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MULTI-PARAMETRIC APPROACH FOR MULTILEVEL MULTI-LEADER-MULTI-FOLLOWER GAMES USING EQUIVALENT REFORMULATIONS

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Abstract: Multilevel multi-leader multi-follower games address compromises among multiple interacting decision agents within a hierarchical system in which multiple followers are involved at each lower-level unit and more than one decision maker (multiple leaders) are involved in the upper-level. The leaders' decisions are affected not only by reactions of the followers but also by various relationships among the leaders themselves. In general, multiple-leaders multiple-followers (MLMF) game serve as an important modeling tool in game theory with many applications in economics, engineering, operations research and other fields. In this paper, we have reformulated a multilevel-MLMF game into an equivalent multilevel single-leader multi-follower (SLMF) game by introducing a suppositional (or dummy) leader, and hence the multiple leaders in the original problem become followers in the second level. If the resulting multilevel-SLMF game consists of separable terms and parameterized common terms across all the followers, then the problem is further transformed into equivalent multilevel programs having a single leader and single follower at each level of the hierarchy. The proposed solution approach can solve multilevel multi-leader multi-follower

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problems whose objective values at all levels have common but having different positive weights of non-separable terms. This result improves the work of Kulkarni and Shanbhag (2015).

Keywords: multilevel programming; multi-leader multi-follower; Nash equilibrium; multi-parametric programming; equivalent reformulation; branch-and-bound.

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1. INTRODUCTION

Nash games model competitive behavior among a set of players that make simultaneous decisions. Nash equilibrium is a set of strategies in which each individual player has chosen an optimal strategy given the strategies chosen by the other players. On the other hand, Stackelberg (single-leader-follower) game arises when one player, called the *leader*, commits to a strategy, while the remaining players, called *followers*, react to the strategy selected by competing among them [2]. That is, the reaction of the followers is a Nash equilibrium parameterized by the decision variables from the leader. The leader chooses an optimal strategy anticipating how the followers will react. This choice of the leader requires a complete knowledge of followers' reaction for each of his/her action. Based on the number of hierarchical levels and based on the number of decision makers at the level of the leader and the followers in the system. If the number of levels is only two, the problem is called a bilevel leader-follower game, and if there is only one leader in the system with multiple decision makers at the follower's levels the problem is named as a single-leader-multiple-followers (SLMF) game.

Multi-leader-follower games are a class of hierarchical games in which a collection of leaders compete in a Nash game constrained by the equilibrium conditions of another Nash game amongst the followers. Generally, in a game, when several players take the position as leaders and the rest of players take the position as followers, it becomes a multi-leader-follower game. The multi-leader-follower game may further be classified into the game which contains only one follower, called the *multi-leader single-follower game*, and the game which contains multiple followers, called the *multi-leader multi-follower game*. The leader-follower Nash equilibrium, a solution

concept for the multi-leader-follower game, can be defined as a set of leaders' and followers' strategies such that no player (leader or follower) can improve his/her status by changing his/her own current strategy unilaterally.

A general k -level multi-leader multi-follower game involving N leaders and multiple followers at each level can be described mathematically as:

$$\begin{aligned}
 & \min_{y_1^n \in Y_1^n} F_1^n(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}), n \in \{1, \dots, N\} \\
 & \text{s. t. } G_1^n(y_1^n, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
 & \quad H_1(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
 & \min_{y_2^i \in Y_2^i} f_2^i(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}), i \in \{1, \dots, I\} \\
 & \text{s. t. } g_2^i(y_1^n, y_1^{-n}, y_2^i, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
 & \quad h_2(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
 & \quad \vdots \\
 & \min_{y_k^l \in Y_k^l} f_k^l(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}), l \in \{1, \dots, L\} \\
 & \text{s. t. } g_k^l(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l) \leq 0 \\
 & \quad h_k(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0
 \end{aligned} \tag{1.1}$$

where $y_1^n \in Y_1^n$ is a decision vector for the leader's optimization problem and y_1^{-n} is a vector of the decision variables for all leaders without the vector of decision variables y_1^n , of the n^{th} leader. i.e., $y_1^{-n} = (y_1^1, \dots, y_1^{n-1}, y_1^{n+1}, \dots, y_1^N)$, $n = 1, 2, \dots, N$. The shared constraint H_1 is the leaders' common constraint set whereas, the constraint G_1^n determines the constraint only for the n^{th} leader. $y_m^c \in Y_m^c$ is a decision vector for the c^{th} follower at level m , and y_m^{-c} is a vector of decision variables for all followers at level m without the vector of decision variables y_m^c , of follower c . i.e., $y_m^{-c} = (y_m^1, \dots, y_m^{c-1}, y_m^{c+1}, \dots, y_m^N)$, where $c = i, j, \dots, l$ and $m \in \{2, 3, \dots, k\}$. The shared constraint h_m is the m^{th} level followers' common constraint set, whereas the constraint g_m^c determines the constraint only for the c^{th} follower at the m^{th} level optimization problem.

One of a mathematical formulation to solve multi-leader-follower type games is the equilibrium problem with equilibrium constraints (EPEC). An EPEC is an equilibrium problem consisting of several parametric mathematical programs with equilibrium constraints (MPECs) which contain

the strategies of other players as parameters. The equilibria of an EPEC can be achieved when all MPECs are solved simultaneously. That means, the equilibrium amongst the followers is compactly captured by an equilibrium constraint in the optimization problem of a leader, whereby each leader faces a MPECs. The equilibrium amongst leaders is captured by an EPEC and its associated equilibria. Multilevel multi-leader-follower problems have the property that the EPECs at the lower level are parametric problems as opposed to the bilevel multi-leader multi-follower problem.

The early study associated with the multi-leader-follower game and EPEC could date back to 1984 by Sherali [16], where a multi-leader-follower game was called a multiple Stackelberg model. While multi-leader generalizations were touched upon by Okuguchi [13] and Sherali [16] presented amongst the first models for multi-leader-follower games in a Cournot regime. Sherali [16] established existence of an equilibrium by assuming that each leader can exactly anticipate the aggregate follower reaction curve. He also showed the uniqueness of equilibrium for a special case where all leaders share an identical cost function and make identical decisions.

As Ehrenmann [3] pointed out, the assumption that all leaders make identical decisions is essential for ensuring the uniqueness result. He also gave a counterexample to show that, when all leaders with identical cost functions make different decisions, the game could reach multiple equilibria. In addition, Su [18] considered a forward market equilibrium model that extended the existence result of Sherali [16] under some weaker assumptions. Pang and Fukushima [14] considered a class of remedial models for the multi-leader-follower game that can be formulated as a generalized Nash equilibrium problem (GNEP) with convexified strategy sets. They further defined a new equilibrium concept called remedial leader-follower Nash equilibrium and presented an existence result with this equilibrium concept. Moreover, in the same paper the authors also formulated some examples about oligopolistic electricity market that lead to the multi-leader-follower games. Based on the strong stationary conditions of each leader in a multi-leader-follower game, Leyffer and Munson [12] derived a family of nonlinear complementarity problem, nonlinear

program, and MPEC formulations of the multi-leader multi-follower games. They also introduced an alternative price-consistent formulation of the multi-leader-follower game that gives rise to a square complementarity problem.

Su [17] proposed a sequential nonlinear complementarity problem (NCP) approach for solving EPECs. The approach is related to the relaxation approach used in MPECs [15] that relaxes the complementarity condition of each leader and drives the relaxation parameter to zero. Hu and Fukushima [7] also considered the EPEC approach in solving bilevel multi-leader-follower games for separable cases and assuming existence of unique Nash equilibrium reaction from the followers.

There are several instances of EPECs for which equilibria have been shown to exist, but there are also fairly simple EPECs which admit no equilibria as shown in [14]. Definitive statements on the existence of equilibria have been obtained mainly for two level multi-leader-follower games with specific structure. In the majority of these settings, the uniqueness of the follower-level equilibrium is assumed to construct an implicit form (such as problems with convex strategy sets) which allows for the application of standard fixed-point theorems of Brouwer and Kakutani [1,5]. Indeed, when the feasible region of the EPEC is convex and compact, the two-level multi-leader multi-follower game can be thought of as a conventional Nash game or a generalized Nash game and the existence of a global equilibrium follows from classical results. But the equilibrium constraint in an EPEC is known for being non-convex and for lacking the continuity properties required to apply fixed point theory. Consequently, most standard approaches fail to apply to EPECs and there currently exists no general mathematical paradigm that could be built upon to make a theory for general EPECs.

