous we have that $\{A_1 z_{n_k}\}$ is bounded. Taking limit of (6.64) as $k \rightarrow \infty$ and since $z_{n_k} - y_{n_k} \rightarrow 0$ we obtain $\liminf_{k \rightarrow \infty} \langle A_1 z_{n_k}, x - z_{n_k} \rangle \geq 0$. Now we assume that $\liminf_{k \rightarrow \infty} \lambda_{n_k} = 0$. Take $\omega_{n_k} = \nabla f^* [\nabla f z_{n_k} - \lambda_{n_k} l^{-1} A_1 z_{n_k}]$. Note that $\lambda_{n_k} l^{-1} > \lambda_{n_k}$. Thus we obtain by Lemma 3.3.1 that

$$\|z_{n_k} - \omega_{n_k}\| \leq \frac{1}{l} \|z_{n_k} - y_{n_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (6.65)$$

As a result, we obtain $\omega_{n_k} \rightharpoonup \tilde{x} \in E_1$. This implies by the uniform continuity of A_1 that

$$\lim_{k \rightarrow \infty} \|A_1 z_{n_k} - A_1 \omega_{n_k}\| = 0. \quad (6.66)$$

By the Armijo line search rule, we have

$$\begin{aligned} \lambda_{n_k} l^{-1} \|A_1 z_{n_k} - A_1 (\nabla f^* (\nabla f z_{n_k} - \lambda_{n_k} l^{-1} A_1 z_{n_k}))\| \\ > \mu \|z_{n_k} - \nabla f^* (\nabla f z_{n_k} - \lambda_{n_k} l^{-1} A_1 z_{n_k})\|, \end{aligned}$$

that is,

$$\frac{1}{\mu} \|A_1 z_{n_k} - A_1 \omega_{n_k}\| > \frac{\|z_{n_k} - \omega_{n_k}\|}{\lambda_{n_k} l^{-1}}. \quad (6.67)$$

Combining (6.66) and (6.67), we obtain $\lim_{k \rightarrow \infty} \frac{\|z_{n_k} - \omega_{n_k}\|}{\lambda_{n_k} l^{-1}} = 0$. Furthermore, we have

$$\langle \nabla f z_{n_k} - \lambda_{n_k} l^{-1} A_1 z_{n_k} - \nabla f \omega_{n_k}, x - \omega_{n_k} \rangle \leq 0, \text{ for all } x \in E_1,$$

and this implies that

$$\begin{aligned} \frac{1}{\lambda_{n_k} l^{-1}} \langle \nabla f z_{n_k} - \nabla f \omega_{n_k}, x - \omega_{n_k} \rangle + \langle A_1 z_{n_k}, \omega_{n_k} - z_{n_k} \rangle \\ \leq \langle A_1 z_{n_k}, x - z_{n_k} \rangle \leq \langle A_1 x, x - z_{n_k} \rangle, \text{ for all } x \in E_1. \end{aligned} \quad (6.68)$$

Taking limit inferior as $k \rightarrow \infty$ in (6.68) we get $\liminf_{k \rightarrow \infty} \langle A_1 z_{n_k}, x - z_{n_k} \rangle \geq 0$. As a consequence, we get $\liminf_{k \rightarrow \infty} \langle A_1 x, x - z_{n_k} \rangle = \langle A_1 x, x - \tilde{x} \rangle \geq 0$, for all $x \in E_1$, or equivalently $\langle A_1 \tilde{x}, x - \tilde{x} \rangle \geq 0$, for all $x \in E_1$ and thus, we obtain that $\tilde{x} \in A_1^{-1}(0)$. Similarly, we have $\tilde{w} \in A_2^{-1}(0)$.

Moreover, by Lemma 3.1, we have

$$\begin{aligned} \|T\tilde{x} - S\tilde{w}\|^2 &= \|Te_{n_k} - Sh_{n_k} + T\tilde{x} - Te_{n_k} + Sh_{n_k} - S\tilde{w}\|^2 \\ &\leq \|Te_{n_k} - Sh_{n_k}\|^2 + 2\langle J_{E_3}(T\tilde{x} - S\tilde{w}), T\tilde{x} - Te_{n_k} + Sh_{n_k} - S\tilde{w} \rangle. \end{aligned} \quad (6.69)$$

From (6.56) and (6.57), we have that $e_{n_k} \rightharpoonup \tilde{x}$ and $h_{n_k} \rightharpoonup \tilde{w}$. Since T and S are bounded linear mappings, they are sequentially weakly continuous and hence we have $Te_{n_k} \rightharpoonup T\tilde{x}$ and $Sh_{n_k} \rightharpoonup S\tilde{w}$. Thus, we obtain from (6.55) and (6.69) that $T\tilde{x} = S\tilde{w}$. Therefore, $(\tilde{x}, \tilde{w}) \in \Pi$.

By (1.14), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (\nabla f x, \nabla s w) - (\nabla f \hat{x}, \nabla s \hat{w}), (x_n, w_n) - (\hat{x}, \hat{w}) \rangle \\ = \lim_{k \rightarrow \infty} \langle (\nabla f x, \nabla s w) - (\nabla f \hat{x}, \nabla s \hat{w}), (x_{n_k}, w_{n_k}) - (\hat{x}, \hat{w}) \rangle \\ = \langle (\nabla f x, \nabla s w) - (\nabla f \hat{x}, \nabla s \hat{w}), (\tilde{x}, \tilde{w}) - (\hat{x}, \hat{w}) \rangle \leq 0. \end{aligned} \quad (6.70)$$

From (6.53), (6.61), (6.62), (6.63), (6.70) and Lemma 3.1.6, we obtain

$$\lim_{n \rightarrow \infty} \Omega_n^* = 0,$$

which implies that $\lim_{n \rightarrow \infty} D_f(\hat{x}, x_n) = \lim_{n \rightarrow \infty} D_s(\hat{w}, w_n) = 0$ and hence we obtain, by Lemma 3.1.13, that $\|x_n - \hat{x}\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (u_n, v_n) = (\hat{u}, \hat{v}) = \hat{x}$, where $\hat{u}$ is a solution of the Hammerstein equation $u + K_1 F_1 u = 0$ with $\hat{v} = F_1 \hat{u}$. Similarly, $\lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} (p_n, q_n) = (\hat{p}, \hat{q}) = \hat{w}$, where $\hat{p}$ is a solution of the Hammerstein equation $p + K_2 F_2 p = 0$ with $\hat{q} = F_2 \hat{p}$.

Case II. Suppose now that there exists a subsequence $\{\Omega_{n_i}^*\}$ of $\{\Omega_n^*\}$ such that $\Omega_{n_i}^* < \Omega_{n_i+1}^*$, for all $i \geq 0$. Then by Lemma 3.1.7, there exists a nondecreasing sequence $\{m_k\}$ of positive integers such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\Omega_{m_k}^* \leq \Omega_{m_k+1}^* \quad \text{and} \quad \Omega_k^* \leq \Omega_{m_k+1}^*. \quad (6.71)$$

We obtain from (6.54) that

$$\begin{aligned} &(1 - \alpha_{m_k}) \left(1 - \frac{2\zeta_{m_k}}{\beta}\right) D_f(e_{m_k}, x_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{\mu}{\beta}\right) D_f(a_{m_k}, y_{m_k}) \\ &+ (1 - \alpha_{m_k}) \left(1 - \frac{\mu}{\beta}\right) D_f(y_{m_k}, z_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{2\zeta_{m_k}}{\beta}\right) D_s(h_{m_k}, w_{m_k}) \\ &+ (1 - \alpha_{m_k}) \left(1 - \frac{\mu}{\beta}\right) D_s(k_{m_k}, t_{m_k}) + (1 - \alpha_{m_k}) \left(1 - \frac{\mu}{\beta}\right) D_s(t_{m_k}, r_{m_k}) \\ &+ \frac{\rho}{2} \|Te_{m_k} - Sh_{m_k}\|^2 \leq \Omega_{m_k}^* - \Omega_{m_k+1}^* + \alpha_{m_k} \left[\Sigma^* + \left(\frac{2\varepsilon}{\beta} - 1\right) \Omega_{m_k}^* + \frac{\zeta_{m_k}}{\alpha_{m_k}} \right]. \end{aligned} \quad (6.72)$$

Taking limit as $k \rightarrow \infty$ of (6.72), making use of (6.71) and the assumptions on α_{m_k} and ζ_{m_k} , we obtain that

$$\lim_{k \rightarrow \infty} \|Te_{m_k} - Sh_{m_k}\| = 0. \quad (6.73)$$

Following similar steps as in Case I, we obtain that

$$\lim_{k \rightarrow \infty} \|x_{m_k+1} - x_{m_k}\| = 0 = \lim_{k \rightarrow \infty} \|w_{m_k+1} - w_{m_k}\| \text{ and } \limsup_{k \rightarrow \infty} (\Delta_{m_k} + \Upsilon_{m_k}) \leq 0. \quad (6.74)$$

From (6.53) and (6.71), we obtain that

$$\begin{aligned} \alpha_{m_k} \left(1 - \frac{2\varepsilon}{\beta}\right) \Omega_{m_k}^* &\leq \Omega_{m_k}^* - \Omega_{m_k+1}^* + \alpha_{m_k} \left(1 - \frac{2\varepsilon}{\beta}\right) (\Delta_{m_k} + \Upsilon_{m_k}) \\ &\leq \alpha_{m_k} \left(1 - \frac{2\varepsilon}{\beta}\right) (\Delta_{m_k} + \Upsilon_{m_k}), \end{aligned}$$

that is, $\Omega_{m_k}^* \leq \Delta_{m_k} + \Upsilon_{m_k}$. This implies by (6.74) that $\lim_{k \rightarrow \infty} \Omega_{m_k}^* = 0$, which in turn implies by (6.53) that $\lim_{k \rightarrow \infty} \Omega_{m_k+1}^* = 0$. Thus, it follows from (6.71) that $\lim_{k \rightarrow \infty} \Omega_k^* = 0$. Therefore, $\lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} (u_k, v_k) = (\hat{u}, \hat{v}) = \hat{x}$, where $\hat{u}$ is a solution of $u + K_1 F_1 u = 0$ with $\hat{v} = F_1 \hat{u}$. Similarly, $\lim_{k \rightarrow \infty} w_k = \lim_{k \rightarrow \infty} (p_k, q_k) = (\hat{p}, \hat{q}) = \hat{w}$, where $\hat{p}$ is a solution of $p + K_2 F_2 p = 0$ with $\hat{q} = F_2 \hat{p}$. Thus, the proof is complete. $\square$

If, in Theorem 6.2.2, we assume that F_1, F_2, K_1 and K_2 are Lipschitz continuous monotone mappings, then we obtain the following corollary.

Corollary 6.2.1. *Assume that the conditions (L1), (L3) – (L7) hold. If $F_1: X \rightarrow X^*, K_1: X^* \rightarrow X, F_2: Y \rightarrow Y^*, K_2: Y^* \rightarrow Y$ are Lipschitz continuous monotone mappings, then the sequences $\{u_n\}, \{v_n\}, \{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.1 converge strongly to u^*, v^*, p^* and q^* , respectively, where $(u^*, p^*) \in \Lambda$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Proof. As every Lipschitz continuous mapping is uniformly continuous, the result follows from Theorem 6.2.2. $\square$

If we assume $g(x) = \frac{1}{2}\|x\|^2$ in Theorem 6.2.2, then $\nabla g = J$ and $\nabla g^* = J^{-1}$. Thus, we get the following corollary.

Corollary 6.2.2. *Assume that the conditions (L1) – (L4), (L6), (L7) are satisfied. Then, the sequences $\{u_n\}, \{v_n\}, \{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.1, with $\nabla g = J$ and $\nabla g^* = J^{-1}$, converge strongly to u^*, v^*, p^* and q^* , respectively, where $(u^*, p^*) \in \Lambda$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

If we assume that X, Y and Z are real Hilbert spaces and $g(x) = \frac{1}{2}\|x\|^2$ and $h(y) = \frac{1}{2}\|y\|^2$ in Theorem 6.2.2, then $\nabla g = J_X = I_X, \nabla g^* = J_X^{-1} = I_{X^*}, \nabla h = J_Y = I_Y$ and $\nabla h^* = J_Y^{-1} = I_{Y^*}$ and hence we obtain the following corollary.

Corollary 6.2.3. *Let X, Y and Z be real Hilbert spaces and assume that the conditions (L2) – (L4), (L6), (L7) are satisfied. Then the sequences $\{u_n\}, \{v_n\}, \{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.1, with $\nabla g = I_X, \nabla g^* = I_{X^*}, \nabla h = I_Y$ and $\nabla h^* = I_{Y^*}$, converge strongly to u^*, v^*, p^* and q^* , respectively, where $(u^*, p^*) \in \Lambda$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

6.2.2 Some Theoretical Applications

This section deals with applications of the main result to specific cases.

Common Solutions of Hammerstein Type Equation Problems

If we consider $X = Y = Z$ and $T_1 = S_1 = I_X$ in Algorithm 6.1, then SEHTEP reduces to *common solutions of the Hammerstein type equation problem*, which is defined as finding two points $u, p \in X$ such that $u + K_1 F_1 u = 0$, $p + K_2 F_2 p = 0$ and $u = p$.

Denote $\Gamma = \{(u, p) \in X \times X : u + K_1 F_1 u = 0, p + K_2 F_2 p = 0 \text{ and } u = p\}$.

