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Leray-Serre Spectral Sequence and Some of its Applications

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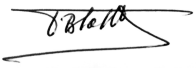
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Abstract

The purpose of this thesis is to look at how the Leray-Serre spectral sequence is constructed and explore some of their applications. For the purpose of this thesis, our spectral sequences will be based on filtered complexes, however one can use exact couples. Spectral sequences have been one of the greatest achievements of Algebraic Topology since their invention by *Leray* in the late 1940's, hence the name *Leray-Serre spectral sequences*. Ever since, they have become a very useful tool in many disciplines like Geometry, Topology and Algebra. For example, for a fibration, spectral sequences play an important role in identifying how the cohomology rings of the total space E , the base space B and the fibre F are related. We discuss the construction in brief and give its applications in computing the cohomology algebra of unitary groups, Stiefel and infinite dimensional complex Grassmannian manifolds, over $\mathbb{Q}$.



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1. Introduction

1.1 Brief Literature Review

The fundamental objects that will be used here are vector spaces over a field $\mathbb{F}$, e.g. $\mathbb{R}, \mathbb{Q}$ or $\mathbb{C}$. In particular, the computation of various (co)homology theories leads to the natural emergence of (co)chain complexes in algebraic topology. A series of vector spaces and linear maps

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \quad (1.1)$$

makes up a *chain complex* C_* if $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{N}$. Hence $\text{Im}(\partial_{n+1}) \subseteq \text{Ker}(\partial_n)$. As a result, the n^{th} -homology is the quotient vector space $H_n(C_*) := \text{Ker}(\partial_n) / \text{Im}(\partial_{n+1})$.

Dualizing sequence 1.1 yields a sequence

$$C^0 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} C^{n-1} \xrightarrow{\delta^n} C^n \xrightarrow{\delta^{n+1}} C^{n+1} \xrightarrow{\delta^{n+2}} \cdots$$

where $C^n = \text{Hom}(C_n; \mathbb{F})$ and δ^n is the dual of $\partial_n : C_n \rightarrow C_{n-1}$. Clearly $\delta^{n+1} \circ \delta^n = 0$, hence $\text{Im}(\delta^n) \subseteq \text{Ker}(\delta^{n+1})$. The sequence is called a *cochain complex* and its cohomology is $H^n(C^*) := \text{Ker}(\delta^{n+1}) / \text{Im}(\delta^n)$.

In algebraic topology, topological spaces are investigated using algebraic invariants like groups: for instance, the homotopy groups $\pi_n(X, x_0)$ of a pointed topological space (X, x_0) or homology groups. For a good pair (X, A) of topological spaces, a sequence $A \hookrightarrow X \rightarrow X/A$ yields a long exact sequence of homology groups

$$\cdots \rightarrow \tilde{H}_n(A) \rightarrow \tilde{H}_n(X) \rightarrow \tilde{H}_n(X/A) \rightarrow \tilde{H}_{n-1}(A) \rightarrow \cdots$$

In relation to space X , a map $p : E \rightarrow B$ is said to have the *homotopy lifting property* if, for any two maps $f : X \rightarrow E$ and $G : X \times I \rightarrow B$ such that $p \circ f = G \circ i$, where $i : X \rightarrow X \times I, x \mapsto (x, 0)$, then there exists a continuous map $\tilde{G} : X \times I \rightarrow E$ lifting G . A map $p : E \rightarrow B$ with the homotopy lifting property with regard to all spaces is thus referred to as a *fibration*. For a fibration $E \rightarrow B$ with fibre F , there is an exact sequence

$$\cdots \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \rightarrow \cdots,$$

relating the homotopy groups of F, E and B . The relationship between the (co)homology rings of the total space E , the base space B , and the fibre F of a fibration is dealt with through the notion of *Leray-Serre spectral sequence*.

Algebraic topology and algebraic geometry both make extensive use of spectral sequences, a key tool in homological algebra. They were initially developed by *Jean Leary* while he was a

prisoner of war during World War II in the 1940's to calculate the (co)homology of a fibration (16; 19; 5).

The Leray-Serre spectral sequence is used, among other things, to compute the *cohomology of infinite complex Grassmannians*. The complex Grassmannian, denoted $G_k(\mathbb{C}^n)$, is a generalization of complex projective spaces, and is defined as the set of all k -planes in $\mathbb{C}^n$. Using the auxiliary space $V_k(\mathbb{C}^n)$, also known as the *Stiefel manifold*, which consists of k -orthonormal n -tuples of vectors in $\mathbb{C}^n$, one can give it a natural topology. There is an obvious inclusion $G_k(\mathbb{C}^n) \hookrightarrow G_k(\mathbb{C}^{n+1})$. And $G_k(\mathbb{C}^\infty)$ is the direct limit of

$$\cdots \longrightarrow G_k(\mathbb{C}^n) \longrightarrow G_k(\mathbb{C}^{n+1}) \longrightarrow \cdots,$$

that is,

$$G_k(\mathbb{C}^\infty) = \bigcup_{n \geq k} G_k(\mathbb{C}^{n+1}).$$

1.2 Objectives of the Study

The objective of the study is to familiarise ourselves with spheres and projective spaces (real and complex), and compute their simplicial (co)homology over $\mathbb{Z}$ and $\mathbb{Q}$. We also familiarise ourselves with manifolds, Stiefel and Grassmannian manifolds, from which we get fibre bundles that are used in computing (co)homology algebra of unitary groups and complex Grassmannian manifolds over $\mathbb{Q}$.

1.3 Outline of the Study

This study is organised in 7 chapters. In Chapter 2 we give basic definitions about homotopy and fundamental groups which lead to the notion of higher homotopy groups. In Chapter 3 we focus on simplicial, singular and cellular homology, and cohomology. We compute the simplicial homology groups of the circle, disk and sphere. And also the simplicial cohomology of the real projective space, $\mathbb{R}P^2$, is computed.

In Chapter 4 we look at fibrations, but first, we look at group actions on topological spaces and fibre bundles. And also we give examples in which fibrations were used to compute the homotopy groups of some topological groups. In Chapter 5 we look at the Leray-Serre spectral sequence and how it is constructed.

And in Chapter 6 we look at some applications of Leray-Serre spectral sequence in computing the cohomology algebra over $\mathbb{Q}$ of unitary groups (Theorem 6.1.1), complex Stiefel manifolds (Theorem 6.2.1) and infinite dimensional complex Grassmannians where we just compute the cohomology of $G_k(\mathbb{C}^\infty)$ using the Leray-Serre spectral sequence (Section 6.3). We also give some information on the cohomology algebra of the Grassmannian $G_2(\mathbb{C}^4)$ (Example 6.4.3). Chapter 5 and Chapter 6 are the main chapters of this study. The conclusion is given in Chapter 7.

2. Homotopy Groups

In this chapter we deal with *continuous deformations* of topological spaces. We recall that the n -disk $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ and the n -sphere $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ are examples of spaces. They are subspaces of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively.

2.1 Homotopy

We first give basic definitions:

Definition 2.1.1. Let X be a topological space. A *pointed space* is defined by a space X and a point $x_0 \in X$, written as (X, x_0) . We call x_0 a *basepoint*.

Definition 2.1.2. Let X and Y be topological spaces and $f, g : X \rightarrow Y$ be continuous maps. A *homotopy* between f and g is a continuous map $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. We write $f \sim g$ and say H is a *homotopy* between f and g .

Example 2.1.3. Let Y be a convex subset of X . Let $f, g : X \rightarrow Y$ be continuous maps. Then the mapping

$$H(x, t) = tg(x) + (1 - t)f(x),$$

is a homotopy from f to g .

Example 2.1.4. Let $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ and the antipodal map $A : S^1 \rightarrow S^1$ be defined by $A(z) = -z$. Then there is a homotopy $H : S^1 \times [0, 1] \rightarrow S^1$ from the identity mapping to the antipodal map defined by

$$H(e^{i\theta}, t) = e^{i(\theta + t\pi)}.$$

All maps which are homotopic to a constant map $c_{y_0} : X \rightarrow Y$ with $c_{y_0}(x) = y_0$ for all $x \in X$ are called *nullhomotopic* and can then be deformed into a single point.

Homotopy defines an equivalence relation. Thus,

Theorem 2.1.5. Let X and Y be topological spaces. On the set of continuous mappings from X to Y , define that f and g are related if $f \simeq g$. This relation is an equivalence relation.

Proof. Let X and Y be topological spaces and let $f, g, h : X \rightarrow Y$ be continuous maps. Then

- (i) $f \simeq f$ by the homotopy $H(x, t) = f(x)$.
- (ii) Suppose $f \simeq g$, that is, there is a homotopy $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Define the map $G : X \times [0, 1] \rightarrow Y$ by

$G(x, t) = H(x, 1 - t)$, that is, $G(x, 0) = H(x, 1) = g(x)$ and $G(x, 1) = H(x, 0) = f(x)$. Therefore, $g \simeq f$.

- (iii) Suppose $f \simeq g$ and $g \simeq h$, that is, there are homotopies $H : X \times [0, 1] \rightarrow Y$ and $G : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, and $G(x, 0) = g(x)$ and $G(x, 1) = h(x)$. Define the map $F : X \times [0, 1] \rightarrow Y$ by

$$F(x, t) = \begin{cases} H(x, 2t); & 0 \leq t \leq \frac{1}{2}, \\ G(x, 2t - 1); & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Therefore, $F(x, 0) = H(x, 0) = f(x)$ and $F(x, 1) = G(x, 1) = h(x)$. Hence, $f \simeq h$. □

Let (X, A) be a topological pair consisting of a topological space X together with a subspace $A \subset X$. A map $f : (X, A) \rightarrow (Y, B)$ is just a continuous map $f : X \rightarrow Y$ such that $f(A) \subset B$.

Definition 2.1.6. Let $f, g : (X, A) \rightarrow (Y, B)$ be maps such that $f|_A = g|_A$. Such maps are called homotopic relative to A , denoted $f \sim g \text{ rel } A$, if there is a homotopy $H : X \times [0, 1] \rightarrow Y$ such that $f \sim g$ and $H(a, t) = f(a) = g(a)$, for all $a \in A$ and $t \in [0, 1]$.

Definition 2.1.7. (Deformation Retraction) Let $A \subset X$, where X is a topological space. If there is a continuous map $r : X \rightarrow A$ such that $r \circ i = I_A$, where $i : A \rightarrow X$ is the inclusion, then we say that A is a retract of X . Additionally, if $i \circ r \sim I_X \text{ rel } A$, then A is a deformation retract of X .

The proposition that follows provides a useful deformation retract.

Proposition 2.1.8. For $n \geq 0$, $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is a deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$ (or $D^{n+1} \setminus \{0\}$).

Proof. Let $i : S^n \rightarrow \mathbb{R}^{n+1} \setminus \{0\}$ be the inclusion. Consider $r : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow S^n$ defined by

$$r(x) = \frac{x}{\|x\|}.$$

Clearly r is continuous. For $y \in S^n$, it follows that $\|y\| = 1$, and we have that $r \circ i = I_{S^n}$. Hence r is a retract. Now we show that $i \circ r$ is homotopic to the identity, relative to S^n . Consider the homotopy $H : (\mathbb{R}^{n+1} \setminus \{0\}) \times [0, 1] \rightarrow \mathbb{R}^{n+1} \setminus \{0\}$ defined by

$$\begin{aligned} H(x, t) &= tx + (1 - t)r(x) \\ &= tx + (1 - t)\frac{x}{\|x\|}. \end{aligned}$$

That is,

$$H(x, 0) = \frac{x}{\|x\|} = (i \circ r)(x) \quad \text{and} \quad H(x, 1) = x.$$

Now if $x \in S^n$, then $H(x, t) = tx + (1 - t)x = x$. So S^n is a deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$. □

From Definition 2.1.7 we get the following definition of homotopy equivalence.

Definition 2.1.9. (11, Page 3) Let X and Y be topological spaces. A continuous map $f : X \rightarrow Y$ is called a homotopy equivalence if there is a map $g : Y \rightarrow X$ such that $f \circ g \simeq I_Y$ and $g \circ f \simeq I_X$. We write $X \simeq Y$.

Definition 2.1.10. (Contractible Spaces) A topological space X is said to be contractible if $X \simeq \{pt\}$.

Example 2.1.11. For all $n \geq 0$, the space $\mathbb{R}^n$ and the disk D^n are contractible.

The definition of a CW complex is as follows:

Definition 2.1.12. (11, Page 5) A CW complex is a space $X = \bigcup_n X^n$ constructed inductively in the following way:

- (i) Let X^0 (the 0-skeleton of X) be a discrete set of points which are regarded as 0-cells.
- (ii) Given the set X^{n-1} , we construct X^n by attaching a set of n -cells $\{e_i^n\}_{i \in I}$, where e_i^n is an open n -disk, to X^{n-1} via maps $\partial D^n = \phi_i : S^{n-1} \rightarrow X^{n-1}$. This means we can define X^n as the quotient space of the disjoint union $X^n = (X^{n-1} \cup \{e_i^n \mid i \in I\}) / \sim$, such that if $x \in X^{n-1}$ and $y \in \partial e_i^n$, $x \sim y$ if $\phi_i(y) = x$. It is noted that no two points of X^{n-1} are collapsed together in their inclusion in X^n .

The above definition works only for finite dimensional CW complexes.

Definition 2.1.13. (11, Page 7) Given a CW complex X , a subcomplex of X is a subspace $A \subset X$ that is a CW complex itself and its cells are some of the cells of X .

Remark 2.1.14. For any CW complex X and any $n \in \mathbb{N}$, X^n is a subcomplex of X , called the n -skeleton.

Example 2.1.15. The sphere S^n has a structure of a CW complex or just one cell complex with e^0 and e^n as the only two cells, the n -cell being attached by the constant map $S^{n-1} \rightarrow e^0$. The sphere S^n here is equivalently considered as the quotient space $D^n / \partial D^n$.

Definition 2.1.16. Let X and Y be CW complexes and, let X^n and Y^n denote the n -skeletons of X and Y respectively. A continuous map $f : X \rightarrow Y$ is called cellular if $f(X^n) \subseteq Y^n$ for all non-negative integers n .

Every continuous map between CW complexes is homotopic to a cellular map, according to the following theorem. We will not give the proof here, but it can be found in (11, Page 349).

Theorem 2.1.17. (Cellular Approximation Theorem) Given that X and Y are CW complexes, let $f : X \rightarrow Y$ be a continuous map. Then f is homotopic to a cellular map.

2.2 The Fundamental Group

The fundamental group is a topological invariant, which is an algebraic object that can be attached to a geometric space. The fundamental group is an important tool for distinguishing between spaces. The fundamental group of a circle, for example, is $\mathbb{Z}$, while the fundamental

group of a point or a line is trivial. This will be explained in greater detail in the following sections. The fundamental group will simply be defined as the collection of all homotopy classes of loops. With that being said, we start with loops:

Definition 2.2.1. A continuous map $f : [0, 1] \rightarrow X$ such that $f(0) = x_0$ and $f(1) = x_1$ is called a path in X from x_0 to x_1 . A loop in a pointed space (X, x_0) is a path $f : [0, 1] \rightarrow X$ such that $f(0) = f(1) = x_0$.

Remark 2.2.2. Definition 2.2.1 simply tells us that a loop in X based at x_0 is a path in X that begins and ends at the same point x_0 .

Example 2.2.3. Consider the pointed space $(S^1, (1, 0))$. Then the map $f : [0, 1] \rightarrow S^1$ which takes t to $(\cos(2\pi t), \sin(2\pi t))$ is a loop.

Definition 2.2.4. (1, Page 48) Two paths $f, g : [0, 1] \rightarrow X$ which share the same initial and final points, x_0 and x_1 respectively, are said to be homotopic if there exists a homotopy

$$H : [0, 1] \times [0, 1] \rightarrow X$$

such that $H(s, 0) = f(s)$, $H(s, 1) = g(s)$ and, $H(0, t) = x_0$ and $H(1, t) = x_1$ for all $s, t \in [0, 1]$.

Here we call H a path homotopy between f and g , and we write $f \sim g$. According to Definition 2.2.4, the path's end points stay fixed during the deformation. From Definition 2.1.6, H is simply a homotopy relative to $\{0, 1\}$.

Definition 2.2.5. A homotopy of loops on (X, x_0) is a continuous map $H : [0, 1] \times [0, 1] \rightarrow X$ with $H(0, t) = H(1, t) = x_0$. We say a loop is nullhomotopic if it is homotopic to the constant loop.

Remark 2.2.6. The equivalence relation of homotopy partitions loops in (X, x_0) into equivalence classes called homotopy classes.

We now define an operation on path homotopy equivalence classes.

Definition 2.2.7. (20, Page 39) Let $f, g : [0, 1] \rightarrow X$ be two paths with $f(1) = g(0)$. Then we define the composition of f and g , denoted $(f \star g) : [0, 1] \rightarrow X$, by

$$(f \star g)(t) := \begin{cases} f(2t); & t \in [0, \frac{1}{2}], \\ g(2t - 1); & t \in [\frac{1}{2}, 1]. \end{cases}$$

Remark 2.2.8.

- (i) If f and g are loops, then $f \star g$ is a loop.
- (ii) The set of all path homotopy equivalence classes of paths in a space is not a group under the operation $\star$ defined above since the operation is only defined between two paths for which the endpoint of the first path is the starting point of the second path.

Proposition 2.2.9. Consider two paths $f_0 \sim f_1 \text{ rel } \{0, 1\}$ and $g_0 \sim g_1 \text{ rel } \{0, 1\}$. If $f_0(1) = g_0(0)$ and $f_1(1) = g_1(0)$, then $f_0 \star g_0 \sim f_1 \star g_1 \text{ rel } \{0, 1\}$.

Proof. See (7, Proposition 8.1) for a detailed proof. $\square$

Corollary 2.2.10. *Let $[f]$ be the homotopy class of a loop f such that $f(0) = f(1) = x_0$. Suppose g is another loop based at x_0 . Then the operation $[f] \star [g] = [f \star g]$ is well defined.*

Lemma 2.2.11. *Let c_{x_0} be a constant loop at x_0 , that is, $c_{x_0}(t) = x_0$, where $t \in [0, 1]$. Let f be a loop based at x_0 . Then $f \star c_{x_0} \sim f \text{ rel } \{0, 1\}$ and $c_{x_0} \star f \sim f \text{ rel } \{0, 1\}$.*

Proof. See (7, Proposition 8.2) for a detailed proof. $\square$

Definition 2.2.12. *Let f be a loop. Then the inverse loop $\tilde{f}$ is defined by $\tilde{f}(t) = f(1-t)$, where $t \in [0, 1]$. This is simply the same physical loop as f but going in the opposite direction.*

Lemma 2.2.13. *Given a loop f , then $f \star \tilde{f} \sim c_{x_0} \text{ rel } \{0, 1\}$ and $\tilde{f} \star f \sim c_{x_0} \text{ rel } \{0, 1\}$.*

Proof. See (7, Proposition 8.3) for detailed proof. $\square$

We can now define the fundamental group.

Definition 2.2.14. *The fundamental group of (X, x_0) , denoted $\pi_1(X, x_0)$, is the group whose underlying set is homotopy classes of loops. On the homotopy classes of loops based at x_0 , define $[f] \star [g] = [f \star g]$. This operation is associative, i.e., $(f \star g) \star h \sim f \star (g \star h) \text{ rel } \{0, 1\}$ and with identity element $id = [c_{x_0}]$, and $[f]^{-1} = [\tilde{f}]$. This gives a group structure on $\pi_1(X, x_0)$.*

Definition 2.2.15. *A space X is path-connected if there is a path joining any two points, that is, for all $x, y \in X$ there is some path $f : [0, 1] \rightarrow X$ with $f(0) = x$ and $f(1) = y$. A space X is simply connected if it is path-connected and $\pi_1(X, x) = \{0\}$, $x \in X$.*

Theorem 2.2.16. (1, Theorem 3.1.18) *Let X be path-connected and $x_0, x_1 \in X$ be two points. Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. See (1, Theorem 3.1.18) for proof. $\square$

In the above theorem we note that the fundamental group of a path-connected space does not depend on the choice of basepoint. We can simply write $\pi_1(X)$ ignoring the basepoint if X is path-connected.

We mentioned that the fundamental group is a tool for distinguishing between spaces and that it is topologically invariant. To demonstrate that it is, consider a continuous map $h : (X, x_0) \rightarrow (Y, y_0)$ that maps $x_0 \in X$ to $y_0 \in Y$. Take any loop f in X that starts at x_0 , then $h \circ f$ is a loop in Y that starts at y_0 . We can then define a homomorphism between the fundamental groups of X and Y induced by the continuous map h because of this relationship between loops in X and loops in Y .

Proposition 2.2.17. *Let $h : (X, x_0) \rightarrow (Y, y_0)$ be a continuous map. Then the map*

$$h_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0),$$

defined by

$$h_*([f]) = [h \circ f]$$

is a group homomorphism.

