

SOME NEW AND GENERALIZED DISTRIBUTIONS FOR RELIABILITY AND LIFETIME
DATA ANALYSIS

BY

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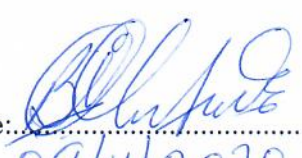
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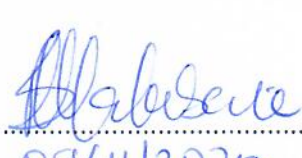
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Dedication

This thesis is wholeheartedly dedicated to the memory of my mother Botsalo Moakofi, who made me the person i am today by providing all forms of support.

To my family (lost and alive). And lastly, i dedicate this work to the Almighty God, thank you for the guidance, power of mind, protection and for the gift of life.

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Abstract

New distributions namely, Lindley-Weibull Power Series (LWPS) class of distributions and the special case called Lindley-Weibull logarithmic (LWL) distribution, a new gamma generalized Lindley log-logistic (GELLLoG) distribution, Marshall-Olkin Lindley log-logistic (MOLLLoG) distribution and half logistic log-logistic Weibull (HLLLoGW) distribution and their sub-models are presented. Structural properties including hazard and reverse hazard functions, quantile function, moments, conditional moments, mean deviations, Bonferroni and Lorenz curves, Rényi entropy, distribution of order statistics and maximum likelihood estimates of the above proposed distributions are presented. Applications of the presented models to real data sets are given in order to illustrate the applicability and usefulness of the proposed distributions. These distributions were also compared to some non-nested models.

List of Publications, Accepted and Submitted Papers

Table 1: List of Publications, Accepted and Submitted Papers

Distribution	Status	Journal
A New Generalized Lindley-Weibull Class of Distributions with Applications	Accepted	Mathematica Slovaca
A New Gamma Generalized Lindley-Log-logistic (GELLLoG) Distribution	Accepted	JAS
Marshall-Olkin Lindley-Log-logistic Distribution	Accepted	Mathematica Slovaca

Chapter 1

1.1 Introduction

The Lindley, log-logistic and Weibull distributions are widely used in several areas including reliability, economics, finance as well as actuarial sciences. However, these models have a limited range of behaviour and do not provide adequate fit to all the practical situations.

Adding parameters to a well-established distribution is one of the well established techniques for obtaining more flexible new families of distributions (Cordeiro and Lemonte [19]). These models constitute flexible family of distributions in terms of the varieties of shapes and hazard functions. In many applied areas such as reliability, lifetime analysis, finance and insurance, there is a clear need for extended and generalized forms of these distributions. For that reason, several methods for generating new families of distributions have been studied. In real life applications, empirical hazard rate curves often exhibit non-monotonic shapes such as a bathtub, upside-down bathtub (uni-modal) and others. So, increased interest in generating new families of distributions that can provide more flexibility in lifetime modeling is desirable.

There are very useful and important generalizations of the Lindley distribution in the literature that are suitable for modeling data with different types of hazard rate functions: increasing, decreasing, bathtub and

unimodal. (Lindley [41]) used a mixture of exponential and length-biased exponential distributions to illustrate the difference between fiducial and posterior distributions. The resulting mixture is called the Lindley (L) distribution. (Oluyede and Yang [64]) developed an extension of the Lindley distribution called the beta generalized Lindley distribution. A generalization of the Lindley distribution called Kumaraswamy Lindley distribution with applications to lifetime data was presented by (Oluyede et al. [65]). (Ghitany et al. [29]) investigated the properties of Lindley distribution. (Nadarajah et al. [52]) studied the mathematical and statistical properties of the exponentiated or generalized Lindley (GL) distribution. The cumulative distribution function (cdf) and probability density function (pdf) of the GL distribution are given by

$$G_{GL}(x; \alpha, \lambda) = \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \exp(-\lambda x) \right]^\alpha, \quad (1.1)$$

and

$$g_{GL}(x; \alpha, \lambda) = \frac{\alpha \lambda^2}{1 + \lambda} (1 + x) \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \exp(-\lambda x) \right]^{\alpha-1} \exp(-\lambda x), \quad (1.2)$$

for $x > 0$, $\lambda > 0$, and $\alpha > 0$. This distribution is the exponentiated Lindley distribution. (Ghitany et al. [28]) presented results on a two-parameter Lindley distribution referred to as power-Lindley distribution. (Zakerzadeh and Dolati [87]) looked at a different generalization of the Lindley distribution.

Lindley distribution is a mixture of exponential and gamma distributions, that is $f(x; \lambda) = (1 - p)f_G(x; \lambda) + pf_E(x; \lambda)$ with $p = \frac{1}{1+\lambda}$, where $f_G(x; \lambda) \equiv GAM(2, \lambda)$, and $f_E(x; \lambda) \equiv EXP(\lambda)$.

The log-logistic distribution (known as the Fisk distribution) plays an important role in income. It is the probability distribution of a random variable whose logarithm has logistic distribution. Some generalization of the log-logistic model have appeared in the literature. For example, beta log-logistic distribution presented by (Lemonte [39]) and the log-logistic Weibull distribution by (Oluyede et al. [66]).

There are several extensions of the Weibull model which are able to depict more complex hazard rates like upside-down bathtub-shaped or unimodal shapes. Generalizations of the Weibull distribution in the literature include

work by (Xie et al. [86]), (Bebbington et al. [9]), (Cordeiro et al. [20]) and (Silva et al.[76]). (Lai et al. [36]) presented modification of Weibull distribution. (Cordeiro et al. [17]) tackled Lindley-Weibull and its applications. (Lai et al. [35]) studied modified Weibull distribution. (Famoye et al. [24]) developed the beta Weibull distribution. (Pal et al. [58]) introduced the exponentiated Weibull distribution. (Mudholkar and Srivastata [49]) developed the generalized Weibull distribution.

The new Lindley-Weibull distribution and power series distributions are compounded, to introduce a new class of distributions called the Lindley-Weibull power series (LWPS) distribution and its sub-model called Lindley-Weibull logarithmic (LWL) distribution. Recently, Several generalized distributions are proposed in the literature including the exponential-power series (EP) distribution by (Chahkandi and Ganjali [12]), Weibull-power series (WPS) distributions by (Morais and Barreto-Souza [48]), generalized exponential-power series (GEP) distribution by (Mahmoudi and Jafari [44]), complementary exponential power series by (Flores et al. [26]), extended Weibull-power series (EWPS) distribution by (Silva et al. [74]), double bounded Kumaraswamy power series by (Bidram and Nekoukhou [11]), Burr XII power series by (Silva and Cordeiro [75]), generalized linear failure rate-power series (GLFRP) distribution by (Alamatsaz and Shams [79]), Birnbaum-Saunders power series distribution by (Bourguignon et al. [10]), linear failure rate-power series by (Mahmoudi and Jafari [45]), and complementary extended Weibull-power series by (Cordeiro and Silva [21]). Similar procedures are used by (Roman et al. [68]), (Lu and Shi [43]), (Nadarajah et al. [53]) and (Louzada et al. [42]). (Oluyede et al. [67]) developed a new Burr XII-Weibull-Logarithmic distribution, (Nekoukhou et al. [60]) introduced a Flexible Skew Generalized Normal Distribution, (Shams Harandi and Alamatsaz [77]) presented Alpha-skew Laplace distribution, (Nekoukhou et al. [61]) looked at an extension of the exponentiated Weibull distribution, (Shams Harandi and Alamatsaz [78]) investigated a complementary generalized linear failure rate-geometric distribution and (Najarzadegan and Alamatsaz [57]) looked at a new generalization of weighted geometric distribution for

survival and lifetime data analysis. For compounding continuous distributions with discrete distributions, (Nadarajah et al. [56]) introduced the package Compounding in R software (R Development Core Team, [69]).

(Marshall and Olkin [47]) developed an important method of including an extra shape parameter to a given baseline model thus defining an extended distribution. The Marshall-Olkin transformation provides a wide range of behaviors with respect to the baseline distribution (Santos-Neo et al. [71]). Marshall-Olkin transformation was applied to several well-known distributions: Weibull (Ghittany et al. [27], Zhang and Xie [88]). More recently, general results were given by (Barreto-Souza et al. [8]) and (Cordeiro and Lemonte [19]). (Krishna et al. [33]) established Marshall-Olkin Fréchet distribution and its applications. (Santos-Nero et al. [53]) introduces a new class of models called the Marshall-Olkin extended Weibull family of distributions which defines at least twenty-one special models. (Lepetu et al. [40]) tackled Marshall-Olkin Log-Logistic Extended Weibull distribution. (Chakraborty and Handique [13]) presented the generalized Marshall-Olkin Kumaraswamy-G distribution. (Lazhar [37]) developed and studied the properties of the Marshall-Olkin extended generalized Gompertz distribution. (Kumar [34]) discussed the ratio and inverse moments of Marshall-Olkin extended Burr Type III distribution based on lower generalized order statistics. (Usman and Haq [82]) studied the Marshall-Olkin extended inverted Kumaraswamy distribution. (Javed et al. [32]) developed the Marshall-Olkin Kappa distribution. (Afify et al. [2]) presented the Marshall-Olkin additive Weibull distribution with variable shapes for the hazard rate. However, these authors do not employ the Marshall-Olkin transformation in extending the Lindley-log-logistic distribution (Oluyede et al. [63]). The combined distribution of Lindley and log-logistic is obtained from the product of the reliability or survival functions of the Lindley and log-logistic distributions via competing risk model. The Marshall-Olkin transformation is then employed to obtain a new model called Marshall-Olkin Lindley log-logistic (MOLLLoG) distribution.

1.2 Some Generators of Distributions

In this section, several generators used in this thesis to extend some standard and new lifetime distributions are discussed.

1.2.1 Competing Risk Method

Consider a series system and assume that the lifetime of the components follow $F_1(x)$ and $F_2(x)$ distributions with reliability functions $R_1(x) = 1 - F_1(x)$ and $R_2(x) = 1 - F_2(x)$, respectively. The reliability $R(x) = P(X > x)$ of the system is given by

$$R(x) = \prod_{i=1}^2 R_i(x). \quad (1.3)$$

The resulting cdf is given by $F(x) = 1 - R(x)$. In the context of reliability, a series model is named a competing risk model. The corresponding pdf is obtained by differentiating $F(x)$.

1.2.2 Compounding Method

Let N be a zero truncated discrete random variable having a power series distribution, whose probability mass function (pmf) is given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, n = 1, 2, 3, \dots, \quad (1.4)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n \geq 1}$ a sequence of positive real numbers. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions (Johnson et.al). Table 1.1 shows some useful quantities including $a_n, C(\theta), C^{-1}(\theta), C'(\theta)$ and $C''(\theta)$ for the binomial, Poisson, geometric and logarithmic distributions.

Table 1.1: Useful Quantities for Some Power Series Distributions

Distribution	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	a_n	Parameter Space
Binomial	$(1 + \theta)^m - 1$	$m(1 + \theta)^{m-1}$	$m(m-1)(1 + \theta)^{m-2}$	$(\theta - 1)^{1/m} - 1$	$\binom{m}{n}$	$(0, 1)$
Poisson	$e^\theta - 1$	e^θ	e^θ	$\log(1 + \theta)$	$(n!)^{-1}$	$(0, \infty)$
Geometric	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(1 + \theta)^{-1}$	1	$(0, 1)$
Logarithmic	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	n^{-1}	$(0, 1)$

1.2.3 Gamma Generator

Let $G(x)$ and $g(x)$ denote the cumulative distribution function (cdf) and probability density function (pdf) of a continuous random variable X , then the cdf and pdf of the Gamma-G (Zografos and Balakrishnan [90]) distribution are given by

$$F(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log(1-G(x))} t^{\delta-1} e^{-t} dt = \frac{\gamma(\delta, G(x))}{\Gamma(\delta)}, \quad (1.5)$$

and

$$f(x) = \frac{1}{\Gamma(\delta)} [G(x)]^{\delta-1} g(x), x \in R, \delta > 0, \quad (1.6)$$

respectively, where $\gamma(x; \delta) = \int_0^x t^{\delta-1} e^{-t} dt$ is the incomplete gamma function.

1.2.4 Marshall-Olkin-G Generator

The Marshall-Olkin (Marshall and Olkin [47]) class of distributions is given by

$$F(x; \delta) = 1 - \frac{\delta \bar{G}(x)}{1 - \delta \bar{G}(x)}, \quad (1.7)$$

for $\delta > 0$ (tilt parameter), $\bar{\delta} = 1 - \delta$ and $\bar{G}(x) = 1 - G(x)$, where $G(x)$ is the baseline cdf. Note that when $\delta = 1$, we have $G(x) = F(x)$, the baseline distribution function. The corresponding pdf is given by

$$f(x; \delta) = \frac{\delta g(x)}{(1 - \delta \bar{G}(x))^2}. \quad (1.8)$$

Marshall and Olkin referred to the shape parameter δ as the tilt parameter due to the fact that for all $x > 0$, the hazard functions are ordered based on the values of δ , that is, $h_G(x; \delta) \leq h_F(x)$ when $\delta \geq 1$, and $h_G(x; \delta) \geq h_F(x)$ when $0 < \delta \leq 1$.

1.2.5 Half-Logistic-G Generator

(Cordeiro [18]) define the cdf of the new type I half logistic-G (TIHL-G) family of distributions by

$$\begin{aligned} F(x; \lambda, \underline{\epsilon}) &= \int_0^{-\ln(1-G(x; \underline{\epsilon}))} \frac{2\lambda e^{-\lambda x}}{(1 + e^{-\lambda x})^2} dx \\ &= \frac{1 - [1 - G(x; \underline{\epsilon})]^\lambda}{1 + [1 - G(x; \underline{\epsilon})]^\lambda}, \end{aligned} \quad (1.9)$$

where $G(x; \underline{\epsilon})$ is the baseline cdf depending on a parameter vector $\underline{\epsilon}$ and an additional shape parameter $\lambda > 0$. If we take $\lambda = 1$ then the TIHL-G reduces to the half logistic-G (HL-G) distribution with cdf

$$F(x; \underline{\epsilon}) = \frac{G(x; \underline{\epsilon})}{1 + \overline{G}(x; \underline{\epsilon})}, \quad (1.10)$$

where $\overline{G}(x; \underline{\epsilon}) = 1 - G(x; \underline{\epsilon})$.

The corresponding pdf to (1.10) is given by

$$f(x; \underline{\epsilon}) = \frac{2g(x; \underline{\epsilon})}{[1 + \overline{G}(x; \underline{\epsilon})]^2}, \quad (1.11)$$

where $g(x; \underline{\epsilon}) = \frac{dG(x; \underline{\epsilon})}{dx}$.

1.3 Some Properties of Distributions

1.3.1 Survival Function

Survival function, also known as the reliability function is defined as the probability that a unit will operate without failure for a given time under specified environmental conditions. Let X be a continuous random variable with cdf $F(x)$ and pdf $f(x)$, then its survival function is obtained as

$$\begin{aligned} S(x) &= \int_t^{\infty} f(x) dx \\ &= 1 - F(x). \end{aligned} \quad (1.12)$$

1.3.2 Quantile Function

Quantiles are an important tool for statistical analysis. They give statistical properties of the random variable X , just like the distribution function. The quantile function is commonly used in statistical aspects such as generating random numbers. Let X be a continuous random variable with cdf $F(x)$ and pdf $f(x)$. The quantile function of X is the generalized inverse of F , that is $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $0 \leq u \leq 1$.

1.3.3 Hazard Rate and Reverse Hazard Rate

The hazard function, denoted $h(x)$ plays an important role in modelling lifetime data. It is obtained as the ratio of the probability density function and the survival function, that is $h(x) = \frac{f(x)}{S(x)}$. The hazard rate function can exhibit different shapes, that is increasing, decreasing, bathtub, upside-down bathtub and a combination of bathtub and upside-down bathtub shapes.

In case the hazard function may not capture such failures, then the reverse hazard rate function becomes a useful tool in such instances. The reverse hazard rate function is obtained by dividing the probability density function by the cumulative distribution function, that is $\tau(x) = \frac{f(x)}{F(x)}$.

1.3.4 Moments

Moments are parameters used to describe the features of a statistical distribution (e.g. mean, variance, skewness and kurtosis). It is very useful to derive the ordinary and incomplete moments of a generalized distribution from a weighted infinite linear combination of established distributions, (for example: moments of the Odd Exponentiated Half-Logistic-G distribution from those of a weighted infinite linear combination for Exp-G distributions [1]). The r^{th} moment of a random variable X can be defined as $\mu'_r = E(X^r) = \int x^r f(x)dx$ and the incomplete moment can be obtained as follows $I_X(t) = \int_0^t x^r f(x)dx$. The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD= σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$

and

$$CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively.

1.3.5 Conditional Moments

When it comes to lifetime models, it is often of interest to find the conditional moments. Let X be a random variable with the cdf $F(x)$, then the r^{th} conditional moment of X is given by

$$E(X^r | X > t) = \frac{1}{\bar{F}(t)} \int_t^\infty f(x) dx. \quad (1.13)$$

1.3.6 Mean Deviations

Let X be a random variable with pdf $f(x)$, the mean deviation about the mean $D(\mu)$ and the mean deviation about the median $D(M)$, are defined as

$$D(\mu) = \int_0^\infty |x - \mu| f(x) dx, \quad \text{and} \quad D(M) = \int_0^\infty |x - M| f(x) dx, \quad (1.14)$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(X) = F^{-1}(\frac{1}{2})$ is the median of $F(x)$. However, the following relationships can be used to evaluate the measures $D(\mu)$ and $D(M)$:

$$D(\mu) = 2\mu F(\mu) - 2\mu + 2 \int_\mu^\infty x f(x) dx, \quad (1.15)$$

and

$$D(M) = -\mu + 2 \int_M^\infty x f(x) dx. \quad (1.16)$$

1.3.7 Lorenz and Bonferroni Curves

The Lorenz and Bonferroni curves have a number of applications in different fields such as medicine, income and poverty, reliability and insurance. The Lorenz and Bonferroni curves are given by

$$L(F(x)) = \frac{\int_0^x t f(t) dt}{E(X)}, \quad \text{and} \quad B(F(x)) = \frac{L(F(x))}{F}(x),$$

or

$$L(p) = \frac{1}{\mu} \int_0^q x f(x) dx, \quad \text{and} \quad B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx, \quad (1.17)$$

respectively.

1.3.8 Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with pdf $f(x)$ and cdf $F(x)$. Suppose $X_{1:n} < X_{2:n}, \dots < X_{n:n}$ denote the corresponding order statistics, then the pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l f(x) [F(x)]^{k+l-1}. \quad (1.18)$$

1.3.9 Rényi Entropy

An entropy is a measure of uncertainty or variation of a random variable. Rényi entropy is an extension of Shannon entropy. (Rényi entropy [70]) is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f(x)]^v dx \right), v \neq 1, v > 0. \quad (1.19)$$

Rényi entropy tends to Shannon entropy as $v \rightarrow 1$.