Corollary 6.2.4. *Assume that conditions (L1), (L2) and (L5)–(L7), with $X = Y = Z$ hold. If $\Gamma \neq \emptyset$, then the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.1 with $T_1 = I_X$ converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Gamma$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Corollary 6.2.5. *Assume that conditions (L1) and (L5)–(L7), with $X = Y = Z$, hold. Let $F_1: X \rightarrow X^*$, $K_1: X^* \rightarrow X$, $F_2: X \rightarrow X^*$, $K_2: X^* \rightarrow X$ be Lipschitz continuous monotone mappings. Assume that $\Gamma \neq \emptyset$. Then the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.1 with $T_1 = S_1 = I_X$ converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Gamma$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Split Hammerstein Type Equation Problems

If we take $Y = Z$ and $S_1 = I_Y$ in Algorithm 6.1, then the SEHTEP reduces to *split Hammerstein type equation problem* which seeks to find two points $u \in X$ and $p \in Y$ such that $u + K_1 F_1 u = 0$, $p + K_2 F_2 p = 0$ and $T_1 u = p$.

Denote $\Gamma^* = \{(u, p) \in X \times Y : u + K_1 F_1 u = 0, p + K_2 F_2 p = 0 \text{ and } T_1 u = p\}$.

Corollary 6.2.6. *Assume that conditions (L1)–(L3) and (L5)–(L7) hold with $Y = Z$, $S_1 = I_Y$. Let $\Gamma^* \neq \emptyset$. Then the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.1 converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Gamma^*$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Corollary 6.2.7. *Assume that conditions (L1), (L3) and (L5)–(L7) with $Y = Z$ and $S_1 = I_Y$ hold. Let $F_1: X \rightarrow X^*$, $K_1: X^* \rightarrow X$, $F_2: Y \rightarrow Y^*$, $K_2: Y^* \rightarrow Y$ be Lipschitz continuous monotone mappings and $\Gamma^* \neq \emptyset$. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.1 converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Gamma^*$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Hammerstein Type Equation Problems

If we take $X = Y = Z$, $F = F_1 = F_2$, $K = K_1 = K_2$, $T_1 = S_1 = 0$ in Algorithm 6.1, then the SEHTEP reduces to *Hammerstein type equation problem* which seeks to find a point $u \in X$ such that $u + K F u = 0$. Denote $\Psi = \{u \in X : u + K F u = 0\}$.

Corollary 6.2.8. *Let X be a reflexive real Banach space with its dual X^* . Let $F: X \rightarrow X^*$ and $K: X^* \rightarrow X$ be uniformly continuous monotone mappings. Assume that $\Psi \neq \emptyset$. Let $g:$*

$X \rightarrow (-\infty, +\infty]$ be a strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre function on bounded subsets of X with strongly convex conjugate g^* . Let the strong convexity constants of g and g^* be β_1 and β_2 , respectively, and let $\beta = \min\{\beta_1, \beta_2\}$. If conditions (L6) – (L7), hold, then the sequence $\{(u_n, v_n)\}$ generated by Algorithm 6.1, with $X = Y = Z$, $F = F_1 = F_2$, $K = K_1 = K_2$ and $T_1 = S_1 = 0$, converges strongly to (u^*, v^*) , where $u^* \in \Psi$ and $v^* = Fu^*$.

6.2.3 Numerical Example

In this section, we give examples of uniformly continuous monotone mappings which satisfy the conditions of Theorem 6.2.2. A numerical experiment is also provided to illustrate the applicability of the algorithm.

Example 6.2.1. Let $X = Y = Z = L_p^{\mathbb{R}}([0, 1])$, where $1 < p < \infty$ with the norm $\|x\|_{L_p} = \left(\int_0^1 |x(t)|^p dt\right)^{\frac{1}{p}}$ and $g, h: X \rightarrow (-\infty, +\infty]$ be defined by $g(x) = h(x) = \frac{1}{p}\|x\|^p$. Then, $X^* = Y^* = Z^* = L_q^{\mathbb{R}}([0, 1])$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Let $F_1, F_2: X \rightarrow X^*$ and $K_1, K_2: X^* \rightarrow X$ be defined by $(F_1u)(t) = 2\nabla gu(t) = 2J_p u(t)$, $(F_2p)(t) = 3\nabla g^*p(t) = 3J_p^{-1}p(t)$, $(K_1v)(t) = \nabla hv(t) - 2 = J_p v(t) - 2$ and $(K_2q)(t) = \nabla h^*q(t) + 1 = J_p^{-1}q(t) + 1$.

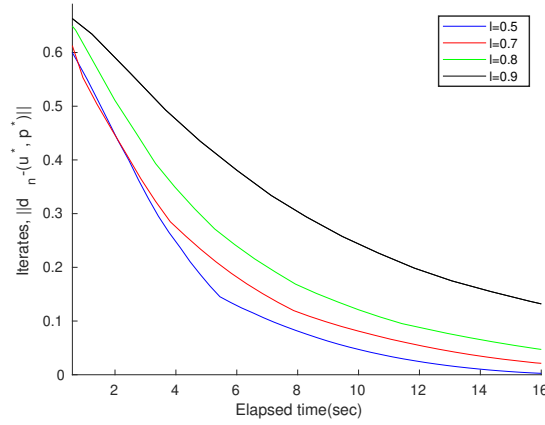
One can show that F_1, K_1, F_2 and K_2 are uniformly continuous monotone mappings and the functions $u^*(t) = \frac{2}{3}$ and $p^*(t) = \frac{-1}{4}$ are solutions of the equations $u(t) + K_1F_1u(t) = 0$ and $p(t) + K_2F_2p(t) = 0$, respectively. Now, define $T_1: X \rightarrow Z$ and $S_1: Y \rightarrow Z$ by

$$(T_1u)(t) = \frac{1}{2}u(t) \quad \text{and} \quad (S_1p)(t) = \left(\frac{-4}{3}\right)p(t).$$

Clearly, T_1 and S_1 are bounded linear mappings with $T_1\left(\frac{2}{3}\right) = \frac{1}{3} = S_1\left(\frac{-1}{4}\right)$. Therefore,

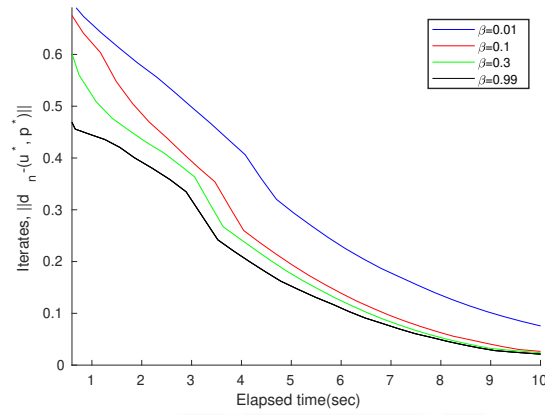
$$(u^*(t), p^*(t)) = \left(\frac{2}{3}, \frac{-1}{4}\right) \in \Lambda = \{(u^*, p^*) \in X \times Y : u^* + K_1F_1u^* = 0, p^* + K_2F_2p^* = 0 \text{ and } T_1u^* = S_1p^*\}.$$

For simplicity of the computation, we take $p = 2$ so that $g(x) = h(x) = \frac{1}{2}\|x\|^2$ and hence $\nabla gx = \nabla g^*x = Jx = Ix = x$ and $\nabla hx = \nabla h^*x = Jx = Ix = x$, where I is the identity mapping on $L_2^{\mathbb{R}}([0, 1])$. Then, the mappings F_1, K_1, F_2 and K_2 reduce to $(F_1u)(t) = 2u(t)$, $(K_1v)(t) = v(t) - 2$, $(F_2p)(t) = 3p(t)$ and $(K_2q)(t) = q(t) + 1$. If we take $\zeta_n = \frac{1}{n^2 + 1000}$, $\alpha_n = \frac{1}{n + 100000}$, $\mu = 0.3$, and $\gamma = 0.7$, $u = 1, v = 0.5, p = 0.5, q = 1$, then the conditions (L1) – (L7) are satisfied and the numerical MATLAB experiments shown below indicate that the sequence $\{d_n\} = \{(u_n, p_n)\}$ generated by Algorithm 6.1 converges strongly to a solution (u^*, p^*) of SE-HTEP with fixed initial point $(u_1, v_1, p_1, q_1) = (1, 0, 1, 0)$, different values of the inertial parameter θ_n and different choices of the parameters l and β as it is shown in the figures below.



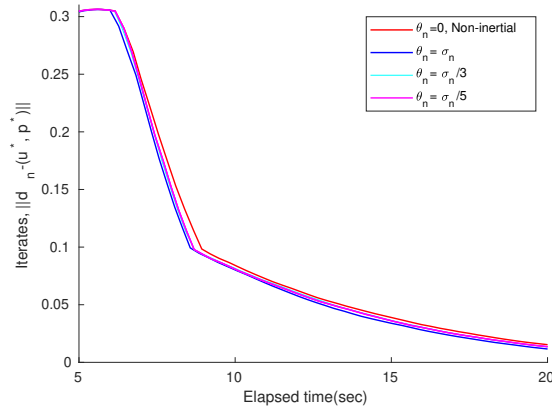
$$(u_0, v_0, p_0, q_0) = (0, 0.5, 0, 0.5), \quad \theta = 0.5, \quad \beta = 0.5.$$

Figure 6.1: Convergence of the sequence for different values of the parameter l .



$$(u_0, v_0, p_0, q_0) = (0.5, 0, 0.5, 0), \quad \theta = 0.5, \quad l = 0.5.$$

Figure 6.2: Convergence of the sequence for different values of the strong convexity constant β .



$$(u_0, v_0, p_0, q_0) = (0.5, 0.5, 0.5, 0.5), \quad l = 0.2, \quad \beta = 0.5.$$

Figure 6.3: Convergence of the sequence for different values of the inertial parameter θ_n

Remark 6.2.2. *Figure 6.1 reveals that the convergence of the method gets faster as l gets closer to zero while all other parameters and initial points are kept fixed. We also observe, from Figure 6.2, that the convergence of the method gets faster as the strong convexity constant, β , of f gets closer to 1 keeping all other parameters and the initial point fixed. From Figure 6.3, we observe that the rate of convergence of the method with inertial algorithm is faster than that of the non-inertial version.*

6.3 Inertial Method for Solving Split Equality Hammerstein Type Equation Problems Involving Continuous Monotone Mappings in Reflexive Real Banach Spaces

In this section, we approximate a solution of the split equality Hammerstein type equation problem involving continuous monotone mappings in reflexive real Banach spaces. We first construct an iterative algorithm and discuss its strong convergence analysis. Finally, we provide numerical examples to demonstrate its efficiency.

6.3.1 Main Results

The following conditions will be assumed to establish the convergence results of our method:

- (M1) Let E_1 , E_2 and E_3 be reflexive real Banach spaces with dual spaces E_1^* , E_2^* and E_3^* , respectively;
- (M2) Let $F_1: E_1 \rightarrow E_1^*$, $K_1: E_1^* \rightarrow E_1$, $F_2: E_2 \rightarrow E_2^*$ and $K_2: E_2^* \rightarrow E_2$ be continuous monotone mappings;
- (M3) Let $T_1: E_1 \rightarrow E_3$ and $S_1: E_2 \rightarrow E_3$ be bounded linear mappings with adjoints T_1^* and S_1^* , respectively;
- (M4) Let $\Pi = \{(u, p) \in E_1 \times E_2 : u + K_1 F_1 u = 0, p + K_2 F_2 p = 0 \text{ and } T_1 u = S_1 p\} \neq \emptyset$;
- (M5) Let $g: E_1 \rightarrow (-\infty, +\infty]$ and $h: E_2 \rightarrow (-\infty, +\infty]$ be strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre functions on bounded subsets with strongly convex conjugates g^* and h^* , respectively. Let the strong convexity constants of g , g^* , h and h^* be μ_1 , μ_2 , β_1 and β_2 , respectively, and let $\mu = \min \{\mu_1, \mu_2, \beta_1, \beta_2\}$;
- (M6) Let $\{\alpha_n\} \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (M7) Let $\{\eta_n\}$ be a positive sequence such that $\eta_n \in \left(0, \frac{\mu}{2}\right)$ and $\frac{\eta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$.

First, we prove the following Lemmas that will be used in the sequel.

Lemma 6.3.1. *Let E be a reflexive real Banach space with dual E^* and let $F: E \rightarrow E^*$ and $K: E^* \rightarrow E$ be continuous monotone mappings. Let $X = E \times E^*$ with norm $\|x\|_X^2 = \|u\|_E^2 + \|v\|_{E^*}^2$, where $x = (u, v) \in X$. Then the mapping $A: X \rightarrow X^*$ defined by $Ax = (Fu - v, Kv + u)$ is continuous monotone.*

Proof. The continuity of A readily follows from the continuity of F and K . It remains to show that A is monotone. We recall that for $z = (z_1, z_2) \in X$ and $z^* = (z_1^*, z_2^*) \in X^*$, we have $\langle z, z^* \rangle = \langle z_1, z_1^* \rangle + \langle z_2, z_2^* \rangle$. If we now take $x_1, x_2 \in X$, with $x_1 = (u_1, v_1)$ and $x_2 = (u_2, v_2)$, then we obtain from the monotonicity of F and K that

$$\begin{aligned}
 \langle Ax_1 - Ax_2, x_1 - x_2 \rangle &= \langle (Fu_1 - v_1, Kv_1 + u_1) - (Fu_2 - v_2, Kv_2 + u_2), (u_1, v_1) - (u_2, v_2) \rangle \\
 &= \langle (Fu_1 - Fu_2 + v_2 - v_1, Kv_1 - Kv_2 + u_1 - u_2), (u_1 - u_2, v_1 - v_2) \rangle \\
 &= \langle Fu_1 - Fu_2 - (v_1 - v_2), u_1 - u_2 \rangle + \langle Kv_1 - Kv_2 + (u_1 - u_2), v_1 - v_2 \rangle \\
 &= \langle Fu_1 - Fu_2, u_1 - u_2 \rangle - \langle v_1 - v_2, u_1 - u_2 \rangle \\
 &\quad + \langle Kv_1 - Kv_2, v_1 - v_2 \rangle + \langle u_1 - u_2, v_1 - v_2 \rangle \\
 &\geq 0,
 \end{aligned}$$

Thus, A is monotone mapping. $\square$

Lemma 6.3.2. *Let E be a reflexive real Banach space with dual E^* . Let $F: E \rightarrow E^*$ and $K: E^* \rightarrow E$ be continuous monotone mappings and let $g: E \rightarrow (-\infty, +\infty]$ be a Legendre function with Fenchel conjugate g^* . Then, for any $r > 0$ and $(x, y) \in X = E \times E^*$, there exists $(w, z) \in X$ such that*

$$\langle Fw - z, u - w \rangle + \langle Kz + w, v - z \rangle + \frac{1}{r} \langle \nabla g w - \nabla g x, u - w \rangle + \frac{1}{r} \langle \nabla g^* z - \nabla g^* y, v - z \rangle \geq 0,$$

for all $(u, v) \in X$.