Proof. To prove this, we follow (20, Theorem 3.4). Here we want to show that $h_*([f] \star [g]) = h_*([f]) \star h_*([g])$. So by definition

$$h_*([f] \star [g]) = h_*([f \star g]) = [h \circ (f \star g)],$$

and

$$h_*([f]) \star h_*([g]) = [h \circ f] \star [h \circ g] = [(h \circ f) \star (h \circ g)].$$

By Definition 2.2.7, we get;

$$\begin{aligned} (h \circ (f \star g))(t) &= h(f \star g)(t) \\ &= \begin{cases} h(f(2t)); & t \in [0, \frac{1}{2}] , \\ h(g(2t - 1)); & t \in [\frac{1}{2}, 1] . \end{cases} \end{aligned}$$

And

$$((h \circ f) \star (h \circ g))(t) = \begin{cases} (h \circ f)(2t) = h(f(2t)); & t \in [0, \frac{1}{2}] , \\ (h \circ g)(2t - 1) = h(g(2t - 1)); & t \in [\frac{1}{2}, 1] . \end{cases}$$

Therefore, indeed $h_*([f] \star [g]) = h_*([f]) \star h_*([g])$. □

Moreover, if $h_1 : (X, x_0) \longrightarrow (Y, y_0)$ and $h_2 : (Y, y_0) \longrightarrow (Z, z_0)$, then $(h_2 \circ h_1)_* = (h_2)_* \circ (h_1)_*$.

Theorem 2.2.18. (20, Theorem 3.10) *If $f : X \longrightarrow Y$ is a homotopy equivalence and, X and Y are path-connected, then $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism.*

Corollary 2.2.19. *If X is contractible, then $\pi_1(X) = \{0\}$.*

Example 2.2.20. *The space $\mathbb{R}^n$ is simply connected with trivial fundamental group since we can get a straight line homotopy $H : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^n$ between any two loops $f, g \in \mathbb{R}^n$ based at x_0 defined by*

$$H(x, t) = tg(x) + (1 - t)f(x).$$

Since S^1 is path-connected, the fundamental group of S^1 does not depend on the choice of basepoint, from Theorem 2.2.16. Hence

Theorem 2.2.21. *Consider the pointed space (S^1, x_0) . Then $\pi_1(S^1, x_0) \cong \mathbb{Z}$ and $\pi_1(S^n) = \{0\}$ for $n \geq 2$.*

Proof. See (11, Page 29) for the proof. □

We will look at higher homotopy groups in the following section.

2.3 Higher Homotopy Groups

Higher homotopy groups are generalizations of the fundamental groups $\pi_1(X, x_0)$ to homotopy groups $\pi_n(X, x_0)$, defined for each positive integer n . They are easier to define but more difficult to compute than lower homotopy groups. We will not go into great detail. The homotopy classes of maps from fixed spaces into the supplied space make up the higher homotopy groups of a space. *Eduard Čech* was the first to define the concept of higher homotopy groups (6, Section 6.1).

Definition 2.3.1. Let (X, x_0) be a pointed space. Then $\pi_n(X, x_0)$ is defined to be

$$\pi_n(X, x_0) = \{[f] \mid f : (I^n, \partial I^n) \longrightarrow (X, x_0), \text{ and } f \text{ is continuous}\},$$

where homotopies f_t satisfy $f_t(\partial I^n) = x_0$ for all t . The sum operation in $\pi_n(X, x_0)$ with $n \geq 2$, is defined by

$$(f + g)(x_1, x_2, \dots, x_n) = \begin{cases} f(2x_1, x_2, x_3, \dots, x_n); & x_1 \in [0, \frac{1}{2}], \\ g(2x_1 - 1, x_2, x_3, \dots, x_n); & x_1 \in [\frac{1}{2}, 1]. \end{cases}$$

It generalizes the composition of loops.

Definition 2.3.2. Let (X, x_0) be a pointed space. Then the homotopy groups $\pi_n(X, x_0)$ are called the higher homotopy groups, defined for each positive integer n .

Remark 2.3.3.

- (i) For $n = 1$, one recovers the fundamental group defined earlier.
- (ii) $\pi_n(X, x_0)$ for $n > 1$ are abelian. This is a very nice property of the higher homotopy groups (7, Page 140).

The concept of relative homotopy groups is the generalization of homotopy groups of a space to a pair (X, A) . They are useful in homotopy group computations because they fit into a long exact sequence of homotopy groups. The homotopy groups of a pair of spaces are referred to as relative homotopy groups.

Definition 2.3.4. The n^{th} -relative homotopy group $\pi_n(X, A, x_0)$ of a pair (X, A) with basepoint $x_0 \in A$ is the set of homotopy classes of maps

$$f : (D^n, S^{n-1}, x_0) \longrightarrow (X, A, f(x_0))$$

where the homotopies are relative to the basepoint x_0 . In other words;

$$\pi_n(X, A, x_0) = \{[f] \mid f : (D^n, S^{n-1}, x_0) \longrightarrow (X, A, f(x_0)), \text{ where } f \text{ is continuous}\}.$$

One defines a morphism

$$\partial_n : \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0)$$

for each $n \geq 0$ defined as the restriction $\partial([f]) = [f|_{S^{n-1}}]$

$$f : (D^n, S^{n-1}, x_0) \longrightarrow (X, A, f(x_0)).$$

Definition 2.3.5. (11, Page 113) A sequence of the form:

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \longrightarrow \cdots,$$

where A_i 's are R -modules and ∂_i 's are R -homomorphisms is called an exact sequence if for all n , $\text{Ker}(\partial_n) = \text{Im}(\partial_{n+1})$.

Remark 2.3.6. We note two things:

- (i) $\text{Im}(\partial_{n+1}) \subset \text{Ker}(\partial_n)$ if and only if $\partial_n \partial_{n+1} = 0$, so an exact sequence is a chain complex.
- (ii) $\text{Ker}(\partial_n) \subset \text{Im}(\partial_{n+1})$ implies that the homology groups of an exact sequence are trivial.

Definition 2.3.7. (11, Page 114) An exact sequence of the form:

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0,$$

is called a short exact sequence.

Proposition 2.3.8. The short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0,$$

is exact if and only if f is injective and g is surjective, and $\text{Ker}(g) = \text{Im}(f)$. Note that the sequence

$$0 \rightarrow A \xrightarrow{f} B \rightarrow 0$$

is exact if and only if f is an isomorphism.

Proof. Immediate. □

The following theorem is presented without proof.

Theorem 2.3.9. The sequence

$$\cdots \xrightarrow{\partial_n} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \cdots \xrightarrow{i_*} \pi_0(X, x_0)$$

is exact where i_* and j_* are induced by the inclusions

$$i : (A, x_0) \hookrightarrow (X, x_0) \quad \text{and} \quad j : (X, x_0) \hookrightarrow (X, A, x_0),$$

This exact sequence is called the long exact sequence of homotopy groups of the pair (X, A) .

Now we will look at the results required for computing the n^{th} -homotopy group of the n -sphere, S^n . We begin by defining the concept of n -connected spaces, which are critical in the formulation of Hurewicz's theorem. The theorem will not be proved.

Definition 2.3.10. (11, Page 346) A pointed space (X, x_0) is called n -connected if $\pi_i(X, x_0) = \{0\}$ for all $i \leq n$. Similarly for pairs.

For $n = 1$, we recover the definition of a simply connected space (see Definition 2.2.15).

Definition 2.3.11.

(i) A map $f : X \rightarrow Y$ is called an n -equivalence or n -connected map if

$$f_* : \pi_i(X, x_0) \rightarrow \pi_i(Y, f(x_0))$$

is an isomorphism for all $i < n$ and epimorphism for $i = n$.

(ii) A map $f : X \rightarrow Y$ is called a weak (homotopy) equivalence if

$$f_* : \pi_i(X, x_0) \rightarrow \pi_i(Y, f(x_0))$$

is an isomorphism for all i .

The following theorem is very useful when showing that a map is a homotopy equivalence.

Theorem 2.3.12. (Whitehead's first theorem) (2, Theorem 2.4.7) Let $f : X \rightarrow Y$ be a map of CW complexes. Then f is a weak equivalence if and only if f is a homotopy equivalence.

Proof. See (2, Theorem 2.4.7) for the proof. □

Theorem 2.3.13. Let $X = S^1$. Then $\pi_p(S^1) = \{0\}$ for all $p \geq 2$.

We will not go over the proof here, but one makes use of the Homotopy Lifting, see (7, Proposition 6.28). All the homotopy groups of S^1 are known since $\pi_1(S^1) \simeq \mathbb{Z}$.

Theorem 2.3.14.

$$\pi_p(S^n, x_0) = \begin{cases} \{0\}; & 1 \leq p < n, \\ \mathbb{Z}; & p = n. \end{cases}$$

The higher homotopy groups $\pi_p(S^n, x_0)$, for $p > n$, are very complicated to compute. The following is an example that clarifies Theorem 2.3.14 above:

Example 2.3.15. Here we want to show that $\pi_p(S^n, x_0) = \{0\}$ when $p < n$. Given S^p and S^n with their canonical CW structures, with one 0-cell each and one p -cell for S^p and one n -cell for

S^n , if $f : S^p \longrightarrow S^n$ is a continuous, base-point preserving map, then by Theorem 2.1.17, it is homotopic to a cellular map g . This map g must map the p -skeleton of S^p into the p -skeleton of S^n . But the p -skeleton of S^p is S^p itself, whereas the p -skeleton of S^n is the 0-cell, i.e. a point, due to the assumption that $p < n$. Thus, map g is a constant map, hence $\pi_p(S^n, x_0) = \{0\}$.



3. Simplicial, Singular and Cellular Homology, and Cohomology

3.1 Homology

Homology is a computable and combinatorial functor that gives us a finite description of the topology of a space. Homology groups can be thought of as an algebraization of the first layer of geometry in cell structures, i.e. how n -dimensional cells attach to $(n-1)$ -dimensional cells. The Fundamental group is not necessarily abelian, whereas homology groups are. The first homology group is the abelianization of the fundamental group.

For a topological space X , $H_n(X)$ is the n^{th} -homology group (for $n \in \mathbb{N}$) which measures number of n -dimensional holes in X .

We first look at the notion of complexes.

3.1.1 Chain Complexes

Definition 3.1.1.1. A chain complex C_* is a sequence

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \quad (3.1)$$

of R -modules C_n and R -homomorphisms ∂_n , called boundary operators, such that $\partial_n \partial_{n+1} = 0$ for all $n \in \mathbb{N}$.

From the definition, $\partial_0 = 0$. The boundary operator allows us to define submodules of C_n , the submodule of *cycles* and the submodule of *boundaries*.

Definition 3.1.1.2. An element $c \in C_n$ is called an n -chain. An n -chain c is called a cycle if $\partial_n(c) = 0$, and a boundary if $c \in \text{Im}(\partial_{n+1})$.

The submodule of n -cycles is denoted by $Z_n = \text{Ker}(\partial_n)$ and the submodule of n -boundaries $B_n = \text{Im}(\partial_{n+1})$. The condition $\partial_n \partial_{n+1} = 0$ means that $B_n \subseteq Z_n$.

We can therefore have the following definition:

Definition 3.1.1.3. The n^{th} -homology module of C_* is the quotient

$$H_n = Z_n / B_n,$$

for all $n \in \mathbb{N}$. For $z_1, z_2 \in Z_n$, then $[z_1] = [z_2]$ if and only if $z_1 - z_2 \in \text{Im}(\partial_{n+1})$.

Definition 3.1.1.4. (11, Page 111) Let A_* and B_* be chain complexes. A chain map $f : A_* \rightarrow$

B_* is a collection $\{f_n : A_n \longrightarrow B_n\}_{n \in \mathbb{N}}$ of R -homomorphisms such that

$$\partial_n f_n = f_{n-1} \partial_n, \quad (3.2)$$

for all $n \in \mathbb{N}$. This is equivalent to saying that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial_{n+1}} & A_n & \xrightarrow{\partial_n} & A_{n-1} \longrightarrow \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial_{n+1}} & B_n & \xrightarrow{\partial_n} & B_{n-1} \longrightarrow \cdots \end{array}$$

Remark 3.1.1.5. The composition $g \circ f : A_* \longrightarrow C_*$ of two chain maps $A_* \xrightarrow{f} B_* \xrightarrow{g} C_*$ is defined by $(g \circ f)_n = g_n \circ f_n$. It is again a chain map.

Definition 3.1.1.6. (19, Page 331) Let A_* and B_* be chain complexes and $f : A_* \longrightarrow B_*$ a chain map. We call $H_n(f) : H_n(A_*) \longrightarrow H_n(B_*)$ the induced map defined by

$$H_n(f)([x]) = [f_n(x)].$$

From Definition 3.1.1.4, the relation $\partial_n f_n = f_{n-1} \partial_n$ implies that f takes cycles to cycles and boundaries to boundaries since $\partial(\alpha) = 0$ implies $\partial(f(\alpha)) = f(\partial(\alpha)) = 0$ and $f(\partial(\alpha)) = \partial(f(\alpha))$ respectively. This shows that $H_n(f)$ induces a well-defined homomorphism between the homologies of A_* and B_* .

Remark 3.1.1.7. We will frequently omit indices when there is no risk of confusion, for example, instead of $\partial_n(x)$ and $f_n(x)$, we will write $\partial(x)$ and $f(x)$.

We will now look at the concept of homotopies between chain maps.

Definition 3.1.1.8. (11, Page 113) Let A_* and B_* be chain complexes, and $f, g : A_* \longrightarrow B_*$ be chain maps. A chain homotopy h between f and g is a collection of homomorphisms $h : A_n \longrightarrow B_{n+1}$ such that

$$\partial h + h \partial = f - g. \quad (3.3)$$

This is depicted below:

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\partial} & A_4 & \xrightarrow{\partial} & A_3 & \xrightarrow{\partial} & A_2 & \xrightarrow{\partial} & A_1 & \longrightarrow & 0 \\ & & \downarrow f & \downarrow g & \swarrow h_3 & \downarrow f & \downarrow g & \swarrow h_2 & \downarrow f & \downarrow g & \swarrow h_1 \\ \cdots & \xrightarrow{\partial} & B_4 & \xrightarrow{\partial} & B_3 & \xrightarrow{\partial} & B_2 & \xrightarrow{\partial} & B_1 & \longrightarrow & 0 \end{array}$$

If such a homotopy exists, we say f and g are homotopic and write $f \sim g$.

Remark 3.1.1.9. We say that f is null-homotopic if $f \sim 0$.

Theorem 3.1.1.10. Let A_* and B_* be chain complexes. If $f, g : A_* \longrightarrow B_*$ are homotopic, then $H_n(f) = H_n(g)$ for all $n \in \mathbb{N}$.

Proof. From Equation 3.3, if $x \in Z_n$ then $f_n(x) - g_n(x)$ is a boundary which, by definition, vanishes in homology. Let x be an arbitrary cycle in A_n and let h be a homotopy that gives $f \sim g$. Then by Definitions 3.1.1.6 and 3.1.1.8,

$$\begin{aligned} H_n(f)([x]) &= [f_n(x)], \\ &= [g_n(x) + \partial_{n+1}(h_n(x)) + h_{n-1}(\partial_n(x))], \\ &= [g_n(x) + \partial_{n+1}(h_n(x))] \quad ; \text{ as } \partial_n(x) = 0, \\ &= H_n(g)([x]). \end{aligned}$$

Therefore, $H_n(f) = H_n(g)$ for all $n \in \mathbb{N}$. □

The basic construction that follows is critical because it establishes a link between homologies of chain complexes connected by some "special" chain maps.

3.1.2 Connecting Homomorphism and Exact Homology Sequence

Definition 3.1.2.1. *An exact sequence of the form:*

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \rightarrow 0,$$

is called a short exact sequence of chain complexes.

Lemma 3.1.2.2. *Let $0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \rightarrow 0$ be a short exact sequence of chain complexes. Then for all n , there exists a canonical homomorphism*

$$\delta : H_n(C_*) \longrightarrow H_{n-1}(A_*)$$

called connecting homomorphism such that the sequence

$$\cdots \rightarrow H_n(A_*) \xrightarrow{H_n(i)} H_n(B_*) \xrightarrow{H_n(j)} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{H_{n-1}(i)} H_{n-1}(B_*) \rightarrow \cdots$$

is exact. Moreover if

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{j} & C_* \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A'_* & \xrightarrow{i'} & B'_* & \xrightarrow{j'} & C'_* \longrightarrow 0 \end{array}$$

is a commutative diagram of chain complexes in which the rows are exact, then $H_{n-1}(\alpha) \circ \delta = \delta' \circ H_n(\gamma)$, i.e. the following diagram commutes,

$$\begin{array}{ccc} H_n(C_*) & \xrightarrow{\delta} & H_{n-1}(A_*) \\ \downarrow H_n(\gamma) & & \downarrow H_{n-1}(\alpha) \\ H_n(C'_*) & \xrightarrow{\delta'} & H_{n-1}(A'_*) \end{array}$$

Proof. See (8, Proposition 2.9) for proof. □

We will now talk about simplicial homology, or spaces made up of simplices.

3.2 Simplicial Homology

3.2.1 Simplicial Complexes

The term simplicial complex refers to *geometric* simplicial complex, which is a geometric object in Euclidean space made up of glued simplices of various dimensions. When it comes to simplices, the following definition is essential.

Definition 3.2.1.1. *The standard n -simplex Δ^n in $\mathbb{R}^{n+1}$ is the convex hull of the standard basis vectors, that is,*

$$\begin{aligned}\Delta^n &= \left\{ (\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n+1} \mid \lambda_i \in [0, 1] \text{ and } \sum_{i=0}^n \lambda_i = 1 \right\} \\ &= \left\{ \sum_{i=0}^n \lambda_i e_i \mid 0 \leq \lambda_i \leq 1 \text{ and } \sum_{i=0}^n \lambda_i = 1 \right\},\end{aligned}$$

where $e_0 = (1, 0, 0, \dots, 0)$, $e_1 = (0, 1, 0, \dots, 0)$, $\dots$, $e_n = (0, 0, 0, \dots, 1)$. The points $\{e_0, e_1, \dots, e_n\}$ are the vertices of Δ^n .

It is obvious that Δ^n is closed and bounded, so it is compact.

Example 3.2.1.2. *The $(n + 1)$ vertices of the standard n -simplex are the points $e_i \in \mathbb{R}^{n+1}$, where $e_0 = (1, 0, 0, \dots, 0)$, $e_1 = (0, 1, 0, \dots, 0)$, $\dots$, $e_n = (0, 0, 0, \dots, 1)$. The following diagram depicts the standard 2-simplex:*

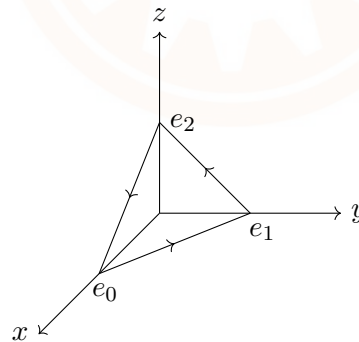


Figure 3.1: Standard 2-simplex.

Definition 3.2.1.3. *Let $\{v_0, v_1, \dots, v_n\}$ be points (or vectors) in $\mathbb{R}^{n+1}$. This set is said to be geometrically independent if the vectors $\{v_1 - v_0, v_2 - v_0, \dots, v_n - v_0\}$ are linearly independent.*

Remark 3.2.1.4. *We will consider singletons to be geometrically independent.*

Definition 3.2.1.5. *An n -simplex in $\mathbb{R}^{n+1}$ is the convex hull of any $(n+1)$ points $\{v_0, v_1, \dots, v_n\}$ in $\mathbb{R}^{n+1}$. We call $\{v_0, v_1, \dots, v_n\} \in \mathbb{R}^{n+1}$ the vertices set of the simplex. We denote the n -*

simplex spanned by these points as

$$\sigma = \langle v_0, v_1, \dots, v_n \rangle = \left\{ \sum_{i=0}^n \lambda_i v_i \mid \sum_{i=0}^n \lambda_i = 1 \right\}. \quad (3.4)$$

Definition 3.2.1.6. If we delete one of the $(n+1)$ points of an n simplex σ , then the remaining n vertices span an $(n-1)$ simplex called a face of σ .

Definition 3.2.1.7. (7, Page 119) A simplicial complex K is a subspace of $\mathbb{R}^n$, together with a finite list of simplices such that:

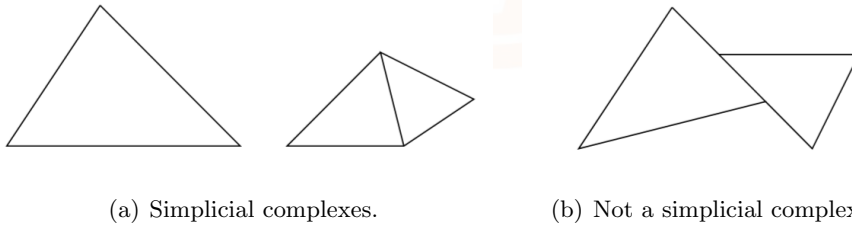
- (i) The union of the simplices is the set K and each point in K lies in the interior of only one simplex.
- (ii) Every face of every simplex in the list is also in the list.

A simplicial complex K is simply a pair, $K = (V, S)$, consisting of a (finite, otherwise it might not be compact) non-empty set V of vertices and a family S of simplices, which are finite subsets of V . Any two simplices are either disjoint or meet at a common face if they do meet. A face is a simplex $P \subseteq S$ with vertices that are also vertices of S .

Remark 3.2.1.8.