1.3.10 Estimation

There are many methods of estimation in the literature such as maximum likelihood estimation (MLE), method of moments (MOM) and Weighted least squares estimation (WLS) to name a few. However, the most commonly used method is the MLE. The maximum likelihood principle of estimation chooses an estimate $\hat{\theta}$ of θ for a given set of observations (data) the value that is most likely to occur. The likelihood function is computed as $L(\theta, \underline{x}) = L(\theta) = \prod_{i=1}^n f(x_i, \theta)$. The maximum likelihood estimates of the parameters $(\theta_1, \theta_2, \dots, \theta_n)$, denoted by $(\hat{\Delta})$ is obtained by solving the non linear equation $\frac{\partial L(\theta)}{\partial \theta_i} = 0$, using a numerical method such as Newton-Raphson procedure since the equations obtained are not in closed form. The Fisher information matrix (FIM) is the $k \times k$ symmetric matrix given by $\mathbf{I}(\Delta) = [\mathbf{I}_{\theta_i \theta_j}]_{k \times k} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4$, can be numerically obtained by NLMIXED in SAS or mle2 package in R, MATLAB or MAPLE software. The total Fisher information matrix $\mathbf{I}_n(\Delta)$ can be approximated by

$$\mathbf{J}_n(\Delta) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \right]_{\Delta=\hat{\Delta}} \Big|_{k \times k}. \quad (1.20)$$

Note that the expectations in the Fisher Information matrix (FIM) can be obtained numerically.

1.3.11 Goodness of Fit Statistics

In order to determine which model fits a certain real life dataset well, the following statistics are used;

- -2log-likelihood statistic (-2log(L))
- Akaike Information Criterion (AIC): $AIC = 2p - 2\ln(L)$
- Consistent Akaike Information Criterion (AICC): $AICC = AIC + 2 \frac{p(p+1)}{n-p-1}$
and
- Bayesian Information Criterion (BIC): $BIC = p\ln(n) - 2\ln(L)$,

where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. The model with smallest values of these statistics is regarded as the best fit model for that particular dataset. The goodness-of-fit statistics: Cramer-von Mises (W^*) and Anderson-Darling (A^*), described by (Chen and Balakrishnan [16]) are also obtained. These statistics can be used to verify which distribution fits better to the data. The smaller the values of W^* and A^* , the better the fit.

Consider $\Delta = (\lambda, c, \delta)$, $\hat{\Delta} = (\hat{\lambda}, \hat{c}, \hat{\delta})$ and $\tilde{\Delta} = (\tilde{\lambda}, \tilde{c}, 1)$, where Δ , $\hat{\Delta}$ and $\tilde{\Delta}$ represent a vector of parameters, a vector of unrestricted estimates and a vector of restricted estimates, respectively. Then, the likelihood ratio (LR) test can be used to compare the fit of the new distribution with its sub-models for a given data set. For example, to test $\delta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\Delta})) - \ln(L(\tilde{\Delta}))]$, where $\tilde{\Delta}$ and $\hat{\Delta}$ are the estimates under the null and alternative hypotheses, respectively. The LR test rejects the null hypothesis if $\omega > \chi_{\epsilon}^2$, where χ_{ϵ}^2 denote the upper 100 ϵ % point of the χ^2 distribution with 1 degrees of freedom.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [14]) are presented for each dataset. For the probability plot, we plotted $G(x_{(j)}; \hat{\lambda}, \hat{c}, \hat{\delta})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the

sum of squares

$$SS = \sum_{j=1}^n \left[G_{(x_{(j)})} - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The plot with the smallest SS is corresponds to the best fitting model.

1.4 Aims and Objectives

The main aim of the study is to develop extensions of distributions via compounding, gamma-G, Marshall-Olkin-G and Half logistic-G approaches. The specific objectives are:

1. To develop the following models:
 - Lindley Weibull class of distributions (LWPS)
 - Gamma generalized Lindely log-logistic distribution (GELLLoG)
 - Marshall-Olkin Lindley log-logistic (MOLLLoG)
 - Half logistic log-logistic Weibull distribution (HLLLoGW)
1. Develop mathematical and statistical properties of these proposed distributions.
2. To determine maximum likelihood estimates of the proposed models using the maximum likelihood estimation (MLE).
3. To assess the accuracy of the MLE via Monte Carlo simulations.
4. To illustrate the applicability and usefulness of the proposed models by using real data sets.

1.5 Outline of Thesis

The thesis is organised as follows. In Chapter 1, the literature review, some generators of distributions and properties of distributions are presented. Chapter 2 deals with the Lindley-Weibull class of distributions obtained via compounding the Lindley Weibull and the power series distributions.

The gamma generalized Lindley log-logistic distribution obtained through the gamma generator is given in chapter 3. In Chapter 4 work on the Marshall-Olkin Lindley log-logistic obtained via the Marshall-Olkin method is presented. The half logistic log-logistic Weibull distribution obtained by combining the half logistic and log-logistic Weibull is contained in chapter 5. Last but not least conclusions and suggestions for future studies are presented in chapter 6.

Chapter 2

A New Generalized Lindley-Weibull Class of Distributions with Applications

In this chapter, a new class of distributions, referred to as the Lindley-Weibull power series (LWPS) distribution and its sub-model Lindley-Weibull logarithmic distribution are presented. The Lindley Weibull logarithmic (LWL) distribution is obtained by mixing Lindley-Weibull distribution and logarithmic distribution. In addition to develop a generalized family of distributions, the basic motivations for developing the LWPS class of distributions in practice include the following:

- to construct and generate distributions with symmetric, left-skewed, right-skewed, reversed-J shapes;
- to define new family of distributions that possesses various types of hazard rate functions including monotonic as well as non-monotonic shapes;
- to construct heavy-tailed distributions for modeling real data;
- to provide consistently better fits and statistical properties compared to other competitive models;

- to obtain more flexible class of distributions by introducing an extra parameter to the Lindley-Weibull distribution and improve goodness-of-fit to real data.

2.1 The Model, Sub-models and Properties

In this section, some properties of the LWPS class of distributions including expansion of the density, hazard and reverse hazard functions, quantile function and sub-models, moments, conditional moments and maximum likelihood estimation of model parameters are derived. Using the method of competing risk, that is, consider a series system and assume that the lifetime of the components follow the Lindley and Weibull distributions with reliability functions $R_1(x) = (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}$ and $R_2(x) = e^{-\alpha x^\beta}$, respectively. The reliability $R(x) = P(X > x)$ of the system is given by $R(x) = \prod_{i=1}^2 R_i(x)$ (see Oluyede et al. [66]). (Asgharzadeh et al. [5]) developed and presented the Weibull-Lindley or Lindley-Weibull (LW) distribution with cdf and pdf given by

$$F_{WL}(x; \lambda, \alpha, \beta) = 1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta}, \quad (2.1)$$

and

$$f_{WL}(x; \lambda, \alpha, \beta) = \frac{e^{-\lambda x - \alpha x^\beta}}{1 + \lambda} ([\lambda^2(1 + x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1}]), \quad (2.2)$$

for $x > 0$, and $\lambda, \alpha, \beta > 0$, respectively.

If a random variable X has the LW distribution, we write $X \sim LW(\lambda, \alpha, \beta)$. Note that

$$g(x; \lambda, \alpha, \beta) = g_L(x; \lambda)\overline{G}_W(x; \alpha, \beta) + \overline{G}_L(x; \lambda)g_W(x; \alpha, \beta), \quad (2.3)$$

where $\overline{G}_W(x; \alpha, \beta) = 1 - G_W(x; \alpha, \beta)$, $\overline{G}_L(x; \lambda) = 1 - G_L(x; \lambda)$ are the survival functions of Weibull and Lindley distributions, respectively and $g_W(x; \alpha, \beta)$ and $g_L(x; \lambda)$ are the pdf's of Weibull and Lindley distributions. In this paper, we propose the LWPS class of distributions which is obtained by mixing the Lindley-Weibull distribution and power series distributions.

Given N , let $X_1, X_2, \dots, X_N$ be identically and independently distributed (iid) random variable following Lindley-Weibull distribution. Let $X_{(n)} =$

$\max(X_1, X_2, \dots, X_n)$, then the cdf of $X_{(n)}|N = n$ is given by

$$F_{X_{(n)}|N=n}(x; \lambda, \alpha, \beta) = \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta}\right)^n, \lambda, \alpha, \beta > 0, n \geq 1. \quad (2.4)$$

The Lindley-Weibull Power Series (LWPS) distribution denoted by $LWPS(\lambda, \alpha, \beta, \theta)$ is defined by the marginal distribution of $X_{(n)}$, that is,

$$\begin{aligned} F_{X_{(n)}}(x) &= \sum_{n=1}^{\infty} P(N = n) P(X_n \leq x | N = n) \\ &= \sum_{n=1}^{\infty} \frac{a_n}{C(\theta)} \left(\theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right)^n \\ &= \frac{C \left(\theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right)}{C(\theta)}, x > 0. \end{aligned} \quad (2.5)$$

The corresponding pdf is given by

$$\begin{aligned} f_{X_{(n)}}(x) &= \frac{\theta e^{-\lambda x - \alpha x^\beta}}{1 + \lambda} \left[\lambda^2(1 + x) + (1 + \lambda + \lambda x) \alpha \beta x^{\beta-1} \right] \\ &\quad \times \frac{C' \left(\theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right)}{C(\theta)}. \end{aligned} \quad (2.6)$$

The hazard function is given in equation (2.7). That is,

$$\begin{aligned} h_F(x) &= \frac{\theta e^{-\lambda x - \alpha x^\beta}}{1 + \lambda} \left[\lambda^2(1 + x) + (1 + \lambda + \lambda x) \alpha \beta x^{\beta-1} \right] C' \left(\theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right) \\ &\quad \times \left(C(\theta) - C \left(\theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right) \right)^{-1}. \end{aligned} \quad (2.7)$$

2.1.1 Some Sub-Classes of the LWPS Distribution

There are several new and well-known sub-classes that can be obtained from the LWPS class of distributions. Note that:

- If $\lambda \rightarrow 0^+$, then we obtain the Weibull Power Series (WPS) class of distribution (Morais and Barreto-Souza [48]).
- If $\alpha \rightarrow 0^+$, then we have the Lindley Power series (LPS) class of distributions (Warahena-Liyanage and Pararai [83]).
- If $\beta = 1$ and $\beta = 2$, then we have the Lindley Exponential Power Series (LEPS) class and Lindley Rayleigh Power Series (LRPS) class of distributions, respectively.

- If $\lambda \rightarrow 0^+$, with $\beta = 1$ or $\beta = 2$, then we obtain the Exponential Power Series (EPS) class of distributions or the Rayleigh Power Series (RPS) class of distributions, which are sub-classes of the LWPS class of distributions.

2.1.2 Quantile Function

Let X be a random variable with cdf as in (2.5). The quantile function $Q_{X_{(n)}}(u)$ is defined by $F_{X_{(n)}}(Q_{X_{(n)}}(u)) = u, 0 \leq u \leq 1$. That is, we solve the non-linear equation

$$C \left(\theta \left(1 - \frac{1 + \lambda + \lambda Q_{X_{(n)}}(u)}{1 + \lambda} e^{-\lambda Q_{X_{(n)}}(u) - \alpha (Q_{X_{(n)}}(u))^\beta} \right) \right) - uC(\theta) = 0, \quad (2.8)$$

equivalently,

$$\lambda Q_{X_{(n)}}(u) + \alpha (Q_{X_{(n)}}(u))^\beta - \ln \left(1 - \frac{1 + \lambda + \lambda Q_{X_{(n)}}(u)}{1 + \lambda} \right) + \ln \left(\frac{C^{-1}(uC(\theta))}{\theta} \right) = 0. \quad (2.9)$$

2.1.3 Expansion of Density

Expansion of the density of the LWPS class of distributions is presented in this sub-section. The LWPS density function can be expressed as a linear combination of the conditional density of $X_{(n)} = \max(X_1, X_2, \dots, X_N)$, given $N = n$. Note that from the derivative of equation (2.5), that is, equation (2.6) and the fact that $C'(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1}$, we have

$$f_{X_{(n)}}(x) = \sum_{n=1}^{\infty} f_{X_{(n)}|N=n}(x; \lambda, \alpha, \beta) P(N = n), \quad (2.10)$$

where $f_{X_{(n)}|N=n}(x; \lambda, \alpha, \beta)$ is the conditional pdf of $X_{(n)} = \max(X_1, X_2, \dots, X_N)$ given $N = n$, that is,

$$\begin{aligned} f_{X_{(n)}|N=n}(x; \lambda, \alpha, \beta) &= \frac{n e^{-\lambda x - \alpha x^\beta}}{1 + \lambda} \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right)^{n-1} \\ &\quad \times \left[\lambda^2 (1 + x) + (1 + \lambda + \lambda x) \alpha \beta x^{\beta-1} \right]. \end{aligned} \quad (2.11)$$

2.1.4 Moments

In order to find the moments of LWPS class of distributions the following Lemma is used.

Lemma 1.2 Let

$$L_1(\lambda, \alpha, \beta, n, z) = \int_0^\infty x^z e^{-\lambda x - \alpha x^\beta} \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta}\right)^{n-1} \times \left[\lambda^2(1+x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1}\right] dx. \quad (2.12)$$

Then

$$L_1(\lambda, \alpha, \beta, n, z) = \sum_{i,j,k=0}^{\infty} \frac{\binom{n-1}{i} \binom{i}{j} \binom{j}{k} (-1)^{(i+l)} \lambda^j (\alpha(i+1))^l}{(1+\lambda)^{il!}} \left[\frac{\lambda^2 \Gamma(z+k+\beta l+1)}{[\lambda(i+1)]^{z+k+\beta l+1}} + \frac{\lambda^2 \Gamma(z+k+\beta l+2)}{[\lambda(i+1)]^{z+k+\beta l+2}} + \frac{\alpha\beta(1+\lambda)\Gamma(z+k+\beta l+\beta)}{[\lambda(i+1)]^{z+k+\beta l+\beta}} + \frac{\alpha\beta\lambda\Gamma(z+k+\beta l+\beta+1)}{[\lambda(i+1)]^{z+k+\beta l+\beta+1}} \right].$$

Proof: Using the series expansions

$$(1-x)^{n-1} = \sum_{i=0}^{\infty} \binom{n-1}{i} (-1)^i x^i \quad \text{and} \quad e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}, \quad (2.13)$$

we obtain the following

$$\begin{aligned} L_1(\lambda, \alpha, \beta, n, z) &= \int_0^\infty x^z e^{-\lambda x - \alpha x^\beta} \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta}\right)^{n-1} \times \left[\lambda^2(1+x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1}\right] dx \\ &= \sum_{i,j,k=0}^{\infty} \frac{\binom{n-1}{i} \binom{i}{j} \binom{j}{k} (-1)^i \lambda^j}{(1+\lambda)^i} \int_0^\infty x^{z+k} e^{-(i+1)\lambda x} e^{-(i+1)\alpha x^\beta} \\ &\quad \times \left[\lambda^2(1+x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1}\right] dx \\ &= \sum_{i,j,k=0}^{\infty} \frac{\binom{n-1}{i} \binom{i}{j} \binom{j}{k} (-1)^{(i+l)} \lambda^j (\alpha(i+1))^l}{(1+\lambda)^{il!}} \int_0^\infty x^{z+k+\beta l} e^{-(i+1)\lambda x} \\ &\quad \times \left[\lambda^2(1+x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1}\right] dx \\ &= \sum_{i,j,k=0}^{\infty} \frac{\binom{n-1}{i} \binom{i}{j} \binom{j}{k} (-1)^{(i+l)} \lambda^j (\alpha(i+1))^l}{(1+\lambda)^{il!}} \left[\lambda^2 \int_0^\infty x^{z+k+\beta l} e^{-(i+1)\lambda x} dx \right. \\ &\quad + \lambda^2 \int_0^\infty x^{z+k+\beta l+1} e^{-(i+1)\lambda x} dx + \alpha\beta(1+\lambda) \\ &\quad \times \int_0^\infty x^{z+k+\beta l+\beta-1} e^{-(i+1)\lambda x} dx + \alpha\beta\lambda \int_0^\infty x^{z+k+\beta l+\beta} e^{-(i+1)\lambda x} dx \left. \right]. \end{aligned}$$

Let $u = (i + 1)\lambda x$, then $du = (i + 1)\lambda dx$, and $x = \frac{u}{(i+1)\lambda}$, so that

$$\begin{aligned} L_1(\lambda, \alpha, \beta, n, z) &= \sum_{i,j,k=0}^{\infty} \frac{\binom{n-1}{i} \binom{j}{j} \binom{k}{k} (-1)^{(i+l)} \lambda^j (\alpha(i+1))^l}{(1+\lambda)^{il}} \left[\frac{\lambda^2 \Gamma(z+k+\beta l+1)}{[\lambda(i+1)]^{z+k+\beta l+1}} \right. \\ &\quad + \frac{\lambda^2 \Gamma(z+k+\beta l+2)}{[\lambda(i+1)]^{z+k+\beta l+2}} + \frac{\alpha \beta (1+\lambda) \Gamma(z+k+\beta l+\beta)}{[\lambda(i+1)]^{z+k+\beta l+\beta}} \\ &\quad \left. + \frac{\alpha \beta \lambda \Gamma(z+k+\beta l+\beta+1)}{[\lambda(i+1)]^{z+k+\beta l+\beta+1}} \right]. \end{aligned} \quad (2.14)$$

The r^{th} moment of the LWPS class of distributions is given by

$$E(X^r) = \int_0^{\infty} x^r f_{X_{(n)}}(x) dx, \quad (2.15)$$

and using equation (2.10), we get

$$\begin{aligned} E(X^r) &= \sum_{n=1}^{\infty} P(N=n) E(X_{(n)}^r) = \sum_{n=1}^{\infty} P(N=n) \int_0^{\infty} x^r f_{X_{(n)}|N=n}(x; \lambda, \alpha, \beta) dx \\ &= \sum_{n=1}^{\infty} P(N=n) \frac{n}{1+\lambda} \int_0^{\infty} x^z e^{-\lambda x - \alpha x^\beta} \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x - \alpha x^\beta} \right)^{n-1} \\ &\quad \times \left[\lambda^2 (1+x) + (1+\lambda+\lambda x) \alpha \beta x^{\beta-1} \right]. \end{aligned}$$

Applying Lemma 1.2, it follows that the r^{th} moment of the LWPS class of distributions is given by

$$\begin{aligned} E(X^r) &= \sum_{n=1}^{\infty} P(N=n) E(X_{(n)}^r) = \sum_{n=1}^{\infty} P(N=n) \int_0^{\infty} x^r f_{X_{(n)}|N=n}(x; \lambda, \alpha, \beta) dx \\ &= \sum_{n=1}^{\infty} P(N=n) \frac{n}{1+\lambda} L_1(\lambda, \alpha, \beta, n, r). \end{aligned} \quad (2.16)$$

