DAGUM POWER SERIES DISTRIBUTIONS: THEORY AND APPLICATIONS

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Certification

This undersigned certifies that she has read and hereby recommends for acceptance by the College of Science a dissertation/thesis titled: DAGUM POWER SERIES DISTRIBUTIONS: THEORY AND APPLICATIONS, in fulfillment of the requirements for the Degree of Master of Science in Statistics of BIUST

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Abstract

A new generalized Dagum distribution called the Dagum power series distributions is introduced and studied. This distribution is obtained by compounding Dagum and other power series distributions. The structural properties of the new distributions are discussed, including explicit algebraic formulas for its survival and hazard functions, quantile function, moments, moment generating function, conditional moments, mean and median deviations, Bonferroni and Lorenz curves, distribution of order statistics and Renyi entropy. Method of maximum likelihood is used for estimating the model parameters. A real data set is used to illustrate the usefulness, applicability, importance and flexibility of the new distribution.

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Chapter 1

INTRODUCTION

1.1 Introduction

There are many ways of generating continuous distributions. In this thesis, we look at the method of compounding. Some methods of generating continuous distributions are explained in the next section.

1.2 Some Methods of Generating Families of Continuous Distributions

1.2.1 T-X Family of Distributions Method

(Alzaatreh, [34]) developed a method to generate a family of continuous distributions called T-X family of distributions. The T-X family is a method for generating generalized distributions of X using T . The random variable X is known as ”the transformer” and the random variable T is known as ”the transformed”. The resulting family has a connection with the hazard functions where each generated distributions is considered as weighted hazard function of the random variable X.

Several known continuous distributions are found to be special cases of this family. This family is defined as follows: Let $r(t)$ be the (pdf) of a random variable $T \in [a, b]$, for $-\infty < a < b < \infty$, where $F(x)$ is the cdf of the random variable X and let $W[F(x)]$ be a function of the cdf of a random variable X such that $W[F(x)] \in [a, b]$, $W[F(x)]$ is differentiable and monotonically non-decreasing and $W[F(x)] \to a$ as $x \to -\infty$ and $W[F(x)] \to b$ as $x \to \infty$. The cdf of the T-X family of distributions is defined by

$$G(x) = \int_a^{W[F(x)]} r(t)dt, \tag{1.1}$$

and the corresponding pdf of the T-X family of distributions is given by

$$g(x) = r\Big(W[F(x)]\Big)\left(\frac{d}{dx}W[F(x)]\right). \tag{1.2}$$

This method has been used by Alzaatreh et al in [34], Aljarrah et al in [35] and Jayakumar et al in [36].

1.2.2 Compounding Family of Distributions Method

Suppose a random variable $X = max(X_1, X_2, ..., X_N)$ is continous with cdf $F(X|N = n)$. Also let N be a discrete random variable with power series distribution (truncated at zero) and probability mass function $P(N = n) = \frac{a_n\theta^n}{C(\theta)}$, where $n = 1, 2, ...,$ and $a_n \geq 0$ depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$ and $\theta \in (0, b]$. The distribution $F(x; n)$, which is a compound distribution of X and N is given by

$$F(x; n) = F_{X|N=n}(x)P(N = n) = \frac{a_n\theta^n F_{X|N}(x)}{C(\theta)}, \tag{1.3}$$

where $x > 0$ and $n = 1, 2, \cdots$. This method has been used by Silva et al in [37], Morais et al in [26] and Lu and Shi in [22].

1.2.3 Competing Risk Family of Distributions Method

Consider paired competing risks data (T_i, δ_i) where T_i is the time to failure and δ_i indicates the cause of failure for subject $i, i = 1.2...., $ in the pair (Brazauskas et at [41]). Without loss of generality, we will consider two competing risks with cause 1 being the cause of interest and cause 2 encompassing all other acting competing risks. Then the event indicator δ_i is define as follows;

$\delta_i = 1$, if i^{th} individual failed from cause 1 and $\delta_i = 2$, if i^{th} individual failed from cause 2.

Note that the data consists of n such independent pairs. In many instances, data subjected to competing risks are summarized in terms of cumulative incidence functions which appropriately qualify the cumulative incidence function for cause k is defined as follows:

$$F_k^i(s) = P(T_i, \delta_i = k); i = 1, 2; k = 1, 2. \tag{1.4}$$

This cumulative incidence function depends on the cause-specific hazards for both competing risks affecting the population. This method has been used by Crowder in [43], Gauss et al in [42] and Brazauskas et al in [41].

1.2.4 Exponentiated Generalized Class of Distributions Method

Given the continous cdf $G(x)$, the cdf and pdf of the exponentiated generalized (EG) class of distributions are given by

$$F(x) = \left[1 - \{1 - G(x)\}^{\alpha}\right]^{\beta} \tag{1.5}$$

and

$$f(x) = \alpha\beta \{1 - G(x)\}^{\alpha-1} \left[1 - \{1 - G(x)\}^{\alpha}\right]^{\beta-1} g(x), \tag{1.6}$$

respectively, where $\alpha, \beta > 0$. This method has been used by Pal et al in [40], Gauss et al in [39] and Makubate et al in [38].

1.3 Statistical Properties of Probability Distributions

1.3.1 Survival, Hazard and Reverse Hazard Functions

Survival Function

In reality and survival analysis, where a random variable represents the random lifetime of a unit, a device or and organism, the survival funtion is the probability to survive beyond x units of time. Given a random variable X with distribution function F, it's survival function, denoted by $\bar{F}$, is defined as

$$\bar{F}(x) = 1 - F(x), \tag{1.7}$$

for all $x \in \mathbb{R}$.

Hazard Function

The hazard function of the pdf $f(x)$ given by

$$h_f(x) = \frac{f(x)}{1 - F(x)}; x \in \mathbb{R} \tag{1.8}$$

Reverse Hazard Function

The reverse hazard function of the pdf $f(x)$ is given by

$$\tau_f(x) \quad = \quad \frac{f(x)}{F(x)}; x \in \mathbb{R} \tag{1.9}$$

1.3.2 Quantile Functions

The q^{th} quantile of the pdf $f(x)$ is obtained by solving the nonlinear equation

$$F(x) = U,$$

where U is a uniform variate on the unit interval $[0, 1]$.

1.3.3 Moments

If the random variable X has a distribution$f(x)$, where $x \in \mathbb{R}$, then the r^{th} moment of X is given by

$$E[X^r] = \int^{\mathbb{R}} x^r f(x) dx \tag{1.10}$$

The r^{th} conditional moment for $f(x)$ is given by

$$E(X^r | X > t) = \frac{1}{\overline{F}(t)} \int_t^{\infty} x^r f(x) dx \tag{1.11}$$

1.3.4 Mean and Median Deviations

If X has the distribution $f(x)$, we can derive the mean deviation about the mean $\mu = E(X)$ and the mean deviation about the median M from

$$\delta_1 = \int_0^{\infty} |x - \mu| f(x) dx \quad and \quad \delta_2 = \int_0^{\infty} |x - M| f(x) dx,$$

respectively.

1.3.5 Bonferroni and Lorenz Curves

Let $I(a) = \int_0^a x f_{_{DaPS}}(x) dx$ and $\mu = E(X)$, then Bonferroni, $B(p)$ and Lorenz, $(L(p)$ curves are given by

$$B(p) = \frac{I(q)}{p\mu} \quad and \quad L(p) = \frac{I(q)}{\mu},$$

respectively, for $0 \leq p \leq 1$ and $q = F^{-1}_{DaPS}(p)$.

1.3.6 Rényi Entropy

Rényi Entropy of a distribution with pdf $f_{DaPS}(x)$ is given by

$$I_R(\tau) = (1-\tau)^{-1} log\left\{\int_{\mathbb{R}} f_{DaPS}^{\tau}(x)dx\right\}, \tau > 0, \tau \neq 0. \tag{1.12}$$

1.3.7 Order Statistics

The pdf of the k^{th} order statistics from a pdf $f(x)$ is

$$\begin{aligned} f_{k:n}(x) &= \frac{f(x)}{B(k, n-k+1)} F^{k-1}(x)[1-F(x)]^{n-k} \\ &= k\binom{n}{k} f(x)F^{k-1}(x)[1-F(x)]^{n-k}, x \in \mathbb{R}. \end{aligned} \tag{1.13}$$

1.3.8 Goodness of Fit Statistic

A goodness-of-fit test refers to measuring how well does the observed data correspond to the fitted model. the test statistic for a goodness-of-fit test is $\sum_k \frac{(O-E)^2}{E}$, where O is the observed values, E is the expected values and k is the number of different data cells of categories. The observed values are the data values and the expected values are the values you expect to get if the null hypothesis were true. There are n terms of the form $\frac{(O-E)^2}{E}$. The number of degrees of freedom is df =(number of categories - 1).

1.3.9 Likelihood Ration Test

Likelihood ratio tests (LRT) are used to compare two models provided that the simpler model is a special case of the more complex model (nested). LRTs can be presented as a difference in log-likelihoods and this is often handy as they can be expressed in terms of deviance. Then

$$LRT = -2(\log(L) - \log(L1)), \tag{1.14}$$

where $\log(L)$ is the log-likelihood of a more complex model and $\log(L1)$ is the log-likelihood of a nested model.

1.4 Some Useful Distributions and Their Statistical Properties

1.4.1 Dagum Distribution

Dagum distribution was proposed by Camilo Dagum in 1970's (Dagum 1977-1980) [12],[13]. Dagum distribution is used to fit empirical income and wealth data, that could accommodate both heavy tails in empirical income and wealth distributions, and also permit interior mode. Dagum distribution is a special case of generalized beta distribution of the second kind (GB2), McDonald [24], McDonald and Xu [25], when the parameter $q = 1,$ where the probability density function (pdf) of the GB2 distribution is given by:

$$f_{GB2}(y; a, b, p, q) = \frac{ay^{ap-1}}{b^{ap}B(p,q)[1 + (\frac{y}{b})^a]^{p+q}} \quad \text{for } y > 0.$$

Note that $a > 0, p > 0, q > 0,$ are the shape parameters and $b > 0$ is the scale parameter and $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the beta function. Kleiber [19] traced the genesis of Dagum distribution and summarized several statistical properties of this distribution. See Kleiber and Kotz [20] for additional results on income and size distributions. Domma et al. [15] obtained the maximum likelihood estimates of the parameters of Dagum distribution for censored data. Domma and Condino [14] presented the beta-Dagum distribution. Huang and Oluyede [17] presented the exponentiated Kumaraswamy Dagum distribution and applied the model to income and lifetime data. Oluyede et al. [27] developed a generalized gamma-Dagum distribution with applications to income and lifetime data.

The probability density function (pdf) and cumulative distribution function (cdf)

of Dagum distribution are given by:

$$f_{Da}(x;\lambda,\beta,\delta) = \beta\lambda\delta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1} \tag{1.15}$$

and

$$F_{Da}(x;\lambda,\beta,\delta) = (1+\lambda x^{-\delta})^{-\beta}, \tag{1.16}$$

for $x > 0$, where λ is a scale parameter, δ and β are shape parameters. Dagum [12] refers to his model as the generalized logistic-Burr distribution. The survival and hazard rate functions of Dagum distribution are $\overline{G}(x;\lambda,\delta,\beta) = 1-(1+\lambda x^{-\delta})^{-\beta}$ and $h(x;\lambda,\delta,\beta) = \frac{\beta\lambda\delta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}}{1-(1+\lambda x^{-\delta})^{-\beta}}$, respectively. The k^{th} raw or non central moments of Dagum distribution are given by

$$E\left(X^k\right) \;=\; \beta\lambda^{\frac{k}{\delta}}B\left(\beta+\frac{k}{\delta},1-\frac{k}{\delta}\right), \tag{1.17}$$

for $\delta > k$, and $\lambda,\beta > 0$, where $B(\cdot,\cdot)$ is the beta function (Dey et al, [44]). The q^{th} percentile (Aslam et al [45])of the Dagum distribution is

$$x_q = \lambda^{\frac{1}{\delta}}\left(q^{\frac{-1}{\beta}}-1\right)^{\frac{-1}{\delta}}. \tag{1.18}$$

Dagum distribution has positive asymmetry, and its hazard rate can be monotonically decreasing, upside-down bathtub and bathtub followed by upside-down bathtub (Domma, [16]). This behavior has led several authors to study the model in different fields. In fact, recently, the Dagum distribution has been studied from a reliability point of view and used to analyze survival data (see Domma et al. [15]). When $\beta = 1,$ Dagum distribution is referred to as the log-logistic distribution. Burr type III distribution is obtained when $\lambda = 1.$

1.4.2 Poisson Distribution

The Poisson distribution truncated at zero, with parameter $\theta \in [0,\infty)$ is a power series distribution relative to the function

$$C(\theta) = e^{\theta} - 1. \tag{1.19}$$

Since the pmf of the Poisson distribution truncated at zero is given by

$$P(N = n) = \frac{\theta^n e^{-\theta}}{n!(1 - e^{-\theta})} = \frac{\frac{1}{n!}\theta^n}{e^{\theta} - 1} = \frac{a_n \theta^n}{C(\theta)}, n = 1, 2, \ldots \tag{1.20}$$

1.4.3 Geometric Distribution

The pmf of the geometric distribution truncated at zero is given by

$$P(N = n) = \theta(1 - \theta)^{n-1}; n = 1, 2, \ldots \tag{1.21}$$

The geometric distribution truncated at zero, with parameter $\theta \in [0, 1]$ is a power series distribution relative to the function

$$C(\theta) = \theta(1 - \theta)^{-1}. \tag{1.22}$$

1.4.4 Logarithmic Distribution

The pmf of the logarithmic distribution truncated at zero is given by

$$P(N = n) = \frac{\theta^n}{-n \log(1 - \theta)}; n = 1, 2, \ldots \tag{1.23}$$

The logarithmic distribution truncated at zero, with parameter $\theta \in [0, 1]$ is a power series distribution relative to the function

$$C(\theta) = -\log(1 - \theta). \tag{1.24}$$

1.4.5 Binomial Distribution

The Binomial distribution truncated at zero, with parameter $\theta \in [0, 1]$ is a power series distribution relative to the function

$$C(\theta) = (1 + \theta)^m - 1 \tag{1.25}$$

1.4.6 Power Series Distributions

Suppose that $(a_1, a_2, a_3, \ldots)$ is a sequence of non-negative real numbers. The power series is the defined by

$$C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n. \tag{1.26}$$

A random variable N has the power series distribution associated with $C(\theta)$, if N has a discrete probability function of the form

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}; n = 1, 2 \ldots. \tag{1.27}$$

1.5 Concluding Remarks

Some methods of generating continuous distributions were introduced, these include the T-X family of distributions, compounding family of distributions, competing risks and exponentiated generalized class of distributions. Some properties of probability distributions were introduced, such as the harzard and reverse hazard, and quantile functions. We also introduced the Power series distributions namely, Poisson, Binomial, Geometric and Logarithmic distributions, which we then use to compute the special cases Dagum Poisson, Dagum Binomial, Dagum Geometric and Dagum Logarithmic distributions.

1.6 Objectives

The intention of this study is to successfully establish a competitive model for empirical income and lifetime data. To achieve this objective, the study is governed by a set of specific objectives below:

1. Develop the Dagum Power Series $(DaPS)$ distribution
2. Study the properties of the $DaPS$ distribution, and
3. Evaluating the $DaPS$ model.

1.7 Motivation

Camilo Dagum (Dagum 1977-1980) develped a new model called the Dagum distribution in order to fit emperical income and wealth data as an alternate to the Pareto and Log-normal models. This distribution has been widely used in various fields including income and wealth data, meteorological data, reliability and survival analysis. The flexibility of this distribution that accomodates both heavy tails in emperical income and wealth distributions and also permits interior mode has motivated this study. A new class of distributions called the Dagum Power series class of distributions will be studied, together with some of its statistical properties.

1.8 Outline of Thesis

This thesis is organized as follows; Chapter 1 outlines the background of the study, objectives of the study, and the motivation of the study. In Chapter 2 we present the new class of distribution, its pdf, cdf, special cases and properties, hazard and reverse hazard functions; mean and median deviations, Bonferroni and Lorenz curve, order statistics and Renyi entropy, estimation and inference. Chapter 3 gives some statistical properties of the new proposed class of distributions which are the Dagum Poisson and Dagum Logarithmic distributions and applications to

lifetime data is performed. In Chapter 4 the a conclusion is made and area of further study is also presented. The reference material used and the appendices are also included.

Chapter 2

THE DAGUM POWER SERIES DISTRIBUTION AND SOME STATISTICAL PROPERTIES

2.1 Introduction

This chapter covers the Dagum power series class of distributions and some statistical propeties of the $DaPS$ class of distributions. The survival, hazard, reverse hazard and quantile functions are introduced. The series expansion of the $DaPS$ distribution is studied. Other properties like the r^{th} moment, coditional moments, mean, mean deviations, Bonferroni and Lorenz curves are also studied. The general forms of these properties are computed. We also introduce the order statistics and Rényi Entropy of the $DaPS$ distribution.

2.2 Dagum Power Series Distribution

Suppose that the random variable X=max$(X_1, \ldots, X_N)$ has the Dagum distribution where its pdf and cdf are given in equations (1.15) and (1.16), respectively. Given

N, let $X_1, ..., X_N$ be independent and identically distributed random variables from Dagum distribution. Let N be a discrete random variable with a power series distribution (truncated at zero) and probability mass function (pmf)

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, ...$$

where $a_n \geq 0$ depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ and $\theta \in (0, b)$ (b can be ∞) is chosen such that $C(\theta)$ is finite and its three derivatives with respect to θ are defined and given by $C'(.), C''(.)$ and $C'''(.)$, respectively. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions. See Table 2.1 for some useful quantities including a_n, $C(\theta)$, $C^{-1}(\theta)$, $C'(\theta)$, and $C''(\theta)$ for the Poisson, geometric, logarithmic and binomial distributions.

Let X=max$(X_1, \ldots, X_N)$, then the cumulative distribution function (cdf) of $X|N = n$ is given by:

$$G_{X|N=n}(x) = [1 + \lambda x^{-\delta}]^{-n\beta}, \tag{2.1}$$

$x > 0, \lambda, \beta, \delta > 0$, which is the Dagum distribution with parameters $\lambda, \delta, n\beta$. The Dagum Power Series $(DaPS)$ class of distributions is defined by the marginal cdf of X which is illustrated below. The joint distribution of X and N is given by

$$\begin{aligned} F(x; n) &= G_{X|N=n}(x) P(N = n) \\ &= [1 + \lambda x^{-\delta}]^{-n\beta} P(N = n) \\ &= [1 + \lambda x^{-\delta}]^{-n\beta} \frac{a_n \theta^n}{C(\theta)} \\ &= \frac{(\theta[1 + \lambda x^{-\delta}]^{-\beta})^n a_n}{C(\theta)}. \end{aligned} \tag{2.2}$$

Now, the marginal cdf of X is given by

Table 2.1: Useful Quantities for Some Power Series Distributions

Distribution	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	a_n	Parameter Space
Poisson	$e^\theta - 1$	e^θ	e^θ	$\log(1+\theta)$	$(n!)^{-1}$	$(0,\infty)$
Geometric	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(1+\theta)^{-1}$	1	$(0,1)$
Logarithmic	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{-\theta}$	n^{-1}	$(0,1)$
Binomial	$(1+\theta)^m - 1$	$m(1+\theta)^{m-1}$	$m(m-1)(1+\theta)^{m-2}$	$(\theta-1)^{1/m} - 1$	$\binom{m}{n}$	$(0,1)$

$$\begin{aligned} F_{DaPS}(x) &= \sum_{n=1}^{\infty} \frac{(\theta[1+\lambda x^{-\delta}]^{-\beta})^n a_n}{C(\theta)} \\ &= \frac{1}{C(\theta)} \sum_{n=1}^{\infty} (\theta[1+\lambda x^{-\delta}]^{-\beta})^n a_n \\ &= \frac{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)}. \end{aligned} \tag{2.3}$$

Consequently, the general form of the cdf of the $DaPS$ class of distributions is given by

$$F_{DaPS}(x;\beta,\delta,\lambda,\theta) = \frac{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)}, \tag{2.4}$$

for $\lambda,\beta,\delta,\theta > 0$ and $x \geq 0$.

The general form of the pdf of the $DaPS$ class of distributions is given by

$$f_{DaPS}(x;\beta,\delta,\lambda,\theta) = \frac{d}{dx}\left[F_{DaPS}(x;\beta,\delta,\lambda,\theta\right] \tag{2.5}$$

$$= \frac{d}{dx}\frac{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} \tag{2.6}$$

$$= \frac{\theta(-\beta)[1+\lambda x^{-\delta}]^{-\beta-1}(-\delta)\lambda x^{-\delta-1} C'(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} \tag{2.7}$$

$$= \frac{\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1} C'(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)}, \tag{2.8}$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$, $x > 0$ and $a_n > 0$ depend only on n.

2.3 Survival, Hazard, Reverse Hazard and Quantile Function

This section presents the survival, hazard and reverse hazard functions as well as the quantile of the $DaPS$ distribution together with its special cases.

2.3.1 Survival Function

The survival function of the $DaPS$ distribution is given by

$$\bar{F}_{DaPS}(x) = \frac{C(\theta) - C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)}, \tag{2.9}$$

$$\tag{2.10}$$

where $x > 0, \beta, \delta, \lambda, \theta > 0$.

2.3.2 Hazard Function

The hazard function of the $DaPS$ distribution is given by

$$\begin{aligned} h_{DaPS}(x;\beta,\delta,\lambda,\theta) &= \frac{f_{DaPS}(x;\beta,\delta,\lambda,\theta)}{1 - F_{DaPS}(x;\beta,\delta,\lambda,\theta)} \\ &= \frac{\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1} C'(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta) - C(\theta[1+\lambda x^{-\delta}]^{-\beta})}. \end{aligned} \tag{2.11}$$

2.3.3 Reverse Harzard Function

The reverse hazard function of the $DaPS$ distribution is given by

$$\begin{aligned} \tau_{DaPS}(x;\beta,\delta,\lambda,\theta) &= \frac{f_{DaPS}(x;\beta,\delta,\lambda,\theta)}{F_{DaPS}(x;\beta,\delta,\lambda,\theta)} \\ &= \frac{\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1} C'(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}. \end{aligned} \tag{2.12}$$

2.3.4 Quantile Function

In this subsection, we present the q^{th} quantile of the $DaPS$ class of distributions. The q^{th} quantile of the $DaPS$ class of distributions is obtained by solving the non-linear equation

$$\frac{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} = U,$$

where U is a uniform variate on the unit interval $[0,1]$. Below is the illustration of the q^{th} quantile of the $DaPS$ class of distributions.

$$\begin{aligned}
\frac{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} &= U \\
C(\theta[1+\lambda x^{-\delta}]^{-\beta}) &= UC(\theta) \\
\theta[1+\lambda x^{-\delta}]^{-\beta} &= C^{-1}[UC(\theta)] \\
[1+\lambda x^{-\delta}]^{-\beta} &= \frac{C^{-1}[UC(\theta)]}{\theta} \\
x^{-\delta} &= \frac{1}{\lambda}\left(\left[\frac{C^{-1}[UC(\theta)]}{\theta}\right]^{-\frac{1}{\beta}} - 1\right) \\
x &= \left(\frac{1}{\lambda}\left(\left[\frac{C^{-1}[UC(\theta)]}{\theta}\right]^{-\frac{1}{\beta}} - 1\right)\right)^{-\frac{1}{\delta}}.
\end{aligned}$$

Therefore, the q^{th} quantile of the $DaPS$ class of distributions is given by,

$$X_q = \left(\frac{1}{\lambda}\left(\left[\frac{C^{-1}[UC(\theta)]}{\theta}\right]^{-1/\beta} - 1\right)\right)^{-1/\delta}. \tag{2.13}$$

Consequently, random number can be generated from the $DaPS$ class of disrtibutions based on equation (2.13).