- (i) The largest dimension of the simplices a complex can contain is its dimension.
- (ii) A vertex is a face of every simplex it belongs to and is the smallest face.

Example 3.2.1.9. Below is an example of a complex which is simplicial and which is not simplicial:



Given a simplicial complex K , we write $|K|$ to denote the geometric realization of K , i.e. the topological space which is the union of all the simplices in K . Geometric realization of K , $|K|$, is compact.

Definition 3.2.1.10. Let X be a topological space. A triangulation of X is a simplicial complex K together with a homeomorphism from $|K|$ to X .

Remark 3.2.1.11. A topological space X is triangulable if it has some triangulation.

The notion of triangulation will be used in computing simplicial homology of the circle, disk and sphere (Examples 3.2.2).

The *boundary* of Δ^n , represented by $\partial\Delta^n$, is the union of all its faces. The *open simplex* $\mathring{\Delta}^n$ is $\Delta^n - \partial\Delta^n$, the interior of Δ^n .

Definition 3.2.1.12. (11, Page 103) A Δ -complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \rightarrow X$, with n depending on α , such that

- (i) Then restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$.
- (ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \rightarrow X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.
- (iii) A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α .

Definition 3.2.1.13. An oriented simplex σ is simply a simplex together with a choice of orientation. If $\{v_0, v_1, \dots, v_n\}$ spans σ , then we shall use $[v_0, v_1, \dots, v_n]$ to denote the oriented simplex. An orientation of a simplex is an equivalence class of choices of orderings of v_i 's, where $[v_{i_0}, v_{i_1}, \dots, v_{i_n}] = [v_{j_0}, v_{j_1}, \dots, v_{j_n}]$ if and only if $(v_{i_0}, v_{i_1}, \dots, v_{i_n}) = \rho(v_{j_0}, v_{j_1}, \dots, v_{j_n})$ for an even permutation ρ . Orderings of v_i 's that differ from this by an odd permutation are then designated as reversed.

Remark 3.2.1.14. Clearly 0-simplices have only one orientation.

Definition 3.2.1.15. (17, Page 28) Let K be a simplicial complex and R a ring with unity. An n -chain x is a formal sum

$$x = \sum_{i=1}^k a_i \sigma_i,$$

where σ_i 's are n -simplices and $a_i \in R$. The set of n -chains is an R -module denoted by $\Delta_n(K; R)$.

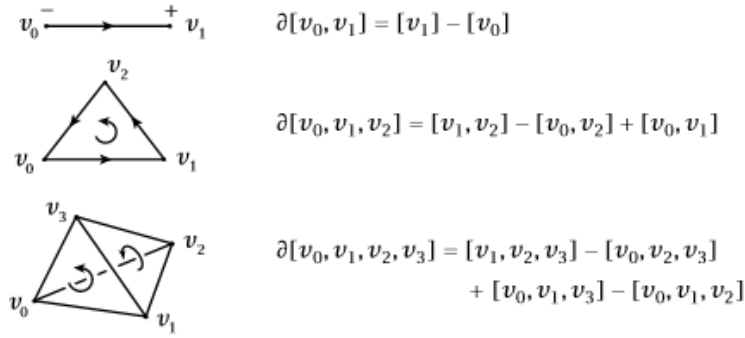
We omit the ring in case $R = \mathbb{Z}$, then $\Delta_n(K, \mathbb{Z})$ is just written $\Delta_n(K)$.

Definition 3.2.1.16. We now define a homomorphism $\partial_n : \Delta_n(K; R) \rightarrow \Delta_{n-1}(K; R)$ called the boundary operator. Let $n > 0$ and $\sigma = [v_0, v_1, \dots, v_n]$ be an oriented simplex. Then the union of all faces of σ is the boundary of σ , defined by

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_n], \quad (3.5)$$

where $\hat{v}_i$ indicates that the vertex v_i is omitted.

Example 3.2.1.17. Let σ be an n -simplex with $n \geq 0$. Consider the following figure as shown in (11, Page 105).


 Figure 3.2: Oriented n -simplices.

From Definition 3.2.1.16 we get the following lemma:

Lemma 3.2.1.18. *For all $n \geq 0$, $\partial_{n-1} \circ \partial_n = 0$.*

Proof. From Equation 3.5, we have $\partial_n(\sigma) = \sum_{i=0}^n (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_n]$. So

$$\begin{aligned} \partial_{n-1} \partial_n(\sigma) &= \sum_{j < i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n] + \sum_{j > i} (-1)^{i+j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n] \\ &= 0. \end{aligned}$$

The two summations cancel out due to switching i and j in the second sum, it becomes the negative of the first sum. $\square$

We illustrate this fact in the following example:

Example 3.2.1.19. *Take an oriented simplex $\sigma = [v_0, v_1, v_2, v_3]$. Then from Equation 3.5,*

$$\partial(\sigma) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2].$$

So

$$\begin{aligned} \partial(\partial(\sigma)) &= \partial[v_1, v_2, v_3] - \partial[v_0, v_2, v_3] + \partial[v_0, v_1, v_3] - \partial[v_0, v_1, v_2] \\ &= ([v_2, v_3] - [v_1, v_3] + [v_1, v_2]) - ([v_2, v_3] - [v_0, v_3] + [v_0, v_2]) \\ &\quad + ([v_1, v_3] - [v_0, v_3] + [v_0, v_1]) - ([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) \\ &= 0. \end{aligned}$$

Definition 3.2.1.20. *The n^{th} -simplicial homology module of a simplicial complex K is defined by*

$$H_n^\Delta(K; R) = Z_n^\Delta / B_n^\Delta, \quad (3.6)$$

where $Z_n^\Delta = \text{Ker}(\partial_n)$ and $B_n^\Delta = \text{Im}(\partial_{n+1})$.

We will denote the "simplicial homology" as in Equation 3.6 in order to distinguish it from the singular homology H_n (to be discussed in the next section (Section 3.3)).

3.2.2 Examples

In the following examples, coefficients are taken in $\mathbb{Z}$. Let $(\Delta_n(K), \partial)$ be the simplicial chain complex of a simplicial complex K (see Definition 3.2.1.15).

We first compute the simplicial homology of the circle.

Example 3.2.2.1. *The following diagram depicts a triangulation of the circle.*

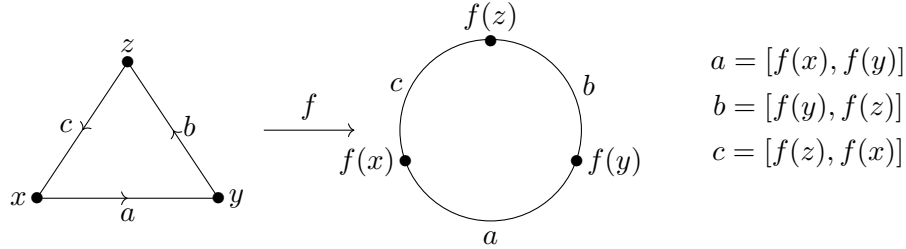


Figure 3.3: Triangulation of the circle: The simplicial complex contains the 0-simplices x, y and z , and the 1-simplices a, b and c . The homeomorphism from the complex to the circle is denoted by f .

The chain groups have the following bases:

$$\Delta_0(K) = \langle x, y, z \rangle, \quad \Delta_1(K) = \langle [x, y], [y, z], [z, x] \rangle \quad \text{and} \quad \Delta_2(K) = \{0\}.$$

Boundaries:

$$\begin{aligned} \partial([x, y]) &= y - x, \\ \partial([y, z]) &= z - y, \\ \partial([z, x]) &= x - z. \end{aligned}$$

Note that $\Delta_2(K) = \{0\}$ because there are no 2-simplices. In order to compute homology we need to compute, at every point/stage, the space of cycles and space of boundary and then we take the quotient. So

$$\begin{aligned} Z_0^\Delta &= \text{Ker}(\partial_0) = \langle x, y, z \rangle, \\ B_0^\Delta &= \text{Im}(\partial_1) = \langle y - x, z - y, x - z \rangle. \end{aligned}$$

Since $x - z = -(y - x) - (z - y)$, then $\text{Im}(\partial_1) = \langle y - x, z - y \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. Hence

$$H_0^\Delta(S^1) = (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) / \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}.$$

To compute $H_1^\Delta(S^1)$, let $l, n, m \in \mathbb{Z}$. So $l[x, y] + m[y, z] + n[z, x]$ is a cycle if and only if

$$\partial(l[x, y] + m[y, z] + n[z, x]) = 0.$$

That is,

$$\begin{aligned}\partial(l[x, y] + m[y, z] + n[z, x]) &= l\partial([x, y]) + m\partial([y, z]) + n\partial([z, x]) \\ &= l(y - x) + m(z - y) + n(x - z) \\ &= (n - l)x + (l - m)y + (m - n)z.\end{aligned}$$

We see that $(n - l)x + (l - m)y + (m - n)z = 0$ if and only if $l = m = n$. So

$$\begin{aligned}Z_1^\Delta &= \text{Ker}(\partial_1) \cong \langle [x, y] + [y, z] + [z, x] \rangle \cong \mathbb{Z}, \\ B_1^\Delta &= \text{Im}(\partial_2) = \{0\}.\end{aligned}$$

Thus,

$$H_1^\Delta(S^1) \cong \mathbb{Z}.$$

As $\Delta_i(K) = \{0\}$ for $i \geq 2$, we then have that $Z_i^\Delta = B_i^\Delta = \{0\}$ for $i \geq 2$. Therefore,

$$H_n^\Delta(S^1) = \begin{cases} \mathbb{Z}; & \text{if } n = 0, 1, \\ \{0\}; & \text{if } n \geq 2. \end{cases}$$

Here we compute the simplicial homology of a disk.

Example 3.2.2.2. The disk is homeomorphic to a solid triangle. Consider the following diagram:

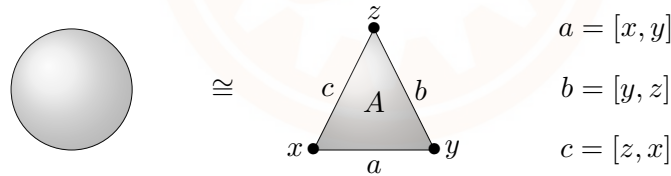


Figure 3.4: Triangulated disk.

The chain groups have the following bases:

$$\Delta_0(K) = \langle x, y, z \rangle, \quad \Delta_1(K) = \langle [x, y], [y, z], [z, x] \rangle \quad \text{and} \quad \Delta_2(K) = \langle A = [x, y, z] \rangle.$$

Boundaries:

$$\begin{aligned}\partial([x, y, z]) &= [y, z] - [x, z] + [x, y], \\ \partial([x, y]) &= y - x, \quad \partial([y, z]) = z - y \quad \text{and} \quad \partial([z, x]) = x - z.\end{aligned}$$

So

$$\begin{aligned} Z_1^\Delta &= \text{Ker}(\partial_1) = \langle a + b + c \rangle \cong \mathbb{Z}, \\ B_1^\Delta &= \text{Im}(\partial_2) = \langle a + b + c \rangle \cong \mathbb{Z}; \quad \text{since} \quad \partial(A) = a + b + c. \end{aligned}$$

Thus,

$$H_1^\Delta(D) = \mathbb{Z}/\mathbb{Z} = \{0\}.$$

Note that Z_2^Δ is trivial. Therefore,

$$H_n^\Delta(D) = \begin{cases} \mathbb{Z}; & \text{if } n = 0, \\ \{0\}; & \text{if } n \geq 1. \end{cases}$$

Now we compute the simplicial homology of a sphere.

Example 3.2.2.3. *The sphere is homeomorphic to the boundary of a tetrahedron. Consider the following diagram:*

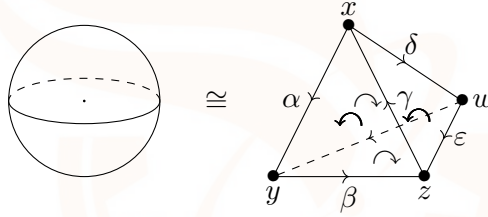


Figure 3.5: Triangulated sphere.

The chain groups have the following bases:

$$\begin{aligned} \Delta_0(K) &= \langle w, x, y, z \rangle, \\ \Delta_1(K) &= \langle [x, y], [y, z], [z, x], [x, w], [w, z], [w, y] \rangle, \\ \Delta_2(K) &= \langle [x, y, z], [x, z, w], [y, w, z], [x, w, y] \rangle, \\ \Delta_3(K) &= \{0\}. \end{aligned}$$

Boundaries:

$$\begin{aligned}
 \partial([x, y, z]) &= [y, z] - [x, z] + [x, y], \\
 \partial([x, z, w]) &= [z, w] - [x, w] + [x, z], \\
 \partial([y, w, z]) &= [w, z] - [y, z] + [y, w], \\
 \partial([x, w, y]) &= [w, y] - [x, y] + [x, w], \\
 \partial([x, y]) &= y - x, \\
 \partial([y, z]) &= z - y, \\
 \partial([z, x]) &= x - z, \\
 \partial([x, w]) &= w - x, \\
 \partial([w, z]) &= z - w, \\
 \partial([w, y]) &= y - w.
 \end{aligned}$$

Note that $\Delta_3(K) = \{0\}$ since there are no 3-simplices. Also note that $H_0^\Delta(\Delta_3(K))$ is spanned by $[w] = [x] = [y] = [z]$. Hence

$$H_0^\Delta(S^2) \cong \mathbb{Z}.$$

To compute $H_1^\Delta(S^2)$, let $\alpha, \beta, \gamma, \delta, \varepsilon, \eta \in \mathbb{Z}$. So $\alpha[x, y] + \beta[y, z] + \gamma[z, x] + \delta[x, w] + \varepsilon[w, z] + \eta[w, y]$ is a cycle if and only if

$$\partial(\alpha[x, y] + \beta[y, z] + \gamma[z, x] + \delta[x, w] + \varepsilon[w, z] + \eta[w, y]) = 0.$$

That is,

$$\begin{aligned}
 \partial(\alpha[x, y] + \beta[y, z] + \gamma[z, x] + \delta[x, w] + \varepsilon[w, z] + \eta[w, y]) &= \alpha\partial([x, y]) + \beta\partial([y, z]) + \gamma\partial([z, x]) \\
 &\quad + \delta\partial([x, w]) + \varepsilon\partial([w, z]) + \eta\partial([w, y]) \\
 &= \alpha(y - x) + \beta(z - y) + \gamma(x - z) \\
 &\quad + \delta(w - x) + \varepsilon(z - w) + \eta(y - w) \\
 &= 0.
 \end{aligned}$$

This is true only if

$$\begin{aligned}
 -\alpha + \gamma - \delta &= 0, \\
 \alpha - \beta + \eta &= 0, \\
 \beta - \gamma + \varepsilon &= 0, \\
 \delta - \varepsilon - \eta &= 0.
 \end{aligned}$$

And we get the matrix to be

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

Writing the the matrix into a row reduced echelon form we get

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And we get infinite solutions since we have a zero row. Let $\gamma = r, \varepsilon = s$ and $\eta = t$. Then $\alpha = r - s - t, \beta = r - s$ and $\delta = s + t$. Thus

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \\ \eta \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = ru_1 + su_2 + tu_3,$$

where $u_1 = [1, 1, 1, 0, 0, 0]^T, u_2 = [-1, -1, 0, 1, 1, 0]^T$ and $u_3 = [-1, 0, 0, 1, 0, 1]^T$ form a basis of the subspace of cycles. Note that,

$$\begin{aligned} u_1 &= [x, y] + [y, z] + [z, x], \\ u_2 &= [x, w] + [w, z] - [x, z], \\ u_3 &= [x, w] + [w, y] + [y, x]. \end{aligned}$$

So $Z_1^\Delta = \text{Ker}(\partial_1) = \langle u_1, u_2, u_3 \rangle$. One can check that each cycle is a closed path. However,

$$u_1 = \partial_2[x, y, z], \quad u_2 = \partial_2[x, w, z] \quad \text{and} \quad u_3 = \partial_2[x, w, y].$$

Hence, $Z_1^\Delta = B_1^\Delta$. Therefore,

$$H_1^\Delta(S^2) = \{0\}.$$

For $H_2^\Delta(S^2)$:

$$\partial([x, y, z] + [x, z, w] + [y, w, z] + [x, w, y]) = 0.$$

That is,

$$\begin{aligned} Z_2^\Delta &= \text{Ker}(\partial_2) = \langle [x, y, z] + [x, z, w] + [y, w, z] + [x, w, y] \rangle \cong \mathbb{Z}, \\ B_2^\Delta &= \text{Im}(\partial_3) = \{0\}. \end{aligned}$$

Thus,

$$H_2^\Delta(S^2) = \mathbb{Z}.$$

In conclusion,

$$H_n^\Delta(S^2) = \begin{cases} \mathbb{Z}; & \text{if } n = 0, 2, \\ \{0\}; & \text{if } n \in \mathbb{N} \setminus \{2\}. \end{cases}$$

Now we look at singular homology which is a theoretical tool.

3.3 Singular Homology

3.3.1 The Singular Complex

Here we construct a functor, called singular complex, from topological spaces to complexes.

Definition 3.3.1.1. (11, Page 108) Let X be a topological space. A singular n -simplex in X is a continuous map $\sigma : \Delta^n \rightarrow X$, where Δ^n denotes the standard n -simplex.

Definition 3.3.1.2. (11, Page 108) Let $S_n(X)$ be the free abelian group with basis $\{\sigma_i : \Delta^n \rightarrow X\}$. Elements of $S_n(X)$ are called singular n -chains. An n -chain x is defined by,

$$x = \sum_{i=1}^m n_i \sigma_i, \tag{3.7}$$

where $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \rightarrow X$.

Define $\partial_n : S_n(X) \rightarrow S_{n-1}(X)$ by

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_n].$$

Definition 3.3.1.3. (8, Page 31) The sequence

$$\cdots \rightarrow S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \rightarrow \cdots,$$

is a complex, i.e. $\partial_n \partial_{n+1} = 0$. It is called the singular complex of X , and is denoted by $S_*(X)$.

Definition 3.3.1.4. (11, Page 108) For all $n \geq 0$, the n^{th} -singular homology of X is defined to be the homology of the chain complex $(S_n(X), \partial_n)$. That is,

$$\begin{aligned} H_n(X) &= Z_n(X)/B_n(X) \\ &= \text{Ker}(\partial_n)/\text{Im}(\partial_{n+1}). \end{aligned}$$

One could take coefficients in a commutative ring R with unity and define in the same way $H_n(X, R)$, the homology of X with coefficients in R .

Proposition 3.3.1.5. If X is a point, then $H_n(X) = \{0\}$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$.

Proof. See (11, Page 110) for proof. □

Definition 3.3.1.6. (11, Page 110) Let X, Y be topological spaces. For a map $f : X \rightarrow Y$, an induced homomorphism

$$f_{\#} : S_n(X) \rightarrow S_n(Y)$$

is defined by composing each singular n -simplex $\sigma : \Delta^n \rightarrow X$ with f to get a singular n -simplex $f(\sigma) = f \circ \sigma : \Delta^n \rightarrow Y$, and extending $f_{\#}$ linearly.

Proposition 3.3.1.7. The map $f_{\#} : S_n(X) \rightarrow S_n(Y)$ satisfy $f_{\#}\partial = \partial f_{\#}$. Equivalently, $f_{\#}$ is a chain map.

Proof. We have, from Equation 3.5, $\partial_n(\sigma) = \sum_{i=0}^n (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_n]$. So

$$\begin{aligned} f_{\#}\partial(\sigma) &= f_{\#} \left(\sum_{i=0}^n (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_n] \right) \\ &= \sum_{i=0}^n (-1)^i f \circ [v_0, v_1, \dots, \hat{v}_i, \dots, v_n] \\ &= \partial f_{\#}(\sigma). \end{aligned}$$

That is, $f_{\#}\partial = \partial f_{\#}$. Thus, the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & S_{n+1}(X) & \xrightarrow{\partial_{n+1}} & S_n(X) & \xrightarrow{\partial_n} & S_{n-1}(X) \longrightarrow \cdots \\ & & \downarrow f_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} \\ \cdots & \longrightarrow & S_{n+1}(Y) & \xrightarrow{\partial_{n+1}} & S_n(Y) & \xrightarrow{\partial_n} & S_{n-1}(Y) \longrightarrow \cdots \end{array}$$

□

From Proposition 3.3.1.7 above, we see that $f_{\#}$ is a map of chain complexes, hence it induces a homomorphism $f_* : H_n(X) \rightarrow H_n(Y)$.

Definition 3.3.1.8. The homomorphism $f_* : H_n(X) \rightarrow H_n(Y)$ is called the induced homomorphism by f in homology.