The mean(μ), variance(σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\mu = E(X) = \sum_{n=1}^{\infty} P(N=n) \frac{n}{1+\lambda} L_1(\lambda, \alpha, \beta, n, 1),$$

$$CV = \frac{\sigma}{\mu} = \sqrt{\frac{\mu'_2 - \mu^2}{\mu}}, CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{\frac{3}{2}}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{\frac{3}{2}}},$$

and

$$CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2}.$$

2.1.5 Conditional Moments

Lemma 1.3. Let

$$L_2(\lambda, \alpha, \beta, n, z, t) = \int_t^\infty x^z e^{-\lambda x - \alpha x^\beta} \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta}\right)^{n-1} \times \left[\lambda^2 (1 + x) + (1 + \lambda + \lambda x) \alpha \beta x^{\beta-1}\right] dx.$$

Then

$$L_2(\lambda, \alpha, \beta, n, z, t) = \sum_{i,j,k=0}^{\infty} \frac{\binom{n-1}{i} \binom{i}{j} \binom{j}{k} (-1)^{i+l} \lambda^j (\alpha(i+1))^l}{(1+\lambda)^{i!}} \left[\frac{\lambda^2 \Gamma(z+k+\beta l+1, (i+1)\lambda t)}{[\lambda(i+1)]^{z+k+\beta l+1}} + \frac{\lambda^2 \Gamma(z+k+\beta l+2, (i+1)\lambda t)}{[\lambda(i+1)]^{z+k+\beta l+2}} \right. \\ \left. + \frac{\alpha \beta (1+\lambda) \Gamma(z+k+\beta l+\beta, (i+1)\lambda t)}{[\lambda(i+1)]^{z+k+\beta l+\beta}} + \frac{\alpha \beta \lambda \Gamma(z+k+\beta l+\beta+1, (i+1)\lambda t)}{[\lambda(i+1)]^{z+k+\beta l+\beta+1}} \right].$$

Proof. Using the series expansions

$$(1-x)^{n-1} = \sum_{i=0}^{\infty} \binom{n-1}{i} (-1)^i x^i \quad \text{and} \quad e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}, \quad (2.17)$$

we obtain the following

$$\begin{aligned}
L_2(\lambda, \alpha, \beta, n, z, t) &= \int_t^\infty x^z e^{-\lambda x - \alpha x^\beta} \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right)^{n-1} \\
&\quad \times \left[\lambda^2 (1 + x) + (1 + \lambda + \lambda x) \alpha \beta x^{\beta-1} \right] dx \\
&= \sum_{i,j,k=0}^\infty \frac{\binom{n-1}{i} \binom{i}{j} \binom{j}{k} (-1)^i \lambda^j}{(1 + \lambda)^i} \int_t^\infty x^{z+k} e^{-(i+1)\lambda x} e^{-(i+1)\alpha x^\beta} \\
&\quad \times \left[\lambda^2 (1 + x) + (1 + \lambda + \lambda x) \alpha \beta x^{\beta-1} \right] dx \\
&= \sum_{i,j,k=0}^\infty \frac{\binom{n-1}{i} \binom{i}{j} \binom{j}{k} (-1)^{(i+l)} \lambda^j (\alpha(i+1))^l}{(1 + \lambda)^i l!} \\
&\quad \times \int_t^\infty x^{z+k+\beta l} e^{-(i+1)\lambda x} \left[\lambda^2 (1 + x) + (1 + \lambda + \lambda x) \alpha \beta x^{\beta-1} \right] dx \\
&= \sum_{i,j,k=0}^\infty \frac{\binom{n-1}{i} \binom{i}{j} \binom{j}{k} (-1)^{(i+l)} \lambda^j (\alpha(i+1))^l}{(1 + \lambda)^i l!} \\
&\quad \times \left[\lambda^2 \int_t^\infty x^{z+k+\beta l} e^{-(i+1)\lambda x} dx + \lambda^2 \int_t^\infty x^{z+k+\beta l+1} e^{-(i+1)\lambda x} dx \right. \\
&\quad + \alpha \beta (1 + \lambda) \int_t^\infty x^{z+k+\beta l+\beta-1} e^{-(i+1)\lambda x} dx \\
&\quad \left. + \alpha \beta \lambda \int_t^\infty x^{z+k+\beta l+\beta} e^{-(i+1)\lambda x} dx \right].
\end{aligned}$$

Using $u = (i + 1)\lambda x$, then $du = (i + 1)\lambda dx$ and $x = \frac{u}{(i+1)\lambda}$. Consequently,

$$\begin{aligned}
L_2(\lambda, \alpha, \beta, n, z, t) &= \sum_{i,j,k=0}^\infty \frac{\binom{n-1}{i} \binom{i}{j} \binom{j}{k} (-1)^{(i+l)} \lambda^j (\alpha(i+1))^l}{(1 + \lambda)^i l!} \\
&\quad \times \left[\frac{\lambda^2 \Gamma(z + k + \beta l + 1, (i + 1)\lambda t)}{[\lambda(i + 1)]^{z+k+\beta l+1}} + \frac{\lambda^2 \Gamma(z + k + \beta l + 2, (i + 1)\lambda t)}{[\lambda(i + 1)]^{z+k+\beta l+2}} \right. \\
&\quad + \frac{\alpha \beta (1 + \lambda) \Gamma(z + k + \beta l + \beta, (i + 1)\lambda t)}{[\lambda(i + 1)]^{z+k+\beta l+\beta}} \\
&\quad \left. + \frac{\alpha \beta \lambda \Gamma(z + k + \beta l + \beta + 1, (i + 1)\lambda t)}{[\lambda(i + 1)]^{z+k+\beta l+\beta+1}} \right].
\end{aligned}$$

Applying equation (2.10) and Lemma 1.3, the r^{th} conditional moment of the LWPS class of distributions is given by

$$\begin{aligned}
E(X^r | X > x) &= \sum_{n=1}^\infty P(N = n) \frac{n}{1 + \lambda} \frac{L_2(\lambda, \alpha, \beta, n, r, x)}{1 - F_{LWPS}(x)} \\
&= \sum_{n=1}^\infty \frac{a_n \theta^n L_2(\lambda, \alpha, \beta, n, r, x)}{C(\theta) - C\left(\theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta}\right)\right)}. \tag{2.18}
\end{aligned}$$

2.1.6 Mean Deviation, Lorenz and Bonferroni Curves

Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni Curves for the LWPS distribution are presented in this section. The application of Lorenz and Bonferroni curves is not limited to economics for the study of income and poverty, but they can also be used to other areas including reliability, demography, insurance and medicine.

Mean Deviations

The mean deviation about the mean $D(\mu)$ and the mean deviation about the median $D(M)$, are defined as

$$D(\mu) = \int_0^\infty |x - \mu| f_{LWPS}(x) dx, \quad \text{and} \quad D(M) = \int_0^\infty |x - M| f_{LWPS}(x) dx, \quad (2.19)$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(X) = F^{-1}(\frac{1}{2})$ is the median of $F_{LWPS}(\cdot)$. However, the following relationships can be used to evaluate the measures $D(\mu)$ and $D(M)$:

$$D(\mu) = 2\mu F_{LWPS}(\mu) - 2\mu + 2 \int_\mu^\infty x f_{LWPS}(x) dx, \quad (2.20)$$

and

$$D(M) = -\mu + 2 \int_M^\infty x f_{LWPS}(x) dx. \quad (2.21)$$

Using Lemma 1.3, we obtain

$$D(\mu) = 2\mu F_{LWPS}(\mu) - 2\mu + 2 \sum_{n=1}^{\infty} P(N = n) \frac{n}{1 + \lambda} L_2(\lambda, \alpha, \beta, n, 1, \mu),$$

and

$$D(M) = -\mu + 2 \sum_{n=1}^{\infty} P(N = n) \frac{n}{1 + \lambda} L_2(\lambda, \alpha, \beta, n, 1, M).$$

Lorenz and Bonferroni Curves

In this subsection, we present the Lorenz and Bonferroni curves for LWPS class of distributions. The Lorenz and Bonferroni curves have a number of applications in different fields such as medicine, income and poverty, reliability and insurance. The Lorenz and Bonferroni curves are given by

$$L(F_{LWPS}(x)) = \frac{\int_0^x t f_{LWPS}(t) dt}{E(X)}, \quad \text{and} \quad B(F_{LWPS}(x)) = \frac{L(F_{LWPS}(x))}{F_{LWPS}(x)},$$

or

$$L(p) = \frac{1}{\mu} \int_0^q x f_{LWPS}(x) dx, \quad \text{and} \quad B(p) = \frac{1}{p\mu} \int_0^q x f_{LWPS}(x) dx, \quad (2.22)$$

respectively, where $q = F_{LWPS}^{-1}(p)$. Applying Lemma (1.3) in (2.22), we obtain

$$L(p) = \frac{1}{\mu} \left(\mu - \sum_{n=1}^{\infty} P(N=n) \frac{n}{1+\lambda} L_2(\lambda, \alpha, \beta, n, 1, q) \right) \quad (2.23)$$

and

$$B(p) = \frac{1}{p\mu} \left(\mu - \sum_{n=1}^{\infty} P(N=n) \frac{n}{1+\lambda} L_2(\lambda, \alpha, \beta, n, 1, q) \right). \quad (2.24)$$

2.1.7 Order Statistics and Rényi Entropy

The concept of entropy plays a very important role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. In this section, we present the distribution of the order statistic and Rényi entropy.

Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from the LWPS distribution and suppose $X_{1:n} < X_{2:n}, \dots < X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l f(x) [F(x)]^{k+l-1}. \quad (2.25)$$

Note, $f(x) [F(x)]^{k+l-1} = \frac{1}{k+l} \frac{d}{dx} [F(x)]^{k+l}$. The corresponding pdf of $f_{k:n}(x)$ is given by

$$\begin{aligned} f_{k:n}(x) &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \frac{\binom{n-k}{l} (-1)^l}{k+l} [F(x)]^{k+l} \\ &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \frac{\binom{n-k}{l} (-1)^l}{k+l} \left[\frac{C \left(\theta \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x - \alpha x^\beta} \right) \right)}{C(\theta)} \right]^{k+l} \\ &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \frac{\binom{n-k}{l} (-1)^l}{k+l} F_{ELWPS}(x; \lambda, \alpha, \beta, \theta, k+l), \end{aligned} \quad (2.26)$$

where $F_{ELWPS}(x; \lambda, \alpha, \beta, \theta, k + l)$ is exponentiated LWPS (ELWPS) distribution function with parameters $\lambda, \alpha, \beta, \theta$ and $k + l$. Thus, the pdf of the k^{th} order statistic can be expressed as a linear combination of the pdf of the ELWPS class of distributions.

Rényi Entropy

In this subsection, Rényi entropy of the LWPS distribution is derived. An entropy is a measure of uncertainty or variation of a random variable. Rényi entropy is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f_{LWPS}(x; \lambda, \alpha, \beta, \theta)]^v dx \right), v \neq 1, v > 0. \quad (2.27)$$

Rényi entropy tends to Shannon entropy as $v \rightarrow 1$. Note that $[f_{LWPS}(x; \lambda, \alpha, \beta, \theta)]^v = f_{LWPS}^v(x)$ can be written as

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left(\int_0^\infty \left[\sum_{n=1}^\infty P(N=n) \frac{n e^{-\lambda x - \alpha x^\beta}}{1+\lambda} \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x - \alpha x^\beta} \right)^{n-1} \right. \right. \\ &\quad \left. \left. \times \left[\lambda^2(1+x) + (1+\lambda+\lambda x)\alpha\beta x^{\beta-1} \right] \right]^v dx \right) \\ &= \frac{1}{1-v} \log \left[\left[\sum_{n=1}^\infty P(N=n) \frac{n}{1+\lambda} \right]^v \int_0^\infty \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x - \alpha x^\beta} \right)^{v(n-1)} \right. \\ &\quad \left. \times \left[\lambda^2(1+x) + (1+\lambda+\lambda x)\alpha\beta x^{\beta-1} \right]^v dx \right]. \end{aligned}$$

Consider the integral

$$\int_0^\infty e^{-v(\lambda x + \alpha x^\beta)} \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x - \alpha x^\beta} \right)^{v(n-1)} \left[\lambda^2(1+x) + (1+\lambda+\lambda x)\alpha\beta x^{\beta-1} \right]^v dx.$$

Applying the series expansion in (2.17) and the binomial expansion

$$(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i},$$

we obtain

$$\begin{aligned} &\sum_{i,j,k,l,m,n,s,p=0}^\infty \binom{v(n-1)}{i} \binom{i}{j} \binom{j}{k} \binom{v}{l} \binom{v-l}{m} \binom{l}{n} \binom{n}{s} (-1)^i (-(i+v))^p \\ &\times \lambda^{j+n+2(v-l)} \alpha^{l+p} \beta^l \int_0^\infty x^{k+l(\beta-1)+m+s+\beta l} e^{-\lambda x(i+v)} dx. \end{aligned}$$

We evaluate the integral $\int_0^\infty x^{k+l(\beta-1)+m+s+\beta l} e^{-\lambda x(i+v)} dx$. Let $u = \lambda x(i+v)$, then $x = \frac{u}{\lambda(i+v)}$ and $dx = \frac{du}{\lambda(i+v)}$. The integral reduces to

$$\begin{aligned} & \frac{1}{[\lambda(v+i)]^{k+l(\beta-1)+m+s+\beta l+1}} \int_0^\infty u^{k+l(\beta-1)+m+s+\beta l} e^{-u} du \\ &= \frac{1}{[\lambda(v+i)]^{k+l(\beta-1)+m+s+\beta l+1}} \Gamma(k+l(\beta-1)+m+s+\beta l+1). \end{aligned}$$

Consequently, Rényi entropy of the LWPS class of distributions reduces to

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left[\left[\sum_{n=1}^\infty P(N=n) \frac{n}{1+\lambda} \right]^v \sum_{i,j,k,l,m,n,s,p=0}^\infty \binom{v(n-1)}{i} \binom{i}{j} \right. \\ &\quad \times \binom{j}{k} \binom{v}{l} \binom{v-l}{m} \binom{l}{n} \binom{n}{s} \frac{(-1)^i (-(i+v))^p \lambda^{j+n+2(v-l)} \alpha^{l+p} \beta^l}{[\lambda(v+i)]^{k+l(\beta-1)+m+s+\beta l+1}} \\ &\quad \left. \times \Gamma(k+l(\beta-1)+m+s+\beta l+1) \right]. \end{aligned}$$

2.1.8 Maximum Likelihood Estimation

Let $X \sim LWPS(\alpha, \beta, \lambda, \theta)$ and $\Delta = (\alpha, \beta, \lambda, \theta)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned} \ell(\Delta) &= n \ln \theta - n \lambda x_i - n \alpha \sum_{i=1}^n x_i^\beta + \sum_{i=1}^n \ln \left[\lambda^2 (1+x_i) + (1+\lambda+\lambda x_i) \alpha \beta x_i^{\beta-1} \right] \\ &\quad + \sum_{i=1}^n \ln \left[C' \left(\theta \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^\beta} \right) \right) \right] \\ &\quad - n \ln C(\theta) - n \ln(1+\lambda). \end{aligned} \tag{2.28}$$

The elements of the score vector are given in the Appendix. The equations obtained by setting the elements of the score vector to zero are not in closed form and the values of the parameters $(\alpha, \beta, \lambda, \theta)$ must be found via iterative methods. The maximum likelihood estimates of the parameters, denoted $\hat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \theta})^T = 0$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix (FIM) is the 4×4 symmetric matrix given by $\mathbf{I}(\Delta) = [\mathbf{I}_{\theta_i \theta_j}]_{4 \times 4} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4$, can be numerically obtained by NLMIXED in SAS or mle2 package in R, MATLAB or MAPLE software. The total Fisher information matrix $\mathbf{I}_n(\Delta)$ can be approximated by

$$\mathbf{J}_n(\Delta) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \right]_{\Delta=\hat{\Delta}} \Big|_{4 \times 4}. \tag{2.29}$$

Note that the expectations in the Fisher Information matrix (FIM) can be obtained numerically. Let $\hat{\Delta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$ be the maximum likelihood estimate of $\Delta = (\lambda, \beta, \theta, \alpha)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} N_4(\underline{0}, \mathbf{I}^{-1}(\Delta))$, where $\mathbf{I}(\Delta)$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $\mathbf{I}(\Delta)$ is replaced by the observed information matrix evaluated at $\hat{\Delta}$, that is $\mathbf{J}(\hat{\Delta})$. The multivariate normal distribution $N_4(\underline{0}, \mathbf{J}(\hat{\Delta})^{-1})$, where the mean vector $\underline{0} = (0, 0, 0, 0)^T$, can be used to construct confidence intervals and confidence regions for the individual parameters and for the survival and hazard rate functions.

2.2 A Sub-Model of LWPS Distribution and Properties

In this section, we derive some properties of the LW-logarithmic (LWL) distribution including expansion of the density, hazard and reverse hazard functions, quantile function and sub-models, moments, conditional moments and maximum likelihood estimates of model parameters. The LWL distribution is obtained from the LWPS distribution given in Equation (2.5) with $a_n = \frac{1}{n}$, and $C(\theta) = -\log(1 - \theta)$, where $0 < \theta < 1$. The cdf and pdf of the LWL distribution are given by

$$F_{LWL}(x; \alpha, \beta, \lambda, \theta) = \frac{\log \left(1 - \theta \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x - \alpha x^\beta} \right) \right)}{\log(1 - \theta)}$$

and

$$\begin{aligned} f_{LWL}(x; \alpha, \beta, \lambda, \theta) &= \frac{\theta e^{-\lambda x - \alpha x^\beta}}{1 + \lambda} \left[\lambda^2(1 + x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1} \right] \\ &\times \frac{\left[1 - \theta \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x - \alpha x^\beta} \right) \right]^{-1}}{-\log(1 - \theta)}, \end{aligned}$$

respectively. Figure 2.1 represent the plots for the pdf of LWL distribution for different values of parameters. The graph takes various shapes including decreasing, almost symmetric, left skewed and right skewed.