Proof. For $a = (a_1, a_2) \in X$, define $f: X \rightarrow (-\infty, +\infty]$ by $f(a) = g(a_1) + g^*(a_2)$. Then, X is a reflexive Banach space and f is a Legendre function with $\nabla f = (\nabla g, \nabla g^*)$. Now, define $A: X \rightarrow X^*$ by $A(p, q) = (Fp - q, Kq + p)$. Then, we have by Lemma 6.3.1 that A is continuous monotone. Thus, we have by Lemma 6.1.4, that for any $(x, y) \in X$ and $r > 0$, there exists a unique $(w, z) \in X$ such that

$$\begin{aligned}
 &\langle A(w, z), (u, v) - (w, z) \rangle + \frac{1}{r} \langle \nabla f(w, z) - \nabla f(x, y), (u, v) - (w, z) \rangle \geq 0, \text{ for all } (u, v) \in X \\
 &\iff \langle (Fw - z, Kz + w), (u, v) - (w, z) \rangle \\
 &\quad + \frac{1}{r} \langle (\nabla g w, \nabla g^* z) - (\nabla g x, \nabla g^* y), (u, v) - (w, z) \rangle \geq 0, \text{ for all } (u, v) \in X \\
 &\iff \langle (Fw - z, Kz + w), (u - w, v - z) \rangle \\
 &\quad + \frac{1}{r} \langle (\nabla g w - \nabla g x, \nabla g^* z - \nabla g^* y), (u - w, v - z) \rangle \geq 0, \text{ for all } (u, v) \in X \\
 &\iff \langle Fw - z, u - w \rangle + \langle Kz + w, v - z \rangle + \frac{1}{r} \langle \nabla g w - \nabla g x, u - w \rangle \\
 &\quad + \frac{1}{r} \langle \nabla g^* z - \nabla g^* y, v - z \rangle \geq 0, \text{ for all } (u, v) \in X,
 \end{aligned}$$

and hence the proof is complete. $\square$

We will use notation

$$F_r^{FK}(x, y) = \left\{ (w, z) \in X : \langle Fw - z, u - w \rangle + \langle Kz + w, v - z \rangle + \frac{1}{r} \langle \nabla gw - \nabla gx, u - w \rangle + \frac{1}{r} \langle \nabla g^* z - \nabla g^* y, v - z \rangle \geq 0 \right\},$$

for all $(u, v) \in X$, in the implementation of the undermentioned algorithm. We notice here that the existence and uniqueness of $F_r^{FK}(x, y)$ are guaranteed by Lemma 6.3.2 and Lemma 6.1.4, respectively.

Algorithm 6.2.

Initialization: Let $u_0, u_1 \in E_1$, $v_0, v_1 \in E_1^*$, $p_0, p_1 \in E_2$, $q_0, q_1 \in E_2^*$ and $\theta \in [0, 1)$ be given. For arbitrary $u \in E_1$, $v \in E_1^*$, $p \in E_2$ and $q \in E_2^*$, compute $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ as follows:

Step 1: Given the iterates $u_{n-1}, u_n \in E_1$, $v_{n-1}, v_n \in E_1^*$, $p_{n-1}, p_n \in E_2$ and $q_{n-1}, q_n \in E_2^*$, choose θ_n such that $0 \leq \theta_n \leq \gamma_n$, where

$$\gamma_n = \begin{cases} \min \left\{ \theta, \frac{\eta_n}{\Theta_n + \Phi_n} \right\}, & \text{if } \Theta_n + \Phi_n \neq 0; \\ \theta, & \text{otherwise,} \end{cases}$$

where

$$\Theta_n = \left\{ \|\nabla gu_n - \nabla gu_{n-1}\|^2 + \|\nabla g^* v_n - \nabla g^* v_{n-1}\|^2 \right\}^{\frac{1}{2}}, \text{ and} \\ \Phi_n = \left\{ \|\nabla hp_n - \nabla hp_{n-1}\|^2 + \|\nabla h^* q_n - \nabla h^* q_{n-1}\|^2 \right\}^{\frac{1}{2}}.$$

Step 2: Compute

$$\begin{aligned} a_{1n} &= \nabla g^* [\nabla gu_n + \theta_n (\nabla gu_n - \nabla gu_{n-1})], \\ a_{2n} &= \nabla g [\nabla g^* v_n + \theta_n (\nabla g^* v_n - \nabla g^* v_{n-1})], \\ b_{1n} &= \nabla h^* [\nabla hp_n + \theta_n (\nabla hp_n - \nabla hp_{n-1})], \\ b_{2n} &= \nabla h [\nabla h^* q_n + \theta_n (\nabla h^* q_n - \nabla h^* q_{n-1})]. \end{aligned}$$

Step 3: Compute

$$\begin{aligned} c_{1n} &= \nabla g^* [\nabla ga_{1n} - \sigma_n T_1^* J_{E_3} (T_1 a_{1n} - S_1 b_{1n})], \\ d_{1n} &= \nabla h^* [\nabla hb_{1n} - \sigma_n S_1^* J_{E_3} (S_1 b_{1n} - T_1 a_{1n})], \end{aligned}$$

where $0 < \rho \leq \sigma_n \leq \rho_n$ with

$$\rho_n = \min \left\{ \rho + 1, \frac{\mu \|T_1 a_{1n} - S_1 b_{1n}\|^2}{2[\|T_1^* J_{E_3} (T_1 a_{1n} - S_1 b_{1n})\|^2 + \|S_1^* J_{E_3} (S_1 b_{1n} - T_1 a_{1n})\|^2]} \right\}$$

for $n \in \mathcal{M} = \{m \in \mathbb{N} : T_1 a_{1m} - S_1 b_{1m} \neq 0\}$, otherwise $\sigma_n = \rho$.

Step 4: Compute

$$(e_{1n}, e_{2n}) = F_{r_n}^{F_1 K_1}(c_{1n}, a_{2n}),$$

$$(w_{1n}, w_{2n}) = F_{r_n}^{F_2 K_2}(d_{1n}, b_{2n}),$$

where $r_n \subset [h_1, \infty)$, for some $h_1 > 0$, and for all $n \geq 1$.

Step 5: Compute

$$u_{n+1} = \nabla g^* [\alpha_n \nabla g u + (1 - \alpha_n) \nabla g e_{1n}],$$

$$v_{n+1} = \nabla g [\alpha_n \nabla g^* v + (1 - \alpha_n) \nabla g^* e_{2n}],$$

$$p_{n+1} = \nabla h^* [\alpha_n \nabla h p + (1 - \alpha_n) \nabla h w_{1n}],$$

$$q_{n+1} = \nabla h [\alpha_n \nabla h^* q + (1 - \alpha_n) \nabla h^* w_{2n}].$$

Set $n = n + 1$ and go to **Step 1**.

Remark 6.3.1. We remark here that the underlying mappings in our algorithm are continuous monotone mappings which are more general than the mappings considered in the literature.

Theorem 6.3.1. Assume that conditions (M1) – (M7) hold. Then, the sequences $\{u_n\} \subset E_1$, $\{v_n\} \subset E_1^*$, $\{p_n\} \subset E_2$ and $\{q_n\} \subset E_2^*$ generated by Algorithm 6.2 are bounded.

Proof. Let $X = E_1 \times E_1^*$ and $x = (u, v) \in X$. Now, define a norm on X by $\|x\|_X^2 = \|u\|_{E_1}^2 + \|v\|_{E_1^*}^2$, then we have by Lemma 6.1.3 that the mapping $A_1: X \rightarrow X^*$ defined by $A_1 t = (F_1 t_1 - t_2, K_1 t_2 + t_1)$, where $t = (t_1, t_2) \in X$, is maximal monotone. Similarly, the mapping $A_2: Y \rightarrow Y^*$ given by $A_2 z = (F_2 z_1 - z_2, K_2 z_2 + z_1)$, where $z = (z_1, z_2) \in Y = E_2 \times E_2^*$, is maximal monotone. Let $Z = E_3 \times E_3^*$ and $T: X \rightarrow Z$, $S: Y \rightarrow Z$ be mappings defined by $T = (T_1, 0)$ and $S = (S_1, 0)$. Then, T and S are bounded linear mappings and Algorithm 6.2 can be rewritten as follows:

Initialization: Let $x_0 = (u_0, v_0), x_1 = (u_1, v_1) \in X$ and $y_0 = (p_0, q_0), y_1 = (p_1, q_1) \in Y$, and $\theta \in [0, 1)$. For $x = (u, v) \in X$ and $y = (p, q) \in Y$, compute $\{x_n\} = \{(u_n, v_n)\}$ and $\{y_n\} = \{(p_n, q_n)\}$ as follows:

Step 1: Given the iterates $x_{n-1}, x_n \in X$ and $y_{n-1}, y_n \in Y$, choose θ_n such that $0 \leq \theta_n \leq \gamma_n$, where

$$\gamma_n = \begin{cases} \min \left\{ \theta, \frac{\eta_n}{\Theta_n + \Phi_n} \right\}, & \text{if } \Theta_n + \Phi_n \neq 0; \\ \theta, & \text{otherwise,} \end{cases} \quad (6.75)$$

where

$$\Theta_n = \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| \text{ and } \Phi_n = \|\nabla f_2 y_n - \nabla f_2 y_{n-1}\|.$$

Step 2: Compute

$$a_n = \nabla f_1^* [\nabla f_1 x_n + \theta_n (\nabla f_1 x_n - \nabla f_1 x_{n-1})], \quad (6.76)$$

$$b_n = \nabla f_2^* [\nabla f_2 y_n + \theta_n (\nabla f_2 y_n - \nabla f_2 y_{n-1})].$$

Step 3: Compute

$$c_n = \nabla f_1^* [\nabla f_1 a_n - \sigma_n T^* J_Z (Ta_n - Sb_n)], \quad (6.77)$$

$$d_n = \nabla f_2^* [\nabla f_2 b_n - \sigma_n S^* J_Z (Sb_n - Ta_n)].$$

where $0 < \rho \leq \sigma_n \leq \rho_n$ with

$$\rho_n = \min \left\{ \rho + 1, \frac{\mu \|Ta_n - Sb_n\|^2}{2[\|T^* J_Z (Ta_n - Sb_n)\|^2 + \|S^* J_Z (Sb_n - Ta_n)\|^2]} \right\},$$

for $n \in \mathcal{M} = \{m \in \mathbb{N} : Ta_m - Sb_m \neq 0\}$, otherwise $\sigma_n = \rho$.

Step 4: Compute

$$e_n = F_{r_n}^{A_1} c_n, \quad (6.78)$$

$$w_n = F_{r_n}^{A_2} d_n,$$

where $F_{r_n}^{A_i} = F_{r_n}^{F_i K_i}$, for $i = 1, 2$, and $r_n \subset [h_1, \infty)$, for some $h_1 > 0$ for all $n \geq 1$.

Step 5: Compute

$$x_{n+1} = \nabla f_1^* [\alpha_n \nabla f_1 x + (1 - \alpha_n) \nabla f_1 e_n], \quad (6.79)$$

$$y_{n+1} = \nabla f_2^* [\alpha_n \nabla f_2 y + (1 - \alpha_n) \nabla f_2 w_n],$$

where $f_1: X \rightarrow (-\infty, +\infty]$ and $f_2: Y \rightarrow (-\infty, +\infty]$ are defined by $f_1(u, v) = g(u) + g^*(v)$ and $f_2(p, q) = h(p) + h^*(q)$. Note that $x_n = (u_n, v_n)$, $y_n = (p_n, q_n)$, and f_1 and f_2 are uniformly Fréchet differentiable Legendre functions which are bounded on bounded subsets of X and Y , respectively. By Lemma 6.1.1, we have that f_1 and f_2 are strongly coercive and strongly convex with common constant μ .