2.4 Series Expansion of The Density Function

By using the fact that $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$, a series representation of the $DaPS$ cdf is given by

$$\begin{aligned} F_{DaPS}(x;\beta,\delta,\lambda,\theta) &= \frac{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} && (2.14)\\ &= \sum_{n=1}^{\infty} \frac{a_{n+1}(\theta[1+\lambda x^{-\delta}]^{-\beta})^{n+1}}{C(\theta)} && (2.15)\\ &= \sum_{n=1}^{\infty} \frac{a_{n+1}\theta^{n+1}[1+\lambda x^{-\delta}]^{-\beta}(n+1)}{C(\theta)} && (2.16)\\ &= \sum_{n=1}^{\infty} \omega(n,\theta) F_{Da}(x,\lambda,\delta,\beta(n+1)), && (2.17) \end{aligned}$$

where $\omega(n,\theta) = \frac{a_{n+1}\theta^{n+1}}{C(\theta)}$, and $F_{Da}(x,\lambda,\delta,\beta(n+1))$ is the Dagum cdf with parameters $\lambda, \beta(n+1), \delta > 0$. Similarly, the $DaPS$ pdf can be written as

$$f_{DaPS}(x;\beta,\delta,\lambda,\theta) = \sum_{n=1}^{\infty} \omega(n,\theta) f_{Da}(x;\lambda,\delta,\beta(n+1)), \quad (2.18)$$

where $f_{Da}(x;\lambda,\delta,\beta(n+1))$ is the Dagum pdf with parameters $\lambda, \beta(n+1), \delta > 0$.

2.5 Moments and Conditional Moments

In this subsection, the r^{th} moment and the conditional moment of $DaPS$ class of distributions are presented. Moments are necessary and crucial in any statistical analysis, especially in applications. Moments can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). If the random variable X has a $DaPS$ distribution, with

parameter vector $\varphi = (\lambda, \delta, \beta, \theta)$, then the r^{th} moment of X is given by

$$\begin{aligned} E[X^r] &= \int_0^\infty x^r f_{DaPS}(x;\beta,\delta,\lambda,\theta)dx \\ &= \sum_{n=1}^{\infty} \omega(n,\theta) \int_0^\infty x^r f_{Da}(x;\beta(n+1),\delta,\lambda,\theta)dx \\ &= \sum_{n=1}^{\infty} \omega(n,\theta)\beta(n+1)\lambda^{\frac{r}{\delta}} B\left(\frac{r}{\delta}+\beta(n+1), 1-\frac{r}{\delta}\right). \end{aligned} \tag{2.19}$$

Note that the r^{th} non-central moment follows readily from the fact that $DaPS$ pdf can be written as a linear combination of Dagum densities with parameters $\lambda, \beta(n+1), \delta > 0$.

The r^{th} conditional moment for $DaPS$ distribution is given by

$$\begin{aligned} E(X^r|X>t) &= \frac{1}{\overline{F}_{_{DaPS}}(t)} \int_t^\infty x^r f_{_{DaPS}}(x)dx \\ &= \frac{1}{\overline{F}_{_{DaPS}}(t)} \sum_{n=0}^{\infty} \omega(n,\theta)\lambda^{\frac{r}{\delta}} \left[B\left(\beta(n+1)+\frac{r}{\delta}, 1-\frac{r}{\delta}\right) \right. \\ &- \left. B_t\left(\beta(n+1)+\frac{r}{\delta}, 1-\frac{r}{\delta}\right)\right], \end{aligned} \tag{2.20}$$

where $B_t(c,d) = \int_0^t u^{c-1}(1-u)^{d-1}du$ and $\delta > r$. The mean residual life function is $E(X|X>t) - t$.

2.6 Mean and Median Deviations, Bonferroni and Lorenz curves

In this section, mean and median deviations, as well as Bonferroni and Lorenz curves of the $DaPS$ distribution are presented. If X has the $DaPS$ distribution, we can derive the mean deviation about the mean $\mu = E(X)$ and the mean deviation about the median M from

$$\delta_1 = \int_0^\infty |x-\mu| f_{_{DaPS}}(x)dx \quad and \quad \delta_2 = \int_0^\infty |x-M| f_{_{DaPS}}(x)dx,$$

respectively. The mean μ is obtained from equation (2.19) with $r = 1$, and the median M is given by equation (2.13) when $q = \frac{1}{2}$.

The measure δ_1 and δ_2 can be calculated by the following relationships:

$$\delta_1 = 2\mu F_{DaPS}(\mu) - 2\mu + 2T(\mu) \quad = 2\mu F_{DaPS}(\mu) - 2\mu + 2\int_{\mu}^{\infty} x f_{DaPS}(x)dx$$

and

$$\delta_2 = 2T(M) - \mu = 2\int_{M}^{\infty} x f_{DaPS}(x)dx,$$

where $T(a) = \int_a^{\infty} x f_{_{DaPS}}(x)dx$ follows from equation (2.20), that is,

$$T(a) = \sum_{n=0}^{\infty} \omega(n,\theta)\lambda^{\frac{1}{\delta}} \left[B\left(\beta(n+1) + \frac{1}{\delta}, 1 - \frac{1}{\delta}\right) - B_a\left(\beta(n+1) + \frac{1}{\delta}, 1 - \frac{1}{\delta}\right)\right].$$

Let $I(a) = \int_0^a x f_{_{DaPS}}(x)dx$ and $\mu = E(X)$, then Bonferroni, $B(p)$ and Lorenz, $(L(p)$ curves are given by

$$B(p) = \frac{I(q)}{p\mu} \quad and \quad L(p) = \frac{I(q)}{\mu},$$

respectively, for $0 \leq p \leq 1$, and $q = F_{DaPS}^{-1}(p)$. The mean of the $DaPS$ distribution is obtained from equation (2.19) with $r = 1$ and the quantile function is given in equation (2.13). Consequently,

$$I(a) = \sum_{n=0}^{\infty} \omega(n,\theta)\lambda^{\frac{1}{\delta}} B_a\left(\beta(n+1) + \frac{1}{\delta}, 1 - \frac{1}{\delta}\right), \tag{2.21}$$

for $\delta > 1$, where $B_{F(x)}(c,d) = \int_0^{F(x)} t^{c-1}(1-t)^{d-1}dt$ for $|F(x)| < 1$ is incomplete Beta function.

2.7 Order Statistics and Renyi Entropy

In this section, the distribution of the k^{th} order statistic and R$\acute{e}$nyi entropy for the $DaPS$ class of distributions are presented. The entropy of a random variable is a

measure of variation of the uncertainty.

2.7.1 Order Statistics

The pdf of the k^{th} order statistics from a $pdf\ f(x)$ is

$$\begin{aligned} f_{k:n}(x) &= \frac{f(x)}{B(k, n-k+1)} F^{k-1}(x)[1-F(x)]^{n-k} \\ &= k\binom{n}{k} f(x) F^{k-1}(x)[1-F(x)]^{n-k}. \end{aligned} \tag{2.22}$$

We apply the series expansion

$$(1-z)^{b-1} = \sum_{j=b-1}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j!} z^j, \tag{2.23}$$

for $b > 1$ and $|z| < 1$, to obtain the series expansion of the distribution of order statistics from $DaPS$ distribution. Using equations (2.23), the pdf of the k^{th} order statistic from $DaPS$ distribution is given by

$$\begin{aligned} f_{k:n}(x) &= \sum_{s=0}^{n-k} k\binom{n}{k} \frac{(-1)^s f(x) \Gamma(n-k+1)}{s! \Gamma(n-k+1-s)} [F(x)]^{s+k-1} \\ &= \sum_{s=0}^{\infty} k\binom{n}{k} \frac{(-1)^s \Gamma(n-k+1)}{s! \Gamma(n-k+1-s)} \frac{\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1} C'(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} \\ &\times \left(\frac{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)}\right)^{s+k-1}. \end{aligned} \tag{2.24}$$

2.7.2 Rényi Entropy

Rényi Entropy of a distribution with pdf f(x) is given by

$$I_R(\tau) = (1-\tau)^{-1} log\left\{\int_{\mathbb{R}} f^{\tau}(x) dx\right\}, \tau > 0, \tau \neq 0. \tag{2.25}$$

For the Dagum Power series distribution,

$$f_{DaPS}^{\tau}(x) = \left(\frac{\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1} C'(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} \right)^{\tau} \tag{2.26}$$

The entropy for the Dagum Power series distribution is therefore, given by

$$I_R(\tau) = \frac{1}{1-\tau} log\left[\int_0^{\infty} \left(\frac{\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1} C'(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} \right)^{\tau} dx \right] \tag{2.27}$$

Rényi Entropy for the speacial cases can be readily obtained by substituting for the respective power series.

2.8 Estimation and Inference

2.8.1 Maximum Likelihood Estimation

Estimation of the unknown parameters of the $DaPS$ distribution by the method of maximum likelihood is addressed in this section. Let $x_1, ..., x_n$ be a random sample of size n from $DaPS$ distribution and $\Theta = (\lambda, \beta, \delta, \theta)^T$ be the parameter vector. The log-likelihood function for Θ based on this sample becomes

$$\begin{aligned}
L &= L_{DaPS}(\Theta) && (2.28)\\
&= \log\left[\prod_{i=1}^{n} f_{DaPS}(x_i; \beta, \delta, \lambda, \theta) \right] && (2.29)\\
&= \log\left[\prod_{i=1}^{n} \frac{\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1} C'(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} \right] && (2.30)\\
&= n\log\frac{\lambda\beta\delta\theta}{C(\theta)} - (\delta+1)\sum_{i=1}^{n}\log x_i - (\beta+1)\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta})\\
&+ \sum_{i=1}^{n}\log C'\Big(\theta(1+\lambda x_i^{-\delta})^{-\beta}\Big). && (2.31)
\end{aligned}$$

The score components corresponding to the parameters in Θ are given by

$$\begin{aligned}
\frac{\partial L}{\partial \lambda} &= \frac{n}{\lambda} - (\beta+1)\sum_{i=1}^{n}\frac{x_i^{-\delta}}{(1+\lambda x_i^{-\delta})} - \sum_{i=1}^{n}\beta\theta x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-\beta-1}\mu_i,\\
\frac{\partial L}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta}) + \sum_{i=1}^{n}\theta(1+\lambda x_i^{-\delta})^{-\beta}\log(1+\lambda x_i^{-\delta})\mu_i,\\
\frac{\partial L}{\partial \delta} &= \frac{n}{\delta} - \sum_{i=1}^{n}\log x_i + (\beta+1)\sum_{i=1}^{n}\frac{\lambda x_i^{-\delta}\log x_i}{(1+\lambda x_i^{-\delta})} + \sum_{i=1}^{n}\theta\lambda\beta(1+\lambda x_i^{-\delta})^{-\beta-1}x_i^{-\delta}\mu_i\log x_i,\\
&and\\
\frac{\partial L}{\partial \theta} &= \frac{n}{\theta} - \frac{nC'(\theta)}{C(\theta)} + \sum_{i=1}^{n}(1+\lambda x_i^{-\delta})^{-\beta}\mu_i, \qquad (2.32)
\end{aligned}$$

where $\mu_i = \frac{C''(\theta[1+\lambda x_i^{-\delta}]^{-\beta})}{C'(\theta[1+\lambda x_i^{-\delta}]^{-\beta})}$. The maximum likehood estimator (MLE), $\hat{\Theta} = (\hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\theta})$ can be obtained by solving simultaneously the nonlinear equations

$$\frac{\partial L}{\partial \lambda} = \frac{\partial L}{\partial \beta} = \frac{\partial L}{\partial \delta} = \frac{\partial L}{\partial \theta} = 0.$$

These equations are not linear in the parameters. Hence iterative methods are required to solve them. The Newton-Raphson method is an iterative method for solving nonlinear equations. To implement the Newton-Raphson method we require the second partial derivatives of log-likelihood function (2.28). The second partial derivatives are obtained as

$$\frac{\partial^2 L_n(\Theta)}{\partial\theta_i\partial\theta_j}. \qquad (2.33)$$

Note that (2.33) is a matrix called the Hessian matrix. In the Newton-Raphson method, provisional estimates for the vector of parameters Θ on iteration i are improved by

$$\Theta_{i+1} = \Theta_i - \frac{\partial L_n(\Theta)}{\partial\Theta_i}\left(\frac{\partial^2 L_n(\Theta)}{\partial\theta_i\partial\theta_j}\right)^{-1}.$$

This iterations continue until the changes in the parameter estimates and/or likelihood value are sufficiently small. At this point the solution is said to have converged, and the large-sample variance-covariance matrix of the maximum like-

lihood estimator is then obtained as the negative inverse of the matrix of second derivatives. Standard errors of the parameter estimates are obtained as the square root values of the diagonal entries of this (negative inverse) matrix. The maximum likelihood estimates and their accompanying standard errors can be used to compute asymptotic z-statistics (i.e., Wald statistics) or construct confidence intervals.

2.8.2 Fisher Information Matrix

In this section, we present a measure for the amount of information. This information can be used to obtain bounds on the variance of estimators, approximate the sampling distribution of an estimator and obtain an approximate confidence interval in the case of a large sample.

Let X be a random variable with the $DaPS$ pdf $f_{DaPS}(x;\boldsymbol{\Theta})$, where $\boldsymbol{\Theta} = (\theta_1,\theta_2,\theta_3,\theta_4)^T = (\lambda,\delta,\beta,\theta)^T$. Then, Fisher information matrix (FIM) is the 4×4 symmetric matrix with elements:

$$\mathbf{I}_{ij}(\boldsymbol{\Theta}) = E_{\boldsymbol{\Theta}}\left[\frac{\partial \log(f_{DaPS}(X;\boldsymbol{\Theta}))}{\partial\theta_i}\frac{\partial \log(f_{DaPS}(X;\boldsymbol{\Theta}))}{\partial\theta_j}\right].$$

If the density $f_{DaPS}(x;\boldsymbol{\Theta})$ has a second derivative for all i and j, then an alternative expression for $\mathbf{I}_{ij}(\boldsymbol{\Theta})$ is

$$\mathbf{I}_{ij}(\boldsymbol{\Theta}) = -E_{\boldsymbol{\Theta}}\left[\frac{\partial^2 \log(f_{DaP}(X;\boldsymbol{\Theta}))}{\partial\theta_i\partial\theta_j}\right].$$

For the $DaPS$ distribution, all second derivatives exist; therefore, the formula above is appropriate and most importantly significantly simplifies the computations. Elements of the FIM can be numerically obtained by R or MATLAB. The total FIM $\mathbf{I}_n(\boldsymbol{\Theta})$ can be approximated by

$$\mathbf{J}_n(\hat{\boldsymbol{\Theta}}) \approx \left[-\frac{\partial^2 \log L}{\partial\theta_i\partial\theta_j}\bigg|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}}\right]_{4\times 4}. \tag{2.34}$$

For real data, the matrix given in Equation (2.34) is obtained after the convergence of the Newton-Raphson procedure in MATLAB or R software.

2.8.3 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the $DaPS$ distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\mathbf{\Theta}} = (\hat{\lambda}, \hat{\delta}, \hat{\beta}, \hat{\theta})$ be the maximum likelihood estimate of $\mathbf{\Theta} = (\lambda, \delta, \beta, \theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \xrightarrow{d} N_4(\underline{0}, \mathbf{I}^{-1}(\mathbf{\Theta}))$, where $\mathbf{I}(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $\mathbf{I}(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $\mathbf{J}(\hat{\mathbf{\Theta}})$. The multivariate normal distribution with mean vector $\underline{0} = (0, 0, 0, 0)^T$ and covariance matrix $\mathbf{I^{-1}}(\mathbf{\Theta})$ can be used to construct confidence intervals for the model parameters. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for λ, δ, β and θ are given by

$$\hat{\lambda} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\lambda\lambda}(\hat{\mathbf{\Theta}})}, \quad \hat{\delta} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\delta\delta}(\hat{\mathbf{\Theta}})}, \quad \hat{\beta} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\beta\beta}(\hat{\mathbf{\Theta}})},$$

and $\hat{\theta} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\theta\theta}(\hat{\mathbf{\Theta}})}$, respectively, where $\mathbf{I}^{-1}_{\lambda\lambda}(\hat{\mathbf{\Theta}}), \mathbf{I}^{-1}_{\delta\delta}(\hat{\mathbf{\Theta}}), \mathbf{I}^{-1}_{\beta\beta}(\hat{\mathbf{\Theta}})$, and $\mathbf{I}^{-1}_{\theta\theta}(\hat{\mathbf{\Theta}})$ are diagonal elements of $\mathbf{I}^{-1}_{n}(\hat{\mathbf{\Theta}}) = (n\mathbf{I}\hat{\mathbf{\Theta}}))^{-1}$ and $Z_{\eta/2}$ is the upper $(\eta/2)^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the $DaPS$ distribution with its sub-models for a given data set. For example, to test $\theta = 0$, the LR statistic is $\omega = 2[\ln(L(\hat{\lambda}, \hat{\delta}, \hat{\beta}, \hat{\theta})) - \ln(L(\tilde{\lambda}, \tilde{\delta}, \tilde{\beta}, 0))]$, where $\hat{\lambda}$, $\hat{\delta}$, $\hat{\beta}$, and $\hat{\theta}$, are the unrestricted estimates, and $\tilde{\lambda}$, $\tilde{\delta}$, and $\tilde{\beta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_{\epsilon}$, where χ^2_{ϵ} denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degrees of freedom.

2.8.4 Concluding Remarks

The Dagum Power Series $(DaPS)$ distribution is introduced and studied. Some of its properties such as the pdf and cdf, harzard and reverse hazard, and quantile functions are also introduced. We also introduced the Power series distributions namely, Poisson, Binomial, Geometric and Logarithmic distributions, which

we then use to compute the special cases Dagum Poisson, Dagum Binomial, Dagum Geometric and Dagum Logarithmic distributions. The pdf's, cdf's, hazard and reverse hazard functions for the special cases are also computed. Lastly we also show that the $DaPS$ distribution can be written as a linear combination of the Dagum densities.

Chapter 3

SOME SPECIAL MODELS

3.1 Dagum Poisson Distributon

The general form of the cdf and pdf of the $DaPS$ class of distributions are given by

$$F_{DaPS}(x;\lambda,\delta,\beta,\theta)=\frac{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)}, \tag{3.1}$$

and

$$f_{DaPS}(x;\lambda,\delta,\beta,\theta)=\frac{\beta\lambda\delta\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}C'(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)}, \tag{3.2}$$

where $C(y)=\sum_{n=1}^{\infty}a_n y^n$ and $a_n>0$ depend only on n. Let N be distributed according to the zero truncated Poisson distribution with pdf

$$P(N=n)=\frac{\theta^n e^{-\theta}}{n!(1-e^{-\theta})},\quad n=1,2,...,\theta>0. \tag{3.3}$$

Note that N can also be taken to follow other discrete distributions, such as binomial, geometric, logarithmic, where the discrete distribution need to be truncated at zero because one must have $N\geq 1$. Let X=max$(Y_1,...,Y_N)$, then the cdf of $X|N=n$ is given by

$$G_{X|N=n}(x) \;=\; \left[1+\lambda x^{-\delta}\right]^{-n\beta},\quad x>0,\,\lambda,\delta,\beta>0, \tag{3.4}$$

which is the Dagum distribution with parameters λ, δ and $n\beta$. The Dagum-Poisson (DaP) distribution denoted by $DaP(\lambda, \delta, \beta, \theta)$ is defined by the marginal cdf of X, that is,

$$F_{DaP}(x; \lambda, \delta, \beta, \theta) = \frac{1 - \exp(\theta[1 + \lambda x^{-\delta}]^{-\beta})}{1 - e^{\theta}}, \tag{3.5}$$

for $x > 0, \lambda, \beta, \delta, \theta > 0$. The DaP density function is given by

$$f_{DaP}(x; \lambda, \delta, \beta, \theta) = \frac{\beta\lambda\delta\theta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1} \exp(\theta[1 + \lambda x^{-\delta}]^{-\beta})}{e^{\theta} - 1}. \tag{3.6}$$

A series representation of the DaP cdf is given by

$$\begin{aligned} F_{DaP}(x; \lambda, \delta, \beta, \theta) &= \frac{1 - \exp(\theta[1 + \lambda x^{-\delta}]^{-\beta})}{1 - e^{\theta}} \\ &= \frac{1}{e^{\theta} - 1}\left[\sum_{k=0}^{\infty} \frac{\theta^k (1 + \lambda x^{-\delta})^{-k\beta}}{k!} - 1\right] \\ &= \frac{1}{e^{\theta} - 1}\left[\sum_{k=1}^{\infty} \frac{\theta^k (1 + \lambda x^{-\delta})^{-k\beta}}{k!}\right] \\ &= \sum_{k=0}^{\infty} \frac{\theta^{k+1}}{(e^{\theta} - 1)(k+1)!}(1 + \lambda x^{-\delta})^{-\beta(k+1)} \\ &= \sum_{k=0}^{\infty} \omega(k, \theta) F_{Da}(x; \lambda, \delta, \beta(k+1)), \end{aligned} \tag{3.7}$$

where $\omega(k, \theta) = \frac{\theta^{k+1}}{(k+1)!(e^{\theta}-1)}$, and $F_{Da}(x; \lambda, \beta(k+1), \delta)$ is the Dagum cdf with parameters $\lambda, \beta(k+1), \delta > 0$.