Proposition 3.3.1.9. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be maps. Then;*

$$(i) \ (g \circ f)_* = g_* \circ f_*,$$

$$(ii) \ (I_X)_* = I_{H_*(X)}, \text{ where } I_X \text{ denotes the identity map.}$$

Theorem 3.3.1.10. (11, Page 111) *Two maps $f, g : X \longrightarrow Y$ will produce the same homomorphism $f_* = g_* : H_n(X) \longrightarrow H_n(Y)$ if they are homotopic.*

Proof. See (11, Page 111 – 113). The proof consists of establishing a chain homotopy between $f_\#$ and $g_\#$. $\square$

Theorem 3.3.1.10 implies that the homology functor factors to the homotopy category of topological spaces with homotopy classes of maps. And it leads us to the following corollary:

Corollary 3.3.1.11. *The maps $f_* : H_n(X) \longrightarrow H_n(Y)$ are isomorphisms for all n if $f : X \longrightarrow Y$ is a homotopy equivalence.*

Proof. Suppose that $f : X \longrightarrow Y$ is a homotopy equivalence and $g : Y \longrightarrow X$ is a homotopy inverse. Then from Proposition 3.3.1.9, we have:

$$(g \circ f)_* = g_* \circ f_* = (I_X)_* = I_{H_n(X)}.$$

And

$$(f \circ g)_* = f_* \circ g_* = (I_Y)_* = I_{H_n(Y)}.$$

Therefore f_* is an isomorphism with an inverse g_* . $\square$

Consider a pair of spaces (X, A) where X is a topological space and $A \subseteq X$. Let the map $i : A \longrightarrow X$ be an inclusion. Then the induced map of singular chain complexes $i_* : S(A) \longrightarrow S(X)$ is injective. We can then consider the quotient $S(X, A) = S(X)/S(A)$.

Definition 3.3.1.12. *The relative singular chain complex $S(X, A)$ of a pair of spaces (X, A) is the quotient complex*

$$S(X, A) = S(X)/S(A).$$

The homology of $S(X, A)$ is the relative singular homology of pair (X, A) , and it is denoted by

$$H_n(X, A) = H_n(S(X, A)),$$

for all $n \geq 0$.

For a pair of spaces (X, A) , we have, by definition, a short exact sequence of chain complexes

$$0 \longrightarrow S(A) \xrightarrow{i_*} S(X) \xrightarrow{p} S(X, A) \longrightarrow 0,$$

where p is the canonical projection. Lemma 3.1.2.2 then yields the following corollary.

Corollary 3.3.1.13. *Let (X, A) be a pair of spaces. Then there exists a connecting homomorphisms*

$$\delta_n : H_n(X, A) \longrightarrow H_{n-1}(A),$$

for $n \geq 1$ such that the sequence

$$\cdots \rightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} H_{n-1}(X) \rightarrow \cdots$$

is exact.

This is the *long exact homology sequence* associated to the pair of spaces (X, A) .

Example 3.3.1.14. *Let $(X, \emptyset)$ be a pair of spaces where X is a topological space. Then the homology of an empty space is trivial in all dimensions. Thus, the inclusion $j : \emptyset \longrightarrow X$ induces an isomorphism*

$$H_n(X) \xrightarrow{\cong} H_n(X, \emptyset),$$

for all $n \in \mathbb{N}$.

The following theorem is about the equivalence of simplicial and singular homology and it applies to a wider category of topological spaces called Δ -complexes (see Definition 3.2.1.12).

Theorem 3.3.1.15. (11, Theorem 2.27) *If X is a Δ -complex, then there are isomorphisms $H_n^\Delta(X) \longrightarrow H_n(X)$ for all n .*

Proof. See (11, Page 128) for proof. □

3.3.2 Reduced Homology

A modified homology that assumes a point has trivial homology groups in all dimensions, including zero, is frequently very helpful. This is possible when we define the *reduced homology groups* $\tilde{H}_n(X)$.

Definition 3.3.2.1. (11, Page 110) *Let X be a topological space, $X \neq \emptyset$. The reduced homology $\tilde{H}_n(X)$ is the homology of the augmented chain complex:*

$$\cdots \rightarrow S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0,$$

where the homomorphism $\varepsilon : S_0(X) \longrightarrow \mathbb{Z}$ is defined by $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$, with $n \in \mathbb{Z}$.

Remark 3.3.2.2. *Since $\varepsilon \partial_1 = 0$, ε vanishes on $\text{Im}(\partial_1)$ and hence induces a surjective map $H_0(X) \longrightarrow \mathbb{Z}$ with kernel $\tilde{H}_0(X)$, so $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$ and $H_n(X) \cong \tilde{H}_n(X)$ for all $n > 0$.*

The following corollary comes from the above remark and Proposition 3.3.1.5.

Corollary 3.3.2.3. *If X is a non-empty contractible space, then $\tilde{H}_n(X) = \{0\}$ for all $n \geq 0$.*

Corollary 3.3.2.4. *Let X be a topological space and $\emptyset \neq A \subset X$. Let $x_0 \in X$ be a basepoint. Then we have the following results:*

- (i) $\tilde{H}_n(\{x_0\}) = \{0\}$ for all n ,
- (ii) $\tilde{H}_n(X) = H_n(X, \{x_0\})$ for all n ,
- (iii) $\tilde{H}_n(X, A) = H_n(X, A)$ for $n \geq 1$.

Short exact sequences are vital in relating the homology groups of a space, a subspace and the resulting quotient space.

Theorem 3.3.2.5. (11, Page 114) *Let X be a space and A be a nonempty closed subspace that is a deformation retract of some neighbourhood in X . Then there is an exact sequence*

$$\cdots \rightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \rightarrow \cdots \xrightarrow{j_*} \tilde{H}_0(X/A) \rightarrow 0,$$

where i is the inclusion $A \hookrightarrow X$ and j is the quotient map $X \rightarrow X/A$.

Good pairs are pairs of spaces (X, A) that satisfy the hypothesis of Theorem 3.3.2.5. If (X, A) is a good pair, then $H_n(X, A) \cong \tilde{H}_n(X/A)$. The pair (D^n, S^{n-1}) is a good pair and $D^n/S^{n-1} \cong S^n$. Using the long exact sequence of a pair, one gets

$$\tilde{H}_i(D^n, S^{n-1}) \cong \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}).$$

Then an induction argument shows that

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z}; & \text{if } i = n, \\ \{0\}; & \text{otherwise.} \end{cases}$$

Definition 3.3.2.6. *If $f : S^n \rightarrow S^n$, then $f_* : H^n(S^n) \rightarrow H^n(S^n)$ is a group homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$ such that $f(1) = k$. The integer k is called the degree of f , denoted $\deg(f)$.*

Now we will look at how homotopy and homology groups are related. But first, let's establish what Hurewicz homomorphism is.

Definition 3.3.2.7. (Hurewicz homomorphism) *Let X be a path-connected space and n be a positive integer. The Hurewicz homomorphism is a morphism from the n^{th} -homotopy group to the n^{th} -homology group, with integer coefficients, and it exists as a group homomorphism*

$$h_* : \pi_n(X) \rightarrow H_n(X).$$

It is given in the following way:

Let $[f] \in \pi_n(X)$, $n \geq 1$. The map $f : S^n \rightarrow X$ induces a homomorphism $f_* : H_n(S^n) \rightarrow H_n(X)$. Choose a generator $\gamma_n \in H_n(S^n)$ and define h_* by $h_*([f]) = f_*(\gamma_n) \in H_n(X)$. We can see that h_* is well-defined (by Theorem 3.3.1.10). Clearly, if $f : X \rightarrow Y$ is a map, then the

following diagram commutes:

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{f_*} & \pi_n(Y) \\ \downarrow h_* & & \downarrow h'_* \\ H_n(X) & \xrightarrow{f_*} & H_n(Y) \end{array}$$

where h_* and h'_* are Hurewicz homomorphisms.

If X is path-connected with abelian fundamental group, then $\pi_1(X, x_0)$ is isomorphic to $H_1(X)$. If X is $(n-1)$ -connected, then $h_* : \pi_n(X) \rightarrow H_n(X)$ is an isomorphism (23, Chapter 4).

3.4 Cellular Homology

Now we look at cellular homology and introduce the cellular boundary formula, which we will use to calculate the homology groups of some CW complexes.

A topological space X 's homology groups, $H_n(X)$, are crucial for understanding the space's properties and structure in relation to other spaces. We will define a kind of homologies of CW complexes, called *cellular homology*. CW complexes were defined in Definition 2.1.12.

The following theorem gives us a few properties about the singular homology groups of a CW complex X 's n -skeletons. As a result, we will be able to construct cellular homology more easily.

Theorem 3.4.1. *If X is a CW complex, then*

- (i) $H_k(X^n, X^{n-1}) = \{0\}$, for $k \neq n$, and a free abelian for $k = n$ with one to one correspondence between its basis and the n -cells of X ,
- (ii) $H_k(X^n) = \{0\}$ for $k > n$. Hence, if X is finite dimensional, then $H_k(X) = \{0\}$, for $k > \dim(X)$.
- (iii) The map $H_k(X^n) \rightarrow H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.

Proof. See (11, Lemma 2.34) for a detailed proof. □

We can interpret $H_n(X^n, X^{n-1})$ as a linear combination of n -cells of the CW complex X , as we did with singular chain groups, using Theorem 3.4.1. The sequence of these groups can then be used to construct a homology theory. Consider pairs (X^{n-1}, X^{n-2}) , (X^n, X^{n-1}) and (X^{n+1}, X^n) with boundary homomorphisms δ and, i_* and j_* are induced by inclusions

$$i : X^n \rightarrow X^{n+1} \quad \text{and} \quad j : (X^n, \phi) \rightarrow (X^n, X^{n-1}).$$

For the sake of clarity, we will not include dimensions here. All of this is depicted in the diagram below:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & \nearrow & \\
 & & & & H_n(X^{n+1}) & \xrightarrow{\cong} & H_n(X) \\
 & & & \nearrow i_* & & & \\
 0 & & & & H_n(X^n) & & \\
 & \searrow \delta & & \nearrow j_* & & & \\
 & & & & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \dots \\
 \dots & \longrightarrow & H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & & & \\
 & & & \searrow \delta & \nearrow j_* & & \\
 & & & & H_{n-1}(X^{n-1}) & & \\
 & & & \nearrow & & & 0
 \end{array}$$

Thus all the sequences in the above diagram are exact and the two triangles commute as $d_n \circ d_{n+1} = 0$. We see that the middle row forms a chain complex which we call the *cellular chain complex* with boundary homomorphism $d_n = j_* \circ \delta$, so triangles commute in the above diagram. We can now formulate the following definition:

Definition 3.4.2. Let X be a CW complex with the cellular chain complex as above in the diagram. Then we define the n^{th} -cellular homology group $H_n(X)$ as

$$H_n(X) = \text{Ker}(d_n) / \text{Im}(d_{n+1}).$$

Remark 3.4.3. Since cellular and singular homology groups are isomorphic for every $n \geq 0$, we use the same notation, $H_n(X)$, for both (11, Theorem 2.35). For this reason, unless otherwise specified, we will refer to cellular homology groups in this section.

We state the following theorem, without proof, known as the *cellular boundary theorem* which simplifies the computations of the homology groups of a CW complex:

Theorem 3.4.4. (Cellular Boundary Formula) (11, Page 140) The cellular boundary homomorphism

$$d_n : H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$$

can be calculated on the basis elements e_i^n by the formula

$$d_n(e_i^n) = \sum_j d_{ij} e_j^{n-1},$$

where for fixed (i, j) , $d_{ij} := \deg(\omega_{ij})$ and $\omega_{ij} = S_i^{n-1} \longrightarrow X^{n-1} \longrightarrow S_j^{n-1}$. That is the composition of the attaching map of the cells e_i^n with the quotient map collapsing $(n-2)$ -skeleton to a

point.

Remark 3.4.5. Theorem 3.4.4 may be extended to $n = 0$ using the idea of reduced homology groups

To compute the cellular homology groups of some CW complexes X , we use Theorem 3.4.4. See the following example:

Example 3.4.6. We use Theorem 3.4.4 to compute cellular homology group of S^n . We can see that S^n is a CW complex with one n -cell (the interior of D^n) attached to a point. Let S_m^n denote the m skeleton of S^n . Then from the Theorem 3.4.1 $H_m(S_m^n, S_{m-1}^n) = \{0\}$ for $m \neq 0, n$ while $H_n(S_n^n, S_{n-1}^n) \cong \mathbb{Z}$ and $H_0(S_0^n) \cong \mathbb{Z}$. So, the cellular chain complex reduces to obtain the chain complex:

$$0 \rightarrow \mathbb{Z} \xrightarrow{d_n} 0 \rightarrow \cdots \rightarrow 0 \xrightarrow{d_1} \mathbb{Z} \rightarrow 0,$$

with homology groups

$$\begin{cases} H_0(S^n) \cong \mathbb{Z}, \\ H_m(S^n) = \{0\}, \text{ when } m \neq 0, n, \\ H_n(S^n) = \text{Ker}(d_{n-1})/\text{Im}(d_n) \cong \mathbb{Z}/\{0\} = \mathbb{Z}. \end{cases}$$

3.5 Cohomology

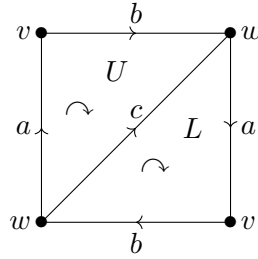
Cohomology is simply the dualization of homology. Homology groups are obtained where one first forms a chain complex $\cdots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots$ of singular, simplicial or cellular chains, then takes the homology groups of this chain complex, that is, $H_n(C_*) = \text{Ker}(\partial_n)/\text{Im}(\partial_{n+1})$. To obtain the cohomology groups, we replace the chain complex C_* by the dual groups $C^n = \text{Hom}(C_n, R)$, where R is a commutative ring with unity, and the boundary maps ∂_n by their dual maps δ^n , then the cohomology groups $H^n(C^*) = \text{Ker}(\delta^{n+1})/\text{Im}(\delta^n)$.

Definition 3.5.1. The dual map $\delta^n : C^{n-1} \rightarrow C^n$, defined by $(\delta^n f)(\sigma) = (-1)^n(f(\partial\sigma))$, for $\sigma \in C_n$ and $f \in C^{n-1}$, defines a cochain complex for which the cohomology is

$$H^n(C^*) = \text{Ker}(\delta^{n+1})/\text{Im}(\delta^n).$$

Consider the following example where we compare the simplicial homology and cohomology groups of the real projective plane using Δ -complexes.

Example 3.5.2. We compute the Δ -cohomology of the projective space, $\mathbb{RP}^2$, taking coefficients in $\mathbb{Z}$. The real projective space is the Δ -complex obtained as the quotient of a rectangle by identifying opposite edges as in the figure below. It has two vertices v and w , three edges a, b and c , and also two 2-simplices U and L . The Δ -complex structure is depicted bellow



The chain groups have the following bases:

$$\Delta_0(K) = \langle v, w \rangle, \quad \Delta_1(K) = \langle a, b, c \rangle \quad \text{and} \quad \Delta_2(K) = \langle U, L \rangle.$$

Boundaries:

$$\partial_1(a) = v - w, \quad \partial_1(b) = w - v \quad \text{and} \quad \partial_1(c) = w - w = 0.$$

Then, $B_0 = \langle v - w \rangle$ and $Z_0 = \langle v, w \rangle$. Therefore, $Z_0/B_0 \cong \mathbb{Z}$. And

$$\partial_2(U) = a + b - c \quad \text{and} \quad \partial_2(L) = a + b + c.$$

One deduces that

$$\begin{aligned} Z_1 &= \langle a + b + c, c \rangle, \\ B_1 &= \langle a + b + c, a + b - c \rangle = \langle a + b + c, 2c \rangle. \end{aligned}$$

Hence $Z_1/B_1 = \mathbb{Z}_2$ and $Z_2/B_2 = \{0\}$ since $Z_2 = \{0\}$. Therefore,

$$H_n^\Delta(\mathbb{R}P^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}; & \text{if } n = 0, \\ \mathbb{Z}_2; & \text{if } n = 1, \\ \{0\}; & \text{if } n \geq 2. \end{cases}$$

We dualize this chain complex to get the following cochain complex:

$$0 \xrightarrow{\delta^0} \langle v^*, w^* \rangle \xrightarrow{\delta^1} \langle a^*, b^*, c^* \rangle \xrightarrow{\delta^2} \langle U^*, L^* \rangle \xrightarrow{\delta^3} 0.$$

And the coboundaries for δ^1 becomes:

$$(\delta^1 v^*)(a) = v^*(\partial_1(a)) = v^*(v - w) = 1.$$

In the same way, $(\delta^1 v^*)(b) = -1$ and $(\delta^1 v^*)(c) = 0$. Therefore, $\delta^1 v^* = a^* - b^*$. Similar computations yield $\delta^1 w^* = -a^* + b^*$ and $\delta^2 a^* = \delta^2 b^* = U^* + L^*$ and $\delta^2 c^* = -U^* + L^*$.

So for $n = 0$:

$$\text{Ker}(\delta^1) = \langle v^* + w^* \rangle \cong \mathbb{Z} \quad \text{and} \quad \text{Im}(\delta^0) = \{0\}.$$

Therefore, $H_{\Delta}^0(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z}/\{0\} \cong \mathbb{Z}$.

For $n = 1$:

$$\text{Ker}(\delta^2) = \langle a^* - b^* \rangle \cong \mathbb{Z} \quad \text{and} \quad \text{Im}(\delta^1) = \langle a^* - b^*, -a^* + b^* \rangle \cong \mathbb{Z}.$$

Therefore, $H_{\Delta}^1(\mathbb{R}P^2; \mathbb{Z}) = \{0\}$.

For $n = 2$:

$$\begin{aligned} \text{Ker}(\delta^3) &= \langle U^*, L^* \rangle \cong \mathbb{Z} \oplus \mathbb{Z}, \\ \text{Im}(\delta^2) &= \langle U^* + L^*, -U^* + L^* \rangle = \langle U^* + L^*, 2L^* \rangle \cong \mathbb{Z} \oplus 2\mathbb{Z}. \end{aligned}$$

Therefore,

$$H_{\Delta}^2(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z}_2.$$

In summary,

$$H_{\Delta}^n(\mathbb{R}P^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}; & \text{if } n = 0 \\ \mathbb{Z}_2; & \text{if } n = 2 \\ \{0\}; & \text{if } n \neq 0, 2. \end{cases}$$

We conclude that $H_{\Delta}^n(\mathbb{R}P^2; \mathbb{Z}) \neq \text{Hom}(H_n^{\Delta}(\mathbb{R}P^2); \mathbb{Z})$ as $\text{Hom}(H_1^{\Delta}(\mathbb{R}P^2); \mathbb{Z}) = \text{Hom}(\mathbb{Z}_2; \mathbb{Z}) \cong \mathbb{Z}_2$ while $H_{\Delta}^1(\mathbb{R}P^2; \mathbb{Z}) = \{0\}$.

This example lead us to discuss the relationship between a chain complex's cohomology groups and homology groups. These relation is given by the following evaluation homomorphism.

Definition 3.5.3. Let (S_*, ∂_*) be a chain complex and R be a commutative ring with unity. Then the evaluation homomorphism is given by the map

$$\begin{aligned} \text{ev} : H^n(S_*; R) &\longrightarrow \text{Hom}(H_n(S_*), R) \\ [\alpha : S_n(X) \rightarrow R] &\mapsto H_n(S_*) \longrightarrow R \\ [c] &\mapsto \alpha(c). \end{aligned}$$

From the fact that $H^2(\mathbb{R}P^2; R) \cong \mathbb{Z}_2$ and $H_2(\mathbb{R}P^2; R) = \{0\}$ we say that the evaluation homomorphism is in general not a monomorphism. Through evaluation homomorphism and Ext-groups $\text{Ext}(H, R)$, set of isomorphism classes of extensions of R by H , that is, a short exact sequence

$$0 \longrightarrow R \longrightarrow J \longrightarrow H \longrightarrow 0,$$

with a natural definition of isomorphism between such exact sequences.

The following theorem connects homology groups and cohomology groups (11, Page 200).

Theorem 3.5.4. *Let (S_*, ∂_*) be a chain complex of free abelian groups and let R be a commutative ring with unity. Then for each $n \in \mathbb{N} \setminus \{0\}$ there exists a short exact sequence*

$$0 \longrightarrow \text{Ext}(H_{n-1}(S_*), R) \longrightarrow H^n(S_*; R) \xrightarrow{ev} \text{Hom}(H_n(S_*), R) \longrightarrow 0$$

which is natural in the chain complex and in the commutative ring R . This short exact sequence splits and there exists an isomorphism

$$H^n(S_*; R) \cong \text{Ext}(H_{n-1}(S_*), R) \oplus \text{Hom}(H_n(S_*), R).$$

We won't be give the proof here but this is further explained in (11, Page 198). The corresponding cohomology is determined by homology groups, but the opposite is true only if the homology groups are finitely generated.

3.5.1 Singular Cohomology

In this section we look at singular cohomology.

Definition 3.5.1.1. (1, Page 375) *Let X be a topological space and R be a commutative ring with unity. The singular n -cochain group $S^n(X; R)$ with coefficients in R is defined to be the dual group given by*

$$S^n(X; R) = \text{Hom}_{\mathbb{Z}}(S_n(X; \mathbb{Z}), R).$$

We define $S^{n-1}(X, R) \xrightarrow{\delta^n} S^n(X, R)$ by $(\delta^n f)(\sigma) = (-1)^n(f(\partial\sigma))$, and obtain a sequence

$$0 \longrightarrow S^0(X, R) \xrightarrow{\delta^1} S^1(X, R) \xrightarrow{\delta^2} \cdots,$$

where $\delta^{n+1} \circ \delta^n = 0$, called a cochain complex.