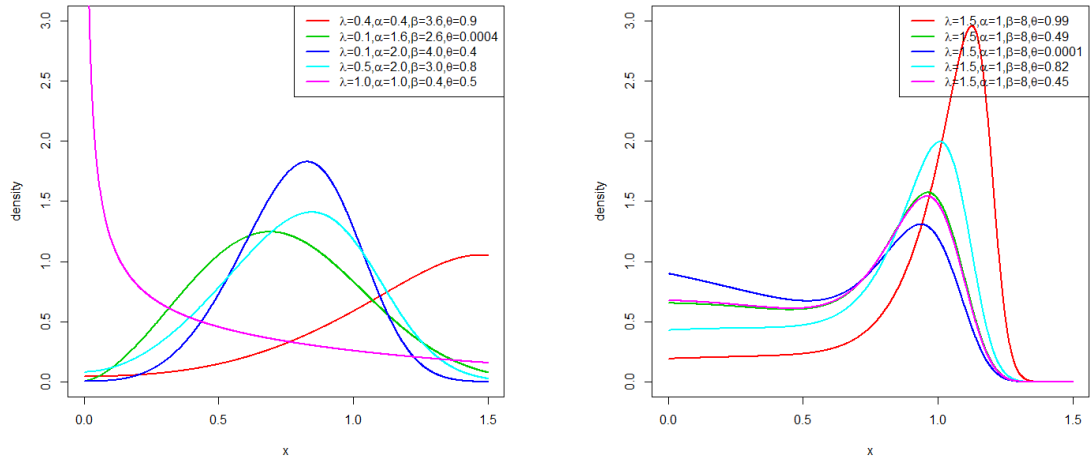


Figure 2.1: Graph of pdf of Lindley-Weibull-logarithmic distribution

2.2.1 Hazard and Reverse Hazard Functions

The hazard and reverse hazard functions of the LWL distribution are respectively given as follows:

$$h_{LWL}(x; \alpha, \beta, \lambda, \theta) = \frac{\theta e^{-\lambda x - \alpha x^\beta}}{1 + \lambda} \left[\lambda^2(1 + x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1} \right] \times \frac{\left[1 - \theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right]^{-1}}{\log \left(1 - \theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right) - \log(1 - \theta)}$$

and

$$\tau_{LWL}(x; \alpha, \beta, \lambda, \theta) = \frac{\theta e^{-\lambda x - \alpha x^\beta}}{1 + \lambda} \left[\lambda^2(1 + x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1} \right] \times \frac{\left[1 - \theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right]^{-1}}{-\log \left(1 - \theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right)}.$$

Figure 2.2 represent the plots for the hazard function of LWL distribution for different parameter values. The graph takes various shapes including decreasing, increasing, bathtub and bathtub followed by upsidedown bathtub.

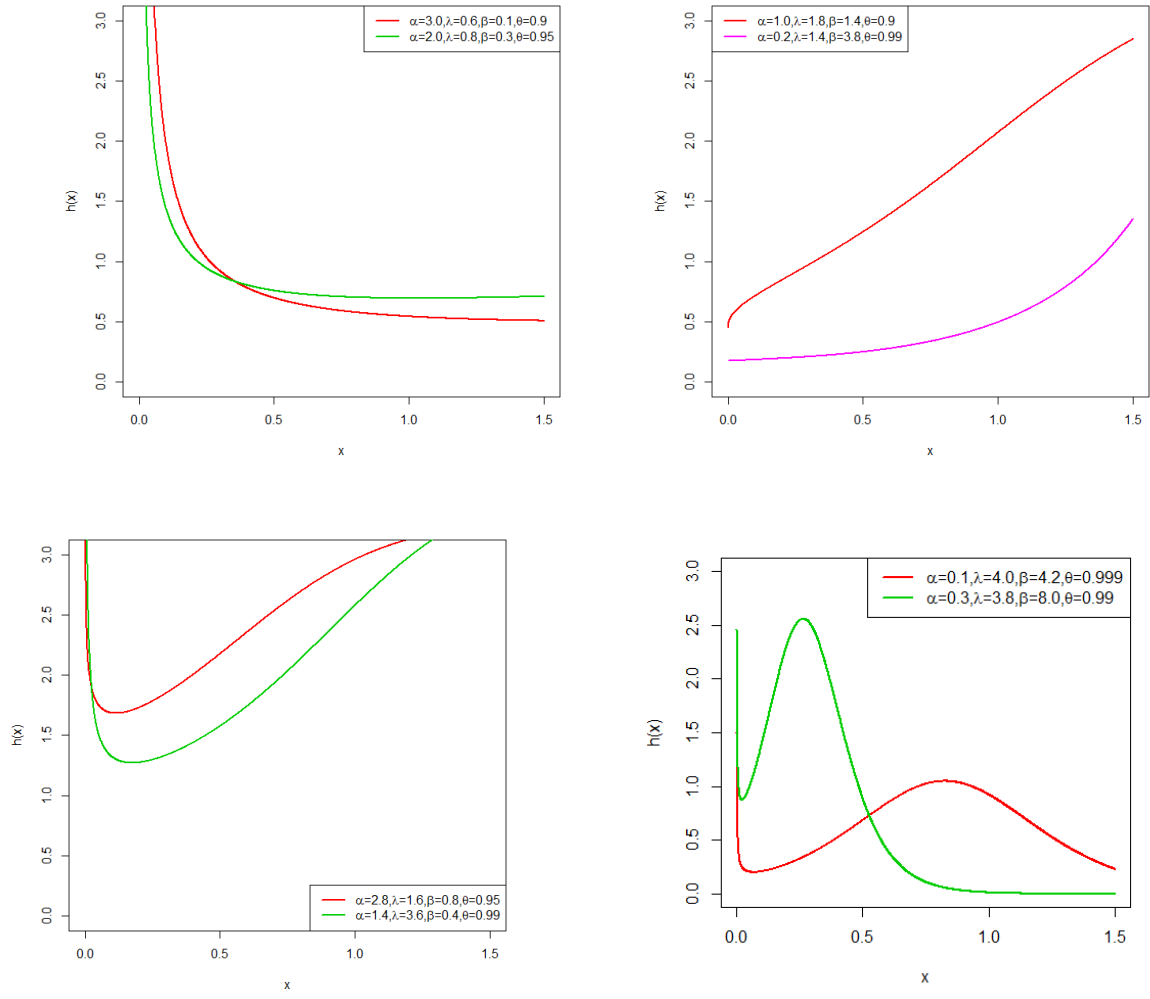


Figure 2.2: Plots of hazard function for different parameter values

2.2.2 Quantile Function

The LWL quantile function can be obtained by substituting $C(\theta) = -\log(1 - \theta)$ in (2.8). That is, we solve the non-linear equation

$$-\log \left(1 - \theta \left(1 - \frac{1 + \lambda + \lambda Q_{X(n)}(u)}{1 + \lambda} e^{-\lambda Q_{X(n)}(u) - \alpha Q_{X(n)}(u)^\beta} \right) \right) + u \ln(1 - \theta) = 0. \quad (2.30)$$

Table 2.1: Table of Quantile for LWL Distribution

$(\alpha, \lambda, \beta, \theta)$					
u	(0.1, 0.5, 2.5, 0.5)	(0.4, 1, 1.2, 0.2)	(1.5, 1, 2.5, 0.5)	(3, 2.5, 3.5, 0.8)	(1.6, 4.5, 6.5, 0.6)
0.1	0.5947	0.1478	0.2136	0.1126	0.04267
0.2	0.9743	0.2911	0.3515	0.2212	0.0882
0.3	1.2812	0.4406	0.4614	0.3206	0.1374
0.4	1.5560	0.6023	0.5586	0.4097	0.1915
0.5	1.8185	0.7826	0.6506	0.4906	0.2523
0.6	2.0835	0.9916	0.7430	0.5666	0.3230
0.7	2.3677	1.2471	0.8415	0.6419	0.4089
0.8	2.6998	1.5883	0.9560	0.7228	0.5202
0.9	3.1592	2.1369	1.1136	0.8252	0.6802

2.2.3 Some Sub-models of the LWL Distribution

- If $\alpha = 0$, $\theta \rightarrow 0^+$ and $\beta = 0$, we obtain Lindley (L) Distribution.
- When $\theta \rightarrow 0^+$, we obtain Lindley Weibull (LW) distribution.
- When $\theta \rightarrow 0^+$ and $\lambda \rightarrow 0^+$, we obtain Weibull (W) distribution.
- When $\beta = 2$, we obtain Lindley-Rayleigh Logarithmic (LRL) distribution.
- When $\alpha \rightarrow 0^+$, we obtain Lindley-logarithmic (LL) distribution.
- When $\lambda \rightarrow 0^+$, we obtain Weibull-logarithmic (WL) distribution.
- When $\lambda \rightarrow 0^+$ and $\beta = 2$, we obtain Rayleigh-Logarithmic (RL) distribution.
- When $\lambda \rightarrow 0^+$ and $\beta = 1$, we obtain Exponential-Logarithmic (EL) distribution.
- When $\beta = 1$, we obtain Lindley-Exponential Logarithmic (LEL) distribution.
- When $\theta \rightarrow 0^+$, $\lambda \rightarrow 0^+$ and $\beta = 1$ we obtain Exponential (E) distribution.
- When $\theta \rightarrow 0^+$, $\lambda \rightarrow 0^+$ and $\beta = 2$ we obtain Rayleigh (R) distribution.

- When $\theta \rightarrow 0^+$ and $\beta = 2$ we obtain lindley-Rayleigh (R) distribution.
- When $\theta \rightarrow 0^+$ and $\beta = 1$ we obtain Lindley-Exponential (LE) distribution.

2.2.4 Moments and Conditional Moments

Substituting $C(\theta) = -\log(1 - \theta)$ in (2.16), the r^{th} moment of a random variable X from the LWL distribution is given by

$$E(X^r) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{-\log(1 - \theta)} \frac{n}{1 + \lambda} L_1(\lambda, \alpha, \beta, n, r). \quad (2.31)$$

Table 2.2 below present the first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the LWL distribution for different parameter values.

Table 2.2: Moments for LWL distribution for some parameter values

	$(\alpha, \beta, \lambda, \theta)$				
	(0.1, 0.2, 0.2, 0.5)	(0.7, 1.0, 1.5, 0.5)	(0.2, 2.0, 2.0, 0.2)	(3.4, 4.0, 0.5, 0.9)	(0.2, 2.0, 1.5, 0.5)
$E(X)$	0.0282	0.3244	0.3286	0.1607	0.3261
$E(X^2)$	0.0178	0.2111	0.1957	0.0563	0.1987
$E(X^3)$	0.0131	0.1547	0.1364	0.0274	0.1404
$E(X^4)$	0.0104	0.1215	0.1037	0.0162	0.1077
$E(X^5)$	0.0086	0.0998	0.0832	0.0108	0.0870
$E(X^6)$	0.0074	0.0845	0.0693	0.0079	0.0728
SD	0.0064	0.0732	0.0593	0.0061	0.0625
CV	0.0057	0.0645	0.0518	0.0049	0.0547
CS	0.0051	0.0577	0.0459	0.0041	0.0486
CK	0.0046	0.0521	0.0412	0.0035	0.0437

Substituting $C(\theta) = -\log(1 - \theta)$ in (2.18), the r^{th} conditional moment of the LWL distribution is given by

$$E(X^r \mid X > x) = \sum_{n=1}^{\infty} \frac{a_n \theta^n L_2(\lambda, \alpha, \beta, n, r, x)}{\log \left(1 - \theta \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x - \alpha x^\beta} \right) \right) - \log(1 - \theta)}. \quad (2.32)$$

2.2.5 Maximum Likelihood Estimation

Let $X \sim LWL(\alpha, \beta, \lambda, \theta)$ and $\Delta = (\alpha, \beta, \lambda, \theta)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned} \ell(\Delta) = & n \ln \theta - n \lambda x_i - \alpha \sum_{i=1}^n x_i^\beta + \sum_{i=1}^n \ln \left[\lambda^2 (1 + x_i) + (1 + \lambda + \lambda x_i) \alpha \beta x_i^{\beta-1} \right] \\ & - \sum_{i=1}^n \ln \left[1 - \theta \left(1 - \frac{1 + \lambda + \lambda x_i}{1 + \lambda} e^{-\lambda x_i - \alpha x_i^\beta} \right) \right] \\ & - \sum_{i=1}^n \ln \left(-\log(1 - \theta) \right) - n \ln(1 + \lambda). \end{aligned} \quad (2.33)$$

The partial derivatives of the log-likelihood function with respect to β, θ, α and λ are given by

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} = & -\alpha \sum_{i=1}^n x_i^\beta \ln x_i + \sum_{i=1}^n \frac{\alpha (1 + \lambda + \lambda x_i) [x_i^{\beta-1} (1 + \beta \ln x_i)]}{\lambda^2 (1 + x_i) + (1 + \lambda + \lambda x_i) \alpha \beta x_i^{\beta-1}} \\ & + \sum_{i=1}^n \frac{\theta \left(\frac{1 + \lambda + \lambda x_i}{1 + \lambda} e^{-\lambda x_i - \alpha x_i^\beta} \right) \alpha x_i^\beta \ln x_i}{\left(1 - \theta \left(1 - \frac{1 + \lambda + \lambda x_i}{1 + \lambda} e^{-\lambda x_i - \alpha x_i^\beta} \right) \right)}, \\ \frac{\partial \ell}{\partial \theta} = & \frac{n}{\theta} + \sum_{i=1}^n \frac{\left(1 - \frac{1 + \lambda + \lambda x_i}{1 + \lambda} e^{-\lambda x_i - \alpha x_i^\beta} \right)}{\left(1 - \theta \left(1 - \frac{1 + \lambda + \lambda x_i}{1 + \lambda} e^{-\lambda x_i - \alpha x_i^\beta} \right) \right)} - \sum_{i=1}^n \frac{(1 - \theta)^{-1}}{-\log(1 - \theta)}, \\ \frac{\partial \ell}{\partial \alpha} = & -\sum_{i=1}^n x_i^\beta + \sum_{i=1}^n \frac{(1 + \lambda + \lambda x_i) (\beta x_i^{\beta-1})}{\lambda^2 (1 + x_i) + (1 + \lambda + \lambda x_i) \alpha \beta x_i^{\beta-1}} \\ & + \sum_{i=1}^n \frac{\theta \left(\frac{1 + \lambda + \lambda x_i}{1 + \lambda} e^{-\lambda x_i - \alpha x_i^\beta} \right) x_i^\beta}{\left(1 - \theta \left(1 - \frac{1 + \lambda + \lambda x_i}{1 + \lambda} e^{-\lambda x_i - \alpha x_i^\beta} \right) \right)}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \lambda} = & -n x_i + \sum_{i=1}^n \frac{2\lambda(1 + x_i) + (1 + x_i) \alpha \beta x_i^{\beta-1}}{\lambda^2 (1 + x_i) + (1 + \lambda + \lambda x_i) \alpha \beta x_i^{\beta-1}} \\ & + \sum_{i=1}^n \frac{\theta e^{-\lambda x_i - \alpha x_i^\beta} \left[x_i \left(\frac{1 + \lambda + \lambda x_i}{1 + \lambda} \right) - \frac{(1 + \lambda)(1 + x_i) - (1 + \lambda + \lambda x_i)}{(1 + \lambda)^2} \right]}{\left(1 - \theta \left(1 - \frac{1 + \lambda + \lambda x_i}{1 + \lambda} e^{-\lambda x_i - \alpha x_i^\beta} \right) \right)}. \end{aligned}$$

The maximum likelihood estimates (MLEs) can be obtained by equating the above partial derivatives to zero and solving. Since the partial derivatives are not in closed form, numerical methods such as Newton-Raphson procedure can be used to obtain the MLE of the model parameters.

2.3 Simulation Study

In this section, we examine the performance of the LWL distribution by conducting various simulations for different sizes ($n=25, 50, 100, 200, 400, 800$) via the R package. We simulate $N = 1000$ samples for the true parameters values given in the Table. The bias and RMSE are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

Table 2.3: Monte Carlo Simulation Results

parameter	Sample Size	(1.5,0.8,0.9,0.7)			(2.5,1.4,0.2,0.5)			(1.0,1.0,0.2,0.5)		
		Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	1.2534	0.8861	-0.2466	2.7644	0.9766	0.2644	0.8346	0.5115	-0.1654
	50	1.2590	0.8629	-0.2409	2.7345	0.7493	0.2344	0.8328	0.4880	-0.1672
	100	1.2634	0.8235	-0.2366	2.6671	0.6222	0.1671	0.8499	0.4668	-0.1501
	200	1.3099	0.7931	-0.1901	2.6507	0.5422	0.1507	0.8694	0.4500	-0.1306
	400	1.4262	0.7434	-0.0737	2.6123	0.4470	0.1123	0.8751	0.4139	-0.1248
	800	1.4968	0.7093	-0.0032	2.5464	0.3784	0.0464	0.9023	0.3694	-0.0976
β	25	0.7874	0.6660	-0.0125	1.5949	1.2565	0.1949	0.8649	0.3586	-0.1350
	50	0.7353	0.5187	-0.0646	1.4882	0.8146	0.0882	0.8669	0.2920	-0.1330
	100	0.7445	0.3177	-0.0554	1.4183	0.2656	0.0183	0.8808	0.2564	-0.1191
	200	0.7486	0.1654	-0.0513	1.4129	0.3984	0.0129	0.8904	0.2249	-0.1095
	400	0.7744	0.1285	-0.0255	1.3987	0.1201	-0.0012	0.9214	0.1782	-0.0785
	800	0.7873	0.0962	-0.0126	1.4032	0.0999	0.0032	0.9504	0.1385	-0.0495
λ	25	1.1371	0.9889	0.2371	0.3351	0.8156	0.1351	0.6159	0.7332	0.4159
	50	1.0800	0.9565	0.1800	0.2437	0.6216	0.0437	0.5867	0.7038	0.3867
	100	1.0398	0.8747	0.1398	0.2305	0.5972	0.0305	0.5192	0.6481	0.3192
	200	0.9939	0.8167	0.0939	0.1914	0.4923	-0.0085	0.4610	0.5887	0.2610
	400	0.8444	0.7475	-0.0555	0.1874	0.4070	-0.0125	0.4071	0.5214	0.2071
	800	0.8548	0.6870	-0.0451	0.1737	0.3361	-0.0262	0.3373	0.4464	0.1373
θ	25	0.6064	0.2644	-0.0935	0.6075	0.2703	0.1075	0.5961	0.2647	0.0961
	50	0.6120	0.2575	-0.0879	0.6133	0.2734	0.1133	0.5955	0.2678	0.0955
	100	0.5902	0.2685	-0.1097	0.5926	0.2665	0.0926	0.5689	0.2562	0.0689
	200	0.5928	0.2678	-0.1071	0.5704	0.2659	0.0704	0.5565	0.2456	0.0565
	400	0.5982	0.2616	-0.1017	0.5532	0.2455	0.0532	0.5134	0.2405	0.0134
	800	0.6176	0.2390	-0.0823	0.5062	0.2336	0.0062	0.4924	0.2267	-0.0075

From the results, we can verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

2.4 Applications

In this section, examples to illustrate the flexibility of the LWL distribution by fitting some special cases, and their sub-models for data modeling are

presented. Estimates of the parameters of each distribution (standard error in parentheses), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*), Andersen-Darling (A^*), Kolmogorov Simirnov (KS) and sum of squares (SS) from the probability plots are presented for each data set. We also compare LWL distribution to the non-nested new modified Weibull (NMW) distribution introduced by (Doostmoradi et al. [22]) and a four parameter beta generalized exponential (BGE) introduced by (Barreto-Souza et al. [7]). The pdf of four parameter NMW and BGE distributions are given in equation (2.34) and equation (2.35), respectively, that is,

$$g_{NMW}(x) = \left(\alpha \gamma x^{\gamma-1} e^{\alpha x^\gamma} + \lambda \beta x^{\lambda-1} e^{-\beta x^\lambda} \right) e^{-e^{\alpha x^\gamma} + e^{-\beta x^\lambda}}, \quad x > 0, \quad (2.34)$$

and

$$g_{BGE}(x) = \frac{\alpha \lambda}{B(a, b)} e^{-\lambda x} \left(1 - e^{-\lambda x} \right)^{\alpha a - 1} \left(1 - \left(1 - e^{-\lambda x} \right)^\alpha \right)^{b-1}, \quad x > 0. \quad (2.35)$$

The maximum likelihood estimates (MLEs) of the LWL parameters are computed by maximizing the objective function via the subroutine NLMIXED in SAS as well as the function nlm in R. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2 \ln(L)$, and Bayesian Information Criterion, $BIC = p \ln(n) - 2 \ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2 \frac{p(p+1)}{n-p-1}$ where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [14]) are given in Figure 2.3 for the fracture toughness of alumina data and in Figure 2.4 for the exceedances of wheaton river flood data, respectively. For the probability plot, we plotted $F_{LWPS}(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^n \left[F_{LWPS}(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The goodness-of-fit statistics W^* and A^* , described by (Chen and Balakrishnan [16]) are also presented in the tables. These statistics can be used

to verify which distribution fits better to the data. In general, the smaller the values of W^* , A^* and KS , the better the fit.

