

EXOTIC NEUTRON STAR EQUATIONS OF STATE

by

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Declaration

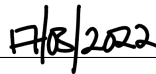
I, **Wazha German**, declare that this thesis is my own original work unless otherwise referenced or acknowledged. It is being submitted for the degree of Master of Science (Physics) in Botswana International University of Science and Technology and has not been presented nor under consideration for any degree or examination in any other university for a similar or any other degree award.

Certification

The undersigned certify that have read and hereby recommend for acceptance by the Faculty of Sciences a dissertation titled: **EXOTIC NEUTRON STAR EQUATIONS OF STATE**, in fulfillment of the requirements for the degree of Master of Science (MSc Physics) of the Botswana International University of Science and Technology.

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ABSTRACT

Neutron stars are dense stellar remnants formed out of the core collapse of giant stars in a supernovae. Due to the inferred extreme densities the neutron star interior is assumed to consist of dense nuclear matter, thus provide a laboratory to study matter at high densities. In this work relativistic mean field (RMF) theory is applied to a relativistic description of nuclear matter known as Quantum Hadrodynamics (QHD), or the $\sigma - \omega$ model, where the strong nuclear force is described by the exchange of mesons. Parameters sets in QHD are defined by the meson-baryon coupling constants: QHD-I is an introductory parameter set whilst NL3 is a more advanced version including more mesons and additional interactions. These QHD parameter sets were then expanded to include the neutral lambda hyperon, as a first approximation of hyperonic matter. RMF is applied as an approximation to calculate the bulk properties of the nuclear matter and to study the neutron star equation of state (EoS). The EoS for pure neutron matter model as well as beta-equilibrated, charge neutral matter model were compared to the EoS of beta-equilibrated, charge neutral matter with lambda hyperons. These were then used to compare and contrast the influence on the neutron star mass-radius relationship. Comparing the calculated results to observational data, the presence of the lambda hyperon in the neutron star cannot be ruled out. However, a more complete model would be to include the full baryon octet in the model for dense nuclear matter.

1. Introduction

This chapter gives a brief overview of neutron stars and the study thereof. The scientific implications of these objects for Botswana as well as the notation and other fundamental concepts used in this work will also be shortly discussed.

1.1 Literature Review: neutron stars and nuclear physics

One of the areas of interest in the study of nuclear matter is the nature of its ground state. In particular whether the lowest energy of an equilibrated system can consist of particles other than that found in the nucleus. In nuclear matter the ground state of matter at the core of most massive stars due to the short range of the nuclear forces is that of ^{56}Fe [1]. However, there has been suggestions that this might not be the ultimate ground state of nuclear matter since it has been found that some stars blow up in type II supernovae to form neutron stars by releasing massive amounts of energy [1]. Currently various experiments studying hot dense matter are being conducted[2]. However, there are also other systems outside the experimental laboratory that can provide a unique insight.

Neutron stars are formed from the remnants of the gravitational collapse of the core of giant stars ($\geq 8M_{\odot}$) [3]. Neutron stars have a mass in the range of 1.0 - 2.0 $M_{\odot}$ [3], a radius of around 12km and a very high central density (5 – 10 times the nuclear equilibrium density) [4]. This means that neutron stars are some of the densest forms of matter in the observable universe [5]. Comparatively, the binding energy of the neutron star is 10% of its mass while ^{56}Fe has a nuclear binding energy of 9 MeV/nucleon which is 1% of its mass. Thus the binding energy of a neutron star is ten times the nuclear binding energy of ^{56}Fe meaning that the star is in a state other than that of ^{56}Fe [1]. Hence neutron stars provide ideal laboratories for studying physical laws as the neutron star conditions cannot be replicated in terrestrial laboratories. These could include conditions such as extreme densities, hyperon dominated matter, quark deconfinement and also superconductivity and superfluidity with excess temperatures.

For a general star gravity pushes inwards on the star. Inside thermal pressure released by nuclear fusion stabilizes the star. In contrast to normal stars as well as nuclei, neutron stars are bound by gravity and stabilised by the nuclear degeneracy pressure (from Pauli's Exclusion Principle suggesting that two or more Fermions cannot occupy the same quantum state) inside

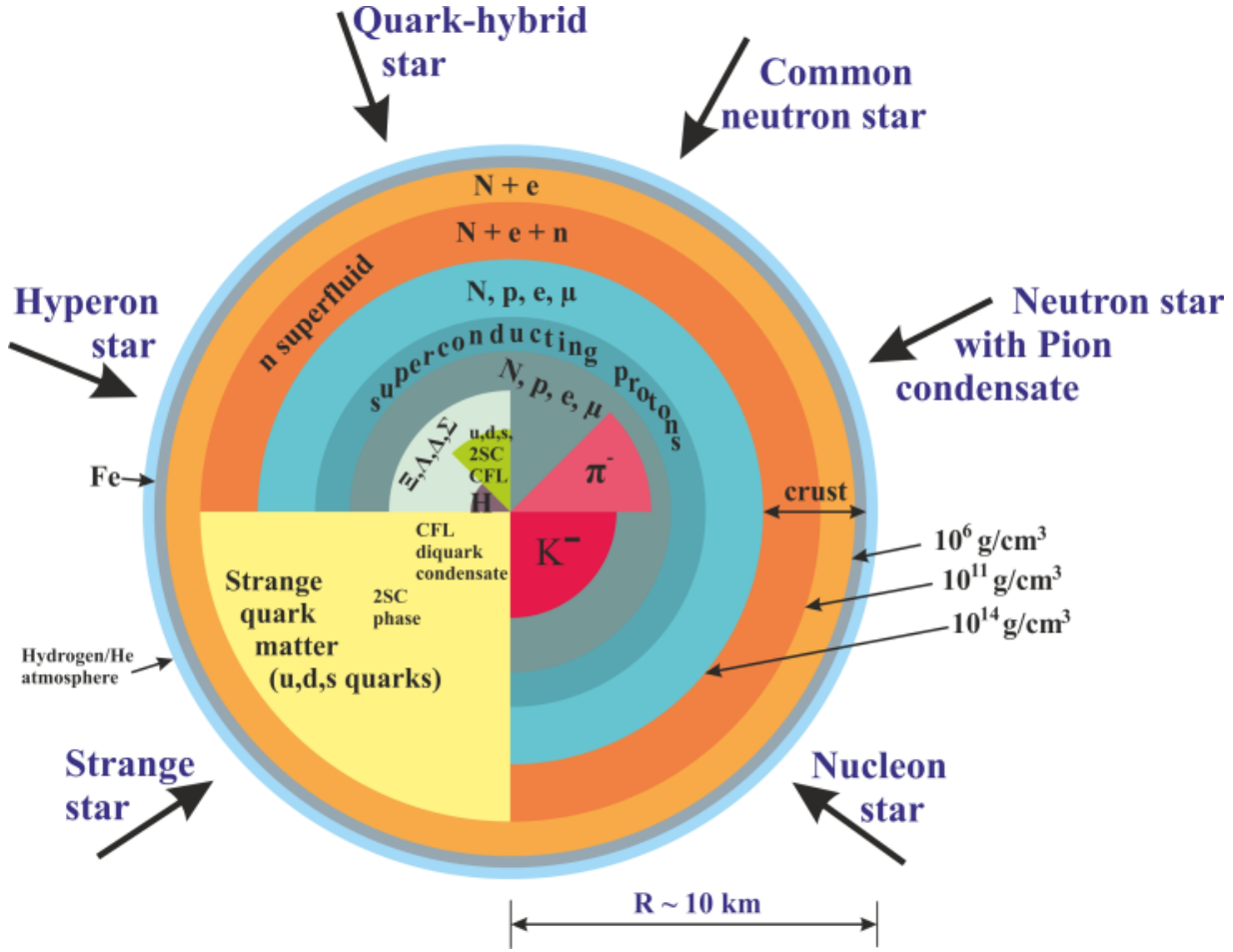


Figure 1.1: A theoretical description of the ideas of the neutron star interior with this project focusing on the hyperon star model [6]

the star. If the nuclear degeneracy pressure is overcome, the neutron star will collapse into a black hole[1].

Since neutron stars cannot be directly probed, not a lot is known about the interior of neutron stars. However, it is assumed they are composed of nuclear matter (protons and neutrons)[1]. As mentioned these nuclear systems are bound by gravity whilst nuclei are bound by the strong nuclear force. At supra-nuclear densities exotic particles such as hyperons and deconfined quarks could also be present [4]. Fig 1.1 illustrates some of the models that are assumed for the neutron star interior. Strange quark matter stars are hypothesised to be self bound as they do not require gravity to bind them. These stars are of interest since quark matter with up, down and strange quarks is said to have a greater binding energy per nucleon at zero pressure than matter composed of nucleons [4].

1.2 Aims and Objectives

The aim is to study exotic neutron star equations of state by calculating the physical attributes such as the relationship between the mass and radius of the star and compared to observed and published results or textbooks and at the end contribute in finding a suitable description for cold dense nuclear matter. Due to the extreme densities of neutron stars the principles of general relativity and special relativity will be used in this work as gravity plays a deep underlying role in neutron stars[1]. The models are hard to solve due to many hadronic degrees of freedom so cannot be solved exactly and have to be approximated[7]. A summary of neutron star observations have been made in published literature like [3], [8] and the implied meaning on the neutron star equation of state, observations of neutron star mass and radius as well as nuclear constraints on the pressure and energy densities for the neutron star as well as discussing how the constraints affects nuclear properties such as the binding energy[3]. The references therein [3] could also provide more details on neutron star observations.

The Tolman-Oppenheimer-Volkoff (TOV) equations are a set of hydrostatic equilibrium equations that provide general relativity solutions (structure of static, spherical objects) for stars[9]. In Chapter 2 the TOV equations will be discussed as well as the conditions for hydrostatic equilibrium in order to properly describe the mass radius relation for the neutron star. The equation of state is an equation that describes the relationship between the pressure and energy density for matter. If an equation of state for nuclear matter at high densities is found treating the neutron star as a laboratory this would help provide a window into the properties of dense nuclear matter[4].

In order to calculate different neutron star properties, an assumption has to be made about the constituent matter of the interior, as well as the imposed equilibrium conditions. Different assumptions result in different equations of state (relationship between energy density and the pressure of the matter inside the star) for a neutron star. The equation of state used in this project is a beta equilibrated matter with the lambda hyperons included as a first approximation for dealing with hyperonic matter.

This project will aim to derive an equation of state for three different nuclear matter models by

applying the relativistic mean field theory to a nuclear model called Quantum Hadrodynamics (QHD) with the aim to try to reproduce the data from already published results from sources such as [1, 10, 11, 7]. QHD is a nuclear field theory (effective theory) that describes mesons and provides a structure for us to describe the nuclear many body problem as a relativistic system of mesons and baryons (hadrons) [10]. The use of QHD as a general model for discussing the many body problem has been chosen because earlier models such as the Brueckner-Bethe-Goldstone approach were well suited for low densities [12]. The properties of nuclear matter at saturation determines the coupling constants and calculations in QHD describe the bulk properties of nuclear matter [12]. Also as the density increases the Fermi momenta of the baryons also increases meaning strong effects of relativity making QHD a great model for describing neutron star calculations [1]. The RMF is an approximation used to describe the ground state properties of nuclear matter [10].

The objectives of this project will be to;

- Use Quantum Hadrodynamics (QHD) to formulate nuclear matter equations of state that will be used to solve the TOV equations,
- Calculate physical attributes such as the mass radius relationship of a neutron star,
- Compare results from this project to various published neutron star observations such as in [8] and [3].

The TOV equations has to be coupled with the equation of state for nuclear matter. To get the equation of state for nuclear matter we will begin in Appendix A by discussing relativistic quantum mechanical concepts. This will provide an introduction into nuclear matter which will show that a multi-particle description (field theory) for nuclear matter is needed in our discussion. Appendix B will discuss classical field theory with the aim of introducing quantum field theory concepts which are described in Appendix C. In Chapter 4 a nuclear field theory called the $\sigma - \omega$ model/Quantum Hadrodynamics is discussed in order to get an equation of state for nuclear matter which is then coupled to the TOV equation to give the mass-radius relation for neutron stars. The project will then go further into discussing several advanced parameters and equations of state in order to improve on the $\sigma - \omega$ model. This offers insight into the high density behaviour of the matter inside neutron stars.

Chapter 5 provides a description of nuclear matter by including hyperons (the lambda hyperon in this thesis) and observing the behaviour of the equation of state. This chapter introduces the concept that at high densities the hyperon states are populated and this will affect the equation of state and the behaviour of the neutron star. The expectation is that the equation of state should soften when hyperons are included in the discussion. The equilibrium conditions inside the neutron star can make it so that nucleons form hyperons contrary to what is observed in terrestrial conditions where hyperons are considered unstable and always decay to nucleons. At high densities the nucleon chemical potential will be large enough to make the conversion of nucleons to hyperons favourable. This conversion of nucleons to hyperons will relieve the Fermi pressure exerted by the baryons which softens the equation of state[13]. The results calculated in this project will be compared to already published results such as [14] and [15].

Three different nuclear matter models will be developed giving three different equations of state for nuclear matter and the results calculated from this project will be compared with observed results with the aim of contributing in finding a suitable model for describing nuclear matter at high densities.

1.3 Botswana Perspective

Botswana has been chosen to be a part of the Square Kilometer Array project (SKA) which aims to build the world's largest radio-telescope. This telescope will observe our Universe with the highest resolution ever and will give more data when studying stars such as pulsars (neutron stars) leading to more theories about dense nuclear matter being developed and solving one of the big nuclear physics questions out there currently

Botswana will host a subset of radio telescope dishes as part of 3000 dishes that stretches across Southern Africa and East Africa. This project will bring in a lot of scientific, economic and infrastructure developments as well as enhancing international research cooperation between scientists involved in the project. More skilled personnel will be needed to man these substations meaning that employment opportunities will be created in the region and as such will undergo training in science and engineering.

1.4 Units and Scales

Here the basic concepts fundamental in our discussion in this project such as units and the scales used will be discussed. In nature there are three natural fundamental dimensional

constants. The speed of light c , Planck's constant (divided by 2π) $\hbar$, and Newton's universal constant G . Throughout this work natural units will be used defined by:

$$c = \hbar = 1 \quad (1.1)$$

This convention simplifies most of the equations they carry the factors of $\hbar$ and c since carrying such factors along is cumbersome. The natural units also allow for all dimensional quantities to be expressed in terms of a single scale which is chosen to be mass or equivalently energy. The choice for the units is the electron volt, in particular MeV.

1.5 Conventions and Notations

Throughout the thesis two length scales will be employed. The nuclear scale which deals with lengths of fm and masses in MeV (microscopic scale) whilst the other scale is the size of the star which is given in km and masses in $M_\odot$ solar masses (macroscopic scale).

1.5.1 Vectors

Vectors in three dimensions will be indicated in bold $\mathbf{v} = v^0, v^1, v^2$. Vectors in four dimensions will be indicated using covariant indices like $x_\mu = x_0, x_1, x_2, x_3$ or contravariant indices like $x^\mu = x^0, x^1, x^2, x^3$ and when summing over the indices the Einstein summation convention will be employed [16].

The d'Alembert operator is

$$\square = \partial_\mu \partial^\mu, \quad (1.2)$$

where ∂_μ is a four dimensional gradient given by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial t}, \nabla \right). \quad (1.3)$$

There is also the Minkowski metric tensor $\eta^{\mu\nu}$ for flat space-time which is

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1.4)$$

Table 1.1 gives the values used for various constants.

Name of Constant	Symbol	Value
Solar Mass	$M_{\odot}$	1.98892×10^{30} kg or 1.10086×10^{60} MeV
Gravitational Constant	G	6.67×10^{-11} m ³ kg ⁻¹ s ⁻² or 1.325×10^{-42} fm/ MeV
Length	1 fm	1×10^{-15} m
Energy	1 MeV	
Energy density	1 MeV/ fm ³	1.7827×10^{12} g/cm ³
Pressure	1 MeV/ fm ³	1.6022×10^{33} dyne/cm ²

Table 1.1: Values of constants used in the project

2. TOV EQUATION

The simplest model for a neutron star is a fluid in hydrostatic equilibrium, where the star is bound by the gravitational pressure, but supported from gravitational collapse by the neutron degeneracy pressure due to the Pauli exclusion principle [1]. In this chapter the description of the neutron star structure is derived, by way of introduction, from a non-relativistic Newtonian point of view. The need to general relativistic description is discussed, which leads to the full Tolman-Oppenheimer-Volkoff equations.

2.1 Newtonian hydrostatic equilibrium

In order to describe the neutron star structure three basic concepts are used to approximate the structure of the neutron star:

1. condition of hydrostatic equilibrium,
2. conservation of mass, and
3. the relationship between the energy density and the pressure of the matter in the interior, the equation of state (EoS).

In this section the first two conditions will be described in detail before getting to the latter one.

2.1.1 Pressure

The first step is to get the equation for hydrostatic equilibrium, see Fig 2.1, adapted from [17]. The conditions above are used to derive the equations for the neutron star structure from Newtonian mechanics.

The first step is look at a small spherical section of the star that is at a distance r away from the center of the star with a smaller mass, Δm (see Fig 2.1) given by

$$\Delta m(r) = \rho(r)dV(r) \tag{2.1}$$

where $\rho(r)$ is the matter density as a function of the radius r . The volume of the enclosed matter is given by

$$dV = dA dr. \tag{2.2}$$

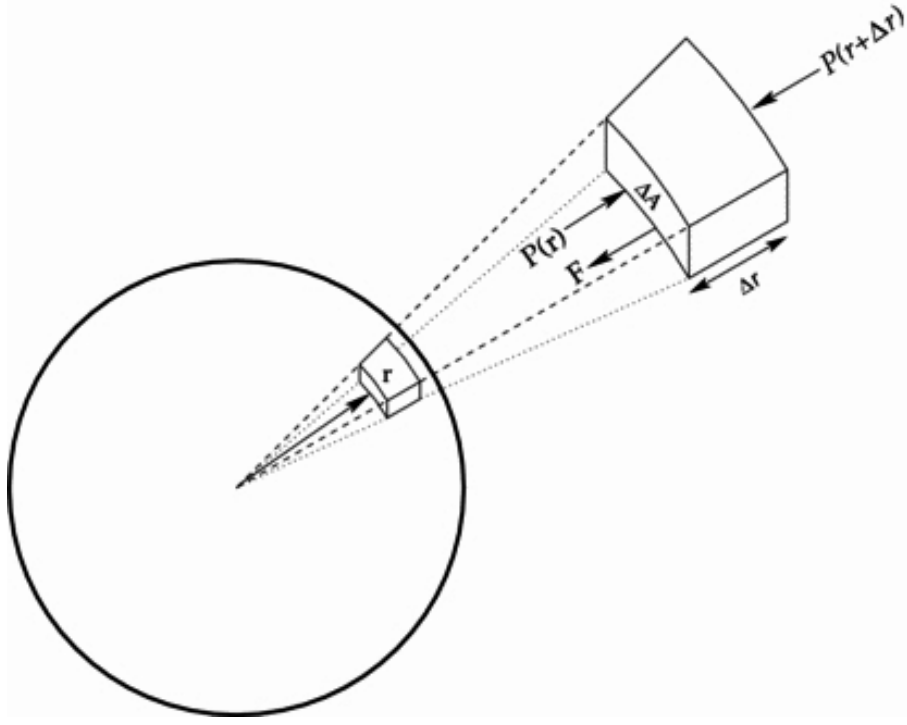


Figure 2.1: The radial force acting on a small mass element a distance r from the center of the star[17]

The mass will then be given by

$$\Delta m(r) = \rho(r) dA dr \quad (2.3)$$

The force of gravity, F_G , by Newton's universal law of gravitation pushes inwards on the star with [17]

$$F_G = \frac{-GM(r)\Delta m(r)}{r^2}, \quad (2.4)$$

where

- G is the universal gravitation constant
- $M(r)$ is the mass enclosed in radius r in the star.

Substituting Eq. (2.3) into Eq. (2.4) into yields

$$F_G = \frac{-GM(r)\rho(r)dA dr}{r^2}. \quad (2.5)$$

For a system in hydrostatic equilibrium the gravitational force is balanced by the pressure P of the enclosed matter. Hence, the next step is to calculate the pressure acting on the spherical

section. The section feels a pressure $P(r + \Delta r)$ due to the gravitational weight of the matter above the spherical section. While the pressure, $P(r)$, is due to the compressibility of the matter below the spherical section, which in the case of a neutron star is a pressure due to the neutron degeneracy pressure which stems from Pauli's Exclusion Principle. The net pressure dP , on the elemental mass will then be given by

$$dP = P(r + \Delta r) - P(r). \quad (2.6)$$

Force due to the difference in the neutron degeneracy pressure and the pressure $P(r + \Delta r)$ due to the matter above it relative to the center of the star is given by $F_P = PA$, thus the force due to pressure will be given by

$$\begin{aligned} dF_P &= (P(r + \Delta r) - P(r))dA \\ &= dPdA. \end{aligned} \quad (2.7)$$

The forces F_P and F_G must balance since the system is assumed to be in equilibrium therefore the force from the degeneracy pressure must equal the force of gravity so that $F_P = F_G$ and thus

$$dPdA = \frac{-GM(r)\rho(r)dAdr}{r^2}. \quad (2.8)$$

Then re-arranging the terms yields the equation for hydrostatic equilibrium as

$$\frac{dP}{dr} = \frac{-GM(r)\rho(r)}{r^2}. \quad (2.9)$$

2.1.2 Enclosed mass

The total mass enclosed by radius r in the star is $M(r)$. For a change in r of step size dr , the change in $M(r)$ is

$$dM(r) = \rho(r)dV. \quad (2.10)$$

The star is spherically symmetric; therefore the volume is given by

$$dV(r) = 4\pi r^2 dr, \quad (2.11)$$

Substituting Eq. (2.11) into Eq. (2.10) gives

$$dM(r) = \rho(r)4\pi r^2 dr \quad (2.12)$$

Then simplifying the terms and re-arranging yields

$$\frac{dM(r)}{dr} = \rho(r)4\pi r^2. \quad (2.13)$$

Thus Eq. (2.9) and Eq. (2.13) can be expressed as

$$\frac{dP}{dr} = \frac{-GM(r)\rho(r)}{r^2}, \quad (2.14)$$

$$\frac{dM(r)}{dr} = \rho(r)4\pi r^2 \quad (2.15)$$

A non-relativistic description (which only uses classical newtonian mechanics) of nuclear matter has been given and it yielded two equations.