Since $\Pi \neq \emptyset$, take any $(u^*, p^*) \in \Pi$. We have that u^* is a solution of the equation $u + K_1 F_1 u = 0$ with $v^* = F_1 u^*$ if and only if $x^* = (u^*, v^*)$ is a solution of the equation $A_1 x = 0$. Thus, the set of null points of A_1 , $N(A_1)$, is nonempty. Similarly, it can be shown that the set of null points of A_2 is nonempty. As $(x^*, y^*) \in \Delta = \{(x, y) \in N(A_1) \times N(A_2) : Tx = Sy\}$, Δ is nonempty. From (6.79) and Lemma 3.1.9, we have

$$\begin{aligned} D_{f_1}(x^*, x_{n+1}) &= D_{f_1}(x^*, \nabla f_1^* (\alpha_n \nabla f_1 x + (1 - \alpha_n) \nabla f_1 e_n)) \\ &\leq \alpha_n D_{f_1}(x^*, x) + (1 - \alpha_n) D_{f_1}(x^*, e_n). \end{aligned} \quad (6.80)$$

From (6.78) and part (iv) of Lemma 6.1.4, we obtain

$$D_{f_1}(x^*, e_n) \leq D_{f_1}(x^*, c_n) - D_{f_1}(e_n, c_n). \quad (6.81)$$

Substituting (6.81) into (6.80), we obtain

$$\begin{aligned} D_{f_1}(x^*, x_{n+1}) &\leq \alpha_n D_{f_1}(x^*, x) + (1 - \alpha_n) [D_{f_1}(x^*, c_n) - D_{f_1}(e_n, c_n)] \\ &\leq \alpha_n D_{f_1}(x^*, x) + (1 - \alpha_n) D_{f_1}(x^*, c_n). \end{aligned} \quad (6.82)$$

From (6.77), (1.21) and (1.22), we obtain

$$\begin{aligned}
 D_{f_1}(x^*, c_n) &= D_{f_1}(x^*, \nabla f_1^*(\nabla f_1 a_n - \sigma_n T^* J_Z(Ta_n - Sb_n))) \\
 &= V_{f_1}(x^*, \nabla f_1 a_n - \sigma_n T^* J_Z(Ta_n - Sb_n)) \\
 &\leq V_{f_1}(x^*, \nabla f_1 a_n) - \langle \sigma_n T^* J_Z(Ta_n - Sb_n), c_n - x^* \rangle \\
 &= D_{f_1}(x^*, a_n) - \langle \sigma_n T^* J_Z(Ta_n - Sb_n), c_n - x^* \rangle.
 \end{aligned} \tag{6.83}$$

Moreover, we have from (6.76) and Definition 1.2.17 that

$$\begin{aligned}
 D_{f_1}(x^*, a_n) &= D_{f_1}(x^*, \nabla f_1^*(\nabla f_1 x_n + \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}))) \\
 &= f_1(x^*) - \langle \nabla f_1 x_n + \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}), x^* - a_n \rangle - f_1(a_n) \\
 &= f_1(x^*) - \langle \nabla f_1 x_n, x^* - a_n \rangle - \langle \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}), x^* - a_n \rangle - f_1(a_n) \\
 &= f_1(x^*) - \langle \nabla f_1 x_n, x^* - x_n \rangle - f_1(x_n) + \langle \nabla f_1 x_n, x^* - x_n \rangle + f_1(x_n) \\
 &\quad - \langle \nabla f_1 x_n, x^* - a_n \rangle - \langle \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}), x^* - a_n \rangle - f_1(a_n) \\
 &= D_{f_1}(x^*, x_n) + \langle \nabla f_1 x_n, a_n - x_n \rangle + f_1(x_n) - f_1(a_n) \\
 &\quad - \langle \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}), x^* - a_n \rangle \\
 &= D_{f_1}(x^*, x_n) - D_{f_1}(a_n, x_n) - \langle \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}), x^* - a_n \rangle.
 \end{aligned} \tag{6.84}$$

Thus, we have from (6.83) and (6.84) that

$$\begin{aligned}
 D_{f_1}(x^*, c_n) &\leq D_{f_1}(x^*, x_n) - D_{f_1}(a_n, x_n) - \langle \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}), x^* - a_n \rangle \\
 &\quad - \langle \sigma_n T^* J_Z(Ta_n - Sb_n), c_n - x^* \rangle.
 \end{aligned} \tag{6.85}$$

Substituting (6.85) into (6.82), we get

$$\begin{aligned}
 D_{f_1}(x^*, x_{n+1}) &\leq \alpha_n D_{f_1}(x^*, x) + (1 - \alpha_n) D_{f_1}(x^*, x_n) - (1 - \alpha_n) D_{f_1}(a_n, x_n) \\
 &\quad - (1 - \alpha_n) \langle \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}), x^* - a_n \rangle \\
 &\quad - (1 - \alpha_n) \langle \sigma_n T^* J_Z(Ta_n - Sb_n), c_n - x^* \rangle.
 \end{aligned} \tag{6.86}$$

Moreover, by the Cauchy–Schwarz Inequality, (3.1.10) and the property of η_n , we have

$$\begin{aligned}
 -\langle \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}), x^* - a_n \rangle &\leq \theta_n \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| \|x^* - a_n\| \\
 &= \theta_n \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| \|(x^* - a_n) \times 1\| \\
 &\leq \frac{\theta_n}{2} \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| [\|x^* - a_n\|^2 + 1] \\
 &= \frac{\theta_n}{2} \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| [\|x^* - x_n + x_n - a_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| [2\|x^* - x_n\|^2 + 2\|x_n - a_n\|^2 + 1] \\
 &\leq \frac{\theta_n}{2} \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| \left[\frac{4}{\mu} D_{f_1}(x^*, x_n) + \frac{4}{\mu} D_{f_1}(a_n, x_n) + 1 \right],
 \end{aligned}$$

that is,

$$\begin{aligned}
 -\langle \theta_n(\nabla f_1 x_n - \nabla f_1 x_{n-1}), x^* - a_n \rangle &\leq \frac{2\theta_n}{\mu} \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| D_{f_1}(x^*, x_n) \\
 &\quad + \frac{2\theta_n}{\mu} \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| D_{f_1}(a_n, x_n) \\
 &\quad + \frac{\theta_n}{2} \|\nabla f_1 x_n - \nabla f_1 x_{n-1}\| \\
 &\leq \frac{2\eta_n}{\mu} D_{f_1}(x^*, x_n) + \frac{2\eta_n}{\mu} D_{f_1}(a_n, x_n) + \frac{\eta_n}{2}.
 \end{aligned} \tag{6.87}$$

Substituting (6.87) into (6.86), we obtain

$$\begin{aligned}
 D_{f_1}(x^*, x_{n+1}) &\leq \alpha_n D_{f_1}(x^*, x) + (1 - \alpha_n) D_{f_1}(x^*, x_n) - (1 - \alpha_n) D_{f_1}(a_n, x_n) \\
 &\quad + (1 - \alpha_n) \left[\frac{2\eta_n}{\mu} D_{f_1}(x^*, x_n) + \frac{2\eta_n}{\mu} D_{f_1}(a_n, x_n) + \frac{\eta_n}{2} \right] \\
 &\quad - (1 - \alpha_n) \langle \sigma_n T^* J_Z(Ta_n - Sb_n), c_n - x^* \rangle \\
 &= \alpha_n D_{f_1}(x^*, x) + (1 - \alpha_n) \left(1 + \frac{2\eta_n}{\mu} \right) D_{f_1}(x^*, x_n) \\
 &\quad - (1 - \alpha_n) \left(1 - \frac{2\eta_n}{\mu} \right) D_{f_1}(a_n, x_n) + (1 - \alpha_n) \frac{\eta_n}{2} \\
 &\quad - (1 - \alpha_n) \langle \sigma_n T^* J_Z(Ta_n - Sb_n), c_n - x^* \rangle.
 \end{aligned} \tag{6.88}$$

Since $\eta_n \in \left(0, \frac{\mu}{2}\right)$, we obtain from (6.88) that

$$\begin{aligned}
 D_{f_1}(x^*, x_{n+1}) &\leq \alpha_n D_{f_1}(x^*, x) + (1 - \alpha_n) \left(1 + \frac{2\eta_n}{\mu} \right) D_{f_1}(x^*, x_n) \\
 &\quad + (1 - \alpha_n) \frac{\eta_n}{2} - (1 - \alpha_n) \langle \sigma_n J_Z(Ta_n - Sb_n), Tc_n - Tx^* \rangle.
 \end{aligned} \tag{6.89}$$

Similarly, we have

$$\begin{aligned}
 D_{f_2}(y^*, y_{n+1}) &\leq \alpha_n D_{f_2}(y^*, y) + (1 - \alpha_n) \left(1 + \frac{2\eta_n}{\mu} \right) D_{f_2}(y^*, y_n) \\
 &\quad + (1 - \alpha_n) \frac{\eta_n}{2} - (1 - \alpha_n) \langle \sigma_n J_Z(Sb_n - Ta_n), Sd_n - Sy^* \rangle.
 \end{aligned} \tag{6.90}$$

Denote $\Omega_n = D_{f_1}(x^*, x_n) + D_{f_2}(y^*, y_n)$ and $\Sigma = D_{f_1}(x^*, x) + D_{f_2}(y^*, y)$. Combining (6.89) and (6.90), we obtain

$$\begin{aligned}
 \Omega_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{2\eta_n}{\mu} \right) \Omega_n + (1 - \alpha_n) \eta_n \\
 &\quad - (1 - \alpha_n) \sigma_n \langle J_Z(Ta_n - Sb_n), Tc_n - Sd_n \rangle.
 \end{aligned} \tag{6.91}$$

By the Cauchy–Schwarz Inequality, we have

$$\begin{aligned}
 -\langle J_Z(Ta_n - Sb_n), Tc_n - Sd_n \rangle &= -\langle J_Z(Ta_n - Sb_n), Ta_n - Sb_n \rangle \\
 &\quad - \langle J_Z(Ta_n - Sb_n), Tc_n - Ta_n \rangle \\
 &\quad - \langle J_Z(Ta_n - Sb_n), Sb_n - Sd_n \rangle \\
 &= -\|Ta_n - Sb_n\|^2 - \langle T^* J_Z(Ta_n - Sb_n), c_n - a_n \rangle \\
 &\quad - \langle S^* J_Z(Ta_n - Sb_n), b_n - d_n \rangle \\
 &\leq -\|Ta_n - Sb_n\|^2 + \|c_n - a_n\| \|T^* J_Z(Ta_n - Sb_n)\| \\
 &\quad + \|b_n - d_n\| \|S^* J_Z(Ta_n - Sb_n)\|.
 \end{aligned} \tag{6.92}$$

Since f_1 is strongly convex, we have from (6.77) that

$$\begin{aligned}
 \|c_n - a_n\| &= \|\nabla f_1^* (\nabla f_1 a_n - \sigma_n T^* J_Z(Ta_n - Sb_n)) - \nabla f_1^* (\nabla f_1 a_n)\| \\
 &\leq \frac{\sigma_n}{\mu} \|T^* J_Z(Ta_n - Sb_n)\|.
 \end{aligned} \tag{6.93}$$

Similarly, we obtain

$$\begin{aligned}
 \|d_n - b_n\| &= \|\nabla f_2^* (\nabla f_2 b_n - \sigma_n S^* J_Z(Sb_n - Ta_n)) - \nabla f_2^* (\nabla f_2 b_n)\| \\
 &\leq \frac{\sigma_n}{\mu} \|S^* J_Z(Sb_n - Ta_n)\|.
 \end{aligned} \tag{6.94}$$

Substituting (6.93) and (6.94) into (6.92), we get

$$\begin{aligned}
 -\sigma_n \langle J_Z(Ta_n - Sb_n), Tc_n - Sd_n \rangle &\leq -\sigma_n \|Ta_n - Sb_n\|^2 + \frac{\sigma_n^2}{\mu} \|T^* J_Z(Ta_n - Sb_n)\|^2 \\
 &\quad + \frac{\sigma_n^2}{\mu} \|S^* J_Z(Sb_n - Ta_n)\|^2 \\
 &\leq -\frac{\rho}{2} \|Ta_n - Sb_n\|^2 - \frac{\sigma_n}{2} \|Ta_n - Sb_n\|^2 \\
 &\quad + \frac{\sigma_n}{2} \left\{ \frac{2\sigma_n}{\mu} [\|T^* J_Z(Ta_n - Sb_n)\|^2 + \|S^* J_Z(Sb_n - Ta_n)\|^2] \right\} \\
 &\leq -\frac{\rho}{2} \|Ta_n - Sb_n\|^2.
 \end{aligned} \tag{6.95}$$

Combining (6.91) and (6.95), we get

$$\begin{aligned}
 \Omega_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{2\eta_n}{\mu} \right) \Omega_n + (1 - \alpha_n) \eta_n \\
 &\quad - (1 - \alpha_n) \frac{\rho}{2} \|Ta_n - Sb_n\|^2 \\
 &\leq \alpha_n \Sigma + (1 - \alpha_n) \left(1 + \frac{2\eta_n}{\mu} \right) \Omega_n + (1 - \alpha_n) \eta_n \\
 &\leq \alpha_n \Sigma + (1 - \alpha_n) \Omega_n + \frac{2\eta_n}{\mu} \Omega_n + \eta_n.
 \end{aligned} \tag{6.96}$$

Since $\frac{\eta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$, for every $\varepsilon \in (0, \frac{\mu}{2})$, there exists a natural number n_0 such that