2.4.1 The Fracture Toughness of Alumina Data

This data consist of 119 observations of the fracture toughness of Alumina (Al_2O_3) (in the units of $\text{MPa m}^{1/2}$), from (Nadarajah and Kotz [54]). These data are: 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

Estimates of the parameters of LWL distribution (standard errors in parenthesis), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) are given in Table 5.1 for the Fracture Toughness of Alumina Data. The sum of squares and the goodness-of-fit statistics W^* and A^* are also given. Plots of histogram and fitted densities, and observed probability versus predicted probability for the Fracture Toughness of Alumina Data are given in Figures 5.1.

Testing the hypothesis $H_0 : L$ against $H_a : LWL$, the LR statistic is 197.6622 ($p\text{-value} < 0.00001$). Since the $p\text{-value}$ is small, we reject the null hypothesis and conclude that there is a significant difference between L and LWL distributions. There are no significant differences between LW and LWL distributions and well as between WL and LWL distributions base on the LR tests, however the values of the goodness-of-fit statistics: W^* , A^* , KS and its $p\text{-value}$ clearly show that the LWL distribution is by far the better fit for the Fracture Toughness of Alumina data. The value of sum of squares

(SS=0.06617918) from the probability plots in Figure 5.1 is smaller for LWL distribution. Also, the goodness-of-fit statistics W^* and A^* are smaller for LWL distribution as compared to nested and non-nested distributions for the Fracture Toughness of Alumina Data, hence we can conclude that the LWL distribution is a better fit.

Table 2.4: MLEs of the parameters, SEs in parenthesis and the goodness-of-fit statistics for Fracture Toughness of Alumina Data

Distribution	Estimates				-2logL	AIC	CAIC	BIC	SS	W^*	A^*	K-S	P-value
	α	β	λ	θ									
LWL	0.0019 (0.0018)	4.3660 (0.4746)	0.0023 (0.1013)	0.8487 (0.1396)	337.1337	345.1337	345.4846	356.2502	0.0662	0.0754	0.4397	0.0595	0.7932
LW	4.5393×10^{-04} (4.1612×10^{-05})	4.9649 (2.9315×10^{-07})	9.3097×10^{-07} (6.4638×10^{-02})	0 -	337.4137	343.4137	343.6224	351.7511	0.0821	0.0919	0.5639	0.0720	0.5677
L	0 -	0 -	0.3967 (0.0263)	0 -	534.7959	536.7959	536.8301	539.5751	4.9667	0.3542	2.1529	0.3353	4.793×10^{-12}
W	0.0005 (4.1612×10^{-05})	4.9649 (31×10^{-08})	0 -	0 -	337.4137	341.4137	341.5171	346.9719	18.2895	0.0919	0.5639	0.0720	0.5677
WL	0.0017 (0.0024)	4.3571 (0.6819)	0 -	0.7411 (0.3963)	336.5831	342.5831	342.7918	350.9204	38.5007	38.6708	237.5106	0.9985	2.2×10^{-16}
NMW	α	β	λ	γ	447.4069	455.4069	455.7578	466.5234	4.6969	0.3063	2.1586	0.3534	2.48×10^{-13}
	0.0157 (0.0172)	0.0001 (0.00001)	5.7082 (0.0005)	0.5195 (0.5195)									
BGE	α	λ	a	b	337.0887	345.0887	345.4396	356.2052	39.6244	0.0862	0.5197	0.0711	0.5835
	8.5389 (3.6507)	0.1009 (0.0434)	0.6171 (0.2928)	2138.5000 (0.0017)									

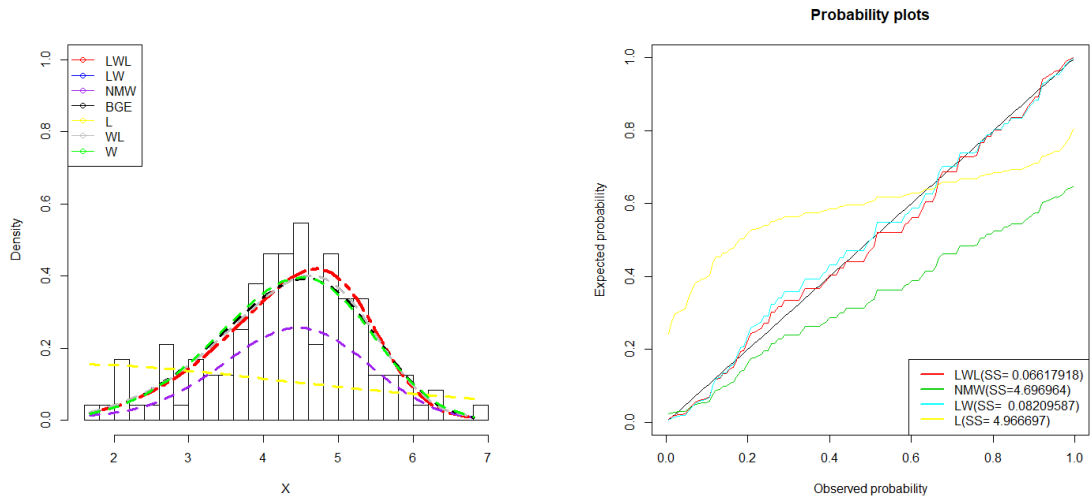


Figure 2.3: Fitted densities and Probability plots of The Fracture Toughness of Alumina Data

2.4.2 Carbon Fibre Data

The second data consist of uncensored data set from (Nichols and Padgett [59]) on the breaking stress of carbon fibers (in Gba). The data is given in Table 5.

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11	3.56
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90	1.57
2.67	2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85
1.25	4.38	1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03
1.89	2.88	2.82	2.05	3.65	3.75	2.43	2.95	2.97	3.39	2.96
2.35	2.55	2.59	2.03	1.61	2.12	3.15	1.08	2.56	1.80	2.53

Table 2.5: Carbon Fiber Data

Estimates of the parameters of LWL distribution (standard errors in parenthesis), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) are given in Table 5.3 for the Carbon fibre data. The sum of squares and the goodness-of-fit statistics W^* and A^* are also given. Plots of histogram and fitted densities, observed probability versus predicted probability for the Carbon fibre data are given in Figures 5.2.

Testing the hypothesis $H_0 : L$ against $H_a : LWL$, the LR statistic is 73.7414 (p-value < 0.00001). Since the p-value is small, we reject the null hypothesis and conclude that there is a significant difference between L and LWL. The value of sum of squares (SS=0.0615221) from the probability plots in Figure 5.2 is smaller for LWL distribution. Also, we can conclude that the LWL distribution is the better fit from the goodness-of-fit statistics W^* and A^* as they are small for LWL distribution as compared to other distributions for the Carbon fibre data.

Table 2.6: MLEs of the parameters, SEs in parenthesis and the goodness-of-fit statistics for Carbon fibre data

Distribution	Estimates				-2logL	AIC	CAIC	BIC	SS	W*	A*	K-S	P-value
	α	β	λ	θ									
LWL	0.0130 (0.0342)	3.7770 (1.4988)	0.1212 (0.0927)	0.0338 (2.8071)	171.0267	179.0267	179.6824	187.7853	0.0615	0.0642	0.4053	0.0805	0.7863
LW	0.0127 (0.0086)	3.7938 (0.4896)	0.1206 (0.0780)	0 -	171.0268	177.0268	177.4139	183.5958	0.0616	0.0642	0.4054	0.0805	0.7858
L	0 -	0 -	0.5903 (0.0532)	0 -	244.7681	246.7681	246.8306	248.9578	2.0688	0.2137	1.1481	0.2977	1.661×10^{-05}
W	0.0213 (0.0091)	3.4412 (0.3309)	0 -	0 -	172.1352	176.1352	176.3256	180.5145	0.0806	0.0929	0.5262	0.0823	0.7625
WL	0.0544 (0.0603)	2.9781 (0.5790)	0 -	0.7750 (0.4164)	171.4563	177.4563	177.8434	184.0253	20.8759	20.9841	131.3425	0.9983	2.2×10^{-16}
	α	β	λ	γ									
NMW	0.0010 (0.0021)	0.0111 (0.0077)	3.9621 (0.6103)	4.4787 (1.3273)	197.1722	205.1722	205.8280	213.9308	0.4725	0.1983	1.041947	0.1890	0.0179
	α	λ	a	b									
BGE	4.7766 (2.0770)	0.0587 (0.0399)	0.6780 (0.3838)	3351.4000 (0.0001)	171.8907	179.8907	180.5465	188.6493	0.0780	0.0856	0.5068	0.0807	0.7836

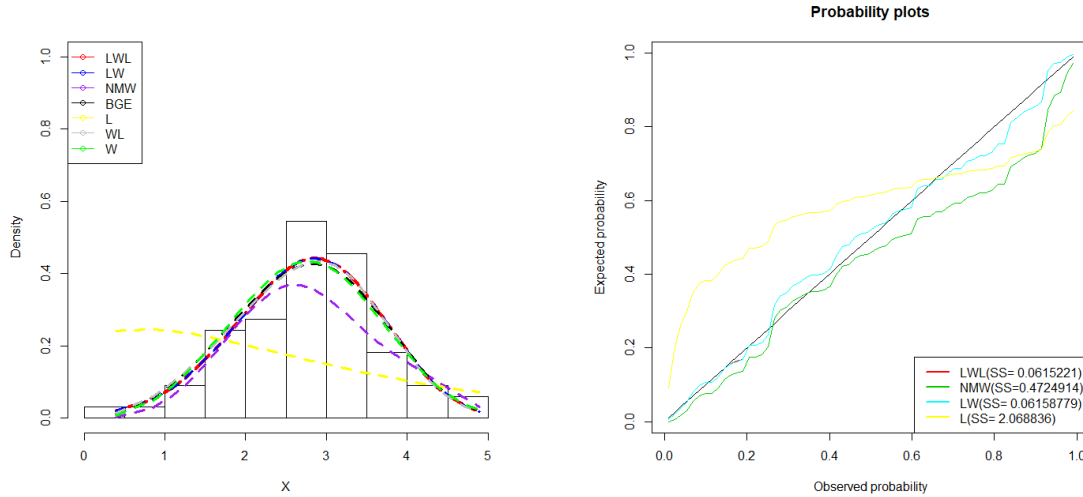


Figure 2.4: Fitted densities and Probability plots of Carbon fibre data

2.5 Conclusions

Presented is a new class of distributions called the Lindley-Weibull power series class of distributions that are suitable for applications in various areas including reliability, survival analysis and actuarial sciences just to mention a few areas. This distribution and some of its structural properties including hazard and reverse hazard functions, quantile function, moments, conditional moments, mean deviations, Bonferroni and Lorenz curves, Rényi entropy, distribution of order statistics and maximum likelihood estimates are presented. Applications

of the special case of the LWL model to real data sets are given in order to illustrate the applicability and usefulness of the proposed class of distributions. These distributions were also compared to some non-nested models.

Chapter 3

A New Gamma Generalized Lindley-Log-logistic (GELLLoG) Distribution

This chapter employs exponentiation, competing risk transformation and ZB-G formulation to obtain a new distribution involving both the Lindley and log-logistic distributions. The new distribution called the gamma exponentiated Lindley log-logistic (GELLLoG) distribution is quite useful, generalizes the Lindley, generalized Lindley and log-logistic distributions, and is more flexible distribution for the description of reliability and lifetime data. The combined distribution of Lindley and log-logistic is obtained from the product of the reliability or survival functions of the Lindley and log-logistic distributions via competing risk model. A motivation for developing this model is the advantages presented by this extended distribution with respect to having a hazard function that exhibits increasing, decreasing and bathtub shapes, as well as the versatility and flexibility of exponentiated distributions in general, as well as the Lindley and log-logistic distributions in modeling lifetime data.

3.1 The Model, Series Expansion of Density Function, Sub-models, Hazard and Quantile Functions

In this section, some properties of the new gamma exponentiated Lindley log-Logistic (GELLLoG) distribution including expansion of the density, hazard function, quantile function, sub-models, moments, conditional moments and maximum likelihood estimation of model parameters are derived.

The cdf, survival function (sf) and pdf of the exponentiated Lindley log-logistic (ELLLoG) distribution (Oluyede et al. [62]) are given by

$$G(x; \lambda, c, \alpha) = \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)} \right]^\alpha, \quad (3.1)$$

$$\bar{G}(x; \lambda, c, \alpha) = 1 - \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)} \right]^\alpha, \quad (3.2)$$

and

$$\begin{aligned} g(x; \lambda, c, \alpha) &= \alpha \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)} \right]^{\alpha-1} \\ &\quad \times \frac{(1 + x^c)^{-1}}{1 + \lambda} e^{-\lambda x} \left[\lambda^2(1 + x) + \frac{(1 + \lambda + \lambda x)cx^{c-1}}{1 + x^c} \right], \end{aligned}$$

respectively, for $\lambda, c, \alpha > 0$. If a random variable X has the ELLLoG distribution, we write $X \sim ELLLoG(\lambda, c, \alpha)$.

The cdf and pdf of the proposed gamma exponentiated Lindley log-logistic (GELLLoG) distribution are given by

$$\begin{aligned} F_{GELLLoG}(x; \lambda, c, \alpha, \delta) &= \frac{1}{\Gamma(\delta)} \int_0^{-\log \left(1 - \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)} \right)^\alpha \right)} t^{\delta-1} e^{-t} dt \\ &= \frac{\gamma \left(-\log \left(1 - \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)} \right)^\alpha \right), \delta \right)}{\Gamma(\delta)}, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} f_{GELLLoG}(x; \lambda, c, \alpha, \delta) &= \frac{1}{\Gamma(\delta)} \left[-\log \left(1 - \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)} \right]^\alpha \right) \right]^{\delta-1} \\ &\quad \times \alpha \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)} \right]^{\alpha-1} \\ &\quad \times \frac{(1 + x^c)^{-1}}{1 + \lambda} e^{-\lambda x} \left[\lambda^2(1 + x) + \frac{(1 + \lambda + \lambda x)cx^{c-1}}{1 + x^c} \right], \end{aligned} \quad (3.4)$$

respectively, for $\lambda, c, \alpha, \delta > 0$, where $\gamma(x, \delta) = \int_0^x t^{\delta-1} e^{-t} dt$ is the lower incomplete gamma function. If a random variable X has the GELLLoG distribution, we write $X \sim GELLLoG(\lambda, c, \alpha, \delta)$. Plots of the GELLLoG pdf below shows different shapes including increasing, right skewed, left skewed, almost symmetric and reverse J shapes.

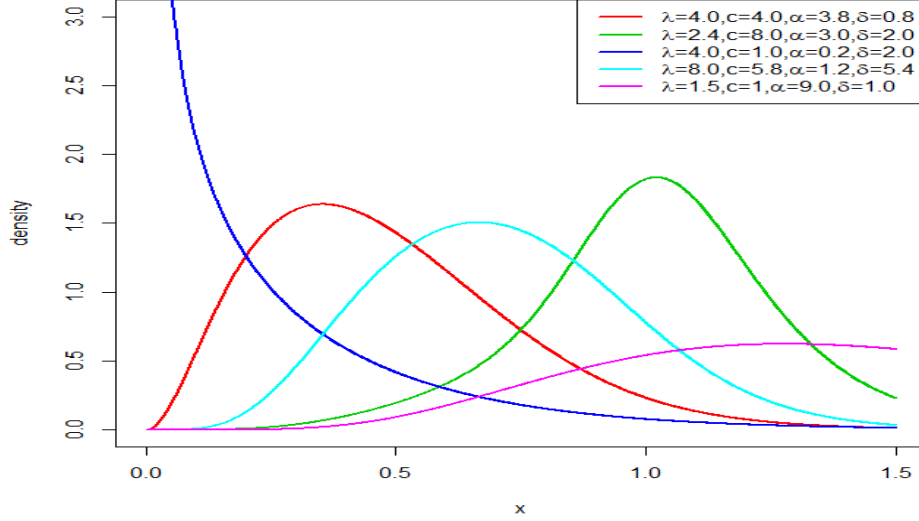


Figure 3.1: Plots of GELLLoG Density Function

3.1.1 Series Expansion of Density Function

In this section, series expansion of the GELLLoG density function is presented. The results allows for the mathematical and statistical properties of the model to be readily obtained.