The plots of the DaP distribution given in Figure 3.1 suggests that the DaP pdf can be right-skewed or decreasing for selected values of the parameters.

3.1.1 Hazard and Reverse Hazard Functions

In this section, hazard and reverse hazard functions of the DaP distribution are presented. The plots of DaP pdf and hazard rate function for selected values of the model parameters λ, δ, β and θ are given in Figure 3.1 and Figure 3.2, respectively.

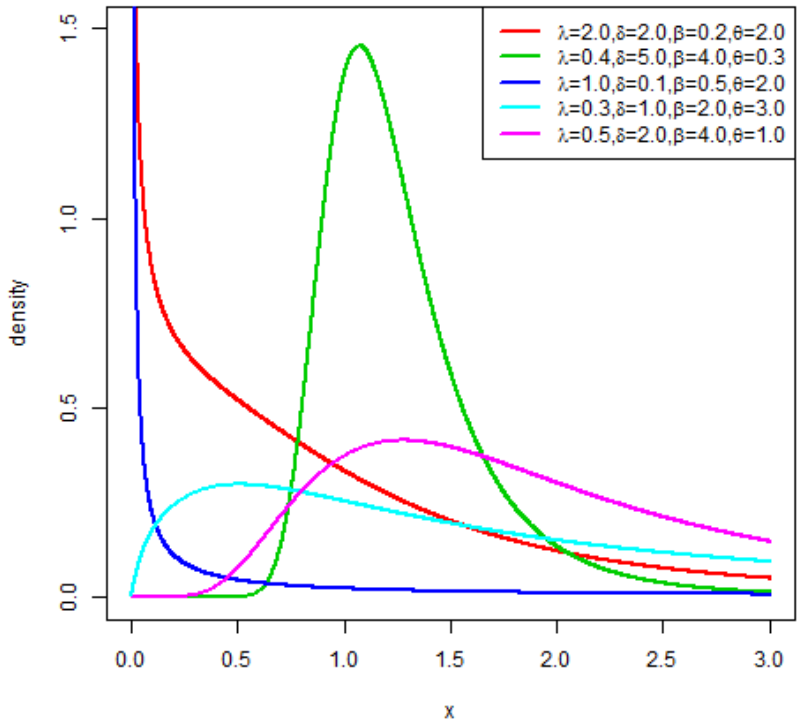


Figure 3.1: DaP Density Functions

The hazard and reverse hazard functions are given by

$$\begin{aligned} h_{F_{DaP}}(x;\lambda,\delta,\beta,\theta) &= \frac{f_{DaP}(x;\lambda,\delta,\beta,\theta)}{1-F_{DaP}(x;\lambda,\delta,\beta,\theta)} \\ &= \frac{\lambda\delta\beta\theta(1+\lambda x^{-\delta})^{-\beta-1}e^{\theta(1+\lambda x^{-\delta})^{-\beta}}}{e^{\theta}-e^{\theta(1+\lambda x^{-\delta})^{-\beta}}} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \tau_{F_{DaP}}(x;\lambda,\delta,\beta,\theta) &= \frac{f_{DaP}(x;\lambda,\delta,\beta,\theta)}{F_{DaP}(x;\lambda,\delta,\beta,\theta)} \\ &= \frac{\lambda\delta\beta\theta(1+\lambda x^{-\delta})^{-\beta-1}e^{\theta(1+\lambda x^{-\delta})^{-\beta}}}{e^{\theta(1+\lambda x^{-\delta})^{-\beta}}-1}, \end{aligned} \tag{3.9}$$

respectively. The plots of the hazard rate function shows various shapes including monotonically decreasing, monotonically increasing, unimodal, upside down bathtub and bathtub followed by upside down bathtub shapes for the combinations of the values of the parameters. This flexibility makes the DaP hazard rate function suitable for both monotonic and non-monotonic empirical hazard behaviors that are likely to be encountered in real life situations. Unfortunately, the analytical analysis of the shape of both the density (except for zero modal when $\beta\delta \leq 1$, and unimodal if $\beta\delta > 1$, both for $\theta \to 0^{+}$) and hazard rate function seems to be very complicated.

3.1.2 Quantile Function

In this subsection, we present the q^{th} quantile of the DaP distribution. The q^{th} quantile of the DaP distribution is obtained by solving the nonlinear equation

$$\frac{e^{\theta(1+\lambda x^{-\delta})^{-\beta}}-1}{e^{\theta}-1} = U,$$

where U is a uniform variate on the unit interval $[0,1]$. It follows that the q^{th} quantile of the DaP distribution is given by

$$X_q = \left(\frac{1}{\lambda}\left(\left[\frac{\ln(U(e^{\theta}-1)+1)}{\theta}\right]^{-1/\beta}-1\right)\right)^{-1/\delta}. \tag{3.10}$$

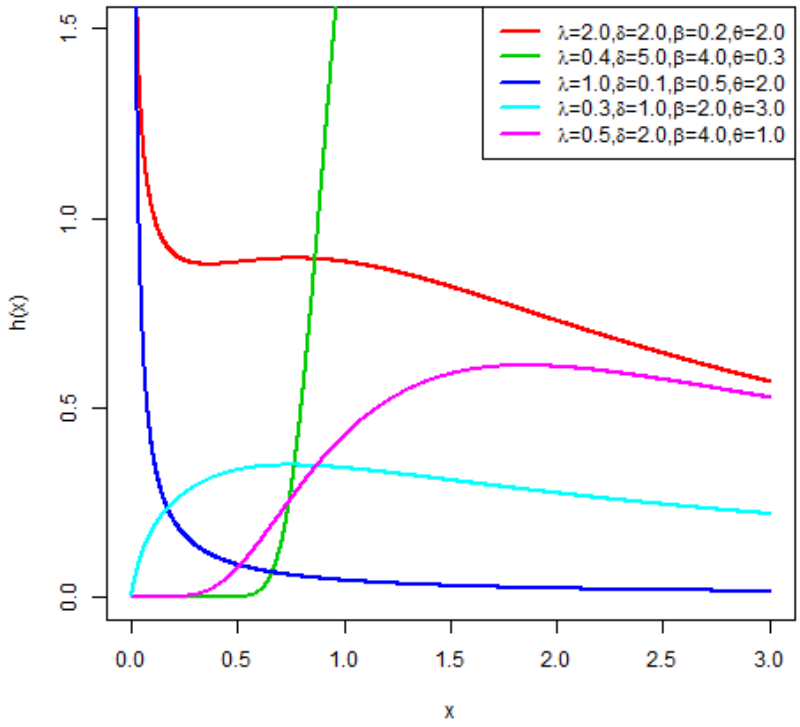


Figure 3.2: Plots of the DaP Hazard Function

Consequently, random number can be generated from the DaP distribution based on equation (3.10).

3.1.3 Some Sub-models of the DaP Distribution

DaP distribution is a very flexible model that has different sub-models when its parameters are changed. DaP distribution contains several sub-models including the following distributions.

- If $\beta = 1$, then DaP distribution reduces to a new distribution called log-logistic Poisson (LLoGP) or Fisk-Poisson (FP) distribution with pdf given by

$$f_{_{LLOGP}}(x;\lambda,\delta,\theta) = \lambda\theta\delta x^{-\delta-1}\frac{(1+\lambda x^{-\delta})^{-2}\exp(\theta(1+\lambda x^{-\delta})^{-1})}{\exp(\theta)-1}, \quad x>0. \tag{3.11}$$

  If in addition to $\beta = 1$, we have $\theta \downarrow 0$, then the resulting distribution is the log-logistic or Fisk distribution.

- When $\lambda = 1$, in the DaP distribution, we obtain Burr-III Poisson distribution given by

$$F_{BIIIP}(x;\beta,\delta,\theta) = \frac{1-exp(\theta[1+x^{-\delta}]^{-\beta})}{1-e^{\theta}}; x>0. \tag{3.12}$$

  If in addition, $\theta \downarrow 0$, Burr-III distribution with parameter $\delta, \beta > 0$ is obtained and is given by

$$F_{BIII}(x;\beta,\delta) = [1+x^{-\delta}]^{-\beta}; x>0. \tag{3.13}$$

- When $\beta = 1$, in the DaP distribution, we obtain the log-logistic or Fisk Power Series distribution.

$$F_{LLOGPS}(x;\delta,\lambda,\theta) = \frac{1-exp(\theta[1+\lambda x^{-\delta}]^{-1})}{1-e^{\theta}}; x>0. \tag{3.14}$$

- When $\theta \downarrow 0$ in the DaP distribution, we obtain Dagum (Da) distribution.

$$F_{Da}(x;\beta,\delta,\lambda) = [1+\lambda x^{-\delta}]^{-\beta}; x>0. \tag{3.15}$$

3.1.4 Expansion of DaP Density

In this section, we provide an expansion of the DaP distribution. Note that, applying the fact that Maclaurin series expansion of $e^x = \sum_{k=0}^{\infty} x^k/k!$, the DP pdf can be written as follows:

$$\begin{aligned} f_{_{DaP}}(x;\lambda,\delta,\beta,\theta) &= \frac{\beta\lambda\delta\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}\exp(\theta(1+\lambda x^{-\delta})^{-\beta})}{\exp(\theta)-1} \\ &= \sum_{k=0}^{\infty}\frac{\theta^k}{k!(e^\theta-1)}\lambda\beta\delta\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta(k+1)-1} \\ &= \sum_{k=0}^{\infty}\omega(k,\theta)f_{_{Da}}(x;\lambda,\delta,\beta(k+1)), \end{aligned} \tag{3.16}$$

where $\omega(k,\theta) = \frac{\theta^{k+1}}{(k+1)!(e^\theta-1)}$, and $f_{_{Da}}(x;\lambda,\beta(k+1),\delta)$ is the Dagum pdf with parameters $\lambda,\beta(k+1),\delta,>0$. The above equation shows that the DaP density is indeed a linear combination of Dagum densities. Hence, most of its mathematical properties can be immediately obtained from those of the Dagum distribution.

3.1.5 Moments

In this subsection, we present the r^{th} moment of DaP distribution. If the random variable X has a DaP distribution, with parameter vector $\varphi = (\lambda,\delta,\beta,\theta)$, and Y is a random variable that follows a Dagum distribution with parameters $\lambda,\beta(k+1),\delta$, then we can obtain the r^{th} moment of X by using Equation 3.16 to is get

$$\begin{aligned} \mu_r^{'} &= \sum_{k=0}^{\infty}\omega(k,\theta)E(Y^r) \\ &= \sum_{k=0}^{\infty}\omega(k,\theta)\beta(k+1)\lambda^{\frac{r}{\delta}}B\left(\frac{r}{\delta}+\beta(k+1),1-\frac{r}{\delta}\right), \quad \delta>r. \end{aligned} \tag{3.17}$$

B(.,.) is a Beta function and can be Note that the r^{th} non-central moment follows readily from the fact that DaP pdf can be written as a linear combination of Dagum densities with parameters $\lambda,\beta(k+1),\delta>0$. Based on the first four moments of the DaP distribution, the measures of skewness (CS) and kurtosis (CK) of the DaP

distribution can obtained from

$$CS = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{\left[\mu_2 - \mu_1^2\right]^{\frac{3}{2}}} \quad \text{and} \quad CK = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{\left[\mu_2 - \mu_1^2\right]^2},$$

respectively. Table 3.1 lists the first six moments of the DaP distribution for selected values of the parameters, by fixing $\delta = 6.5$. Table 3.2 lists the first six moments of the DaP distribution for selected values of the parameters, by fixing $\lambda = 0.3$ and $\beta = 0.8$, and Table 3.3 lists the first six moments of the DaP distribution for selected values of the parameters, by fixing $\delta = 8.5$ and $\beta = 1.5$. These values can be determined numerically using R and MATLAB.

Table 3.1: Moments of the DaP distribution for some parameter values; $\delta = 6.5$.

μ'_s	$\lambda = 0.5, \beta = 2.5, \theta = 2.5$	$\lambda = 2.0, \beta = 0.5, \theta = 0.5$	$\lambda = 3.0, \beta = 1.5, \theta = 3.5$	$\lambda = 0.1, \beta = 0.2, \theta = 0.3$
μ'_1	1.30548	1.01330	1.65702	0.45015
μ'_2	1.81639	1.15863	2.93049	0.26502
μ'_3	2.74198	1.49281	5.62721	0.18674
μ'_4	4.65550	2.22301	12.15746	0.15704
μ'_5	9.74269	4.16566	32.37752	0.16950
μ'_6	38.13338	14.84150	161.26470	0.35254
SD	0.33484	0.36311	0.42985	0.24977
CV	0.25649	0.35835	0.25941	0.55485
CS	2.07731	1.07708	2.00161	0.72369
CK	15.69083	8.48089	15.25227	5.09487

Table 3.2: Moments of the DaP distribution for some parameter values; $\lambda = 0.3$ and $\beta = 0.8$.

μ'_s	$\delta = 7.0, \theta = 1.0$	$\delta = 10.5, \theta = 0.3$	$\delta = 7.5, \theta = 0.1$	$\delta = 6.5, \theta = 3.0$
μ'_1	**0.89281**	**0.88817**	**0.84321**	**1.01339**
μ'_2	**0.85993**	**0.81665**	**0.76169**	**1.10774**
μ'_3	**0.90057**	**0.77797**	**0.74088**	**1.32478**
μ'_4	**1.04747**	**0.76984**	**0.78724**	**1.79292**
μ'_5	**1.42600**	**0.79537**	**0.94477**	**3.00491**
μ'_6	**2.69499**	**0.86581**	**1.40007**	**9.45480**
SD	**0.25065**	**0.16674**	**0.22515**	**0.28424**
CV	**0.28074**	**0.18773**	**0.26702**	**0.28049**
CS	**1.31050**	**0.70110**	**1.15070**	**1.67466**
CK	**9.60878**	**5.71218**	**8.24167**	**12.95428**

Table 3.3: Moments of the DaP distribution for some parameter values; $\delta = 8.5$ and $\beta = 1.5$.

μ'_s	$\lambda = 1.5, \theta = 1.5$	$\lambda = 1.5, \theta = 0.5$	$\lambda = 0.5, \theta = 1.5$	$\lambda = 0.5, \theta = 0.5$
μ'_1	1.24022	1.17897	1.08985	1.03602
μ'_2	1.60034	1.44651	1.23580	1.11701
μ'_3	2.16080	1.85768	1.46628	1.26059
μ'_4	3.08248	2.52160	1.83811	1.50366
μ'_5	4.72733	3.68135	2.47717	1.92907
μ'_6	8.07349	5.98802	3.71766	2.75735
SD	0.24939	0.23780	0.21915	0.20897
CV	0.20108	0.20170	0.20108	0.20170
CS	1.40204	1.40918	1.40197	1.40905
CK	8.97403	8.95440	8.97488	8.95504

Similarly, the moment generating function (mgf) of X is given by

$$\begin{aligned} M_X(t) &= \sum_{k=0}^{\infty} \omega(k,\theta) M_Y(t) \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \omega(k,\theta)\frac{t^j}{j!}\lambda^{\frac{j}{\delta}} B\left(\frac{j}{\delta}+\beta(k+1), 1-\frac{j}{\delta}\right), \quad \delta > j. \end{aligned} \tag{3.18}$$

This follows from the well known definition of the moment generating function given by $M_{_X}(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x;\phi)dx$. Since $\sum_{j=0}^{\infty} \frac{t^j}{j!} x^j f(x)$ converges and each term is integrable for all t close to 0, then we can rewrite the moment generating function as $M_{_X}(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} E(X^j)$ by replacing $E(X^j)$ by the right side of Equation (3.17) to obtain the desired result.

The plots of the CS and CK vs shape parameters β and δ are given in Figures 5.3-5.6. The results in Table 5.1 are used to plot the Figures 5.3-5.6. In Figure 5.3, for λ constant at $\lambda = 0.5$, 2.0 and 3.0 when β increases from $\beta = 1.0$ to 3.0, skewness decreases. The same observation is made for Figure 5.4 for the same parameter value choices.

From Figures 5.5, for $\lambda = 0.5$ and 3.0, when δ increases from $\delta = 1.0$ to 3.0, the skewness decreases and for $\lambda = 2.0$, when δ increases from $\delta = 1.0$ to 3.0, skewness decreases and eventually increases forming a V-shape. In Figure 5.6, for λ constant at $\lambda = 2.0$ and 3.0, when δ increases from $\delta = 1.0$ to 3.0, the kurtosis decreases and for $\lambda = 0.5$, the kurtosis decreases and eventually increases forming a U-shape.

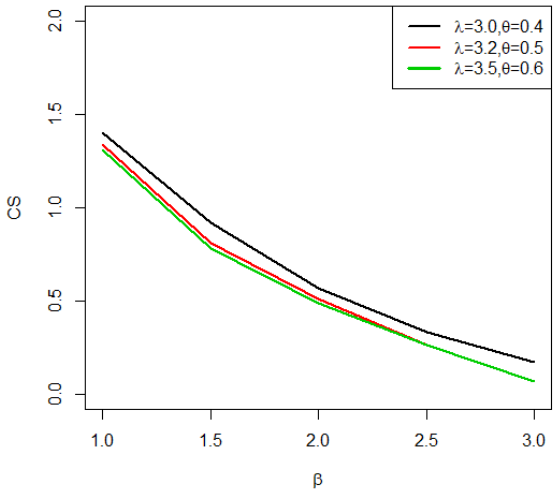


Figure 3.3: Plot of CS vs β.

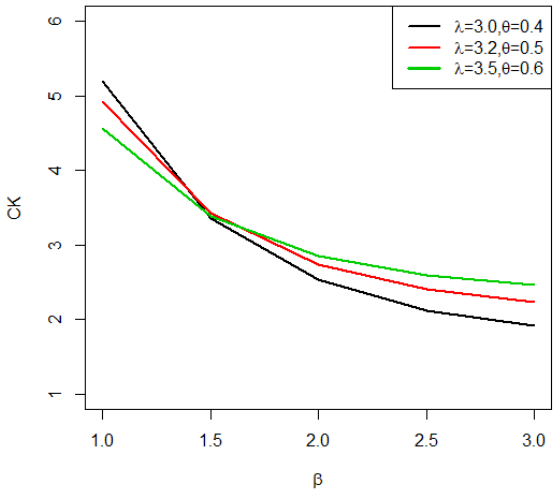


Figure 3.4: Plot of CK vs β.

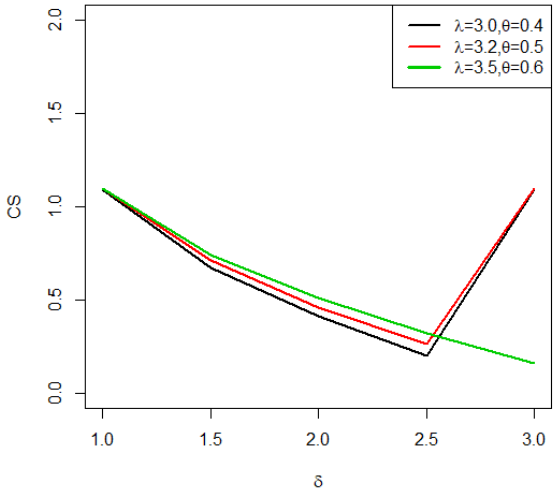


Figure 3.5: Plot of CS vs δ.

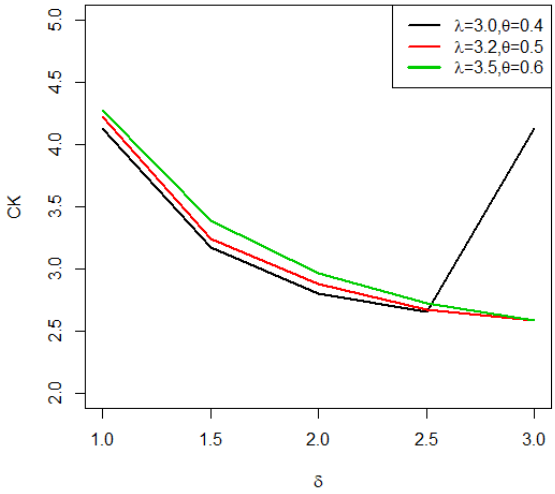


Figure 3.6: Plot of CK vs δ.

3.1.6 Conditional Moments

For income and lifetime distributions, it is of interest to obtain the conditional moments and mean residual life function. The r^{th} conditional moment for DaP

distribution is given by

$$\begin{aligned} E(X^r|X>t) &= \frac{1}{\overline{F}_{_{DaP}}(t)}\int_t^\infty x^r f_{_{DaP}}(x)dx \\ &= \frac{1}{\overline{F}_{_{DaP}}(t)}\sum_{k=0}^{\infty}\omega(k,\theta)\left[B\left(\beta(k+1)+\frac{r}{\delta},1-\frac{r}{\delta}\right)\right. \\ &- \left. B_{_{t(a)}}\left(\beta(j+1)+\frac{r}{\delta},1-\frac{r}{\delta}\right)\right], \end{aligned} \tag{3.19}$$

where $t(a) = (1+\lambda a^{-\delta})^{-1}$, $\delta > r$, and $B_{_{t(a)}}(c,d) = \int_0^{t(a)} u^{c-1}(1-u)^{d-1}du$. The mean residual life function is $E(X|X>t) - t$.

3.1.7 Mean and Median Deviations

The amount of dispersion in a population can be measured to some extent by the totality of deviations from the mean and the median. If X has the DaP distribution, we can derive the mean deviation about the mean $\mu = E(X)$ and the mean deviation about the median M from

$$\delta_1 = \int_0^\infty |x-\mu| f_{_{DaP}}(x)dx \quad \text{and} \quad \delta_2 = \int_0^\infty |x-M| f_{_{DaP}}(x)dx, \tag{3.20}$$

respectively. The mean μ is obtained from equation (3.17) with $r = 1$, and the median M is given by equation (3.10) when $q = \frac{1}{2}$.