Kulkarni [10] identified subclasses of the non-shared constraint multi-leader multi-follower games for which the existence of equilibria can be guaranteed and showed that when the leader-level problem admits a potential function, the set of global minimizers of the potential function over the shared constraint are the equilibria of the multi-leader multi-follower game; in effect, this reduces to a question of the existence of an equilibrium to that of the existence of a solution to an MPEC.

Equilibrium to MPEC exists under standard conditions and the existence of a global equilibrium was seen to follow. The same authors further showed that local minima, B-stationary points, strong-stationary points and second-order strong stationary points of this MPEC are respectively, local Nash equilibria, Nash B-stationary points, Nash strong-stationary points and Nash second-order strong-stationary points of the shared constraint multi-leader multi-follower game.

Motivated by quasi-potential games through an application in communication networks, Kulkarni and Shanbhag [11] also showed that, under the assumption that the objectives of the leaders admit a quasi-potential function, the global and local minimizers of a suitably defined optimization problem are the global and local equilibria of the game. In effect, existence of equilibria can be guaranteed by the solvability of an optimization problem which holds under mild conditions. Their result was a general existence result for equilibria for this class of games. Because they impose no single-valuedness assumption on the equilibrium of the follower level game.

A reformulation of the generalized Nash equilibrium problem into a special bilevel programming problem is studied in [19]. In this study it is shown that the generalized Nash equilibrium problem can be transformed into an equivalent bilevel programming problem having a single decision maker at both levels.

Single-leader and multiple followers (SLMF) games are considered in [9, 20] using different approaches. Tharakunnel & Bhattacharyya [20] used the so called “reinforcement learning” approach with Q-learning scheme to propose an algorithm to solve bilevel-SLMF games. Kassa and Kassa [9] have reformulated the class of multilevel programs with single leader and multiple followers that consist of separable terms and parameterized common terms across all the followers, into equivalent multilevel programs having single follower at each level. Then the resulting (possibly non-convex) multilevel problem is solved by a solution strategy, called a branch-and-bound multi-parametric programming approach, they have developed in [8]. The method works through successive convex relaxation of the inner level problems for fixed parameters followed by the application of MPP procedures.

The solution approaches for multilevel multi-leader-follower problems that are proposed so far are sensitive to the way the criteria functions at each of the levels are formulated and most of them work only for two level of decisions. Moreover, the development of implementable solution algorithms for multilevel-MLMF games is at its infancy, and researchers are still working on this direction. In this paper we will focus on equivalent reformulations of multi-leader problems into single-leader problem type, apply the solution approaches for SLMF problems.

The remaining part of the paper is organized as follows. In the next section the mathematical description of the problem is presented, and Section 3 establishes the proposed framework for equivalent reformulations. Section 4 outlines the solution procedure of multilevel-MLMF problems using the reformulation approach and with the use of multiparametric programming approach. Some concrete examples are used to demonstrate the proposed method in Section 5 followed by conclusive remarks in Section 6.

2. EQUIVALENT REFORMULATIONS OF BILEVEL-MLMF GAMES

In this section, we will consider the possibilities of reformulating bilevel games with multiple leaders and multiple followers first as bilevel games having a single leader and multiple followers.

Consider a bilevel multi-leader multi-follower (bilevel-MLMF) game involving N leaders in the upper level and M followers at lower level which is defined as:

$$\begin{aligned}
 & \min_{x^i \in X^i} F_i(x^i, x^{-i}, y^j, y^{-j}) \\
 & \text{s. t. } G_i(x^i, y^j, y^{-j}) \leq 0 \\
 & \quad H(x^i, x^{-i}, y^j, y^{-j}) \leq 0 \\
 & \quad \min_{y^j \in Y^j} f_j(x^i, x^{-i}, y^j, y^{-j}) \\
 & \quad \text{s. t. } g_j(x^i, x^{-i}, y^j) \leq 0 \\
 & \quad \quad h(x^i, x^{-i}, y^j, y^{-j}) \leq 0
 \end{aligned} \tag{2.1}$$

Let us assume that $F_i, G_i, H, h, f_j, g_j, i = 1, 2, \dots, N, j = 1, 2, \dots, M$ are twice continuously differentiable functions and that the followers' constraint functions satisfy the Guignard constraint qualifications conditions and let us define some relevant sets related to problem (1) as follows:

(i) The feasible set of problem (2.1) is given by:

$$\mathcal{A} = \{(x^i, x^{-i}, y^j, y^{-j}): g_j(x^i, x^{-i}, y^j) \leq 0, h(x^i, x^{-i}, y^j, y^{-j}) \leq 0, G_i(x^i, y^j, y^{-j}) \leq 0, H(x^i, x^{-i}, y^j, y^{-j}) \leq 0, i = 1, \dots, N, j = 1, \dots, M\}.$$

(ii) The feasible set for the j^{th} follower (for any leaders strategy $x = (x^i, x^{-i})$) can be defined as $\mathcal{A}_j(x^i, x^{-i}, y^{-j}) = \{y^j \in Y^j: g_j(x^i, x^{-i}, y^j) \leq 0, h(x^i, x^{-i}, y^j, y^{-j}) \leq 0\}$

(iii) The Nash rational reaction set for the j^{th} follower is defined by the set of parametric solutions,

$$\mathcal{B}_j(x^i, x^{-i}, y^{-j}) = \{\bar{y}^j \in Y^j: \bar{y}^j \in \operatorname{argmin}\{f_j(x^i, x^{-i}, y^j, y^{-j}): y^j \in \mathcal{A}_j(x^i, x^{-i}, y^{-j})\}, j = 1, \dots, M\}.$$

(iv) The feasible set for the i^{th} leader is defined as

$$\mathcal{A}_i(x^{-i}) = \{(x^i, y^j, y^{-j}) \in X^i \times Y^j \times Y^{-j}: G_j(x^i, y^j, y^{-j}) \leq 0, H(x^i, x^{-i}, y^j, y^{-j}) \leq 0, g_j(x^i, x^{-i}, y^j) \leq 0, h(x^i, x^{-i}, y^j, y^{-j}) \leq 0, y^j \in \mathcal{B}_j(x^i, x^{-i}, y^{-j}), j = 1, \dots, M\}$$

(v) The Nash rational reaction set for the i^{th} leader is defined as

$$\mathcal{B}_i(x^{-i}) = \{(x^i, y^j, y^{-j}) \in X^i \times Y^j \times Y^{-j}: x^i \in \operatorname{argmin}\{F_i(x^i, x^{-i}, y^j, y^{-j}): (x^i, y^j, y^{-j}) \in \mathcal{A}_i(x^{-i})\}, i = 1, \dots, N\}.$$

(vi) The set of Nash equilibrium points (optimal solutions) of problem (2.1) is given by

$$\mathcal{S} = \{(x^i, x^{-i}, y^j, y^{-j}): (x^i, x^{-i}, y^j, y^{-j}) \in \mathcal{A}, (x^i, y^j, y^{-j}) \in \mathcal{B}_i(x^{-i}), i = 1, \dots, N\}$$

Now we shall formulate an equivalent trilevel single-leader multi-follower (trilevel-SLMF) programming problem for (2.1) and we will show their equivalence.