Let $y = \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right)^\alpha$, $0 < y < 1$, $\alpha, \lambda, c > 0$, then using the series representation $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, we have

$$\left[-\log(1-y)\right]^{\delta-1} = y^{\delta-1} \left[\sum_{m=0}^{\infty} \binom{\delta-1}{m} y^m \left(\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^m \right],$$

and applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s \right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \quad (3.5)$$

where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^s [m(l+1) - s] a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (Gradshteyn and Ryzhik [30]), the GELLLoG pdf can be written as

$$\begin{aligned}
f_{\text{GELLLoG}}(x) &= \frac{\alpha}{\Gamma(\delta)} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} y^{m+s+\delta-1} \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^{\alpha-1} \\
&\quad \times \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \right] \\
&= \frac{\alpha}{\Gamma(\delta)} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \frac{(m+s+\delta)}{(m+s+\delta)} \\
&\quad \times \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^{\alpha(m+s+\delta)-1} \\
&\quad \times \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \right] \\
&= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\delta-1}{m} \frac{\alpha b_{s,m}}{(m+s+\delta)\Gamma(\delta)} g_{\text{ELLLoG}}(x; c, \lambda, \alpha^*), \tag{3.6}
\end{aligned}$$

where $g_{\text{ELLLoG}}(x; c, \lambda, \alpha^*)$ is the exponentiated Lindley-log-logistic (ELLLoG) pdf with parameters c , λ , and $\alpha^* = \alpha(m+s+\delta) > 0$. Let $D = \{(m, s) \in \mathbf{Z}_+^2\}$, then the weights in the GELLLoG pdf are

$$\omega_\nu = \binom{\delta-1}{m} \frac{\alpha b_{m,s}}{(m+s+\delta)\Gamma(\delta)}, \tag{3.7}$$

and

$$f_{\text{GELLLoG}}(x) = \sum_{\nu \in D} \omega_\nu g_{\text{ELLLoG}}(x; c, \lambda, \alpha(m+s+\delta)). \tag{3.8}$$

It follows therefore that the GELLLoG density is an infinite linear combination of the ELLLoG pdfs. The statistical and mathematical properties of the GELLLoG distribution can be readily obtained from those of the ELLLoG distribution.

3.1.2 Sub-models of GELLLoG Distribution

In this subsection, some useful and important sub-models are presented.

- When $\lambda \rightarrow 0^+$, the resulting distribution is the gamma exponentiated log-logistic (GELLoG) distribution.
- When $\lambda \rightarrow 0^+$, and $\alpha = 1$, we obtain the gamma log-logistic (GLLoG) distribution.

- We obtain gamma Lindley log-logistic (GLLLoG) distribution with $\alpha = 1$.
- When $\delta = 1$, we obtain the baseline exponentiated Lindley log-logistic (ELLLoG) distribution.
- When $\delta = \alpha = 1$, we obtain the Lindley log-logistic (LLLoG) distribution.
- When $\lambda \rightarrow 0^+$, and $\delta = 1$, we obtain the exponentiated log-logistic (ELLoG) distribution.
- If $\lambda \rightarrow 0^+$ and $\alpha = \delta = 1$, we obtain log-logistic (LLoG) distribution.
- If $\delta = c = 1$, and $\lambda \rightarrow 0^+$, we obtain one parameter distribution denoted by $GEULLoG(1, 1, \alpha, 1)$, with the cdf

$$F(x; \alpha) = \left[1 - \frac{1}{(1+x)} \right]^\alpha, \quad \alpha > 0. \quad (3.9)$$

- If $\delta = c = 1$, we obtain the two parameter distribution denoted by $GEULLoG(\lambda, 1, \alpha, 1)$, with the cdf

$$F(x; \lambda, \alpha) = \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1+x)} \right]^\alpha, \quad \lambda, \alpha > 0. \quad (3.10)$$

- If $\delta = c = \alpha = 1$, we obtain the one parameter distribution denoted by $GEULLoG(\lambda, 1, 1, 1)$, with the cdf

$$F(x; \lambda) = 1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1+x)}, \quad \lambda > 0. \quad (3.11)$$

- If $\alpha = c = 1$, we obtain the two parameter distribution denoted by $GEULLoG(\lambda, 1, 1, \delta)$, with the cdf

$$F(x; \lambda, \delta) = \frac{1}{\Gamma(\delta)} \gamma \left(-\log \left(1 - \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{1+x} \right) \right), \delta \right), \quad \lambda, \delta > 0. \quad (3.12)$$

- If $c=1$, we obtain the three parameter distribution denoted by $GEULLoG(\lambda, 1, \alpha, \delta)$, with the cdf

$$F(x; \lambda, \alpha, \delta) = \frac{1}{\Gamma(\delta)} \gamma \left(-\log \left(1 - \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{1+x} \right)^\alpha \right), \delta \right), \quad \lambda, \delta > 0. \quad (3.13)$$

3.1.3 Hazard and Quantile Functions

In this section, the hazard and quantile functions of the GELLLoG distribution are presented. Plots of the hazard function for selected values of the model parameters are presented in Figure 3.2. The hazard rate function of the GELLLoG distribution is given by

$$\begin{aligned}
 h_{F_{GELLLoG}}(x) &= \frac{f_{GELLLoG}(x)}{\bar{F}_{GELLLoG}(x)} \\
 &= \left[-\log \left(1 - \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^\alpha \right) \right]^{\delta-1} \\
 &\quad \times \alpha \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^{\alpha-1} \\
 &\quad \times \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \right] \\
 &\quad \times \left[\Gamma(\delta) - \gamma \left(-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right), \delta \right) \right]^{-1}.
 \end{aligned}$$

Plots of the GELLLoG hazard below shows different shapes including decreasing, increasing, bathtub followed by upside down, upside down bathtub, and bathtub shapes.

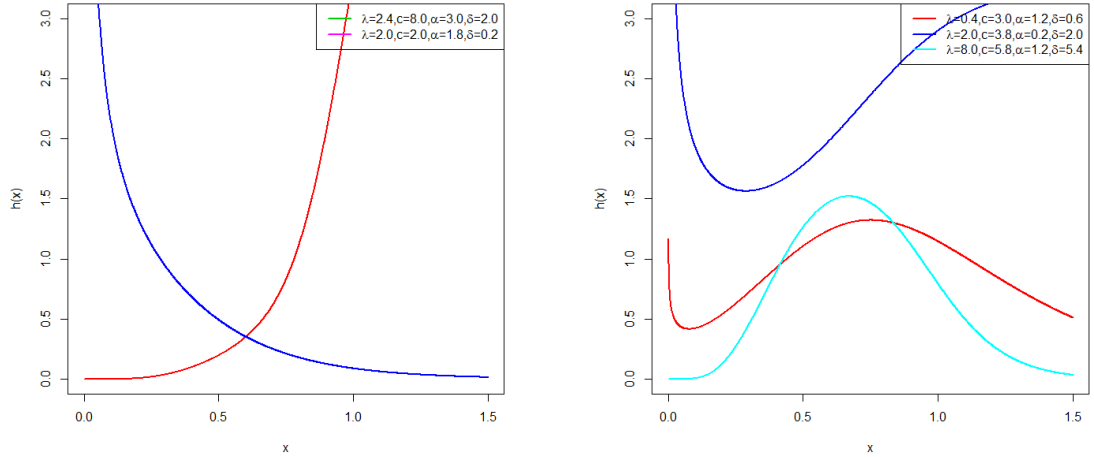


Figure 3.2: Plots of GELLLoG Hazard Function

The quantile function of the GELLLoG distribution is obtained by solving the non-linear equation:

$$\frac{\gamma \left(-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right), \delta \right)}{\Gamma(\delta)} = u, \quad (3.14)$$

$0 \leq u \leq 1$, that is,

$$\left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)}\right)^\alpha = 1 - e^{-\gamma^{-1}(u\Gamma(\delta), \delta)}, \quad (3.15)$$

so that

$$\frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)} = 1 - \left(1 - e^{-\gamma^{-1}(u\Gamma(\delta), \delta)}\right)^{1/\alpha}. \quad (3.16)$$

Consequently, random numbers can be generated for the GELLLoG distribution by numerically solving the nonlinear equation

$$\lambda x + \log(1 + x^c) - \log\left(1 + \frac{\lambda x}{1 + \lambda}\right) + \log\left(1 - \left(1 - e^{-\gamma^{-1}(u\Gamma(\delta), \delta)}\right)^{1/\alpha}\right) = 0. \quad (3.17)$$

Table 3.1 presents quantiles of the GELLLoG distribution for selected values of the model parameters λ , c , α and δ .

Table 3.1: Table of Quantiles for GELLLoG Distribution

u	$(\lambda, c, \alpha, \delta)$				
	(1.2, 1.2, 1, 1.8))	(0.2, 1.5, 2.2, 2)	(0.8, 1.2, 2, 1)	(2, 1, 1, 2.2)	(1, 1.8, 2.6, 2)
0.1	0.2943	1.4788	0.3456	0.2970	0.9711
0.2	0.4582	2.0185	0.5276	0.4557	1.2086
0.3	0.6130	2.5516	0.7021	0.6062	1.4150
0.4	0.7726	3.1391	0.8868	0.7636	1.6199
0.5	0.9460	3.8300	1.0944	0.9395	1.8390
0.6	1.1430	4.6905	1.3409	1.1496	2.0886
0.7	1.3780	5.8378	1.6538	1.4253	2.3939
0.8	1.6776	7.5312	2.0931	1.8588	2.8079
0.9	2.1043	10.6012	2.8509	3.4573	3.4967

3.2 Moments, Conditional Moments, Mean and Median Deviations

In this section, we present the moments, moment generating function, mean and median deviations for the GELLLoG distribution. Moments are very important and necessary in any statistical analysis, especially in applications. Moments can be used to study the most important features and characteristics of a distribution (e.g., central tendency, dispersion, skewness and kurtosis). These measures (moments, moment generating function, mean and median deviations) can be readily obtained for the sub-models given in section 3.1.

3.2.1 Moments and Moment Generating Function

Let $\alpha^* = \alpha(m + s + \delta)$, and $Y \sim \text{ELLLoG}(c, \lambda, \alpha^*)$. Note that the k^{th} moment of the ELLLoG random variable Y is obtained as follows. The k^{th} raw moment, μ'_k of the ELLLoG distribution is given by:

$$\begin{aligned}
E(Y^k) &= \int_0^\infty y^k g_{\text{ELLLoG}}(y; \lambda, c, \alpha(m + s + \delta)) dy \\
&= \int_0^\infty y^k \alpha(m + s + \delta) \left[1 - \frac{1 + \lambda + \lambda y}{1 + \lambda} \frac{e^{-\lambda y}}{(1 + y^c)} \right]^{\alpha(m + s + \delta) - 1} \\
&\quad \times \frac{(1 + y^c)^{-1}}{1 + \lambda} e^{-\lambda y} \left[\lambda^2(1 + y) + \frac{(1 + \lambda + \lambda y)cy^{c-1}}{1 + y^c} \right] dy \\
&= \sum_{t=0}^\infty \binom{\alpha(m + s + \delta) - 1}{t} (-1)^t \alpha(m + s + \delta) \int_0^\infty y^k \left[\frac{1 + \lambda + \lambda y}{1 + \lambda} \frac{e^{-\lambda y}}{(1 + y^c)} \right]^t \\
&\quad \times \frac{(1 + y^c)^{-1}}{1 + \lambda} e^{-\lambda y} \left[\lambda^2(1 + y) + \frac{(1 + \lambda + \lambda y)cy^{c-1}}{1 + y^c} \right] dy \\
&= \sum_{t,p=0}^\infty \frac{\alpha(m + s + \delta) (-1)^{t+p} [\lambda(t+1)]^p}{(1 + \lambda)^{t+1} p!} \binom{\alpha(m + s + \delta) - 1}{t} \\
&\quad \times \left[\lambda^2 \sum_{q=0}^\infty \binom{t}{q} \lambda^q (1 + \lambda)^{t-q} \int_0^\infty y^{k+p+q} (1 + y) (1 + y^c)^{-t-1} dy \right. \\
&\quad \left. + c \sum_{q=0}^\infty \binom{t+1}{q} \lambda^q (1 + \lambda)^{t+1-q} \int_0^\infty y^{k+p+c+q-1} (1 + y^c)^{-t-2} dy \right]. \quad (3.18)
\end{aligned}$$

We note that by applying $(1 + \lambda + \lambda y)^{t+1} = \sum_{q=0}^\infty \binom{t+1}{q} (\lambda y)^q (1 + \lambda)^{t+1-q}$, $(1 + \lambda + \lambda y)^t = \sum_{q=0}^\infty \binom{t}{q} (\lambda y)^q (1 + \lambda)^{t-q}$, and the substitution $w = (1 + y^c)^{-1}$, we have

$$\begin{aligned}
E(Y^k) &= \sum_{t,p=0}^\infty \frac{\alpha(m + s + \delta) (-1)^{t+p} [\lambda(t+1)]^p}{(1 + \lambda)^{t+1} p!} \binom{\alpha(m + s + \delta) - 1}{t} \\
&\quad \times \left[\sum_{q=0}^\infty \binom{t}{q} \lambda^{q+2} \frac{(1 + \lambda)^{t-q}}{c} \int_0^1 w^{t+1 - \frac{k+p+q+1}{c} - 1} (1 - w)^{\frac{k+p+q+1}{c} - 1} dw \right. \\
&\quad \left. + \int_0^1 w^{t+1 - \frac{k+p+q+2}{c} - 1} (1 - w)^{\frac{k+p+q+2}{c} - 1} dw \right. \\
&\quad \left. + c \sum_{q=0}^\infty \binom{t+1}{q} \lambda^q (1 + \lambda)^{t+1-q} \int_0^1 w^{t+2 - \frac{k+p+c+q}{c} - 1} (1 - w)^{\frac{k+p+c+q}{c} - 1} dt \right]. \quad (3.19)
\end{aligned}$$

Consequently,

$$\begin{aligned}
E(Y^k) &= \sum_{t,p=0}^{\infty} \frac{\alpha(m+s+\delta)(-1)^{t+p}[\lambda(t+1)]^p}{(1+\lambda)^{t+1}p!} \binom{\alpha(m+s+\delta)-1}{t} \\
&\times \left[\sum_{q=0}^{\infty} \binom{t}{q} \lambda^{q+2} \frac{(1+\lambda)^{t-q}}{c} \left(B\left(t+1 - \frac{k+p+q+1}{c}, \frac{k+p+q+1}{c}\right) \right. \right. \\
&+ B\left(t+1 - \frac{k+p+q+2}{c}, \frac{k+p+q+2}{c}\right) \Bigg) \\
&+ c \sum_{q=0}^{\infty} \binom{t+1}{q} \lambda^q (1+\lambda)^{t+1-q} \\
&\left. \left(t+2 - \frac{k+p+c+q}{c}, \frac{k+q+p+c+q}{c} \right) \right]. \tag{3.20}
\end{aligned}$$

Thus, the k^{th} moments of the GELLLoG distribution is given by

$$E(X^k) = \sum_{\nu \in D} \omega_{\nu} \Delta(t, p, q, c, \lambda, k), \tag{3.21}$$

where $\Delta(t, p, q, c, \lambda, k)$ is given by equation (3.20). The moment generating function of the GELLLoG class of distribution is given by $E(e^{tX}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k)$, where $E(X^k)$ is given by the equation (3.21). The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation ($SD=\sigma$), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$

and

$$CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively. Some moments for selected parameters values are given in Table 3.2 and plots of CS and CK versus the shape parameters, α , c and δ are presented in Figure 3.3, Figure 3.4 and Figure 3.5. Plots of skewness and kurtosis for choices of the model parameters reveal that skewness and kurtosis depend on the parameters α , c , and δ .

Table 3.2: Moments for Selected Parameters Values of GELLLoG Distribution

	$(\lambda, c, \alpha, \delta)$				
	(0.1,0.2,0.2,0.5)	(1.8,1.5,2.2,0.5)	(0.8,1.0,2.2,1.0)	(2.0,2.2,0.2,1.8)	(0.1,1.0,2.0,0.5)
$E(X)$	0.0080	0.2415	0.2861	0.1690	0.3272
$E(X^2)$	0.0039	0.1321	0.1611	0.0727	0.2062
$E(X^3)$	0.0025	0.0889	0.1088	0.0422	0.1474
$E(X^4)$	0.0019	0.0665	0.0811	0.0285	0.1135
$E(X^5)$	0.0015	0.0530	0.0642	0.0210	0.0918
$E(X^6)$	0.0012	0.0439	0.0529	0.0164	0.0769
SD	0.0011	0.0375	0.0450	0.0133	0.0660
CV	0.0009	0.0327	0.0390	0.0112	0.0577
CS	0.0008	0.0290	0.0344	0.0096	0.0513
CK	0.0007	0.02607	0.0308	0.0084	0.0461