The measure δ_1 and δ_2 can be calculated by the following relationships:

$$\delta_1 = 2\mu F_{DaP}(\mu) - 2\mu + 2\int_\mu^\infty x f_{DaP}(x)dx \tag{3.21}$$

and

$$\delta_2 = 2\int_M^\infty x f_{DaP}(x)dx - \mu, \tag{3.22}$$

where $T(a) = \int_a^\infty x \cdot f_{_{DaP}}(x)dx$ follows from equation (3.19), that is,

$$T(a) = \sum_{k=0}^{\infty}\omega(k,\theta)\left[B\left(\beta(k+1)+\frac{1}{\delta},1-\frac{1}{\delta}\right) - B_{t(a)}\left(\beta(k+1)+\frac{1}{\delta},1-\frac{1}{\delta}\right)\right].$$

3.1.8 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves are widely used tool for analyzing and visualizing income inequality. Lorenz curve, $L(p)$ can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume a, and Bonferroni curve, $B(p)$ is the scaled conditional mean curve, that is, ratio of group mean income of the population. Let $I(a) = \int_0^a x f_{_{DaP}}(x)dx$ and $\mu = E(X)$, then Bonferroni and Lorenz curves are given by $B(p) = \frac{I(q)}{p\mu}$ and $L(p) = \frac{I(q)}{\mu}$, respectively, for $0 \leq p \leq 1$, and $q = F^{-1}_{DaP}(p)$. The mean of the DaP distribution is obtained from equation (3.17) with $r = 1$ and the quantile function is given in equation (3.10). Consequently,

$$I(a) = \sum_{k=0}^{\infty} \omega(k,\theta) B_{_{t(a)}}\left(\beta(k+1) + \frac{1}{\delta}, 1 - \frac{1}{\delta}\right), \tag{3.23}$$

for $\delta > 1$, where $t(a) = (1+\lambda a^{-\delta})^{-1}$, and $B_{F(x)}(c,d) = \int_0^{F(x)} t^{c-1}(1-t)^{d-1}dt$ for $|F(x)| < 1$ is incomplete Beta function.

3.1.9 Order Statistics

The pdf of the k^{th} order statistics from a pdf $f(x)$ is

$$\begin{aligned} f_{k:n}(x) &= \frac{f(x)}{B(k, n-k+1)} F^{k-1}(x)[1-F(x)]^{n-k} \\ &= k\binom{n}{k} f(x) F^{k-1}(x)[1-F(x)]^{n-k}. \end{aligned} \tag{3.24}$$

We apply the series expansion

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} z^j, \tag{3.25}$$

for $b > 0$ and $|z| < 1$, to obtain the series expansion of the distribution of order statistics from DaP distribution. Using equations (3.25), the pdf of the k^{th} order

statistic from DaP distribution is given by

$$\begin{aligned} f_{k:n}(x) &= \sum_{s=0}^{\infty} k\binom{n}{k}\frac{(-1)^s f(x)\Gamma(n-k+1)}{s!\Gamma(n-k+1-s)}[F(x)]^{s+k-1} \\ &= \sum_{s,t,w=0}^{\infty} k\binom{n}{k}\frac{(-1)^{s+t+1}\theta^w(t+1)^w\Gamma(n-k+1)\Gamma(s+k)}{(1-e^{\theta})^{s+k}s!t!(w+1)!\Gamma(n-k+1-s)\Gamma(s+k-t)} \\ &\times \quad \lambda\beta(w+1)\delta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta(w+1)-1}. \end{aligned}$$

That is,

$$f_{k:n}(x) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{w=0}^{\infty} H(s,t,w,k)\cdot f_{_{Da}}(x;\lambda,\beta(w+1),\delta),$$

where $H(s,t,w,k) = k\binom{n}{k}\frac{(-1)^{s+t+1}\theta^w(t+1)^w\Gamma(n-k+1)\Gamma(s+k)}{(1-e^{\theta})^{s+k}s!t!(w+1)!\Gamma(n-k+1-s)\Gamma(s+k-t)}$. Thus, the pdf of the i^{th} order statistic from the DaP distribution is a linear combination of Dagum $pdfs$ with parameters $\lambda, \beta(w+1)$ and $\delta > 0$. The r^{th} moment of the distribution of the i^{th} order statistic is given by

$$E(X_{i:n}^r) = \sum_{s,t,w=0}^{\infty} H(s,t,w,k)\beta(w+1)\lambda^{\frac{r}{\delta}}B\left(\beta(w+1)+\frac{r}{\delta}, 1-\frac{r}{\delta}\right), \quad \delta > r. \tag{3.26}$$

3.1.10 Rényi Entropy

An entropy is a measure of uncertainty or variation of a variable. Rényi entropy of a DaP distribution is defined as

$$I_R(\tau) = (1-\tau)^{-1}\log\left\{\int_0^{\infty} f_{DaP}^{\tau}(x;\beta,\delta,\lambda,\theta)dx\right\}, \tau > 0, \tau \neq 1.$$

$f_{DaP}^{\tau}(x) = [f_{DaP}(x;\beta,\delta,\lambda,\theta)]^{\tau}$ and note that by using equation (3.25), we have

$$f_{_{DaP}}^{\tau}(x) = \sum_{k=0}^{\infty}\frac{(\tau\theta)^k(\lambda\beta\delta\theta)^{\tau}}{k!(e^{\theta}-1)^{\tau}}x^{-\delta\tau-\tau}(1+\lambda x^{-\delta})^{-\beta\tau-\beta k-\tau}. \tag{3.27}$$

Consequently, Rényi entropy of DaP distribution is given by

$$I_R(\tau) = \frac{1}{1-\tau}\log\left[\sum_{k=0}^{\infty}\frac{(\tau\theta)^k(\lambda\beta\delta\theta)^{\tau}}{k!(e^{\theta}-1)^{\tau}}\frac{\lambda^{1/\delta}}{\delta}B\left(\beta(k+\tau)-\frac{\tau-1}{\delta}, \tau+\frac{\tau-1}{\delta}\right)\right].$$

for $\beta(k+\tau)-\frac{\tau-1}{\delta}>0$ and $\tau+\frac{1-\tau}{\delta}>0$. Renyi entropy for the sub-models can be readily obtained.

3.1.11 Maximum Likelihood Estimation

In this section, we consider the maximum likelihood estimators (MLE's) of the parameters of the DaP distribution. Let $x_1,...,x_n$ be a random sample of size n from DaP distribution and $\mathbf{\Theta}=(\lambda,\delta,\beta,\theta)^T$ be the parameter vector. The log-likelihood function can be written as

$$\begin{aligned} L &= n\log\theta+n\log\beta+n\log\delta+n\log\lambda-(\delta+1)\sum_{i=1}^{n}\log x_i \\ &- (\beta+1)\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta})+\theta\sum_{i=1}^{n}(1+\lambda x_i^{-\delta})^{-\beta}-n\log(e^{\theta}-1). \end{aligned} \tag{3.28}$$

The associated score function is given by

$$U_n(\mathbf{\Theta})=\left(\frac{\partial L}{\partial\lambda},\frac{\partial L}{\partial\delta},\frac{\partial L}{\partial\beta},\frac{\partial L}{\partial\theta}\right)^T,$$

where

$$\frac{\partial L}{\partial\lambda}=\frac{n}{\lambda}-(\beta+1)\sum_{i=1}^{n}\frac{x_i^{-\delta}}{(1+\lambda x_i^{-\delta})}-\sum_{i=1}^{n}\beta\theta x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-\beta-1}, \tag{3.29}$$

$$\begin{aligned} \frac{\partial L}{\partial\delta} &= \frac{n}{\delta}-\sum_{i=1}^{n}\log x_i+(\beta+1)\sum_{i=1}^{n}\frac{\lambda x_i^{\delta}\log x_i}{(1+\lambda x_i^{-\delta})} \\ &+ \sum_{i=1}^{n}\theta\lambda\beta(1+\lambda x_i^{-\delta})^{-\beta-1}x_i^{-\delta}\log x_i, \end{aligned} \tag{3.30}$$

$$\frac{\partial L}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta})+\sum_{i=1}^{n}\theta(1+\lambda x_i^{-\delta})^{-\beta}\log(1+\lambda x_i^{-\delta}), \tag{3.31}$$

and

$$\frac{\partial L}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}(1+\lambda x_i^{-\delta})^{-\beta}-n\log\left[\frac{e^{\theta}}{e^{\theta}-1}\right]. \tag{3.32}$$

The maximum likelihood estimate (MLE) of $\boldsymbol{\Theta}$, say $\widehat{\boldsymbol{\Theta}}$, is obtained by solving the nonlinear system $U_n(\boldsymbol{\Theta}) = 0$. The solution of this nonlinear system of equation is not in a closed form. These equations cannot be solved analytically, and statistical software can be used to solve them numerically via iterative methods. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the estimate $\widehat{\boldsymbol{\Theta}}$.

3.1.12 Monte Carlo Simulation Study

In this section, a simulation study is conducted to assess the performance and examine the mean estimate, average bias, root mean square error of the maximum likelihood estimators and width of the confidence interval for each parameter. We study the performance of the DaP distribution by conducting various simulations for different sample sizes and different parameter values. Equation (2.13) is used to generate random data from the DaP distribution. The simulation study is repeated for $N = 5,000$ times each with sample size $n = 25, 50, 75, 100, 200, 400, 800$ and parameter values $I : \lambda = 3.0, \delta = 1.2, \beta = 0.2, \theta = 0.3$ and $II : \lambda = 3.5, \delta = 1.0, \beta = 0.2, \theta = 0.4$. Five quantities are computed in this simulation study.

(a) Mean estimate of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \lambda, \delta, \beta, \theta$:

$$\frac{1}{N}\sum_{i=1}^{N}\hat{\vartheta}.$$

(b) Average bias of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \lambda, \delta, \beta, \theta$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta).$$

(c) Root mean squared error (RMSE) of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \lambda, \delta, \beta, \theta$:

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta)^2}.$$

(d) Coverage probability (CP) of 95% confidence intervals of the parameter $\vartheta = \lambda, \delta, \beta, \theta$, i.e., the percentage of intervals that contain the true value of parameter ϑ.

(e) Average width (AW) of 95% confidence intervals of the parameter $\vartheta = \lambda, \delta, \beta, \theta$.

Table (3.4) presents the Mean, Average Bias, RMSE, CP and AW values of the parameters λ, δ, β and θ for different sample sizes. From the results in Table 3.4, we can verify that as the sample size n increases, the RMSEs decay toward zero. We also observe that for all the parametric values, the biases decrease as the sample size n increases. The table shows that the coverage probabilities of the confidence intervals are reasonably close to the nominal level of 95% and the average confidence widths decrease as the sample size n increases. Consequently, the MLE's and their asymptotic results can be used for estimating and constructing confidence intervals even for reasonably small sample sizes.

Table 3.4: Monte Carlo Simulation Results: Mean, Average Bias, RMSE, CP and AW

		I					II				
Parameter	n	Mean	Average Bias	RMSE	CP	AW	Mean	Average Bias	RMSE	CP	AW
λ	25	4.8530	1.8530	7.2589	0.8132	44.1472	5.9531	2.4531	9.1470	0.8146	55.0576
	50	4.2227	1.2227	5.6644	0.8214	27.9452	5.4153	1.9153	7.4619	0.8358	37.5203
	75	3.9258	0.9258	4.8932	0.8286	21.7068	4.9140	1.4140	6.3977	0.8372	27.8844
	100	3.5974	0.5974	4.1241	0.8278	16.7771	4.5233	1.0233	5.2756	0.8378	22.3011
	200	3.0985	0.0985	2.5313	0.8330	10.0894	3.7666	0.2666	3.0828	0.8480	12.6877
	400	2.8628	-0.1372	1.4204	0.8570	6.5562	3.5132	0.0132	2.0123	0.8640	8.3644
	800	2.8558	-0.1442	1.0060	0.8882	4.6937	3.4562	-0.0438	1.2749	0.8954	5.8443
δ	25	1.4116	0.2116	0.5876	0.9606	2.5486	1.1663	0.1663	0.4842	0.9676	2.0609
	50	1.3148	0.1148	0.4152	0.9628	1.7464	1.1036	0.1036	0.3433	0.9642	1.4579
	75	1.2825	0.0825	0.3494	0.9628	1.4374	1.0681	0.0681	0.2795	0.9622	1.1698
	100	1.2580	0.0580	0.2989	0.9592	1.2210	1.0498	0.0498	0.2477	0.9594	1.0090
	200	1.2143	0.0143	0.2085	0.9456	0.8563	1.0140	0.0140	0.1718	0.9528	0.7063
	400	1.1944	-0.0056	0.1468	0.9468	0.6080	0.9973	-0.0027	0.1226	0.9468	0.5021
	800	1.1909	-0.0091	0.1052	0.9488	0.4356	0.9967	-0.0033	0.0878	0.9488	0.3607
β	25	0.1600	-0.0400	0.1253	0.8940	0.5016	0.1635	-0.0365	0.1425	0.9132	0.5677
	50	0.1558	-0.0442	0.0803	0.8822	0.3540	0.1565	-0.0435	0.0827	0.8914	0.3546
	75	0.1605	-0.0395	0.0702	0.8760	0.2937	0.1610	-0.0390	0.0706	0.8858	0.2981
	100	0.1646	-0.0354	0.0638	0.8836	0.2581	0.1640	-0.0360	0.0631	0.8884	0.2613
	200	0.1782	-0.0218	0.0470	0.9104	0.1858	0.1793	-0.0207	0.0464	0.9142	0.1876
	400	0.1895	-0.0105	0.0314	0.9392	0.1291	0.1900	-0.0100	0.0321	0.9414	0.1316
	800	0.1950	-0.0050	0.0219	0.9452	0.0902	0.1953	-0.0047	0.0224	0.9472	0.0914
θ	25	1.7827	1.4827	1.9319	0.9788	10.2716	1.8219	1.4219	1.8842	0.9794	10.4265
	50	1.6338	1.3338	1.8923	0.9788	8.4984	1.6645	1.2645	1.8594	0.9824	8.5670
	75	1.4764	1.1764	1.8148	0.9810	7.2930	1.5348	1.1348	1.7991	0.9778	7.3682
	100	1.3709	1.0709	1.7215	0.9772	6.3903	1.4247	1.0247	1.6870	0.9782	6.4487
	200	1.0276	0.7276	1.3219	0.9816	4.4662	1.0617	0.6617	1.2299	0.9854	4.4305
	400	0.7053	0.4053	0.7313	0.9794	2.9341	0.7702	0.3702	0.7428	0.9798	2.9895
	800	0.5335	0.2335	0.4320	0.9754	2.0481	0.5935	0.1935	0.4372	0.9488	2.0669

3.1.13 Applications of The Model to Real Life Data

In this section, we present examples to illustrate the flexibility of the DaP distribution and its sub-models for data modeling. Estimates of the parameters of DaP distribution (standard error in parentheses), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), sum of squares (SS) from the probability plots, Cramér-von Mises W^* and Anderson Darling A^* goodness-of-fit statistics are presented for the data set.

The table in the example contain the goodness-of-fit statistics W^* and A^*, and are described by Chen and Balakrishnan [10]. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit. In order to compare the models, we use the criteria stated above. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred, for AIC, AICC and BIC, smaller values are preferred and for the Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics, smaller values are preferred.

The DaP distribution is compared with other distributions including Exponentiated Weibull-Poisson (EWP) [23] distribution. The cdf of the EWP distribution is given by

$$F_{_{EWP}}(x;\delta,\beta,\gamma,\theta)=\left(\frac{e^{\theta(1-e^{-(\beta x)^{\gamma}})^{\delta}}-1}{e^{\theta}-1}\right),\quad x>0. \tag{3.33}$$

Plots of the fitted densities and the histogram of the data are given in Figure (3.7). The probability plots (Chambers et al. [8]) are also presented in Figure (3.8). For the probability plot, we plotted $G_{_{DaP}}(x_{(j)};\hat{\lambda},\hat{\delta},\hat{\beta},\hat{\theta})$ against $\frac{j-0.375}{n+0.25}, j=1,2,\cdots,n$, where $x_{(j)}$ are the ordered values of the observed data. A measure of closeness of each plot to the diagonal line is also computed. This measure of closeness is given by the sum of squares $SS=\sum_{j=1}^{n}\left[G_{_{DaP}}(x_{(j)};\hat{\lambda},\hat{\delta},\hat{\beta},\hat{\theta})-\left(\frac{j-0.375}{n+0.25}\right)\right]^2$.

3.1.14 Example 1: Breaking Stress of Carbon Fibres

This data set consists of 100 observations corresponding to the breaking stress of carbon fibres. The data is presented in Table (3.5).

Table 3.5: Breaking Stress of Carbon Fibres

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90
3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67	2.93	3.22
3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85	2.56	3.56
3.15	2.35	2.55	2.59	2.38	2.81	2.77	2.17	2.83	1.92
1.41	3.68	2.97	1.36	0.98	2.76	4.91	3.68	1.84	1.59
3.19	1.57	0.81	5.56	1.73	1.59	2.00	1.22	1.12	1.71
2.17	1.17	5.08	2.48	1.18	3.51	2.17	1.69	1.25	4.38
1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03	1.80
1.57	1.08	2.03	1.61	2.12	1.89	2.88	2.82	2.05	3.65

Table 3.6: Estimates of Models for Breaking Stress of Carbon Fibres Data

	Estimates				Statistics						
Model	λ	δ	β	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	SS
DaP	52.2546	1.7411	1.8116	55.0514	282.5882	290.5853	291.0064	301.0060	0.0707	0.4045	0.0687
	(195.3528)	(2.1672)	(2.4594)	(208.5243)							
FP	40.6911	4.1147		0.1242	292.5596	298.5597	298.8097	306.3752	0.2390	1.2406	0.1661
	(75.6047)	(0.3490)		(3.5492)							
BIIIP		2.8604	2.0099	4.6540	306.0291	312.0291	312.2791	319.8446	0.3634	1.9627	0.3688
		(0.1914)	(0.7089)	(1.4134)							
	λ	β	δ								
Dagum	161.6524	0.6431	4.8981		285.7197	293.1009	293.3509	301.916	0.1538	0.7824	0.1812
	(79.5109)	(0.0991)	(0.3709)								
	α	β									
Fisk	4.1180	0.4003			292.5591	296.5591	296.6828	301.7697	0.2390	1.2409	0.1666
	(0.3441)	(0.0169)									
		c	k								
Burr-III		5.2212	2.3192		321.9463	325.9463	326.07	331.1566	0.5690	3.1733	0.5745
		(0.5984)	(0.1496)								
	δ	β	γ	θ							
EWP	223.0874	118.1239	0.3123	8.9081	420.6519	425.6519	426.0729	436.0725	0.3727	2.0209	25.8314
	(0.3679)	(0.7394)	(0.2408)	(1.2007)							

Estimates of the parameters of the distributions, standard errors (in parentheses), -2log-likelihood statistic, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC) are given in the table below. The results obtained from fitting the DaP distribution and its sub-models FP and $BIIIP$ distributions and other alternatives including EWP distribution presented for the data set. We also use the LR test to compare the DaP distribution and its sub-models.

The estimated variance-covariance matrix for the DaP distribution is given by:

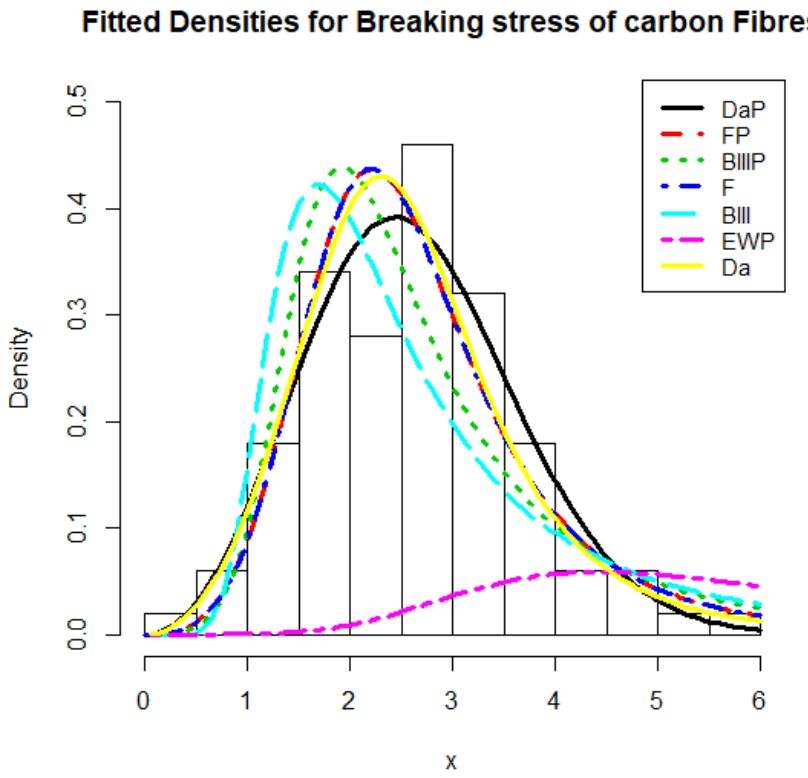


Figure 3.7: Fitted Densities

$$\begin{pmatrix} 4.8945 & -4.3347 & -383.2632 & -334.4601 \\ -4.3347 & 3.9055 & 330.4860 & 316.1448 \\ -383.2632 & 330.4860 & 32219.9762 & 21984.7567 \\ -334.4601 & 316.1448 & 21984.7567 & 31168.2274 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for λ, δ, β and θ are given by $52.2545 \pm 38.2891, 1.7411 \pm 0.4248, 1.8116 \pm 0.4820,$ and 55.0514 ± 40.8708, respectively.

Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 3.7 and Figure 3.8.

The LR test statistics of the hypothesis $H_0 : FP$ vs $H_a : DaP$ and $H_0 : BIIIP$ vs $H_a : DaP$ are 10.0014 (p-value $= 0.0016$) and 23.4709 (p-value < 0.0001). The DaP distribution is significantly better than FP and $BIIIP$ distributions. The DaP distribution is also significantly better that Dagum, Fisk and Burr-III distributions.

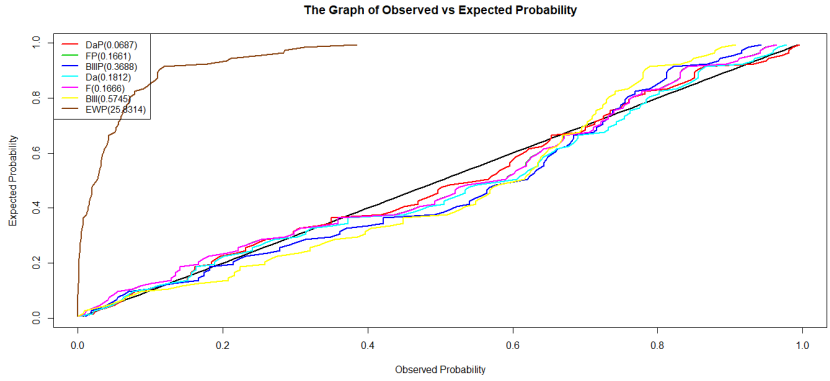


Figure 3.8: Probability Plots for Kevlar Strands Failure Times Data

The goodness of fit statistics W^* and A^* and the SS statistic from the probability plots clearly show that DaP distribution is better than all other distributions that are compared to it for the data. We also compared the DaP distribution to non-nested EWP distribution using the AIC, AICC, BIC, W^*, A^* and SS statistics. The model with the smallest value for each of the statistics will be the best one to be used in fitting the data. When the DaP distribution is compared to the non-nested EWP distribution, it is clear that it is superior based on the AIC, AICC and BIC values. The DaP distribution has the smallest goodness of fit statistic W^* and A^* values as well as the smallest SS value among all the models that were fitted. Clearly, the DaP model has points closer to the diagonal line corresponding to the smallest SS value for the probability plots when compared to the non-nested EWP distribution. Hence, the DaP distribution is the "best" fit for the data when compared to all the other models that were considered.