Let us add an upper level decision maker, a suppositional (or dummy) leader, to the problem (2.1) with the corresponding decision variable z , where $z = (x, y) = (x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^m)$, and objective function equal to a constant α . Then the multiple leaders in the upper level problem of (2.1) become middle-level followers and the multiple followers in the lower level problem of (2.1) become bottom-level followers in the second level and we will get the following trilevel-SLMF programming:

$$\begin{aligned}
 & \min_z \alpha \\
 & \text{s. t. } z = (x, y) \\
 & \quad \min_{x^i \in X^i} F_i(x^i, x^{-i}, y^j, y^{-j}) \\
 & \quad \text{s. t. } G_i(x^i, y^j, y^{-j}) \leq 0 \\
 & \quad \quad H(x^i, x^{-i}, y^j, y^{-j}) \leq 0 \\
 & \quad \quad \min_{y^j \in Y^j} f_j(x^i, x^{-i}, y^j, y^{-j}) \\
 & \quad \quad \text{s. t. } g_j(x^i, x^{-i}, y^j) \leq 0 \\
 & \quad \quad \quad h(x^i, x^{-i}, y^j, y^{-j}) \leq 0
 \end{aligned} \tag{2.2}$$

Let us assume that each of the objective functions is convex with respect to its own decision variable for the second and third level followers and the Guignard constraint qualifications condition hold for the follower's constraints and let us define some relevant sets related to problem (2.2) as follows:

(i) The feasible set for the third level follower's problem is defined as:

$$\Omega_3(x^i, x^{-i}, y^j) = \{y^j \in Y^j : g_j(x^i, x^{-i}, y^j) \leq 0, h(x^i, x^{-i}, y^j, y^{-j}) \leq 0\}.$$

(ii) The rational reaction set for the third level followers problem is given by a set of parametric

$$\text{solutions, } \Psi_3(x^i, x^{-i}, y^{-j}) = \{\bar{y}^j \in Y^j : \bar{y}^j \in \operatorname{argmin}\{f_j(x^i, x^{-i}, y^j, y^{-j}) : y^j \in$$

$$\Omega_3(x^i, x^{-i}, y^j)\}, j = 1, \dots, M\}.$$

(iii) The feasible set for the second level problem is given by

$$\begin{aligned}\Omega_2(x^{-i}) &= \{(x^i, y^j, y^{-j}) \in X^i \times Y^j \times Y^{-j} : G_j(x^i, y^j, y^{-j}) \leq 0, H(x^i, x^{-i}, y^j, y^{-j}) \\ &\leq 0, g_j(x^i, x^{-i}, y^j) \leq 0, h_j(x^i, x^{-i}, y^j, y^{-j}) \leq 0, y^j \in \Psi_3(x^i, x^{-i}, y^{-j}), j \\ &= 1, \dots, M\}\end{aligned}$$

(iv) The rational reaction set for the second level followers' problem is defined as:

$$\begin{aligned}\Psi_2(x^{-i}) &= \{(x^i, y^j, y^{-j}) \in X^i \times Y^j \times Y^{-j} : x^i \\ &\in \operatorname{argmin}\{F_i(x^i, x^{-i}, y^j, y^{-j}) : (x^i, y^j, y^{-j}) \in \Omega_2(x^{-i})\}, i = 1, \dots, N\}.\end{aligned}$$

(v) The feasible set of problem (2.2) is given by:

$$\begin{aligned}\Phi &= \{(z, x^i, x^{-i}, y^j, y^{-j}) : z = (x, y), G_j(x^i, y^j, y^{-j}) \leq 0, H(x^i, x^{-i}, y^j, y^{-j}) \\ &\leq 0, g_j(x^i, x^{-i}, y^j) \leq 0, h_j(x^i, x^{-i}, y^j, y^{-j}) \leq 0, i = 1, \dots, N, j \\ &= 1, \dots, M\}\end{aligned}$$

(vi) The inducible region of problem (2.2) is given by

$$\begin{aligned}\mathcal{IR} &= \{(z, x^i, x^{-i}, y^j, y^{-j}) : (z, x^i, x^{-i}, y^j, y^{-j}) \in \Phi, (x^i, y^j, y^{-j}) \in \Psi_2(x^{-i}), i \\ &= 1, \dots, N\}\end{aligned}$$

With these definitions of sets, problem (2.2) could be equivalently rewritten as:

$$\begin{aligned}\min_{\substack{z \\ \text{s.t.}}} \alpha \\ \text{s.t. } (z, x^i, x^{-i}, y^j, y^{-j}) \in \mathcal{IR}\end{aligned} \quad (2.3)$$

Since every feasible point of (2.3) is an optimal point, the optimal set of (2.3) is given by

$$\mathcal{S}^* = \mathcal{IR} = \{(z, x^i, x^{-i}, y^j, y^{-j}) : (z, x^i, x^{-i}, y^j, y^{-j}) \in \Phi, (x^i, y^j, y^{-j}) \in \Psi_2(x^{-i}), i = 1, \dots, N\}$$

Once we have established relations between the bilevel-SLMF problem (2.1) and the trilevel-SLMF problem (2.2). We will describe their equivalence with the following theorem.

Theorem 2.1 A point $(x^{*,i}, x^{*-i}, y^{*,j}, y^{*,-j})$ is an optimal solution to (2.1) if and only if $(z^*, x^{*,i}, x^{*-i}, y^{*,j}, y^{*,-j})$ is an optimal solution to (2.3).

Proof: Suppose that $(x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j})$ is an optimal solution to (2.1), i.e., $(x^i, x^{-i}, y^j, y^{-j}) \in \mathcal{S}$ which implies that $(x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \in \mathcal{A}, (x^i, y^j, y^{-j}) \in \mathcal{B}_i(x^{*-i}), i = 1, \dots, N$. That means,

$$(x^{*,i}, y^{*,j}, y^{*, -j}) \in \Psi_2(x^{*-i}), g_j(x^{*,i}, x^{*-i}, y^{*,j}) \leq 0, h(x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \leq 0,$$

$$G_i(x^i, y^j, y^{-j}) \leq 0, H(x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \leq 0, i = 1, \dots, N, j = 1, \dots, M.$$

Then for any point $(z^*, x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j})$ where $z^* = (x^*, y^*)$ and $(x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \in S$, we have

$$(x^{*,i}, y^{*,j}, y^{*, -j}) \in \Psi_2(x^{*-i}), z^* = (x^*, y^*),$$

$$g_j(x^{*,i}, x^{*-i}, y^{*,j}) \leq 0, h(x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \leq 0, j = 1, \dots, M$$

$$G_i(x^{*,i}, y^{*,j}, y^{*, -j}) \leq 0, H(x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \leq 0, i = 1, \dots, N.$$

This implies that $(z^*, x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \in \Phi$ and $(x^{*,i}, y^{*,j}, y^{*, -j}) \in \Psi_2(x^{*-i})$. Therefore, $(z^*, x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \in \mathcal{IR} = \mathcal{S}^*$ and hence $(z^*, x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j})$ is an optimal solution to (2.3).

Conversely, suppose that $(z^*, x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j})$ is an optimal solution to (2.3), i.e., $(z^*, x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \in \mathcal{S}^*$ then we have $(z^*, x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \in \Phi$ and $(x^{*,i}, y^{*,j}, y^{*, -j}) \in \Psi_2(x^{*-i})$. This implies the following

$$(x^{*,i}, y^{*,j}, y^{*, -j}) \in B_i(x^{*-i}),$$

$$g_j(x^{*,i}, x^{*-i}, y^{*,j}) \leq 0, h(x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \leq 0, j = 1, \dots, M$$

$$G_i(x^{*,i}, y^{*,j}, y^{*, -j}) \leq 0, H(x^{*,i}, x^{*-i}, y^{*,j}, y^{*, -j}) \leq 0, i = 1, \dots, N.$$

That is, $(x^{*,i}, x^{*-i}, y^{*,j}, y^{*-j}) \in \mathcal{A}$, $(x^{*,i}, y^{*,j}, y^{*-j}) \in \mathcal{B}_i(x^{*, -i}), i = 1, \dots, N$.

Therefore $(x^{*,i}, x^{*-i}, y^{*,j}, y^{*-j}) \in \mathcal{S}$ and hence $(x^{*,i}, x^{*-i}, y^{*,j}, y^{*-j})$ is an optimal solution to (2.1).

Remark 2.1: The idea described above can be extended to any finite k -level multi-leader multi-follower programming problem. By adding a dummy upper decision maker, problem (1.1) can be equivalently reformulated as $(k + 1)$ -level SLMF game. As a result, leaders in the upper-level problem of (1.1) become followers at the second-level and followers at m^{th} -level problem of (1.1) become followers at $(m + 1)^{th}$ -level, where $m \in \{2, \dots, k\}$.