Definition 3.5.1.2. *Consider the singular cohomology with coefficients in a commutative ring R (e.g in $\mathbb{Z}$). For cochains $\alpha \in S^k(X; R)$ and $\beta \in S^l(X; R)$, the cup product $\alpha \cup \beta \in S^{k+l}(X; R)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+l} \longrightarrow X$ is given by*

$$(\alpha \cup \beta)(\sigma) = \alpha(\sigma|_{[v_0, \dots, v_k]})\beta(\sigma|_{[v_k, \dots, v_{k+l}]}).$$

The cup product of cochains is bilinear and associative. A cup product of cohomology classes is generated by the cup product of cochains and is as follows:

$$\cup : H^k(X; R) \times H^l(X; R) \longrightarrow H^{k+l}(X; R).$$

This product is bilinear, associative and distributive since the product possesses these characteristics at the level of cochains. If R has an identity element, then there is an identity element $1 \in H^0(X; R)$ for the cup product.

Remark 3.5.1.3. *The cup product is commutative in the "graded sense" (see Section 5.3 and*

(16)). If $\alpha \in H^k(X; R)$ and $\beta \in H^l(X; R)$, then we have

$$\beta \cup \alpha = (-1)^{kl} \alpha \cup \beta.$$

Definition 3.5.1.4. Define the cohomology ring $H^*(X; R)$ as the graded ring

$$H^*(X; R) := \bigoplus_{n \geq 0} H^n(X; R).$$

If R is a field $\mathbb{F}$, then $H^*(X; \mathbb{F})$ is a graded algebra.



4. Fibrations

4.1 Topological Groups and Group Action

We start by looking at topological groups as they play a central role in bundle theory. Simply put, a topological group is a combination of the terms "group" and "topological space". Algebraically speaking, topological groups are groups together with continuous group operations. In other words, the underlying topological space should be compatible with the group structure.

Definition 4.1.1. *A topological group G is a set G together with a group structure and a topology on G such that the map*

$$\mu : G \times G \longrightarrow G, (g_1, g_2) \mapsto g_1 g_2,$$

and the inverse map

$$i : G \longrightarrow G, g \mapsto g^{-1},$$

are continuous.

One can combine the two conditions above into one.

Proposition 4.1.2. *A group G is a topological group if and only if $G \times G \longrightarrow G$ defined by $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is continuous.*

Proof. Suppose G is a topological group, that is, the multiplication $\mu : G \times G \longrightarrow G$ and the inverse $i : G \longrightarrow G$ are continuous. Then the composition $G \times G \xrightarrow{(id, i)} G \times G \xrightarrow{\mu} G$ is continuous, that is, $(g_1, g_2) \mapsto (g_1, g_2^{-1}) \mapsto g_1 g_2^{-1}$ is continuous.

Conversely, assume that the map $\tilde{\mu} : G \times G \longrightarrow G$ defined by $\tilde{\mu}(g_1, g_2) = g_1 g_2^{-1}$ is continuous. Then the composition $G \rightarrow G \times G \xrightarrow{\tilde{\mu}} G$, defined by $g \mapsto (1, g) \mapsto 1 \cdot g^{-1} = g^{-1}$ is continuous. Hence the inverse is continuous. As the inverse map is continuous, then $(id, i) : G \times G \longrightarrow G \times G$ is continuous. Composing with $\tilde{\mu}$ yields that μ is continuous. $\square$

Example 4.1.3. *Consider $\mathbb{R}^n$ (under the usual addition) and $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ (under the usual multiplication of complex numbers). They are examples of topological groups.*

The following examples are of some classical topological groups. Let $M(n, \mathbb{F})$ be the set of $n \times n$ matrices with entries in $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$).

Definition 4.1.4. *The set $GL(n, \mathbb{R})$ of all $n \times n$ nonsingular matrices with entries in $\mathbb{R}$, is a group under usual multiplication of matrices. It is called the General Linear Group over $\mathbb{R}$, and it is a topological group. The topology is the subset topology from the inclusion $GL(n, \mathbb{R}) \subseteq M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$.*

Definition 4.1.5. The set $SL(n, \mathbb{R})$ defined by

$$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det(A) = 1\}$$

is a topological subgroup of $GL(n, \mathbb{R})$, called the *Special Linear Group*.

Definition 4.1.6. The set $O(n, \mathbb{R})$ defined by

$$O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid AA^T = I_n\}$$

is a topological subgroup of $GL(n, \mathbb{R})$, called the *Orthogonal Group*.

Definition 4.1.7. The set $SO(n, \mathbb{R})$ defined by

$$SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) \mid \det(A) = 1\}$$

is a topological subgroup of $O(n, \mathbb{R})$, called the *Special Orthogonal Group*.

Definition 4.1.8. A topological space X is Hausdorff if for any $x, y \in X$ there exist $U, V \subset X$, U and V open sets, such that $x \in U, y \in V$ and $U \cap V = \emptyset$. When each open cover in a topological space X has a finite subcover, the space is said to be compact, that is, if

$$X = \bigcup_{i \in I} U_i$$

then we also have

$$X = \bigcup_{i \in I'} U_i$$

for some finite subset I' of I .

Remark 4.1.9. A subset $X \subset \mathbb{R}^n$ (with its usual topology) is compact if and only if X is closed and bounded (4, Corollary 8.7).

Example 4.1.10. We show that $O(n, \mathbb{R})$ is compact. To show that $O(n, \mathbb{R})$ is closed, from Definition 4.1.6, $A \in O(n, \mathbb{R})$ if and only if $AA^T = I_n$. That is, the entries of A satisfy n^2 polynomial equations, one for each entry of I_n . Take, for instance, $n = 2$, let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

So $AA^T = I_2$ implies that

$$\begin{bmatrix} a_{11}^2 + a_{12}^2 & a_{11}a_{21} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{12} & a_{21}^2 + a_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

And so, the entries of A satisfy $2^2 = 4$ polynomial equations. So we can view $O(n, \mathbb{R})$ as a subset of $\mathbb{R}^{n^2}$ which is the intersection of the set of solutions of these n^2 polynomial equations. Thus,

$O(n, \mathbb{R})$ is a closed subset of $\mathbb{R}^{n^2}$.

To show that $O(n, \mathbb{R})$ is bounded, let $A \in O(n, \mathbb{R})$. Let $\vec{v} = [v_1, v_2, \dots, v_n]^T$ be a column vector of A . Then we have $\|\vec{v}\|^2 = \sum_{i=1}^n v_i^2 = 1$. That is, each entry a_{ij} of A satisfy $|a_{ij}| \leq 1$. Hence, $O(n, \mathbb{R})$ is a bounded subset of $\mathbb{R}^{n^2}$. Therefore, $O(n, \mathbb{R})$ is compact.

Remark 4.1.11. Since $SO(n, \mathbb{R})$ is a closed subgroup of the compact group $O(n, \mathbb{R})$, it is itself compact.

Theorem 4.1.12. (1, Appendix A, Theorem A.1.8) The topological space $GL(n, \mathbb{R})$ is neither connected nor compact.

Proof. (1, Appendix A, Theorem A.1.8) The determinant function, $Det : M(n, \mathbb{R}) \rightarrow \mathbb{R}$ is continuous. Moreover, $GL(n, \mathbb{R}) = Det^{-1}(\mathbb{R} \setminus \{0\})$, hence open. Therefore $GL(n, \mathbb{R})$ is an open subspace of $M(n, \mathbb{R})$, it is not compact. Moreover $GL(n, \mathbb{R})$ has 2 components namely,

$$GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid Det(A) > 0\}$$

and

$$GL^-(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid Det(A) < 0\}.$$

□

Definition 4.1.13. The set $GL(n, \mathbb{C})$ is the set of all $n \times n$ nonsingular matrices with complex entries. It is a group under the usual multiplication of matrices, called the Complex Linear Group.

Theorem 4.1.14. The topological group $GL(n, \mathbb{C})$ is not compact.

Proof. As in Theorem 4.1.12, $GL(n, \mathbb{C})$ is open in $M(n, \mathbb{C})$.

□

Proposition 4.1.15. The set $U(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) \mid A\bar{A}^T = I\}$ is a compact subgroup of $GL(n, \mathbb{C})$, and it is called the unitary group.

Proof. Similar to Example 4.1.10.

□

Definition 4.1.16. Let G and H be topological groups. An isomorphism of topological groups is a homeomorphism $f : G \rightarrow H$ which is also a group homomorphism.

Example 4.1.17. $SO(2, \mathbb{R})$ and S^1 are isomorphic topological groups under the isomorphism f defined by

$$f : SO(2, \mathbb{R}) \rightarrow S^1$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mapsto e^{i\theta}.$$

The idea of actions for topological groups is introduced in the following definition.

Definition 4.1.18. A right G -space, for a topological space G , is a topological space X together with a continuous map

$$\begin{aligned}\mu : X \times G &\longrightarrow X \\ (x, g) &\longmapsto xg\end{aligned}$$

satisfying the following basic conditions

- (i) $xe = x$ for all $x \in X$,
- (ii) $x(g_1g_2) = (xg_1)g_2$ for all $x \in X$ and $g_1, g_2 \in G$.

Remark 4.1.19.

- (i) Suppose we have a right G -space $\mu : X \times G \longrightarrow X$. Then for every $g \in G$, we have a function $\mu_g : X \longrightarrow X$, defined by

$$\mu_g(x) = xg,$$

for all $x \in X$. Note that μ_g has the inverse function as $\mu_{g^{-1}}$ since

$$\mu_{g^{-1}}(\mu_g(x)) = \mu_{g^{-1}}(xg) = xgg^{-1} = x.$$

Similarly, $\mu_g \circ \mu_{g^{-1}} = \text{id}$. Hence μ_g is bijective, consequently μ_g is a homeomorphism.

- (ii) The group action μ defines a map $\tilde{\mu} : G \longrightarrow \text{Homeo}(X)$, where $\text{Homeo}(X)$ denotes the group of homeomorphisms of X , defined by

$$\tilde{\mu}(g)(x) = xg.$$

Condition (ii) in Definition 4.1.18 is equivalent to the requirement that $\tilde{\mu} : G \longrightarrow \text{Homeo}(X)$ be a group homomorphism.

A left G -space is defined in a similar way.

Example 4.1.20. The space $\mathbb{R}^n$ is a left $GL(n, \mathbb{R})$ -space or a left $O(n)$ -space, where the G -space action is given by matrix multiplication.

Definition 4.1.21. (14, Page 42) A G -space X is called free if it has the property that $xg = x$, for all $x \in X$, implies that $g = 1$. In particular, $\tilde{\mu} : G \longrightarrow \text{Homeo}(X)$ is injective.

Definition 4.1.22. A group G acts transitively on X if for any $x_1, x_2 \in X$ there exists an element $g \in G$ such that $x_2 = x_1g$. Fix an element $x_0 \in X$ and let $x \in X$. Then there exists $g \in G$ such that $x_0g = x$. Hence the map $G \longrightarrow X, g \mapsto x_0g$ is surjective.

Definition 4.1.23. Let X be a left G -space and $x \in X$. Then the isotropy subgroup G_x is the set $\{g \in G \mid gx = x\}$.

Proposition 4.1.24. (15, Page 6) *If G acts transitively on X and G_x is the isotropy subgroup of $x \in X$, then G/G_x and X are homeomorphic.*

Example 4.1.25. *The space $GL(2, \mathbb{R})$ acts on $\mathbb{R}^2 \setminus \{0\}$ transitively. To see this, let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{v}$ non-zero. We want $A \in GL(2, \mathbb{R})$ such that $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. If $a \neq 0$, then let $A = \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$.*

If $b \neq 0$, then let $A = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$. These matrices are invertible and they send $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

Now given $v, w \in \mathbb{R}^2$, there exists $A, B \in GL(2, \mathbb{R})$ such that $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = v$ and $B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w$, then $BA^{-1}v = w$. Hence the action is transitive.

In general, $GL(n, \mathbb{R})$ (respectively $GL(n, \mathbb{C})$) acts transitively on $\mathbb{R}^n \setminus \{0\}$ ($\mathbb{C}^n \setminus \{0\}$ respectively).

We now look at manifolds. Manifolds are merely extensions of curves and surfaces to objects of any arbitrary dimension.

Definition 4.1.26. (1, Page 38) *A Hausdorff topological space with a countable open covering $\{U_i\}$ is known as a n -dimensional smooth manifold M , such that*

- (i) *for each U_i , there is a homeomorphism $\psi_i : U_i \rightarrow V_i$, where V_i is an open set in $\mathbb{R}^n$,*
- (ii) *if $U_i \cap U_j \neq \emptyset$, then the homeomorphisms $\psi_{ji} : \psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \rightarrow \psi_j(U_i \cap U_j)$ are C^∞ differentiable.*

The pair (U_i, ψ_i) is called a chart of M and $\{(U_i, \psi_i)\}$ is the set of charts of M known as an atlas.

A complex manifold is defined in a similar way, the only difference being that V_i is an open set of $\mathbb{C}^n$.

Example 4.1.27. *Consider the complex projective space, $\mathbb{CP}^n$, the set of all complex lines inside $\mathbb{C}^{n+1}$. It is a smooth manifold with coordinate charts $\phi_j : U_j \rightarrow \mathbb{C}^n$ where*

$$U_j = \{[z_1, z_2, \dots, z_{n+1}] \mid z_j \neq 0\}$$

and

$$\phi_j([z_1, z_2, \dots, z_{n+1}]) = \left(\frac{z_1}{z_j}, \frac{z_2}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_{n+1}}{z_j} \right).$$

Example 4.1.28. *The space $\mathbb{R}^n$ and S^n are examples of n -dimensional differentiable manifolds.*

We now look at Stiefel and Grassmannian manifolds.

Definition 4.1.29. *The real Stiefel manifold, denoted $V_k(\mathbb{R}^n)$, is the set of all orthonormal*

k -frames in $\mathbb{R}^n$, i.e.

$$V_k(\mathbb{R}^n) = \left\{ (x_1, x_2, \dots, x_k) \in (\mathbb{R}^n)^k \mid x_i \cdot x_j = \delta_{ij} \right\}.$$

In a similar way, the complex Stiefel manifold, denoted $V_k(\mathbb{C}^n)$, is

$$V_k(\mathbb{C}^n) = \left\{ (x_1, x_2, \dots, x_k) \in (\mathbb{C}^n)^k \mid x_i \cdot x_j = \delta_{ij} \right\}.$$

One can show that $V_k(\mathbb{R}^n)$ and $V_k(\mathbb{C}^n)$ are indeed smooth manifolds (24, Section 3.2).

Remark 4.1.30. Consider the Stiefel manifold $V_k(\mathbb{F}^n)$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then:

(i) $V_n(\mathbb{R}^n) \cong O(n)$ and $V_n(\mathbb{C}^n) \cong U(n)$.

(ii) For $k = 1$:

$$V_1(\mathbb{R}^n) \cong S^{n-1} \quad \text{and} \quad V_1(\mathbb{C}^n) \cong S^{2n-1}.$$

(iii) For $k = n - 1$:

$$V_{n-1}(\mathbb{R}^n) \cong SO(n) \quad \text{and} \quad V_{n-1}(\mathbb{C}^n) \cong SU(n).$$

(iv) For $n > k$: The orthogonal group $O(n)$ acts on $V_k(\mathbb{R}^n)$ by left multiplication. Let $(e_1, \dots, e_n)$ be the canonical basis in $\mathbb{R}^n$ and take $z_0 = (e_1, \dots, e_k) \in V_k(\mathbb{R}^n)$, we compute the isotropy subgroup of z_0 . Then $Az_0 = z_0$ is equivalent to saying that the matrix A has the form

$$\begin{bmatrix} I_k & 0 \\ 0 & B \end{bmatrix},$$

where I_k is the identity matrix in $O(k)$ and $B \in O(n - k)$. From Proposition 4.1.24,

$$V_k(\mathbb{R}^n) \cong O(n)/O(n - k) \cong SO(n)/SO(n - k),$$

In a similar manner,

$$V_k(\mathbb{C}^n) \cong U(n)/U(n - k) \cong SU(n)/SU(n - k).$$

Definition 4.1.31. The real Grassmannian manifold $G_k(\mathbb{R}^n)$ is the set whose points are k -dimensional subspaces of $\mathbb{R}^n$. In the same way, the complex Grassmannian manifold $G_k(\mathbb{C}^n)$ is the set whose points are k -dimensional subspaces of $\mathbb{C}^n$. In particular, $G_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$.

According to (16, Page 211), there is a mapping $f : V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n)$ which sends each set of k -linear independent orthonormal vectors in $\mathbb{R}^n$ to the k -dimensional linear subspace it spans. Let $A \in G_k(\mathbb{R}^n)$. Then $f^{-1}(A)$ is the set of ordered orthonormal bases for A , any two differing by an element of $O(k)$. Indeed the group $O(k)$ is isomorphic to the subgroup H of $O(k)$ of

elements of the form

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & I_{n-1} \end{array} \right],$$

where $A \in O(k)$. This provides an action of $O(k)$ on $G_k(\mathbb{R}^n)$ by left multiplication. Hence

$$\begin{aligned} G_k(\mathbb{R}^n) &\cong V_k(\mathbb{R}^n)/O(k); && \text{by Proposition 4.1.24,} \\ &\cong O(n)/(O(k) \times O(n-k)), \end{aligned}$$

Similarly,

$$G_k(\mathbb{C}^n) \cong U(n)/(U(k) \times U(n-k))$$

We now look at fibre bundles.

4.2 Fibre Bundles

Definition 4.2.1. Let B be a pointed, path-connected space and $p : E \rightarrow B$ be a continuous mapping. Then the map p is said to be a fibre bundle with fibre F if

- (i) $p^{-1}(b) = F$, where b is the basepoint of B ,
- (ii) p is surjective,
- (iii) there exists an open covering $\{U_i\}$ of B and homeomorphisms $\psi_i : p^{-1}(U_i) \rightarrow U_i \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{p} & U_i \\ \downarrow \psi_i & \nearrow p_1 & \\ U_i \times F & & \end{array}$$

where p_1 is the projection on the first factor.

Definition 4.2.2. A section of a fibre bundle $p : E \rightarrow B$ is a continuous map $s : B \rightarrow E$ such that $p \circ s = I_B$.

Definition 4.2.3. A fibre bundle $p : E \rightarrow B$ has local sections if it has an open cover $\{U_i, i \in I\}$ such the restriction $p : p^{-1}(U_i) \rightarrow U_i$ has a section.

Example 4.2.4. The projection map $p_1 : X \times F \rightarrow X$ is a fibre bundle over X with fibre F . It is called the trivial bundle.

Example 4.2.5. Let $\exp : \mathbb{R} \rightarrow S^1$ be the map defined by $\exp(t) = e^{2\pi it} \in S^1$. Then $\exp$ is a fibre bundle with fibre $\mathbb{Z}$.

Example 4.2.6. The canonical projection $S^n \rightarrow \mathbb{R}P^n$ is a fibre bundle with fibre $\mathbb{Z}_2$.

Example 4.2.7. Consider $S^{2n+1} = \{(z_1, z_2, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \text{ such that } \sum_{i=1}^{n+1} |z_i|^2 = 1\}$. Then $p : S^{2n+1} \rightarrow \mathbb{C}P^n$ defined by $p(z_1, z_2, \dots, z_{n+1}) = [z_1, z_2, \dots, z_{n+1}]$ is a fibre bundle with fibre S^1 .

Example 4.2.8. Let $E = \{(x, v) \in S^{n-1} \times \mathbb{R}^n \mid \langle x, v \rangle = 0\}$. Then the map $p : E \rightarrow S^{n-1}$ defined by $p(x, v) = x$ is a fibre bundle with fibre $F \cong \mathbb{R}^n$.

Remark 4.2.9. Let $p : E \rightarrow B$ be a fibre bundle and, (U_i, ψ_i) and (U_j, ψ_j) be trivialization maps. If $x \in U_i \cap U_j$, then consider mappings

$$(U_i \cap U_j) \times F \xrightarrow{\psi_i} p^{-1}(U_i \times F) \xrightarrow{\psi_j^{-1}} (U_i \cap U_j) \times F$$

defined by $(\psi_j^{-1} \circ \psi_i)(x, f) = (x, g_{ij}(x)(f))$, where $g_{ij}(x)$ is a homeomorphism of F . Then there is a map $g_{ij} : U_i \cap U_j \rightarrow \text{Homeo}(F)$.

Definition 4.2.10. Let G be a group. A fibre bundle $F \hookrightarrow E \xrightarrow{p} B$ has a structure group G if G acts on F and $g_{ij}(x) : F \rightarrow F$ is the multiplication by an element of G . We say that p is a G -bundle.

Definition 4.2.11. (16, Page 209) A morphism between G -bundles $\mu' = (F' \hookrightarrow E' \rightarrow B')$ and $\mu = (F \hookrightarrow E \rightarrow B)$, is given by maps $f : B' \rightarrow B$ and $\tilde{f} : E' \rightarrow E$ such that

(i) the following diagram commutes

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ \downarrow p' & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

(ii) the restriction of $\tilde{f} : F' \rightarrow F$ is G -equivariant, that is, if $x \in F'$ and $g \in G$, then

$$\tilde{f}(gx) = g\tilde{f}(x).$$

Remark 4.2.12. If $B = B'$, $f = I_B$ and $\tilde{f}$ is a homeomorphism, then we say that the G -bundles μ' and μ are isomorphic.