3.1.15 Example 2: Failure Times of a Particular Windshield Device

In the second example, we consider the data set representing the failure times of a particular windshield device. This data set consists of 84 observations corresponding to the failure times of a particular windshield device. The data is presented in Table (3.7). Estimates of the parameters of the distributions, standard errors (in parentheses), -2log-likelihood statistic, Akaike Information Criterion (AIC), Con-

Table 3.7: Failure times of a Particular Windshield Device

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899
2.610	3.478	0.557	1.911	2.625	3.578	0.943	1.912	2.632	3.595
1.070	1.914	2.646	3.699	1.124	1.981	2.661	3.779	1.248	2.010
2.688	3.924	1.281	2.038	2.823	4.035	1.281	2.085	2.890	4.121
1.303	2.089	2.902	4.167	1.432	2.097	2.934	4.240	1.480	2.135
2.962	4.255	1.505	2.154	2.964	4.278	1.506	2.190	3.000	4.305
1.568	2.194	3.103	4.376	1.615	2.223	3.114	4.449	1.619	2.224
3.117	4.485	1.652	2.229	3.166	4.570	1.652	2.300	3.344	4.602
1.757	2.324	3.376	4.663						

Table 3.8: Estimates of Models for Failure Times of a Particular Windshield Device

	Estimates				Statistics						
Model	λ	δ	β	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	SS
DaP	55.7918	4.2639	0.2096	4.3341	260.1817	268.6462	269.1526	278.3695	0.0637	0.5581	0.0536
	(44.1084)	(0.4944)	(0.1125)	(1.9205)							
FP	91.7873	2.3564		9.5376	264.7527	272.2921	272.5921	279.5845	0.0823	0.8138	0.1790
	(34.3913)	(0.2391)		(3.9803)							
BIIIP		2.3809	0.9556	5.8312	286.3653	292.3653	292.6653	299.6578	0.2043	1.5790	0.2248
		(0.1790)	(0.2333)	(1.2532)							
	λ	β	δ								
Dagum	112.7111	4.2537	0.5146		265.4501	272.6841	272.9841	279.9765	0.0768	0.7727	0.0894
	(73.2446)	(0.4418)	(0.0947)								
	α	β									
Fisk	0.4182	3.2238			279.1625	283.1625	283.3106	288.0241	0.1865	1.5201	0.0714
	(0.0240)	(0.2971)									
		c	k								
Burr-III		2.8829	1.7537		319.0933	323.0933	323.2415	327.955	0.7278	4.6982	0.8676
		(0.3205)	(0.1313)								
	δ	β	γ	θ							
EWP	34.0676	103.4831	0.3163	6.5145	378.951	385.951	386.4573	395.6742	0.3559	2.5707	0.3891
	(14.9820)	(47.5602)	(0.0197)	(1.5419)							

sistent Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC) are given in the table below. The results obtained from fitting the DaP distribution and its sub-models FP and $BIIIP$ distributions and other alternatives including EWP distribution are presented for the data set. We also use the LR test to compare the DaP distribution and its sub-models.

The estimated variance-covariance matrix for the DaP distribution is given by:

$$\begin{pmatrix} 0.0167 & -0.0252 & -4.9582 & -0.2743 \\ -0.0252 & 0.3612 & 42.2948 & -0.0115 \\ -4.9582 & 42.2948 & 5877.9252 & 19.4695 \\ -0.2743 & -0.0115 & 19.4695 & 5.7964 \end{pmatrix}$$

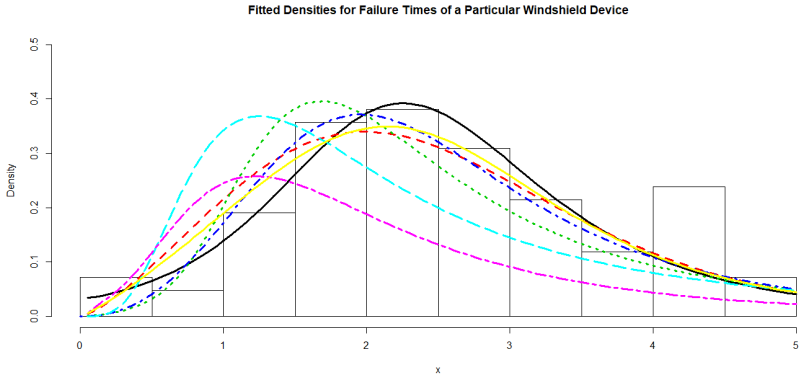


Figure 3.9: Fitted Densities

and the 95% two-sided asymptotic confidence intervals for λ, δ, β and θ are given by $55.7918 \pm 9.4327, 4.2639 \pm 0.1057, 0.2096 \pm 0.0241,$ and 4.3341 ± 0.4107, respectively.

Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 3.9 and Figure 3.10.

The LR test statistics of the hypothesis $H_0 : FP$ vs $H_a : DaP$ and $H_0 : BIIIP$ vs $H_a : DaP$ are 4.571 (p-value $= 0.0325$) and 26.1836(p-value < 0.0001). The DaP distribution is significantly better than the Fisk Poisson and $BIIIP$ distributions. The DaP distribution is also significantly better that Dagum, Fisk and Burr-III distributions. The goodness of fit statistics W^* and A^* and the SS statistic from the probability plots clearly shows that DaP distribution is better than all other distributions that are compared to it for the data. We also compared the DaP distribution to non-nested EWP distribution using the AIC, AICC, BIC, W^*, A^* and SS statistics. The model with the smallest value for each of the statistics will be the best one to be used in fitting the data. When the DaP distribution is compared to the non-nested EWP distribution, it is clear that it is superior based on the AIC, AICC and BIC values. The DaP distribution has the smallest goodness of fit statistic W^* and A^* values as well as the smallest SS value among all the models that were fitted. Clearly, the DaP model has points closer to the diagonal line corresponding to the smallest SS value for the probability plots when compared to the non-nested EWP distribution. Hence, the DaP distribution is the "best" fit for the data when compared to all the other models that were considered.

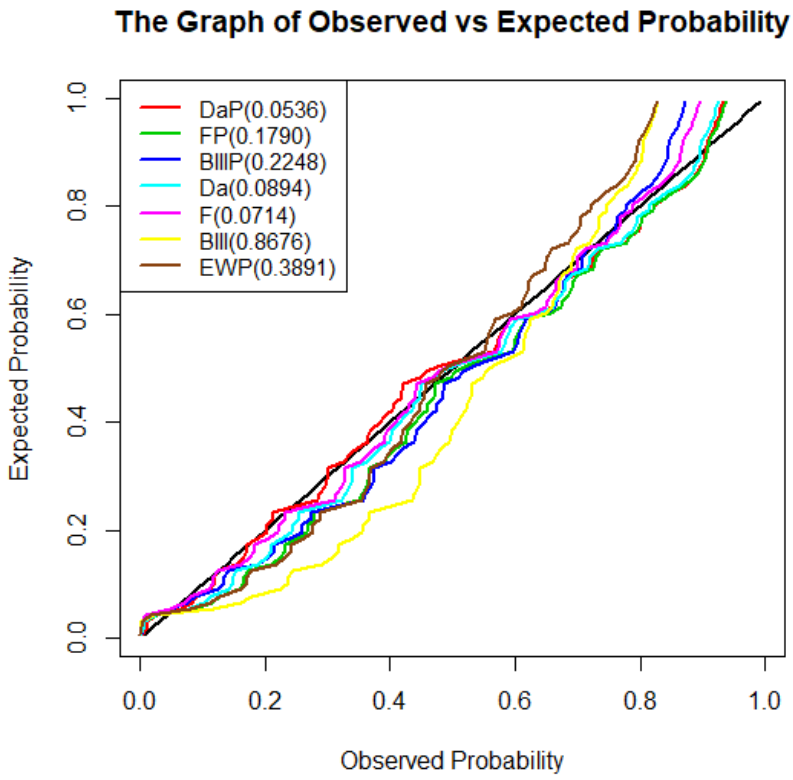


Figure 3.10: Probability Plots for Failure times of a Particular Windshield Device Data

3.1.16 Concluding Remarks

A new generalized Dagum distribution called the Dagum Poisson (DaP) distribution is studied in detail. The DaP distribution has the Fisk or log-logistic-Poisson, Fisk or log-logistic, Burr III-Poisson, Burr III and Dagum distributions as special cases. The distribution possesses hazard rate function with flexible behavior. We also obtain closed form expressions for the moments, conditional moments, mean deviations, Lorenz and Bonferroni curves, distribution of order statistics and Rényi entropy. Maximum likelihood estimation technique was used to estimate the model parameters. Finally, the DaP model was fitted to a real data set in order to illustrate the applicability and usefulness of the distribution.

3.2 Dagum Logarithmic Distribution

Let N be distributed according to the zero truncated logarithmic distribution with $a_n = n^{-1}$ and $c(\theta) = -log(1-\theta)$, where $\theta \in (0,1)$. The cdf of the Dagum Logarithmic $(DaLn)$ distribution is given by

$$F_{DaLn}(x;\lambda,\delta,\beta,\theta) = \frac{C(\theta[1+\lambda x^{-\delta}]^{-\beta})}{C(\theta)} \tag{3.34}$$

$$= \frac{log\Big(1-\theta(1+\lambda x^{-\delta})^{-\beta}\Big)}{log(1-\theta)}, \tag{3.35}$$

and the corresponding pdf is given by

$$f_{DaLn}(x;\lambda,\delta,\beta,\theta) = \frac{\lambda\beta\delta\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}}{\Big(\theta(1+\lambda x^{-\delta})^{-\beta}-1\Big)log(1-\theta)}. \tag{3.36}$$

Where $x > 0$, $\beta, \delta, \lambda > 0$ and $\theta \in (0,1)$.

3.2.1 Hazard and Reverse Hazard Function

In this section, hazard and reverse hazard functions of the $DaLn$ distribution are presented. The plots of DL pdf and hazard rate function for selected values of the

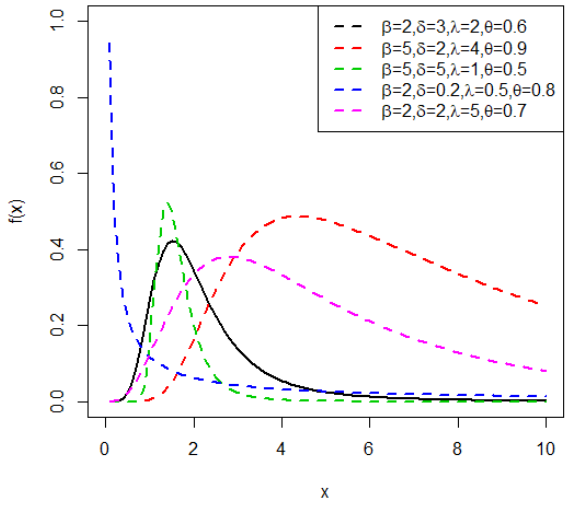


Figure 3.11: DaLn Density Functions

model parameters λ, δ, β and θ are given in Figure 3.11 and Figure 3.12, respectively. The hazard and reverse hazard functions are given by

$$h_{F_{DaLn}}(x;\lambda,\delta,\beta,\theta) = \frac{-\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}(1-\theta[1+\lambda x^{-\delta}]^{-\beta})^{-1}}{log(1-\theta)-log(1-\theta[1+\lambda x^{-\delta}]^{-\beta})} \quad (3.37)$$

and

$$\tau_{F_{DaLn}}(x;\lambda,\delta,\beta,\theta) = \frac{-\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}(1-\theta[1+\lambda x^{-\delta}]^{-\beta})^{-1}}{log(1-\theta[1+\lambda x^{-\delta}]^{-\beta})}, \quad (3.38)$$

respectively. Where $x > 0$, $\beta, \delta, \lambda > 0$ and $\theta \in (0, 1)$.

The plots of the DaLn pdf and hazard rate function for selected values of the model parameters β, δ, λ and θ are given in (3.11) and (3.12). These plots of the hazard function shows various shapes including monotonically decreasing, monotonically increasing, and increasing and decreasing shapes for the combinations of the values of the parameters. This flexibility makes the DL hazard rate function suitable for both monotonic and non-monotonic empirical hazard behaviors that are likely to be encountered in real life situations.

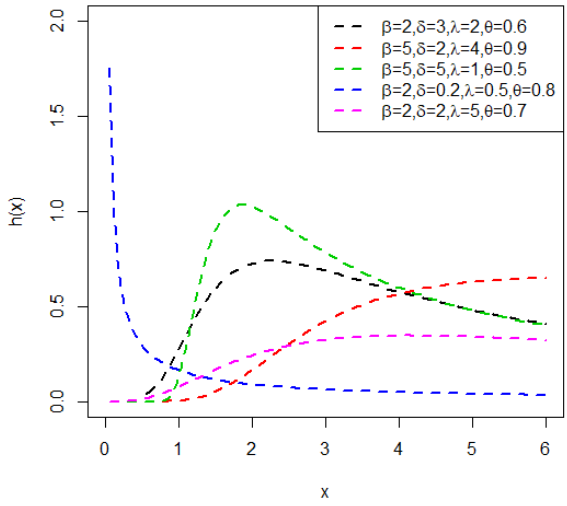


Figure 3.12: Plots of the DaLn Hazard Function

3.2.2 Quantile Function

In this subsection, we present the q^{th} quantile of the $DaLn$ distribution. The q^{th} quantile of the $DaLn$ distribution is obtained by solving the nonlinear equation

$$\frac{log(1-\theta(1+\lambda x^{-\delta})^{-\beta})}{log(1-\theta)} = U,$$

where U is a uniform variate on the unit interval $[0,1]$. It follows that the q^{th} quantile of the $DaLn$ distribution is given by

$$X_q = \left(\frac{1}{\lambda}\left(\left[\frac{1-(e^{Ulog(1-\theta)})}{\theta}\right]^{-1/\beta} - 1\right)\right)^{-1/\delta}. \tag{3.39}$$

Consequently, random number can be generated from the $DaLn$ distribution based on equation (3.39).

3.2.3 Some Sub-models of the DaLn Distribution

The $DaLn$ distribution is a very flexible model that has different sub-models when its parameters are changed. $DaLn$ distribution contains several sub-models including the following distributions.

- If $\beta = 1$, then $DaLn$ distribution reduces to a new distribution called Fisk-Logarithmic (FL) distribution with cdf given by

$$F_{FL}(x;\lambda,\delta,\theta) = \frac{log(1-\theta[1+\lambda x^{-\delta}]^{-1})}{log(1-\theta)}. \tag{3.40}$$

- When $\lambda = 1$, we obtain Burr-III Logarithmic $(BIIIL)$ distribution.

3.2.4 Expansion of DaLn Density

In this section, we provide an expansion of the $DaLn$ distribution. Note that, by using the fact that if $|z| < 1$ and $k > 0$, we have the series representation

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^j, \tag{3.41}$$

and the maclaurin series representation

$$-log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \tag{3.42}$$

the DaLn pdf can be written as follows:

$$\begin{aligned} f_{DaLn}(x;\lambda,\delta,\beta,\theta) &= \frac{\lambda\beta\delta\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}}{-\left(1-\theta(1+\lambda x^{-\delta})^{-\beta}\right)log(1-\theta)} \\ &= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \omega(j,n,\theta) f_D a(x;\delta,\lambda,\beta(j+1)), \end{aligned} \tag{3.43}$$

where $\omega(j,n,\theta) = \frac{(n+1)\Gamma(j+1)}{j!\theta^{n+1}}$, and $f_{Da}(x;\delta,\lambda,\beta(j+1))$ is the Dagum pdf with parameters $\lambda, \beta(j+1), \delta, > 0$. The above equation shows that the $DaLn$ density is indeed a linear combination of Dagum densities. Hence, most of its mathematical properties can be immediately obtained from those of the Dagum distribution.

3.2.5 Moments

In this subsection, we present the r^{th} moment of $DaLn$ distribution. If the random variable X has a $DaLn$ distribution, with parameter vector $\varphi = (\lambda, \delta, \beta, \theta)$, and Y is a random variable that follows a Dagum distribution with parameters $\lambda, \beta(j+1), \delta$, then the r^{th} moment of X is given by

$$\begin{aligned} \mu_r^{'} &= \sum_{j=0}^{\infty} \omega(j,n,\theta) E(Y^r) \\ &= \sum_{j=0}^{\infty} \omega(j,n,\theta) \beta \lambda^{\frac{r}{\delta}} B\left(\frac{r}{\delta} + \beta(j+1), 1 - \frac{r}{\delta}\right), \quad \delta > r. \end{aligned} \tag{3.44}$$

Note that the r^{th} non-central moment follows readily from the fact that DaLn pdf can be written as a linear combination of Dagum densities with parameters $\lambda, \beta(j+1), \delta > 0$. Based on the first four moments of the $DaLn$ distribution, the measures of skewness (CS) and kurtosis (CK) of the $DaLn$ distribution can obtained from

$$CS = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{[\mu_2 - \mu_1^2]^{\frac{3}{2}}} \quad \text{and} \quad CK = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{[\mu_2 - \mu_1^2]^2},$$

respectively. Table 3.9 lists the first six moments of the $DaLn$ distribution for selected values of the parameters, by fixing $\delta = 6.5$ and Table 3.10 lists the first six moments of the $DaLn$ distribution for selected values of the parameters, by fixing $\lambda = 9.0$ and $\beta = 0.1$,. These values can be determined numerically using R and MATLAB.

Table 3.9: Moments of the $DaLn$ distribution for some parameter values; $\delta = 6.5$.

μ'_s	$\lambda = 3.5, \beta = 1.0, \theta = 0.4$	$\lambda = 3.0, \beta = 0.9, \theta = 0.5$	$\lambda = 3.5, \beta = 0.8, \theta = 0.6$	$\lambda = 3.2, \beta = 1.2, \theta = 0.6$
μ'_1	1.3120	1.2702	1.2903	1.3866
μ'_2	1.8701	1.7602	1.8259	2.0805
μ'_3	2.9366	2.6954	2.8650	3.4305
μ'_4	5.2521	4.7114	5.1443	6.4333
μ'_5	11.7016	10.2761	11.5487	15.0185
μ'_6	49.2086	42.3642	49.0772	66.1632
SD	0.3858	0.3833	0.4013	0.3972
CV	0.2940	0.3018	0.3110	0.2864
CS	1.6136	1.5348	1.4474	1.7256
CK	12.0247	11.4623	10.8555	12.7280

Table 3.10: Moments of the $DaLn$ distribution for some parameter values; $\beta = 0.1$ and $\delta = 9.0$.

μ'_s	$\lambda = 3.5, \theta = 0.1$	$\lambda = 3.0, \theta = 0.5$	$\lambda = 3.5, \theta = 0.2$	$\lambda = 3.2, \theta = 0.4$
μ'_1	**0.5758**	**0.6285**	**0.5884**	**0.6133**
μ'_2	**0.4726**	**0.5369**	**0.4884**	**0.5184**
μ'_3	**0.4581**	**0.5317**	**0.4770**	**0.5107**
μ'_4	**0.4991**	**0.5859**	**0.5224**	**0.5615**
μ'_5	**0.6036**	**0.7125**	**0.6344**	**0.6826**
μ'_6	**0.8191**	**0.9687**	**0.8638**	**0.9287**
SD	**0.3756**	**0.3768**	**0.3772**	**0.3772**
CV	**0.6523**	**0.5995**	**0.6410**	**0.6149**
CS	**0.4454**	**0.2959**	**0.4142**	**0.3404**
CK	**2.7270**	**2.6664**	**2.7083**	**2.6767**

Similarly, the moment generating function (mgf) of X is given by

$$\begin{aligned} M_X(t) &= \sum_{j=0}^{\infty} \omega(j, n, \theta) M_Y(t) \\ &= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \omega(j, n, \theta) \frac{t^n}{n!} \lambda^{\frac{n}{\delta}} B\left(\frac{n}{\delta} + \beta(j+1), 1 - \frac{n}{\delta}\right), \quad \delta > n. \end{aligned} \tag{3.45}$$

This follows from the well known definition of the moment generating function given by $M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x; \phi) dx$. Since $\sum_{n=0}^{\infty} \frac{t^n}{n!} x^n f(x)$ converges and each term is integrable for all t close to 0, then we can rewrite the moment generating function as $M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$ by replacing $E(X^n)$ by the right side of Equation (3.44) to obtain the desired result.

The plots of the CS and CK vs shape parameters β and $\delta$are given in Figures 6.3-6.6. The results in Table 6.1 are used to plot the Figures 6.3-6.6. In Figure 6.3,

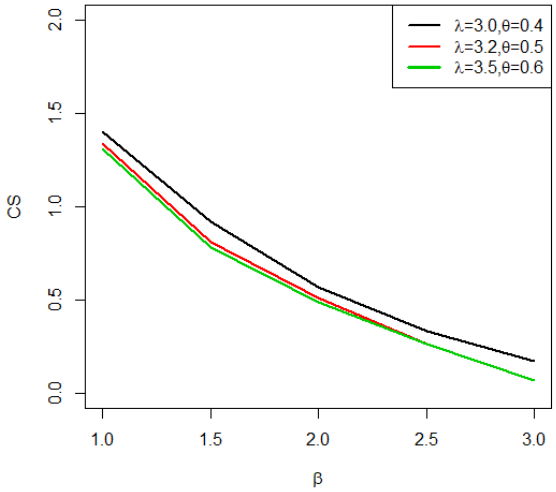


Figure 3.13: Plot of CS vs β.

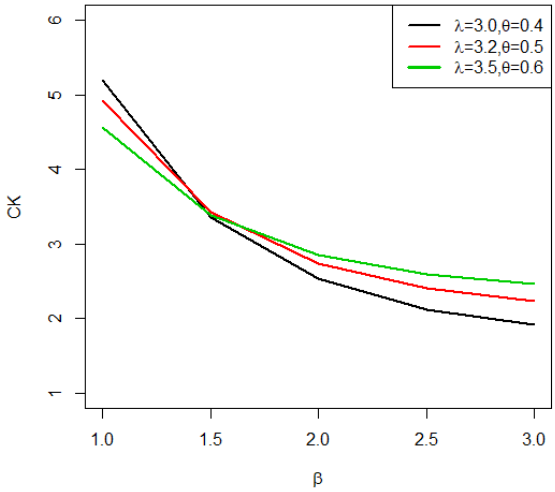


Figure 3.14: Plot of CK vs β.

for λ constant at $\lambda = 3.0$, 3.2 and 3.5 when β increases from $\beta = 1.0$ to 3.0, skewness decreases. The same observation is made for Figure 6.4 for the same parameter value choices.