3. EQUIVALENT REFORMULATION OF MULTILEVEL-MLMF GAMES

In this section we extend the reformulation discussed in Section 2 to the general k -level case. Consider a class of problem (1.1) with a property that each leaders' objective function consisting of separable terms and parameterized common terms across all leaders with positive weights $\rho_1 \in \mathbb{R}_+^N$, i.e., for each n^{th} leader we have:

$$\begin{aligned} F_1^n(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \\ = \hat{F}_1^n(y_1^n, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) + F_1^n(y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \\ + \rho_1^n \tilde{F}_1^n(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}), \end{aligned}$$

with $\rho_1^n > 0$ for each $n = 1, 2, \dots, N$, and with a property that at all levels in the hierarchy each followers' objective function consisting of separable terms and parameterized common terms across all followers of the same level with positive weights $\rho_m^c \in \mathbb{R}_+^N$, i.e., for the c^{th} follower at m^{th} -level, $m \in \{2, 3, \dots, k\}$, we have

$$f_m^c(y_1^n, y_1^{-n}, y_m^c, y_m^{-c}) = \hat{f}_m^c(y_1^n, y_1^{-n}, y_m^c) + \bar{f}_m^c(y_1^n, y_1^{-n}, y_m^{-c}) + \rho_m^c \tilde{f}_m^c(y_1^n, y_1^{-n}, y_m^c, y_m^{-c})$$

with $\rho_m^c > 0$, for each $m = 2, 3, \dots, k$ and $c = i, j, \dots, l$.

With these assumptions and using Remark 2.1, problem (1.1) can be reformulated as:

$$\begin{aligned}
& \min_x \alpha \\
& \text{s.t. } x = (y_1, y_2, \dots, y_n) \\
& \min_{y_1^n \in Y_1^n} \{ \hat{F}_1^n(y_1^n, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) + \bar{F}_1^n(y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \\
& \quad + \rho_1^n \tilde{F}_1^n(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \}, n \in \{1, \dots, N\} \\
& \text{s.t. } G_1^n(y_1^n, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
& \quad H_1(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
& \min_{y_2^i \in Y_2^i} \{ \hat{f}_2^i(y_1^n, y_1^{-n}, y_2^i, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) + \bar{f}_2^i(y_1^n, y_1^{-n}, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \\
& \quad + \rho_2^i \tilde{f}_2^i(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \}, i \in \{1, \dots, I\} \\
& \text{s.t. } g_2^i(y_1^n, y_1^{-n}, y_2^i, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
& \quad h_2(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
& \quad \vdots \\
& \min_{y_k^l \in Y_k^l} \{ \hat{f}_k^l(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l) + \bar{f}_k^l(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^{-l}) \\
& \quad + \rho_k^l \tilde{f}_k^l(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \}, l \in \{1, \dots, L\} \\
& \text{s.t. } g_k^l(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l) \leq 0 \\
& \quad h_k(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0
\end{aligned} \tag{3.1}$$

Then as it was shown in [9], problem (3.1) can be reformulated into its equivalent hierarchical multilevel programming problem having single decision maker at each decision level:

$$\begin{aligned}
& \min_x \alpha \\
& \text{s.t. } x = (y_1, y_2, \dots, y_n) \\
& \min_{y_1^n \in Y_1^n} \left\{ \frac{1}{\rho_1^n} \hat{F}_1^n(y_1^n, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) + \tilde{F}_1^n(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \right\} \\
& \text{s.t. } G_1^n(y_1^n, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
& \quad H_1(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
& \min_{y_2^i \in Y_2^i} \left\{ \frac{1}{\rho_2^i} \hat{f}_2^i(y_1^n, y_1^{-n}, y_2^i, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) + \tilde{f}_2^i(y_1^n, y_1^{-n}, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \right\} \\
& \text{s.t. } g_2^i(y_1^n, y_1^{-n}, y_2^i, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
& \quad h_2(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0 \\
& \quad \vdots \\
& \min_{y_k^l \in Y_k^l} \left\{ \frac{1}{\rho_k^l} \hat{f}_k^l(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l) + \tilde{f}_k^l(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \right\} \\
& \text{s.t. } g_k^l(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l) \leq 0 \\
& \quad h_k(y_1^n, y_1^{-n}, y_2^i, y_2^{-i}, y_3^j, y_3^{-j}, \dots, y_k^l, y_k^{-l}) \leq 0
\end{aligned} \tag{3.2}$$

Proposition 3.1: A point $(y_1^{*,n}, y_1^{*,-n}, y_2^{*,i}, y_2^{*,-i}, y_3^{*,j}, y_3^{*,-j}, \dots, y_k^{*,l}, y_k^{*,-l})$ is an optimal solution to (3.1) if and only if $(x^*, y_1^{*,n}, y_1^{*,-n}, y_2^{*,i}, y_2^{*,-i}, y_3^{*,j}, y_3^{*,-j}, \dots, y_k^{*,l}, y_k^{*,-l})$ is an optimal solution to (3.2).

Proof: Follows from Theorem 2.1 and the equivalence of (3.1) and (3.2).

4. SOLUTION APPROACH FOR SPECIAL CLASSES OF MULTILEVEL-MLMF GAMES

In this section we suggest an appropriate solution method to solve a multilevel program with multiple leader and multiple followers at each decision level. And we introduce a pseudo algorithmic approach to solve some classes of multilevel program with multiple leader and multiple followers.

The basic steps of the proposed algorithm were as follows:

- (1) Reformulate the given multilevel program with multiple leaders and multiple followers into equivalent multilevel program with single leader and multiple followers as discussed in Section 2 and 3.
- (2) If the resulting problems in step (1) above have a property that at all levels in the hierarchy each follower's objective function consisting of separable terms and parameterized common terms across all followers of the same level, then it can be reformulated into equivalent multilevel program having a single follower over the hierarchy as discussed in Section 4.
- (3) Then to solve the resulting problem in step (2) above, one can apply any known method that can effectively solve multilevel single-leader-follower problems.

For the examples below, in particular, we apply the multi-parametric programming methods proposed in [6] and [9].

5. ILLUSTRATIVE EXAMPLES

Example 1. Consider the following nonlinear bilevel-MLMF programming problem:

$$\begin{aligned}
 & \left\{ \min_{x_1} F_1(x_1, x_2, y_1, y_2) = x_1^2 - x_1 x_2 - x_1 + x_1 y_1 \right. \\
 & \left. \min_{x_2} F_2(x_1, x_2, y_1, y_2) = x_2^2 - \frac{1}{2} x_1 x_2 - 2x_2 + y_2 \right\} \\
 & \text{s. t. } x_1 + x_2 \leq 1.5 \\
 & \left\{ \min_{y_1} f_1(x_1, x_2, y_1, y_2) = \frac{1}{2} y_1^2 + y_1 y_2 + y_1 - x_1 y_1 \right. \\
 & \left. \min_{y_2} f_2(x_1, x_2, y_1, y_2) = \frac{1}{2} y_2^2 + y_1 y_2 + y_2 - x_2 y_2 \right\} \\
 & \text{s. t. } 2y_1 + y_2 + x_1 - 2x_2 \leq 3, y_1 + y_2 \leq 2.5 \\
 & 0 \leq x_1, x_2 \leq 1, 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 1
 \end{aligned} \tag{5.1}$$

An equivalent tri-level single-leader multi-follower problem for (5.1) is given by:

$$\begin{aligned}
 & \min_z \alpha \\
 & \text{s. t. } z = (x, y) \\
 & \left\{ \min_{x_1} F_1(x_1, x_2, y_1, y_2) = x_1^2 - x_1 x_2 - x_1 + x_1 y_1 \right. \\
 & \left. \min_{x_2} F_2(x_1, x_2, y_1, y_2) = x_2^2 - \frac{1}{2} x_1 x_2 - 2x_2 + y_2 \right\} \\
 & \text{s. t. } x_1 + x_2 \leq 1.5 \\
 & \left\{ \min_{y_1} f_1(x_1, x_2, y_1, y_2) = \frac{1}{2} y_1^2 + y_1 y_2 + y_1 - x_1 y_1 \right. \\
 & \left. \min_{y_2} f_2(x_1, x_2, y_1, y_2) = \frac{1}{2} y_2^2 + y_1 y_2 + y_2 - x_2 y_2 \right\} \\
 & \text{s. t. } 2y_1 + y_2 + x_1 - 2x_2 \leq 3, y_1 + y_2 \leq 2.5 \\
 & 0 \leq x_1, x_2 \leq 1, 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 1
 \end{aligned} \tag{5.2}$$