$\eta_n < \alpha_n \varepsilon$ for all $n \geq n_0$. Thus, (6.96) becomes

$$\begin{aligned} \Omega_{n+1} &\leq \alpha_n \Sigma + (1 - \alpha_n) \Omega_n + \frac{2\varepsilon \alpha_n}{\mu} \Omega_n + \varepsilon \alpha_n \\ &= \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\mu} \right) \right] \Omega_n + \alpha_n [\Sigma + \varepsilon] \\ &= \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\mu} \right) \right] \Omega_n + \alpha_n \left(1 - \frac{2\varepsilon}{\mu} \right) \left[\frac{\mu \Sigma}{\mu - 2\varepsilon} + \frac{\mu \varepsilon}{\mu - 2\varepsilon} \right] \\ &\leq \max \left\{ \Omega_n, \frac{\mu \Sigma}{\mu - 2\varepsilon} + \frac{\mu \varepsilon}{\mu - 2\varepsilon} \right\}, \text{ for all } n \geq n_0, \end{aligned}$$

which implies by the Principle of Mathematical Induction that

$$\Omega_n \leq \max \left\{ \Omega_{n_0}, \frac{\mu \Sigma}{\mu - 2\varepsilon} + \frac{\mu \varepsilon}{\mu - 2\varepsilon} \right\}, \text{ for all } n \geq n_0,$$

from which we obtain

$$\Omega_n \leq \max \left\{ M, \Omega_{n_0}, \frac{\mu \Sigma}{\mu - 2\varepsilon} + \frac{\mu \varepsilon}{\mu - 2\varepsilon} \right\}, \text{ for all } n \geq 1,$$

where $M = \max \{\Omega_1, \Omega_2, \Omega_3, \dots, \Omega_{n_0-1}\}$. Thus, we conclude that $\{\Omega_n\}$ is a bounded sequence and so are the sequences $\{D_{f_1}(x^*, x_n)\}$ and $\{D_{f_2}(y^*, y_n)\}$. By Lemma 3.1.11 we have that $\{x_n\}$ and $\{y_n\}$ are bounded, and hence the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ are bounded. $\square$

Theorem 6.3.2. *Assume that conditions (M1) – (M7) hold. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.2 converge strongly to $\hat{u}$, $\hat{v}$, $\hat{p}$ and $\hat{q}$, respectively, where $(\hat{u}, \hat{p}) \in \Pi$ with $\hat{v} = F_1 \hat{u}$ and $\hat{q} = F_2 \hat{p}$.*

Proof. Let Δ , A_1 , A_2 , T and S be defined as in the proof of Theorem 6.3.1. Let $(\hat{x}, \hat{y}) = P_{\Delta}^f(x, y)$, where $f = (f_1, f_2)$ with $\hat{x} = (\hat{u}, \hat{v})$ and $\hat{y} = (\hat{p}, \hat{q})$. Then, by (1.18), we have

$$\langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (s, t) - (\hat{x}, \hat{y}) \rangle \leq 0, \text{ for all } (s, t) \in \Delta. \quad (6.97)$$

From (6.79), (1.21), (1.22), (6.81) and Lemma 3.1.9, we have

$$\begin{aligned} D_{f_1}(\hat{x}, x_{n+1}) &= D_{f_1}(\hat{x}, \nabla f_1^*(\alpha_n \nabla f_1 x + (1 - \alpha_n) \nabla f_1 e_n)) \\ &= V_{f_1}(\hat{x}, \alpha_n \nabla f_1 x + (1 - \alpha_n) \nabla f_1 e_n) \\ &\leq V_{f_1}(\hat{x}, \alpha_n \nabla f_1 x + (1 - \alpha_n) \nabla f_1 e_n - \alpha_n (\nabla f_1 x - \nabla f_1 \hat{x})) \\ &\quad + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_{n+1} - \hat{x} \rangle \\ &= V_{f_1}(\hat{x}, \alpha_n \nabla f_1 \hat{x} + (1 - \alpha_n) \nabla f_1 e_n) + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_{n+1} - \hat{x} \rangle \\ &= D_{f_1}(\hat{x}, \nabla f_1^*(\alpha_n \nabla f_1 \hat{x} + (1 - \alpha_n) \nabla f_1 e_n)) \\ &\quad + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_{n+1} - \hat{x} \rangle \\ &\leq (1 - \alpha_n) D_{f_1}(\hat{x}, e_n) + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_{n+1} - \hat{x} \rangle \\ &\leq (1 - \alpha_n) D_{f_1}(\hat{x}, c_n) - (1 - \alpha_n) D_{f_1}(e_n, c_n) \\ &\quad + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_{n+1} - \hat{x} \rangle. \end{aligned} \quad (6.98)$$

Substituting (6.85) into (6.98) and making use of (6.87), we get

$$\begin{aligned}
 D_{f_1}(\hat{x}, x_{n+1}) &\leq (1 - \alpha_n) D_{f_1}(\hat{x}, x_n) - (1 - \alpha_n) D_{f_1}(a_n, x_n) - (1 - \alpha_n) D_{f_1}(e_n, c_n) \\
 &\quad + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_{n+1} - \hat{x} \rangle - (1 - \alpha_n) \langle \theta_n (\nabla f_1 x_n - \nabla f_1 x_{n-1}), \hat{x} - a_n \rangle \\
 &\quad - (1 - \alpha_n) \langle \sigma_n T^* J_Z (Ta_n - Sb_n), c_n - \hat{x} \rangle \\
 &\leq (1 - \alpha_n) D_{f_1}(\hat{x}, x_n) - (1 - \alpha_n) D_{f_1}(a_n, x_n) - (1 - \alpha_n) D_{f_1}(e_n, c_n) \\
 &\quad + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_{n+1} - \hat{x} \rangle - (1 - \alpha_n) \langle \sigma_n T^* J_Z (Ta_n - Sb_n), c_n - \hat{x} \rangle \\
 &\quad + (1 - \alpha_n) \frac{2\eta_n}{\mu} D_{f_1}(\hat{x}, x_n) + (1 - \alpha_n) \frac{2\eta_n}{\mu} D_{f_1}(a_n, x_n) + (1 - \alpha_n) \frac{\eta_n}{2} \\
 &= (1 - \alpha_n) \left(1 + 2\frac{\eta_n}{\mu}\right) D_{f_1}(\hat{x}, x_n) - (1 - \alpha_n) \left(1 - 2\frac{\eta_n}{\mu}\right) D_{f_1}(a_n, x_n) \\
 &\quad - (1 - \alpha_n) D_{f_1}(e_n, c_n) + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_{n+1} - \hat{x} \rangle + (1 - \alpha_n) \frac{\eta_n}{2} \\
 &\quad - (1 - \alpha_n) \langle \sigma_n T^* J_Z (Ta_n - Sb_n), c_n - \hat{x} \rangle.
 \end{aligned} \tag{6.99}$$

Thus, we obtain from (6.99) by the Cauchy–Schwarz Inequality and the condition on α_n that

$$\begin{aligned}
 D_{f_1}(\hat{x}, x_{n+1}) &\leq (1 - \alpha_n) \left(1 + 2\frac{\eta_n}{\mu}\right) D_{f_1}(\hat{x}, x_n) - (1 - \alpha_n) \left(1 - 2\frac{\eta_n}{\mu}\right) D_{f_1}(a_n, x_n) \\
 &\quad - (1 - \alpha_n) D_{f_1}(e_n, c_n) + \alpha_n \|\nabla f_1 x - \nabla f_1 \hat{x}\| \|x_{n+1} - x_n\| \\
 &\quad + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_n - \hat{x} \rangle + \frac{\eta_n}{2} \\
 &\quad - (1 - \alpha_n) \langle \sigma_n T^* J_Z (Ta_n - Sb_n), c_n - \hat{x} \rangle.
 \end{aligned} \tag{6.100}$$

Similarly, we have

$$\begin{aligned}
 D_{f_2}(\hat{y}, y_{n+1}) &\leq (1 - \alpha_n) \left(1 + 2\frac{\eta_n}{\mu}\right) D_{f_2}(\hat{y}, y_n) - (1 - \alpha_n) \left(1 - 2\frac{\eta_n}{\mu}\right) D_{f_2}(b_n, y_n) \\
 &\quad - (1 - \alpha_n) D_{f_2}(w_n, d_n) + \alpha_n \|\nabla f_2 y - \nabla f_2 \hat{y}\| \|y_{n+1} - y_n\| \\
 &\quad + \alpha_n \langle \nabla f_2 y - \nabla f_2 \hat{y}, y_n - \hat{y} \rangle + \frac{\eta_n}{2} \\
 &\quad - (1 - \alpha_n) \langle \sigma_n S^* J_Z (Sb_n - Ta_n), d_n - \hat{y} \rangle.
 \end{aligned} \tag{6.101}$$

Denote $\hat{\Omega}_n = D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n)$. Then combining (6.100) and (6.101), and making use of (6.95) gives

$$\begin{aligned}
 \hat{\Omega}_{n+1} &\leq (1 - \alpha_n) \left(1 + 2\frac{\eta_n}{\mu}\right) \hat{\Omega}_n - (1 - \alpha_n) \left(1 - 2\frac{\eta_n}{\mu}\right) D_{f_1}(a_n, x_n) - (1 - \alpha_n) D_{f_1}(e_n, c_n) \\
 &\quad - (1 - \alpha_n) \left(1 - 2\frac{\eta_n}{\mu}\right) D_{f_2}(b_n, y_n) - (1 - \alpha_n) D_{f_2}(w_n, d_n) - (1 - \alpha_n) \frac{\rho}{2} \|Ta_n - Sb_n\|^2 \\
 &\quad + \alpha_n \|\nabla f_1 x - \nabla f_1 \hat{x}\| \|x_{n+1} - x_n\| + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_n - \hat{x} \rangle + \eta_n \\
 &\quad + \alpha_n \|\nabla f_2 y - \nabla f_2 \hat{y}\| \|y_{n+1} - y_n\| + \alpha_n \langle \nabla f_2 y - \nabla f_2 \hat{y}, y_n - \hat{y} \rangle,
 \end{aligned}$$

that is,

$$\begin{aligned}
 \widehat{\Omega} \leq & (1 - \alpha_n) \widehat{\Omega}_n + 2 \frac{\varepsilon \alpha_n}{\mu} \widehat{\Omega}_n - (1 - \alpha_n) \left(1 - 2 \frac{\eta_n}{\mu} \right) D_{f_1}(a_n, x_n) - (1 - \alpha_n) D_{f_1}(e_n, c_n) \\
 & - (1 - \alpha_n) \left(1 - 2 \frac{\eta_n}{\mu} \right) D_{f_2}(b_n, y_n) - (1 - \alpha_n) D_{f_2}(w_n, d_n) - (1 - \alpha_n) \frac{\rho}{2} \|Ta_n - Sb_n\|^2 \\
 & + \alpha_n \|\nabla f_1 x - \nabla f_1 \hat{x}\| \|x_{n+1} - x_n\| + \alpha_n \langle \nabla f_1 x - \nabla f_1 \hat{x}, x_n - \hat{x} \rangle + \alpha_n \frac{\eta_n}{\alpha_n} \\
 & + \alpha_n \|\nabla f_2 y - \nabla f_2 \hat{y}\| \|y_{n+1} - y_n\| + \alpha_n \langle \nabla f_2 y - \nabla f_2 \hat{y}, y_n - \hat{y} \rangle.
 \end{aligned} \tag{6.102}$$

From (6.102) and the conditions on α_n and η_n , we obtain

$$\begin{aligned}
 \widehat{\Omega}_{n+1} \leq & (1 - \alpha_n) \left(1 + 2 \frac{\varepsilon \alpha_n}{\mu} \right) \widehat{\Omega}_n + \alpha_n \|\nabla f_1 x - \nabla f_1 \hat{x}\| \|x_{n+1} - x_n\| \\
 & + \alpha_n \|\nabla f_2 y - \nabla f_2 \hat{y}\| \|y_{n+1} - y_n\| \\
 & + \alpha_n \langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (x_n, y_n) - (\hat{x}, \hat{y}) \rangle + \alpha_n \frac{\eta_n}{\alpha_n} \\
 \leq & \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\mu} \right) \right] \widehat{\Omega}_n + \alpha_n \|\nabla f_1 x - \nabla f_1 \hat{x}\| \|x_{n+1} - x_n\| \\
 & + \alpha_n \|\nabla f_2 y - \nabla f_2 \hat{y}\| \|y_{n+1} - y_n\| \\
 & + \alpha_n \langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (x_n, y_n) - (\hat{x}, \hat{y}) \rangle + \alpha_n \frac{\eta_n}{\alpha_n} \\
 = & \left[1 - \alpha_n \left(1 - \frac{2\varepsilon}{\mu} \right) \right] \widehat{\Omega}_n + \alpha_n \left(1 - \frac{2\varepsilon}{\mu} \right) R_n, \text{ for all } n \geq n_0,
 \end{aligned} \tag{6.103}$$

where

$$\begin{aligned}
 R_n = & \frac{\mu}{\mu - 2\varepsilon} [\|\nabla f_1 x - \nabla f_1 \hat{x}\| \|x_{n+1} - x_n\| + \|\nabla f_2 y - \nabla f_2 \hat{y}\| \|y_{n+1} - y_n\|] \\
 & + \frac{\mu}{\mu - 2\varepsilon} \left[\langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (x_n, y_n) - (\hat{x}, \hat{y}) \rangle + \frac{\eta_n}{\alpha_n} \right].
 \end{aligned}$$

Rearranging the terms in (6.102), we get

$$\begin{aligned}
 & (1 - \alpha_n) \left(1 - 2 \frac{\eta_n}{\mu} \right) D_{f_1}(a_n, x_n) + (1 - \alpha_n) D_{f_1}(e_n, c_n) + (1 - \alpha_n) \left(1 - 2 \frac{\eta_n}{\mu} \right) D_{f_2}(b_n, y_n) \\
 & + (1 - \alpha_n) D_{f_2}(w_n, d_n) + (1 - \alpha_n) \frac{\rho}{2} \|Ta_n - Sb_n\|^2 \leq \widehat{\Omega}_n - \widehat{\Omega}_{n+1} + 5\alpha_n P,
 \end{aligned} \tag{6.104}$$

where

$$\begin{aligned}
 P = & \max \left\{ \frac{2\varepsilon}{\mu} \widehat{\Omega}_n, \|\nabla f_1 x - \nabla f_1 \hat{x}\| \|x_{n+1} - x_n\|, \|\nabla f_2 y - \nabla f_2 \hat{y}\| \|y_{n+1} - y_n\|, \right. \\
 & \left. \langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (x_n, y_n) - (\hat{x}, \hat{y}) \rangle, \frac{\eta_n}{\alpha_n} \right\}.
 \end{aligned}$$

The rest of the proof is divided into two cases.