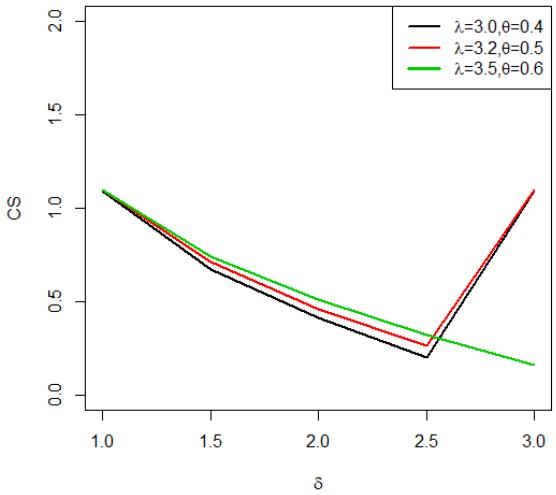


Figure 3.15: Plot of CS vs δ.

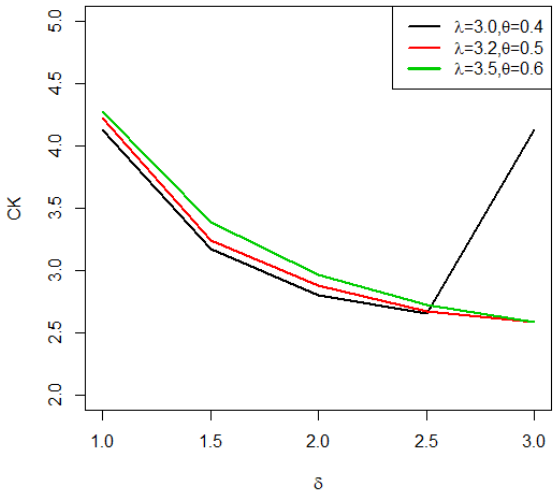


Figure 3.16: Plot of CK vs δ.

From Figures 6.5 and 6.6, for $\lambda = 3.0$ and 3.2, when δ increases from $\delta = 1.0$ to

3.0, the distribution is heavily tailed and with a peaked kurtosis. Conversely, for $\lambda = 3.5$, Figures 6.5 and 6.6 suggest that the tail of the distribution is short with a kurtosis.

3.2.6 Conditional Moments

For income and lifetime distributions, it is of interest to obtain the conditional moments and mean residual life function. The r^{th} conditional moment for $DaLn$ distribution is given by

$$\begin{aligned} E(X^r|X>t) &= \frac{1}{\overline{F}_{_{DaLn}}(t)}\int_t^\infty x^r f_{_{DaLn}}(x)dx \\ &= \frac{1}{\overline{F}_{_{DaLn}}(t)}\sum_{j=0}^{\infty}\omega(j,n,\theta)\left[B\left(\beta(j+1)+\frac{r}{\delta},1-\frac{r}{\delta}\right)\right. \\ &- \left. B_{_{t(a)}}\left(\beta(k+1)+\frac{r}{\delta},1-\frac{r}{\delta}\right)\right], \end{aligned} \tag{3.46}$$

where $t(a) = (1+\lambda a^{-\delta})^{-1}$, $\delta > r$, and $B_{_{t(a)}}(c,d) = \int_0^{t(a)} u^{c-1}(1-u)^{d-1}du$. The mean residual life function is $E(X|X>t) - t$.

3.2.7 Mean and Median Deviations

The amount of dispersion in a population can be measured to some extent by the totality of deviations from the mean and the median. If X has the $DaLn$ distribution, we can derive the mean deviation about the mean $\mu = E(X)$ and the mean deviation about the median M from

$$\delta_1 = \int_0^\infty |x-\mu| f_{_{DaLn}}(x)dx \quad \text{and} \quad \delta_2 = \int_0^\infty |x-M| f_{_{DaLn}}(x)dx, \tag{3.47}$$

respectively. The mean μ is obtained from equation (3.44) with $r = 1$, and the median M is given by equation (3.39) when $q = \frac{1}{2}$.

The measure δ_1 and δ_2 can be calculated by the following relationships:

$$\delta_1 = 2\mu F_{_{DaLn}}(\mu) - 2\mu + 2T(\mu) \quad \text{and} \quad \delta_2 = 2T(M) - \mu, \tag{3.48}$$

where $T(a) = \int_a^\infty x \cdot f_{_{DaLn}}(x)dx$ follows from equation (3.46), that is,

$$T(a) = \sum_{k=0}^{\infty} \omega(k,\theta)\left[B\left(\beta(k+1)+\frac{1}{\delta}, 1-\frac{1}{\delta}\right) - B_{t(a)}\left(\beta(k+1)+\frac{1}{\delta}, 1-\frac{1}{\delta}\right)\right].$$

3.2.8 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves are widely used tool for analyzing and visualizing income inequality. Lorenz curve, $L(p)$ can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume a, and Bonferroni curve, $B(p)$ is the scaled conditional mean curve, that is, ratio of group mean income of the population. Let $I(a) = \int_0^a x f_{_{DL}}(x)dx$ and $\mu = E(X)$, then Bonferroni and Lorenz curves are given by $B(p) = \frac{I(q)}{p\mu}$ and $L(p) = \frac{I(q)}{\mu}$, respectively, for $0 \leq p \leq 1$, and $q = F^{-1}_{DaLn}(p)$. The mean of the DL distribution is obtained from equation (3.44) with $r = 1$ and the quantile function is given in equation (3.39). Consequently,

$$I(a) = \sum_{j=0}^{\infty} \omega(j,n,\theta) B_{_{t(a)}}\left(\beta(j+1)+\frac{1}{\delta}, 1-\frac{1}{\delta}\right), \tag{3.49}$$

for $\delta > 1$, where $t(a) = (1+\lambda a^{-\delta})^{-1}$, and $B_{F(x)}(c,d) = \int_0^{F(x)} t^{c-1}(1-t)^{d-1}dt$ for $|F(x)| < 1$ is incomplete Beta function.

3.2.9 Order Statistics

The pdf of the k^{th} order statistics from a pdf $f(x)$ is

$$\begin{aligned} f_{k:n}(x) &= \frac{f(x)}{B(k,n-k+1)} F^{k-1}(x)[1-F(x)]^{n-k} \\ &= k\binom{n}{k} f(x) F^{k-1}(x)[1-F(x)]^{n-k}. \end{aligned} \tag{3.50}$$

We apply the series expansion

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} z^j, \tag{3.51}$$

for $b > 0$ and $|z| < 1$, to obtain the series expansion of the distribution of order statistics from DaLn distribution. Using equations (3.51), the pdf of the k^{th} order statistic from DaLn distribution is given by

$$\begin{aligned} f_{k:n}(x) &= \sum_{s=0}^{\infty} k\binom{n}{k}\frac{(-1)^s f(x)\Gamma(n-k+1)}{s!\Gamma(n-k+1-s)}[F(x)]^{s+k-1} \\ &= \sum_{s=0}^{\infty} k\binom{n}{k}\frac{(-1)^s\Gamma(n-k+1)}{s!\Gamma(n-k+1-s)}\frac{\beta\delta\lambda\theta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}}{(\theta[1+\lambda x^{-\delta}]^{-\beta}-1)\log(1-\theta)} \\ &\times \left(\frac{\log(1-\theta[1+\lambda x^{-\delta}]^{-\beta})}{\log(1-\theta)}\right)^{s+k-1}. \end{aligned} \tag{3.52}$$

That is,

$$f_{k:n}(x) = \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} H(s,n,k)\cdot\omega(j,n,\theta) f_{Da}(x;\delta,\lambda,\beta(j+1)),$$

where $\omega(j,n,\theta) = \frac{(n+1)\Gamma(j+1)}{j!\theta^{n+1}}$ and $H(s,n,k) = k\binom{n}{k}\frac{(-1)^s f(x)\Gamma(n-k+1)}{s!\Gamma(n-k+1-s)} \times \frac{\sum_{n=0}^{\infty}\frac{(\theta[1+\lambda x^{-\beta}]^{-\beta})^{n+1}}{n+1}}{\sum_{n=0}^{\infty}\frac{\theta^{n+1}}{n+1}}$. Thus, the pdf of the i^{th} order statistic from the $DaLn$ distribution is a linear combination of Dagum $pdf s$ with parameters $\lambda, \delta, \beta(j+1) > 0$.

3.2.10 Rényi Entropy

Rényi entropy of a distribution with pdf $f(x)$ is defined as

$$I_R(\tau) = (1-\tau)^{-1}\log\left\{\int_{\mathbb{R}} f^{\tau}(x)dx\right\}, \tau > 0, \tau \neq 1.$$

For the $DaLn$ distribution,

$$f^{\tau}_{DaLn}(x) = \left(\frac{\lambda\beta\delta\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}}{(\theta(1+\lambda x^{-\delta})^{-\beta}-1)log(1-\theta)}\right)^{\tau}. \tag{3.53}$$

The Rényi entropy of $DaLn$ distribution is given by

$$I_R(\tau) = \frac{1}{1-\tau}\log\left[\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\omega(j,n,\theta)^{\tau} f^{\tau}_{Da}(x;\delta,\lambda,\beta(j+1))\right].$$

where $\omega(j,n,\theta)=\frac{(n+1)\Gamma(j+1)}{j!\theta^{n+1}}$, and $f_{Da}(x;\delta,\lambda,\beta(j+1))$ is the Dagum pdf with parameters $\lambda,\beta(j+1),\delta,>0$. Renyi entropy for the sub-models can be readily obtained.

3.3 Maximum Likelihood Estimators

In this section, we consider the maximum likelihood estimators (MLE's) of the parameters of the $DaLn$ distribution. Let $x_1,...,\ x_n$ be a random sample of size n from $DaLn$ distribution and $\mathbf{\Theta}=(\lambda,\delta,\beta,\theta)^T$ be the parameter vector. The log-likelihood function for the $DaLn$ distribution can be written as

$$\begin{aligned} L &= n\log\beta+n\log\delta+n\log\lambda+n\log\theta-(\delta+1)\sum_{i=1}^{n}\log x_i \\ &- (\beta+1)\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta})-\sum_{i=1}^{n}\log(\theta(1+\lambda x_i^{-\delta})^{-\beta}-1) \\ &- n\log(\log(1-\theta)). \end{aligned} \tag{3.54}$$

The associated score function is given by

$$U_n(\mathbf{\Theta})=\left(\frac{\partial L}{\partial\lambda},\frac{\partial L}{\partial\delta},\frac{\partial L}{\partial\beta},\frac{\partial L}{\partial\theta}\right)^T,$$

where

$$\frac{\partial L}{\partial\lambda}=\frac{n}{\lambda}-(\beta+1)\sum_{i=1}^{n}\frac{x_i^{-\delta}}{(1+\lambda x_i^{-\delta})}+\beta\theta\sum_{i=1}^{n}\frac{x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-\beta-1}}{(\theta(1+\lambda x_i^{-\delta})^{-\beta}-1)}, \tag{3.55}$$

$$\begin{aligned} \frac{\partial L}{\partial\delta} &= \frac{n}{\delta}-\sum_{i=1}^{n}\log x_i+(\beta+1)\lambda\sum_{i=1}^{n}\frac{x_i^{\delta}\log x_i}{(1+\lambda x_i^{-\delta})} \\ &- \beta\lambda\theta\sum_{i=1}^{n}\frac{x_i^{\delta}(1+\lambda x_i^{-\delta})^{-\beta-1}\log x_i}{\left\{\theta(1+\lambda x_i^{-\delta})^{-\beta}-1\right\}}, \end{aligned} \tag{3.56}$$

$$\frac{\partial L}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta})-\theta\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-\beta}\log(1+\lambda x_i^{-\delta})}{\left\{\theta(1+\lambda x_i^{-\delta})^{-\beta}-1\right\}} \tag{3.57}$$

and

$$\frac{\partial L}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-\beta}}{\left\{\theta(1+\lambda x_i^{-\delta})^{-\beta}-1\right\}}+\frac{n}{(1-\theta)\log(1-\theta)}. \tag{3.58}$$

The maximum likelihood estimate (MLE) of $\boldsymbol{\Theta}$, say $\widehat{\boldsymbol{\Theta}}$, is obtained by solving the nonlinear system $U_n(\boldsymbol{\Theta}) = 0$. The solution of this nonlinear system of equation is not in a closed form. These equations cannot be solved analytically, and statistical software can be used to solve them numerically via iterative methods. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the estimate $\widehat{\boldsymbol{\Theta}}$.

3.4 Applications of The Model to Real Life Data

In this section, we present examples to illustrate the flexibility of the $DaLn$ distribution and its sub-models for data modeling. Estimates of the parameters of DaLn distribution (standard error in parentheses), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), sum of squares (SS) from the probability plots, Cramér-von Mises and Anderson Darling goodness-of-fit statistics W^* and A^* are presented for the data set.

The table in the example contain the goodness-of-fit statistics W^* and A^*, and are described by Chen and Balakrishnan [10]. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit. In order to compare the models, we use the criteria stated above. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred, for AIC, AICC and BIC, smaller values are preferred and for the Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics, smaller values are preferred.

The $DaLn$ distribution is compared with other distributions including the Beta Weibull Logarithmic distribution [33]. The cdf of the BWL distribution is given by

$$F_{BWL}(x;\beta,\gamma,\theta) = 1 - \frac{log(1-\theta e^{-(\beta x)^{\gamma}})}{\log(1-\theta)}, \tag{3.59}$$

where $x > 0$, $\beta, \gamma > 0$, and $\theta \in (0, 1)$.

Plots of the fitted densities and the histogram of the data are given in Figure (3.17). The probability plots (Chambers et al. [8]) are also presented in Figure

(3.18). For the probability plot, we plotted $G_{DaLn}(x_{(j)}; \hat{\lambda}, \hat{\delta}, \hat{\beta}, \hat{\theta})$ against $\frac{j-0.375}{n+0.25}, j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. A measure of closeness of each plot to the diagonal line is also computed. This measure of closeness is given by the sum of squares $SS = \sum_{j=1}^{n} \left[G_{DaLn}(x_{(j)}; \hat{\lambda}, \hat{\delta}, \hat{\beta}, \hat{\theta}) - \left(\frac{j-0.375}{n+0.25}\right) \right]^2$.

3.4.1 Example 1: Repair Times Data

This data set consists of 46 observations corresponding to the maintenance on active repair times (in hours) for an airborne communication transceiver from Leiva et al. The data is presented in Table (3.11).

Table 3.11: Repair times data of 46 communication transceiver

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7
0.8	0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5	1.5	1.5
1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3	4.0
4.0	4.5	4.7	5.0	5.4	5.4	7.0	7.5	8.8	9.0	10.3
22.0	24.5									

Table 3.12: Estimates of Models for Repair Times Data

	Estimates				Statistics						
Model	β	δ	λ	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	SS
DaLn	4.0312	2.0090	0.0545	0.9958	198.3143	206.339	207.3146	213.6536	0.0283	0.2016	0.2282
	(7.1738)	(0.3907)	(0.1073)	(0.0059)							
FL		2.3029	0.2188	0.9979	198.8044	205.3501	205.9215	210.836	0.0418	0.2772	0.3619
		(0.2497)	(0.0918)	(0.0008)							
Dagum	3.6937	1.1833	0.3880		200.5575	206.5575	207.1289	212.0434	0.0577	0.3491	0.0528
	(5.0394)	(0.2486)	(0.6538)								
Fisk		0.1539	0.5349		202.3421	206.3421	206.6211	209.993	0.0750	0.4519	0.0890
		(0.1858)	(0.0939)								
	β	γ		θ							
BWL	0.0837	1.8745		0.9977	203.3227	209.2582	209.8297	214.7442	0.0608	0.5062	1.2163
	(0.0138)	(0.1816)		(0.0012)							

Estimates of the parameters of the distributions, standard errors (in parentheses), -2log-likelihood statistic, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC) are given in the table below. The results obtained from fitting the $DaLn$ distribution and its sub-models FL, Fisk and Dagum distributions and one other alternative, and BWL distribution are presented for the data set. We also use the LR test to compare the $DaLn$ distribution and its sub-models.

The estimated variance-covariance matrix for the $DaLn$ distribution is given by:

$$\begin{pmatrix} 0.6952 & -0.3140 & -6.3861 & -5.4278 \\ -0.3140 & 0.2014 & 3.9845 & -17.0344 \\ -6.3861 & 3.9845 & 96.1580 & -603.3669 \\ -5.4278 & -17.0344 & -603.3669 & 12131.6773 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for β, δ, λ and θ are given by $4.0312 \pm 14.0606, 2.0090 \pm 0.7658, 0.0545 \pm 0.2103,$ and 0.9958 ± 0.0116, respectively.

Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 3.17 and Figure 3.18.

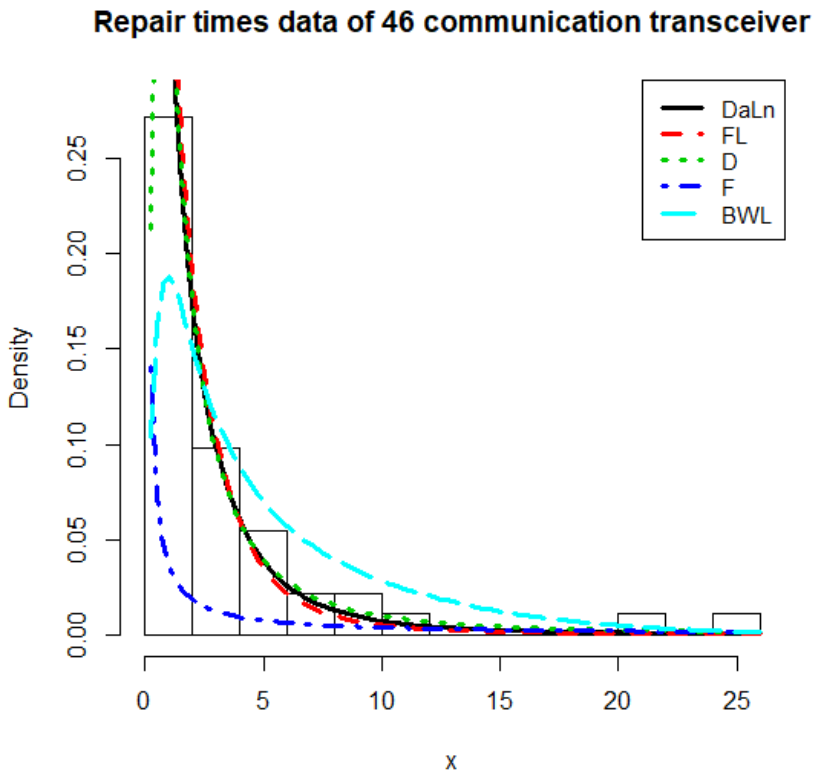


Figure 3.17: Fitted Densities

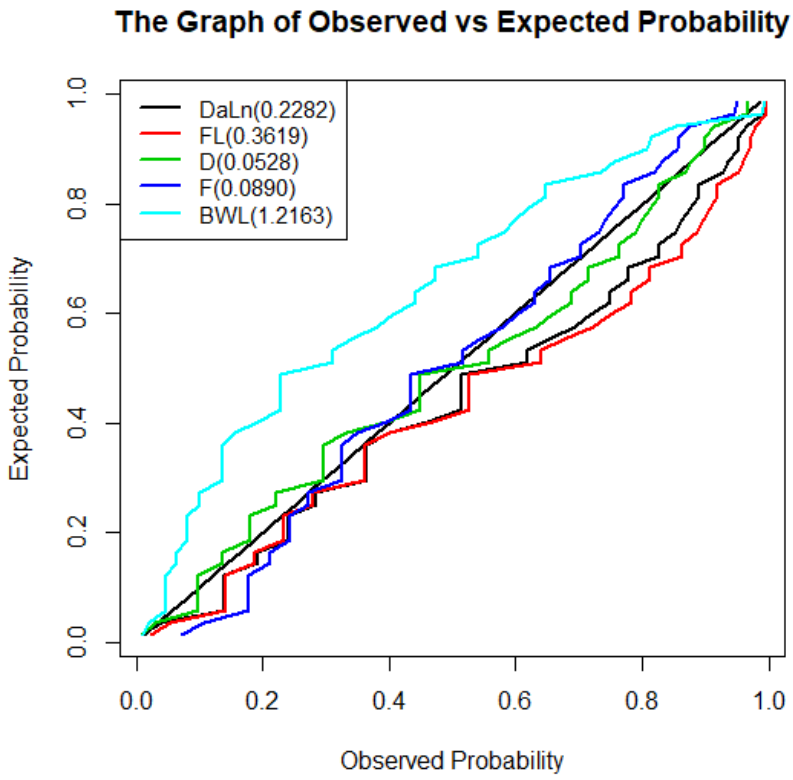


Figure 3.18: Probability Plots for Repair Times of 46 Communication Transceiver Data

There is no difference between the $DaLn$, FL and Dagum distributions according to the LR test. However, the values of the goodness-of-fit statistics A^* and W^* shows that $DaLn$ distribution is better than the nested FL, Dagum and Fisk distributions. We also compared the $DaLn$ distribution to non-nested BWL distribution using the AIC, AICC, BIC, W^*, A^* and SS statistics. The model with the smallest value for each of the statistics will be the best one to be used in fitting the data. When the $DaLn$ distribution is compared to the non-nested BWL distribution, it is clear that it is superior based on the AIC, AICC and BIC values. The $DaLn$ distribution has the smallest goodness of fit statistic W^* and A^* values. Clearly, the $DaLn$ model has points closer to the diagonal line corresponding to the smallest SS value for the probability plots when compared to the non-nested BWL distribution. Hence, the $DaLn$ distribution is the "best" fit for the data when compared to all the other models that were considered.

3.4.2 Example 2: Single Carbon Fibers of 20 mm in Gauge

The data set from the second example was originally reported by Badar and Priest [34], which represents the strength measured in GPa for single carbon fibers and impregnated at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 100 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. Here, we consider the data set of single fibers of 20 mm in gauge with a sample of size 63. The data is presented in Table (3.13). Estimates of the parameters of the distributions, standard errors (in parentheses), -2log-likelihood statistic, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC) are given in the table below. The results obtained from fitting the $DaLn$ distribution and its sub-models FL, Dagum and Fisk distributions and one other alternative, BWL distribution are presented for the data set.