Then (5.2) is transformed into the following tri-level programming problem:

$$\begin{aligned}
 & \min_z \alpha \\
 & \text{s. t. } z = (x, y) \\
 & \min_{x_1, x_2} F(x_1, x_2, y_1, y_2) = x_1^2 - x_1 + x_1 y_1 + 2x_2^2 - 4x_2 - x_1 x_2 \\
 & \text{s. t. } x_1 + x_2 \leq 1.5 \\
 & \min_{y_1, y_2} f(x_1, x_2, y_1, y_2) = \frac{1}{2} y_1^2 + y_1 - x_1 y_1 + \frac{1}{2} y_2^2 + y_2 - x_2 y_2 + y_1 y_2 \\
 & \text{s. t. } 2y_1 + y_2 + x_1 - 2x_2 \leq 3, y_1 + y_2 \leq 2.5 \\
 & 0 \leq x_1, x_2 \leq 1, 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 1
 \end{aligned} \tag{5.3}$$

Then the third level problem in (5.3) can be considered as a MPP problem with parameter $x = (x_1, x_2)$:

$$\begin{aligned} \min_{y_1, y_2} f(x_1, x_2, y_1, y_2) &= \frac{1}{2}y_1^2 + y_1 - x_1y_1 + \frac{1}{2}y_2^2 + y_2 - x_2y_2 + y_1y_2 \\ \text{s. t. } 2y_1 + y_2 + x_1 - 2x_2 &\leq 3, y_1 + y_2 \leq 2.5 \\ 0 \leq x_1, x_2 \leq 1, 0 \leq y_1 &\leq 2, 0 \leq y_2 \leq 1 \end{aligned} \quad (5.4)$$

Problem (5.4) have a bilinear term $b_{12}y_1y_2 = y_1y_2$, a concave function $c(y) = 0, h_1(x) = -x_1y_1 - x_2y_2$ and $h_2(y) = y_1 + y_2$ at the objective function. This can result in multiple Nash equilibrium reaction for at least one feasible choice of the leader's problem. So, we should apply a mathematical procedure described in [8]. The convex envelope of the bilinear terms y_1y_2 taken over the rectangle $R = \{(y_1, y_2): 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 1\}$ is denoted by $VexR[b_{12}y_1y_2]$ and can be obtained as follows: $b_{12} = 1 > 0 \Rightarrow l_{12}^1(y_1, y_2) = 0, l_{12}^2(y_1, y_2) = y_1 + 2y_2 - 2$,

$$VexR[b_{12}y_1y_2] = \max \{l_{12}^1(y_1, y_2), l_{12}^2(y_1, y_2)\} = \max \{0, y_1 + 2y_2 - 2\} = y_1 + 2y_2 - 2$$

Therefore, the under-estimator of the objective function in (6.8) is equal to:

$$\begin{aligned} &\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + VexR[b_{12}y_1y_2] + Vex_c + h_1(x) + h_2(y) \\ &= \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + 2y_1 + 3y_2 - x_1y_1 - x_2y_2 - 2 \end{aligned}$$

Thus, the under-estimator problem of (6.4) is formulated as:

$$\begin{aligned} \min_{y_1, y_2} f(x_1, x_2, y_1, y_2) &= \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + 2y_1 + 3y_2 - x_1y_1 - x_2y_2 - 2 \\ \text{s. t. } 2y_1 + y_2 + x_1 - 2x_2 &\leq 3, y_1 + y_2 \leq 2.5 \\ 0 \leq x_1, x_2 \leq 1, 0 \leq y_1 &\leq 2, 0 \leq y_2 \leq 1 \end{aligned} \quad (5.5)$$

The Lagrangian of the problem (5.5) is given by, $L(x, y, \lambda) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + 2y_1 + 3y_2 - x_1y_1 - x_2y_2 - 2 + \lambda_1(2y_1 + y_2 + x_1 - 2x_2 - 3) + \lambda_2(y_1 + y_2 - 2.5)$.

We can apply the multi-parametric programming (MPP) approach to solve (5.5).

$$\begin{pmatrix} y(x) \\ \lambda(x) \end{pmatrix} = \begin{pmatrix} y_0 \\ \lambda_0 \end{pmatrix} - M_0^{-1}N_0(x - x_0)$$

where $(y_0, \lambda_0) = (y(x_0), \lambda(x_0))$, $M_0 = M(x_0)$ and $N_0 = N(x_0)$.

If we take a point $x_0 = (0.5, 0.5)$ which is feasible for the second level problem we have,

$$\begin{pmatrix} y_0 \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, N_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, M_0 = \begin{pmatrix} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 3.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & 0 & 0 \end{pmatrix}$$

And we have got the following parametric solutions,

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} x_1 - 0.5 \\ x_2 - 0.5 \end{pmatrix}$$

with the corresponding critical region $CR = CR_I = \begin{cases} y^*(x) = \begin{bmatrix} x_1 - 0.5 \\ x_2 - 0.5 \end{bmatrix} \\ 0 \leq x_1, x_2 \leq 1 \end{cases}$

Incorporating the solution into the second level follower's problem of (6.3) and we obtain:

$$\begin{aligned} \min_{x_1, x_2} F(x_1, x_2, y_1, y_2) &= 2x_1^2 + 2x_2^2 - x_1x_2 - 1.5x_1 - 4x_2 \\ \text{s.t. } x_1 + x_2 &\leq 1.5, 0 \leq x_1, x_2 \leq 1 \end{aligned} \quad (5.6)$$

Solving (5.6) we obtain the solution $(x_1, x_2) = (0.5, 1)$. Then it is incorporated into the leader's problem of (5.3) and solved to obtain the solution $z = (x_1, x_2, y_1, y_2) = (0.5, 1, 0, 0.5)$. Therefore, the optimal solution to the bilevel multi-leader multi-follower programming problem (5.1) is $(x_1, x_2, y_1, y_2) = (0.5, 1, 0, 0.5)$ with the corresponding objective values $F_1 = -0.75, F_2 = -0.75, f_1 = 0$ and $f_2 = 0.125$.

Example 2: Consider the following trilevel-MLMF programming problem:

$$\begin{aligned}
& \left\{ \min_{x_1} F_1(x_1, x_2, y_1, y_2, z_1, z_2) = x_1^2 + y_1 x_2 + 2z_1 \right. \\
& \left. \min_{x_2} F_2(x_1, x_2, y_1, y_2, z_1, z_2) = e^{x_2} - 3x_1 y_2^2 - z_2 \right\} \\
& \text{s. t. } \left\{ \min_{y_1} f_1^2(x_1, x_2, y_1, y_2, z_1, z_2) = (y_1 - x_1)^2 + z_1^2 \right. \\
& \left. \min_{y_2} f_2^2(x_1, x_2, y_1, y_2, z_1, z_2) = x_2^2 + (y_2 - 2)^2 + z_2^2 \right\} \\
& \text{s. t. } \left\{ \min_{z_1} f_1^3(x_1, x_2, y_1, y_2, z_1, z_2) = x_1 - y_1 z_2 + 2z_1^2 \right. \\
& \left. \min_{z_2} f_2^3(x_1, x_2, y_1, y_2, z_1, z_2) = x_2 + y_2 z_1 + 3z_2^2 \right\} \\
& \text{s. t. } 2x_1 + 2y_1 + 3z_1 \geq 6, x_2 + y_2 + z_2 \geq 1, \\
& \quad 2x_1 + x_2 + 3y_1 + z_1 \geq 3, x_1 + x_2 + y_2 + z_2 \geq 1, \\
& \quad x_1 + 5x_2 + y_1 + y_2 + z_1 + 2z_2 \geq 4, \\
& \quad 0 \leq x_1, x_2 \leq 1, 0 \leq y_1, y_2 \leq 1, 0 \leq z_1, z_2 \leq 3
\end{aligned} \tag{5.7}$$