2.1.3 Newtonian stars

The mass density $\rho(r)$ can be written in terms of its relation to the energy density, $\epsilon(r)$ (in natural units) which is[1]

$$\epsilon(r) = \rho(r), \quad (2.16)$$

This is done to get a relation where the energy density can be substituted in Eq. (2.14) and (2.15). A relation $P(\epsilon)$ between the energy density and pressure is called the equation of state. Once the equation of state is found then it is used along with the hydrostatic equilibrium equations to calculate the mass-radius relationship of a neutron star.

To solve equations (2.14) and (2.15) for $P(r)$ and $M(r)$ an integral is taken from the centre (centre of the star) going outwards until the point where the pressure is zero[9]. The point where the pressure is zero defines $r = R$, the radius of the star[18]. The two equations (2.14) and (2.15) indicate how the pressure decreases from the origin ($r = 0$) until at $r = R$ where it is zero[9]. A set of initial values for the initial pressure (P_0) where $r = 0$ are needed, the radius of the star R and the mass of the star $M = M(R)$ will depend on the central pressure of the star[18].

2.2 General Relativity Corrections

Equations (2.14) and (2.15) are useful only for stars where general relativity effects can be ignored, where the stars experience less significant gravity and thus its effects can be neglected. It is the value $2GM(r)/c^2r$ that tells us if it is important to consider general relativity effects. It

can be neglected if it is small. This value $2GM(r)/c^2$ is essentially the Schwarzschild radius of the enclosed mass $M(r)$. If a star or any mass for that matter is compressed to a value below the Schwarzschild radius it will collapse into a black hole[1]. Estimated for a typical neutron star of $1.5 M_\odot$ to be 4.38 km, therefore of the order of the neutron stars' expected radius. For neutron stars general relativity effects cannot be ignored since $2GM(r)/c^2r$ is close to 1. Neutron stars experience a high gravity effects and therefore gravity impacts the star structure therefore corrections due to these effects have to be included to the equations derived above [17]. This is done to properly describe the neutron star structure. From relativity a set of correction terms are added to this equations to correct the equations and this yields what is called the Tolman-Oppenheimer-Volkoff, (TOV) equations [19]. These correction terms reduce to 1 if we go to the non-relativistic limit.

Adding general relativity corrections to Eq. (2.14) and Eq. (2.15) will give the TOV equations as

$$\frac{dP(r)}{dr} = -\frac{G\epsilon(r)M(r)}{c^2r^2} \left[1 + \frac{P(r)}{\epsilon(r)} \right] \left[1 + \frac{4\pi r^3 P(r)}{M(r)c^2} \right] \left[1 - \frac{2GM(r)}{c^2r} \right]^{-1}, \quad (2.17)$$

where

$$\frac{dM(r)}{dr} = \frac{4\pi\epsilon(r)r^2}{c^2}. \quad (2.18)$$

The equations in Eq. (2.17) and Eq. (2.18) for the pressure $P(r)$ and mass $M(r)$ are integrated from $r = 0$ for the initial condition for the pressure til the point where $P(r) = 0$. $P(r) = 0 = P(R)$ defines the point where $r = R$ and R will be the radius of the star and $M(R)$ the mass of the star.

A bigger star will have a larger pressure at the centre and if the central pressure increases the smaller the radius R of the star becomes. This is because the force of gravity is always pushing inwards and the central pressure has to increase in order to balance the gravity until a point where the force of gravity overcomes degeneracy pressure. The equations imply there exists a maximum mass that a star can have beyond which the star collapses due to its own gravity [19].

To solve the TOV equations boundary conditions are imposed. The first boundary condition is that the pressure at the centre is not zero; the pressure at the centre of the star has to be some value P_0 as discussed in the previous section, whilst the mass at the centre of the star is

zero [17]

$$P(r = 0) = P_0, \quad (2.19)$$

$$M(r = 0) = 0. \quad (2.20)$$

The starting point is the centre of the star and as the radius increases the mass increases. The pressure decreases gradually until it is zero at the surface($r = R$) of the star where the boundary conditions do not hold.

The TOV equations describes the pressure as it varies from the centre towards the surface of the star.

2.3 The Equation of State

The TOV equations provide two differential equations Eq. (2.17) and (2.18) for the mass and pressure in a star. Both equations depend on the energy density ϵ . A relation between the energy density and pressure $P(\epsilon)$ is needed. This is called the equation of state.

There are different equations of state depending on the composition of what the neutron star is made up of and the interaction of the neutron star constituents. The TOV equations along with the equation of state are used to derive the mass radius relationship for a neutron star however before getting to solve the TOV equations there is need to rescale some of the terms used and all this is explained in the next step.

2.4 Rescaling the TOV equations

When dealing with the TOV equations the order of some terms like the radius of a nucleon (order of fm) are very small and some terms like the mass of the neutron star (the order of km) are very large and this will create problems in terms of coding the equation.

A way to solve this problem is called rescaling where the terms are made dimensionless by absorbing out some of the powers that are involved. The mass is simplified by recording it in terms of the solar masses (mass of the sun $M_\odot$) and the distance in kilometres, thus

$$\bar{M}(r) = \frac{M(r)}{M_\odot}. \quad (2.21)$$

The energy density and the pressure also carry dimensions. To make them dimensionless they are written in the following way

$$P(r) = \epsilon_0 \bar{P}(r), \quad (2.22)$$

$$\epsilon(r) = \epsilon_0 \bar{\epsilon}(r). \quad (2.23)$$

The re-scaled quantities are fitted back into the equation of state and TOV equations and then an integral is taken from $r = 0$ to $r = R$ where the pressure is zero.

The rescaled TOV Eq. (2.17 and 2.18) will be given by

$$\frac{dP(r)}{dr} = -\frac{G\epsilon(r)M_\odot\bar{M}(r)}{r^2} \left[1 + \frac{P(r)}{\epsilon(r)}\right] \left[1 + \frac{4\pi r^3 P(r)}{M_\odot\bar{M}(r)}\right] \left[1 - \frac{2GM_\odot\bar{M}(r)}{r}\right]^{-1}, \quad (2.24)$$

where

$$\frac{d\bar{M}(r)}{dr} = \frac{4\pi\epsilon(r)r^2}{M_\odot}. \quad (2.25)$$

2.5 Solving the TOV equations

A python code is written which solves the TOV equation. The Runge-Kuta method is used to solve the TOV equations by numerically approximating the coupled differential equations [20]. It involves the use of weighted average values a function at four different points of the interval $r_n \leq r \leq r_{n+1}$ [20].

For Eq. (2.24) and Eq. (2.25) let

$$\frac{dP}{dr} = F(r, P, M) \quad (2.26a)$$

$$\frac{dM}{dr} = G(r, P, M). \quad (2.26b)$$

The weighted averages for the pressure and the mass are defined by single integration steps of size h and are given by

$$P_{n+1} = P_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 \quad (2.27a)$$

$$M_{n+1} = M_n + \frac{1}{6}d_1 + \frac{1}{3}d_2 + \frac{1}{3}d_3 + \frac{1}{6}d_4. \quad (2.27b)$$

The sum of the k -terms gives the change in pressure with the radius. The k_1 term is the pressure gradient at the left end of the radius interval, k_2 is the midpoint between r_n and $r_n + \frac{h}{2}$, k_3 is the pressure gradient at the midpoint on the right side of the interval $r_n + \frac{h}{2}$ and $r_n + h$ and finally k_4 is the pressure gradient at the point $r_n + h$. The d -terms also follow the same concept. The sum of the d -terms gives the change in the mass with the radius. The terms follow the same concept as the k -terms but for the mass gradient in this case. All this can be expressed in the following way:

$$\begin{aligned}
 k_1 &= hF(r_n, P_n, M_n) \\
 k_2 &= hF(r_n + \frac{h}{2}, P_n + \frac{k_1}{2}, M_n + \frac{d_1}{2}) \\
 k_3 &= hF(r_n + \frac{h}{2}, P_n + \frac{k_2}{2}, M_n + \frac{d_2}{2}) \\
 k_4 &= hF(r_n + h, P_n + k_3, M_n + d_3),
 \end{aligned} \tag{2.28}$$

$$\begin{aligned}
 d_1 &= hG(r_n, P_n, M_n) \\
 d_2 &= hG(r_n + \frac{h}{2}, P_n + \frac{k_1}{2}, M_n + \frac{d_1}{2}) \\
 d_3 &= hG(r_n + \frac{h}{2}, P_n + \frac{k_2}{2}, M_n + \frac{d_2}{2}) \\
 d_4 &= hG(r_n + h, P_n + k_3, M_n + d_3)
 \end{aligned} \tag{2.29}$$

As already indicated the results of the TOV equations depends on the chosen EoS. In the next chapter we discuss relativistic quantum mechanics as a build up to studying several nuclear equation of states that will be used to solve the TOV equations in this work.

3. Quantum Hadrodynamics

The framework for describing quantum field theory and a multi-particle description for nuclear matter is needed in the description of relativistic particles. Appendix A to C chapters provides a short summary of the concepts that lay this framework. In this chapter the application of quantum field theory to nuclear matter will be described in order to give a nuclear field theory that provides a description of the many-body states for nuclear matter. The nuclear field theory that will be used is called Quantum Hadrodynamics (QHD)[1]. This nuclear field theory is modelled around the properties of saturated nuclear matter. The behaviour of the strong nuclear force is that it is a long range attractive force and short range repulsive force mediated by a one pion (π meson) exchange (OPE)[21]. The point where the two forces balance is the point where there is a saturation in nuclear matter[21]. The strong nuclear force can also be described by the exchange of mesons[21]. QHD uses the exchange of mesons to describe the behaviour of the observed strong nuclear force and the properties of nuclear matter[10]. The first step is to discuss the properties of nuclear matter which are given in the following section.

3.1 Properties Of Nuclear Matter

The basic property of nuclear matter is that there is a short range repulsive force and a long range attractive force. Thus they are saturated systems because of this balance. The saturated system is due to the short range nuclear force and the Pauli exclusion principle which states that no two similar particles can occupy the same quantum state. The saturation density is defined by the density where, even if more nucleons are added to the nucleus, the density of the central region of the nucleus will remain constant. The value of the saturation density, ρ_{sat} , for nuclear matter is taken as [1]

$$\rho_{sat} = 0.153 \text{ fm}^{-3}. \quad (3.1)$$

The binding energy per nucleon (B/A) is the energy that is needed to keep the nucleons bound together inside the nucleus. At saturation density the system is in its most stable configuration and therefore the binding energy will be at its lowest value. For nuclear matter the binding energy per nucleon (B/A) is assumed to be $B/A = -16.3 \text{ MeV}$ and to be calculated as [1]

$$B/A = \frac{\epsilon}{\rho} - M \quad (3.2)$$

where

- ϵ is the energy density,
- ρ is the baryon density, and,
- M is the nucleon mass.

3.2 The $\sigma - \omega$ model

The effective nuclear field theory discussed in this section is called Quantum Hadrodynamics (QHD)[10], also known as the $\sigma - \omega$ model. This model of nuclear field theory describes the nuclear force based on the exchange of mesons between the baryons that mediate the strong nuclear force. The scalar meson fields from the σ meson give rise to a strong attractive central force whilst the vector meson gives rise to a strong repulsive central force and thus the interaction of the nucleons is described by the fields of the baryons, scalar meson and the omega vector meson. The baryons repel each other at short distances because of the repulsion from the ω vector meson and attract each other at long distances because of the attraction from the σ meson which is why this model is a good one to describe nuclear matter as it also behaves in this way [1].

The $\sigma - \omega$ model takes the mass M of protons and neutrons to be equal to

$$M = 939 \text{ MeV.} \quad (3.3)$$

As the intention is to describe neutron star matter consisting of many relativistic particles the model has to be Lorentz covariant with quantum field theory governing the dynamics of the system. All interactions are to be included in the Lagrangian which must be a Lorentz scalar. To make the Lagrangian a scalar the scalar meson is coupled to the baryon scalar current $\bar{\psi}\psi$ and the vector meson is coupled to the baryon four current $\bar{\psi}\gamma^\mu\psi$. Including these interactions to the free nucleon Lagrangian, the total Lagrangian is [1, 10]

$$\begin{aligned} \mathcal{L} = & \bar{\psi}(x) \left[\gamma_\mu \left(i\partial^\mu - g_v V^\mu(x) \right) - \left(M - g_s \phi(x) \right) \right] \psi(x) \\ & + \frac{1}{2} \left(\partial_\mu \phi(x) \partial^\mu \phi(x) - m_s^2 \phi^2(x) \right) - \frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{1}{2} m_\omega^2 V_\mu(x) V^\mu(x) \end{aligned}$$

where

- $V^\mu(x)$ denotes the vector meson field operator,
- $\phi(x)$ denotes the scalar meson field operator,
- m_ω and m_s the different meson masses and M denotes the nucleon mass,

- g_v and g_s are the vector and scalar coupling constants and are not fixed, they are larger than 1 [10] and
- $V_{\mu\nu} = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)$.
- $x = x^\mu = (x^0, x^1, x^2, x^3)$

The next step is to calculate the Euler-Lagrange equations of motion (B.9) for the given Lagrangian. The equations of motion are given by

$$\partial_\mu \partial^\mu \phi(x) + m_s^2 \phi(x) = g_s \bar{\psi}(x) \psi(x) \quad (3.4a)$$

$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu(x) = g_v \bar{\psi}(x) \gamma^\nu \psi(x) \quad (3.4b)$$

$$\left[\gamma_\mu (i\partial^\mu - g_v V^\mu(x)) - (M - g_s \phi(x)) \right] \psi(x) = 0. \quad (3.4c)$$

Eq. (3.4a) is the equation of motion for a free scalar field which is the Klein-Gordon equation (C.9) with an extra terms for the baryon scalar density) with $\bar{\psi}(x)\psi(x)$ being the source term.

Eq. (3.4b) is analogous to the equation of motion for massive QED also known as the Proca equation with an added conserved baryon four current as the source term.

Lastly Eq. (3.4c) is the equation of motion for baryons (which is the Dirac Equation (C.22) with the scalar and vector fields introduced). As it is assumed the proton and nucleon have equal mass, $\psi(x)$ this gives an eight component spinor as in Eq. (C.39).

The three equations of motion are coupled and non-linear differential equations which are difficult to solve. To solve this equations is by an approximation method called the relativistic mean-field approximation method which is described in the next section [10].

3.3 Relativistic mean-field theory (RMF)

RMF is an approximation method that has become increasingly valid for models that have increase in densities. Unlike in Quantum Electrodynamics (QED) where a perturbative approach can be applied and all the terms can solved exactly, in QHD the higher order terms diverge and thus perturbation theory cannot be applied [10]. The equations in Eq. (3.4a -3.4c) are coupled, non-linear differential equations and the coupling constants are large and therefore perturbation theory cannot be applied to this theory [1].

The coupling constants in QHD cannot be solved exactly[1]. It has been observed that at high baryon density the scalar and vector meson field operators can be replaced by their expectation values[10]. The expectation values can be solved exactly and they provide a good description of the high density limit of the nuclear equation of state[10]. The source terms are (3.4a) and (3.4b), the meson fields are replaced by their ground state expectation values which are classical fields [10]

$$\phi \longrightarrow \langle \Phi | \phi | \Phi \rangle = \langle \phi \rangle = \phi_0 \quad (3.5a)$$

$$V_\mu \longrightarrow \langle \Phi | V_\mu | \Phi \rangle = \langle V_\mu \rangle = \delta_{\mu 0} V_0. \quad (3.5b)$$

The vector meson $V_\mu = (V_0, V_1, V_2, V_3)$ is a four-vector but the spatial components vanish since now we are in the rest frame of the mean-field so there is no preferred direction [1]. The integral taken is over all occupied momentum states.

The meson field operators have been replaced by their ground state expectation values to give us exact solutions but the nucleon field operators remain operators. Only the ground state expectation values of the meson field operators are considered in the mean-field approximation which is why the baryon four-current and the baryon scalar current are evaluated by operating on the ground state and hence in the equation of motion for the meson fields they will be replaced by their normal ordered ground state expectation values given by [10]

$$\bar{\psi}(x)\psi(x) \longrightarrow \langle \Phi | :\bar{\psi}(x)\psi(x): | \Phi \rangle = \langle \bar{\psi}\psi \rangle \quad (3.6a)$$

$$\bar{\psi}(x)\gamma^\mu\psi(x) \longrightarrow \langle \Phi | :\bar{\psi}(x)\gamma^\mu\psi(x): | \Phi \rangle = \langle \bar{\psi}\gamma^0\psi \rangle. \quad (3.6b)$$

The equations of motion for the meson fields (3.4a) and (3.4b) then reduce to simpler forms as a result of taking the expectations to give

$$m_s^2 \phi_0 = g_s \langle \bar{\psi}\psi \rangle \quad (3.7a)$$

$$m_v^2 V_0 = g_v \langle \bar{\psi}\gamma_0\psi \rangle. \quad (3.7b)$$

$$(3.7c)$$

When these expectation values are substituted into the Dirac field Equation (3.4c)

$$[i\gamma_\mu \partial^\mu - g_v \gamma_0 V_0 - (M - g_s \phi_0)] \psi = 0. \quad (3.8)$$

This equation is linear and can be solved directly. The solutions that are of interest should be written in a way that there is no x -dependence (thus looking for stationary-state solutions for the uniform system). Therefore the Dirac fields in (C.26) are written in the following way [1]

$$\psi(x) = \sum_k \psi(\mathbf{k}, s) e^{-i\mathbf{k}\cdot\mathbf{x} - ie(\mathbf{k})t} \quad (3.9)$$

where

- $\psi(\mathbf{k}, s)$ is a four component Dirac spinor,
- $e(\mathbf{k})$ is the energy associated with the momentum state k .

Substituting Eq. (3.9) into Eq. (3.8) leads to

$$(\gamma_i k^i + \gamma_0 \epsilon(\mathbf{k}) - g_v \gamma^0 V_0 - (M - g_s \phi_0)) \psi(\mathbf{k}, s) = 0. \quad (3.10)$$

Then multiplying by γ^0 and reverting back to the α and β matrices notation in Sec. A.4 yields (α is an eight component spinor)

$$(\beta^2(e(\mathbf{k}) - g_v V_0)) \psi(\mathbf{k}, s) = (\alpha \cdot \mathbf{k} + \beta m^*) \psi(\mathbf{k}, s) \quad (3.11)$$

where m^* is the effective mass and is given by

$$m^* = M - g_s \phi_0. \quad (3.12)$$

Thus the scalar meson shifts the mass of the baryons and the vector meson shifts the energy of the solutions. When Eq. (3.11) is squared and also keeping the properties of the Dirac equation this will give the eigenvalue equation

$$e^{(\pm)}(\mathbf{k}) = g_v V_0 \pm \sqrt{\mathbf{k}^2 + m^{*2}}. \quad (3.13)$$

This yields both positive and negative energy solutions which are a characteristic of the Dirac equation. From this result the baryon field operator can then be written as

$$\hat{\psi}(x) = \sum_s \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\hat{b}(\mathbf{k}, s) u(\mathbf{k}, s) e^{i(\mathbf{k}\cdot\mathbf{x}) - ie^{(+)}(\mathbf{k})t} + \hat{d}^\dagger(\mathbf{k}, s) v(\mathbf{k}, s) e^{-i(\mathbf{k}\cdot\mathbf{x}) - ie^{(+)}(\mathbf{k})t} \right) \quad (3.14)$$

where

- t is the time coordinate of x^μ

- $\mathbf{x}$ is the spatial components of x^μ

The creating and annihilation operators for the particle and antiparticles are the same as the ones used in the Dirac fields section of the previous chapter.

In the mean-field theory approximation the Lagrangian $\mathcal{L}_{RMF}$ is given by [10]

$$\mathcal{L}_{RMF} = \bar{\psi} \left[i\gamma_\mu \partial^\mu - g_v \gamma^0 V_0 - (M - g_s \phi_0) \right] \psi - \frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_\omega^2 V_0^2.$$

The energy density and the pressure are

$$\begin{aligned} \epsilon &= \langle T^{00} \rangle \\ &= \left\langle i\bar{\psi} \gamma_0 \partial_0 \psi - \left(-\frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_\omega^2 V_0^2 \right) \right\rangle \\ &= \langle i\bar{\psi} \gamma_0 \partial_0 \psi \rangle + \frac{1}{2} m_s^2 \phi_0^2 - \frac{1}{2} m_\omega^2 V_0^2, \end{aligned} \quad (3.15a)$$

$$\begin{aligned} P &= \frac{1}{3} \langle T^{ii} \rangle \\ &= \frac{1}{3} \left\langle i\bar{\psi} \gamma^i \partial_i \psi + \left(-\frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_\omega^2 V_0^2 \right) \right\rangle \\ &= \frac{1}{3} \langle i\bar{\psi} \gamma_i \partial_i \psi \rangle - \frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_\omega^2 V_0^2. \end{aligned} \quad (3.15b)$$

$\epsilon = \langle T^{00} \rangle$ is the energy density expression and $P = \frac{1}{3} \langle T^{ii} \rangle$ is the pressure expression given in [10]. The expressions are derived from the energy momentum tensor given in [10].

The Dirac momentum eigenvalues have been given in terms of the meson field operators ϕ and V but their expectation values of $\langle \phi \rangle$ and $\langle V \rangle$ [1] are yet to be found. To find the expectation values there is no need to create the spinors as this method is more rigorous but rather a trick is used by defining what is meant by the expectation value of an operator (Γ) in the ground state of the many nucleon system. This expectation value is written in the following way [1]

$$\langle \bar{\psi} \Gamma \psi \rangle = \sum_k \int \frac{d\mathbf{k}}{(2\pi)^3} (\bar{\psi} \Gamma \psi)_{\mathbf{k},s} \Theta[\mu - e(\mathbf{k})], \quad (3.16)$$

where

- μ is the chemical potential/Fermi energy level
- k is the sum over all the spin-isospin states
- Θ is the step function that has unity for $x \geq 0$ and 0 for negative values.