Example 4.2.13. Let $F \hookrightarrow E \xrightarrow{p} B$ be a bundle and A be a subspace of B . Then the restriction of E to A , written $p|_A : E|_A := p^{-1}(A) \rightarrow A$ is a bundle over A and the inclusion $E|_A \hookrightarrow E$ induces a bundle morphism

$$\begin{array}{ccc} E|_A & \hookrightarrow & E \\ \downarrow p|_A & & \downarrow p \\ A & \hookrightarrow & B \end{array}$$

Definition 4.2.14. Let $\eta = (F \hookrightarrow E \rightarrow B)$ be a fibre bundle and $f : B' \rightarrow B$ be a continuous map. Then the induced bundle or pullback bundle of η by f is the bundle $f^*(\eta) = (F' \hookrightarrow E' \rightarrow B')$ consisting of total space $E' = \{(b, e) \in B' \times E \mid p(e) = f(b)\}$ and the projection $p'(b, e) = b$.

Proposition 4.2.15. If f and g are homotopic, then $f^*(\eta)$ and $g^*(\eta)$ are isomorphic.

Proof. See (18, Theorem 2.0.2) for proof. $\square$

We now define principal bundles which are basically parameterized families of topological groups.

Definition 4.2.16. (16, Page 209) A G -bundle is called *principal* if it has structure group G and fibre G with G acting on itself by left multiplication.

Example 4.2.17. Take $S^{2n+1} \subset \mathbb{C}^{n+1}$ as defined in Example 4.2.7. We show that fibre bundle $S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$ is principal. Recall the trivialization map $U_j = \{[z_1, z_2, \dots, z_{n+1}] \in \mathbb{C}P^n \mid z_j \neq 0\}$ and the chart maps defined in Example 4.2.7. By definition we have

$$\pi^{-1}(U_j) = \{z = (z_1, z_2, \dots, z_{n+1}) \in S^{2n+1} \mid z_j \neq 0\}$$

and the local trivialization map can be taken as

$$\begin{aligned} \psi_j : \pi^{-1}(U_j) &\longrightarrow U_j \times S^1 \\ z &\mapsto \left([z_1, z_2, \dots, z_{n+1}], \frac{z_j}{|z_j|} \right). \end{aligned}$$

It is an homeomorphism with inverse map

$$\psi_j^{-1} \left([z_1, z_2, \dots, z_{n+1}], e^{i\theta} \right) = \frac{e^{i\theta}|z_j|}{z_j \sqrt{\sum_{j=1}^{n+1} |z_j|^2}} (z_1, z_2, \dots, z_{n+1}).$$

For $j \neq k$, we have

$$\psi_k \psi_j^{-1} = \left([z_1, z_2, \dots, z_{n+1}], e^{i\theta} \right) = \left([z_1, z_2, \dots, z_{n+1}], e^{i\theta} \frac{z_k |z_j|}{z_j |z_k|} \right).$$

So the transition map $g_{kj} : U_k \cap U_j \longrightarrow \text{Homeo}(S^1)$ is the multiplication by $\frac{z_k |z_j|}{z_j |z_k|}$. Therefore, S^{2n+1} is indeed a principal S^1 -bundle over $\mathbb{C}P^n$. It is called the Hopf bundle.

Example 4.2.18. One also get principal fibre bundles

$$O(k) \hookrightarrow V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n) \quad \text{and} \quad U(k) \hookrightarrow V_k(\mathbb{C}^n) \longrightarrow G_k(\mathbb{C}^n).$$

For $k = 1$, one retrieves principal fibre bundles

$$\mathbb{Z}/2\mathbb{Z} \hookrightarrow S^n \longrightarrow \mathbb{R}P^n = G_1(\mathbb{R}^{n+1}) \quad \text{and} \quad S^1 \hookrightarrow S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$$

respectively.

Example 4.2.19. If a group G acts freely on X and the projection $p : X \longrightarrow X/G$ has local sections, then p is a principal G -bundle. If G is a group and H is a closed subgroup, then the projection $G \longrightarrow G/H$ is a principal H -bundle (see (5, Section 1)).

Definition 4.2.20. (16, Page 210) A principal G -bundle $G \hookrightarrow E \longrightarrow B$ is called n -universal if $\pi_i(E) = \{0\}$, for $i \leq n$.

Definition 4.2.21. (16, Page 210) If G is a topological group, then there exists a principal G -bundle $\eta_\infty = (G \hookrightarrow EG \rightarrow BG)$, such that EG is contractible (Definition 2.1.10). The condition that EG is contractible determines BG up to homotopy. In particular, $\pi_i(EG) = \{0\}$, for all i . The G -bundle η_∞ is called the universal G -bundle of G .

The space BG is unique up to homotopy equivalence and is called the classifying space for G . Moreover, if G and B have the homotopy type of a CW complex, each principal G -bundle $\eta = (G \hookrightarrow E \rightarrow B)$ is classified by a continuous map $f : B \rightarrow BG$ such that $f^*(\eta_\infty)$ is isomorphic to η .

Example 4.2.22. (16, Page 212) Let G be a topological group and let H be a closed subgroup of G . If $p : EG \rightarrow BG$ is the universal principal bundle for G , then

$$EG \rightarrow EG/H$$

is the universal bundle for H since EG is contractible by assumption. Hence $BH \cong EG/H$.

Example 4.2.23. Take $G = \mathbb{Z}_2$. Then for each n , we have the bundle $\mathbb{Z}_2 \hookrightarrow S^n \rightarrow \mathbb{R}P^n$ is $(n-1)$ -universal. Taking inclusions

$$\begin{array}{ccccccc} S^n & \longrightarrow & S^{n+1} & \longrightarrow & \dots & \longrightarrow & S^\infty \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathbb{R}P^n & \longrightarrow & \mathbb{R}P^{n+1} & \longrightarrow & \dots & \longrightarrow & \mathbb{R}P^\infty \end{array}$$

yields the universal $\mathbb{Z}_2$ -bundle $\mathbb{Z}_2 \hookrightarrow S^\infty \rightarrow \mathbb{R}P^\infty$. Here $\mathbb{R}P^\infty$ is the classifying space for $\mathbb{Z}_2$.

4.3 Fibrations

Definition 4.3.1. (Homotopy Lifting Property) Given two topological spaces E and B along with a continuous map $p : E \rightarrow B$, we say that p has the homotopy lifting property with respect to X if, for any two maps $f : X \rightarrow E$ and $G : X \times I \rightarrow B$ such that $p \circ f = G \circ i$, where $i : X \rightarrow X \times I, x \mapsto (x, 0)$, there is a continuous map $\tilde{G} : X \times I \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ \downarrow i & \nearrow \tilde{G} & \downarrow p \\ X \times I & \xrightarrow{G} & B \end{array}$$

Definition 4.3.2. We say that $p : E \rightarrow B$ is a Hurewicz fibration if it has the homotopy lifting property with respect to all spaces X . We say that $p : E \rightarrow B$ is a Serre fibration if it has homotopy lifting property with respect to CW-complexes.

We will only consider Serre fibrations. The following theorem is about when a fibre bundle is a fibration.

Theorem 4.3.3. (21, Page 96) If $p : E \rightarrow B$ is a fibre bundle with paracompact base B , then it is also a fibration.

Proof. See (16, Proposition 6.32) for sketch of proof. $\square$

Theorem 4.3.4. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration, then we have a long exact sequence*

$$\cdots \xrightarrow{\partial} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{i_*} \cdots$$

This is called the long exact sequence of a fibration or homotopy exact sequence of a fibration.

Proof can be found in (4, Chapter VII, Section 6). Using Theorem 4.3.4 above, the long exact sequence of homotopy groups formed by the Hopf fibration, we can now compute $\pi_3(S^2)$.

Proposition 4.3.5. $\pi_3(S^2) \cong \mathbb{Z}$.

Proof. Recall that $S^1 \hookrightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$ is a principal G -bundle, hence it is a fibration. For $n = 1$, we get the Hopf fibration $S^1 \hookrightarrow S^3 \rightarrow \mathbb{C}P^1 \cong S^2$ and thus induces the following long exact sequence of homotopy groups (basepoint omitted plus n -spheres are path-connected)

$$\cdots \xrightarrow{\partial} \pi_n(S^1) \xrightarrow{i_*} \pi_n(S^3) \xrightarrow{p_*} \pi_n(S^2) \xrightarrow{\partial} \pi_{n-1}(S^1) \xrightarrow{i_*} \cdots$$

From the long exact sequence, we get (from Theorem 2.3.13):

$$\{0\} = \pi_3(S^1) \rightarrow \pi_3(S^3) \rightarrow \pi_3(S^2) \rightarrow \pi_2(S^1) = \{0\}.$$

Thus, $\pi_3(S^2) \cong \mathbb{Z}$. $\square$

In the following example we use fibrations to compute the homotopy groups of some unitary group $U(n)$.

Example 4.3.6. *Recall that*

$$V_k(\mathbb{C}^n) \cong U(n)/U(n-k).$$

Taking $k = 1$,

$$S^{2n+1} = V_1(\mathbb{C}^{n+1}) \cong U(n+1)/U(n).$$

The projection

$$p : U(n+1) \rightarrow U(n+1)/U(n)$$

is a fibre bundle with fibre $U(n)$. So the long exact sequence of homotopy groups of the fibration

$$U(n) \hookrightarrow U(n+1) \rightarrow S^{2n+1}$$

is of the form

$$\cdots \rightarrow \pi_{i+1}(S^{2n+1}) \rightarrow \pi_i(U(n)) \rightarrow \pi_i(U(n+1)) \rightarrow \pi_i(S^{2n+1}) \rightarrow \cdots$$

Therefore by Theorem 2.3.14, $\pi_i(S^{2n+1}) = \{0\}$ for $i \leq 2n$. That is,

$$\{0\} \longrightarrow \pi_{i < 2n}(U(n)) \longrightarrow \pi_{i < 2n}(U(n+1)) \longrightarrow \{0\}.$$

Thus, there is an isomorphism between $\pi_{i < 2n}(U(n))$ and $\pi_{i < 2n}(U(n+1))$. Thus,

$$\pi_{2n}(U(n)) \longrightarrow \pi_{2n}(U(n+1))$$

is surjective.

Taking $n = 1$ yields the fibration $U(1) \simeq S^1 \hookrightarrow U(2) \longrightarrow S^3$, and one deduces that

$$\begin{aligned}\pi_1(U(2)) &= \mathbb{Z} \\ \pi_2(U(2)) &= \{0\} \\ \pi_3(U(2)) &= \mathbb{Z}.\end{aligned}$$

We now compute homotopy groups of Stiefel manifolds.

Theorem 4.3.7. For $k \neq n$, $\pi_i(V_k(\mathbb{R}^n)) = \{0\}$ for $i < n - k - 1$.

Proof. We prove by induction on k . Let $k = 1$. Then $V_1(\mathbb{R}^n) = S^{n-1}$ is a sphere and the theorem holds true. The projection $p : V_{k+1}(\mathbb{R}^n) \longrightarrow V_k(\mathbb{R}^n)$ is a locally trivial fibration with fibre $V_1(\mathbb{R}^{n-k}) = S^{n-k-1}$. We then get its long exact sequence of homotopy groups to be

$$\cdots \longrightarrow \pi_i(S^{n-k-1}) \longrightarrow \pi_i(V_{k+1}(\mathbb{R}^n)) \longrightarrow \pi_i(V_k(\mathbb{R}^n)) \longrightarrow \cdots$$

By Theorem 2.3.14, $\pi_i(S^{n-k-1}) = \{0\}$ for $i < n - k - 1$. By assumption, $\pi_i(V_k(\mathbb{R}^n)) = \{0\}$ for $i < n - k - 1$. Thus, $\pi_i(V_k(\mathbb{R}^n)) = \{0\}$ for $k \neq n$ and $i < n - (k + 1) = n - k - 1$. From the homotopy exact sequence $\pi_i(V_{k+1}(\mathbb{R}^n)) = \{0\}$ for $n - k - 2 = n - (k + 1) - 1$. $\square$

Theorem 4.3.8. $\pi_i(V_k(\mathbb{C}^n)) = \{0\}$ for $k \neq n$ and $i < 2(n - k)$.

Proof. Proved in a similar manner as Theorem 4.3.7. $\square$

Corollary 4.3.9.

- (i) The principal $O(k)$ -bundle $O(k) \hookrightarrow V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n)$ is $(n - k - 1)$ -universal.
- (ii) In the same way, the bundle $U(k) \hookrightarrow V_k(\mathbb{C}^n) \longrightarrow G_k(\mathbb{C}^n)$ is $2(n - k)$ -universal.

Example 4.3.10. From Definition 4.1.31 and Theorem 4.3.7 we get that $G_k(\mathbb{R}^n)$ is an $(n - k - 1)$ -classifying space for $O(k)$. Taking inclusions

$$\begin{array}{ccccccc} V_k(\mathbb{R}^n) & \longrightarrow & V_k(\mathbb{R}^{n+1}) & \longrightarrow & \cdots & \longrightarrow & V_k(\mathbb{R}^\infty) \\ \downarrow & & \downarrow & & & & \downarrow \\ G_k(\mathbb{R}^n) & \longrightarrow & G_k(\mathbb{R}^{n+1}) & \longrightarrow & \cdots & \longrightarrow & G_k(\mathbb{R}^\infty) \end{array}$$

yields the universal $O(k)$ -bundle $O(k) \hookrightarrow V_k(\mathbb{R}^\infty) \longrightarrow G_k(\mathbb{R}^\infty)$. Hence $G_k(\mathbb{R}^\infty)$ is the classifying space $BO(k)$. In the same way, $G_k(\mathbb{C}^\infty)$ is the classifying space $BU(k)$ of $U(k)$.

Our aim is to compute the cohomology algebra of $BU(k) = G_k(\mathbb{C}^\infty)$ using the Serre spectral sequences.



5. Leray-Serre Spectral Sequence

Let $F \hookrightarrow E \longrightarrow B$ be a fibration. The purpose of this section is to discuss the relationship between the cohomology rings of the total space E , the base space B , and the fibre F . To deal with this relationship we need a computational tool called *spectral sequence*. Spectral sequences have been one of the greatest achievements of Algebraic Topology since their invention by *Leray* in the late 1940's. Since then, they have become a very useful tool in many disciplines, including Geometry, Topology and Algebra.

In this chapter we present the notion of spectral sequence and some of its application. Our main references for this chapter are (19; 16). The first application of spectral sequences was to calculate the homology of fibrations by Serre (5, Page 152).

5.1 Bicomplexes

Spectral sequences are algebraic tools for working with (co)homology, in that they consist of an array of long exact sequences that are arranged in a systematic pattern and applied in a similar way (5). We first introduce some concepts of homological algebra which are very useful in understanding how spectral sequences work. All vector spaces will be over a fixed field $\mathbb{F}$, although there is a general version on modules.

Definition 5.1.1. A graded vector space M over $\mathbb{F}$ is an indexed family of vector spaces M^p with $p \geq 0$. We write $M = \bigoplus_{p \geq 0} M^p$. An element $x \in M^p$ is said to be homogeneous of degree p . A differential graded vector space M is a graded vector space together with linear maps $d : M^p \longrightarrow M^{p+1}$ such that $d^2 = 0$. The cohomology is then

$$H^*(M) = \bigoplus_{p \geq 0} H^p(M), \quad \text{where,} \quad H^p(M) = Z^p / B^p.$$

Example 5.1.2. Consider a cochain complex of vector spaces $C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} C^3 \xrightarrow{d} \dots$. Then $C = \bigoplus_{n \geq 0} C^n$ is a differential graded vector space.

Definition 5.1.3. A graded map $f : N \longrightarrow M$, where N and M are graded vector spaces, is a family of linear maps $f = \{f^p : N^p \longrightarrow M^{p+q}\}_{p,q \geq 0}$. We say that the degree of f is q .

Let M, N and P be graded vector spaces and $M \xrightarrow{f} N \xrightarrow{g} P$ be graded maps of degree a and b respectively. Then $g \circ f$ is a graded map of degree $a + b$.

Definition 5.1.4. A bigraded vector space is a doubly indexed family of vector spaces $M^{p,q}$ with $p, q \geq 0$.

Similarly to Definition 5.1.3, we can define a bigraded map as follows:

Definition 5.1.5. A bigraded map $f : N \longrightarrow M$, where N and M are bigraded vector spaces,

is a family of homomorphisms $f = \{f^{p,q} : N^{p,q} \longrightarrow M^{p+a,q+b}\}_{p,q \geq 0}$. The bidegree of f is then (a, b) .

We noticed that for graded vector spaces, degrees add with composition maps, so is bidegrees when composing bigraded maps.

In conjunction with differentials, a bigraded vector space becomes a double cochain complex, as defined below.

Definition 5.1.6. (16, Page 47) A double cochain complex is a bigraded vector space

$$C = \bigoplus_{p,q \geq 0} C^{p,q},$$

with differentials $d_h : C^{p,q} \longrightarrow C^{p+1,q}$ and $d_v : C^{p,q} \longrightarrow C^{p,q+1}$ such that $d_h^2 = d_v^2 = 0$ and $d_h \circ d_v + d_v \circ d_h = 0$.

Definition 5.1.7. (16, Page 47) Let (C, d) be a double complex. Then its total complex $T((C, d))$ is a complex satisfying

$$T((C, d))^n = \bigoplus_{n=p+q} C^{p,q},$$

and the coboundary map $d : T((C, d))^n \longrightarrow T((C, d))^{n+1}$ is defined by

$$d(c^{p,q}) = d_h(c^{p,q}) + d_v(c^{p,q}), \tag{5.1}$$

for all $c^{p,q} \in C^{p,q}$.

Remark 5.1.8. Note that due to the relations demanded of d_h and d_v ,

$$\begin{aligned} d^2 &= (d_h + d_v)(d_h + d_v) \\ &= d_h^2 + d_v^2 + d_h d_v + d_v d_h \\ &= 0. \end{aligned}$$

The following figure depicts a total complex:

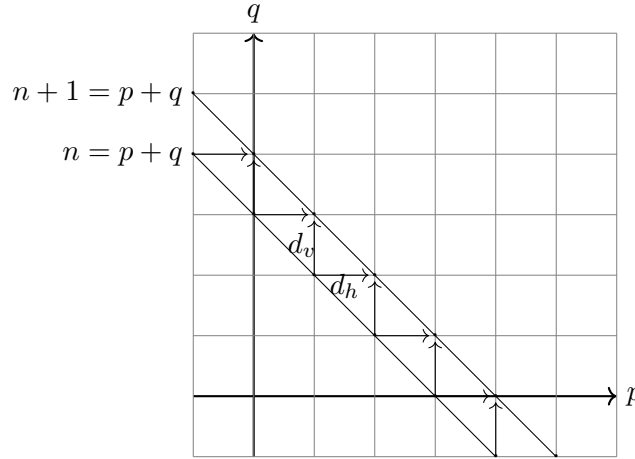


Figure 5.1: Total complex.

Example 5.1.9. If (A, d_A) , with $A = \bigoplus_{i \geq 0} A^i$, and (B, d_B) , with $B = \bigoplus_{j \geq 0} B^j$, are cochain complexes, then $A \otimes B$, where

$$(A \otimes B)^n = \bigoplus_{n=i+j} A^i \otimes B^j$$

with $d(a \otimes b) = d_A a \otimes b + (-1)^{|a|} a \otimes d_B b$, is a double complex, where $d_h = d_A \otimes 1$ and $d_v = 1 \otimes d_B$.

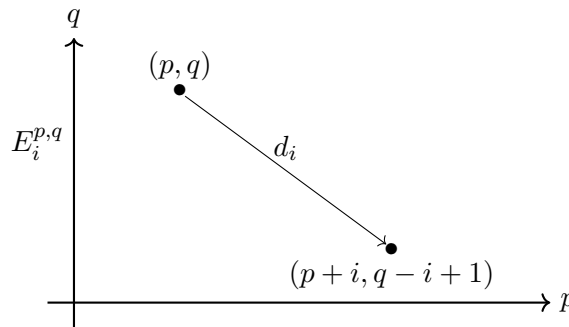
Our introduction of spectral sequences will be based on filtered complexes, however one can also use exact couples. See for instance (22, Section 5.9).

5.2 Spectral Sequence

We now give a formal definition of a cohomological spectral sequence:

Definition 5.2.1. A first quadrant cohomological spectral sequence is a sequence of bigraded vector spaces (E_i, d_i) , for $i \geq 0$, where

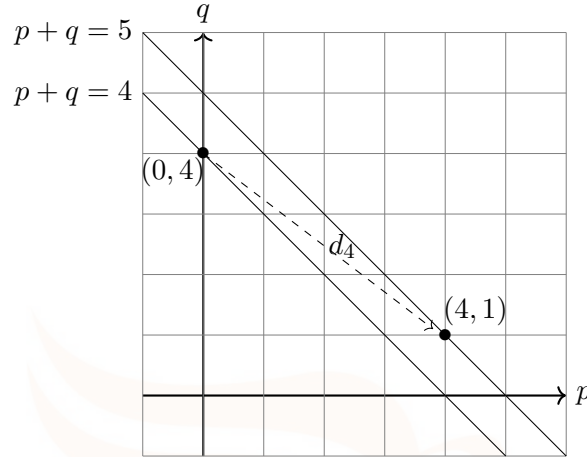
- (i) $E_i = \bigoplus_{p,q \geq 0} E_i^{p,q}$,
- (ii) $d_i : E_i^{p,q} \longrightarrow E_i^{p+i, q-i+1}$ are maps of bidegree $(i, -i+1)$ such that $d_i^2 = 0$,
- (iii) $H^*(E_i, d_i) \cong E_{i+1}$.