The estimated variance-covariance matrix for the $DaLn$ distribution is given by:

Table 3.13: Single Carbon Fibers of 20 mm in Gauge

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397
2.445	2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614
2.616	2.618	2.624	2.659	2.675	2.738	2.740	2.856	2.917
2.928	2.937	2.937	2.977	2.996	3.030	3.125	3.139	3.145
3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.332	3.346
3.377	3.408	3.435	3.493	3.501	3.537	3.554	3.562	3.628
3.852	3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020

Table 3.14: Estimates of Models for Single Carbon Fibers of 20 mm in Gauge Data

	Estimates				Statistics						
Model	β	δ	λ	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	SS
DaLn	46.9763	9.2680	47.3694	0.9887	111.2343	121.617	122.3067	130.1896	0.0484	0.2923	0.0464
	(222.5449)	(2.3007)	(204.1055)	(0.0259)							
FL		6.5485	248.3746	0.9429	111.4843	135.0814	135.4882	141.5108	0.064	0.3569	0.3714
		(0.7148)	(100.9608)	(0.0731)							
Dagum	14.6830	5.6209	20.0986		116.8173	122.8172	123.2240	129.2466	0.1093	0.5937	0.0866
	(23.8733)	(0.5764)	(40.8144)								
Fisk		8.6457	0.3341		115.6897	119.6897	119.8897	123.976	0.0889	0.5018	0.0986
		(0.8902)	(0.0086)								
	β	γ		θ							
BWL	0.2449	9.0954		0.9931	119.4741	125.4739	125.8806	131.9033	0.0677	0.5062	0.6108
	(0.0118)	(1.7186)		(0.0112)							

$$\begin{pmatrix} 4.5606 & -0.6655 & -220.3471 & -1.2283 \\ -0.6655 & 0.2778 & 64.0683 & -5.4984 \\ -220.3471 & 64.0683 & 13318.6911 & 506.6136 \\ -1.2283 & -5.4984 & 506.6136 & -482.5255 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for β, δ, λ and θ are given by $46.9763 \pm 436.1880, 9.2680 \pm 4.5094, 47.3694 \pm 400.047$ and 0.9887 ± 0.0508, respectively.

Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 3.19 and Figure 3.20.

The LR test show that the $DaLn$ distribution is significantly better than Fisk and Dagum distributions. There is no difference between $DaLn$ and FL distributions based on the LR test. The goodness of fit statistics W^* and A^* and the SS statistic from the probability plots shows that $DaLn$ distribution is better than FL distribution for the data. We also compared the $DaLn$ distribution to its sub-models and one non-nested model BWL using the AIC, AICC, BIC, W^*, A^* and SS statistics. The model with the smallest value for each of the statistics will be the best

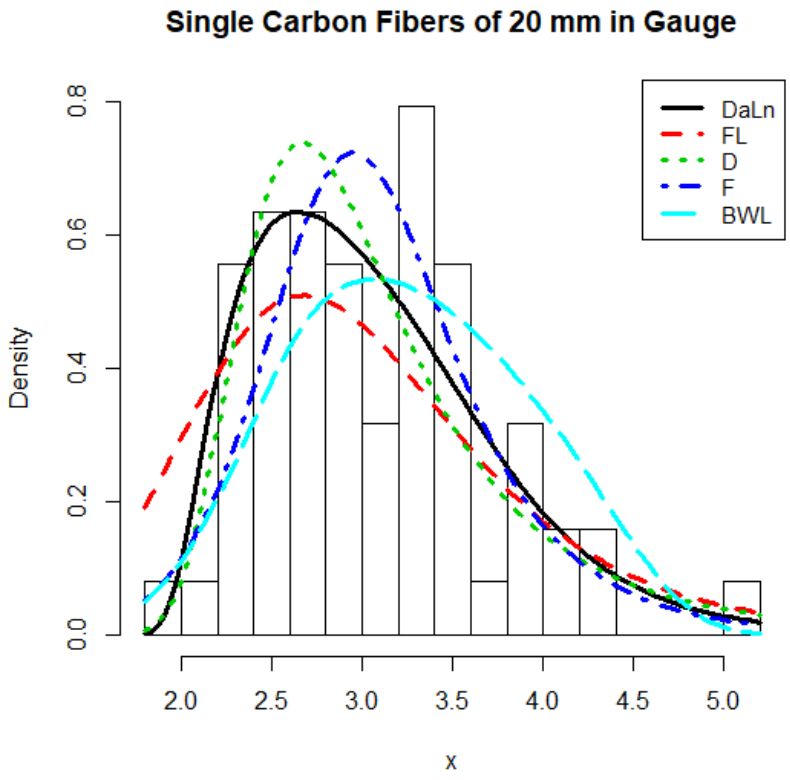


Figure 3.19: Fitted Densities

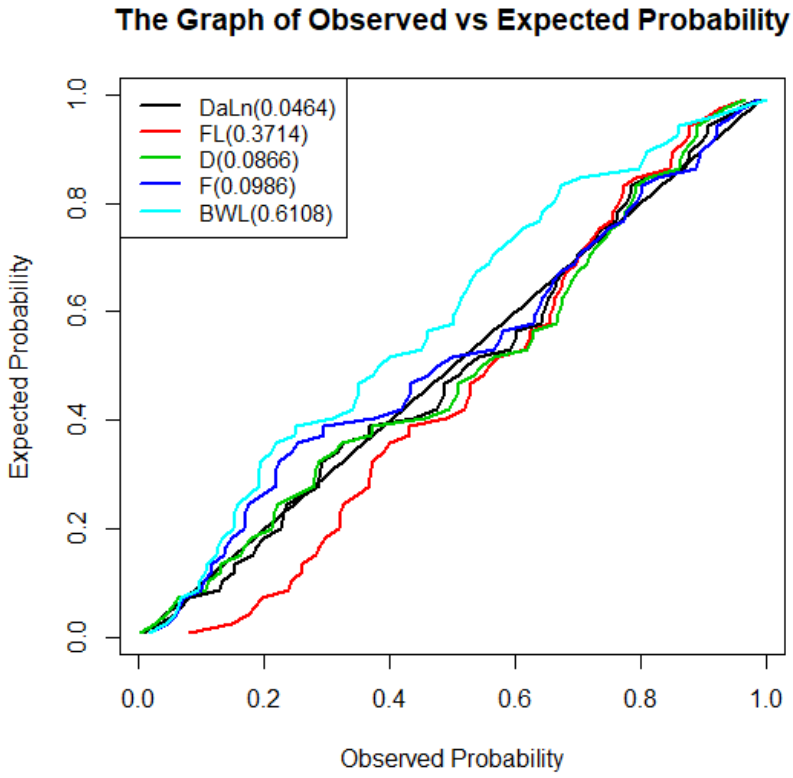


Figure 3.20: Probability Plots for Single Carbon Fibers of 20 mm in Gauge Data

one to be used in fitting the data. It is clear that the DL distribution is superior based on the AIC, AICC and BIC values. The $DaLn$ distribution has the smallest goodness of fit statistic W^* and A^* values as well as the smallest SS value among all the models that were fitted. Clearly, the $DaLn$ model has points closer to the diagonal line corresponding to the smallest SS value for the probability plots when compared to the non-nested BWL distribution. Hence, the $DaLn$ distribution is the "best" fit for the data when compared to all the other models that were considered.

3.5 Concluding Remarks

A new generalized Dagum distribution called the Dagum Logarithmic $(DaLn)$ distribution is studied in detail. The $DaLn$ distribution has the Fisk Logarithmic, Fisk and Dagum distributions as special cases. The distribution possesses hazard rate function with flexible behavior. We also obtain closed form expressions for the moments, conditional moments, mean deviations, Lorenz and Bonferroni curves, distribution of order statistics and Rényi entropy. Maximum likelihood estimation technique was used to estimate the model parameters. Finally, the $DaLn$ model was fitted to a real data set in order to illustrate the applicability and usefulness of the distribution.

Chapter 4

CONCLUSION AND AREA OF FURTHER STUDY

4.1 Conlusion

A new generalized class of distributions called the Dagum Power Series class of distributions is studied in detail. The $DaPS$ distribution is studied together with some of its properties including hazard and reverse hazard functions, quantile and series expansion. We also obtain closed form expressions for the moments, conditional moments, mean deviations, Lorenz and Bonferroni curves, distribution of order statistics and Rényi entropy. Maximum likelihood estimation technique was used to estimate the model parameters. We further studied two special cases of the DPS distribution namely the Dagum Poisson (DaP) and Dagum Logarithmic $(DaLn)$ distributions. Some properties of the DaP and $DaLn$ distributions are studied, application to real life data is carried out in both cases. We can conclude that the DaP and $DaLn$ distributions performed better than other models that were compared to with them as evident from the graphs and tables in Chapter 3. Therefore, the $DaPS$ distribution proved to be an applicable and useful distribution.

4.2 Area of further Study

Area of future study may include the study of the Dagum Binomial (DaB) and Dagum Geometric (DaG) distributions, the applications of the $(DaPS)$ special case models to different censoring schemes, the use of other methods of estimation such as Least Square, Weighted Least Square, Maximum Product of Spacing and Bayesian estimation for parameter estimation.

Chapter 5

Appendix

5.1 R Codes

```
# DP: Dagum Poisson

    rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
# Define Function #
# Define DP pdf

    DP_ pdf=function(beta,lambda,delta,theta,x){ (1-exp(theta*(1+lambda*x^ {
(-delta)} )^ { (-beta)} ))/(1-exp(-theta))}

    # unimodal for values of beta=5,delta=2,lambda=4,theta=0.1(600-604)
# L-shape beta=2,delta=0.2,lambda=0.5,theta=0.8(605-609)
# skewness to the right beta=5,delta=0.5,lambda=1,theta=0.5(610-613)
```

```
# pic2
x=seq(0,3,by=0.5)
y=seq(0,1.5,by=0.5)
curve(DP_ pdf(x,beta=0.2,delta=2.0,lambda=2.0,theta=2.0),xlim=c(0,3),
ylim=c(0,1.5),lwd=2,xlab="x",ylab="density" )
curve(DP_ pdf(x,beta=4.0,delta=5.0,lambda=0.4,theta=0.3),lty=2, col=2, lwd=2,
add=TRUE)
curve(DP_ pdf(x,beta=0.5,delta=0.1,lambda=1.0,theta=2.0),lty=2, col=3, lwd=2,
add=TRUE)
curve(DP_ pdf(x,beta=2.0,delta=1.0,lambda=0.3,theta=3.0),lty=2, col=4, lwd=2,
add=TRUE)
curve(DP_ pdf(x,beta=4.0,delta=2.0,lambda=0.5,theta=1.0),lty=2, col=6, lwd=2,
add=TRUE)
legend("topright",c(
expression(paste(beta,'=0.2,',delta,'=2.0,',lambda,'=2.0,',theta,'=2.0
expression(paste(beta,'=4.0,',delta,'=5.0,',lambda,'=0.4,',theta,'=0.3')),
expression(paste(beta,'=0.5,',delta,'=0.1,',lambda,'=1.0,',theta,'=2.0')),
expression(paste(beta,'=2.0,',delta,'=1.0,',lambda,'=0.3,',theta,'=3.0')),
expression(paste(beta,'=4.0,',delta,'=2.0,',lambda,'=0.5,',theta,'=1.0'))),
lty=c(2,2,2,2,2),col=c(1,2,3,4,6),lwd=c(2,2,2,2))

    # define DL pdf
DL_ pdf=function(beta,lambda,delta,theta,x){ (beta*delta*lambda*theta*x^ (-
delta-1)
*(1+lambda*x^ (-delta))^ (-beta-1))/(theta*(1+lambda*x^ (-delta))^
(-beta)-1)*log(1-theta)}

    # unimodal for values of beta=5,delta=2,lambda=4,theta=0.1(600-604)
# L-shape beta=2,delta=0.2,lambda=0.5,theta=0.8(605-609)
# skewness to the right beta=5,delta=0.5,lambda=1,theta=0.5(610-613)
```

```
    # pic2
x=seq(0,10,by=2)
y=seq(0,1,by=0.1)
curve(DL_pdf(x,beta=2,delta=3,lambda=2,theta=0.6),xlim=c(0,10),ylim=c(0,1),lwd=2,
xlab="x",ylab="f(x)")
curve(DL_ pdf(x,beta=5,delta=2,lambda=4,theta=0.9),lty=2, col=2, lwd=2,
add=TRUE)
curve(DL_ pdf(x,beta=5,delta=5,lambda=1,theta=0.5),lty=2, col=3, lwd=2,
add=TRUE)
curve(DL_ pdf(x,beta=2,delta=0.2,lambda=0.5,theta=0.8),lty=2, col=4, lwd=2,
add=TRUE)
curve(DL_ pdf(x,beta=2,delta=2,lambda=5,theta=0.7),lty=2, col=6, lwd=2,
add=TRUE)

    legend("topright",c(
expression(paste(beta,'=2,',delta,'=3,',lambda,'=2,',theta,'=0.6')),
expression(paste(beta,'=5,',delta,'=2,',lambda,'=4,',theta,'=0.9')),
expression(paste(beta,'=5,',delta,'=5,',lambda,'=1,',theta,'=0.5')),
expression(paste(beta,'=2,',delta,'=0.2,',lambda,'=0.5,',theta,'=0.8')),
expression(paste(beta,'=2,',delta,'=2,',lambda,'=5,',theta,'=0.7'))),
lty=c(2,2,2,2,2),col=c(1,2,3,4,6),lwd=c(2,2,2,2,2))

    # define DL pdf
DL_ pdf- function(beta,lambda,delta,theta,x){ (beta*delta*lambda*theta*x^ (-
delta-1)
 (1+lambda*x^ (-delta))^ (-beta-1))/(theta*(1+lambda*x^ (-delta))^ (-beta)-1)*log(1-
theta)}

    # define DL cdf
DL_ cdf- function(beta,lambda,delta,theta,x){ log(1 - theta * (1 + lambda *
x^(-delta))^(-beta))/ log(1 - theta)}
```

```
    # define DL hazard
DL_ hazard=function(beta,lambda,delta,theta,x){
DL_ pdf(beta,lambda,delta,theta,x)/(1-(DL_ cdf(beta,lambda,delta,theta,x)))}


    # bathtub beta=2,delta=3,lambda=2,theta=0.6(614-617)
# cubic beta=0.9,delta=3,lambda=2,theta=0.9(531-535)
# bathtub beta=1,delta5,lambda=3,theta=0.9(526-530)
# upsidedown bathtub beta=1,delta=4,lambda=1,theta=0.9(536-540)
# decreasing beta=0.9,delta=3,lambda=3,theta=0.9


    x=seq(0,6,by=1)
y=seq(0,2,by=0.5)
curve(DL_ hazard(x,beta=2,delta=3,lambda=2,theta=0.6),xlim=c(0,6),ylim=c(0,2),
lty=2,lwd=2,xlab="x",ylab="h(x)")
curve(DL_ hazard(x,beta=5,delta=2,lambda=4,theta=0.9),lty=2, col=2, lwd=2,
add=TRUE)
curve(DL_ hazard(x,beta=5,delta=5,lambda=1,theta=0.5),lty=2, col=3, lwd=2,
add=TRUE)
curve(DL_ hazard(x,beta=2,delta=0.2,lambda=0.5,theta=0.8),lty=2, col=4, lwd=2,
add=TRUE)
curve(DL_ hazard(x,beta=2,delta=2,lambda=5,theta=0.7),lty=2, col=6, lwd=2,
add=TRUE)
```

5.1.1 Simulation codes

```
library(numDeriv)
library(Matrix)
delta=1.0
beta=0.2
lambda=3.5
```

```
theta=0.4

    # Define the quantile of DP
quantile=function(delta,beta,lambda,theta,u)[result< −
((((log(u*(exp(theta)-1)+1))/
theta)^ (-1/beta)-1)
/lambda)^ (-1/delta)]

    DP_ LL< −function(par)[-sum(log(par[2]*par[3]*par[1]*par[4]*(x^ (-par[1]-1))
*((1+par[3]*x^ (-par[1]))^ (-par[2]-1))
*exp(par[4]*(1+par[3]*x^ (-par[1]))^ (-par[2]))/(exp(par[4])-1)))]

    n1=c(25,50,75,100,200,400,800)
If you want to check one sample at a time then use n1< −c(sample size)

    for (j in 1:length(n1))[n=n1[j]
N=5000
mle-lambda< −c(rep(0,N))
mle-beta< −c(rep(0,N))
mle-delta< −c(rep(0,N))
mle-theta< −c(rep(0,N))

    LC-lambda< −c(rep(0,N))
UC-lambda< −c(rep(0,N))
LC-beta< −c(rep(0,N))
UC-beta< −c(rep(0,N))
LC-delta< −c(rep(0,N))
UC-delta< −c(rep(0,N))
LC-theta< −c(rep(0,N))
UC-theta< −c(rep(0,N))
```

```
    count-lambda=0
count-beta=0
count-delta=0
count-theta=0
]
temp=1
HH1< −matrix(c(rep(2,16)),nrow=4,ncol=4)
HH2< −matrix(c(rep(2,16)),nrow=4,ncol=4)
for (i in 1:N)
    print(i)
flush.console()
repeat[x< −c(rep(0,n))


]


    Generate a random variable from uniform distribution
u< −0
u< −runif(n,min=0,max=1)


    for (k in 1:n)[
x[k]< −quantile(delta,beta,lambda,theta,u[k])
]


    # Maximum likelihood estimation
mle.result< −nlminb(c(delta,beta,lambda,theta),DP_ LL,lower=0,upper=Inf)


    temp=mle.result$ convergence
if(temp==0)[
temp-lambda< −mle.result$ par[3]
temp-beta< −mle.result$ par[2]
temp-delta< −mle.result$ par[1]
```

```
temp-theta< −mle.result$ par[4]

    HH1< −hessian(DP-LL,c(temp-delta,temp-beta,temp-lambda,temp-theta))
if(( rcond(HH1)>1e-8) & sum(is.nan(HH1))=0 & (diag(HH1)[1]>0) &
(diag(HH1)[2]>0) & (diag(HH1)[3]>0) & (diag(HH1)[4]>0) )[
HH2< −solve(HH1)
print(det(HH1))
]
else[
temp=1]
]

    if ((temp==0) & (diag(HH2)[1]>0) & (diag(HH2)[2]>0) & (diag(HH2)[3]>0)
& (diag(HH2)[4]>0) & (sum(is.nan(HH2))==0))[
break
]

    # print(temp)
temp=1
mle-lambda[i]< −mle.result$ par[3]
mle-beta[i]< −mle.result$ par[2]
mle-delta[i]< −mle.result$ par[1]
mle-theta[i]< −mle.result$ par[4]

    HH< −hessian(DP-LL,c(mle-delta[i],mle-beta[i],mle-lambda[i],mle-theta[i]))
H< −solve(HH) LC_ lambda[i]< −mle_ lambda[i]-1.96*sqrt(diag(H)[3])
UC_ lambda[i]< −mle_ lambda[i]+1.96*sqrt(diag(H)[3])
if( (LC_ lambda[i]<=lambda) & (lambda<=UC_ lambda[i]))[
count_ lambda=count_ lambda+1
]
    LC_ beta[i]< −mle_ beta[i]-1.96*sqrt(diag(H)[2])
```

```
UC_ beta[i]< −mle_ beta[i]+1.96*sqrt(diag(H)[2])
if( (LC_ beta[i]<=beta) & (beta<=UC_ beta[i]))[
count_ beta=count_ beta+1
]

    LC_ delta[i]< −mle_ delta[i]-1.96*sqrt(diag(H)[1])
UC_ delta[i]< −mle_ delta[i]+1.96*sqrt(diag(H)[1])
if( (LC_ delta[i]<=delta) & (delta<=UC_ delta[i]))[
count_ delta=count_ delta+1
]

    LC_ theta[i]< −mle_ theta[i]-1.96*sqrt(diag(H)[4])
UC_ theta[i]< −mle_ theta[i]+1.96*sqrt(diag(H)[4])
if( (LC_ theta[i]<=theta) & (theta<=UC_ theta[i]))[
count_ theta=count_ theta+1
]

    # Calculate Average Bias
Bias_ lambda<-sum(mle_ lambda-lambda)/N
Bias_ beta<-sum(mle_ beta-beta)/N
Bias_ delta<-sum(mle_ delta-delta)/N
Bias_ theta<-sum(mle_ theta-theta)/N

    print(cbind(Bias_ lambda,Bias_ beta,Bias_ delta,Bias_ theta))

    # Calculate RMSE
RMSE_ lambda<-sqrt(sum((lambda-mle_ lambda)^ 2)/N)
RMSE_ beta<-sqrt(sum((beta-mle_ beta)^ 2)/N)
RMSE_ delta<-sqrt(sum((delta-mle_ delta)^ 2)/N)
RMSE_ theta<-sqrt(sum((theta-mle_ theta)^ 2)/N)
print(cbind(RMSE_ lambda,RMSE_ beta,RMSE_ delta,RMSE_ theta))
```

```
    # Converge Probability
CP_ lambda<-count_ lambda/N
CP_ beta<-count_ beta/N
CP_ delta<-count_ delta/N
CP_ theta<-count_ theta/N
print(cbind(CP_ lambda,CP_ beta,CP_ delta,CP_ theta))

    # Average Width
AW_ lambda<-sum(abs(UC_ lambda-LC_ lambda))/N
AW_ beta<-sum(abs(UC_ beta-LC_ beta))/N
AW_ delta<-sum(abs(UC_ delta-LC_ delta))/N
AW_ theta<-sum(abs(UC_ theta-LC_ theta))/N
print(cbind(AW_ lambda,AW_ beta,AW_ delta,AW_ theta))
```