The ground state is defined by the Fermi wavenumber, k_F which denotes the filled positive energy states and the result is that the energy levels of the states can be filled up to the Fermi level.

To find the single particle expectation states of the mean-field approximation in Eq. (3.7a) and Eq. (3.7b) using Eq. (3.16).

The Dirac Hamiltonian is given by [1]

$$H_D = \gamma_0 [\boldsymbol{\gamma} \cdot \mathbf{k} + g_v \gamma_\mu V^\mu + m^*]. \quad (3.17)$$

Then taking the single particle expectation yields

$$\left(\psi^\dagger H_D \psi \right)_{\mathbf{k},s} = e(\mathbf{k}). \quad (3.18)$$

Taking the derivative of the above equation of the single particle expectation with respect to any variable ζ

$$\frac{\partial}{\partial \zeta} \left(\psi^\dagger H_D \psi \right)_{\mathbf{k},s} = \left(\psi^\dagger \frac{\partial}{\partial \zeta} H_D \psi \right)_{\mathbf{k},s} + e(\mathbf{k}) \frac{\partial}{\partial \zeta} \left(\psi^\dagger \psi \right)_{\mathbf{k},s}. \quad (3.19)$$

The last term on the right vanishes as ψ is an eigenfunction of the Dirac Hamiltonian.

Using this result any expectation value can be derived using this method by first taking the derivative of the Dirac Hamiltonian with respect to V_0 . This method yields the normalization

$$\left(\psi^\dagger \psi \right)_{\mathbf{k},s} = 1. \quad (3.20)$$

This method is used to determine the expectation values for the baryon density

$$\langle \psi^\dagger \psi \rangle = \gamma \int \frac{d\mathbf{k}}{(2\pi)^3} \Theta[\mu - e(\mathbf{k})] = \frac{\gamma}{6\pi^2} k_F^3 = \rho, \quad (3.21)$$

where γ is the degeneracy of the nucleon spin state [1].

The energy density and the pressure can also be evaluated using this method. The expectation value in the expression for the energy density in Eq. (3.15a) when using Eq. (3.12) and Eq.

(3.14) will be

$$\begin{aligned}
\epsilon &= \left\langle \psi^\dagger (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m^* + g_v V_0) \psi \right\rangle \\
&= \left\langle \psi^\dagger (\sqrt{\vec{k}^2 + m^{*2}} + g_v V_0) \psi \right\rangle \\
&= \langle \psi^\dagger e(\mathbf{k}) \psi \rangle.
\end{aligned} \tag{3.22}$$

Simplifying by using the value for the single particle expectation of the Dirac Hamiltonian Eq. (3.18) the expectation value for the energy density yields

$$\begin{aligned}
\left\langle \psi^\dagger (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m^* + g_v V_0) \psi \right\rangle &= \sum \int \frac{d\mathbf{k}}{(2\pi)^3} e(\mathbf{k}) \Theta[\mu - e(\mathbf{k})] \\
&= \sum \int \frac{d\mathbf{k}}{(2\pi)^3} (\sqrt{\mathbf{k}^2 + m^{*2}} + g_v V_0) \Theta[\mu - e(\mathbf{k})] \\
&= g_v V_0 \rho + \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d\mathbf{k} \sqrt{\mathbf{k}^2 + m^{*2}}.
\end{aligned} \tag{3.23}$$

The expectation value in the equation for the pressure is given by

$$\begin{aligned}
\left\langle \psi^\dagger (-i\vec{\alpha} \cdot \vec{\nabla}) \psi \right\rangle &= \langle \psi^\dagger (\vec{\alpha} \cdot \mathbf{k}) \psi \rangle \\
&= \langle \bar{\psi} (\vec{\gamma} \cdot \mathbf{k}) \psi \rangle.
\end{aligned} \tag{3.24}$$

To find the single particle expectation value take the derivative of the Dirac Hamiltonian Eq. (3.17) with respect to $\mathbf{k}$

$$\begin{aligned}
(\bar{\psi} (\vec{\gamma} \cdot \mathbf{k}) \psi)_{\mathbf{k},s} &= (\bar{\psi} \vec{\gamma} \psi) \cdot \mathbf{k} \\
&= \left(\bar{\psi} \frac{\partial}{\partial \mathbf{k}} H_D \psi \right) \cdot \mathbf{k} \\
&= \frac{\partial}{\partial \mathbf{k}} \epsilon(\mathbf{k}) \cdot \mathbf{k} \\
&= \frac{\mathbf{k}^2}{\sqrt{\mathbf{k}^2 + m^{*2}}}.
\end{aligned} \tag{3.25}$$

The general expression is then given by

$$\left\langle \psi^\dagger (-i\vec{\alpha} \cdot \vec{\nabla}) \psi \right\rangle = \sum \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\mathbf{k}^2}{\sqrt{\mathbf{k}^2 + m^{*2}}} \Theta[\mu - e(\mathbf{k})] \tag{3.26}$$

$$= \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d\mathbf{k} \frac{\mathbf{k}^2}{\sqrt{\mathbf{k}^2 + m^{*2}}}. \tag{3.27}$$

The scalar density $\langle \bar{\psi} \psi \rangle$ is also needed. Taking the derivative with respect to the nucleon mass

M the single particle expectation value is given by

$$(\bar{\psi}\psi)_{\mathbf{k},s} = \left(\psi^\dagger \gamma^0 \psi\right)_{\mathbf{k}} \quad (3.28)$$

$$= \left(\psi^\dagger \frac{\partial H_D}{\partial M} \psi\right)_{\mathbf{k}} \quad (3.29)$$

$$= \frac{\partial \epsilon(\mathbf{k})}{\partial M} \quad (3.30)$$

$$= \frac{m^*}{\sqrt{\mathbf{k}^2 + m^{*2}}}. \quad (3.31)$$

This gives the general expression for the expectation value as

$$\langle \bar{\psi}\psi \rangle = \sum \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{m^*}{\sqrt{\mathbf{k}^2 + m^{*2}}} \Theta[\mu - e(\mathbf{k})] \quad (3.32)$$

$$= \frac{\gamma}{(2\pi)^2} \int_0^{k_F} d\mathbf{k} \frac{\mathbf{k}^2 m^*}{\sqrt{\mathbf{k}^2 + m^{*2}}}. \quad (3.33)$$

This method is very powerful as it allows for evaluation of the expectation values of any operator appearing in the Dirac Hamiltonian without constructing Dirac spinors. These expectation values are then substituted into the equations of motion for simplicity in the next section.

3.4 The Nuclear Matter Equation of State

In the previous section the expectation values of the operators appearing in the Dirac Hamiltonian were evaluated. This section will look to describe the nuclear matter equation of state which is given in parametric form by the pressure and energy density equations [10]. When substituting the expectation values the equations for the meson fields will be given by (the integrals are over all occupied nucleon momentum states)[1]

$$\phi_0 = \frac{g_s}{m_s^2} \langle \bar{\psi}\psi \rangle = \frac{g_s}{m_s^2} \frac{\gamma}{2\pi^2} \int_0^{k_F} d\mathbf{k} \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}}, \quad (3.34a)$$

$$V_0 = \frac{g_v}{m_\omega^2} \langle \psi^\dagger \psi \rangle = \frac{g_v}{m_\omega^2} \rho, \quad (3.34b)$$

where ρ is the baryon density given in Eq. (3.21).

The above equations can be rearranged to yield expressions needed to calculate the energy

density and the pressure given by

$$\epsilon = \frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2V_0^2 + \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d\mathbf{k} \sqrt{\mathbf{k}^2 + (M - g_s\phi_0)^2}, \quad (3.35a)$$

$$P = -\frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2V_0^2 + \frac{1}{3} \left(\frac{\gamma}{(2\pi)^3} \int_0^{k_F} d\mathbf{k} \frac{\mathbf{k}^2}{\sqrt{\mathbf{k}^2 + (M - g_s\phi_0)^2}} \right). \quad (3.35b)$$

The two expressions in Eq.(3.35a) and Eq. (3.35b) can be simplified further when taking into account spherical symmetry (the integral is over all occupied nucleon momentum states and its a sphere since $e(\mathbf{k}_F)$ depends only on $k_F = |\mathbf{k}_F|$) to give the energy density and the pressure as [1]

$$\epsilon = \frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2V_0^2 + \frac{\gamma}{2\pi^2} \int_0^{k_F} dk k^2 \sqrt{k^2 + m^{*2}} \quad (3.36a)$$

$$P = -\frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2V_0^2 + \frac{1}{3} \left(\frac{\gamma}{2\pi^2} \int_0^{k_F} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}} \right). \quad (3.36b)$$

The equation of state for nuclear matter is given by these two expressions. The effective mass is determined by minimizing the energy density.

$$m^* = M - \frac{g_s^2}{m_s^2} \frac{\gamma}{2\pi^2} \int_0^{k_F} dk \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}}. \quad (3.37)$$

Solving this equation as in [10] yields the reduced mass as a self consistency equation:

$$m^* = M - \frac{g_s^2}{m_s^2} \frac{\gamma m^*}{4\pi^2} \left[k_F \mu - m^{*2} \ln \left(\frac{k_F + \mu}{m^*} \right) \right], \quad (3.38)$$

where $\mu = e(\mathbf{k}_F) = (k_F^2 + m^{*2})^{\frac{1}{2}} + g_\nu V_0$ is the chemical potential at zero temperature.

The equation of state is in the parametric form $\epsilon = \epsilon(\rho)$ and $P = P(\rho)$. However, ρ is eliminated to yield the equation of state as $P = P(\epsilon)$. Eq. (3.35a) and (3.35b) are solved numerically in order to evaluate the pressure and energy density as functions of the nucleon density [1]. So far the $\sigma - \omega$ model has been used for describing nuclear matter but it can also be used in neutron stars. If in the case of neutron matter we will be dealing with (degeneracy = 2) instead of nuclear matter (degeneracy = 4). The equations still remain the same except for the degeneracy.

The model has two parameters $\left(\frac{g_\sigma}{m_\sigma}\right)$ and $\left(\frac{g_\omega}{m_\omega}\right)$. The saturation density and the binding energy per nucleon can be fitted exactly using these parameter sets. The $\sigma - \omega$ model provides a

much simpler introduction into the description of nuclear matter. However other nuclear matter properties such as the symmetry energy, compression modulus and the effective mass give values are not in agreement with the theoretical values[1] which means the model yields wrong values for these nuclear parameters[1]. Nonetheless the $\sigma - \omega$ model is a good first approximation for introducing techniques used to describe nuclear matter. In the next chapter more advanced parameter sets of QHD will be discussed in order to improve on the framework layed so far with the $\sigma - \omega$ model.

4. Advanced QHD parameters

In Chapter 3 the $\sigma - \omega$ provided an introduction into the tools used to describe the properties of nuclear matter as in Sec. 3.1. The $\sigma - \omega$ model had two free parameters managed to fix two nuclear matter properties being the binding energy and the saturation density. However, in order to give a more complete description of nuclear matter additional mesons as well as couplings between the various fields are introduced to improve on the $\sigma - \omega$. In this chapter a extended parameter set called NL3 where self-interactions of the σ field as well as the isospin symmetry force will be introduced.

The advanced form of the Lagrangian density that describes both the QHD-I, NL3 parameter sets in QHD, is,

$$\begin{aligned}
\mathcal{L} = & \bar{\psi}(x) \left[\gamma^\mu (i\partial_\mu - g_v V_\mu(x)) - \frac{g_\rho}{2} \boldsymbol{\tau} \cdot \boldsymbol{\gamma}^\mu \cdot \mathbf{b}_\mu(x) - (M - g_s \phi(x)) \right] \psi(x) \\
& + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi(x) - \frac{1}{2} m_s^2 \phi^2(x) - \frac{\kappa}{3!} (g_s \phi(x))^3 - \frac{\lambda}{4!} (g_s \phi(x))^4 \\
& - \frac{1}{4} V^{\mu\nu} V_{\mu\nu} + \frac{1}{2} m_\omega^2 V^\mu(x) V_\mu(x) \\
& - \frac{1}{4} \mathbf{b}^{\mu\nu} \mathbf{b}_{\mu\nu} + \frac{1}{2} m_\rho^2 \mathbf{b}^\mu(x) \cdot \mathbf{b}_\mu(x)
\end{aligned} \tag{4.1}$$

where the field tensors are defined as

$$V_{\mu\nu} = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) \tag{4.2}$$

$$\mathbf{b}_{\mu\nu} = \partial_\mu \mathbf{b}_\nu(x) - \partial_\nu \mathbf{b}_\mu(x) \tag{4.3}$$

with

- $\psi(x)$ is the baryon field spinor
- $\phi(x)$ is the sigma field scalar meson
- $\mathbf{V}^\mu(x)$ is the omega vector meson field.
- $\mathbf{b}^\mu(x)$ is the Lorentz vector field for the three isospin components of the rho meson fields,

$$\mathbf{b}^\mu = (b_1^\mu, b_2^\mu, b_3^\mu). \tag{4.4}$$

The charged rho meson field $\rho^\pm$ can be constructed in terms of the first two components

of $\mathbf{b}^\mu$ as[1]

$$b_\pm^\mu = (b_1^\mu \pm b_2^\mu) \quad (4.5)$$

- $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ is the isospin operator. The isospin operator is described in terms of the Pauli 2×2 spin matrices Eq. A.15c as

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \quad (4.6)$$

The nucleon field ψ is an isodoublet thus

$$\psi = \begin{bmatrix} \psi_p \\ \psi_n \end{bmatrix} \quad (4.7)$$

where

- ψ_p is the proton field
- ψ_n is the neutron field and each field consists of 4×1 Dirac spinors making ψ an 8×1 component spinor.

Next we explain the physical meaning behind the terms introduced in the Lagrangian.

4.1 Self-interactions

The self-interactions extend the Lagrangian of the scalar field by adding self-interactions to the sigma meson field and in doing this it helps in improving the $\sigma - \omega$ model by giving two more nuclear matter properties being the compression modulus and the reduced mass at saturation[1]. The self-interactions of the scalar meson field are included so they improve the long range attraction in the nuclear force produced by the model [22]. The self-interaction terms are given by

$$U(s) = \frac{\kappa}{3!} (g_s \phi(x))^3 + \frac{\lambda}{4!} (g_s \phi(x))^4, \quad (4.8)$$

where

- κ and λ are the coupling strengths.

4.2 Introducing the isospin force

In this section the isospin interaction is included to distinguish between neutrons and protons; the isospin interaction will serve the purpose of improving on the $\sigma - \omega$ model which assigned the same mass (939 MeV) to them. The project is dealing with a neutron star and thus isospin will have an effect on its equation of state[1]. The $\sigma - \omega$ model can be improved by adding self-interactions to the $\sigma - \omega$ model. The addition of the charged isospin vector rho meson helps to distinguish between protons and neutrons since they have different isospin projections. The charged vector rho meson is a three component isovector given by[1]

$$\mathbf{b}^\mu = (b_1^\mu, b_2^\mu, b_3^\mu). \quad (4.9)$$

The charged rho meson is a vector meson and is similar to the ω vector meson in that only the three component of the $\mathbf{b}^\mu$ field has an exact mean value and the Lorentz three current will vanish in the ground state similar. It will also obey the Klein-Gordon Equation (C.9)[1]. The rho meson are described as a triplet of mesons that form an isospin triplet with the isospin current[1].

The equations of motion can be given by applying the Euler-Lagrange Equation (B.9) to the Lagrangian to give

$$\partial_\mu \partial^\mu \phi(x) + m_s^2 \phi(x) + \frac{\kappa}{2!} g_s^3 \phi^2(x) + \frac{\lambda}{3!} g_s^4 \phi^3(x) = g_s \bar{\psi}(x) \psi(x) \quad (4.10a)$$

$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu(x) = g_v \bar{\psi}(x) \gamma^\nu \psi(x) \quad (4.10b)$$

$$\partial_\mu \mathbf{b}^{\mu\nu} + m_\rho^2 \mathbf{b}^\nu = \frac{g_\rho}{2} \bar{\psi}(x) \gamma^\nu \boldsymbol{\tau} \psi(x) \quad (4.10c)$$

$$\left[\gamma^\mu (i\partial_\mu - g_v V_\mu(x)) - \frac{g_\rho}{2} \boldsymbol{\tau} \cdot \boldsymbol{\gamma}^\mu \cdot \mathbf{b}_\mu - (M - g_s \phi(x)) \right] \psi(x) = 0. \quad (4.10d)$$

4.3 Relativistic mean-field theory

A continuation of the mean-field theory from Sec. 3.3 is given in this model. The meson fields are replaced by their expectation values again as in Sec. 3.3[10].

$$\phi \longrightarrow \langle \phi(x) \rangle = \phi_0 \quad (4.11a)$$

$$V_\mu \longrightarrow \langle V_\mu \rangle = \delta_{\mu 0} V_0. \quad (4.11b)$$

$$b_\mu \longrightarrow \langle b_\mu \rangle = g^{\mu 0} \delta_{a3} b_0. \quad (4.11c)$$

In the ground state the expectation value of the raising and lowering operators would be zero [1]. The first two components of $\mathbf{b}^\mu$ can be written in terms of the raising and lowering operators of

the charged rho meson fields described in Eq. (4.5). Thus in the uniform static ground state only the $\mathbf{b}_3^\mu$ term in $\mathbf{b}$ will be non vanishing[23]. The Lorentz three current part (spatial components) also vanishes which gives $\mathbf{b}_0$ because we are in the rest frame of the mean-field so there is no preferred direction.

In addition, just as in Sec. 3.3, the baryon sources are treated to the same normal order operation so they are replaced by the expectation values since when we are dealing with the mean-field it means we take the average of the ensemble that is prepared in the system

$$\bar{\psi}(x)\psi(x) \longrightarrow \langle \Phi | : \bar{\psi}(x)\psi(x) : | \Phi \rangle = \langle \bar{\psi}\psi \rangle \quad (4.12a)$$

$$\bar{\psi}(x)\gamma^\mu\psi(x) \longrightarrow \langle \Phi | : \bar{\psi}(x)\gamma^\mu\psi(x) : | \Phi \rangle = \langle \bar{\psi}\gamma^0\psi \rangle, \quad (4.12b)$$

$$\bar{\psi}(x)\gamma^\mu\tau_a\psi(x) \longrightarrow \langle \Phi | : \bar{\psi}(x)\gamma^\mu\tau_a\psi(x) : | \Phi \rangle = \langle \bar{\psi}\gamma^0\tau_3\psi \rangle. \quad (4.12c)$$

In the RMF approximation the Lagrangian (5.14) simplifies to

$$\begin{aligned} \mathcal{L}_{RMF} &= \bar{\psi}(x) \left[\gamma^\mu (i\partial_\mu - g_v V_0(x)) - \frac{g_\rho}{2} \tau_3 \gamma^0 b_0 - (M - g_s \phi_0) \right] \psi(x) \\ &- \frac{1}{2} m_s^2 \phi_0^2 - \frac{\kappa}{3!} (g_s \phi_0)^3 - \frac{\lambda}{4!} (g_s \phi_0)^4 + \frac{1}{2} m_\omega^2 V_0^2 \\ &+ \frac{1}{2} m_\rho^2 b_0^2. \end{aligned} \quad (4.13)$$

which in turn simplifies the equations of motion (4.10) to

$$g_s \phi_0 = \frac{g_s^2}{m_s^2} \left[\langle \bar{\psi}\psi \rangle - \frac{\kappa}{2} (g_s \phi_0)^2 - \frac{\lambda}{6} (g_s \phi_0)^3 \right] \quad (4.14a)$$

$$g_v V_0 = \frac{g_v^2}{m_\omega^2} \left[\langle \psi^\dagger \psi \rangle \right] \quad (4.14b)$$

$$g_\rho b_0 = \frac{g_\rho^2}{m_\rho^2} \left[\frac{1}{2} \langle \psi^\dagger \tau_3 \psi \rangle \right] \quad (4.14c)$$

$$0 = \left[i\gamma^\mu \partial_\mu - g_v \gamma^0 V_0 - \frac{g_\rho}{2} \tau_3 \gamma^0 b_0 - (M - g_s \phi) \right] \psi. \quad (4.14d)$$

The expectation value of the Lagrangian $\langle \mathcal{L} \rangle$ is calculated, the energy density and pressure and it is given by

$$\langle \mathcal{L} \rangle = -\frac{1}{2} m_s^2 \phi_0^2 - \frac{\kappa}{3!} (g_s \phi_0)^3 - \frac{\lambda}{4!} (g_s \phi_0)^4 + \frac{1}{2} m_\omega^2 V_0^2 + \frac{1}{2} m_\rho^2 b_0^2. \quad (4.15)$$

The energy density and the pressure is given by the general expression in Eq. (3.15a) and (3.15b)

$$\epsilon = -\langle \mathcal{L} \rangle + \langle \bar{\psi} \gamma_0 k_0 \psi \rangle \quad (4.16a)$$

$$P = \langle \mathcal{L} \rangle + \frac{1}{3} \langle \bar{\psi} \gamma_i k_i \psi \rangle. \quad (4.16b)$$

The equations for the mean values of the rho meson and Dirac equations will be given by

$$g_\rho b_3^0 = \frac{1}{2} \left(\frac{g_\rho}{m_\rho} \right)^2 \langle \bar{\psi} \gamma^0 \tau_3 \psi \rangle = \left(\frac{g_\rho}{m_\rho} \right)^2 \frac{1}{2} (\rho_p - \rho_n), \quad (4.17a)$$

$$g_\rho b_3^i = \frac{1}{2} \left(\frac{g_\rho}{m_\rho} \right)^2 \langle \bar{\psi} \gamma^i \tau_3 \psi \rangle = 0 \text{ and,} \quad (4.17b)$$

$$\left[\gamma^\mu (i\partial_\mu - g_v V_0(x)) - \frac{g_\rho}{2} \tau_3 \gamma^0 b_0 - (M - g_s \phi_0) \right] \psi(x) = 0. \quad (4.17c)$$

The ρ_p and ρ_n terms are the isospin baryon densities of the proton and neutron with $\rho_{p,n}$ as the baryon density[1] given as

$$\rho_n = \frac{1}{3\pi^2} k_n^3 \quad (4.18a)$$

$$\rho_p = \frac{1}{3\pi^2} k_p^3 \quad (4.18b)$$

The Dirac eigenvalues are obtained by rationalizing the Dirac operator as in Sec. 3.3 to give

$$e(\mathbf{k}) = g_v V_0 + \frac{1}{2} \tau_3 g_\rho b_0 + \sqrt{k^2 + m^{*2}}. \quad (4.19)$$

where

- τ_3 here and from this point onwards represent the eigenvalues of the τ_3 matrix Eq. (4.6) and have values of +1 for protons and -1 for neutrons.