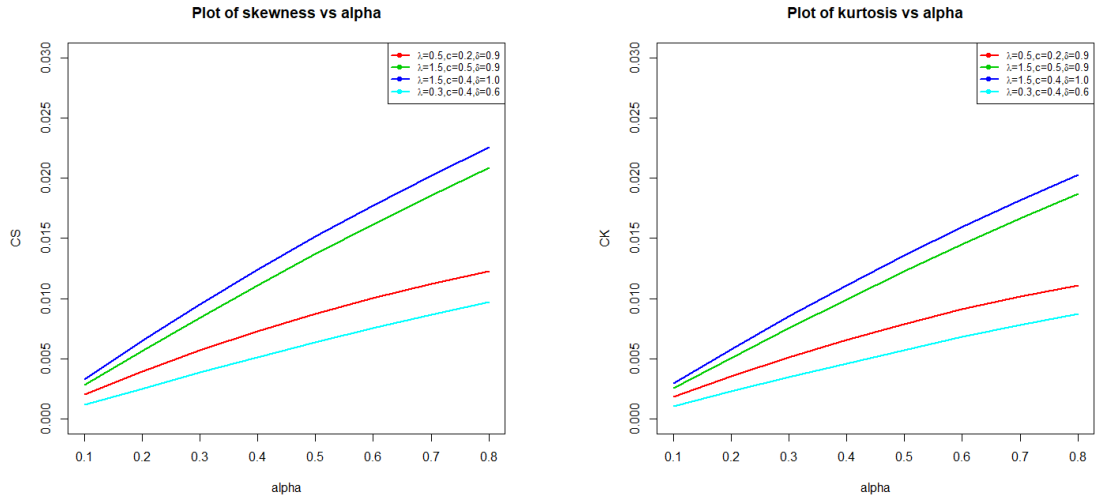


Figure 3.3: Plots of Skewness and Kurtosis for parameter alpha

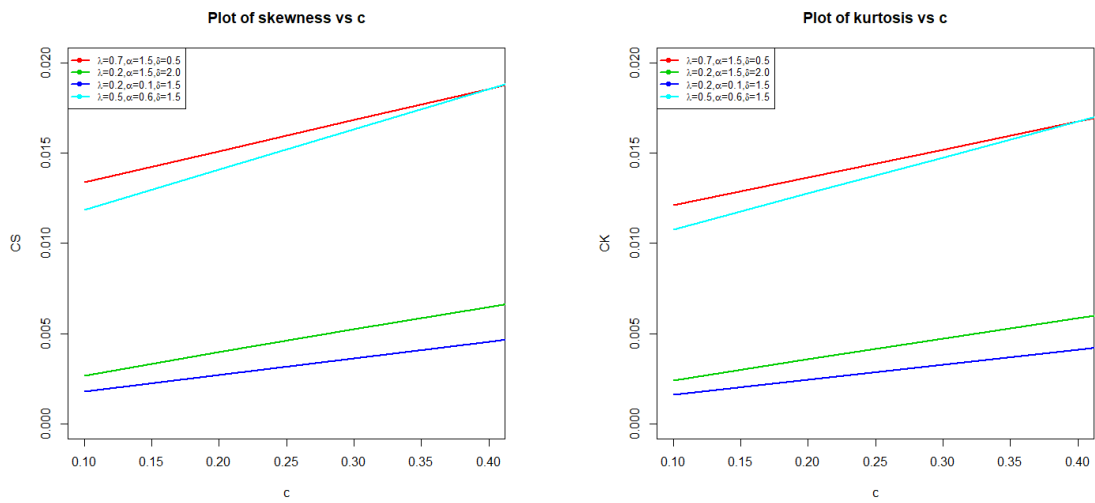


Figure 3.4: Plots of Skewness and Kurtosis for parameter c

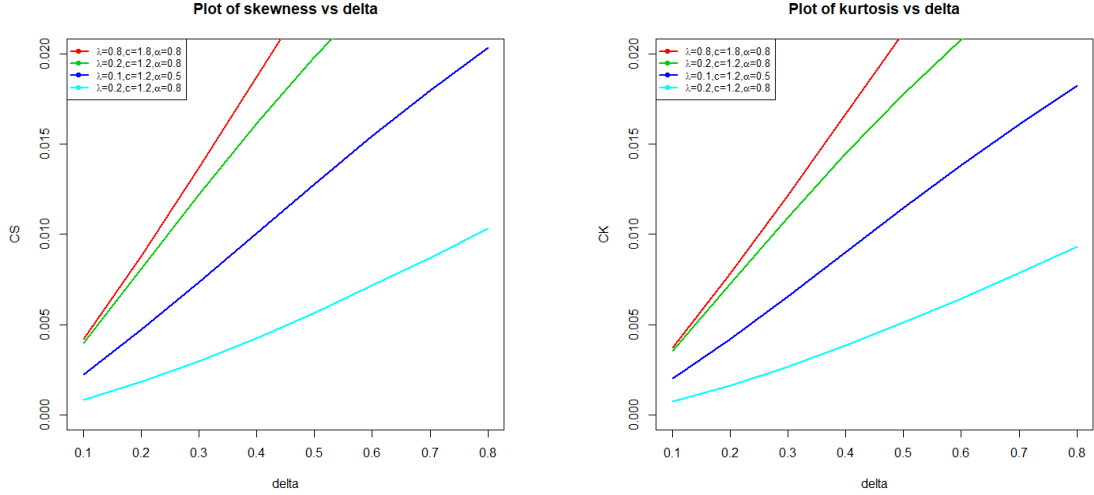


Figure 3.5: Plots of Skewness and Kurtosis for parameter delta

3.2.2 Conditional Moments

The mean residual life function, vitality function and related reliability measures can be readily obtained from the conditional moments of a distribution.

The k^{th} conditional moments for the GELLoG distribution is given by

$$\begin{aligned}
 E(X^k | X > a) &= \frac{1}{\overline{F}_{GELLoG}(a)} \int_t^\infty x^k f_{GELLoG}(x; c, \lambda, \alpha, \delta) dx \\
 &= \frac{1}{\overline{F}_{GELLoG}(a)} \sum_{\nu \in D} \sum_{t,p=0}^\infty \omega_\nu \frac{\alpha(m+s+\delta)(-1)^{t+p} [\lambda(t+1)]^p}{(1+\lambda)^{t+1} p!} \\
 &\quad \times \binom{\alpha(m+s+\delta)-1}{t} \left[\sum_{q=0}^\infty \binom{t}{q} \lambda^{q+2} \frac{(1+\lambda)^{t-q}}{c} \right. \\
 &\quad \times \left(B_{(1+a^c)^{-1}} \left(t+1 - \frac{k+p+q+1}{c}, \frac{k+p+q+1}{c} \right) \right. \\
 &\quad \left. + B_{(1+a^c)^{-1}} \left(t+1 - \frac{k+p+q+2}{c}, \frac{k+p+q+2}{c} \right) \right) \\
 &\quad + c \sum_{q=0}^\infty \binom{t+1}{q} \lambda^q (1+\lambda)^{t+1-q} \\
 &\quad \left. \times B_{(1+a^c)^{-1}} \left(t+2 - \frac{k+p+c+q}{c}, \frac{k+q+p+c+q}{c} \right) \right],
 \end{aligned}$$

where $B_{(1+a^c)^{-1}}(a, b)$ is the incomplete beta function.

3.2.3 Mean Deviation, Lorenz and Bonferroni Curves

Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the GELLLoG distribution are presented in this subsection.

Mean Deviations

The mean deviation about the mean and the mean deviation about the median are defined by

$$\delta_1(x) = \int_0^\infty |x - \mu| f_{GELLLoG}(x) dx \quad \text{and} \quad \delta_2(x) = \int_0^\infty |x - M| f_{GELLLoG}(x) dx, \quad (3.22)$$

respectively, where $\mu = E[X]$ and $M = \text{Median}(X)$ denotes the median. We note that $\delta_1(x)$ and $\delta_2(x)$ can be expressed as $\delta_1(x) = 2\mu F_{GELLLoG}(\mu) - 2\mu + 2 \int_\mu^\infty f_{GELLLoG}(x) dx$ and $\delta_2(x) = -\mu + 2 \int_M^\infty x f_{GELLLoG}(x) dx$, respectively, that is,

$$\delta_1(x) = 2\mu F_{GELLLoG}(\mu) - 2\mu + 2T(\mu) \quad \text{and} \quad \delta_2(x) = 2T(M) - \mu, \quad (3.23)$$

where

$$\begin{aligned} T(\mu) &= \int_\mu^\infty x f_{GELLLoG}(x) dx \\ &= \sum_{\nu \in D} \sum_{t,p=0}^\infty \omega_\nu \frac{\alpha(m+s+\delta)(-1)^{t+p}[\lambda(t+1)]^p}{(1+\lambda)^{t+1}p!} \\ &\quad \times \left(\binom{\alpha(m+s+\delta)-1}{t} \left[\sum_{q=0}^\infty \binom{t}{q} \lambda^{q+2} \frac{(1+\lambda)^{t-q}}{c} \right. \right. \\ &\quad \times \left(B_{(1+\mu^c)^{-1}} \left(t+1 - \frac{1+p+q+1}{c}, \frac{1+p+q+1}{c} \right) \right. \\ &\quad \left. \left. + B_{(1+\mu^c)^{-1}} \left(t+1 - \frac{1+p+q+2}{c}, \frac{1+p+q+2}{c} \right) \right) \right. \\ &\quad \left. + c \sum_{q=0}^\infty \binom{t+1}{q} \lambda^q (1+\lambda)^{t+1-q} \right. \\ &\quad \left. \left. (1+\mu^c)^{-1} \left(t+2 - \frac{1+p+c+q}{c}, \frac{1+q+p+c+q}{c} \right) \right] \right). \end{aligned} \quad (3.24)$$

Lorenz and Bonferroni Curves

Lorenz and Bonferroni curves are applicable to economics for the study of income and poverty, and are also usefull in other areas such as reliability,

demography, insurance and medicine. Bonferroni and Lorenz curves for the GELLLoG distribution are given as

$$B(p) = \frac{1}{p\mu} \int_0^q x f_{GELLLoG}(x) dx = \frac{1}{p\mu} [\mu - T(q)],$$

and

$$L(p) = \frac{1}{\mu} \int_0^q x f_{GELLLoG}(x) dx = \frac{1}{\mu} [\mu - T(q)],$$

respectively, where $T(q) = \int_q^\infty x f_{GELLLoG}(x) dx$ is given by equation (3.24), $q = F_{GELLLoG}^{-1}(p)$, $0 \leq p \leq 1$.

3.3 Order Statistics and Rényi Entropy

Order statistics play an important role in probability and statistics, particularly in reliability and lifetime data analysis. The concept of entropy plays a vital role in information theory. In this section, we present the distribution of the i^{th} order statistics and Rényi entropy for the GELLLoG distribution.

3.3.1 Order Statistics

In this subsection, the pdf of the i^{th} order statistic and the corresponding moments are presented. Let $X_1, X_2, \dots, X_n$ be independent and identically distributed GELLLoG random variables. Using the binomial expansion $(1 - G_{GELLLoG}(x))^{n-i} = \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j [G_{GELLLoG}(x)]^j$, the pdf of the i^{th} order statistic from the GELLLoG pdf $f_{GELLLoG}(x)$ can be written as

$$\begin{aligned} f_{i:n}(x) &= \frac{n! f_{GELLLoG}(x)}{(i-1)!(n-i)!} [F_{GELLLoG}(x)]^{i-1} [1 - F_{GELLLoG}(x)]^{n-i} \\ &= \frac{n! f_{GELLLoG}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \\ &\quad \times \left[\frac{\gamma \left(-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right), \delta \right)}{\Gamma(\delta)} \right]^{i+j-1}. \end{aligned} \quad (3.25)$$

Now, let $0 < y = \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha < 1$, $x > 0$, $c, \lambda, \alpha > 0$. Using the fact that $\gamma(x, \delta) = \sum_{m=0}^\infty \frac{(-1)^m x^{m+\delta}}{(m+\delta)m!}$, and setting $c_m = (-1)^m / ((m+\delta)m!)$, we can write the

pdf of the i^{th} order statistic from the GELLoG distribution as follows:

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!f_{GELLoG}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(-1)^j}{[\Gamma(\delta)]^{i+j-1}} \\
&\quad \times \left[-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right) \right]^{\delta(i+j-1)} \\
&\quad \times \left[\sum_{m=0}^{\infty} \frac{(-1)^m \left(\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right) \right)^m}{(m+\delta)m!} \right]^{i+j-1} \\
&= \frac{n!f_{GELLoG}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(-1)^j}{[\Gamma(\delta)]^{i+j-1}} \\
&\quad \times \left[-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right) \right]^{\delta(i+j-1)} \\
&\quad \times \sum_{m=0}^{\infty} d_{m,i+j-1} \left[-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right) \right]^m, \quad (3.26)
\end{aligned}$$

where $d_0 = c_0^{(i+j-1)}$, $d_{m,i+j-1} = (mc_0)^{-1} \sum_{l=1}^m [(i+j-1)l - m + l] c_l d_{m-l,i+j-1}$. We note that the pdf of the i^{th} order statistic from the GELLLOG distribution can be written as

$$\begin{aligned}
f_{i:n}(x) &= \frac{n! f_{GELLLoG}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\
&\quad \times \left[-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right) \right]^{\delta(i+j-1)+m} \\
&= \frac{n! \left[-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right) \right]^{\delta-1}}{(i-1)!(n-i)! \Gamma(\delta)} \alpha \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^{\alpha-1} \\
&\quad \times \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \right] \\
&\quad \times \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\
&\quad \times \left[-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right) \right]^{\delta(i+j-1)+m} \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\
&\quad \times \frac{\Gamma(\delta(i+j-1)+m+\delta)}{\Gamma(\delta(i+j-1)+m+\delta)} \frac{\left[-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right) \right]^{\delta(i+j-1)+m+\delta-1}}{\Gamma(\delta)} \\
&\quad \times \alpha \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^{\alpha-1} \\
&\quad \times \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \right] \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \\
&\quad \times \frac{(-1)^j d_{m,i+j-1} \Gamma(\delta(i+j-1)+m+\delta)}{[\Gamma(\delta)]^{i+j}} f_{GELLLoG}(x),
\end{aligned}$$

where

$$\begin{aligned}
f_{GELLLoG}(x) &= \frac{\left[-\log \left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right)^\alpha \right) \right]^{\delta(i+j-1)+m+\delta-1}}{\Gamma(\delta(i+j-1)+m+\delta)} \\
&\quad \times \alpha \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^{\alpha-1} \\
&\quad \times \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \right] \quad (3.27)
\end{aligned}$$

is the GELLLoG pdf with parameters $c, \lambda, \alpha > 0$, and shape parameter $\delta^* = \delta(i+j) + m > 0$. It follows therefore that the t^{th} moment of the i^{th} order statistic from the GELLLoG density is given by

$$E(X_{i:n}^t) = \sum_{\nu \in D} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \omega_\nu \ell_{i,j,m} E(X^t), \quad (3.28)$$

where $E(X^t)$ is the t^{th} moment of the GELLLoG distribution given by (3.21) with the parameters c, α, λ and $\delta(i+j) + m > 0$,

$$\ell_{i,j,m} = \frac{n!}{(i-1)!(n-i)!} \frac{(-1)^j d_{m,i+j-1} \Gamma(\delta(i+j) + m)}{[\Gamma(\delta)]^{i+j}}.$$

We note that these moments are often used in several areas including reliability, survival analysis, biometry, engineering, insurance and quality control for the prediction of future failures times from a set of past or previous failures.

3.3.2 Rényi Entropy

Rényi entropy (Rényi [70]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f_{GELLLoG}(x; c, \alpha, \lambda, \delta)]^v dx \right), v \neq 1, v > 0. \quad (3.29)$$

Rényi entropy tends to Shannon entropy as $v \rightarrow 1$. Note that

$$\begin{aligned} \int_0^\infty f_{GELLLoG}^v(x) dx &= \left(\frac{1}{\Gamma(\delta)} \right)^v \int_0^\infty \left[-\log \left(1 - \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^\alpha \right) \right]^{v(\delta-1)} \\ &\quad \times \alpha^v \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^{v(\alpha-1)} \\ &\quad \times \frac{(1+x^c)^{-v}}{(1+\lambda)^v} e^{-\lambda v x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \right]^v dx. \end{aligned} \quad (3.30)$$

Let $0 < y = \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^\alpha < 1$. Note that

$$\begin{aligned} \left(\lambda^2(1+x) + (1+\lambda+\lambda x)(1+x^c)^{-1}cx^{c-1} \right)^v &= \sum_{p=0}^\infty \binom{v}{p} \lambda^{2(v-p)} (1+x)^{v-p} c^p x^{cp-p} \\ &\quad \times \frac{(1+\lambda+\lambda x)^p}{(1+x^c)^p}, \end{aligned}$$

and

$$\left[-\log \left(1 - \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} \right]^\alpha \right) \right]^{v(\delta-1)} = \sum_{m,s=0}^\infty \binom{v\delta-v}{m} d_{s,m} y^{m+s+\delta-1},$$

by applying the result on power series raised to a positive integer, with $c_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^\infty c_s y^s \right)^m = \sum_{s=0}^\infty d_{s,m} y^s, \quad (3.31)$$

where $d_{s,m} = (sc_0)^{-1} \sum_{l=1}^s [m(l+1) - s] c_l d_{s-l,m}$, and $d_{0,m} = c_0^m$, (Gradshteyn and Ryzhik [30]), so that

$$\begin{aligned} \int_0^\infty f_{GELLLoG}^v(x) dx &= \left(\frac{1}{\Gamma(\delta)} \right)^v \sum_{m,s,p,k,q,t,w=0}^\infty d_{s,m} \binom{v\delta-v}{m} \binom{v}{p} \binom{v-p}{t} \binom{k+p}{w} \\ &\times \frac{\Gamma(\alpha(m+s+\delta+v-1) - v + 1)}{\Gamma(\alpha(m+s+\delta+v-1) - v + 1 - k)k!} \\ &\times \frac{c^p \lambda^{2(v-p)+w} (-1)^q [\lambda(k+v)]^q}{q!(1+\lambda)^{v-p+w}} \\ &\times \int_0^\infty x^{cp-p+q+w+t} (1+x^c)^{-v-k-p} dx. \end{aligned}$$

Now, with $y = (1+x^c)^{-1}$, Rényi entropy for the GELLLoG distribution reduces to

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left[\left(\frac{1}{c\Gamma(\delta)} \right)^v \sum_{m,s,p,k,q,t,w=0}^\infty d_{s,m} \binom{v\delta-v}{m} \binom{v}{p} \binom{v-p}{t} \binom{k+p}{w} \right. \\ &\times \frac{\Gamma(\alpha(m+s+\delta+v-1) - v + 1)}{\Gamma(\alpha(m+s+\delta+v-1) - v + 1 - k)k!} \frac{c^p \lambda^{2(v-p)+w} (-1)^q [\lambda(k+v)]^q}{q!(1+\lambda)^{v-p+w}} \\ &\left. \times \mathcal{B} \left(v+k+p - \frac{cp+q+w+t-p+1}{c}, \frac{cp+q+w+t-p+1}{c} \right) \right], \end{aligned}$$

for $v > 0$, $v \neq 1$, where $\mathcal{B}(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1}$ is the beta function.

3.4 Maximum Likelihood Estimation

Let $X \sim GELLLoG(c, \alpha, \lambda, \delta)$ and $\Delta = (c, \alpha, \lambda, \delta)^T$ be the parameter vector. The log-likelihood $\ell_n = \ell_n(\Delta)$ based on a random sample of size n from the GLLoGW distribution is given by

$$\begin{aligned} \ell_n(\Delta) &= -n \ln \Gamma(\delta) + (\delta-1) \sum_{i=1}^n \ln \left[-\ln \left(1 - \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)} \right]^\alpha \right) \right] \\ &+ n \ln(\alpha) + (\alpha-1) \sum_{i=1}^n \ln \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)} \right] - \sum_{i=1}^n \ln(1+x_i^c) \\ &- n \ln(1+\lambda) - \sum_{i=1}^n \lambda x_i + \sum_{i=1}^n \ln \left[\lambda^2 (1+x_i) + \frac{(1+\lambda+\lambda x_i) c x_i^{c-1}}{(1+x_i^c)} \right] \quad (3.32) \end{aligned}$$

The first derivative of the log-likelihood function with respect to each component of the parameter vector $\Delta = (c, \alpha, \lambda, \delta)^T$ can be readily obtained. The equations obtained by setting the partial derivatives to zero are not in closed form and the

values of the parameters c, α, λ , and δ must be found by using iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial c}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \lambda}, \frac{\partial \ell_n}{\partial \delta})^T = 0$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $I(\Delta) = [I_{\theta_i, \theta_j}]_{4 \times 4} = E(-\frac{\partial^2 \ell_n}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4$ can be numerically obtained by MATLAB, SAS or R software. The total Fisher information matrix $nI(\Delta)$ can be approximated by

$$J_n(\hat{\Delta}) \approx \left[-\frac{\partial^2 \ell_n}{\partial \theta_i \partial \theta_j} \Big|_{\Delta=\hat{\Delta}} \right]_{4 \times 4}, \quad i, j = 1, 2, 3, 4. \quad (3.33)$$

For a given set of observations, the matrix given in equation (3.33) is obtained after the convergence of the Newton-Raphson procedure. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\Delta} = (\hat{c}, \hat{\alpha}, \hat{\lambda}, \hat{\delta})$ be the maximum likelihood estimate of $\Delta = (c, \alpha, \lambda, \delta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} N_4(\underline{0}, I^{-1}(\Delta))$, where $I(\Delta)$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\Delta)$ is replaced by the observed information matrix evaluated at $\hat{\Delta}$, that is $J(\hat{\Delta})$. The multivariate normal distribution $N_4(\underline{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\underline{0} = (0, 0, 0, 0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1 - \eta)\%$ two-sided confidence intervals for c, α, λ and δ are given by:

$$\hat{c} \pm Z_{\frac{\eta}{2}} \sqrt{I_{cc}^{-1}(\hat{\Delta})}, \quad \hat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\Delta})}, \quad \hat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\Delta})}, \quad \hat{\delta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\delta\delta}^{-1}(\hat{\Delta})},$$

respectively, where $I_{cc}^{-1}(\hat{\Delta})$, $I_{\alpha\alpha}^{-1}(\hat{\Delta})$, $I_{\lambda\lambda}^{-1}(\hat{\Delta})$, and $I_{\delta\delta}^{-1}(\hat{\Delta})$ are the diagonal elements of $I_n^{-1}(\hat{\Delta}) = (nI(\hat{\Delta}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

We maximize the likelihood function using NLMixed in SAS as well as the function nlm in R (R Development Core Team [69]). These functions were applied and executed for wide range of initial values. This process often results or lead to more than one maximum, however, in these cases, we take the MLEs corresponding to the largest value of the maxima. In a few cases, no maximum was identified for the selected initial values. In these cases, a new initial value

was tried in order to obtain a maximum. The issues of existence and uniqueness of the MLEs are of theoretical interest and has been studied by several authors for different distributions including (Seregin [73]), (Santos Silva and Tenreiro [72]), (Zhou [89]), and (Xia et al. [85]). At this point we are not able to address the theoretical aspects (existence, uniqueness) of the MLE of the parameters of the GELLLoG distribution.