Case I. Suppose that the sequence $\{\widehat{\Omega}_n\}$ is nonincreasing. It then follows from the Monotone

Convergence Theorem that the sequence $\{\widehat{\Omega}_n\}$ is convergent, and thus $\lim_{n \rightarrow \infty} (\widehat{\Omega}_n - \widehat{\Omega}_{n+1}) = 0$. Taking limit on both sides of (6.104) as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \|Ta_n - Sb_n\| = 0. \quad (6.105)$$

Moreover, we have

$$\lim_{n \rightarrow \infty} D_{f_1}(a_n, x_n) = \lim_{n \rightarrow \infty} D_{f_1}(e_n, c_n) = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} D_{f_2}(b_n, y_n) = \lim_{n \rightarrow \infty} D_{f_2}(w_n, d_n) = 0,$$

which implies by Lemma 3.1.13 that

$$\lim_{n \rightarrow \infty} \|a_n - x_n\| = \lim_{n \rightarrow \infty} \|e_n - c_n\| = 0, \quad (6.106)$$

and

$$\lim_{n \rightarrow \infty} \|b_n - y_n\| = \lim_{n \rightarrow \infty} \|w_n - d_n\| = 0. \quad (6.107)$$

From (6.77) and (6.105), we obtain

$$\begin{aligned} \|\nabla f_1 c_n - \nabla f_1 a_n\| &= \sigma_n \|T^* J_Z(Ta_n - Sb_n)\| \\ &\leq (\rho + 1) \|T^* J_Z(Ta_n - Sb_n)\| \\ &\leq (\rho + 1) \|T^*\| \|J_Z(Ta_n - Sb_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

which implies by part (iii) of Lemma 3.1.14 that

$$\lim_{n \rightarrow \infty} \|c_n - a_n\| = 0. \quad (6.108)$$

From (6.79) and the condition on α_n , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\nabla f_1 x_{n+1} - \nabla f_1 e_n\| &= \lim_{n \rightarrow \infty} \|\alpha_n \nabla f_1 x + (1 - \alpha_n) \nabla f_1 e_n - \nabla f_1 e_n\| \\ &= \lim_{n \rightarrow \infty} \alpha_n \|\nabla f_1 x - \nabla f_1 e_n\| = 0, \end{aligned}$$

which implies by Lemma 3.1.14 that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - e_n\| = 0. \quad (6.109)$$

Consequently, we obtain from (6.106), (6.108) and (6.109) that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - e_n\| + \|e_n - c_n\| + \|c_n - a_n\| \\ &\quad + \|a_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (6.110)$$

Similarly,

$$\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0. \quad (6.111)$$

Since $X \times Y$ is reflexive and $\{(x_n, y_n)\}$ is bounded, there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of

$\{(x_n, y_n)\}$ such that $(x_{n_k}, y_{n_k}) \rightharpoonup (\bar{x}, \bar{y})$ as $k \rightarrow \infty$ for some $(\bar{x}, \bar{y}) \in X \times Y$ and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (x_n, y_n) - (\hat{x}, \hat{y}) \rangle \\ = \lim_{k \rightarrow \infty} \langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (x_{n_k}, y_{n_k}) - (\hat{x}, \hat{y}) \rangle. \end{aligned}$$

Moreover, we have $x_{n_k} \rightharpoonup \bar{x}$ and $y_{n_k} \rightharpoonup \bar{y}$ as $k \rightarrow \infty$. We have by Lemma 3.1.12 that ∇f_1 is uniformly continuous and hence we obtain from (6.106) that

$$\nabla f_1 c_{n_k} - \nabla f_1 e_{n_k} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (6.112)$$

Similarly, we obtain

$$\nabla f_2 d_{n_k} - \nabla f_2 w_{n_k} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

We now show that $(\bar{x}, \bar{y}) \in \Delta$. From the definition of e_{n_k} , we have

$$\langle A_1 e_{n_k}, z - e_{n_k} \rangle + \left\langle \frac{\nabla f_1 e_{n_k} - \nabla f_1 c_{n_k}}{r_{n_k}}, z - e_{n_k} \right\rangle \geq 0, \text{ for all } z \in X. \quad (6.113)$$

Now, set $z_t = tz + (1-t)\bar{x}$ for all $t \in (0, 1]$ and fixed $z \in X$. We obtain using (6.113) that

$$\begin{aligned} \langle A_1 z_t, z_t - e_{n_k} \rangle &\geq \langle A_1 z_t, z_t - e_{n_k} \rangle - \langle A_1 e_{n_k}, z_t - e_{n_k} \rangle + \left\langle \frac{\nabla f_1 c_{n_k} - \nabla f_1 e_{n_k}}{r_{n_k}}, z_t - e_{n_k} \right\rangle \\ &= \langle A_1 z_t - A_1 e_{n_k}, z_t - e_{n_k} \rangle + \left\langle \frac{\nabla f_1 c_{n_k} - \nabla f_1 e_{n_k}}{r_{n_k}}, z_t - e_{n_k} \right\rangle, \end{aligned} \quad (6.114)$$

which implies by the monotonicity of A_1 and (6.112) that

$$0 \leq \lim_{k \rightarrow \infty} \langle A_1 z_t, z_t - e_{n_k} \rangle = \langle A_1 z_t, z_t - \bar{x} \rangle,$$

and thus

$$\langle A_1 z_t, z - \bar{x} \rangle \geq 0, \text{ for all } z \in X. \quad (6.115)$$

Taking limit as $t \rightarrow 1$ in (6.115) and using the continuity of A_1 , we have that $\langle A_1 z, z - \bar{x} \rangle \geq 0$, for all $z \in X$, and this implies by the maximality of A_1 that $\bar{x} \in N(A_1)$. Similarly, we can show that $\bar{y} \in N(A_2)$.

Moreover, by Lemma 3.1, we have

$$\begin{aligned} \|T\bar{x} - S\bar{y}\|^2 &= \|Ta_{n_k} - Sb_{n_k} + T\bar{x} - Ta_{n_k} + Sb_{n_k} - S\bar{y}\|^2 \\ &\leq \|Ta_{n_k} - Sb_{n_k}\|^2 + 2\langle J_Z(T\bar{x} - S\bar{y}), T\bar{x} - Ta_{n_k} + Sb_{n_k} - S\bar{y} \rangle. \end{aligned} \quad (6.116)$$

From (6.106), (6.107) and weak sequential continuity of T and S , we have $Ta_{n_k} \rightharpoonup T\bar{x}$ and $Sb_{n_k} \rightharpoonup S\bar{y}$. Thus, we obtain from (6.105) and (6.116) that $T\bar{x} = S\bar{y}$. Therefore, $(\bar{x}, \bar{y}) \in \Delta$.

Thus, we obtain by (1.14) that

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (x_n, y_n) - (\hat{x}, \hat{y}) \rangle \\
 = \lim_{k \rightarrow \infty} \langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (x_{n_k}, y_{n_k}) - (\hat{x}, \hat{y}) \rangle \\
 = \langle (\nabla f_1 x, \nabla f_2 y) - (\nabla f_1 \hat{x}, \nabla f_2 \hat{y}), (\bar{x}, \bar{y}) - (\hat{x}, \hat{y}) \rangle \leq 0.
 \end{aligned} \tag{6.117}$$

From (6.110), (6.111) and (6.117), we obtain

$$\limsup_{n \rightarrow \infty} R_n \leq 0. \tag{6.118}$$

From (6.103), (6.118) and Lemma 3.1.6, we conclude that $\lim_{n \rightarrow \infty} \hat{\Omega}_n = 0$ which implies that $\lim_{n \rightarrow \infty} D_{f_1}(\hat{x}, x_n) = \lim_{n \rightarrow \infty} D_{f_2}(\hat{y}, y_n) = 0$ and hence we obtain, by Lemma 3.1.13, that $\lim_{n \rightarrow \infty} \|x_n - \hat{x}\| = \lim_{n \rightarrow \infty} \|y_n - \hat{y}\| = 0$. Therefore, $\lim_{n \rightarrow \infty} x_n = \hat{x} = (\hat{u}, \hat{v}) = \lim_{n \rightarrow \infty} (u_n, v_n)$ and $\lim_{n \rightarrow \infty} y_n = \hat{y} = (\hat{p}, \hat{q}) = \lim_{n \rightarrow \infty} (p_n, q_n)$, where $\hat{u} + K_1 F_1 \hat{u} = 0$ and $\hat{p} + K_2 F_2 \hat{p} = 0$ with $\hat{v} = F_1 \hat{u}$ and $\hat{q} = F_2 \hat{p}$. Moreover, $T_1 \hat{u} = S_1 \hat{p}$.

Case II. Suppose now that there exists a subsequence $\{\hat{\Omega}_{n_i}\}$ of $\{\hat{\Omega}_n\}$ such that $\hat{\Omega}_{n_i} < \hat{\Omega}_{n_i+1}$, for all $i \geq 0$. By Lemma 3.1.7, there exists an increasing sequence $\{m_k\}$ of positive integers such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\hat{\Omega}_{m_k} \leq \hat{\Omega}_{m_k+1} \text{ and } \hat{\Omega}_k \leq \hat{\Omega}_{m_k+1}. \tag{6.119}$$

With this m_k , (6.104) becomes

$$\begin{aligned}
 (1 - \alpha_{m_k}) \left(1 - 2 \frac{\eta_{m_k}}{\mu} \right) D_{f_1}(a_{m_k}, x_{m_k}) + (1 - \alpha_{m_k}) D_{f_1}(e_{m_k}, c_{m_k}) \\
 + (1 - \alpha_{m_k}) \left(1 - 2 \frac{\eta_{m_k}}{\mu} \right) D_{f_2}(b_{m_k}, y_{m_k}) + (1 - \alpha_{m_k}) D_{f_2}(w_{m_k}, d_{m_k}) \\
 + (1 - \alpha_{m_k}) \frac{\rho}{2} \|Ta_{m_k} - Sb_{m_k}\|^2 \leq \hat{\Omega}_{m_k} - \hat{\Omega}_{m_k+1} + 5\alpha_{m_k} P.
 \end{aligned} \tag{6.120}$$

Taking limit as $k \rightarrow \infty$ of both sides of (6.120) and taking the conditions on α_{m_k} and η_{m_k} in to account, we obtain that $\lim_{k \rightarrow \infty} \|Ta_{m_k} - Sb_{m_k}\| = 0$.

Following similar steps as in Case I, we obtain that

$$\lim_{k \rightarrow \infty} \|x_{m_k+1} - x_{m_k}\| = 0 = \lim_{k \rightarrow \infty} \|y_{m_k+1} - y_{m_k}\| \text{ and } \limsup_{k \rightarrow \infty} R_{m_k} \leq 0. \tag{6.121}$$

From (6.103) and (6.119), we obtain

$$\alpha_{m_k} \left(1 - \frac{2\varepsilon}{\mu} \right) \hat{\Omega}_{m_k} \leq \hat{\Omega}_{m_k} - \hat{\Omega}_{m_k+1} + \alpha_{m_k} \left(1 - \frac{2\varepsilon}{\mu} \right) R_{m_k} \leq \alpha_n \left(1 - \frac{2\varepsilon}{\mu} \right) R_{m_k},$$

which implies with the help of (6.121) that $\lim_{k \rightarrow \infty} \hat{\Omega}_{m_k} = 0$. From this and (6.103) we get $\lim_{k \rightarrow \infty} \hat{\Omega}_{m_k+1} = 0$. Thus, we obtain from (6.119) that $\lim_{k \rightarrow \infty} \hat{\Omega}_k = 0$. Therefore, $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (u_n, v_n) = (\hat{u}, \hat{v})$, where $\hat{u}$ is a solution of $u + K_1 F_1 u = 0$ with $\hat{v} = F_1 \hat{u}$. Similarly,

$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} (p_n, q_n) = (\hat{p}, \hat{q})$, where $\hat{p}$ is a solution of $p + K_2 F_2 p = 0$ with $\hat{q} = F_2 \hat{p}$. Thus, the proof is complete. $\square$

If, in Theorem 6.3.2, we assume that F_1 , F_2 , K_1 and K_2 are uniformly continuous monotone mappings, then we obtain the following corollary.

Corollary 6.3.1. *Let E_1 , E_2 and E_3 be reflexive real Banach spaces with dual spaces E_1^* , E_2^* and E_3^* , respectively and let $F_1 : E_1 \rightarrow E_1^*$, $K_1 : E_1^* \rightarrow E_1$, $F_2 : E_2 \rightarrow E_2^*$, $K_2 : E_2^* \rightarrow E_2$ be uniformly continuous monotone mappings. Assume that the conditions (M3) – (M7) hold. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.2 converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Pi$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Proof. As every uniformly continuous mapping is continuous, the result follows from Theorem 6.3.2. $\square$

If we assume $g(\cdot) = \frac{1}{2} \|\cdot\|^2$ and $h(\cdot) = \frac{1}{2} \|\cdot\|^2$ in Theorem 6.3.2, then $\nabla g = J_{E_1}$, $\nabla h = J_{E_2}$, $\nabla g^* = J_{E_1^*}^{-1}$ and $\nabla h^* = J_{E_2^*}^{-1}$. Thus, we get the following corollary.