An equivalent four-level single-leader multi-follower problem for (5.7) is given by:

$$\begin{aligned}
& \min_u \alpha \\
& \text{s. t. } u = (x, y, z) \\
& \left\{ \min_{x_1} F_1(x_1, x_2, y_1, y_2, z_1, z_2) = x_1^2 + y_1 x_2 + 2z_1 \right. \\
& \left. \min_{x_2} F_2(x_1, x_2, y_1, y_2, z_1, z_2) = e^{x_2} - 3x_1 y_2^2 - z_2 \right\} \\
& \text{s. t. } \left\{ \min_{y_1} f_1^2(x_1, x_2, y_1, y_2, z_1, z_2) = (y_1 - x_1)^2 + z_1^2 \right. \\
& \left. \min_{y_2} f_2^2(x_1, x_2, y_1, y_2, z_1, z_2) = x_2^2 + (y_2 - 2)^2 + z_2^2 \right\} \\
& \text{s. t. } \left\{ \min_{z_1} f_1^3(x_1, x_2, y_1, y_2, z_1, z_2) = x_1 - y_1 z_2 + 2z_1^2 \right. \\
& \left. \min_{z_2} f_2^3(x_1, x_2, y_1, y_2, z_1, z_2) = x_2 + y_2 z_1 + 3z_2^2 \right\} \\
& \text{s. t. } 2x_1 + 2y_1 + 3z_1 \geq 6, x_2 + y_2 + z_2 \geq 1, \\
& \quad 2x_1 + x_2 + 3y_1 + z_1 \geq 3, x_1 + x_2 + y_2 + z_2 \geq 1, \\
& \quad x_1 + 5x_2 + y_1 + y_2 + z_1 + 2z_2 \geq 4, \\
& \quad 0 \leq x_1, x_2 \leq 1, 0 \leq y_1, y_2 \leq 1, 0 \leq z_1, z_2 \leq 3
\end{aligned} \tag{5.8}$$

Then (5.8) is transformed into the following tri-level programming problem:

$$\begin{aligned}
& \min_u \alpha \\
& s. t. u = (x, y, z) \\
& \left\{ \min_x F(x, y, z) = x_1^2 + e^{x_2} + x_2 y_1 - 3x_1 y_2^2 \right\} \\
& s. t. \left\{ \min_y f^2(x, y, z) = (y_1 - x_1)^2 + (y_2 - 2)^2 \right\} \\
& s. t. \left\{ \min_z f^3(x, y, z) = 2z_1^2 + 3z_2^2 - y_1 z_2 + y_2 z_1 \right\} \\
& s. t. 2x_1 + 2y_1 + 3z_1 \geq 6, x_2 + y_2 + z_2 \geq 1, \\
& 2x_1 + x_2 + 3y_1 + z_1 \geq 3, x_1 + x_2 + y_2 + z_2 \geq 1, \\
& x_1 + 5x_2 + y_1 + y_2 + z_1 + 2z_2 \geq 4, \\
& 0 \leq x_1, x_2 \leq 1, 0 \leq y_1, y_2 \leq 1, 0 \leq z_1, z_2 \leq 3
\end{aligned} \tag{5.9}$$

Then the fourth level problem in (5.9) can be considered as a MPP problem with parameter $(x, y) = (x_1, x_2, y_1, y_2)$:

$$\begin{aligned}
& \min_{z_1, z_2} f^3(x_1, x_2, y_1, y_2, z_1, z_2) = 2z_1^2 + 3z_2^2 - y_1 z_2 + y_2 z_1 \\
& s. t. 2x_1 + 2y_1 + 3z_1 \geq 6, x_2 + y_2 + z_2 \geq 1, \\
& 2x_1 + x_2 + 3y_1 + z_1 \geq 3, x_1 + x_2 + y_2 + z_2 \geq 1, \\
& x_1 + 5x_2 + y_1 + y_2 + z_1 + 2z_2 \geq 4, \\
& 0 \leq x_1, x_2 \leq 1, 0 \leq y_1, y_2 \leq 1, 0 \leq z_1, z_2 \leq 3
\end{aligned} \tag{5.10}$$

Applying multi-parametric programming (MPP) approach to solve (5.10).

$$\begin{pmatrix} z(x, y) \\ \lambda(x, y) \end{pmatrix} = \begin{pmatrix} z_0 \\ \lambda_0 \end{pmatrix} - M_0^{-1} \cdot N_0 \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}, \text{ where } \begin{pmatrix} z_0 \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} z(x_0, y_0) \\ \lambda(x_0, y_0) \end{pmatrix}, M_0 = M(x_0, y_0) \text{ and } N_0 = N(x_0, y_0)$$

we have got the following parametric solutions,

$$\begin{aligned}
& \begin{bmatrix} z_1(x, y) \\ z_2(x, y) \end{bmatrix} = \begin{bmatrix} 2 - 0.6667x_1 - 0.6667y_1 \\ 0.1667y_1 \end{bmatrix} \text{ in CR}_1 \text{ where} \\
& \text{CR}_1 = \begin{cases} -0.7023x_2 - 0.1170y_1 - 0.7023y_2 \leq -0.7023 \\ -0.4650x_1 - 0.3487x_2 - 0.8137y_1 \leq -0.3487 \\ -0.0647x_1 - 0.9703x_2 - 0.1294y_1 + y_2 \leq -0.3881 \\ 0 \leq x_1 \leq 1, x_2 \leq 1 \\ 0 \leq y_1, y_2 \leq 1 \end{cases} \\
& \begin{bmatrix} z_1(x, y) \\ z_2(x, y) \end{bmatrix} = \begin{bmatrix} 2 - 0.6667x_1 - 0.6667y_1 \\ 1 - y_1 - y_2 \end{bmatrix} \text{ in CR}_2 \text{ where}
\end{aligned}$$

$$CR_2 = \begin{cases} 0.7023x_2 + 0.1170y_1 + 0.7023y_2 \leq 0.7023 \\ -0.4650x_1 - 0.3487x_2 - 0.8137y_1 \leq -0.3487 \\ -0.1043x_1 - 0.9383x_2 - 0.1043y_1 + 0.3128y_2 \leq 0 \\ 0 \leq x_1 \leq 1, x_2 \geq 0 \\ 0 \leq y_1 \leq 1, y_2 \geq 0 \end{cases}$$

$$\begin{bmatrix} z_1(x, y) \\ z_2(x, y) \end{bmatrix} = \begin{bmatrix} 3 - 2x_1 - x_2 - 3y_1 \\ 1 - x_2 - y_2 \end{bmatrix} \text{ in } CR_3 \text{ where}$$

$$CR_3 = \begin{cases} 0.7023x_2 + 0.1170y_1 + 0.7023y_2 \leq 0.7023 \\ 0.4650x_1 + 0.3487x_2 + 0.8137y_1 \leq 0.3487 \\ 0.3162x_1 - 0.6325x_2 + 0.6325y_1 + 0.3162y_2 \leq 0.3162 \\ x_1 \geq 0, x_2 \geq 0 \\ y_1 \geq 0, y_2 \geq 0 \end{cases}$$

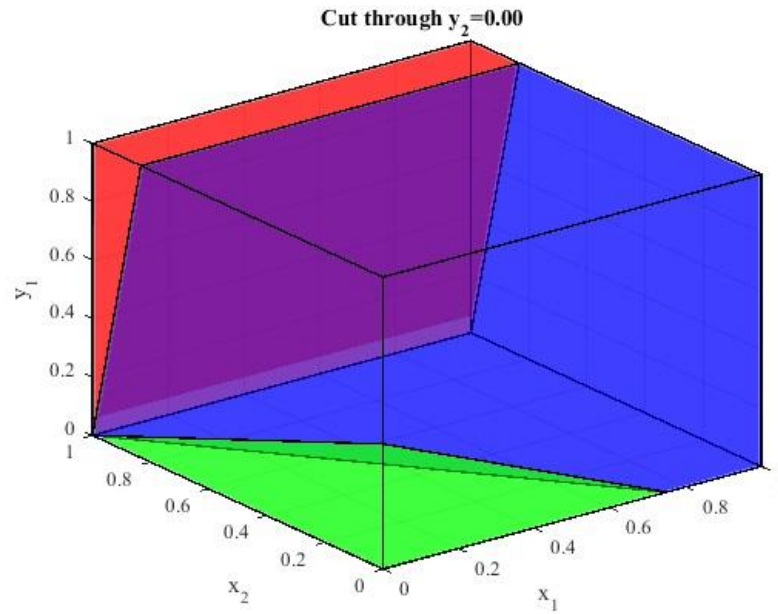


Fig. 1: Critical regions for the problem (5.10)