A spectral sequence is like an "infinite" book and it is described by objects E_i called *pages* (or *sheets*), while endomorphisms d_i determine how E_i have their structure of a cochain complex. The cohomology vector spaces of these cochain complexes are just the vector spaces which appear on the next page.

We will consider only spectral sequences $E_i^{p,q}$, where $p, q \geq 0$. They are called *first quadrant spectral sequences*.

Example 5.2.2. The following diagram depicts $E_4^{p,q}$ -term, so the bidegree $(4, -3)$:



5.2.1 Filtration

Definition 5.2.1.1. Let $M = \bigoplus_{n \geq 0} M^n$ be a differential graded vector space together with a differential $d : M^n \rightarrow M^{n+1}$. A differential subspace $N \subseteq M$ is a graded subspace of M such that $dN \subseteq N$. Then a filtration F^* of M is a family of differential subspaces $\{F^p M\}_{p \geq 0}$ with

$$\dots \subseteq F^{n+1} M \subseteq F^n M \subseteq \dots \subseteq F^1 M \subseteq F^0 M = M.$$

Remark 5.2.1.2.

- (i) A filtration F^* on M means a filtration on each M^n , that is, there is a filtration given by $F^p M = \bigoplus_{n \geq 0} F^p M^n$.
- (ii) The inclusion $i^p : F^p M \rightarrow M$ induces $H^*(i^p) : H^*(F^p M) \rightarrow H^*(M)$. This yields a filtration $F^p H^*(M) = \text{Im}(H^*(i^p))$ on $H^*(M)$.

Definition 5.2.1.3. Let M be a differential graded vector space with a filtration F^* . Then the associated graded vector space $Gr M$ is defined by

$$Gr M = \bigoplus_{p \geq 0} Gr^p M,$$

where $Gr^p M = E_0^p(M) = F^p M / F^{p+1} M$. If M^n is finite dimensional for each n , then

$$M = \bigoplus_{p \geq 0} F^p M / F^{p+1} M.$$

Consider a filtered differential graded vector space M and the corresponding filtration $F^* H^*(M)$ on $H^*(M)$, for which the associated graded vector space is

$$Gr^p H^{p+q}(M) = F^p H^{p+q}(M) / F^{p+1} H^{p+q}(M).$$

If M^n is finite dimensional, then

$$H^*(M) = \bigoplus_{p,q} Gr^p H^{p+q}(M).$$

5.2.2 Convergence

When we recursively take the cohomology of successive complexes, under certain conditions, we eventually end up having each vector space stabilizing to the E_∞ term, from which we interpolate our desired cohomology (under certain conditions). Since we are considering cases where $p, q \geq 0$, it follows that if we continue turning successive pages $i \gg 1$, for a fixed p and q , while focusing on the point (p, q) , in this case, the differentials d_i are eventually equal to zero, that is, have trivial codomain, and they become the zero maps in which (p, q) is not traversed. So for larger i we get the following definition:

Definition 5.2.2.1. We say a spectral sequence $(E_i^{p,q}, d_i)$ of differential graded vector spaces converges to a limit page, denoted $E_\infty^{p,q}$, if for some sufficiently large positive integer N , all differentials leaving and entering $E_i^{p,q}$ are zero for $i \geq N$. That is,

$$E_N^{p,q} \cong E_{N+1}^{p,q} \cong E_{N+2}^{p,q} \cong \dots,$$

which we denote by $E_\infty^{p,q}$. We write $E_i^{p,q} \implies E_\infty^{p,q}$. The spectral sequence $(E_i^{p,q}, d_i)$ converges to H^* , and we write $E_i^{p,q} \implies H^{p+q}$, if there is a filtration F^* on H^* such that

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}.$$

If p and q are fixed, then $d_{q+2} : E_{q+2}^{p+1,q} \longrightarrow E_{q+2}^{p+q+3,-1} = \{0\}$ is the zero map. The differential $d_{p+1} : E_{p+1}^{-1,q+p} \longrightarrow E_{p+1}^{p,q}$ is also zero. Hence, for $s = \max(p+1, q+2)$

$$E_s^{p,q} \cong E_{s+1}^{p,q} \cong E_{s+2}^{p,q} \cong \dots$$

which we denote $E_\infty^{p,q}$. In particular, a first quadrant spectral sequence converges.

Definition 5.2.2.2. A spectral sequence $(E_i^{p,q}, d_i)$ is said to degenerate (or collapse) at the N^{th} term if $d_i = 0$ for all $i \geq N$ and hence $E_N = E_\infty$.

We state the following theorem without providing the proof.

Theorem 5.2.2.3. (9, Page 440) *Given a non-negative filtered differential graded vector space C^* , there exists a spectral sequence $(E_i^{p,q}, d_i)$ with*

$$\begin{aligned} E_0^{p,q} &= F^p C^{p+q} / F^{p+1} C^{p+q} = Gr^p(C^*), \\ E_1^{p,q} &= H^{p+q}(Gr^p(C^*)), \\ E_\infty^{p,q} &= Gr^p(H^{p+q}(C^*)). \end{aligned}$$

Example 5.2.2.4. *From Theorem 5.2.2.3, we have a convergent spectral sequence*

$$E_1^{p,q} = H^{p+q}(Gr^p(C^*)) \implies H^{p+q}(C^*).$$

If C^n is finite dimensional, one retrieves

$$H^n(C^*) \cong \bigoplus_{n=p+q} E_\infty^{p,q}.$$

The following examples illustrate how two different filtrations on a double complex (C, d) result in two spectral sequences linked with the cohomology of $T((C, d))$.

Example 5.2.2.5. *Assume that there is a filtration on (A, d) such that the resulting spectral sequence satisfies $E_2^{p,q} = V^p \otimes W^q$ where V^* is concentrated in odd degree and W^* is concentrated in even degree. Then $E_2 = E_\infty$ (See (16, Page 8)).*

Example 5.2.2.6. (Filtration by columns) *Let (C, d) be a double complex. Then the total complex $T((C, d))$ can be filtered by the columns of C and it is given by*

$$'F^p T((C, d))^n = \bigoplus_{\substack{n=p'+q \\ p' \geq p}} T((C, d))^{p',q}, \quad \text{hence,} \quad 'F^p C = \bigoplus_{p' \geq p} C^{p',q}.$$

By Theorem 5.2.2.3, the filtration $'F^p T((C, d))^n$ determines a spectral sequence $('E_i^{p,q})$. Observe that

$$'E_0^{p,q} = (C^{p,q} \oplus C^{p+1,q-1} \oplus C^{p+2,q-2} \oplus \dots) / (C^{p+1,q-1} \oplus C^{p+2,q-2} \oplus \dots) \cong C^{p,q},$$

and the differential maps d_0 are just the vertical differentials $d_v : C^{p,q} \longrightarrow C^{p,q+1}$. Hence

$$'E_1^{p,q} = H^q(C^{p,*}, d_v) = H^*(C^{p,*}, d_v),$$

which is the q^{th} -cohomology of the complex

$$\dots \xrightarrow{d_v} C^{p,q-1} \xrightarrow{d_v} C^{p,q} \xrightarrow{d_v} C^{p,q+1} \xrightarrow{d_v} \dots$$

The differentials $d_1 : H^q(C^{p,*}, d_v) \longrightarrow H^q(C^{p+1,*}, d_v)$ are induced by the horizontal differential

d_h of C . Hence,

$${}^1E_2^{p,q} = H^*({}^1E_1^{p,q}, d_1) \cong H^p(H^q(C^{*,*}, d_v), d_h),$$

where $H^p(H^q(C^{*,*}, d_v), d_h)$ is the cohomology of

$$\dots \xrightarrow{H^*(d_h)} H^q(C^{p-1,*}, d_v) \xrightarrow{H^*(d_h)} H^q(C^{p,*}, d_v) \xrightarrow{H^*(d_h)} H^q(C^{p+1,*}, d_v) \xrightarrow{H^*(d_h)} \dots$$

The above sequence has a meaning since $d_h \circ d_v + d_v \circ d_h = 0$, i.e. $d_h : C^{p,q} \rightarrow C^{p+1,q}$ is a cochain morphism of degree 1. We then have the convergent spectral sequence

$${}^1E_2^{p,q} = H^p(H^q(C^{p,*}, d_v), d_h) \implies H^{p+q}(T(C, d))$$

since it is a first quadrant spectral sequence.

Example 5.2.2.7. (Filtration by rows) Let (C, d) be a double complex. Then the total complex $T((C, d))$ can be filtered by the rows of C , and it is given by

$${}^nF^q T((C, d))^n = \bigoplus_{\substack{n=p+q'' \\ q'' \geq q}} T((C, d))^{p,q''}.$$

In a similar manner one gets a spectral sequence such that

$${}^nE_2^{p,q} = H^p(H^q(C^{*,p}, d_h), d_v),$$

which converges to $H^{p+q}(T(C, d))$.

5.3 The Leray-Serre Spectral Sequence

We can now look at a cohomological spectral sequence known as the *Serre (or Leray-Serre) spectral sequence*. Here we want to understand the relationship between the cohomology of the three spaces (total space, base space and fibre) of a given fibration $F \hookrightarrow E \rightarrow B$. Every Serre fibration gives rise to a spectral sequence called *Serre (or Leray-Serre) spectral sequence*.

We start by looking at graded (bigraded) algebras:

Definition 5.3.1. (16, Page 10) Suppose A is a graded vector space over a field $\mathbb{F}$. We call A a graded algebra if there is a map $\psi : A \otimes A \rightarrow A$, that is, a linear map $\psi : A^p \otimes A^q \rightarrow A^{p+q}$ for all p and q , which is associative and has the identity $1 \in A^0$. The associativity is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\psi \otimes id} & A \otimes A \\ \downarrow id \otimes \psi & & \downarrow \psi \\ A \otimes A & \xrightarrow{\psi} & A \end{array}$$

Example 5.3.2. $(S^*(X; \mathbb{F}), d)$ is a graded algebra, with the usual cup product (see Section 3.5.1).

Definition 5.3.3. A graded algebra is called commutative if $xy = (-1)^{pq}yx$, for $x \in A^p$ and $y \in A^q$. In other words, the following daigram commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\psi} & A \\ \downarrow \tau & \nearrow \psi & \\ A \otimes A & & \end{array}$$

where $\tau : A \otimes A \longrightarrow A \otimes A$ is defined by $\tau(x \otimes y) = (-1)^{pq}y \otimes x$.

Example 5.3.4. The cohomology $H^*(X; \mathbb{F})$ of a space X is a graded commutative algebra.

Definition 5.3.5. (5, Page 30) Let V be a vector space. The tensor algebra on V is the graded algebra

$$T(V) = \bigoplus_{i=0}^{\infty} T^i(V)$$

where each $T^i(V)$ consists of the tensors of rank i , that is,

$$T^i(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{i \text{ factors}}$$

and $T^0(V) = \mathbb{F}$. By convention $x \in T^i(V)$ has degree i . The multiplication

$$T^n(V) \otimes T^m(V) \longrightarrow T^{n+m}(V)$$

is defined by concatenation:

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m.$$

Hence $T(V)$ is a graded algebra.

Example 5.3.6. If V is generated by a single element x , then $T(V)$ is isomorphic to the polynomial algebra $\mathbb{F}[x]$.

Remark 5.3.7. If V is a graded vector space, then $T(V)$ is defined in the same fashion as a graded algebra. However, if $u, v \in V$ are of respective degree m and n , then

$$\deg(u \otimes v) = |u| + |v| = m + n.$$

Definition 5.3.8. Assume $\text{char}(\mathbb{F}) \neq 2$. If V is a graded algebra and I is the ideal of $T(V)$ generated by elements $u \otimes v - (-1)^{pq}v \otimes u$, with $u \in T^p(V)$ and $v \in T^q(V)$. Then $\Lambda V = T(V)/I$ is called the free commutative graded algebra generated by V .

Definition 5.3.9. (16, Page 10) Let $E^{*,*}$ be a bigraded vector space. Let $E^{*,*} \otimes E^{*,*}$ be the

bigraded vector space defined by

$$(E^{*,*} \otimes E^{*,*})^{p,q} = \bigoplus_{\substack{p=m+n \\ q=r+s}} E^{m,r} \otimes E^{n,s}.$$

We call $E^{*,*}$ a bigraded algebra if there is a map $\psi : E^{m,n} \otimes E^{r,s} \longrightarrow E^{m+r,n+s}$, such that

(i) The product ψ has a unit $1 \in E^{0,0}$, that is, for all $\alpha \in E_i^{*,*}$,

$$\psi(1 \otimes \alpha) = \alpha = \psi(\alpha \otimes 1).$$

(ii) The product is associative, that is, the following diagram commutes:

$$\begin{array}{ccc} E^{*,*} \otimes E^{*,*} \otimes E^{*,*} & \xrightarrow{\psi \otimes id} & E^{*,*} \otimes E^{*,*} \\ \downarrow id \otimes \psi & & \downarrow \psi \\ E^{*,*} \otimes E^{*,*} & \xrightarrow{\psi} & E^{*,*} \end{array}$$

Definition 5.3.10. A bigraded algebra $(E^{*,*}, \psi)$ is called commutative if the following diagram commutes:

$$\begin{array}{ccc} E^{*,*} \otimes E^{*,*} & \xrightarrow{\psi} & E^{*,*} \\ \downarrow \tau & \nearrow \psi & \\ E^{*,*} \otimes E^{*,*} & & \end{array}$$

where

$$\tau(\alpha \otimes \beta) = (-1)^{(p+q)(r+s)} \beta \otimes \alpha$$

for $\alpha \in E_i^{p,q}$ and $\beta \in E_i^{r,s}$.

We now give the definition of the differential graded (bigraded) algebra. Let us denote $\psi(x, y)$ by xy .

Definition 5.3.11. (16, Page 11) A differential graded algebra (A, d) is a graded algebra with a degree 1 linear map $d : A^n \longrightarrow A^{n+1}$ such that

$$d(xy) = d(x)y + (-1)^{\deg(x)}xd(y) \quad \text{and} \quad d^2 = 0. \quad (5.2)$$

Remark 5.3.12. Condition 5.2 assures that $H^*(A)$ is a graded algebra.

Definition 5.3.13. A differential bigraded algebra $(E^{*,*}, d)$ is a bigraded algebra with a total degree 1 linear map

$$d : \bigoplus_{n=p+q} E^{p,q} \longrightarrow \bigoplus_{r+s=n+1} E^{r,s}$$

such that for $\alpha \in E^{p,q}$ and $\beta \in E^{r,s}$,

$$d(\alpha\beta) = d(\alpha)\beta + (-1)^{p+q}\alpha d(\beta) \quad \text{and} \quad d^2 = 0.$$

With the above definitions, we can now give the definition of spectral sequence of algebras.

Definition 5.3.14. (16, Page 12) We say $\{(E_i^{*,*}, d_i)\}$ is a spectral sequence of algebras if, for each i , $(E_i^{*,*}, d_i)$ is a differential bigraded algebra and the map $\psi_i : E_i^{*,*} \otimes E_i^{*,*} \rightarrow E_i^{*,*}$ induces ψ_{i+1} on $E_{i+1}^{*,*}$, that is,

$$\psi_{i+1} : H(E_i^{*,*}, d_i) \otimes H(E_i^{*,*}, d_i) \cong H(E_i^{*,*} \otimes E_i^{*,*}, d_i) \xrightarrow{H(\psi_i)} H(E_i^{*,*}, d_i).$$

Definition 5.3.15. Let (A, d) be a graded differential algebra with a filtration

$$\cdots \subseteq F^n A \subseteq \cdots \subseteq F^1 A \subseteq F^0 A = A$$

such that

$$(i) \quad F^p A \cdot F^q A \subseteq F^{p+q} A.$$

$$(ii) \quad dF^p \subseteq F^p.$$

Then (A, d) is called a filtered differential graded algebra.

Consider the filtration $F^* H^n(A) := \text{Im}(H^n(F^p A) \rightarrow H^n(A))$ on $H^*(A)$. Then there is a spectral sequence (E_i, d_i) of algebras which converges to $H^*(A)$, i.e. E_∞ and $Gr(H^*(A))$ are isomorphic as graded vector spaces. However, E_∞ and $H^*(A, d)$ may not be isomorphic as algebras. The next result provides sufficient conditions to get an isomorphism of algebras.

Theorem 5.3.16. Let (A, d) be a differential graded algebra and

$$\cdots \subseteq F^n A \subseteq \cdots \subseteq F^1 A \subseteq F^0 A$$

be a filtration. If the resulting sequence converges to a free graded commutative algebra $E_\infty^{p,q} = \Lambda V$, where V is a graded vector space and Λ is the symmetric graded algebra over V , then

$$H^*(A, d) = E_\infty^{p,q}.$$

Proof. See (16, Section 2.2). □

5.4 Construction of the Leray-Serre Spectral Sequence

We work on a field $\mathbb{F}$ so that $H^*(A \otimes B) = H^*(A) \otimes H^*(B)$ for differential graded algebras A and B . Here we follow (16, Section 5.1). Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibration with B path-connected finite CW complex (refer to Definition 2.1.12 for definition of a CW complex). Let $S^*(E)$ be

the singular cochain complex of E , with coefficients in a field $\mathbb{F}$. Filter B by skeleta

$$B^0 \subset B^1 \subset \dots \subset B^{n-1} \subset B^n \subset \dots \subset B.$$

Let $J^n := \pi^{-1}(B^n)$. Then we get a filtration

$$J^0 \subset J^1 \subset \dots \subset J^{n-1} \subset J^n \subset \dots \subset J,$$

on E . The canonical inclusions $i_n : J^n \hookrightarrow E$ induce maps

$$S^*(i_n) : S^*(E; \mathbb{F}) \longrightarrow S^*(J^n; \mathbb{F}).$$

And so we get a filtration

$$F^n S^*(E; \mathbb{F}) := \text{Ker}(S^*(E; \mathbb{F}) \xrightarrow{S^*(i_{n-1})} S^*(J^{n-1}; \mathbb{F})).$$

We denote by $i_{n-1,n}$ the inclusion $J^{n-1} \hookrightarrow J^n$. Thus, the following diagram commutes:

$$\begin{array}{ccc} S^*(E; \mathbb{F}) & \xrightarrow{S^*(i_{n-1})} & S^*(J^{n-1}; \mathbb{F}) \\ & \searrow S^*(i_n) & \uparrow S^*(i_{n-1,n}) \\ & & S^*(J^n; \mathbb{F}) \end{array}$$

Therefore, one gets a filtration

$$\dots \subseteq F^{n+1} S^*(E; \mathbb{F}) \subseteq F^n S^*(E; \mathbb{F}) \subseteq \dots \subseteq F^1 S^*(E; \mathbb{F}) \subseteq F^0 S^*(E; \mathbb{F}) = S^*(E; \mathbb{F}).$$

It is clear that

- (i) $d(F^n S^*(E; \mathbb{F})) \subseteq F^n S^*(E; \mathbb{F})$,
- (ii) $F^n S^*(E; \mathbb{F}) \cdot F^m S^*(E; \mathbb{F}) \subseteq F^{n+m} S^*(E; \mathbb{F})$.

Hence $F^* S^*(E; \mathbb{F})$ is a filtration of graded differential algebras. This is a decreasing filtration, so there is a spectral sequence with E_1 -term given by

$$E_1^{p,q} = H^{p+q}(F^p S^*(E; \mathbb{F}) / F^{p+1} S^*(E; \mathbb{F})).$$

We therefore get the following theorem, where the proof can be found in (12, Theorem 5.15). But first we explain how $\pi_1(B)$ acts on $H^*(F)$. Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a Serre fibration where B is path-connected. Let $b_0 \in B$ be point with fibre $F := \pi^{-1}(b_0)$. Let γ be a loop in B based at b_0 , i.e. $\gamma(0) = \gamma(1) = b_0$. Let $x \in F = \pi^{-1}(b_0)$. Lift γ into $\tilde{\gamma} : [0, 1] \longrightarrow E$ such that $\tilde{\gamma}(0) = x$ and denote $y = \tilde{\gamma}(1) \in E$.

$$\begin{array}{ccc}
 & & F \\
 & & \downarrow \\
 & & E \\
 & \nearrow \tilde{\gamma} & \downarrow \pi \\
 [0, 1] & \xrightarrow{\gamma} & (B, b_0)
 \end{array}$$

This defines a map $L_\gamma : F \longrightarrow F$ such that if $\gamma \simeq \gamma'$, then $L_\gamma \simeq L_{\gamma'}$ and we get a well-defined map $H^*(L_\gamma) : H^*(F) \longrightarrow H^*(F)$. Hence, $\pi_1(B, b_0)$ acts on $H^*(F)$.