Corollary 6.3.2. *Assume that the conditions (M1) – (M4), (M6) and (M7) are satisfied. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.2, with $\nabla g = J_{E_1}$, $\nabla h = J_{E_2}$, $\nabla g^* = J_{E_1^*}^{-1}$ and $\nabla h^* = J_{E_2^*}^{-1}$ converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Pi$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

If we assume that E_1, E_2 and E_3 are real Hilbert spaces, and $g(\cdot) = \frac{1}{2} \|\cdot\|^2$ and $h(\cdot) = \frac{1}{2} \|\cdot\|^2$ in Theorem 6.3.2, then $\nabla g = J_{E_1} = I_{E_1}$, $\nabla h = J_{E_2} = I_{E_2}$, $\nabla g^* = J_{E_1^*}^{-1} = I_{E_1^*}$ and $\nabla h^* = J_{E_2^*}^{-1} = I_{E_2^*}$, and hence we obtain the following corollary.

Corollary 6.3.3. *Let E_1, E_2 and E_3 be real Hilbert spaces and assume that the conditions (M2) – (M4), (M6) and (M7) are satisfied. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.2, with $\nabla g = I_{E_1}$, $\nabla h = I_{E_2}$, $\nabla g^* = I_{E_1^*}$ and $\nabla h^* = I_{E_2^*}$, converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Pi$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

6.3.2 Some Theoretical Applications

This section deals with applications of the main result to specific cases.

Common Solutions of Hammerstein Type Equation Problems

If we take $E = E_1 = E_2 = E_3$ and $T_1 = S_1 = I_E$, in Algorithm 6.2, then the SEHTEP reduces to *common solutions of the Hammerstein type equation problems*, which is defined as finding two points $u, p \in E$ such that $u + K_1 F_1 u = 0$, $p + K_2 F_2 p = 0$ and $u = p$. Denote

$$\Gamma = \{(u, p) \in E \times E : u + K_1 F_1 u = 0, p + K_2 F_2 p = 0 \text{ and } u = p\}.$$

Corollary 6.3.4. *Assume that conditions (M1), (M2) and (M5) – (M7), with $E = E_1 = E_2 = E_3$ hold. If $\Gamma \neq \emptyset$, then the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.2 with $T_1 = S_1 = I_E$ converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Gamma$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Corollary 6.3.5. *Assume that conditions (M1) and (M5) – (M7), with $E = E_1 = E_2 = E_3$, hold. Let $F_1, F_2: E \rightarrow E^*$ and $K_1, K_2: E^* \rightarrow E$, be uniformly continuous monotone mappings. Assume that $\Gamma \neq \emptyset$. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.2 with $T_1 = S_1 = I_E$ converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Gamma$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Split Hammerstein Type Equation Problems

If we take $E_2 = E_3$ and $S_1 = I_{E_2}$ in Algorithm 6.2, then the SEHTEP reduces to *split Hammerstein type equation problem* which seeks to find two points $u \in E_1$ and $p \in E_2$ such that $u + K_1 F_1 u = 0$, $p + K_2 F_2 p = 0$ and $T_1 u = p$. Denote

$$\Gamma^* = \{(u, p) \in E_1 \times E_2 : u + K_1 F_1 u = 0, p + K_2 F_2 p = 0 \text{ and } T_1 u = p\}.$$

Corollary 6.3.6. *Assume that conditions (M1) – (M3) and (M5) – (M7) with $E_2 = E_3$ hold. Let $\Gamma^* \neq \emptyset$. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.2, with $S_1 = I_{E_2}$, converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Gamma^*$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Corollary 6.3.7. *Assume that conditions (M1), (M3) and (M5) – (M7) with $E_2 = E_3$ hold. Let $F_1: E_1 \rightarrow E_1^*$, $K_1: E_1^* \rightarrow E_1$, $F_2: E_2 \rightarrow E_2^*$, $K_2: E_2^* \rightarrow E_2$ be uniformly continuous monotone mappings and $\Gamma^* \neq \emptyset$. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 6.2, with $S_1 = I_{E_2}$, converge strongly to u^* , v^* , p^* and q^* , respectively, where $(u^*, p^*) \in \Gamma^*$ with $v^* = F_1 u^*$ and $q^* = F_2 p^*$.*

Hammerstein Type Equation Problems

If we take $E = E_1 = E_2 = E_3$, $F = F_1 = F_2$, $K = K_1 = K_2$, $T_1 = S_1 = 0$ in Algorithm 6.2, then the SEHTEP reduces to *Hammerstein type equation problem* which seeks to find a point $u \in E$ such that $u + K F u = 0$. Denote $\Psi = \{u \in E : u + K F u = 0\}$.

Corollary 6.3.8. *Let E be a reflexive real Banach space with its dual E^* . Let $F: E \rightarrow E^*$ and $K: E^* \rightarrow E$ be continuous monotone mappings. Assume that $\Psi \neq \emptyset$. Let $g: E \rightarrow (-\infty, +\infty]$ be a strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre function on bounded subsets of E with strongly convex conjugate g^* . Let the strong convexity constants of g and g^* be μ_1 and μ_2 , respectively, and let $\mu = \min\{\mu_1, \mu_2\}$. If conditions (M6) and (M7) hold, then the sequence $\{(u_n, v_n)\}$ generated by Algorithm 6.2, with $E = E_1 = E_2 = E_3$, $F = F_1 = F_2$, $K = K_1 = K_2$ and $T_1 = S_1 = 0$, converges strongly to (u^*, v^*) , where $u^* \in \Psi$ and $v^* = F u^*$.*

Corollary 6.3.9. *Let E be a reflexive real Banach space with its dual E^* . Let $F: E \rightarrow E^*$ and $K: E^* \rightarrow E$ be uniformly continuous monotone mappings. Assume that $\Psi \neq \emptyset$. Let $g: E \rightarrow (-\infty, +\infty]$ be a strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre function on bounded subsets of E with strongly convex conjugate g^* . Let the strong convexity constants of g and g^* be μ_1 and μ_2 , respectively, and let $\mu = \min\{\mu_1, \mu_2\}$. If conditions (M6) and (M7) hold, then the sequence $\{(u_n, v_n)\}$ generated*

by Algorithm 6.2, with $E = E_1 = E_2 = E_3$, $F = F_1 = F_2$, $K = K_1 = K_2$ and $T_1 = S_1 = 0$, converges strongly to (u^*, v^*) , where $u^* \in \Psi$ and $v^* = Fu^*$.

6.3.3 Numerical Examples

Example 6.3.1. Let $E = E_1 = E_2 = E_3 = L_p^{\mathbb{R}}([0, 1])$, where $1 < p < \infty$ with norm $\|x\|_{L_p} = \left(\int_0^1 |x(t)|^p dt\right)^{\frac{1}{p}}$. Then, $E^* = E_1^* = E_2^* = E_3^* = L_q^{\mathbb{R}}([0, 1])$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let $g, h: E \rightarrow (-\infty, +\infty]$ be defined by $g(z) = h(z) = \frac{1}{2}\|z\|^2$. Let $F_1: E \rightarrow E^*$, $K_1: E^* \rightarrow E$, $F_2: E \rightarrow E^*$ and $K_2: E^* \rightarrow E$ be defined by $(F_1u)(t) = 5J_p u(t)$, $(K_1v)(t) = \frac{1}{3}J_p^{-1}v(t) + 1$, $(F_2p)(t) = \frac{3}{2}J_p p(t)$ and $(K_2q)(t) = 8J_p^{-1}q(t) + 3$. Clearly, g and h are strongly convex functions satisfying condition (M5) (see, [180]). Moreover, we note that F_1, K_1, F_2 and K_2 are continuous monotone mappings. Since $J_q(\lambda x) = \lambda^{q-1}J_q(x)$ for any $\lambda > 0$ and $J_q = J_p^{-1}$, we have that

$$\begin{aligned} u(t) + K_1 F_1 u(t) = 0 &\iff u(t) + \frac{1}{3}J_p^{-1}(5J_p u(t)) + 1 = 0 \\ &\iff u(t) + \frac{1}{3}J_q(5J_p u(t)) + 1 = 0 \\ &\iff u(t) + \frac{5^{q-1}}{3}u(t) + 1 = 0 \iff u(t) = \frac{-3}{3 + 5^{q-1}}, \end{aligned}$$

that is, the function $u^*(t) = \frac{-3}{3 + 5^{q-1}}$ is the solution of the equation $u(t) + K_1 F_1 u(t) = 0$. Similarly, one obtains that the function $p^*(t) = \frac{-3(2^{q-1})}{2^{q-1} + 8(3^{q-1})}$ is the solution of the equation $p(t) + K_2 F_2 p(t) = 0$. Let $T_1, S_1: E \rightarrow E$ be defined by $(T_1u)(t) = \left(\frac{3 + 5^{q-1}}{3}\right)u(t)$ and $(S_1p)(t) = \left(\frac{2^{q-1} + 8(3^{q-1})}{3(2^{q-1})}\right)p(t)$. Clearly, $(T_1u^*)(t) = -1 = (S_1p^*)(t)$ and hence the point $(u^*(t), p^*(t)) = \left(\frac{-3}{3 + 5^{q-1}}, \frac{-3(2^{q-1})}{2^{q-1} + 8(3^{q-1})}\right)$ is a solution of (1.34). We observe here that the solutions depend on the values of p and q . For simplicity of the computation, we take $p = 2$ so that our space will be $E = L_2^{\mathbb{R}}([0, 1])$ and the mappings F_1, K_1, F_2 and K_2 reduce to $(F_1u)(t) = 5u(t)$, $(K_1v)(t) = \frac{1}{3}v(t) + 1$, $(F_2p)(t) = \frac{3}{2}p(t)$ and $(K_2q)(t) = 8q(t) + 3$. The solutions $u^*(t)$ and $p^*(t)$ reduce to $u^*(t) = \frac{-3}{8}$ and $p^*(t) = \frac{-3}{13}$. Moreover, the mappings T_1 and S_1 reduced to $(T_1u)(t) = \frac{8}{3}u(t)$ and $(S_1p)(t) = \frac{13}{3}p(t)$. Clearly, T_1 and S_1 are bounded linear mappings with adjoints $(T_1u)^*(t) = \frac{8}{3}u(t)$ and $(S_1p)^*(t) = \frac{13}{3}p(t)$. Moreover, $T_1\left(\frac{-3}{8}\right) = -1 = S_1\left(\frac{-3}{13}\right)$. Therefore, we get $(u^*(t), p^*(t)) = \left(\frac{-3}{8}, \frac{-3}{13}\right) \in \Pi$. Let $\eta_n = \frac{1}{n^2 + 1000}$, $\alpha_n = \frac{1}{n + 100000}$, $\mu = 0.9$, $r_n = 1$, $(u, v, p, q) = (1, 0.5, 0.5, 1)$ and $(u_0, v_0, p_0, q_0) = (0, 1, 0, 1)$. Then, conditions (M1) – (M7) are satisfied. We have conducted a numerical experiment for these values to validate our theoretical part. Denote $E_n = (u_n, p_n)$ for each n . Then, the figures below show that the error term sequence $\{E_n - (u^*, p^*)\}$ converges strongly to zero for different choices of initial points and different values of the inertial parameter θ_n , where (u^*, p^*) is the solution of Problem 5.1.

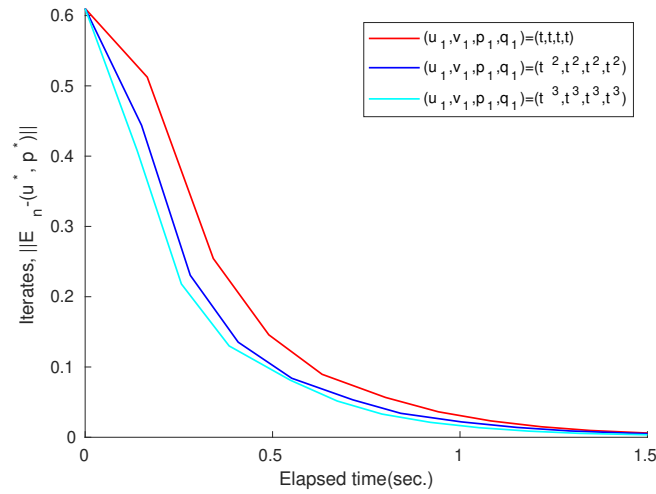


Figure 6.4: Rate of convergence of the sequence $\{(u_n, p_n)\}$ for different initial points and $\theta = 0.6$.

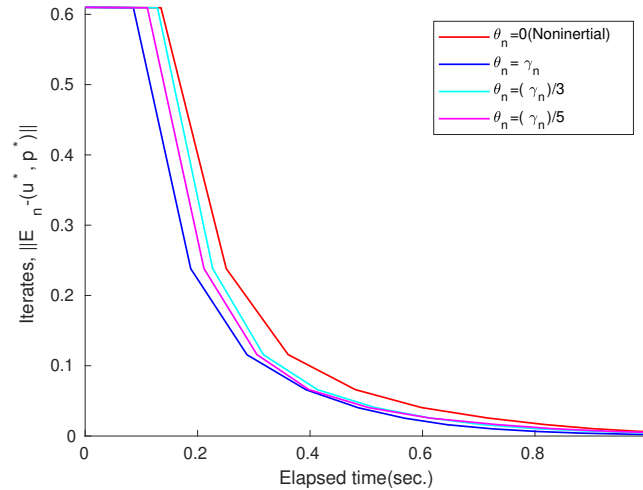


Figure 6.5: Rate of convergence of $\{(u_n, p_n)\}$ for initial point $(u_1, v_1, p_1, q_1) = (1, 0, 1, 0)$.