From the above the expression $\langle \psi^\dagger \psi \rangle$ will be different. The Fermi momenta of the protons will be different from those of the neutrons. Hence the particle density will be

$$\begin{aligned} \langle \psi^\dagger \psi \rangle &= \sum_s \int_0^{k_p} \frac{d\mathbf{k}}{(2\pi)^3} + \sum_s \int_0^{k_n} \frac{d\mathbf{k}}{(2\pi)^3} \\ &= \frac{1}{3\pi^2} k_p^3 + \frac{1}{3\pi^2} k_n^3 \\ &= \rho_p + \rho_n. \end{aligned} \quad (4.20)$$

The same will apply to the values for the expectation value in the pressure and the expression $\langle \bar{\psi}\psi \rangle$. There will be an integral for the protons and another for the neutrons. The expression in the energy density expression will vary slightly and will be given by

$$\begin{aligned}
\langle i\bar{\psi}\gamma_0\partial_0\psi \rangle &= \sum_s \int \frac{d\mathbf{k}}{(2\pi)^3} e(\mathbf{k}) \Theta[\mu - e(\mathbf{k})] \\
&= \sum_s \int \frac{d\mathbf{k}}{(2\pi)^3} (g_v V_0 + \frac{1}{2}g_\rho \gamma^0 \tau_3 b_0 + \sqrt{\mathbf{k}^2 + m^{*2}}) \Theta[\mu - e(\mathbf{k})] \\
&= g_v V_0 \rho + \frac{1}{2}g_\rho b_0 (\rho_p - \rho_n) + \sum_s \int \frac{d\mathbf{k}}{(2\pi)^3} \sqrt{\mathbf{k}^2 + m^{*2}} \Theta[\mu - e(\mathbf{k})] \\
&= g_v V_0 \rho + \frac{1}{2}g_\rho b_0 (\rho_p - \rho_n) \\
&\quad + \frac{1}{4\pi^2} \int_0^{k_p} d\mathbf{k} \sqrt{\mathbf{k}^2 + m^{*2}} + \frac{1}{4\pi^2} \int_0^{k_n} d\mathbf{k} \sqrt{\mathbf{k}^2 + m^{*2}}. \tag{4.21}
\end{aligned}$$

4.4 Parameter Sets

The parameter sets considered in the thesis will be QHD-I which are taken from [10] and the NL3 parameter set which are taken from [24]. This work follows the notation given in [10].

To best describe the bulk properties of nuclear matter the coupling constants of the parameter sets are used. This is done by fitting the coupling constants of the different parameter sets to these properties. The parameter sets differ in their coupling constants. There are many different parameter sets that are fitted to best describe dense nuclear matter. For this project the QHD-I and NL3 parameter sets will be used. The NL3 parameter set is more advanced and improves on the QHD-I lagrangian as it accounts for contributions of the rho vector meson and self interactions of the scalar field. It has been found to better describe the bulk properties of nuclear matter.

The table 4.1 gives the various coupling constants while table 4.2 gives the masses used in the two parameter sets.

Model	g_s^2	g_v^2	g_ρ^2	κ	λ
QHD-I	109.6000	190.4000	0.00	0.00	0.00
NL3	104.3871	165.5854	79.6000	3.8599	-0.01591

Table 4.1: Coupling constants for QHD-I and NL3 with all of them being dimensionless except for κ which has units of MeV (Walecka Convention)

Model	M_p	M_n	m_s	m_ω	m_ρ
QHD-I	939	939	520	783	763
NL3	939	939	508.1940	782.5	763

Table 4.2: Particle masses for QHD-I and NL3 in MeV for the Walecka Convention

4.5 Equation of state

Using the equations of motion for the different meson fields in Eq. (4.14), since now the expectation values have been evaluated the equations of motion simplify to

$$g_s \phi_0 = \frac{g_s^2}{m_s^2} \left(\frac{1}{\pi^2} \int_0^{k_p} dk \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}} + \frac{1}{\pi^2} \int_0^{k_n} dk \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}} \right) + \frac{g_s^2}{m_s^2} \left(-\frac{\kappa}{2} (g_s \phi_0)^2 - \frac{\lambda}{6} (g_s \phi_0)^3 \right) \quad (4.22a)$$

$$g_v V_0 = \frac{g_v^2}{m_\omega^2} (\rho_p + \rho_n) \quad (4.22b)$$

$$g_\rho b_0 = \frac{g_\rho^2}{m_\rho^2} \left(\frac{1}{2} (\rho_p - \rho_n) \right). \quad (4.22c)$$

The equation of state can now be constructed to give the pressure and energy density as

$$\begin{aligned} \epsilon = & \frac{1}{2} m_s^2 \phi_0^2 + \frac{\kappa}{3!} (g_s \phi_0)^3 + \frac{\lambda}{4!} (g_s \phi_0)^4 - \frac{1}{2} m_\omega^2 V_0^2 \\ & - \frac{1}{2} m_\rho^2 b_0^2 + g_v V_0 (\rho_n + \rho_p) + \frac{1}{2} g_\rho b_0 (\rho_p - \rho_n) \\ & + \frac{1}{\pi^2} \int_0^{k_p} dk k^2 \sqrt{k^2 + m^{*2}} + \frac{1}{\pi^2} \int_0^{k_n} dk k^2 \sqrt{k^2 + m^{*2}} \end{aligned} \quad (4.23a)$$

$$\begin{aligned} P = & -\frac{1}{2} m_s^2 \phi_0^2 - \frac{\kappa}{3!} (g_s \phi_0)^3 - \frac{\lambda}{4!} (g_s \phi_0)^4 + \frac{1}{2} m_\omega^2 V_0^2 + \frac{1}{2} m_\rho^2 b_0^2 \\ & + \frac{1}{3\pi^2} \int_0^{k_p} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}} + \frac{1}{3\pi^2} \int_0^{k_n} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}}. \end{aligned} \quad (4.23b)$$

In the next chapter all that has been learnt from this chapter is applied to the interior of the star as the different particles that make up the neutron star are discussed.

5. Exotic neutron star matter

In this chapter assumptions of equilibrium conditions and constraints for the different models of the neutron star interior that are considered in this work will be introduced and discussed.

5.1 Beta-equilibrium

This section will further extend my model of neutron star matter by discussing beta-equilibrium and charge neutrality. Free neutrons are unstable and have a short half life of 14 minutes compared to a neutron star which has a longer lifespan. Free neutrons beta-decay to protons, electrons and antineutrinos [5]. Neutron stars have much longer lifespans that are greater which is why inside the star a significant number of neutrons will beta-decay to give protons, electrons and antineutrinos. However the neutron star is considered an equilibrated system. Therefore the protons and electrons (that attain relativistic energies due to high densities) can also combine to give neutrons and neutrinos. The decay process for a neutron and electron capture process can be represented in the following way

$$n \longrightarrow p + e^- + \bar{\nu}_e \quad (5.1a)$$

$$p + e^- \longrightarrow n + \nu_e. \quad (5.1b)$$

where the first process (Eq. (5.1a)) describes the beta-decay of a neutron to give a proton, electron and an antineutrino. Eq. (5.1b) describes the inverse process.

The two processes will go on inside the star driven by high energies and excess neutrons on inside the star until a state is reached where as many neutrons decay into protons and electrons there is an equal number of electron captures from protons, resulting in new neutrons. This state is called the beta-equilibrated state and it is a state where the energy density of the system will be a minimum. The chemical potential μ_x for a system is given by

$$\mu_x = \frac{\partial \epsilon}{\partial \rho_x}. \quad (5.2)$$

In the beta-equilibrium state the effect of neutrinos will be ignored because their contribution to the equation of state is negligible when considering uniform nuclear matter which means the

state can be mathematically written as

$$\mu_n = \mu_p + \mu_e. \quad (5.3)$$

The chemical potentials Eq. (4.19) for the different species can be written in terms of the energy eigenvalues as

$$\mu_n = \sqrt{k_n^2 + m^{*2}} + g_v V_0 - \frac{1}{2} g_\rho b_0 \quad (5.4)$$

$$\mu_p = \sqrt{k_p^2 + m^{*2}} + g_v V_0 + \frac{1}{2} g_\rho b_0 \quad (5.5)$$

$$\mu_e = \sqrt{k_e^2 + m^2}. \quad (5.6)$$

5.2 Muons

In ultra-dense matter it is energetically favourable for electrons to populate muon states if they attain ultra-relativistic energies. Muons and electrons have the same charge but muons have a larger mass ($m_\mu = 105.658389$ MeV whilst the electron mass is $m_e = 0.5109989461$ MeV) [25]. The muon states can only be populated once electrons reach energies comparable to the muon mass. Muons are very unstable particles and decay to electrons for low densities. The decay process can be written as

$$\mu^- \longrightarrow e^- + \bar{\nu}_e + \nu_\mu. \quad (5.7)$$

The above process in Eq. (5.7) implies that

$$\mu_e = \mu_\mu \quad (5.8)$$

where the energy eigenvalues for the muons can be given by

$$\mu_\mu = \sqrt{k_\mu^2 + m_\mu^2}. \quad (5.9)$$

5.3 Charge neutrality

Another equilibrium condition is charge neutrality. To enforce charge neutrality the number of protons must be equal to the number of electrons.

From Eq. (3.21) the number density of a particle ρ_x can be expressed in terms of the parti-

cle's Fermi momenta

$$k_{F_x} = (3\pi^2 \rho_x)^{\frac{1}{3}}. \quad (5.10)$$

and in terms of the Fermi momenta of the different particles it can be written as

$$k_{F_e} = k_{F_p}. \quad (5.11)$$

If the muon states are populated, to enforce charge neutrality the number of protons must be equal to the combined number of the electrons and muons. This statement can be written mathematically as

$$k_{F_p} = (k_{F_e}^3 + k_{F_\mu}^3)^{\frac{1}{3}}. \quad (5.12)$$

5.4 General neutron star equilibrium

When the condition stipulated by Eq. (5.3) for beta-equilibrium, (5.8) for including muons and (5.12) for charge neutrality are satisfied then a neutron star can be said to be in general equilibrium.

The corresponding neutron star equation of state can be derived from the one in Chapter 4 by adding the leptons as well as the equilibrium conditions Eq. (5.3) for beta equilibrium, (5.8) for the inclusion of muons if the chemical potential of the electrons exceeds the rest mass of the muons and (5.12) for charge neutrality to ensure that the neutron star is charge neutral else it will be ripped apart by the Coulombs force which is stronger than the gravity that binds the

star to give the energy density and the pressure of the system as

$$\begin{aligned}
\epsilon = & \frac{1}{2}m_s^2\phi_0^2 + \frac{\kappa}{3!}(g_s\phi_0)^3 + \frac{\lambda}{4!}(g_s\phi_0)^4 - \frac{1}{2}m_\omega^2V_0^2 \\
& - \frac{1}{2}m_\rho^2b_0^2 + g_vV_0(\rho_n + \rho_p) + \frac{1}{2}g_\rho b_0(\rho_p - \rho_n) \\
& + \frac{1}{\pi^2} \int_0^{k_{Fp}} dk k^2 \sqrt{k^2 + m^{*2}} + \frac{1}{\pi^2} \int_0^{k_{Fn}} dk k^2 \sqrt{k^2 + m^{*2}} \\
& + \sum_{l=e,\mu} \frac{1}{\pi^2} \int_0^{k_{Fl}} dk k^2 \sqrt{k^2 + m_l^2}
\end{aligned} \tag{5.13a}$$

$$\begin{aligned}
P = & -\frac{1}{2}m_s^2\phi_0^2 - \frac{\kappa}{3!}(g_s\phi_0)^3 - \frac{\lambda}{4!}(g_s\phi_0)^4 + \frac{1}{2}m_\omega^2V_0^2 \\
& + \frac{1}{2}m_\rho^2b_0^2 \\
& + \frac{1}{3\pi^2} \int_0^{k_{Fp}} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}} + \frac{1}{3\pi^2} \int_0^{k_{Fn}} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}} \\
& + \sum_{l=e,\mu} \frac{1}{3\pi^2} \int_0^{k_{Fl}} dk \frac{k^4}{\sqrt{k^2 + m_l^2}}.
\end{aligned} \tag{5.13b}$$

where $l = e, \mu$ stands for the leptons being the electrons and muons.

5.5 The inclusion of lambda hyperons in the neutron star EoS

In Sec. 3.1 the properties for nuclear matter were described to extend our description of relativistic nuclear matter. The $\sigma - \omega$ model was then used as an introduction into relativistic nuclear matter. Chapter 4 then modified the $\sigma - \omega$ model by describing a theory so as to align with the properties of nuclear matter at saturation and thus ensuring that the equation of state is normalised and assures extrapolation to higher densities is reasonable.

Neutron stars are modelled as nuclear matter systems that are in general equilibrium (beta equilibrated and charge neutral with muons). Due to the large densities in the star there is allowance for new hadronic degrees of freedom and a possible hadronic degree of freedom is the inclusion of hyperons (strange baryons) [1, 7]. Generally hyperons have an effect of softening the neutron star equation of state and thus reducing the maximum mass of the star[8].

The system is at high densities and the other particles will attain relativistic energies/quantum states. When the baryon density exceeds a certain point (threshold) in the system it will be energetically favourable for the nucleons to populate the low energy hyperon states which relieve

the fermi pressure exerted by the baryons on the system[13]. This has the effect of softening the neutron star equation of state and thus the maximum mass too.

As a first approximation for such a dense matter system only the lambda (Λ^0) hyperon will be considered in this work. A more complete approach will be to include all hyperons [1, 7], but the goal here is to simply establish some of the qualitative properties and expand to include other hyperons at a later stage. Therefore the model discussed here will be for an interacting system with the Λ^0 hyperon included in the model as a first approximation for hyperons in the neutron star interior. Next an extension of the models discussed in of Chapter 4 is expanded by including the lambda hyperons. After which the general equilibrium EoS including lambda hyperons will be described.

5.5.1 Appearance of lambda hyperons in the system

The Fermi momentum and the Fermi energy are dependant on the baryon density so as it increases the two quantities also increase. As the Fermi energy of each baryon species (k_{BF}) increases it will surpass the bare mass of the hyperons (for interacting systems chemical potential at $k_{BF} = 0$) of the heavier baryons and this will decrease the energy and pressure (equation of state) of the system due to the conversion of some of the occupied baryons to other species[7]. The equation of state depends on the baryon density through the Fermi momenta. The baryons considered in this project is the lambda hyperon Λ^0 (its bare mass is 1115 MeV so this project will be considering energies upwards of 1000 MeV) [1].

Hyperons are essentially highly energetic heavy nucleons (protons and neutrons) in the sense that they are fermions with rest mass energies over 1000 MeV[7]. They are relativistic and move at speeds near to the speed of light and are also produced in the timeframe it takes a particle travelling at the speed of light to cross the diameter of a subatomic particle[1]. The hyperon states can only be populated if the energies of the nucleons exceed the chemical potential of the hyperons at $k_{BF} = 0$. The lightest hyperon is the Λ^0 hyperon and it is populated when highly energetic nucleons (via the reaction $N + N \rightarrow N + \Lambda + K$) exceed it's chemical potential at $k_{nF} = 0$. The bare masses are mentioned in Table 5.5.1. It is important to note that the timescale for the reactions is very small (10^{-10} s) [1].

Therefore the more energetic the nucleons are the more hyperon states can be populated. Highly energetic nucleons mean that the Fermi momenta and baryon density of each baryon species is

Baryon	m (MeV)	charge, q	Isospin projection, I_3	strangeness, s
p	939	1	$\frac{1}{2}$	0
n	939	0	$-\frac{1}{2}$	0
Λ^0	1115	0	0	-1

Table 5.1: List of baryons used in this project the particle masses (in MeV), charge, isospin projection and strangeness.

also high. These states therefore bring about our discussion of dense nuclear matter whose best laboratory is the neutron star. Hyperons form inside the core of neutron star and therefore there is a need to include them in the discussion of relativistic nuclear matter.

To give one an idea of how relativistic these particles are one can use the saturation density (ρ_{sat}) mentioned in Sec. 3.1.

5.6 The lambda hyperon Lagrangian

The Lagrangian discussed in Chapter 4 for matter in general equilibrium is modified further to give a Lagrangian for the model which includes the lambda hyperon in this project and this is given by

$$\begin{aligned}
\mathcal{L} = & \bar{\psi}(x) \left[\gamma^\mu (i\partial_\mu - g_v V_\mu(x)) - \frac{g_\rho}{2} \boldsymbol{\tau} \cdot \mathbf{b}_\mu(x) - (M - g_s \phi(x)) \right] \psi(x) \\
& + \bar{\psi}_\Lambda(x) \left[\gamma^\mu (i\partial_\mu - g_v V_\mu(x)) - (M_\Lambda - g_s \phi(x)) \right] \psi_\Lambda(x) \\
& + \sum_{l=e,\mu} \bar{\psi}_l(x) \left[i\gamma^\mu \partial_\mu - m_l \right] \psi_l(x) \\
& + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi(x) - \frac{1}{2} m_s^2 \phi^2(x) - \frac{\kappa}{3!} (g_s \phi(x))^3 - \frac{\lambda}{4!} (g_s \phi(x))^4 \\
& - \frac{1}{4} V^{\mu\nu} V_{\mu\nu} + \frac{1}{2} m_\omega^2 V^\mu(x) V_\mu(x) \\
& - \frac{1}{4} \mathbf{b}^{\mu\nu} \mathbf{b}_{\mu\nu} + \frac{1}{2} m_\rho^2 \mathbf{b}^\mu(x) \cdot \mathbf{b}_\mu(x)
\end{aligned} \tag{5.14}$$

where

- ψ is the Dirac spinor for the protons and neutrons,
- ψ_Λ is the Dirac spinor for the lambda hyperon, and,
- ψ_l is the spinor for the leptons.

Applying the Euler-Lagrange equations to the Lagrangian gives the Dirac equations for the baryons as

$$\partial_\mu \partial^\mu \phi(x) + m_s^2 \phi(x) = g_s (\bar{\psi}(x)\psi(x) + \bar{\psi}_\Lambda(x)\psi_\Lambda(x)) \quad (5.15a)$$

$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu(x) = g_v (\bar{\psi}(x)\gamma^\nu\psi(x) + \bar{\psi}_\Lambda(x)\gamma^\nu\psi_\Lambda(x)) \quad (5.15b)$$

$$0 = \left[\gamma^\mu (i\partial_\mu - g_v V_0(x)) - \frac{g_\rho}{2} \tau_3 b_0 - (M - g_s \phi_0) \right] \psi(x), \quad (5.15c)$$

$$0 = \left[\gamma^\mu (i\partial_\mu - g_v V_0(x)) - (M_\Lambda - g_s \phi_0) \right] \psi_\Lambda(x). \quad (5.15d)$$

While it has the same charge as the neutron, the difference is in its isospin. Therefore the lambda hyperon will not couple to the isospin meson b_0 , but couples to the V_0 and ϕ_0 mesons in the same way the neutron couples to these mesons. Its chemical potential will also be similar to the neutron chemical potential in Eq. (5.4) given as

$$\mu_\Lambda = \sqrt{k_\Lambda^2 + m_\Lambda^{*2}} + g_v V_0. \quad (5.16)$$

The energy density and pressure for our interacting model including the lambda hyperon will then be given by

$$\begin{aligned} \epsilon = & \frac{1}{2} m_s^2 \phi_0^2 + \frac{\kappa}{3!} (g_s \phi_0)^3 + \frac{\lambda}{4!} (g_s \phi_0)^4 - \frac{1}{2} m_\omega^2 V_0^2 \\ & - \frac{1}{2} m_\rho^2 b_0^2 + g_v V_0 (\rho_n + \rho_p + \rho_\Lambda) + \frac{1}{2} g_\rho b_0 (\rho_p - \rho_n) \\ & + \frac{1}{\pi^2} \int_0^{k_{Fp}} dk k^2 \sqrt{k^2 + m^{*2}} + \frac{1}{\pi^2} \int_0^{k_{Fn}} dk k^2 \sqrt{k^2 + m^{*2}} \\ & + \frac{1}{\pi^2} \int_0^{k_{F\Lambda}} dk k^2 \sqrt{k^2 + m^{*2}} \\ & + \sum_l \frac{1}{\pi^2} \int_0^{k_{Fl}} dk k^2 \sqrt{k^2 + m^2} \end{aligned} \quad (5.17a)$$

$$\begin{aligned} P = & -\frac{1}{2} m_s^2 \phi_0^2 - \frac{\kappa}{3!} (g_s \phi_0)^3 - \frac{\lambda}{4!} (g_s \phi_0)^4 + \frac{1}{2} m_\omega^2 V_0^2 \\ & + \frac{1}{2} m_\rho^2 b_0^2 \\ & + \frac{1}{3\pi^2} \int_0^{k_{Fp}} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}} + \frac{1}{3\pi^2} \int_0^{k_{Fn}} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}} \\ & + \sum_l \frac{1}{3\pi^2} \int_0^{k_{Fl}} dk \frac{k^4}{\sqrt{k^2 + m^2}} \\ & + \frac{1}{3\pi^2} \int_0^{k_{F\Lambda}} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}}. \end{aligned} \quad (5.17b)$$

where Λ stands for the lambda hyperon.

To describe the model for hyperons in a neutron star, the equation of state above will not be enough alone. A set of equilibrium conditions are added as constraints to help describe the model. The next section will introduce the equilibrium conditions for the model described in this thesis.

5.7 General equilibrium with lambda hyperons

In this section since hyperons are now considered the charge neutrality and beta-equilibrium conditions in Sec. 5.1 will have to be modified to give a careful method for arriving at equilibrium for hyperons is given. The baryons considered in this project are the proton, neutron and the lambda hyperon for energies upwards of 1000 MeV.