Remark 5.4.1. *If $\pi_1(B) = \{0\}$, then the action of $\pi_1(B) = \{0\}$ on $H^*(F)$ is trivial.*

Theorem 5.4.2. *(12, Page 542) Given a fibration $F \hookrightarrow E \xrightarrow{\pi} B$, with connected fibre F , B path-connected CW complex and $\pi_1(B)$ acting trivially on $H^*(F; \mathbb{F})$, there exists a spectral sequence of algebras such that*

$$E_2^{p,q} = H^p(B; \mathbb{F}) \otimes H^q(F; \mathbb{F}) \text{ and converges to } H^{p+q}(E; \mathbb{F}).$$

The cohomological spectral sequence has an internal product induced by the cup product on singular cohomology, hence more powerful.

6. Some Applications of the Serre Spectral Sequence

We now use the Leray-Serre spectral sequence to compute the cohomology algebra over $\mathbb{Q}$ of unitary groups, Stiefel and complex Grassmannian manifolds. In all examples, the base space of the fibration $F \hookrightarrow E \xrightarrow{\pi} B$ is simply connected and hence hypotheses of Theorem 5.4.2 are satisfied.

6.1 Cohomology Ring of Unitary Groups

Theorem 6.1.1. $H^*(U(n); \mathbb{Q}) = \Lambda(x_1, x_3, \dots, x_{2n-1})$ with $\deg(x_i) = i$.

Proof. Consider the Leray-Serre cohomology spectral sequence for the fibration

$$U(n) \hookrightarrow U(n+1) \longrightarrow S^{2n+1}.$$

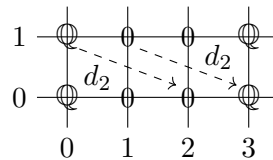
For $n = 1$, we have $U(1) = S^1 \hookrightarrow U(2) \longrightarrow S^3$. The cohomology of the fibre and the base space are $H^*(U(1)) = \Lambda(x_1)$ with $\deg(x_1) = 1$ and $H^*(S^3) = \Lambda(x_3)$ with $\deg(x_3) = 3$ respectively. From Theorem 5.4.2 we get

$$\begin{aligned} E_2^{p,q} &= H^p(B) \otimes H^q(F) \\ &= H^p(S^3) \otimes H^q(S^1) \\ &= \Lambda(x_1, x_3), \end{aligned}$$

where the subscripts indicate the degree. We then have

$$E_2^{p,q} = \begin{cases} \mathbb{Q}; & \text{if } (p, q) = (0, 0), (0, 1), (3, 0), (3, 1), \\ \{0\}; & \text{if Otherwise.} \end{cases}$$

Thus,



Examining d_2 on the figure above, one concludes that $d_2 = 0$. Hence the spectral sequence collapses at $E_2^{p,q} = \Lambda(x_1, x_3)$. Thus, by Theorem 5.3.16,

$$H^*(U(2)) = E_\infty^{p,q} = \Lambda(x_1, x_3).$$

We now look at the general case. By induction, suppose that $H^*(U(n); \mathbb{Q}) = \Lambda(x_1, x_3, \dots, x_{2n-1})$ for some n . Then we consider the fibration $U(n) \hookrightarrow U(n+1) \longrightarrow S^{2n+1}$. Thus,

$$\begin{aligned} E_2^{p,q} &= E_2^{p,0} \otimes E_2^{0,q} \\ &= H^p(S^{2n+1}) \otimes H^q(U(n)). \end{aligned}$$

But $H^p(S^{2n+1}) = \Lambda(x_{2n+1})$. So we get $E_2^{p,q} = \Lambda(x_{2n+1}) \otimes \Lambda(x_1, \dots, x_{2n-1}) = \Lambda(x_1, \dots, x_{2n+1})$. Now we have that the total degree of a class $\alpha \in E_2^{p,q}$ is $p+q$. Let $x_1 \in E_2^{0,1}$. Then $d_2(x_1) = 0$ since $E_2^{2,0} = \{0\}$. Same for $x_3 \in E_2^{0,3}$, we have that $d_2(x_3) = 0$. In a similar way, for $x_{2n+1} \in E_2^{0,2n+1}$, $d_2(x_{2n+1}) = 0$. Hence $d_2 = 0$. Thus, $E_\infty^{p,q} = E_2^{p,q} \cong \Lambda(x_1, \dots, x_{2n+1})$. Therefore,

$$H^*(U(n+1); \mathbb{Q}) = \Lambda(x_1, x_3, \dots, x_{2n+1}),$$

which concludes the induction. $\square$

In a similar way, using the fibration $SU(n-1) \hookrightarrow SU(n) \longrightarrow S^{2n-1}$, and starting with $n=2$, i.e., $SU(2) \cong S^3$, one shows that $H^*(SU(n); \mathbb{Q}) = \Lambda(x_3, \dots, x_{2n-1})$ (see (16), Page 150) for details).

6.2 Cohomology Ring of Complex Stiefel Manifolds

Theorem 6.2.1. $H^*(V_k(\mathbb{C}^n); \mathbb{Q}) = \Lambda(x_{2(n-k)+1}, \dots, x_{2n-1})$.

Proof. Consider the Serre spectral sequence for the fibration

$$S^{2(n-k)-1} \hookrightarrow V_{k+1}(\mathbb{C}^n) \longrightarrow V_k(\mathbb{C}^n).$$

For $k=1$, we have $H^*(V_1(\mathbb{C}^n); \mathbb{Q}) = H^*(S^{2n-1}; \mathbb{Q}) = \Lambda(x_{2n-1})$. Suppose the statement is true for k , i.e., $H^*(V_k(\mathbb{C}^n); \mathbb{Q}) = \Lambda(x_{2(n-k)+1}, \dots, x_{2n-1})$. As $V_k(\mathbb{C}^n)$ is simply connected, from Theorem 4.3.7, then the Serre spectral sequence of the fibration yields

$$E_2^{p,q} = \Lambda(x_{2(n-k)+1}, \dots, x_{2n-1}) \otimes \Lambda(x_{2(n-k)-1}).$$

We then have the differential $d_2(x_{2(n-k)+1}) = \dots = d_2(x_{2n-1}) = d_2(x_{2(n-k)-1}) = 0$ for degree reasons. Similarly, $d_3 = d_4 = \dots = 0$ for degree reasons. Therefore the spectral sequence collapses at $E_2^{p,q}$ and

$$E_\infty^{p,q} = \Lambda(x_{2(n-k)+1}, x_{2(n-k)+3}, \dots, x_{2n-1}).$$

Therefore, we get $H^*(V_{k+1}(\mathbb{C}^n); \mathbb{Q}) = \Lambda(x_{2(n-k)-1}, \dots, x_{2n-1})$. $\square$

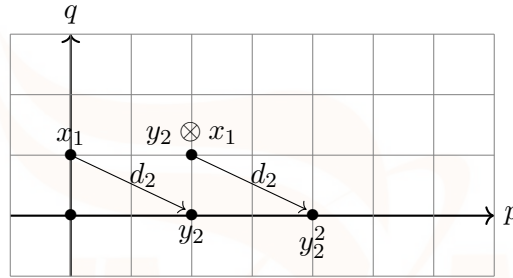
6.3 Cohomology of Infinite Dimensional Complex Grassmannians

In this section we just compute the cohomology of infinite dimensional complex Grassmannian $G_k(\mathbb{C}^\infty)$ using the Leray-Serre spectral sequence.

Taking coefficients in $\mathbb{Q}$, we show that $H^*(BU(k); \mathbb{Q}) = \Lambda(y_2, y_4, \dots, y_{2k})$, where $y_{2k} \in H^{2k}(BU(k))$ with k being positive integer. Consider the universal fibre bundle $U(k) \hookrightarrow EU(k) \rightarrow BU(k)$. Since $EU(k)$ is contractible, one get a spectral sequence such that

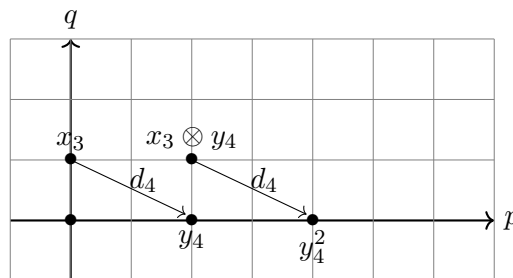
$$E_2^{p,q} = H^*(BU(k)) \otimes H^*(U(k))$$

converges to $E_\infty^{p,q}$ where $E_\infty^{p,q} = \{0\}$ for $p, q \neq 0$ and $E_\infty^{0,0} \cong \mathbb{Q}$. We proceed by induction. Setting $k = 1$, we have the universal fibre bundle as $U(1) \hookrightarrow EU(1) \rightarrow BU(1)$. From Theorem 6.1.1, $H^*(U(1)) = \Lambda(x_1)$. Then the $E_2^{p,q}$ -term is depicted by the following diagram:



Observe that $d_2(1 \otimes x_1) \neq 0$, otherwise the class $[1 \otimes x_1] \in E_3^{0,1} = E_\infty^{0,1}$, for degree reasons. This will contradict that $EU(k)$ is contractible. So there exists $y_2 \in H^2(BU(1))$ such that $d_2(1 \otimes x_1) = y_2 \otimes 1$. And $d_2(x_1 \otimes y_2) = y_2^2 \otimes 1$. Hence, $H^*(BU(1)) = \Lambda(y_2)$.

In a similar way, setting $k = 2$, there exists $y_4 \in H^4(BU(2))$ such that $d_4(1 \otimes x_3) = y_4 \otimes 1$. And $d_4(x_3 \otimes y_4) = y_4^2 \otimes 1$.



Therefore, $H^*(BU(2)) = \Lambda(y_2, y_4)$. For the general case, one applies a result of Borel to conclude that $H^*(BU(k); \mathbb{Q})$ is generated by powers of $y_2, y_4, \dots, y_{2k}$. Therefore, $H^*(BU(k); \mathbb{Q}) = \Lambda(y_2, y_4, \dots, y_{2k})$, where subscripts indicate degree (3).

6.4 Cohomology Algebra of Complex Grassmannians

We compute the cohomology algebra of $G_1(\mathbb{C}^{n+1}) \cong \mathbb{C}P^n$ and attempt to compute $H^*(G_2(\mathbb{C}^4); \mathbb{Q})$.

Example 6.4.1. Consider the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$ which we are going to use to compute $H^*(\mathbb{C}P^n; \mathbb{Q})$. But $H^*(S^1) = \Lambda(x_1)$. From Theorem 5.4.2 we get

$$\begin{aligned} E_2^{p,q} &= H^p(B) \otimes H^q(F) \\ &= H^p(\mathbb{C}P^n) \otimes H^q(S^1), \end{aligned}$$

which converges to $\Lambda(y_{2n+1})$. Let x_1 be a generator of $E_2^{0,1} \cong H^1(S^1)$. As $H^*(S^1; \mathbb{Q}) = \Lambda(x_1)$, so $E_2^{p,q} = \{0\}$, for $q > 1$. Hence, $d_3 = d_4 = \dots = 0$ and $E_3 \cong E_4 \cong \dots \cong E_\infty$. In particular, $E_\infty^{p,q} = E_3^{p,q} = \{0\}$ unless $p + q = 0, 2n + 1$ (See (10, Section 5.5)). As $E_\infty^{0,1} = \{0\}$, there exists $y_2 \in H^2(\mathbb{C}P^n)$ such that $d_2(1 \otimes x_1) = y_2 \otimes 1$. And

$$\begin{aligned} d_2(y_2 \otimes x_1) &= d_2(y_2 \otimes 1) \cdot (1 \otimes x_1) + d_2(1 \otimes x_1) \cdot (y_2 \otimes 1) \\ &= y_2^2 \otimes 1. \end{aligned}$$

Similarly, $d_2(y_2^i \otimes x_1) = y_2^{i+1} \otimes 1$, for $i = 1, 2, \dots, n-1$. And $d_2(y_2^{2n} \otimes x_1) = 0$, as $E_\infty \cong \Lambda(x_{2n+1})$. Therefore, $H^*(\mathbb{C}P^n; \mathbb{Q}) = \mathbb{Q}[y_2]/(y_2^{n+1})$.

Example 6.4.2. (Gysin Sequence) Consider a fibration $F \hookrightarrow E \xrightarrow{\pi} B$ where B is simply connected. Suppose that $H^*(F; \mathbb{Q}) \cong H^*(S^n; \mathbb{Q})$. We show that there exists a long exact sequence

$$\dots \rightarrow H^p(B; \mathbb{Q}) \xrightarrow{\gamma} H^{p+n+1}(B; \mathbb{Q}) \xrightarrow{\pi^*} H^{p+n+1}(E; \mathbb{Q}) \xrightarrow{\pi_*} H^{p+1}(B; \mathbb{Q}) \rightarrow \dots \quad (6.1)$$

Since B is simply connected, there is a spectral sequence $E_*^{p,q}$ such that $E_2^{p,q} = H^p(B; \mathbb{Q}) \otimes H^q(S^n; \mathbb{Q})$ which converges to $H^*(E; \mathbb{Q})$. Recall that $H^*(F; \mathbb{Q}) \cong H^*(S^n; \mathbb{Q}) \cong \mathbb{Q}[h]/(h^2)$, where h has degree n . Then $E_2 = E_3 = \dots = E_{n+1}$ and $E_{n+2} = E_\infty = H(E_{n+1}, d_{n+1})$. Let $z \in H^{n+1}(B; \mathbb{Q})$ such that $d_{n+1}(1 \otimes h) = z \otimes 1$. Note that for $x \in H^{n+1}(B; \mathbb{Q})$, $d_{n+1}(x \otimes 1) = 0$ for degree reasons. Then

$$\begin{aligned} d_{n+1}(x \otimes h) &= (-1)^n(x \otimes 1) \cdot d_{n+1}(1 \otimes h) \\ &= (-1)^n(x \otimes 1) \cdot (z \otimes 1) \\ &= (-1)^n(xz \otimes 1). \end{aligned}$$

We define $\gamma : H^p(B; \mathbb{Q}) \longrightarrow H^{p+n+1}(B; \mathbb{Q})$ by $\gamma(x) = (-1)^n xz$. We have an exact sequence

$$0 \rightarrow E_\infty^{p,n} \rightarrow E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \rightarrow E_\infty^{p+n+1,0} \rightarrow 0. \quad (6.2)$$

If n is even, then

$$0 = d_{n+1}(1 \otimes h^2) = 2(z \otimes h).$$

Hence $z = 0$. Therefore γ is the zero map.

Moreover, as $E_i^{p,q}$ converges to $H^*(E; \mathbb{Q})$, then

$$H^{p+n}(E; \mathbb{Q}) = F^0 H^{p+n}(E; \mathbb{Q}) \supseteq F^1 H^{p+n}(E; \mathbb{Q}) \supseteq \dots \supseteq F^{p+n} H^{p+n}(E; \mathbb{Q}) \supseteq \{0\}, \quad (6.3)$$

and

$$E_\infty^{p,n} = F^p(H^{p+n}(E; \mathbb{Q})) / F^{p+1}(H^{p+n}(E; \mathbb{Q})).$$

From the layout of the spectral sequence, Equation 6.3 yields

$$\begin{aligned} H^{p+n}(E; \mathbb{Q}) &= F^0 H^{p+n}(E; \mathbb{Q}) = F^1 H^{p+n}(E; \mathbb{Q}) = \dots = F^n H^{p+n}(E; \mathbb{Q}) \\ &\supseteq F^{n+1} H^{p+n}(E; \mathbb{Q}) = F^{n+2} H^{p+n}(E; \mathbb{Q}) = \dots = F^{p+n} H^{p+n}(E; \mathbb{Q}) \supseteq \{0\}. \end{aligned}$$

This yields short exact sequences

$$0 \rightarrow E_\infty^{p+n,0} \rightarrow H^{p+n} \rightarrow E_\infty^{p,n} \rightarrow 0. \quad (6.4)$$

Putting Equation 6.3 and Sequence 6.4 together yields

$$\begin{array}{ccccccc} & & \downarrow & & & & \\ & & H^{p+n}(E; \mathbb{Q}) & & & & \{0\} \\ & & \downarrow & \searrow & & & \downarrow \\ \{0\} & \longrightarrow & E_\infty^{p,n} & \longrightarrow & E_2^{p,n} & \xrightarrow{d_{n+1}} & E_2^{p+n+1,0} \longrightarrow E_\infty^{p+n+1,0} \longrightarrow \{0\} \\ & & \downarrow & & & \searrow & \downarrow \\ & & \{0\} & & & & H^{p+n+1}(E; \mathbb{Q}) \\ & & & & & & \downarrow \\ & & & & & & \{0\} \longrightarrow E_\infty^{p+1,n} \longrightarrow E_2^{p+1,n} \\ & & & & & & \downarrow \\ & & & & & & \{0\} \end{array}$$

which yields the Gysin sequence

$$\dots \rightarrow H^{p+n}(E; \mathbb{Q}) \rightarrow H^p(B; \mathbb{Q}) \xrightarrow{\gamma} H^{p+n+1}(B; \mathbb{Q}) \xrightarrow{\pi^*} H^{p+n+1}(E; \mathbb{Q}) \rightarrow \dots$$

In particular, one recovers results of Example 6.4.1, where we deduce from the corresponding Gysin sequence that $H^*(\mathbb{C}P^n; \mathbb{Q}) = \mathbb{Q}[x_2]/(x_2^{n+1})$.

Next we attempt to compute the cohomology of finite dimensional Grassmannian $G_k(\mathbb{C}^n)$.

Example 6.4.3. Here we attempt to compute the cohomology of the Grassmannian $G_2(\mathbb{C}^4)$. Consider the Serre spectral sequence for the fibration $U(k) \hookrightarrow V_k(\mathbb{C}^n) \rightarrow G_k(\mathbb{C}^n)$. Set $n = 4$

and $k = 2$. Then we obtain the fibration $U(2) \hookrightarrow V_2(\mathbb{C}^4) \rightarrow G_2(\mathbb{C}^4)$. From Theorems 6.1.1 and 6.2.1,

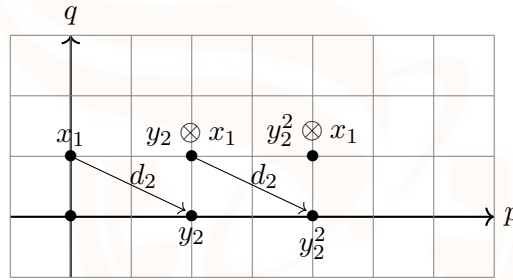
$$H^*(U(2)) = \Lambda(x_1, x_3) \quad \text{and} \quad H^*(V_2(\mathbb{C}^4)) = \Lambda(x_5, x_7).$$

The Serre spectral sequence of the fibration yields

$$\begin{aligned} E_2^{p,q} &= H^p(G_2(\mathbb{C}^4)) \otimes H^q(U(2)) \\ &= H^p(G_2(\mathbb{C}^4)) \otimes \Lambda(x_1, x_3), \end{aligned}$$

which converges to $\Lambda(x_5, x_7)$. We work backwards to determine $H^*(G_2(\mathbb{C}^4))$. Let $x_1 \in E_2^{0,1}$. As $E_\infty = \Lambda(x_5, x_7)$, there exists $y_2 \in H^2(G_2(\mathbb{C}^4))$ such that $d_2(1 \otimes x_1) = y_2 \otimes 1$. Hence

$$\begin{aligned} d_2(y_2 \otimes x_1) &= d_2(y_2 \otimes 1) \cdot (1 \otimes x_1) + (y_2 \otimes 1) \cdot d_2(1 \otimes x_1) \\ &= (y_2 \otimes 1) \cdot (y_2 \otimes 1) \\ &= y_2^2 \otimes 1. \end{aligned}$$



In the same way, there exists $y_4 \in H^4(G_2(\mathbb{C}^4))$ such that $d_4(1 \otimes x_3) = y_4 \otimes 1$. From these computations we conclude that $H^*(G_2(\mathbb{C}^4))$ has at least two generators, y_2 and y_4 . We are unable to determine the relation between generators in $H^*(G_2(\mathbb{C}^4))$ using spectral sequences. However $H^*(G_2(\mathbb{C}^4); \mathbb{Q}) = \Lambda(y_2, y_4)/I$, where I is the ideal generated by $\{y_2^3 - 2y_2y_4, y_4^2 + 3y_2^2y_4 + y_2^4\}$ (See (13)).

7. Conclusion

The aim of this thesis was to discuss the construction of the Leray-Serre spectral sequence and its applications in computing the cohomology algebra over $\mathbb{Q}$ of unitary groups, Stiefel and infinite dimensional complex Grassmannian manifolds. We showed how the Leray-Serre spectral sequence is constructed using filtered complexes. We computed the cohomology ring of unitary groups, complex Stiefel and infinite dimensional complex Grassmannian manifolds. We attempted to compute the cohomology of finite dimensional Grassmannian $G_2(\mathbb{C}^4)$ and we failed to determine the relation in $H^*(G_2(\mathbb{C}^4))$ using the spectral sequences.

In the future work, we will attempt to use (13) to determine $H^*(G_k(\mathbb{C}^n); \mathbb{Q})$ for $k \geq 2$.



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