5.1.2 Applications

```
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(AdequacyModel)

    # DAGUM POISSON
# LIfetime of 20 Electronic Components

    rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(AdequacyModel)
```

```
# This data set consists of 100 observations corresponding to the breaking stress of
carbon fibres

data1=c(3.70, 2.74 ,2.73, 2.50 ,3.60, 3.11, 3.27 ,2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22,
1.69 ,3.28, 3.09 ,1.87, 3.15, 4.90, 3.75 ,2.43, 2.95, 2.97 ,3.39 ,2.96 ,2.53, 2.67,
2.93, 3.22, 3.39, 2.81, 4.20 ,3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35,
2.55, 2.59 ,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41 ,3.68, 2.97 ,1.36, 0.98 ,2.76,
4.91 ,3.68, 1.84, 1.59, 3.19, 1.57 ,0.81 ,5.56, 1.73 ,1.59, 2.00, 1.22, 1.12, 1.71,
2.17, 1.17, 5.08, 2.48 ,1.18, 3.51 ,2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48,
0.85, 1.61, 2.79 ,4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88 ,2.82,
2.05 ,3.65)

# This data set consists of 84 observations correspond-ing to the failure times of a
particular windshield device.

data2=c(0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478,
0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699,
1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035,
1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240,
1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305,
1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485,
1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663
)
DP_ cdf=function(par,x){
beta=par[1]
delta=par[2]
```

```
lambda=par[3]
theta=par[4]
((1-exp(theta*(1+lambda*x^ (-delta))^ (-beta)))/(1-exp(theta)))}

    DP_ pdf=function(par,x){
beta=par[1]
delta=par[2]
lambda=par[3]
theta=par[4]
((beta*delta*lambda*theta*x^ (-delta-1)*(1+lambda*x^ (-delta))^ (-beta-1)
*exp(theta*(1+lambda*x^ (-delta))^ (-beta)))/
(exp(theta)-1))}

    goodness.fit(pdf=DP_ pdf, cdf=DP_ cdf,starts= c(1.5,0.1,0.5,1), data = data1,
method = "BFGS",domain = c(0,Inf),mle = NULL, lim_ inf = c(0,0,0,0),
lim_ sup = c(Inf,Inf,Inf,1))

    # FISK POISSON

    FP_ cdf=function(par,x){
delta=par[1]
lambda=par[2]
theta=par[3]
((1-exp(theta*(1+lambda*x^ (-delta))^ (-1)))/(1-exp(theta)))}

    FP_ pdf=function(par,x){
delta=par[1]
lambda=par[2]
theta=par[3]
((delta*lambda*theta*x^ (-delta-1)*(1+lambda*x^ (-delta))^ (-2)
*exp(theta*(1+lambda*x^ (-delta))^ (-1)))/
```

```
(exp(theta)-1))}

    goodness.fit(pdf=FP_ pdf, cdf=FP_ cdf, starts = c(15,1.0,0.5), data = data1,
method = "BFGS",domain = c(0,Inf),mle = NULL, lim_ inf = c(0,0,0),
lim_ sup = c(Inf,Inf,1))

    # BURR III POISSON

    BIIIP_ cdf=function(par,x){
beta=par[1]
delta=par[2]
theta=par[3]
((exp(theta*(1+x^ (-delta))^ (-beta)))/(exp(theta)-1))}

    BIIIP_ pdf=function(par,x){
beta=par[1]
delta=par[2]
theta=par[3]
((beta*delta*theta*x^ (-delta-1)*(1+x^ (-delta))^ (-beta-1)
*exp(theta*(1+x^ (-delta))^ (-beta)))/
(exp(theta)-1))}

    goodness.fit(pdf=BIIIP_ pdf, cdf=BIIIP_ cdf,starts= c(1.5,0.1,0.5), data = data1,
method = "BFGS",domain = c(0,Inf),mle = NULL, lim_ inf = c(0,0,0),
lim_ sup = c(Inf,Inf,1))

    # DAGUM

    D_ cdf=function(par,x){
beta=par[1]
delta=par[2]
```

```
lambda=par[3]
(1+lambda*x^ (-delta))^ (-beta)}

    D_ pdf=function(par,x){
beta=par[1]
delta=par[2]
lambda=par[3]
(beta*delta*lambda*x^ (-delta-1)*(1+lambda*x^ (-delta))^ (-beta-1))}

    goodness.fit(pdf=D_ pdf, cdf=D_ cdf,starts= c(10,1.5,0.5), data = data1,
method = "BFGS",domain = c(0,Inf),mle = NULL, lim_ inf = c(0,0,0),
lim_ sup = c(Inf,Inf,Inf))

    # FISK

    F_ cdf=function(par,x){
delta=par[1]
lambda=par[2]
(((lambda*x)^ (delta))/(1+(lambda*x)^ (delta)))}

    F_ pdf=function(par,x){
delta=par[1]
lambda=par[2]
((delta*lambda*(lambda*x)^ (delta-1))/
(1+(lambda*x)^ (delta))^ 2)}

    goodness.fit(pdf=F_ pdf, cdf=F_ cdf,starts= c(1.5,1), data = data1,
method = "BFGS",domain = c(0,1),mle = NULL, lim_ inf = c(0,0),
lim_ sup = c(Inf,Inf))

    # BURR III
```

```
    BIII_ cdf=function(par,x){
beta=par[1]
delta=par[2]
(1+x^ (-delta))^ (-beta)}

    BIII_ pdf=function(par,x){
beta=par[1]
delta=par[2]
(beta*delta*x^ (-delta-1)*(1+x^ (-delta))^ (-beta-1))}

    goodness.fit(pdf=BIII_ pdf, cdf=BIII_ cdf,starts= c(2,1), data = data1,
method = "BFGS",domain = c(0,Inf),mle = NULL, lim_ inf = c(0,0),
lim_ sup = c(Inf,1))

    # EXPONENTIATED WEIBULL POISSON

    EWP_ cdf=function(par,x){
beta=par[1]
delta=par[2]
gamma=par[3]
theta=par[4]
((exp(theta*(1-exp(-(beta*x)^ (gamma)))^ delta)-1)/(exp(theta)-1))}

    EWP_ pdf=function(par,x){ beta=par[1]
delta=par[2]
gamma=par[3]
theta=par[4]
(beta^ (gamma)*delta*gamma*theta*x^ (-gamma-1)*exp(-(beta*x)^ (gamma))
*(1-exp(-(beta*x)^  (gamma)))^  (delta-1)*exp(theta*(1-exp(-(beta*x)^  (gamma)))^
(delta)))
```

*(1/(exp(theta)-1))}

goodness.fit(pdf=EWP_ pdf, cdf=EWP_ cdf,starts= c(1.5,0.1,0.5,1), data = data1, method = ”BFGS”,domain = c(0,Inf),mle = NULL, lim_ inf = c(0,0,0,0), lim_ sup = c(Inf,Inf,Inf,1))

Dagum Logarithmic

Repair lifetimes of an airborne transceiver.
These data correspond to maintenance on active repair times (in hours) for an airborne
communication transceiver with size n =46 cited by

Leiva et al. [27] and Chhikara and Folks [29] are given by:

data1 ¯ c(0.2,0.3,0.5,0.5,0.5,0.5,0.6,0.6,0.7,0.7,0.
7,0.8,0.8,1.0,1.0,1.0,1.0,1.1,1.3,1.5,1.5,1.5,1.5,2.0,
2.0,2.2,2.5,2.7,3.0,3.0,3.3,3.3,4.0,4.0,4.5,4.7,5.0,
5.4,5.4,7.0,7.5,8.8,9.0,10.3,22.0,24.5)

Recidivism Data (Failure times in days (61 cases) by Stollmark and Harris-Operations research, 22(6), 1192-1205, 1974.

data2=c(1,6,9,29,30,34,39,41,44,45,49,56,84,89,91,100,103,
104,115,119 ,124,138, 141,146,156,162,168,183,185,
198,209,217,217,228,233,238,241,252,258,271,275,276,
279,282,305,313,329,331,334,336,336,362,384,404,408,
422,438,441,465,486,556)

DAGUM LOGARITHMIC

DL_ pdf=function(par,x){
beta=par[1]
delta=par[2]
lambda=par[3]
theta=par[4]
(beta*delta*lambda*theta*xˆ (-delta-1)*(1+lambda*xˆ (-delta))ˆ
(-beta-1))/((theta*(1+lambda*xˆ (-delta))ˆ (-beta)-1)*log(1-theta))}

DL_ cdf=function(par,x){
beta=par[1]
delta=par[2]
lambda=par[3]
theta=par[4]
(log(1-theta*(1+lambda*xˆ (-delta))ˆ (-beta))/log(1-theta))}

goodness.fit(pdf=DL_ pdf, cdf=DL_ cdf,starts= c(2.0,1.0,0.5,0.1), data = data1, method = ”BFGS”,domain = c(0,Inf),mle = NULL, $\lim_i nf = c(0,0,0,0)$, $lim_s up = c(Inf, Inf, Inf, 1))$

FISK LOGARITHMIC

FL_ pdf=function(par,x){
delta=par[1]
lambda=par[2]
theta=par[3]
(delta*lambda*theta*xˆ (-delta-1)*(1+lambda*xˆ (-delta))ˆ (-2))/((theta*(1+lambda*xˆ (-delta))ˆ (-1)-1)*log(1-theta))}

FL_ cdf=function(par,x){
delta=par[1]
lambda=par[2]

theta=par[3]
(log(1-theta*(1+lambda*x^ (-delta))^ (-1))/log(1-theta))}

goodness.fit(pdf=FL_ pdf, cdf=FL_ cdf,starts= c(1.0,0.5,0.2), data = data1, method = "BFGS",domain = c(0,Inf),mle = NULL, lim$_i nf = c(0,0,0)$, $lim_s up = c(Inf, Inf, 1)$)

DAGUM

D_ cdf=function(par,x){
beta=par[1]
delta=par[2]
lambda=par[3]
(1+lambda*x^ (-delta))^ (-beta)}

D_ pdf=function(par,x){
beta=par[1]
delta=par[2]
lambda=par[3]
(beta*delta*lambda*x^ (-delta-1)*(1+lambda*x^ (-delta))^ (-beta-1))}

goodness.fit(pdf=D_ pdf, cdf=D_ cdf,starts= c(5,0.8,0.5), data = data1, method = "BFGS",domain = c(0,Inf),mle = NULL, lim$_i nf = c(0,0,0)$, $lim_s up = c(Inf, Inf, 1)$)

FISK

F_ cdf=function(par,x){
delta=par[1]
lambda=par[2]
(((lambda*x)^ (delta))/(1+(lambda*x)^ (delta)))}

F_ pdf=function(par,x){
delta=par[1]
lambda=par[2]
((delta*lambda*(lambda*x)^ (delta-1))/
(1+(lambda*x)^ (delta))^ 2)}

goodness.fit(pdf=F_ pdf, cdf=F_ cdf,starts= c(1.5,1), data = data1,
method = "BFGS",domain = c(0,1),mle = NULL, lim$_inf = c(0,0)$,
$lim_sup = c(Inf, Inf)$)

BETA WEIBULL LOGARITHMIC

BWL_ cdf=function(par,x){
beta=par[1]
gamma=par[2]
theta=par[3]
(1-(log(1-theta*exp(-(beta*x)^ (gamma)))/log(1-theta)))}

BWL_ pdf=function(par,x){
beta=par[1]
gamma=par[2]
theta=par[3]
(-beta^ (gamma)*theta*gamma*x^ (gamma-1)*exp(-(beta*x)^ (gamma)))/((1-
theta*exp(-(beta*x)^ (gamma)))*log(1-theta))}

goodness.fit(pdf=BWL_ pdf, cdf=BWL_ cdf,starts= c(1.5,1.0,0.5), data = data1,
method = "BFGS",domain = c(0,Inf),mle = NULL, lim$_inf = c(0,0,0)$,
$lim_sup = c(Inf, Inf, 1)$)

5.1.3 Probability Plots

```
# Dagum Poisson
Obsevered Probability vs Predicted Probability
# png(filename="Kevlar_ 49_ op.png")
pp=(seq(1,length(x),1)-0.375)/(length(x)+0.25)
ss = (pp-op)^ 2
sum(ss)
plot(pp,pp,lwd=2,'l',lty=1,xlab='Observed Probability',ylab='Expected Probability',
main='The Graph of Observed vs Expected Probability',col='black')
lines(DP_ cdf(beta=0.2,delta=3.37,lambda=7.28,theta=0.31,
sort(x)),pp, col=2,lty=1, add=TRUE,lwd=2)
lines(FP_ cdf(delta=0.64,lambda=0.01,theta=39.48,sort(x)),pp, col=3,lty=1,
add=TRUE,lwd=2)
lines(BIIIP_ cdf(beta=0.9,delta=2.13,theta=1.7,sort(x)),pp, col=4,lty=1,lwd=2)
lines(D_ cdf(beta=0.25,delta=3.43,lambda=9.04,sort(x)),pp, col=5,lty=1, lwd=2)
lines(F_ cdf(delta=0.75,lambda=1.62,sort(x)),pp, col=6,lty=1, lwd=2)
lines(BIII_ cdf(beta=0.6,delta=1.17,sort(x)),pp, col=7,lty=1,lwd=2)
lines(EWP_ cdf(beta=1.3,delta=0.86,gamma=0.87,theta=1.27,sort(x)),pp,
col='chocolate4',
lty=1,lwd=2)

    legend("topleft", cex=0.9, c("DP(0.0699)","FP(1.0084)","BIIIP(0.2319)","D(0.0809)","F(0.389
"BIII(0.4741)","EWP(0.1347)"), col = c(2,3,4,5,6,7,'chocolate4','blueviolet'),
lty = c(1,1,1,1,1,1,1,1),lwd=c(2,2,2,2,2,2,2,2))

    # Dagum Logarithmic
# png(filename="Repair_ Lifetimess_ op.png")
pp=(seq(1,length(x),1)-0.375)/(length(x)+0.25)
plot(pp,pp,lwd=2,'l',lty=1,xlab='Observed Probability',ylab='Expected Probability',
main='The Graph of Observed vs Expected Probability',col='black')
```

```
lines(DL_ cdf(beta=4.03,delta=2.01,lambda=0.05,theta=0.99,
sort(x)),pp, col=1,lty=1, lwd=2)
op=DL_ cdf(beta=4.03, delta=2.01, lambda=0.05, theta=0.99,sort(x))
ss = (pp-op)^ 2
sum(ss)
lines(FL_ cdf(delta=2.3,lambda=0.22,theta=0.99,
sort(x)),pp, col=2,lty=1, lwd=2)
op1=FL_ cdf(delta=2.3,lambda=0.22,theta=0.99,sort(x))
ss1= (pp-op1)^ 2
sum(ss1)
lines(D_ cdf(beta=3.69,delta=1.18,lambda=0.39,
sort(x)),pp, col=3,lty=1, lwd=2)
op3=D_ cdf(beta=3.69,delta=1.18,lambda=0.39,sort(x))
ss3= (pp-op3)^ 2
sum(ss3)
lines(F_ cdf(delta=1.15,lambda=0.53,
sort(x)),pp, col=4,lty=1, lwd=2)
op4=F_ cdf(delta=1.15,lambda=0.53,sort(x))
ss4= (pp-op4)^ 2
sum(ss4)
lines(BWL_ cdf(beta=0.08,gamma=1.87,theta=0.99,sort(x)),pp, col=5,lty=1,lwd=2)
op6=BWL_ cdf(beta=0.08,gamma=1.87,theta=0.99,sort(x))
ss6= (pp-op6)^ 2
sum(ss6)
legend("topleft", cex=0.9, c("DL(0.2282)","FL(0.3619)","D(0.0528)","F(0.0890)","BWL(1.2163)")
col = c(1,2,3,4,5),lty = c(1,1,1,1,1),lwd=c(2,2,2,2,2))
```

Bibliography

[1] Adamidis K., Dimitrakopoulou, T., and Loukas, S. (2005). *On a generalization of the exponential-geometric distribution*, Statistics & probability Letters, 73, 259-269.

[2] Adamidis K., and Loukas, S. (1998). *A lifetime distribution with decreasing failure rate*, Statistics & probability Letters, 39, 35- 42.

[3] Andrews, D.F. and Herzberg, A.M. (1985). *Data: A collection of problems from many fields for the student and research Worker*, New York: Springer Series in Statistics.

[4] Barlow, R.E., Toland, R.H. and Freeman, T. (1984). *A Bayesian analysis of stress-rupture life of kevlar 49/epoxy spherical pressure vessels*, In: Proc. Canad. Conf. Appl. Statist. New York: Marcel Dekker.

[5] Barreto-Souza, W., Silva, R.B., and Cordeiro, G. M (2010). *A new distribution with decreasing, increasing and upside-down bathtub failure rate*, Comput. Statist. Data Anal. 54 pp. 935-944.

[6] Barreto-Souza, W., de Morais, A.L., and Cordeiro, G.M. (2011). *The Weibull-Geometric distribution*, J. Stat. Comput. Simul. 81 , pp. 645-657.

[7] Barreto-Souza, W., and Cribari-Neto, F. (2009). *A generalization of the Exponential-Poisson distribution*, Statistics & probability Letters, 79, 2493-2500.

[8] Chambers, J.M., Cleveland, W.S., Kleiner, B., and Tukey, P.A. (1983). *Graphical methods for data analysis*, Belmont, CA: Wadsworth, (1983).

[9] Chahkandi, M., Ganjali, M. (2009). *On some lifetime distributions with decreasing failure rate*, Computational Statistics & Data Analysis, 53, 4433-4440.

[10] Chen, G., and Balakrishnan, N. (1995). *A general purpose approximate goodness-of-fit test*, Journal of Quality Technology, Vol. 27, No. 2, 154-161.

[11] Cooray, K. and Ananda, M.M.A. (2008). *A generalization of the half-normal distribution with applications to lifetime data*, Commun. Statist Theory and Methods, 37(9), 1323-1337.

[12] Dagum, C. (1977). *A new model of personal income distribution: specification and estimation*, Economie Applique'e, 30, 413 - 437.

[13] Dagum, C. (1980). *The generation and distribution of income, the Lorenz curve and the Gini ratio*, Economie Applique'e, 33, 327-367.

[14] Domma, F., and Condino, F. (2013). *The Beta-Dagum distribution: definition and properties,* Communications in statistics-theory and methods, in press.

[15] Domma, F., Giordano, S., and Zenga, M. (2011). *Maximum likelihood estimation in Dagum distribution from censored samples*, Journal of Applied Statistics V(38)12, 2971-2985.

[16] Domma, F., (2002). *L'andamento della hazard function nel modello di Dagum a tre parametri*, Quaderni di Statistica, 4, pp. 103-114 .

[17] Huang, S., and Oluyede, B. O. (2014). *Exponentiated Kumaraswamy Dagum distribution with applications to income and lifetime data*, Journal of Statistical Distributions and Applications, 1:8, pages 1-20.

[18] Johnson, N.L, Kotz, S., and Balakrishnan, N. (1994). *Continuous univariate distributions*, volume 1, John Wiley & Sons, New York.

[19] Kleiber, C., (2008). *A guide to the Dagum distribution. In: Duangkamon,C. (ed.) Modeling income distributions and Lorenz curves series: Economic studies in inequality, social exclusion and well-Being*, 5, Springer, New York.

[20] Kleiber, C., and Kotz, S. (2003). *Statistical size distribution in economics and actuarial sciences*, John Wiley & Sons, Inc.

[21] Kus, C., (2007). *A new lifetime distribution*, Computational Statistics & Data Analysis, 51, 4497-4509.

[22] Lu, W., and Shi, D. (2011). *A new compounding life distribution: the Weibull-Poisson distribution*, Journal of Applied Statistics, DOI:10.1080/02664763.2011.575126.

[23] Mahmoudi, E., and Sepahdar, A. (2013). *Exponentiated Weibull-Poisson distribution: model, properties and applications*, Mathematics and Computers in Simulation, 92, 76-97.

[24] McDonald, B., *Some generalized functions for the size distribution of income,* Econometrica, **52**(3), 647-663, (1984).

[25] McDonald, B., and Xu, J., *A generalization of the Beta distribution with application,* Journal of Econometrics, **69**(2), 133 - 152, (1995).

[26] Morais, A.L., and Barreto-Souza, W. (2011). *A compound class of Weibull and power series distributions*, Computational Statistics & Data Analysis, 55, 1410-1425.

[27] Oluyede, B. O., Pararai, M., and Huang, S. (2014). *A new class of generalized Dagum distribution with applications to income and lifetime data*, Journal of Statistical and Econometric Methods, Vol. 3, No. 2, 125-151.

[28] Pararai, M. Warahena-Liyanage, G. and Oluyede, B. O., *Exponentiated Power Lindley-Poisson distribution: properties and applications*, submitted (2014).

[29] Rezaei, S., Nadarajah, S., and Tahghighnia, N. (2011). *A new three-parameter lifetime distribution*, Statistics 0, 1-26.

[30] Swain. J., Venkatraman, S., and Wilson, J. (1988). *Least squares estimation of distribution function in Johnson's translation system*, Journal of Statistical Computation and Simulation, 29 , 271 - 297.

[31] Tahmasbi, R., and Rezaei, S. (2008). *A two-parameter lifetime distribution with decreasing failure rate*, Computational Statistics & Data Analysis, 52, 3889-3901.

[32] Zakerzadeh, H., and Mahmoudi, E. (2012). *A new two parameter lifetime distribution: model and properties*, arXiv: 1204. 4248 [stat. CO].

[33] Shahrastani, S. Y., Shadrokh, A., Yarmohammadi, M. (2016). *The Beta-Weibull-Logarithmic distribution: some properties and applications*, Department of Statistics , 15, 1, 91-108

[34] Alzaatreh, A., Lee, C., Famoye, F. (2013). *On generating families of continuous distributions*, Metron, 71: 63-79

[35] Aljarrah, M., Lee, C., Famoye, F. (2014). *On generating T-X family of distributions using quantile funtions*

[36] Jayakumar, K., Babu, M. G. (2017). *T-transmuted X family of distributions*, Statistica, anno LXXVII, 3

[37] Silva, R. B., Gauss, M. C. (2013). *The Burr XII power series distribution: A new compounding family*, Brazilian Journal of Probability and Statistics

[38] Makubate, B., Dingalo, N., Oluyede, B., Fagbamigbe, A. F. (2018). *Exponentiated Burr XII Weibull distribution: Model, properties and applications*,Journal of Data Science, 431-462

[39] Gauss, M. C., Ortega, E. M. M., Cunha, D. C. C. (2013). *The exponentiated generalized class of distributions*, Jornal of Data Science, 1-27

[40] Pal, M., Ali, M. M., Woo, J. (2006). *Exponentiated weibull distribution*, Statistica, anno LXVI

[41] Brazauskas, R., Le-Rademacher, J. (2017). *Methods for generating paired competing risks data*, Computer Methods and Programs in Biomedicine, 135: 199-207

[42] Gauss, M. C., Ahmed, Z. A., Haitham, M. Y., Selen, C., Gamze, O. (2017). *The lindley weibull distribution: Properties and applications*, Anais da Academia Brasileira de Ciencias, 90(3): 2579-2598

[43] Crowder, M. J. (2001). *Classical competing risks*, Chapman and Hall/CRC

[44] Dey, S.,Al-Zahrani, B. and Basloom, S. (2017). *Dagum Distribution:Properties and different methods of estimation*, Canadian Center of Science and Education

[45] Aslam, M., Shoaib, M. and Khan, H. (2011). *Improved group acceptance sampling plan for Dagum Distribution under percentiles lifetime*, Research Gate