The equilibrium conditions in Sec. 5.1 were achieved by considering the beta-equilibrium of neutrons, protons and electrons along with the charge neutrality condition $\rho_p = \rho_e$, as the electron density increased muons were added (only when $\mu_e = \mu_\mu$) and this altered the charge neutrality condition to $\rho_p = \rho_\mu + \rho_e$ [1]. In a similar way for nucleons, hyperons are essentially heavier energetic nucleons because they can only form when the chemical potential of the nucleons is higher than the individual hyperon masses [7]. The total baryon conservation and charge neutrality conditions determine the appearance and proportions of these particles inside a neutron star [26]:

$$\sum_B q_B \rho_B + \sum_{l=e,\mu} q_l \rho_l = 0 \quad (5.18)$$

The equilibrium conditions are enforced by considering several coupling constraints such as the baryon density since the lambda hyperon density adds to the total baryon density. The baryon density is conserved. The other constraints are beta-equilibrium and charge neutrality. In an equilibrium state these constraints come in the form of chemical potentials because they are conserved attributes. The equilibrium conditions (chemical potential of the lambda hyperon) depends on the electric charge q_Λ (the lambda hyperon charge is zero) and is given by [7]

$$q_\Lambda = 0 \quad \mu_\Lambda = \mu_n \quad (5.19)$$

Next three different nuclear matter systems will be modelled giving us three different equations of state for a neutron star and a python script will be written in order to compute observable results such as the mass radius relationship for a neutron star.

6. Results and Discussion

The results in this chapter were generated by writing a computer code in the python programming language. The results will be discussed in the context of this work as well as comparing them to the available literature sources.

6.1 Nuclear Properties

In this project two properties of nuclear matter (discussed in Sec. 3.1), the binding energy and saturation density are described using two parameter sets; QHD-I and NL3. The calculations for the binding energy were done using Eq. (3.2). The values shown in table 6.1 are in agreement with what is observed in published results at the saturation density ρ_{sat} Eq. (3.1). It is an important check for my code to reproduce these values as they are fundamental and hence anyone doing the calculation should get exactly the same results for all the nuclear matter properties at saturation.

The published QHD-I parameter sets were from Ref. [1] whilst the NL3 parameter sets were from Ref. [24]. Fig. 6.1 is a binding energy per nucleon of symmetric nuclear matter plot for the QHD-I and NL3 parameter sets. Symmetric nuclear matter is a hypothetical matter composed of equal fractions of protons and neutrons. The binding energy is a minimum at the saturation density which is at a nuclear fermi momentum of (1.30 fm^{-1}) . The binding energy values for both parameter sets are almost the same at saturation and they differ beyond the saturation density showing the high density behaviour of the two parameter sets behave differently. This differences in the binding energy at high densities shows that more information is required to further study nuclear matter at high densities. The NL3 parameter set is more advanced in that it has more coupling constants compared to QHD-I. The NL3 parameter set uses the QHD-I lagrangian density and improves on it by adding the rho mesons and self interactions of the scalar field in order to better describe nuclei with a large isospin asymmetry[23].

Parameter	Published ρ_{sat} (fm^{-3})	Calculated ρ_{sat} (fm^{-3})	Published B/A (MeV)	Calculated B/A (MeV)
QHD-I[1]	0.148	0.148	-15.75	-15.75
NL3[24]	0.148	0.148	-16.25	-16.25

Table 6.1: Comparing the published and calculated saturation density ρ_{sat} and binding energy per nucleon B/A for the two parameter sets.

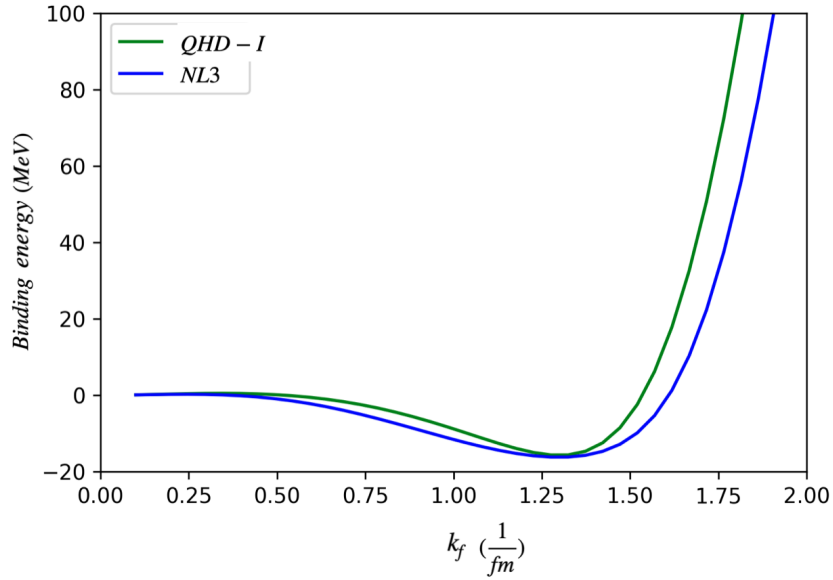


Figure 6.1: Binding energy per nucleon of symmetric nuclear matter of QHD-I and NL3 parameter sets. The binding energy is at a minimum at the saturation density.

6.2 Reduced Mass

Fig 6.2 and 6.3 are QHD-I and NL3 parameter sets plots of the effective reduced mass m^* as a function of the baryon density with the baryon density ranging from $(0.1 - 8.0) \rho_{sat}$ for three different nuclear matter systems being a pure neutron matter system, general equilibrated matter system and finally a general equilibrated matter system with the lambda hyperon included. The plots follow Eq. (3.12) and the self consistency equation Eq. (3.38) which when solved yields the reduced mass (m^*). Plots given in Fig 6.2 and 6.3 gives the reduced mass of each system as a fraction of the nucleon mass, here taken as $M = 939$ MeV as per Eq. 3.3, which is $\frac{m^*}{M}$. In both plots the effective reduced mass of each model is less than one (unity) and this is due to the scalar meson field contribution in the reduced mass calculation in Eq. (3.12).

Comparing (effective reduced mass plots in Fig 6.2 and 6.3) a pure neutron matter model to a general equilibrated matter system there is a decrease in the gradient when the same systems are compared for QHD-I and NL3 parameter sets.

When comparing the general equilibrated matter plots and the general equilibrated with the lambda hyperon included plots the decrease in the gradient for the reduced mass is more in

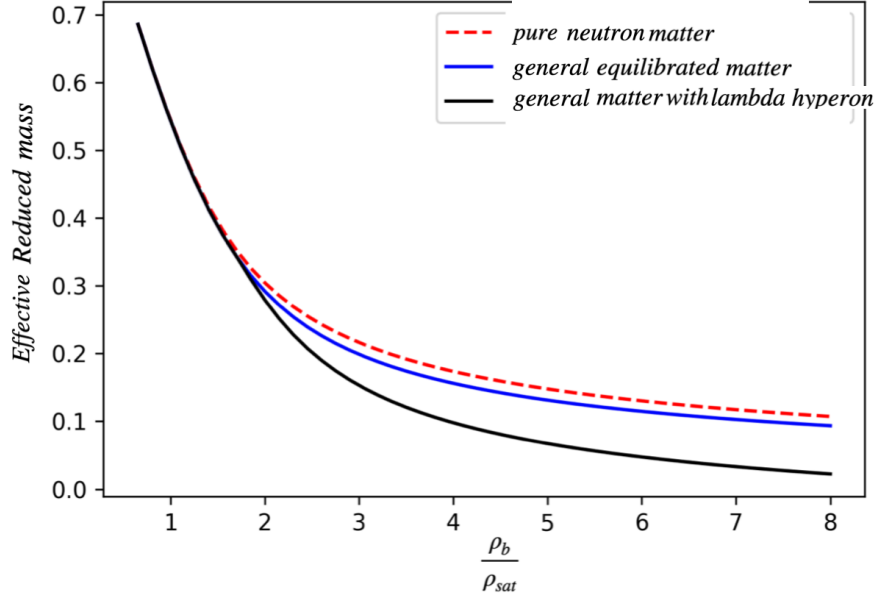


Figure 6.2: The Reduced Mass as a function of the baryon density for general equilibrated matter and general equilibrated matter with the lambda hyperon included. The baryon density ranges from $(0.1 - 8.0)\rho_{sat}$ for QHD-I parameter set.

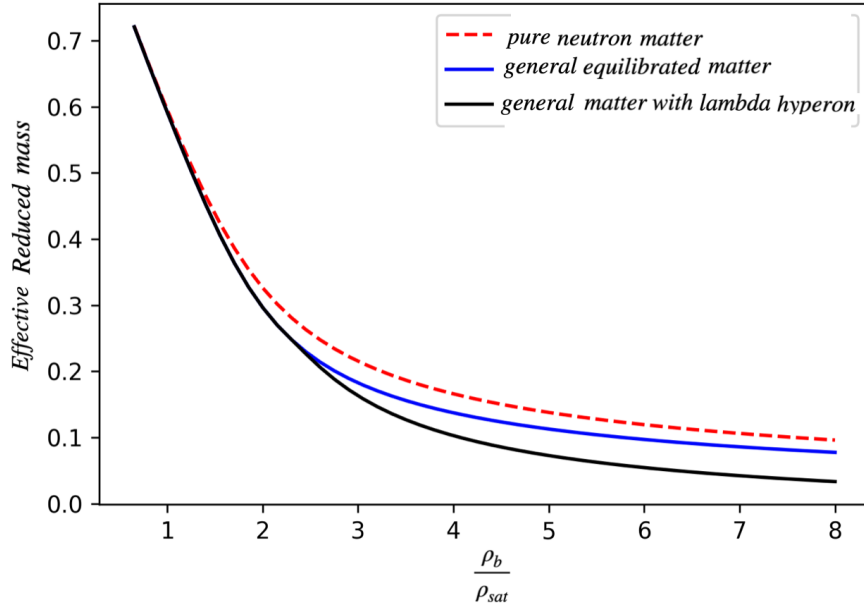


Figure 6.3: The Reduced Mass as a function of the baryon density for general equilibrated matter and general equilibrated matter with the lambda hyperon included. The baryon density ranges from $(0.1 - 8.0)\rho_{sat}$ for NL3 parameter set.

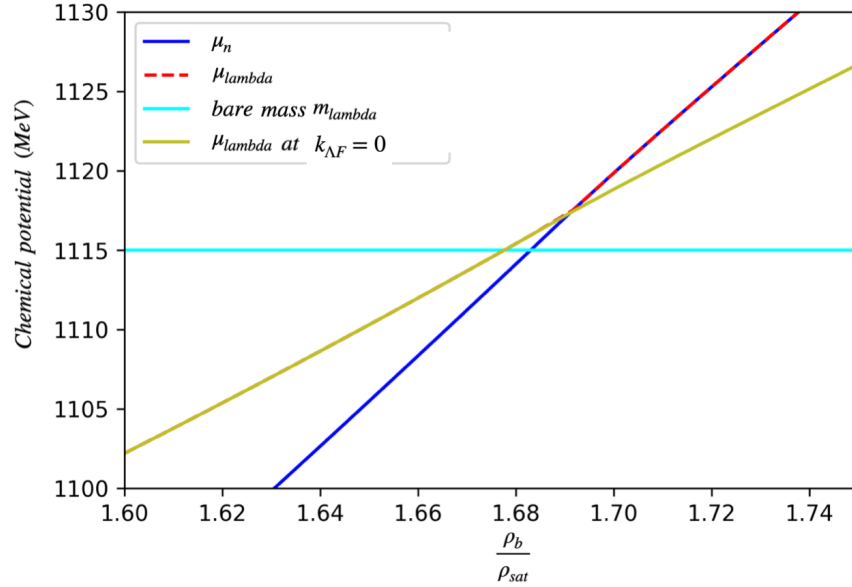


Figure 6.4: the threshold chemical potentials vs baryon density (ranging from $(1.60 - 1.75)\rho_{sat}$) for the lambda hyperon (in red dashed) and the neutrons (in blue) for the QHD-I parameter set. Where the lines two join up is where the hyperons come into the system at a baryon density of $1.69\rho_{sat}$. The aqua line represents the bare mass for the lambda hyperon = 1115 MeV and the yellow line represents the chemical potential of the lambda hyperon at $k_{\Lambda F} = 0$ serving as the threshold energy to be exceeded in order to populate the lambda hyperon states.

QHD-I than in NL3 meaning the model including the lambda hyperon decreases more due to the hyperon states being populated. The reduced mass will influence when the lambda hyperon states begin to be populated.

In the next section a discussion of the chemical potential and how the lambda hyperon states are populated will be discussed to show at what density the lambda hyperon states begin to be populated for each parameter set. In the equation of state discussion later on in Sec. 6.5 it will be shown that the reduced mass has an effect on the behaviour of the equation of state.

6.3 Chemical potentials

In order for hyperon states to be populated, the chemical potentials of the hyperons should exceed a certain threshold energy. For non-interacting system it will be the bare hyperon mass and the hyperon states will be populated once $\mu_{\Lambda} > M_{\Lambda}$. For interacting system the chemical potential is modified and thus the lowest energy state of the hyperons will be when $k_{\Lambda F} = 0$; thus from Eq. (5.16) implies that the lambda hyperons will populate the system when the chemical

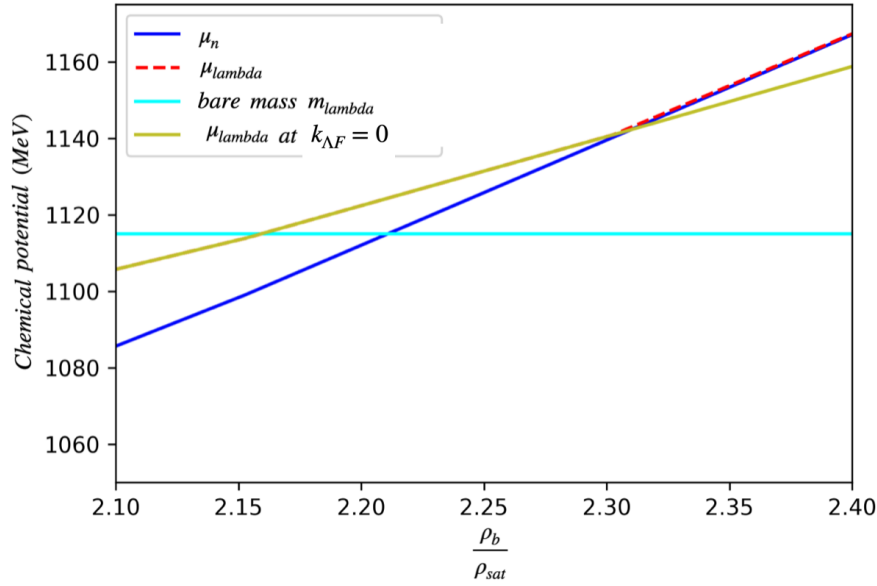


Figure 6.5: the threshold chemical potentials vs baryon density (ranging from $(2.10 - 2.40)\rho_{sat}$) for the lambda hyperon (in red dashed) and the neutrons (in blue) for the NL3 parameter set. Where the lines two join up is where the hyperons come into the system at a baryon density of $2.31\rho_{sat}$. The aqua line represents the bare mass for the lambda hyperon = 1115 MeV and the yellow line represents the chemical potential of the lambda hyperon at $k_{\Lambda F} = 0$ serving as the threshold energy to be exceeded in order to populate the lambda hyperon states.

potential is

$$\mu_{\Lambda}(k_{F\Lambda} = 0) = m_{\Lambda}^{*2} + g_v V_0 \quad (6.1)$$

Fig 6.4 and 6.5 are plots of the threshold chemical potentials for neutrons (blue line) and the lambda hyperon (in red dashed) for baryon density ranges from $(1 - 3)\rho_{sat}$ for QHD-I and NL3 respectively. The area of interest (because in the reduced plot in Fig 6.2 and 6.3 this density range is the density where the plots for the three models begin to diverge) in this plot is at a baryon density between $(1 - 3)\rho_{sat}$ so as to see when the lambda hyperon chemical potential joins up with the neutron chemical potential. The expected behaviour given in [7] is that as the baryon density increases the lambda hyperon states should be populated following Eq. 5.19 where the chemical potential of the neutrons should exceed the chemical potential of the lambda

hyperon at $k_{\Lambda F} = 0$ (which happens at Eq. (6.1)). The lambda hyperon states begin to be populated when the chemical potential of the neutrons (Eq. (5.4)) exceeds the chemical potential of the lambda hyperon at $k_{\Lambda F} = 0$ (Eq. (6.1)).

In Fig 6.4 and 6.5 where the line representing the chemical potential of the lambda hyperon μ_{Λ} joins up with the chemical potential of the neutrons μ_n represents the density where the hyperons states start being populated in the system as discussed in Chapter 5 and Eq. (5.19). Else if the hyperon states were not being populated the chemical potential of the lambda hyperon would not exceed the yellow line representing the chemical potential of the lambda hyperon at $k_{\Lambda F} = 0$. The lambda hyperon states begin to be populated at a baryon density of $1.69 \rho_{sat}$ for QHD-I and $2.31 \rho_{sat}$ for NL3 parameter set. The differences in the density at which the lambda hyperon states begin to be populated for the two parameter sets is expected looking at the reduced plot in Fig 6.2 and 6.3. The plots representing the three systems began to diverge at different densities for both QHD-I and NL3 parameter sets. Looking at the differences in the high density behaviour of the binding energies for the two parameter sets in Fig 6.1 and also the differences in the reduced masses in Fig 6.2 and 6.3 shows the difference in the high density behaviour of the two parameter sets. The reduced mass and the vector meson which contributes to Eq. (6.1) will influence when the lambda hyperon states begin to be filled as it contributes in Eq. (6.1) and the behaviour of the reduced mass plot showed that the lambda hyperon states are populated earlier in QHD-I compared to NL3. From the chemical potential plots Fig 6.4 and 6.5 the lambda hyperon states are indeed populated earlier in QHD-I ($1.69 \rho_{sat}$) compared to NL3 ($2.31 \rho_{sat}$).

The chemical potentials plots in Fig 6.4 and 6.5 showed the baryon density at which the lambda hyperon states began to be populated in both parameter sets. The densities at which the lambda hyperon states were filled are different for the two parameter sets. This will imply that the fractions of the particles for each parameter sets will also be different. Next we look at the fractions of particles in the model to look at the fractions the different particles occupy for each parameter set.

6.4 Particle Fractions

Fig 6.6 and 6.7 are particle fractions plots for the general equilibrium with the inclusion of the lambda hyperon nuclear matter system for a density range of $(0.1 - 8.0) \rho_{sat}$ for QHD-I and NL3 respectively. The particles in this model are discussed in Sec. 5.6 and they are protons,

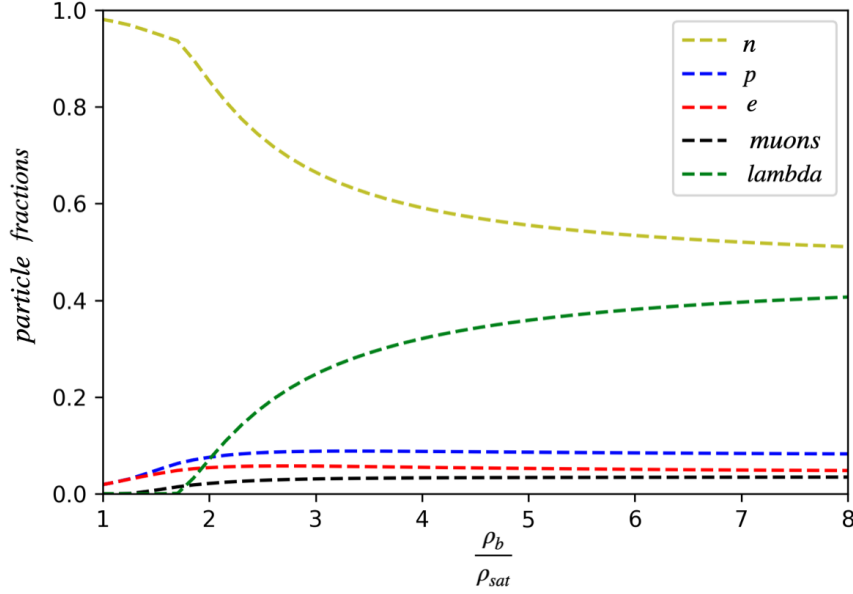


Figure 6.6: Particle fractions vs $\frac{\rho_b}{\rho_{sat}}$ (expressed as a multiple of the saturation density) for neutrons (n), electrons (e), protons (p), muons, and the lambda hyperon. The lambda hyperon appears at a density above the ρ_{sat} of $1.69\rho_{sat}$ for QHD-I parameter set for a system in general equilibrium with the lambda hyperon included.

neutrons, electrons, muons, and the lambda hyperon. The particle fractions plots show that as the baryon density increases the fractions of neutrons decrease and the other particle fractions gradually increase due to beta-equilibrium and charge neutrality conditions constraining the system (Eq. (5.3), (5.8), (5.12)). The lambda hyperon states (represented by the green line) begin to be populated at a baryon density of $1.69 \rho_{sat}$ for QHD-I and $2.31 \rho_{sat}$ for NL3 parameter set. The lambda hyperon states begin to be populated at a low density for QHD-I thus allowing more lambda hyperon states to be filled compared to NL3 where they begin to be populated at $2.31 \rho_{sat}$. This is in agreement with the plots in Fig 6.2 and 6.3 showing that the lambda hyperon states decrease the gradient of the reduced mass more in QHD-I than in NL3.