Remark 6.3.2. Figure 6.4 shows that the rate of convergence of the method is different for different choices of initial points. We observe that the higher the degree of t in the initial point results in faster rate of convergence. Figure 6.5 shows the comparison of rate of convergence of the algorithm with inertia and its non-inertial version. It reveals that the rate of convergence of the method with inertial algorithm is faster than that of the non-inertial one.

Example 6.3.2. $E = E_1 = E_2 = E_3 = [0, \infty)$ with the absolute value norm. Let $g, h: E \rightarrow E$ be defined by $g(\cdot) = h(\cdot) = \frac{1}{2}|\cdot|^2$. Define the mappings $F_1, F_2, K_1, K_2: E \rightarrow E$ by $F_1(u) = u^2$, $K_1(v) = v - 6$, $F_2(p) = p^3$ and $K_2(q) = 4q - 5$. One can easily show that the given mappings are continuous monotone with F_1 and F_2 not uniformly continuous. Moreover, $u^* = 2$ solves the equation

$$u + K_1 F_1(u) = 0 \iff u + K_1(u^2) = 0 \iff u + u^2 - 6 = 0.$$

Thus, $u^* = 2$ is the solution of the Hammerstein type equation $u + K_1 F_1(u) = 0$. Similarly, we can show that $p^* = 1$ is the only non-negative solution of the equation $p + K_2 F_2(p) = 0$. Now, let us define the mappings $T_1, S_1: E \rightarrow E$ by $T_1(u) = \frac{8}{3}u$ and $S_1(p) = \frac{16}{3}p$. Clearly, T_1 and S_1 are bounded linear mappings with adjoints $T_1^*(u) = \frac{8}{3}u$ and $S_1^*(p) = \frac{16}{3}p$. Moreover, $T_1(2) = \frac{16}{3} = S_1(1)$. Thus, $(u^*, p^*) = (2, 1) \in \Pi$. To compute the numerical experiment, we chose $\eta_n = \frac{1}{n^2 + 1000}$, $\alpha_n = \frac{1}{n + 100000}$, $\mu = 0.5$, $\theta = 0.6$, $r_n = 1$, $(u, v, p, q) = (1, 0.5, 0.5, 1)$, $(u_0, v_0, p_0, q_0) = (0, 1, 0, 1)$ and $(u_1, v_1, p_1, q_1) = (1, 0, 1, 0)$. Then, the figure below shows that the error term sequence $\{E_n - (u^*, p^*)\}$ converges strongly to zero for different choices of the inertial parameter θ_n . Here $\theta_n = 0$ corresponds to the non inertial version of the algorithm.

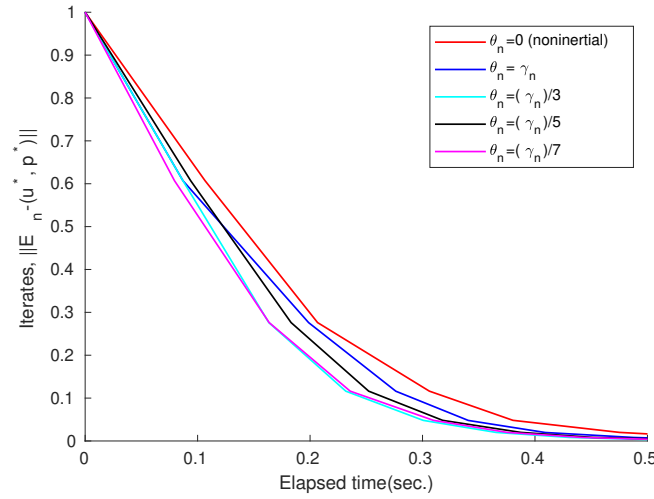


Figure 6.6: Convergence rate for different choices of the inertial parameter θ_n .

Remark 6.3.3. From Figure 6.6, we observe that the sequence $\{(u_n, p_n)\}$ converges strongly to the solution (u^*, p^*) . Moreover, we observe that the rate of convergence of Algorithm 6.2 with inertial parameter ($\theta_n \neq 0$) is relatively faster than that of its non-inertial version ($\theta_n = 0$).

Now, we compare Algorithm 6.1 and Algorithm 6.2 by taking the same control parameters, the same mappings and the same initial points $(u_0, v_0, p_0, q_0) = (0, 1, 0, 1)$, $(u_1, v_1, p_1, q_1) = (1, 0, 1, 0)$.

Algorithm 6.2		Algorithm 6.1
Error	Time(sec.)	Time(sec.)
0.6093	0.0914	87.0000
0.0660	0.3489	185.4800
0.0260	0.5262	214.3448
0.0067	0.7835	236.1087
0.0028	0.9455	247.3149
0.0005	1.3008	293.3143

Table 6.1: Approximate comparison of rates of convergence.

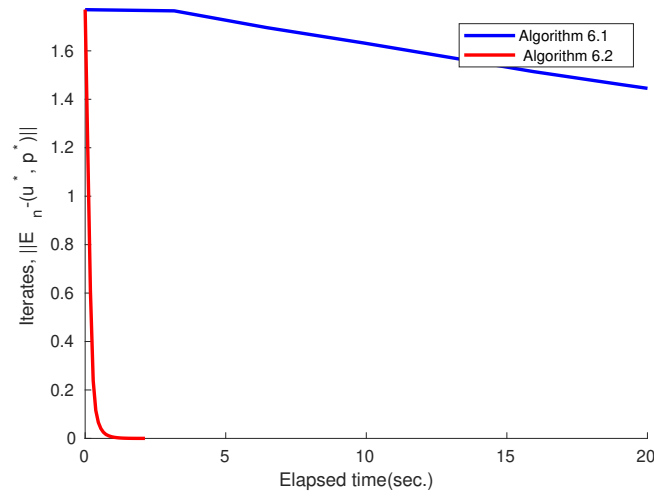


Figure 6.7: Comparison of Alg. 6.1 and Alg. 6.2 with equal parameters and equal initial points.

Remark 6.3.4. Table 6.1 reveals comparison of the rates of convergence of Algorithm 6.2 against Algorithm 6.1. It compares the time taken by each algorithm to reach the same error value. Taking $(u, v, p, q) = (1, 0.5, 0.5, 1)$, $(u_0, v_0, p_0, q_0) = (0, 1, 0, 1)$ and $(u_1, v_1, p_1, q_1) = (1, 0, 1, 0)$, the time taken for Algorithm 6.2 to reach the error value $E = 0.0005$ is 1.3008 seconds, whereas the time taken by Algorithm 6.1 to reach the same error value $E = 0.0005$ is about 293.3143 seconds. Thus, our method is faster than that of 6.1. Observing Figure 6.7, one can give a conclusion which supplements the conclusion drawn in Table 6.1.

7. Conclusion, Contribution to the Current Knowledge and Upcoming Research Works

In this chapter, we put some conclusions, contribution to the existing knowledge and future research works.

7.1 Conclusion and Contribution to the Existing Knowledge

In this section, we draw some conclusions and contributions of this thesis to the existing knowledge.

- In Chapter 3, we introduced inertial iterative algorithms for approximating solutions of split equality fixed point problems involving uniformly continuous pseudocontractive and uniformly continuous f, g -pseudocontractive mappings both in Hilbert and Banach spaces. In Section 3.2, we constructed an inertial iterative algorithm for solving the split equality fixed point problem involving uniformly continuous pseudocontractive mappings in real Hilbert spaces. A convergence analysis was discussed and numerical examples are provided to demonstrate the effectiveness of the algorithm. In Section 3.3, we introduced a strongly convergent inertial method for approximating a solution of the split equality common f, g -fixed point problem in reflexive real Banach spaces. A numerical example was provided to validate the theoretical part of the method. The results obtained extend the result of Osilike [124] to a more general problem with uniformly continuous pseudocontractive mappings which are more general than Lipschitz strict pseudocontractive mappings. Moreover, the results generalize the works of Moudafi and Al-Shemas [116], Che and Li [42] and Chang et al. [40] in the sense that the class of mappings is extended to the class of uniformly continuous pseudocontractive and uniformly continuous f, g -pseudocontractive mappings. The results obtained in Section 3.2 are accepted for publication by the “*Fixed Point Theory Journal*” and the outcomes found in Section 3.3 are published in “*The Journal of Analysis*”, Volume 31, Number 2, (2023), Pages 963–1000.
- In Chapter 4, we constructed an iterative algorithm for solving split equality quasimonotone Minty variational inequality problems. A strong convergence analysis of the method was discussed along with illustrative numerical examples. The results obtained in this chapter extend the works in [87, 101, 169] from variational inequality problem to a more general problem; from Hilbert spaces to more general classes of Banach spaces, and from weak convergence to strong convergence results. The outcomes in this chapter are published in the journal “*Rendiconti del Circolo Matematico di Palermo Series 2*”, 2024, 1–31.
- In Chapter 5, we introduced iterative algorithms for solving variational inequality and

bilevel variational inequality problems involving fixed points of certain mappings. In Section 5.2, we have constructed an inertial method for solving the bilevel variational inequality problem with fixed point constraints involving a demimetric mapping in its lower level problem. A strong convergence result is obtained under the assumption that the underlying mappings in the lower level VIP is uniformly continuous pseudomonotone. Finally, a numerical example was provided to demonstrate the effectiveness of the algorithm. Section 5.3 dealt with an algorithm for approximating a solution of the split equality of variational inequality and f, g -fixed point problems in reflexive real Banach spaces. A strong convergence result was obtained under the assumption that the underlying mappings are uniformly continuous pseudomonotone in the variational inequality part and Bregman relatively f, g -nonexpansive in the f, g -fixed point part. Finally, a numerical example was provided to support the main result. The results obtained in Section 5.2 generalize that of [10, 37, 79, 102, 160] in the sense that the mapping considered in the lower level VIP is a uniformly continuous not just Lipschitz continuous. It also generalizes [12] since the mappings of the lower level problem are extended from inverse strongly monotone to a wider class of pseudomonotone mappings. Moreover, it generalizes the result in [11] as it involves a fixed point problem as additional constraint in the lower level problem. Moreover, it extends the results in [79] because the mapping $T: H_2 \rightarrow H_2$ considered in this work is demimetric which is more general than demicontractive mapping employed in [79]. The results obtained in Section 5.3 improve the result obtained by Wega and Zegeye [172] in the sense that it extends the result from finding solution of variational inequality and f -fixed point problems to a more general problem of finding a solution of split equality of variational inequality and f, g -fixed point problems. It also extends the results obtained by Boikanyo and Zegeye [23] in the sense that it addresses f, g -fixed point problems on top of the split equality variational inequality problems and the setting in this result is a more general reflexive Banach space unlike that of Boikanyo and Zegeye [23] which is just uniformly convex Banach space. The results of Section 5.2 are submitted to the journal of “*Annali dell’Università di Ferrara*” for possible publication. The outcomes found in Section 5.3 are published in the journal “*Nonlinear Functional Analysis and its Applications*”, Volume 28, Number 1, (2023), Pages 135–173.

- In Chapter 6, we proposed and studied iterative algorithms for approximating solutions of split equality Hammerstein type equation problems in reflexive real Banach spaces. In Section 6.2, we constructed an inertial algorithm for solving the split equality Hammerstein type equation problem in reflexive real Banach spaces. A strong convergence result was obtained under the assumption that the underlying mappings are uniformly continuous and monotone. A numerical example was also provided to demonstrate the effectiveness of the proposed method.

In Section 6.3, a strong convergence result was obtained for the approximation of solutions of the split equality Hammerstein type equation problems involving continuous monotone mappings. The results obtained in this chapter generally extend that of Chidume and Zegeye [56], Uba et al. [166] and Bello et al. [18] as the strong convergence of our results

do not depend on the existence of a constant γ_0 , unlike the later ones. Besides this, the results in Section 6.2 and Section 6.3 extend the results of Chidume and Zegeye [56], Uba et al. [166] and Bello et al. [18] from Hilbert space setting to the more general reflexive Banach space. The outcomes of Section 6.2 are published in the “*Carpathian Journal of Mathematics*”, Volume 39, Number 1, Pages 45–72, (2023). The outcomes found in Section 6.3 are published in “*Optimization*”, Pages 1–34, (2023).

7.2 Future Research Works

As far as our knowledge is concerned, the following are still open problems for further research works.

- The strong convergence results obtained in Chapter 3 are under the assumption that the underlying mappings are uniformly continuous. Is it possible to extend these results to a more general class of mappings in more general settings?
- In Chapter 4, we introduced a strongly convergent inertial algorithm for solving the split equality quasimonotone Minty variational inequality problems in reflexive Banach spaces, where the mappings are Lipschitz continuous. Can we extend this result to a more general class of mappings than the class of Lipschitz continuous mappings?
- In Chapter 5, we introduced a strongly convergent inertial algorithm for solving the bilevel variational inequality problem with fixed point constraints in real Hilbert spaces, where the underlying mapping in the lower level variational inequality is uniformly continuous. Can we extend this result to the Banach space settings with mappings more general than uniformly continuous?

We have also discussed the strong convergence analysis of split equality of variational inequality and f, g -fixed point problems in reflexive Banach spaces, where the underlying mappings are uniformly continuous pseudomonotone mappings for the VIP and Bregman relatively f, g -nonexpansive mappings for the fixed point part. Is it possible to extend this result to more general class of mappings than uniformly continuous pseudomonotone mappings?

- Can we construct iterative algorithms which minimize the conditions of all methods we introduced?

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