Corresponding to the first critical set of fourth level-problem, we have got the following parametric solutions with parameter $x = (x_1, x_2)$ to the third-level followers problem of (5.9),

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ 1 \end{bmatrix}, CR_{1-1} = \begin{cases} -0.9648x_1 - 0.2631x_2 \leq -0.2631 \\ -0.1962x_1 - 0.9806x_2 \leq -0.1961 \\ x_1 \leq 1, 0 \leq x_2 \leq 1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.5714x_1 - 0.4285x_2 + 0.4285 \\ -0.0001x_2 + 1 \end{bmatrix}, CR_{1-2} = \begin{cases} 0.0101x_1 - 0.9999x_2 \leq -0.1514 \\ 0.9648x_1 + 0.2613x_2 \leq 0.2631 \\ x_1 \geq 0 \end{cases}$$

$$\begin{aligned} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} 0.5385x_1 - 2.3072x_2 + 0.6686 \\ -0.6923x_1 - 3.4608x_2 + 1.5538 \end{bmatrix}, CR_{1-3} = \begin{cases} 0.1962x_1 + 0.9806x_2 \leq 0.1961 \\ -0.9648x_1 - 0.2613x_2 \leq -0.2631 \\ x_2 \geq 0 \end{cases} \\ \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} 0.5385x_1 - 2.3072x_2 + 0.7724 \\ -0.6923x_1 - 3.4608x_2 + 1.4845 \end{bmatrix}, CR_{1-4} = \begin{cases} -0.0101x_1 + 0.9999x_2 \leq 0.1514 \\ 0.9648x_1 + 0.2613x_2 \leq 0.2631 \\ x_1 \geq 0 \end{cases} \end{aligned}$$

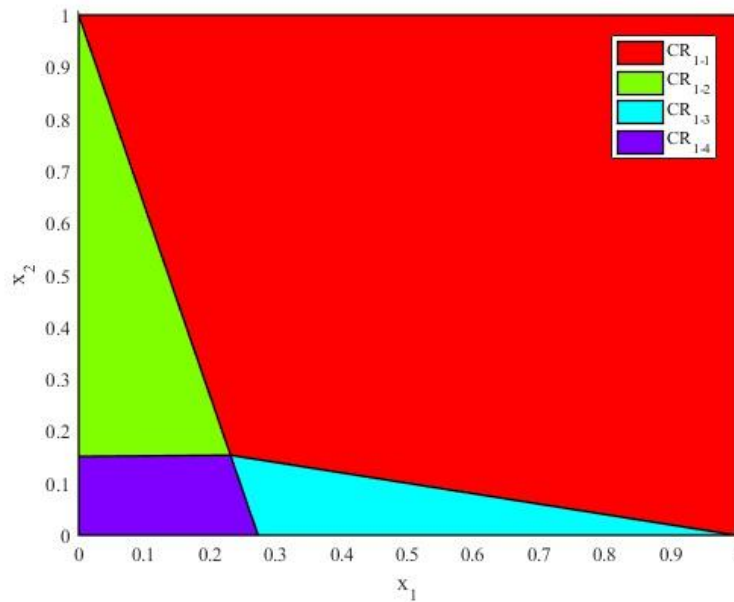


Fig. 2: Critical regions for the third-level problem of (5.9) that corresponds to CR_1 (red region in Fig. 1)

Corresponding to the second critical set of fourth level-problem, we have got the following parametric solutions with parameter $x = (x_1, x_2)$ to the third-level followers problem of (5.9),

$$\begin{aligned} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} 0.9730x_1 - 0.1621x_2 - 0.1621 \\ -0.1621x_1 - 0.9730x_2 + 1.0270 \end{bmatrix}, CR_{2-1} = \begin{cases} -0.9854x_1 - 0.1700x_2 \leq -0.3769 \\ -0.2048x_1 - 0.9788x_2 \leq -0.2700 \\ x_1 \leq 1, x_2 \leq 1 \end{cases} \\ \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} -0.5715x_1 - 0.4285x_2 + 0.4285 \\ 0.0952x_1 - 0.9286x_2 + 0.9286 \end{bmatrix}, CR_{2-2} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3769 \\ -0.0126x_1 - 0.9999x_2 \leq -0.2076 \\ 0 \leq x_1, x_2 \leq 1 \end{cases} \\ \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} 0.7999x_1 - 0.9002x_2 + 0.2001 \\ 0.6001x_1 + 2.6995x_2 + 0.0667 \end{bmatrix}, CR_{2-3} = \begin{cases} -0.9854x_1 - 0.1700x_2 \leq -0.3769 \\ 0.2023x_1 + 0.9793x_2 \leq 0.2483 \\ x_1 \leq 1, 0 \leq x_2 \end{cases} \\ \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} 0.7999x_1 - 0.9001x_2 + 0.6002 \\ 0.6002x_1 + 2.6995x_2 + 0.2001 \end{bmatrix}, CR_{2-4} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3769 \\ 0.2023x_1 + 0.9793x_2 \leq 0.1931 \\ 0 \leq x_1, 0 \leq x_2 \end{cases} \end{aligned}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.6667x_1 - 7.9979x_2 + 1.9996 \\ 0.1111x_1 + 0.3324x_2 + 0.6669 \end{bmatrix}, CR_{2-5} = \begin{cases} -0.9854x_1 - 0.1700x_2 \leq -0.3769 \\ -0.2023x_1 - 0.9793x_2 \leq -0.2483 \\ 0.2048x_1 + 0.9788x_2 \leq 0.2700 \\ x_1 \geq 1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.6668x_1 - 7.9988x_2 + 1.9999 \\ 0.1111x_1 + 0.3326x_2 + 0.6668 \end{bmatrix}, CR_{2-6} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3769 \\ 0.2023x_1 + 0.9793x_2 \leq -0.2221 \\ 0.0126x_1 + 0.9999x_2 \leq 0.2076 \\ -0.2023x_1 - 0.9793x_2 \leq -0.1931 \\ 0 \leq x_1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.6668x_1 - 7.9986x_2 + 1.9998 \\ 0.1111x_1 + 0.3326x_2 + 0.6668 \end{bmatrix}, CR_{2-7} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3769 \\ -0.2023x_1 - 0.9793x_2 \leq -0.2221 \\ 0.0126x_1 + 0.9999x_2 \leq 0.2076 \end{cases}$$

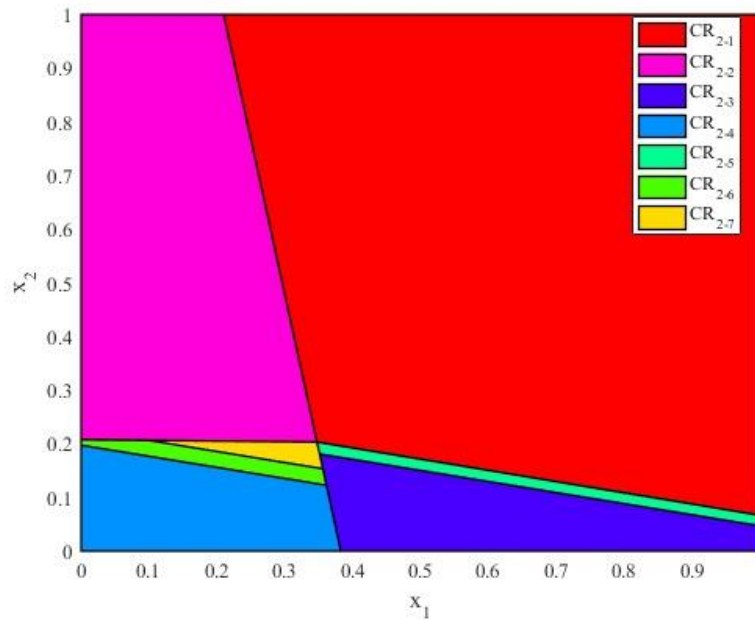


Fig. 3: Critical regions for the third-level problem of (5.9) that corresponds to CR_2 (blue region in Fig. 1)