Fig 6.8 and 6.9 takes a closer look at the particle fractions in Fig 6.6 and 6.7 respectively but focusing only on the leptons and the protons and this is compared to a model where the lambda hyperon states are present in the system. The plots in Fig 6.8 and 6.9 shows that the fractions of particles (leptons and proton) reduce when comparing a general equilibrated system to a general equilibrium system with the lambda hyperon populated. The decrease in other particle fractions is due to the equilibrium conditions (Eq. (5.3), (5.8), (5.12)) constraining the system. The line representing the hyperons is steeper in QHD-I compared to NL3 showing more hyperon states

are populated in QHD-I than in NL3 since the QHD-I lambda hyperon fractions are larger. The difference in QHD-I and NL3 is due to factors such as the contribution to from the vector meson contribution in Eq. (6.1), the contribution to Eq. (6.1) due to the reduced mass as it influences when hyperons appear in the system. The QHD-I plot (Fig 6.6) is steeper and larger because the QHD-I reduced mass decreases more when going from a general equilibrated matter system to a general equilibrated matter with the lambda hyperon included system compared to the NL3 parameter set. This means that more lambda hyperon states are being filled in QHD-I than in NL3 due to this difference in the reduced mass and this is why the fractions of the lambda hyperon particles are more in QHD-I than in NL3.

In the next section a discussion of the different equations of state for the neutron star is given as the difference in the particle fractions due to the reduced mass will affect the neutron star equation of state.

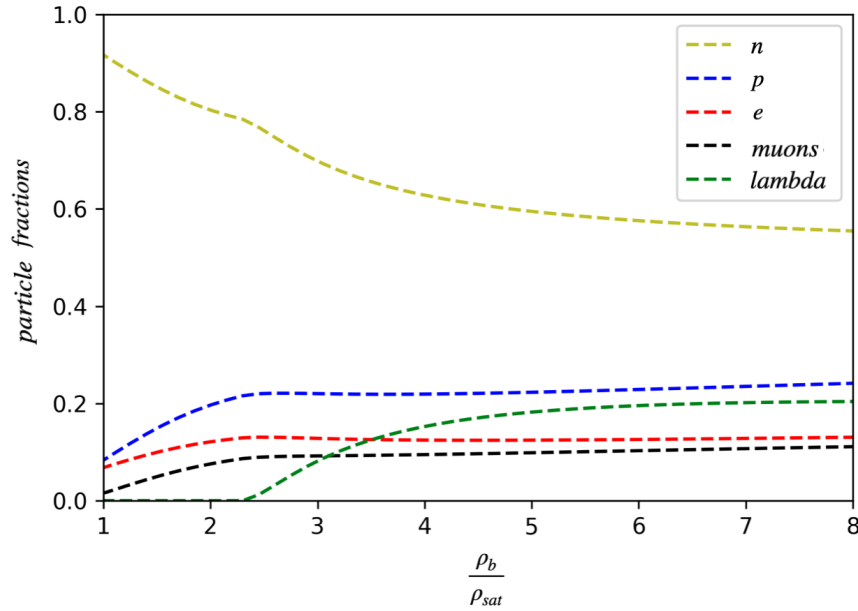


Figure 6.7: Particle fractions vs $\frac{\rho_b}{\rho_{sat}}$ (expressed as a multiple of the saturation density) for neutrons (n), electrons (e), protons (p), muons, and the lambda hyperon. The lambda hyperon appears at a density above the ρ_{sat} of $2.31\rho_{sat}$ for NL3 parameter set for a system in general equilibrium with the lambda hyperon included.

6.5 Neutron star equation of state

The equation of state for a neutron star is important to note as it gives us information on whether an equation of state is stiff (energy density increases rapidly when pressure is increased and such a material is harder to compress as it offers more support against gravity) or soft (gradual increase in the energy density as the pressure increases and is easily compressed by gravity). A stiff equation of state represents a star of large radius and mass as it is harder to compress. A soft equation of state represents a star of smaller radius. For a neutron star there are different equations of states and data from these different equations of states can be compared by plotting a mass radius relationship.

For this project three nuclear matter systems were considered meaning three different equations of state for a neutron star, to derive results such as the mass radius relationship for a neutron star. The first nuclear matter model is assuming the star is composed entirely of neutrons (pure neutron matter system) which obey the equilibrium conditions. Another model to be used is a general equilibrated matter system (constrained by Eq. (5.3 for the chemical potentials), (5.8 for

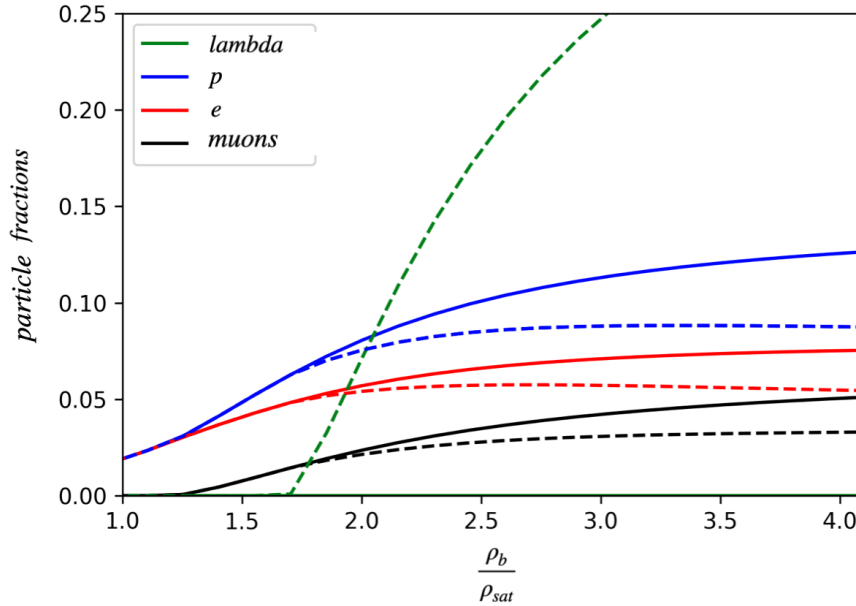


Figure 6.8: A close look at Fig 6.6 with the area of interest showing only the particle fractions for protons, electrons and muons before hyperons appear in the system and after adding hyperons for QHD-I. The solid lines represent general equilibrated matter and the dashed lines represents general equilibrated matter with the lambda (Λ) hyperon included.

the muons), (5.12 for charge neutrality)). Finally the last system will be a general equilibrium with the lambda hyperon included in the system in order to observe the effect the inclusion of the lambda hyperon have on the equation of state.

In Fig 6.10 and 6.11 we have plots of the pressure as a function of the energy density for three nuclear matter systems for a baryon density range of $(0.1 - 8.0) \rho_{sat}$ for QHD-I and NL3 parameter sets respectively.

6.5.1 Pure neutron matter

The first case is to assume the star is made up entirely of pure neutrons. The equation of state for pure neutron matter is given by the dashed red lines in Fig 6.10 and 6.11. The behaviour of the equation of state for pure neutron matter is in agreement with published results in [3] showing a gradual increase in the energy density as the pressure increases.

As noted in Chapter 5 neutron stars should obey the conditions of beta-equilibrium and charge neutrality. For a pure neutron matter EoS, charge neutrality is observed in this case (overall

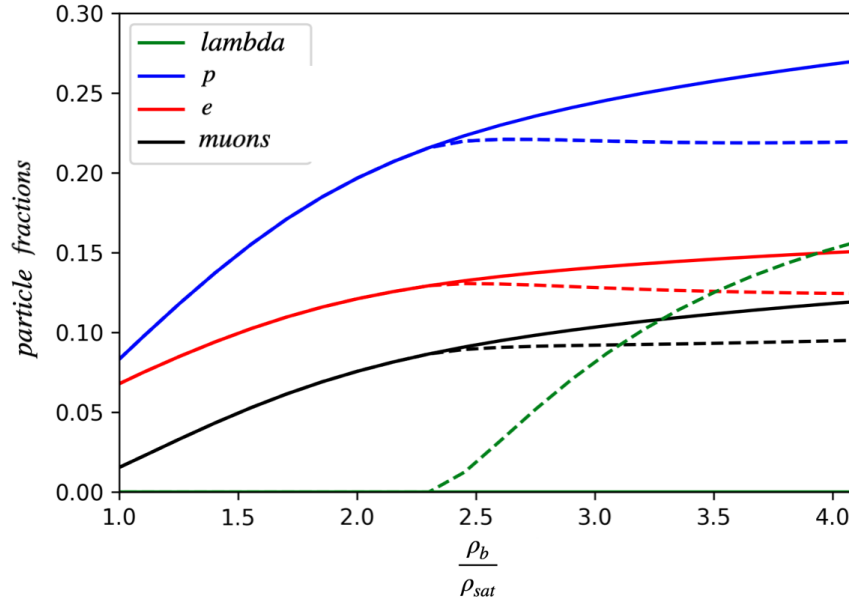


Figure 6.9: A close look at Fig 6.7 with the area of interest showing only the particle fractions for protons, electrons and muons before hyperons appear in the system and after adding hyperons for NL3. The solid lines represent general equilibrated matter and the dashed lines represent general equilibrated matter with the lambda (Λ) hyperon included.

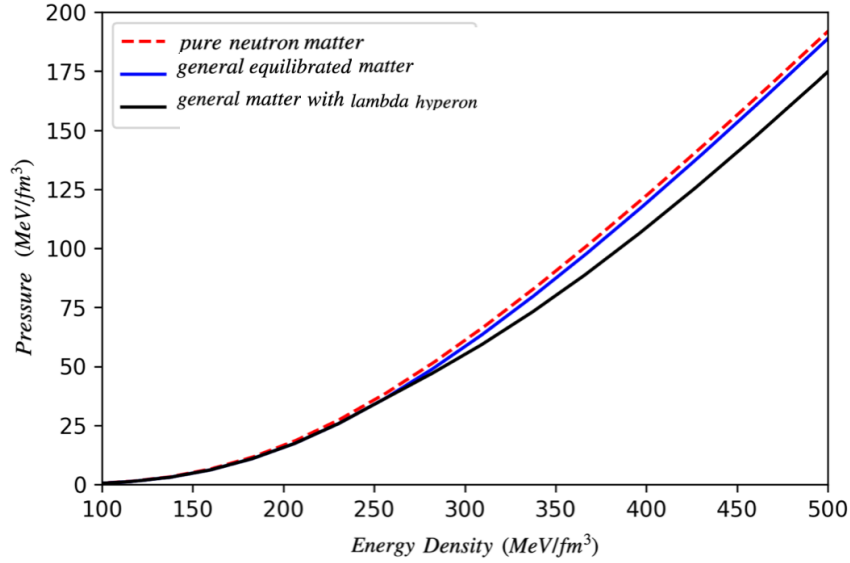


Figure 6.10: Pressure as function of energy density (EoS) for nuclear matter systems with the indicated equilibrium conditions in the QHD-I parameter set. The range of energy density roughly spans the baryon density range of $(0.1 - 8.0) \rho_{sat}$

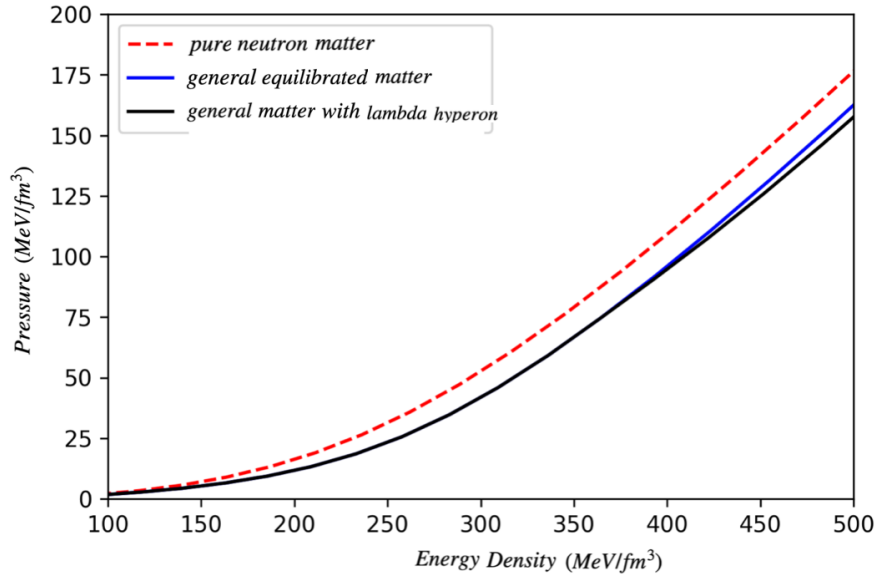


Figure 6.11: Pressure as function of energy density (EoS) for nuclear matter systems with the indicated equilibrium conditions in the NL3 parameter set. The range of energy density roughly spans the baryon density range of $(0.1 - 8.0) \rho_{sat}$

charge is zero since neutrons have zero charge). However beta equilibrium is not observed since this model considers the star to be made up of only neutrons and there are no protons or electrons in the system which is why a second model that accounts for both charge neutrality and beta equilibrium (Chapter 5) is discussed next.

6.5.2 General equilibrated matter

A general equilibrated state (Chapter 5) is a state that accounts for the fact that neutrons have short life cycles compared to the neutron star and thus neutrons inside the star will beta decay (Eq. (5.1a)) to give protons, electrons and anti-neutrinos. The reverse process (Eq. (5.1b)) where protons and electrons combine to give neutrons occurs (if electrons attain relativistic energies) until a state in the system is reached where the proton densities and neutron densities are such that the energy density of the system is at its lowest point (called the beta-equilibrated system constrained by Eq. (5.3 for the chemical potentials), (5.8 for the muons), (5.12 for charge neutrality)).

The plot representing the equation of state for general equilibrated matter is given in Fig 6.10 and 6.11. The general equilibrated matter plot gets less steep compared to the pure neutron matter plot in both Fig 6.10 and 6.11, The equation of state softens for both the QHD-I and NL3 parameter sets. The softening is due to factors such as including different particles as it becomes energetically favoured to populate the lowest energy muons states rather than the electron states (5.8) and this has the effect of reducing the energy density as the pressure increases. The softening of the equation of state when a pure neutron matter model is compared to a general equilibrated matter model is more in NL3 than in QHD-I. The reduced mass plot for general equilibrated matter for NL3 in Fig 6.3 also shows that the decrease due to the reduced mass when compared to pure neutron matter is more in NL3 than in QHD-I. This means that the NL3 parameter set equation of state will soften more than when compared to QHD-I since it is advanced. The NL3 parameter set is more advanced and includes contributions from the rho vector meson and self interactions of the scalar meson field. The NL3 parameter set has more coupling constants meaning it can be fitted to best describe the bulk properties of nuclear matter.

When comparing a pure neutron matter system to a general equilibrated matter system the behaviour of the EoS looking at the reduced mass plots in Fig 6.2 and 6.3 is that the equation of state of the star softens when general equilibrated matter is considered in the system (due to the scalar field contribution in the reduced mass and other particles). The reduced mass

decreases because of the contribution of the scalar field, contribution from the self interactions of the scalar meson field. These contributions (contribution of the scalar field, contribution from the self interactions of the scalar meson field) also have the effect of softening the equation of state. The inner core of the star is at a high baryon density and as it increases (due to factors such as gravity continuously pushing inwards on the star) the nucleons will populate the massive hyperon states [1]. In the next section the inclusion of the lambda hyperon in the EoS will be discussed.

6.5.3 Hyperon Matter

In Chapter 5 the conditions that might lead to hyperon states being populated were discussed. The hyperon states are more massive energy states and can only be populated (the lambda hyperon in this project) if the chemical potential of the nucleons exceeds a certain threshold energy being the chemical potential of the lambda hyperon when $k_{\Lambda F} = 0$ Eq. (6.1).

For a system that is at higher densities the nucleons will populate the hyperon states. The expected behaviour is that the addition of hyperons should soften the equation of state of the system [7], [8].

Here a model that includes the lambda hyperon was studied as a first approximation to explore the effects hyperons have on the equation of state for a neutron star. In Fig 6.10 and 6.11 the expected softening of the neutron star EoS is clearly observed when the lambda hyperon is included. Due to the different densities at which the lambda hyperon will populate the system in the different parameters, as discussed in section 6.4, the amount of softening of the EoS is more pronounced for the QHD-1 parameter set, since the lambda hyperon appears at much lower densities.

In this project (6.10 and 6.11) the plot representing equation of state for the inclusion of the lambda hyperon becomes less steep compared to the general equilibrated matter line meaning that the equation of state further softens which is the effect the inclusion of the lambda hyperon has on the equation of state. The softening is also due factors such as the inclusion of different particles as it will become energetically favourable to populate the hyperon states and muon states. The muon states are also populated due to the equilibrium conditions (Eq. (5.3), (5.8), (5.12)) constraining the system. This has the effect of reducing the energy density as the pressure increases. It is also due to the reduced mass, contribution from the vector meson such as in Eq.

(6.1) which contributes to the equation of state. The reduced mass and the vector meson in Eq. (6.1) influences when the lambda hyperon states appear in the system and in QHD-I the scalar field contribution was more in QHD-I than in NL3 in Fig 6.2 and 6.3.

Next a discussion of the properties of the neutron star for the three nuclear matter systems is given in the following section.

6.6 Neutron Star Properties

In this section the equations of state previously discussed are used to calculate neutron star mass radius plots. The equations of state are used to solve Eq. (2.24) and (2.25) which are the rescaled TOV equations and Sec. 2.5 is used in solving the TOV equation using the Runge-Kuta step method.

Solutions of Eq. (2.24) and (2.25) gives a point of a possible mass and radius for a certain central pressure of the neutron star. The initial conditions are $M = 0$ and $P = P_0$ at $r = 0$. The initial conditions discussed in Sec. 2.2 are needed to solve the TOV equation. The initial conditions for M, r and P at an increased pressure $P(r' = r + dr)$ are calculated using Eq. (2.24) and the corresponding energy density is calculated $\epsilon(r')$ [23]. To get $M(r')$ Eq. (2.25) is used. This cycle continues until the point at which $r' = R$ where $P(r' = 0)$ is the stars boundary at the radius R and $M(R)$ is the mass of the star[23]. This will give a sequence of points (mass and radius for each central pressure) that fall in the possible range (based on the boundary conditions) of mass and radius relations possible for a neutron star.

The TOV equation describes how the pressure decreases from the center of the star to the edge in terms of the energy density and pressure[23]. The energy density and pressure are related in terms of the equation of state of the matter inside the neutron star. If the star is large, the pressure inside the star will be large too because the star would be hard to compress. Thus the more massive a star is the smaller the value of r (Schwarzschild radius) at $P = 0$ meaning there exists a value (called the maximum mass) for the mass of the star above which the stars gravity will collapse the star. The maximum mass represents the maximum possible mass a neutron star can have before the star collapses under its own gravity. It is the maximum mass of an observable neutron star. The average mass (in solar masses) of a neutron star has been found to be around $1.5M_{\odot}$ [1]. The highest discovered maximum mass is for the recently discovered pulsar J0740 + 6620 with a maximum mass of around $2.2M_{\odot}$ [3]. Next maximum masses of the different equation of states discussed in this project (pure neutron matter model, general

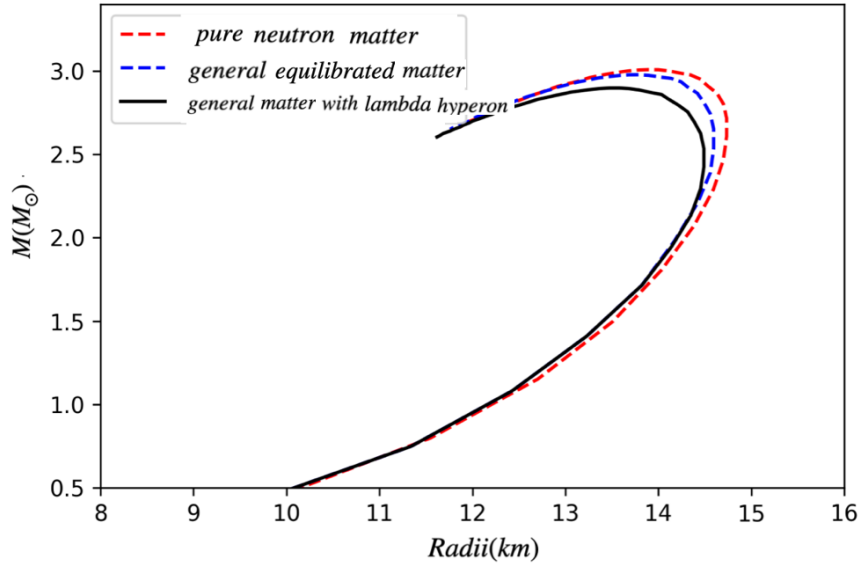


Figure 6.12: The mass-radius relationship for pure neutrons, general equilibrated matter and general equilibrated with the lambda hyperon included Plot for QHD-I.

equilibrated matter model and general equilibrated matter with the lambda hyperon included) will be discussed for NL3 and QHD-I parameter sets.

6.6.1 Pure neutron matter

The maximum masses calculated by applying the equation of state to the TOV equations for the pure neutron matter equation of state are given in Table 6.2. The maximum mass for QHD-I is (The QHD-I mass radius sequence is given in Fig 6.12) is $3.007M_{\odot}$ and for NL3 its $2.901M_{\odot}$. The NL3 mass radius plot is given in Fig 6.13. The NL3 maximum mass is lower, the vector meson and the reduced mass also in plots (Fig 6.2 and 6.3) showed NL3 softens the equation of state more compared to QHD-I.

6.6.2 General equilibrated matter

The maximum mass for general equilibrated matter is given in Table 6.2. The QHD-I maximum mass is $2.977 M_{\odot}$ and $2.761 M_{\odot}$ for NL3. The maximum mass values when compared to the pure neutron matter model decrease because of factors mentioned in Sec. 6.5. The maximum mass values decrease compared to what was observed in the pure neutron matter plot meaning that the equation of state has softened or there is a gradual increase in the energy density as the pressure increases and the star can be bound by gravity. The softening of the equation of state is expected when general equilibrated matter is discussed due to the equilibrium conditions

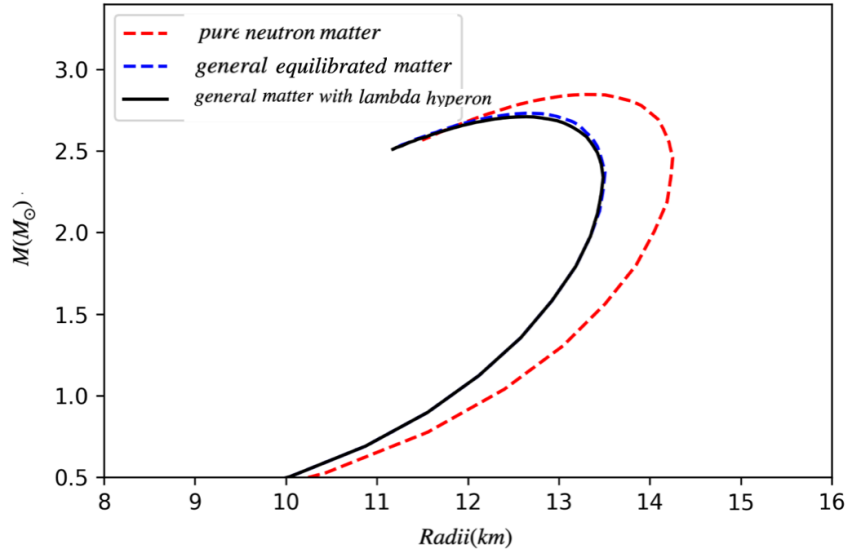


Figure 6.13: The mass-radius relationship for pure neutrons, general equilibrated matter and general equilibrated with the lambda hyperon included Plot for NL3.

for general equilibrium (Eq. (5.3 for the chemical potentials), (5.8 for the muons), (5.12 for charge neutrality)). General equilibrated matter includes more particles which has the effect of softening the neutron star equation of state.

6.6.3 Hyperon Matter

The maximum masses for the general equilibrated matter with the lambda hyperon included is given in Table 6.2. The maximum mass is $2.897M_{\odot}$ for QHD-I and $2.760M_{\odot}$ for NL3. The maximum masses are lower compared to the general equilibrated matter model because the equation of state for general equilibrated matter with the lambda hyperon included softened when the two equation of states (general equilibrated matter and general equilibrated matter with the lambda hyperon included) were compared in Sec. 6.5. The decrease in the values for the maximum masses do show that indeed the equation of state does soften when the lambda hyperon states begin to be populated when the two equation of states are compared.

Maximum Mass ($M_{\odot}$)			
Parameter	Pure Neutron Matter	general equilibrated matter	matter with lambda hyperon
QHD-I	3.007	2.977	2.897
NL3	2.900	2.761	2.760

Table 6.2: The Behaviour of the maximum mass for pure neutrons only, general equilibrated and general equilibrated matter with the lambda hyperon included.

Radius (km)			
Parameter	Pure Neutron Matter	general equilibrated matter	matter with lambda hyperon
QHD-I	13.895	13.790	13.475
NL3	13.370	12.675	12.674

Table 6.3: The Behaviour of the radii for pure neutrons only, general equilibrated and general equilibrated matter with the lambda hyperon included.

6.7 Discussion

From this project the behaviour of the equations of states for the three different nuclear matter models shows that different equation of states are there for a neutron star. The discrepancies in the models shows a need for more observations on high density behaviour of nuclear matter so that the best possible model which best describes high density nuclear matter can be found. The construction of radiotelescopes like the Square Kilometer Array project will offer more observations which can be compared to the theory. Research and neutron star observations have been found in literature such as [11] and [3] about the neutron star EoS and the effect of hyperons on the nuclear equation of state. The mass radius relationships for each model gives maximum masses that are realistic and represent possible neutron star masses. Observed results such as [11] and [3] place the average neutron star mass at $1.5M_{\odot}$ and this mass was well within the mass radius sequences for each equation of state discussed. Thus in the mass radius sequences in the mass radius plots should have this value and indeed from our maximum mass values it does show the mass radius sequence does pass through the average $1.5M_{\odot}$ value.

This project showed that as the baryon density increases the inclusion of the lambda hyperon would have the effect of softening the equation of state of the system. Thus a reduction in the maximum mass of the star as discussed in Sec. 6.5 and 6.6.3. In research papers such as [8] also showed the same behaviour that the inclusion of hyperons does soften the equation of state of a neutron star. A further study would be to include the other hyperons and observing their effect on the equation of state. This would give a full behaviour of the mass radius relationship of a neutron star if the star includes other hyperons. This work provides a first approximation to

studying the effect hyperons have on the neutron star equation of state. The expectation is that the addition of more hyperons should further soften the equation of state[7]. Hyperons relieve the Fermi pressure exerted by the baryons and will thus lead to the softening of the equation of state and the maximum mass.

7. Conclusions

In this project the equation of state was described by applying the relativistic mean-field theory approximation to a description of nuclear matter called Quantum Hadrodynamics [10]. The relativistic mean field approximation was used for the QHD-I and NL3 parameter sets in QHD. A mass-radius relationship for the two parameter sets was calculated for three versions of the equation of state of a neutron star. The expected behaviour is that the neutron star equation of state should soften when hyperons are included. In this project when the lambda hyperon is included in the equation of state it does soften and the maximum mass of the neutron star decrease when the lambda hyperon is included. The results for the pure neutron matter model and matter in general equilibrium (beta-equilibrated matter including muons) model showed agreement with other calculations as in [1] and [7]. The inclusion of the lambda hyperon to the equation of state of a neutron star produced results that are in agreement with other results [7]. While the sole inclusion of the lambda hyperon to the description of dense nuclear matter found in neutron star is not unrealistic, a more complete approach would be to include the full baryon octet in the description of nuclear matter inside a neutron star.

APPENDIX A

Relativistic quantum mechanics

In Chapter 2 the calculations of the mass-radius relationship of a neutron star were found by solving the TOV equations in conjunction with the EoS. Due to the neutron star densities, a relativistic description of dense nuclear matter EoS has to be adopted. The aim of this chapter is to describe the attempts to merge the principles of relativity and quantum mechanics. Whilst most of the work in this chapter is already known all of it was self studied as it was not covered in the undergraduate program.

A.1 Non-relativistic quantum mechanics

In the next section several concepts from quantum mechanics such as the Schrödinger equation will be discussed.

The aim of quantum mechanics is to predict the probabilities for measurements on a particle, often on short length scales. Dynamics in Quantum mechanics is done by solving the Schrödinger equation to get the particle's wave function $\Psi(\mathbf{x}, t)$. A classical particle is localized at a point whilst the wave function $\Psi(\mathbf{x}, t)$ is a function that is spread out in space which shows the wave-particle duality of quantum mechanics. To study this duality a statistical interpretation of the position x and momentum p is made. To give the statistical interpretation the wave function Ψ is needed along with the particles equation of motion. The wave function Ψ represents the state of the particle. The equation of motion is derived from the statistical interpretation using the position x and momentum p as all dynamical variables can be explained in terms of the two.[27]

The non-relativistic Schrödinger equation is given by [27]

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi, \tag{A.1}$$

where

- $\Psi(\mathbf{x}, t)$ is the wave function,
- V is the potential energy,
- m is the mass of the particle, and,

- $\hbar$ is the Planck's constant divided by 2π .

The wave function represents the state of the particle and according to Born's statistical interpretation the mod square of the wave function $|\Psi|^2$ also gives the probability of finding the particle at the point x in one dimension at time t [27].

The statistical interpretation only holds for physically realisable states which are square integrable solutions to the Schrödinger equation [27].

Still following the statistical interpretation the momentum and energy operators are given by

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad (\text{A.2a})$$

$$\mathbf{p} \rightarrow -i\hbar \nabla, \quad (\text{A.2b})$$

where $\nabla = \frac{\partial}{\partial x}$. This physical quantities E and $\mathbf{p}$ are replaced by operators that are an instruction to act on the function that follows it [27].

The Schrödinger equation can be connected to classical mechanics in the following way by using the classical relationship between kinetic energy, momentum relation [27] given by

$$E = \frac{\mathbf{p}^2}{2m}. \quad (\text{A.3})$$

when substituting the operators for energy and momentum from Eq. (A.2a) and Eq. (A.2b) Which gives the Schrödinger equation as

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2}. \quad (\text{A.4})$$

A.2 Relativistic quantum mechanics

Relativistic formulation of quantum mechanics requires the wave function to be Lorentz invariant [28]. An equation is Lorentz covariant under a Lorentz transformation [29]

$$x^\mu \longleftrightarrow (x'^\mu) = \Lambda^\mu_\nu x^\nu. \quad (\text{A.5})$$

Lorentz invariance is one of the core principles for relativistic particles. The Schrödinger equation however is not Lorentz invariant due to the fact that it does not represent a relativistic particle.

In the next section the Klein-Gordon equation is introduced and it will be shown that it is Lorentz invariant and can therefore be a possible candidate for describing relativistic quantum mechanics.

A.3 The Klein-Gordon Equation

The Klein-Gordon equation is motivated in this section from the relativistic energy momentum relationship (in natural units) [28],

$$E^2 = m^2 + \mathbf{p}^2. \quad (\text{A.6})$$

Substituting Eq. (A.2a) and (A.2b) into Eq. (A.6) the result is

$$\frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi - m^2 \phi, \quad (\text{A.7})$$

where $\phi(x^\mu)$ is the wave function.

Eq. (A.7) is the Klein-Gordon equation [28]. The Klein-Gordon equation in its form is not evident that it is Lorentz covariant and to write it manifestly in a Lorentz covariant manner, four-vector notation is employed to give

$$(\partial_\mu \partial^\mu + m^2)\phi = 0, \quad (\text{A.8})$$

where,

$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \quad (\text{A.9})$$

gives the Lorentz invariant scalar product of two vectors.

For Eq. (A.8) to be a covariant equation, ϕ has to describe a scalar particle. The Klein-Gordon equation has plane wave solutions [28] and its solution in four-vector notation is

$$\phi(x^\mu) \propto e^{-ip_\mu x^\mu}, \quad (\text{A.10})$$

where $p^\mu = (E, \mathbf{p})$ and $p_\mu x^\mu = Et - \mathbf{p} \cdot \mathbf{x}$.

Substituting Eq. (A.10) into the expression in Eq. (A.8) the solution is

$$E^2 = \mathbf{p}^2 + m^2 \quad (\text{A.11})$$

where $\mathbf{p} \cdot \mathbf{p} = p_\mu p^\mu$. Then the possible energies are

$$E = \pm \sqrt{\mathbf{p}^2 + m^2}. \quad (\text{A.12})$$

Hence the Klein-Gordon equation gives solutions that are both positive and negative energy solutions. The negative energy solutions posed a problem since there was no physical interpretation for them at the time it was written down[1]. It was not clear at the time what the negative energy solutions represented and they cannot be discarded because in quantum mechanics all solutions are required to give a complete set of states[28]. Another problem was that when it came to calculating the probability densities of the negative solutions, they are not positive definite[28].

The Klein-Gordon equation was deemed as incomplete because of its inability to explain the meaning of the negative energy solutions and also because the probability densities of the negative solutions are not positive definite, The Klein-Gordon equation was redeemed when the physical interpretation of these negative energy solutions was given[28]. An interpretation of the negative energy solutions is provided in the next section.

A.4 The Dirac Equation

P. A. M. Dirac came up with a description of relativistic quantum mechanics which could account for the negative energy solutions that were encountered in Eq. (A.12) and clearly explain their physical interpretation[28].

The Klein-Gordon is of second order; Dirac instead looked for an equation that is of first order in space and time derivatives and could also satisfy the relativistic energy-momentum relation by Einstein (for the solutions to represent relativistic particles they need to satisfy the relation). Dirac then postulated that

$$H\psi = (-i\boldsymbol{\alpha} \cdot \nabla + \beta m)\psi = E\psi = i\frac{\partial\psi}{\partial t} \quad (\text{A.13})$$

where H is the Hamiltonian and α and β are the matrices [28]

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad (\text{A.14a})$$

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (\text{A.14b})$$

where I is the 2×2 matrix and σ_i are the Pauli 2×2 matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (\text{A.15a})$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad (\text{A.15b})$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (\text{A.15c})$$

This means α and β are 4×4 matrices as such $\psi(x)$, the Dirac spinor, in Eq. (A.13) has to

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad (\text{A.16})$$

but due the form of α and β $\psi(x)$ is written as

$$\psi(x) = \omega e^{-i\mathbf{p} \cdot \mathbf{x}}, \quad (\text{A.17})$$

where ω is a column vector (four-vector) with

$$\omega = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \quad (\text{A.18})$$

Correspondingly ϕ and χ are two component spinors and ω should satisfy Eq. (A.13) as

$$E\omega = (\alpha \cdot \mathbf{p} + \beta m)\omega. \quad (\text{A.19})$$

Writing Eq. (A.19) in component form we have

$$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} m & \boldsymbol{\sigma} \cdot \mathbf{p} \\ -\boldsymbol{\sigma} \cdot \mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \quad (\text{A.20})$$

Then the result is

$$\begin{pmatrix} E\phi \\ E\chi \end{pmatrix} = \begin{pmatrix} m\phi + \boldsymbol{\sigma} \cdot \mathbf{p}\chi \\ -\boldsymbol{\sigma} \cdot \mathbf{p}\phi - m\chi \end{pmatrix}. \quad (\text{A.21})$$

Which has yielded two equations which are

$$E\phi = m\phi + \boldsymbol{\sigma} \cdot \mathbf{p}\chi \quad (\text{A.22a})$$

$$E\chi = -\boldsymbol{\sigma} \cdot \mathbf{p}\phi + m\chi. \quad (\text{A.22b})$$

From Eq. (A.22b) making χ subject

$$\chi = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}\phi}{E + m}. \quad (\text{A.23})$$

Substituting Eq. (A.23) into Eq. (A.22a)

$$E\phi = m\phi + \boldsymbol{\sigma} \cdot \mathbf{p} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}\phi}{E + m} \right). \quad (\text{A.24})$$

Re-arranging the terms and eliminating the denominator yields

$$(E - m)(E + m)\phi - \boldsymbol{\sigma} \cdot \mathbf{p}(\boldsymbol{\sigma} \cdot \mathbf{p})\phi = 0. \quad (\text{A.25})$$

Note that

$$(\boldsymbol{\sigma} \cdot \mathbf{p})(\boldsymbol{\sigma} \cdot \mathbf{p}) = \mathbf{p} \cdot \mathbf{p} + i\boldsymbol{\sigma} \cdot (\mathbf{p} \times \mathbf{p}) = \mathbf{p}^2. \quad (\text{A.26})$$

Using Eq. (A.26), Eq. (A.25) becomes

$$(E^2 - m^2 - \mathbf{p}^2)\phi = 0. \quad (\text{A.27})$$

Equation (A.27) implies that

$$E^2 - m^2 - \mathbf{p}^2 = 0. \quad (\text{A.28})$$

Therefore solving for E we get

$$E = \pm \sqrt{m^2 + \mathbf{p}^2}. \quad (\text{A.29})$$

Clearly the Dirac equation satisfies the relativistic energy-momentum relationship, but Eq. (A.29) has yielded solutions that are both positive and negative. The meaning of these two eigenvalues will now be interpreted first starting with positive solutions.

A.4.1 Positive Eigenvalues

It's easier to postulate that

$$\phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{A.30a})$$

Then Eq. (A.26) becomes

$$\chi = \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \chi = \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{A.31})$$

Getting to the dot product we see that

$$\boldsymbol{\sigma} \cdot \mathbf{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}. \quad (\text{A.32})$$

Therefore

$$\chi = \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \quad \text{or} \quad \chi = \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \quad (\text{A.33})$$

which implies that the solution for ω is

$$\omega = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \quad \text{or} \quad \omega = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}. \quad (\text{A.34})$$

When $p = 0$ the last two terms of the matrix disappear and the energy will just be $E = mc^2$ and the time dependant wave function becomes

$$\psi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{imt} \quad \text{or} \quad \psi = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{imt}. \quad (\text{A.35})$$

These corresponds to particle solutions moving forward in time [28]. Further interpretation of this will be provided after dealing with the second case.

A.4.2 Negative Eigenvalues

This time the starting equation is

$$\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{A.36a})$$

or

$$\chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\text{A.36b})$$

then substituting

$$\phi = \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E-m} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \phi = \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E-m} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{A.37})$$

Therefore the solution for ω are

$$\omega = \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \omega = \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}. \quad (\text{A.38})$$

The first two terms of the matrix disappear and the energy will just be $E = mc^2$ and the time dependant wave function becomes

$$\psi = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imt} \quad \text{or} \quad \psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imt}. \quad (\text{A.39})$$

The meaning is that the negative energy solutions describe particles moving backwards in time. These particles are called antiparticles and what the results of the two solutions in Eq. (A.35) and Eq. (A.39) mean is that the Dirac equation gives information on a particle and its anti-particle pair and the spin too. The two positive energy solutions correspond to spin up and spin down positive energy solutions of the Dirac equation whilst the two negative energy solutions correspond to spin and spin down negative energy solutions of the Dirac equation [28].

The Dirac equation can be written in covariant form by introducing what is called γ matrices which are defined by [28]:

$$\gamma^0 = \beta \tag{A.40a}$$

$$\gamma^i = \beta \alpha^i = \gamma^0 \alpha^i. \tag{A.40b}$$

The γ matrices have properties that are derived from the properties of α and β [28].

Multiplying Eq. (A.13) on the left by γ^0 gives the Dirac equation in covariant form as

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{A.41}$$

A.5 Interpreting the Klein-Gordon and Dirac equation

The negative energy solutions having been accounted for by the Dirac equation, an interpretation of the particles described by the two equations needs to be given.

A.5.1 Klein-Gordon equation

The Klein-Gordon equation had difficulty in explaining the meaning of the negative energy solutions and the positive definiteness of the probability densities. The Klein-Gordon equation was then redeemed and it was interpreted as an equation that describes spin-0 particles (bosons) and they do not obey the Pauli Exclusion Principle. Feynman interpreted the negative energy state as an antiparticle propagating forward in time or a positive energy particle propagating backwards in time [28].

A.5.2 Dirac Equation

The Dirac equation solved the two difficulties of first interpreting the meaning of the negative energy solutions and second the positive definiteness of the probability densities that were in the Klein-Gordon equation. The probability density of the Dirac equation is positively definite. The Dirac equation also accounts for the negative energy solutions. Dirac postulated that there is a sea of infinite negative particles called a Dirac Sea in which the holes are the positrons. The Dirac equation describes spin- $\frac{1}{2}$ particles (Fermions)[28].

The first two solutions Eq. (A.35) represents spin up and spin down positive solutions whilst the solutions by Eq. (A.39) represent spin up and spin down negative energy solutions to the Dirac

equation [28].

Dirac further postulated about the existence of the Dirac Sea in which the vacuum is a state in which all the negative energy states have been filled and in the Dirac Sea the Pauli exclusion principle prevents positive energy electrons from occupying the already filled negative energy states and also an electron from the negative energy state can be excited by a particle with energy $E > 2m_e$ leaving behind a hole which has a positive charge relative to its surroundings. The hole is interpreted as the positive energy anti-particle with an opposite charge to the particle states [28]. The Dirac equation has provided four possible solutions for one energy [28]. The result means that a linear combination of these states is also a possible solution (and there are infinite solutions), this is called a field which is why the next chapter gives a description of classical field theory.

APPENDIX B

Classical Mechanics and Field Theory

This chapter will serve as an introduction to the multi-particle description of nuclear matter by introducing concepts from classical mechanics in order to formally introduce classical field theory. The previous chapter showed that the interpretation from the Dirac equation meant that a multi-particle description (Quantum Field Theory) is needed to describe relativistic particles.

In order to properly give a description of field theory first a description of classical mechanics is given as a build up to classical field theory.

B.1 Principle of least action

The principle of least action is an extremization function in non-relativistic mechanics on the difference of the kinetic energy T and the potential energy V added continuously on all the points in a given trajectory over a time interval $[t_1, t_2]$ [30]. This principle means that the trajectory that has the least energy (minimized kinetic energy, T coupled with a maximized potential energy, V) is the one with the least action. Everything in nature and all equations in physics can be explained through the principle of least action [16]. Writing this statement in terms of a mathematical expression gives the action as

$$S = \int_{t_1}^{t_2} (T - V) dt \tag{B.1}$$

where S is the action. The expression in brackets in Eq. (B.1) is called the Lagrangian (L) and it is discussed in detail in the next section.

B.2 Lagrangian Mechanics

Lagrangian mechanics is the description of a system in terms of the degrees of freedom of the system [31]. It is equivalent to using the Newton's second law to determine the motion of a particle/mass. It is more useful since it can be applied even in relativistic systems and all the information about the system is in a function called the Lagrangian making it an extremely

convenient tool for changing coordinates and its formula in simple terms is given by

$$L = T - V \quad (\text{B.2})$$

where,

- T is the kinetic energy
- V is the potential energy of the system.

To properly depict the various degrees of freedom each degree of freedom is described by a generalised coordinate given by $q_i (i = 1, 2, \dots, n)$ for n degrees of freedom. Note that a Lagrangian can be written for each degree of freedom.

The Lagrangian is derived from the action in Eq. (B.1). Then the general definition of the action is given by [30]

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt. \quad (\text{B.3})$$

The general rule (Hamilton's principle) is that the action (S) is stationary which implies that the variation of the action (S) between the time interval $[t_1, t_2]$ will be zero. The variation of the action is given by

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt. \quad (\text{B.4})$$

This result can be used to derive the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (\text{B.5})$$

which gives the equation of motion for a certain degree of freedom q_i [16]. For given a L , the Euler-Lagrange equation has to be solved in order to determine the equations of motion.

B.3 Hamiltonian Formulation

The Hamiltonian is an alternative to the Lagrangian. It is also based on the principle of least action and uses the concept of the conjugate momentum, p_i , corresponding to the generalised coordinate q_i

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad (\text{B.6})$$

instead of a generalised velocity, $\dot{q}_i$. Using p_i instead of q_i is especially important in quantum mechanics and relativity where momentum plays a big role. The Hamiltonian is defined by [16]

$$H = \sum p_i \dot{q}_i - L. \quad (\text{B.7})$$

According to [16] the Hamiltonian gives the expression for the total energy which, unlike the Lagrangian, it is a conserved quantity only if the system doesn't explicitly depend on time. The Lagrangian and the Hamiltonian however are connected by a Legendre transformation.

B.4 Field Theory

B.4.1 The Dynamics of Fields

To begin this discussion a clear definition of what a field is and how it differs from the generalized coordinates of classical mechanics is needed. A field is a quantity that is defined at every space and time $(\mathbf{x}, t)$. It differs from generalized coordinates of classical mechanics in the sense that field theory deals with the dynamics of fields $\phi(\mathbf{x}, t)$ and also deals with infinite number of degrees of freedom [29].