Corresponding to the third critical set of fourth level-problem we have got the following parametric solutions with parameter $x = (x_1, x_2)$ to the third-level followers problem of (5.9),

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0.9730x_1 - 0.1622x_2 - 0.1614 \\ -0.1621x_1 - 0.9730x_2 + 1.0269 \end{bmatrix}, CR_{3-1} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3764 \\ 0.6451x_1 - 0.7641x_2 \leq 0.0686 \\ x_1 \geq 0, 0 \leq x_2 \leq 1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.2000x_1 + 0.8001x_2 + 0.0504 \\ -0.5999x_1 + 0.4000x_2 + 0.8992 \end{bmatrix}, CR_{3-2} = \begin{cases} 0.2894x_1 + 0.9572x_2 \leq 0.2902 \\ 0.9854x_1 + 0.1700x_2 \leq 0.3764 \\ x_2 \geq 0 \end{cases}$$

$$\begin{aligned}
\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} -0.5712x_1 - 0.4285x_2 + 0.4284 \\ 0.0952x_1 - 0.9286x_2 + 0.9286 \end{bmatrix}, CR_{3-3} = \begin{cases} -0.0125x_1 - 0.9999x_2 \leq -0.2075 \\ 0.9854x_1 + 0.1700x_2 \leq 0.4660 \\ -0.9854x_1 - 0.1700x_2 \leq -0.3764 \\ x_2 \leq 1 \end{cases} \\
\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} -0.5715x_1 - 0.4285x_2 + 0.4285 \\ 0.0952x_1 - 0.9286x_2 + 0.9286 \end{bmatrix}, CR_{3-4} = \begin{cases} 0.9847x_1 + 0.1743x_2 \leq 0.8362 \\ -0.0125x_1 - 0.9999x_2 \leq -0.2075 \\ -0.9854x_1 - 0.1700x_2 \leq -0.4660 \\ x_1 \leq 1, x_2 \geq 0 \end{cases} \\
\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} -0.2000x_1 + 0.8000x_2 + 0.0560 \\ -0.5999x_1 + 0.4000x_2 + 0.8880 \end{bmatrix}, CR_{3-5} = \begin{cases} 0.2894x_1 + 0.9572x_2 \leq 0.2902 \\ -0.9854x_1 - 0.1700x_2 \leq -0.3764 \\ x_1 \leq 1, x_2 \geq 0 \end{cases} \\
\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} -0.5453x_1 + 1.6362x_2 \\ 0.0909x_1 - 1.2726x_2 + 1 \end{bmatrix}, CR_{3-6} = \begin{cases} -0.9854x_1 - 0.1700x_2 \leq -0.3764 \\ 0.0125x_1 + 0.9999x_2 \leq 0.2075 \\ -0.2894x_1 - 0.9572x_2 \leq -0.2902 \end{cases} \\
\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} &= \begin{bmatrix} -0.2008x_1 + 0.7973x_2 + 0.0185 \\ -0.5983x_1 + 0.4054x_2 + 0.9630 \end{bmatrix}, CR_{3-7} = \begin{cases} 0.2894x_1 + 0.9572x_2 \leq 0.3202 \\ -0.2894x_1 - 0.9572x_2 \leq -0.2902 \\ -0.6436x_1 + 0.7654x_2 \leq -0.3080 \\ x_1 \leq 1 \end{cases}
\end{aligned}$$

and no feasible solution in CR_{3-8} and CR_{3-9} .

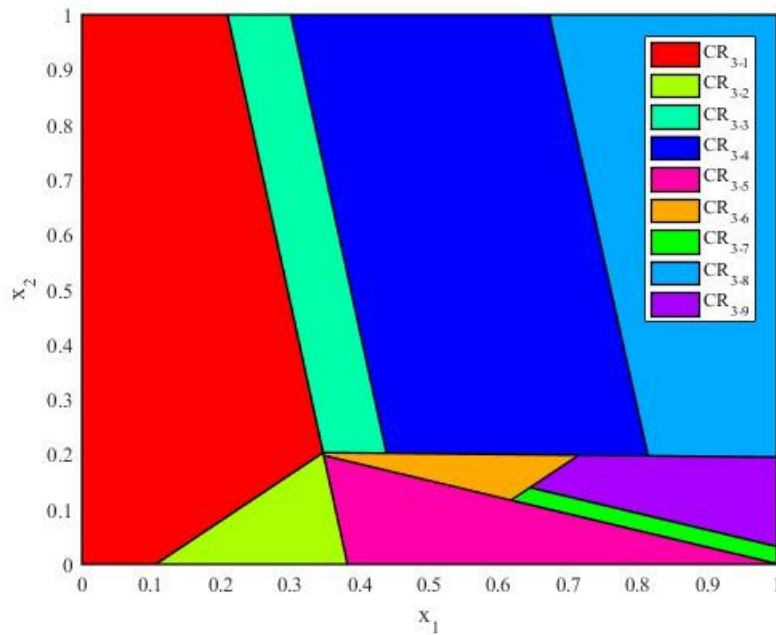


Fig. 4: Critical regions for the third-level problem of (5.9) that corresponds to CR_3 (green region in Fig.1)

Substituting the above parametric solutions into the second-level followers problem of (5.9) and

solving the resulting problems in each critical regions we obtain

$$(x_1, x_2) = \begin{cases} (1,0), (0.2308,0.1537), (0.7006,0.0199), (0.2481,0.0904), \\ (0.2475,0.1864), (0.3474,0.2032), (0.3606,0.1269), (0.1426,0), \\ (0.0918,0.2103), (0.4274,0.1864), (0.3474,0.2032), (0.2033,0.0819), \\ (0.3115,0.0149), (0.4380,0.2020), (0.5998,0.2), (0.3778,0.0244), \\ (0.4768,0.1590), (0.6438,0.1389) \end{cases}$$

in the critical region CR_{1-1} , CR_{1-2} , CR_{1-3} , CR_{1-4} , CR_{2-1} , CR_{2-2} , CR_{2-3} , CR_{2-4} , CR_{2-5} , CR_{2-6} , CR_{2-7} , CR_{3-1} , CR_{3-2} , CR_{2-3} , CR_{3-4} , CR_{3-5} , CR_{3-6} , CR_{3-7} , respectively. But the point $(x_1, x_2) = (1,0)$ provides a best solution with respect to the second-level problem of (5.9), as it was decided by the upper-level decision makers we take this point as an optimal solution. Therefore, the optimal solution to the trilevel multi-leader multi-follower programming problem (5.7) is $u = (x_1, x_2, y_1, y_2, z_1, z_2) = (1, 0, 1, 1, 0.6666, 0.1667)$ on CR_{1-1} with optimal objective values $F_1 = 2.3332$, $F_2 = -2.1667$, $f_1^2 = 0.4444$, $f_2^2 = 1.1667$, $f_1^3 = 1.7220$ and $f_2^3 = 0.7500$.

6. CONCLUSION

In multilevel multi-leader-follower games, various relationships among multiple leaders in the upper-level and multiple followers at the lower-levels would generate different decision processes. To support decision in such problems, this work considered a class of multilevel multi-leader multi-follower games, which consist of separable terms and parameterized common terms across all objective functions of the followers and leaders. We applied two levels of equivalent transformations on such problems; first transforming them into multilevel single-leader multi-follower programming problem, and then the resulting formulations are further transformed into an equivalent multilevel program having only single follower at each level of the hierarchy. Finally, this single leader-follower hierarchical problem is solved using the solution procedure proposed in [6,9]. The proposed solution approach can solve multilevel multi-leader multi-follower problems whose objective values at all levels of the decision hierarchy have common, but having different positive weights of, non-separable terms and with the constraints at each level are

polyhedral. However, much more research is needed in order to provide algorithmic tools to effectively solve the procedures. In this regard we feel it deserves further investigations.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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