B.4.2 Lagrangian Mechanics

Lagrangian mechanics is the description of a physical system in terms of the degrees of freedom of the system [31]. Therefore to understand the dynamics of a system with infinite degrees of freedom Lagrangian mechanics is used. In field theory the Lagrangian is called the Lagrangian density $\mathcal{L}$ and it is of the form [29]

$$L(t) = \int \mathcal{L}(\phi_i, \dot{\phi}_i, \nabla \phi_i) d\mathbf{x}. \quad (\text{B.8})$$

The Lagrangian governs the dynamics of fields. To derive the equations of motion we use the principle of least action as in [29]. The principle of least action gives the Euler-Lagrange equations from the Lagrangian density as

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0. \quad (\text{B.9})$$

The conjugate momentum for the field ϕ_i is

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}. \quad (\text{B.10})$$

The Hamiltonian density $\mathcal{H}$, is derived in the following way [16]

$$\begin{aligned}
 H &= \int \mathcal{H} d\mathbf{x} \\
 &= \int \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \dot{\phi}_i - \mathcal{L} \right) d\mathbf{x} \\
 &= \int \left(\pi^i \dot{\phi}_i - \mathcal{L} \right) d\mathbf{x}.
 \end{aligned} \tag{B.11}$$

E.q B.11 gives the Hamiltonian density $\mathcal{H}$, as [16]

$$\mathcal{H} = \left(\pi^i \dot{\phi}_i - \mathcal{L} \right). \tag{B.12}$$

B.4.3 Noether's Theorem

In this section an introduction into one of the most useful theorems in physics is given due to its use in the various fields in physics.

Noether's theorem states that for each symmetry of the Lagrangian there is a conserved quantity [30]. The proof of Noether's theorem is given in [30]. The Lagrangian be invariant to small, first order changes in coordinates

$$q_i \longrightarrow q_i + \epsilon K_i(q_i); \tag{B.13}$$

where K_i is a function of all the q_i . If L does not change in the first order in ϵ , then

$$0 = \frac{dL}{d\epsilon} = \sum_i \frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial \epsilon} + \frac{\partial L}{\partial \dot{q}_i} \frac{\dot{q}_i}{\partial \epsilon} \tag{B.14}$$

$$= \sum_i \frac{\partial L}{\partial q_i} K_i + \frac{\partial L}{\partial \dot{q}_i} \dot{K}_i. \tag{B.15}$$

Using the Euler-Lagrange equations

$$0 = \sum_i \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) K_i + \frac{\partial L}{\partial \dot{q}_i} \dot{K}_i \right) \tag{B.16}$$

$$= \frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} K_i \right). \tag{B.17}$$

Define the quantity

$$P(q, \dot{q}) = \sum_i \frac{\partial L}{\partial \dot{q}_i} K_i \tag{B.18}$$

$P(q, \dot{q})$ is called the conserved momentum and $P(q, \dot{q})$ is conserved as it does not change with time. This chapter was aimed at laying down tools such as (the Euler-Lagrange Equation in (B.9), and the conjugate momentum Equation in (B.10)) that will be used in the next chapter which elaborates more on Quantum Field Theory which is a an application of field theory to quantum mechanics.

APPENDIX C

Quantum Field Theory

The Dirac equation interpretation in Appendix A showed that there is a need to discuss multi-particle states descriptions for describing relativistic particles and also a proper formalism for classical field theory was given in Appendix B. Appendix B also laid down the tools that would be used to discuss multi-particle states, and this laid the framework for discussing what will follow in this chapter. This chapter will extend the description of multi-particle states by discussing what is called quantum field theory.

C.1 Introduction

Quantum field theory is the application of quantum mechanics to fields [29]. In the first section canonical quantization will be discussed.

C.2 Canonical Quantization

Canonical quantization is a procedure that takes generalised coordinates q^i and their conjugate momenta π^i and promotes them to operators. In field theory the quantum fields $\phi_a(x)$ and their conjugate momenta $\pi_a(x)$ are taken and promoted to operators [29]

$$\phi_a(x) \longrightarrow \hat{\phi}_a(x) = 0 \tag{C.1a}$$

$$\pi_a(x) \longrightarrow \hat{\pi}_a(x). \tag{C.1b}$$

A quantum field $\phi_a(\vec{x})$ is an operator valued function of space. The following equal-time commutation relations are imposed [29]

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = [\pi^a(\mathbf{x}), \pi^b(\mathbf{y})] \tag{C.2a}$$

$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})\delta_a^b. \tag{C.2b}$$

$\delta^3(\mathbf{x} - \mathbf{y})$ is the Dirac delta function and δ_a^b is the Kronecker delta function.

The time component is separated from the space component meaning that the dynamics of the fields will be determined using the Schrödinger picture as our operators do not depend on time, t (only depend on space $\mathbf{x}$). In the Schrödinger picture all the time dependence sits on the

states which evolve as [29]

$$i \frac{\partial}{\partial t} |\mathcal{O}\rangle = \mathcal{H} |\mathcal{O}\rangle, \quad (\text{C.3})$$

where the Hamiltonian H is given by,

$$H = \int \mathcal{H} d\mathbf{x}. \quad (\text{C.4})$$

The Hamiltonian H ; is a function of ϕ and π and will therefore become an operator. In a quantum theory finding the spectrum of the Hamiltonian H will be the next task since because of the infinite degrees of freedom it is very hard to get the spectrum of H . There is what is called free theories in which the dynamics are written in a way such that each degree of freedom evolves independently from all the others. The Lagrangian for free field theories are quadratic so that the equations of motion are linear.

C.3 Free Scalar Field

The simplest example of a relativistic free scalar field is the Klein-Gordon equation (A.8) which describes spin-0 particles (bosons) whose Lagrangian is [29]

$$\mathcal{L} = \frac{1}{2} \frac{\partial \phi(x^\nu)}{\partial x_\mu} \frac{\partial \phi(x^\nu)}{\partial x^\mu} - \frac{1}{2} m^2 \phi(x^\nu)^2 \quad (\text{C.5})$$

$$= \frac{1}{2} \partial_\mu \phi(x^\nu) \partial^\mu \phi(x^\nu) - \frac{1}{2} m^2 \phi(x^\nu)^2. \quad (\text{C.6})$$

where $\phi(x^\nu)$ is a real scalar field operator of particles that have mass m .

The equation of motion is given by applying the Euler-Lagrange equations to Eq. (C.6). The equation of motion is given by

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \quad (\text{C.7})$$

To solve Eq. C.7 take the Fourier transform of the above equation to give the Klein-Gordon field as [29].

$$\phi(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i(\mathbf{p} \cdot \mathbf{x})} \phi(\mathbf{p}, t). \quad (\text{C.8})$$

The Klein-Gordon equation then becomes

$$\left[\frac{\partial^2}{\partial t^2} + (\mathbf{p}^2 + m^2) \right] \phi(\mathbf{p}, t) = 0. \quad (\text{C.9})$$

This expression is the equation of motion for a simple harmonic oscillator with frequency

$$\omega_{\mathbf{p}} = \sqrt{(\mathbf{p}^2 + m^2)}. \quad (\text{C.10})$$

The solution to the Klein-Gordon equation is the superposition of harmonic oscillators. Thus to quantize a field $\phi(\mathbf{x}, t)$ one only needs to use the harmonic oscillator as an analogy. The general spectrum of the Hamiltonian for the harmonic oscillator is well understood and it is provided in [29]. Hence $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ are written as a linear sum of an infinite number of creation and annihilation operators $a_{\mathbf{p}}^\dagger$ and $a_{\mathbf{p}}$ indexed by the 3-momentum $\mathbf{p}$ [29]

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} [a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}], \quad (\text{C.11})$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} [a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}]. \quad (\text{C.12})$$

Using Eq. (C.11) and (C.12) the commutation relation for the creation $a_{\mathbf{p}}^\dagger$ and annihilation operators $a_{\mathbf{p}}$ will be given by [29]

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}), \quad (\text{C.13})$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0. \quad (\text{C.14})$$

The next task will then be calculating the Hamiltonian H in terms of the creation and annihilation operators. The Hamiltonian as a function of ϕ and π is given by [29]

$$H = \frac{1}{2} \int d^3x (\pi^2 + (\nabla\phi)^2 + m^2\phi^2). \quad (\text{C.15})$$

Then substituting Eq. (C.11) and (C.12) into (C.15) will give

$$\begin{aligned} H &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6} \left[-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{2} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}) (a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} + a_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}}) \right. \\ &\quad + \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} (i\mathbf{p} a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + i\mathbf{p} a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}) \cdot (i\mathbf{q} a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} + i\mathbf{q} a_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}}) \\ &\quad \left. + \frac{m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}) \cdot (a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} + a_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}}) \right] \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} [a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger]. \end{aligned} \quad (\text{C.16})$$

$$(\text{C.17})$$

Following that will give the Hamiltonian as

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} [a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + \frac{1}{2}(2\pi)^3 \delta^3(0)]. \quad (\text{C.18})$$

The result from this last term is in two parts, first the result yields an infinite energy and second a delta function which when evaluated at zero gives an infinite spike. This is one of the manifestations of what is called the vacuum energy [29]. This is attributed to filled negative energy states. The result is that the Klein-Gordon field describes spin-0 particles (bosons).

C.4 The Dirac Fields

The Klein-Gordon field deals only with spin-0 particles; however, most particles in nature have an intrinsic spin- $\frac{1}{2}$ which arises naturally when considering fields which transform non-trivially under the Lorentz group[29]. This section uses the Dirac field to describe the spectrum of the Hamiltonian H for spin- $\frac{1}{2}$ particles (fermions).

To quantize the Dirac field the first step is to define the Lagrangian density which is given by [29]

$$\mathcal{L} = \bar{\psi}(x) [i\gamma^\mu \partial_\mu - m] \psi(x) \quad (\text{C.19})$$

where

- $\bar{\psi}$ is the Dirac adjoint,
- ψ is the Dirac spinor,
- $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$,
- $\gamma^0 = \beta$,
- $\gamma_i = \beta\alpha_i$ are gamma matrices (A.40b).

Eq. (C.19) explicitly shows the Lorentz covariant manner of the Dirac equation and the gamma matrices satisfy the following fundamental rule

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \quad (\text{C.20})$$

where $\eta^{\mu\nu}$ is the Minkowski metric tensor for flat space-time

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (\text{C.21})$$

From the Dirac Lagrangian the equation of motion is derived by using the Euler-Lagrange Equation (B.9) to give

$$[i\gamma^\mu \partial_\mu - m] \psi(x) = 0. \quad (\text{C.22})$$

The next step is to determine the conjugate momentum for the field

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger. \quad (\text{C.23})$$

This is needed as the same method that was applied in the Sec. C.2 is applied here. In this case the Dirac spinor ψ and its conjugate momentum are promoted to field operators.

$$\psi(x) \longrightarrow \hat{\psi}(x) \quad (\text{C.24a})$$

$$\psi^\dagger(x) \longrightarrow \hat{\psi}^\dagger(x). \quad (\text{C.24b})$$

The field operators satisfy the following equal-time anticommutation relation [29]

$$[\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})] = [\psi_\alpha^\dagger(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})] = 0 \quad (\text{C.25a})$$

$$[\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})] = \delta^3(\mathbf{x} - \mathbf{y}) \delta_{\alpha\beta}. \quad (\text{C.25b})$$

α and β denote the spinor components of ψ .

As in the previous section the quantum operators are expressed in terms of the creation and annihilation operators which are

$$\hat{\psi}(t, \mathbf{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\hat{b}(\mathbf{p}, s) u(\mathbf{p}, s) e^{-i(\mathbf{p}\cdot\mathbf{x})} + \hat{c}^\dagger(\mathbf{p}, s) v(\mathbf{p}, s) e^{+i(\mathbf{p}\cdot\mathbf{x})} \right), \quad (\text{C.26})$$

$$\hat{\psi}^\dagger(t, \mathbf{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\hat{b}^\dagger(\mathbf{p}, s) \bar{u}(\mathbf{p}, s) e^{+i(\mathbf{p}\cdot\mathbf{x})} + \hat{c}(\mathbf{p}, s) \bar{v}(\mathbf{p}, s) e^{-i(\mathbf{p}\cdot\mathbf{x})} \right). \quad (\text{C.27})$$

Such that $\hat{b}(\mathbf{p})$ destroys particles associated the $\bar{u}(\mathbf{p})$ spinors and $\hat{c}^\dagger(\mathbf{p})$ creates particles associated to $\bar{v}(\mathbf{p})$ spinors.

Using Eq. (C.26) and (C.27) the equal-time anti-commutation relations for the creation and annihilation operators is given by

$$[\hat{b}_{\mathbf{p}}^r, \hat{b}_{\mathbf{q}}^{s\dagger}] = (2\pi)^3 \delta^{rs} \delta^3(\mathbf{p} - \mathbf{q}) \quad (\text{C.28a})$$

$$[\hat{c}_{\mathbf{p}}^r, \hat{c}_{\mathbf{q}}^{s\dagger}] = (2\pi)^3 \delta^{rs} \delta^3(\mathbf{p} - \mathbf{q}). \quad (\text{C.28b})$$

In contrast the anti-commutation relations of the Dirac field operators are designed such that Fermi-Dirac statistics is obeyed. All that remains is to construct the spectrum of the Hamiltonian in terms of the creation and annihilation operators.

The Hamiltonian density $\mathcal{H}$ Eq. (B.11) is given by [29]

$$\mathcal{H} = \pi \dot{\psi} - \mathcal{L} \quad (\text{C.29})$$

$$= \bar{\psi} (-i\gamma^i \partial_i + m) \psi. \quad (\text{C.30})$$

The Hamiltonian $H = \int d^3x \mathcal{H}$ is a function of the operators $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^\dagger(\mathbf{x})$ and will therefore be an operator. Substitute Eq. (C.30) into (C.4) to get

$$\begin{aligned} H &= \int \psi^\dagger \gamma^0 (-i\gamma^i \partial_i + m) \psi \\ &= \frac{d^3p d^3q d^3x}{(2\pi)^6} \sqrt{\frac{E_{\mathbf{p}}}{4E_{\mathbf{q}}}} \left(\hat{b}^\dagger(\mathbf{p}, s) \bar{u}(\mathbf{p}, s) e^{+i(\mathbf{p} \cdot \mathbf{x})} + \hat{c}(\mathbf{p}, s) \bar{v}(\mathbf{p}, s) e^{-i(\mathbf{p} \cdot \mathbf{x})} \right) \\ &\quad \left(\hat{b}(\mathbf{p}, s) u(\mathbf{p}, s) e^{-i(\mathbf{p} \cdot \mathbf{x})} + \hat{c}^\dagger(\mathbf{p}, s) v(\mathbf{p}, s) e^{+i(\mathbf{p} \cdot \mathbf{x})} \right) \\ &= \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} [\hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} - \hat{c}_{\mathbf{p}} \hat{c}_{\mathbf{p}}^\dagger] \\ &= \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} [\hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} - \hat{c}_{\mathbf{p}}^\dagger \hat{c}_{\mathbf{p}} + (2\pi)^3 \delta^3(0)]. \end{aligned} \quad (\text{C.31})$$

The $\hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}}$ term is familiar and positive energy states given by

$$[H, \hat{b}_{\mathbf{p}}^\dagger] = E_{\mathbf{p}} \hat{b}_{\mathbf{p}}^\dagger. \quad (\text{C.32})$$

The negative sign in front of the $\hat{c}_{\mathbf{p}}^\dagger \hat{c}_{\mathbf{p}}$ term makes its interpretation different. The $\hat{c}_{\mathbf{p}}^\dagger$ creates positive energy states given by

$$[H, \hat{c}_{\mathbf{p}}^\dagger] = E_{\mathbf{p}} \hat{c}_{\mathbf{p}}^\dagger. \quad (\text{C.33})$$

Due to the commutation relation in Eq. (C.28b) the $\hat{c}$ is interpreted as a creation operator but this would mean the Hamiltonian is unbounded and this violates the Pauli exclusion principle, spin- $\frac{1}{2}$ particles (fermions) obey Fermi-Dirac statistics and the spinor fields anti-commute in order to preserve causality[29], postulating anti-commutation relations (for Dirac fields) will solve the negative energy problem in Eq. (C.28b). The *equal-time anti-commutation relations* are given by

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = \{\psi_\alpha^\dagger(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = 0 \quad (\text{C.34a})$$

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = \delta^3(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta} \quad (\text{C.34b})$$

where in this instance α and β denote the spinor components of ψ .

Expanding to the creation and annihilation operators the *equal-time anti-commutation* relation is given by:

$$[\hat{b}_\mathbf{p}^r, \hat{b}_\mathbf{q}^{s\dagger}] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \delta^{rs} \quad (\text{C.35a})$$

$$[\hat{c}_\mathbf{p}^r, \hat{c}_\mathbf{q}^{s\dagger}] = -(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \delta^{rs}. \quad (\text{C.35b})$$

The Hamiltonian H in Eq. (C.31) will be

$$H = \frac{d^3p}{(2\pi)^3} E_\mathbf{p} [\hat{b}_\mathbf{p}^\dagger \hat{b}_\mathbf{p} - \hat{c}_\mathbf{p} \hat{c}_\mathbf{p}^\dagger] \quad (\text{C.36})$$

$$= \frac{d^3p}{(2\pi)^3} E_\mathbf{p} [\hat{b}_\mathbf{p}^\dagger \hat{b}_\mathbf{p} + \hat{c}_\mathbf{p}^\dagger \hat{c}_\mathbf{p} - (2\pi)^3 \delta^3(0)] \quad (\text{C.37})$$

This will mean that the Hamiltonian is bounded. The second term of Eq. C.37 is the Dirac Sea that was discussed in Sec. A.5.2. The Dirac fields describes spin- $\frac{1}{2}$ particles (fermions).

C.4.1 Neutron and proton

The neutron and the proton are also taken to be Dirac particles of the same mass m . This means that we can construct a Lagrangian of the sum of the two particles given by[1]

$$\mathcal{L} = \bar{\psi}_p (i\gamma_\mu \partial^\mu - m) \psi_p + \bar{\psi}_n (i\gamma_\mu \partial^\mu - m) \psi_n \quad (\text{C.38})$$

Where $\bar{\psi}_p$ is the adjoint spinor for the protons and ψ_p is the spinor for the protons. $\bar{\psi}_n$ is the adjoint spinor for the neutrons and ψ_n is the spinor for the neutrons.

The spinors can be combined into one to give an eight component Dirac spinor [1]

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}. \quad (\text{C.39})$$

The Lagrangian can then be given by [1]

$$\mathcal{L}_{nuc} = \bar{\psi} (i\gamma_\mu \partial^\mu - m) \psi. \quad (\text{C.40})$$

This chapter described how the application of classical field theory is important to quantum mechanics in order to properly describe relativistic multi-particle systems. Quantum field theory has proven to be a very useful tool. Next, an example of quantum field theory will be given for nuclear matter in order to build up to getting a nuclear equation of state that we can use to solve the TOV equations.

APPENDIX D

Evaluation of Integrals

In this chapter an evaluation of the integrals used is given.

D.1 Energy Density

The energy density described in this work has an integral of the form

$$\frac{1}{\pi^2} \int_0^{k_F} dk \sqrt{k^2 + m^2} \quad (\text{D.1})$$

where

- k_F is the Fermi momentum of the particle and
- m is the mass of the particle.

The integral can be solved by making the following assumption

$$\frac{k}{m} \rightarrow u, \quad (\text{D.2})$$

$$\frac{k_F}{m} \rightarrow x_F. \quad (\text{D.3})$$

Substituting into Eq. (D.1) gives

$$\frac{1}{\pi^2} \int_0^{k_F} dk \sqrt{k^2 + m^2} = \frac{m^4}{\pi^2} \int_0^{k_F/m} du u^2 \sqrt{u^2 + 1} \quad (\text{D.4})$$

$$= \frac{m^4}{\pi^2} \int_0^{x_F} du u^2 \sqrt{u^2 + 1}. \quad (\text{D.5})$$

Applying intergration by parts the solution will be

$$\frac{m^4}{\pi^2} \int_0^{x_F} du u^2 \sqrt{u^2 + 1} = \frac{m^4}{8\pi^2} [(2x_F^3 + x_F)(1 + x_F^2)^1 - \sinh^{-1}(x_F)]. \quad (\text{D.6})$$

D.2 Pressure

The pressure described in this work has an integral of the form

$$\frac{1}{3\pi^2} \int_0^{k_F} dk \frac{k^4}{\sqrt{k^2 + m^2}} \quad (\text{D.7})$$

Applying the same assumption from Eq. (D.2) and (D.3) we get

$$\frac{1}{3\pi^2} \int_0^{k_F} dk \frac{k^4}{\sqrt{k^2 + m^2}} = \frac{m^4}{3\pi^2} \int_0^{k_F/m} du \frac{u^4}{\sqrt{u^2 + 1}} \quad (\text{D.8})$$

$$= \frac{m^4}{3\pi^2} \int_0^{x_F} du \frac{u^4}{\sqrt{u^2 + 1}}. \quad (\text{D.9})$$

The solution to the integral is given by

$$\frac{m^4}{3\pi^2} \int_0^{x_F} du \frac{u^4}{\sqrt{u^2 + 1}} = \frac{1}{8} \frac{m^4}{3\pi^2} [(2x_F^3 - 3x_F)(1 + x_F^2)^{1/2} + 3 \sinh^{-1}(x_F)]. \quad (\text{D.10})$$

D.3 The MFT scalar meson field

Eq. (3.34b) for the scalar meson field has the integral

$$\frac{1}{\pi^2} \int_0^{k_F} dk \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}} \quad (\text{D.11})$$

The solution to the integral is given by[10]

$$\frac{1}{\pi^2} \int_0^{k_F} dk \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}} = \frac{1}{\pi^2} [k_F (k_F^2 + m^2)^{1/2} - m^{*2} \ln \left(\frac{k_F + (k_F^2 + m^2)^{1/2}}{m^*} \right)] \quad (\text{D.12})$$

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