The maximum likelihood estimates (MLEs) of the GELLLoG parameters c , α , λ , and δ are computed by maximizing the objective function via the subroutine NLmixed in SAS and the function nlm in R. The estimated values of the parameters (standard error in parenthesis), $-2\log$ -likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), and Consistent Akaike Information Criterion ($AICC = AIC + 2\frac{p(p+1)}{n-p-1}$), where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented. In order to compare the models, we use the criteria stated above. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred, and for AIC, AICC and BIC, smaller values are preferred. The GELLLoG distribution is fitted to the data sets and these fits are compared to the fits of the nested gamma exponentiated log-logistic (GELLoG), Lindley-log-logistic (LLLoG), and log-logistic distributions (LLoG), and several non-nested distributions given in section 7.

The likelihood ratio (LR) test is applied to compare the fit of the GELLLoG distribution with its sub-models for a given data set. For example, to test $\delta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{c}, \hat{\alpha}, \hat{\lambda}, \hat{\delta})) - \ln(L(\tilde{c}, \tilde{\alpha}, \tilde{\lambda}, 1))]$, where $\hat{c}$, $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\delta}$ are the unrestricted estimates, and $\tilde{c}$, $\tilde{\alpha}$, and $\tilde{\lambda}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi_{\epsilon}^2$, where χ_{ϵ}^2 denote the upper 100 ϵ % point of the χ^2 distribution with 1 degree of freedom.

3.5 Simulation Study

In this section, we examine the performance of the GELLLoG distribution by conducting various simulations for different sizes ($n=25, 50, 100, 200, 400, 800$) via the R package. We simulate $N = 2000$ samples for the true parameters values given in the Table 3.3. The table lists the mean MLEs of the four model parameters along with the respective root mean squared errors (RMSEs). From the results, we can verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero. The bias and RMSE for the estimated parameter $\hat{\theta}$, say, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^n \hat{\theta}_i}{n} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^n (\hat{\theta}_i - \theta)^2}{n}},$$

respectively.

Table 3.3: Monte Carlo Simulation Results

parameter	Sample Size	(1.0,2.5,1.0,2.0)			(1.5,2.0,1.0,2.0)			(2.0,1.0,3.0,1.0)		
		Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
λ	35	1.6155	1.3790	0.6155	2.0458	1.3208	0.5458	2.7723	1.4651	0.7723
	50	1.4916	1.1719	0.4916	1.9560	1.2516	0.4560	2.5528	1.2286	0.5528
	100	1.3178	0.9065	0.3178	1.8398	1.0126	0.3397	2.4102	0.9590	0.4102
	200	1.1820	0.6118	0.1820	1.71647	0.7932	0.2164	2.2497	0.6901	0.2497
	400	1.1122	0.4292	0.1122	1.6470	0.6060	0.1470	2.1503	0.5352	0.1503
	800	1.0550	0.2932	0.0550	1.5893	0.4428	0.0893	2.0678	0.4055	0.0678
c	35	2.0212	1.2877	-0.4787	1.6426	1.1469	-0.3573	1.1900	0.9087	0.1900
	50	2.0901	1.2225	-0.4098	1.6338	1.0348	-0.3661	1.1487	0.6768	0.1487
	100	2.1640	1.0116	-0.3351	1.6559	0.9061	-0.3440	1.0954	0.4732	0.0954
	200	2.2644	0.7613	-0.2355	1.6734	0.7595	-0.3265	1.0722	0.3848	0.0722
	400	2.3433	0.5544	-0.1566	1.7840	0.6140	-0.2159	1.0628	0.3169	0.0628
	800	2.4305	0.3781	-0.0694	1.8432	0.4621	-0.1567	1.0459	0.2826	0.0459
α	35	2.0618	2.4833	1.0618	1.8239	2.0295	0.8239	2.4286	2.1427	-0.5713
	50	1.9298	2.3119	0.9298	1.7459	1.7429	0.7459	2.5856	2.1084	-0.4143
	100	1.8306	2.0975	0.8306	1.6569	1.6916	0.6569	2.6287	1.9137	-0.3712
	200	1.6500	1.7575	0.6500	1.6298	1.5404	0.6298	2.7726	1.6800	-0.2273
	400	1.4112	1.4684	0.4112	1.3464	1.0195	0.3464	2.9444	1.6574	-0.0555
	800	1.1481	0.7029	0.1481	1.2392	0.9252	0.2392	3.0005	1.3830	0.0005
δ	35	2.6765	2.1522	0.6765	2.5907	2.0466	0.5907	2.1088	2.0959	1.1088
	50	2.5350	1.7651	0.5350	2.4733	1.9415	0.4733	1.8387	1.7348	0.8387
	100	2.2934	1.4306	0.2934	2.3002	1.5411	0.3002	1.6543	1.3991	0.6543
	200	2.1168	1.0218	0.1168	2.0839	1.1625	0.0839	1.4012	0.9425	0.4012
	400	2.0405	0.7137	0.0405	2.0596	0.8639	0.0596	1.2569	0.7230	0.2569
	800	2.0349	0.4869	0.0349	2.0282	0.6477	0.0282	1.1451	0.5387	0.1451

3.6 Applications

In this section, examples to illustrate the flexibility and usefulness of the GELLLoG distribution and its sub-models for data modeling are presented. We also compare GELLLoG distribution to the non-nested new modified Weibull (NMW) distribution introduced by (Doostmoradi et al. [22]), a four parameter beta generalized exponential (BGE) distribution introduced by (Barreto-Souza et al. [7]), beta generalized Lindley (BGL) distribution by (Oluyede and Yang [64]) and exponentiated modified Weibull distribution by (Elbatal [23]). The pdf of four parameter NMW, BGE, BGL and EMW distributions are given in equation (3.34), (3.35), (3.36) and (3.37), respectively, that is,

$$g_{NMW}(x) = \left(\alpha \gamma x^{\gamma-1} e^{\alpha x^\gamma} + \lambda \beta x^{\lambda-1} e^{-\beta x^\lambda} \right) e^{-e^{\alpha x^\gamma} + e^{-\beta x^\lambda}}, \quad x > 0, \quad (3.34)$$

$$g_{BGE}(x) = \frac{\alpha \lambda}{B(a, b)} e^{-\lambda x} \left(1 - e^{-\lambda x} \right)^{\alpha a - 1} \left(1 - \left(1 - e^{-\lambda x} \right)^\alpha \right)^{b-1}, \quad x > 0. \quad (3.35)$$

$$\begin{aligned} g_{BGL}(x) &= \frac{\alpha \lambda^2}{B(a, b)(1 + \lambda)} (1 + x) e^{-\lambda x} \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x} \right]^{\alpha a - 1} \\ &\quad \times \left[1 - \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x} \right)^a \right]^{b-1}, \quad x > 0, \end{aligned} \quad (3.36)$$

and

$$g_{EMW}(x) = \gamma \left[\delta + \lambda \theta^\alpha x^{\lambda-1} \right] e^{-(\delta x + (\theta x)^\lambda)} \left[1 - e^{-(\delta x + (\theta x)^\lambda)} \right]^{\delta-1}, \quad x > 0. \quad (3.37)$$

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [14]) are given in Figure 3.6 and Figure 3.7 for the two datasets considered in this section. For the probability plot, we plotted

$F_{GELLLoG}(x_{(j)}; \hat{c}, \hat{\alpha}, \hat{\lambda}, \hat{\delta})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares (SS)

$$SS = \sum_{j=1}^n \left[F_{GELLLoG}(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The goodness-of-fit statistics W^* and A^* , described by (Chen and Balakrishnan [16]) as well as Kolmogorov-Smirnov (KS) statistic, its P-value

and SS are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^* , the better the fit.

3.6.1 Lifetime data

(Gross and Clark [31]) presented the following data for lifetime data. The data are:

1.1,1.4,1.3,1.7,1.9,1.8,1.6,2.2,1.7,2.7,4.1,1.8,1.5,1.2,1.4,3,1.7,2.3,1.6,2.

Estimates of the parameters of GELLLoG distribution and its related sub-models (standard error in parentheses), AIC, BIC, and the goodness-of-fit statistics W^* , A^* , KS and its P-value as well as SS are given in Table 3.4. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 3.6.

Table 3.4: Estimates of Models for Lifetime Data

Model	Estimates				Statistics								
	λ	c	α	δ	$-2 \log L$	AIC	AICC	BIC	W^*	A^*	KS	P-value	SS
GELLLoG	0.1243 (1.6299)	5.1200 (6.8941)	3.4885 (18.2812)	2.0262 (7.8205)	30.8287	38.8287	41.4954	42.8117	0.0262	0.1512	0.0960	0.9928	0.0216
GELLoG	0 -	1.1287 (2.4417 $\times 10^{-01}$)	1.2996 $\times 10^{-01}$ (8.1022 $\times 10^{-05}$)	1.0504 (3.8729 $\times 10^{-03}$)	109.5000	115.4986	116.9986	118.4858	0.0623	0.3667	1.0000	2.2 $\times 10^{-16}$	6.6216
ELLLoG	2.3230 $\times 10^{-09}$ (1.4863 $\times 10^{-01}$)	2.3964 (4.4268 $\times 10^{-01}$)	0.5000 (3.1192 $\times 10^{-01}$)	1 -	63.0817	69.0818	70.5818	72.0690	0.0548	0.3218	0.7462	4.241 $\times 10^{-10}$	3.9409
LLLoG	4.7922 $\times 10^{-09}$ (1.5407 $\times 10^{-01}$)	2.4917 (4.4792 $\times 10^{-01}$)	1 -	1 -	65.6049	69.6049	70.3100	71.5964	0.0492	0.2878	0.5616	6.621 $\times 10^{-06}$	2.3313
LLoG	0 -	2.4916 (0.4479)	1 -	1 -	65.6049	67.6049	67.8271	68.6007	0.0492	0.2878	0.5616	6.6210 $\times 10^{-06}$	2.3313
	α	λ	a	b									
BGE	28.4888 (215.8902)	5.4032 (3.7126)	28.9646 (217.1584)	0.2951 (0.2620)	30.8779	38.8779	41.5446	42.8609	0.0348	0.1958	0.1012	0.9865	0.0262
	α	γ	λ	β									
NMW	0.0090 (0.0163)	3.4110 (1.2816)	6.6101 (1.4907)	0.0169 (0.0156)	37.8087	406.7058	409.3725	410.6888	1.1163	5.6637	0.6348	1.992 $\times 10^{-07}$	1.9879
	α	θ	a	b									
BGL	1.2026 $\times 10^{-01}$ (1.2389 $\times 10^{-02}$)	2.9054 $\times 10^{-07}$ (4.2756 $\times 10^{-06}$)	3.0161 $\times 10^{-01}$ (1.1499 $\times 10^{-03}$)	1.0100 $\times 10^{01}$ (3.5923 $\times 10^{-05}$)	162.3258	170.33	172.9967	174.313	0.0872	0.5162	0.4939	0.0001	1.5551
	γ	δ	λ	θ									
EMW	3.6683 $\times 10^{01}$ (2.5254 $\times 10^{01}$)	2.2352 (4.3602 $\times 10^{-01}$)	1.7001 $\times 10^{01}$ (2.6643 $\times 10^{-13}$)	1.0000 $\times 10^{-04}$ (4.5297 $\times 10^{-08}$)	32.5212	40.5212	43.1878	44.5041	0.0542	0.3184	0.1343	0.8633	0.0431

The Likelihood ratio (LR) test statistic for testing H_0 : GELLoG against H_a : GELLLoG, H_0 : LLLoG against H_a : GELLLoG and H_0 : ELLLoG against H_a : GELLLoG are 78.6712 (p-value < 0.0001), 34.7761 (p-value < 0.0001) and 32.2529 (p-value < 0.0001). We can conclude that there are significant differences between GELLoG and GELLLoG distributions, LLLoG and GELLLoG distributions as well between GELLLoG and ELLLoG distributions, respectively based on the LR tests at 5% level. The values of AIC and BIC are smallest for the GELLLoG distribution, when compared

to the corresponding values for the non-nested BGE, NMW, BGL and EMW distributions. The values of the goodness-of-fit-statistics W^* , A^* , KS and its p-value show that the GELLLoG distribution is the “best” fit for the lifetime data.

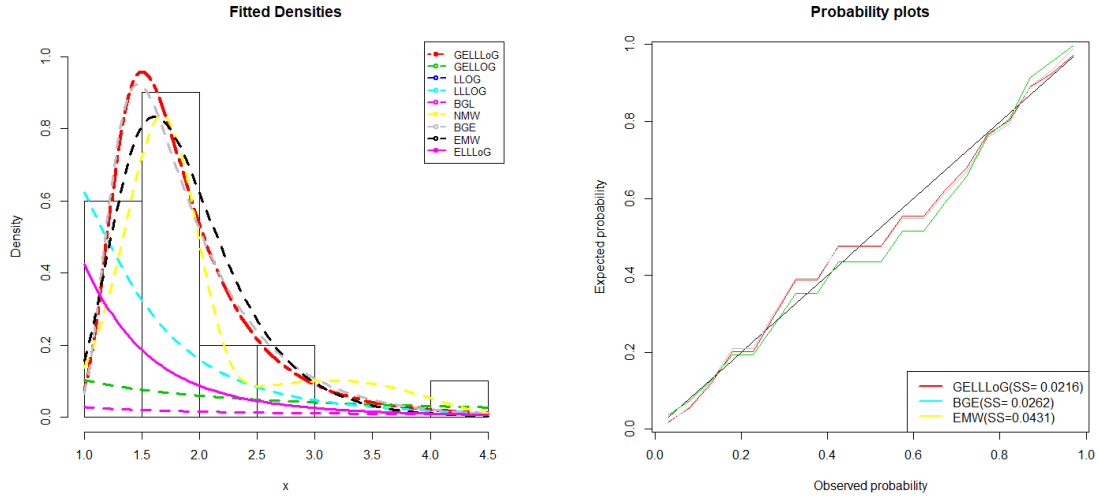


Figure 3.6: Fitted Densities and Probability Plots of the Lifetime Data

3.6.2 Repair lifetimes of an airborne transceiver

These data correspond to maintenance on active repair times (in hours) for an airborne communication transceiver with size $n=46$ from (Leiva et al. [38]) and (Chhikara and Folks [15]). These data are:

0.2,0.3,0.5,0.5,0.5,0.5,0.6,0.6,0.7,0.7,0.7,0.8,0.8,1.0,1.0,1.0,1.0,1.1,1.3,1.5,1.5,
1.5, 1.5,2.0,2.0,2.2,2.5,2.7,3.0,3.0,3.3,3.3,4.0,4.0,4.5,4.7,5.0,5.4,5.4,7.0,7.5,
8.8,9.0,10.3, 22.0,24.5.

Estimates of the parameters of GELLLoG distribution and its related sub-models (standard error in parentheses), AIC, BIC, W^* , A^* , KS and its P-value as well as SS are given in Table 3.5. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 3.7.

Table 3.5: Estimates of Models for repair lifetimes of an airborne transceiver Data

Model	Estimates				Statistics								
	λ	c	α	δ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
GELLLoG	0.0909 (0.0754)	1.2382 (0.3801)	1.8733 (3.2566)	1.0064 (1.3790)	199.7542	207.7542	208.7298	215.0688	0.0489	0.3168	0.0929	0.8216	0.0546
GELLoG	0 -	1.1267 (1.5154 $\times 10^{-01}$)	1.4823 $\times 10^{-04}$ (6.2240 $\times 10^{-05}$)	1.0504 $\times 10^{01}$ (3.1145 $\times 10^{-03}$)	266.6504	272.6513	273.2227	278.1372	0.0676	0.4027	1.0000	2.2 $\times 10^{-16}$	15.2901
ELLLoG	0.0454 (0.0576)	1.3410 (0.1853)	0.5000 (0.1417)	1 -	202.0737	208.0737	209.3084	214.2229	0.0593	0.3655	0.4893	5.419 $\times 10^{-10}$	4.5084
LLLoG	0.0525 (0.0591)	1.3205 (0.1826)	1 -	1 -	214.5014	218.5014	218.7805	222.1587	0.0564	0.3441	0.2435	0.0085	1.1386
LLoG	0 -	1.3643 (0.1670)	1 -	1 -	214.9587	216.9587	217.0496	218.7873	0.0624	0.3725	0.2362	0.0117	1.1276
	α	λ	a	b									
BGE	10.9759 (20.9289)	1.2799 (0.7340)	0.1848 (0.3193)	0.1855 (0.0620)	201.8082	209.8083	210.7839	217.1229	0.0563	0.4558	0.1109	0.6228	0.0890
	α	γ	λ	β									
NMW	0.1280 (0.0680)	0.3343 (0.2015)	1.2957 (0.1966)	0.1810 (0.0619)	237.2661	245.2661	246.2417	252.5807	0.1341	0.8364	0.2340	0.0129	0.6827
	α	θ	a	b									
BGL	1.1793 $\times 10^{-01}$ (7.3103 $\times 10^{-03}$)	1.7823 $\times 10^{-06}$ (2.8513 $\times 10^{-06}$)	3.0140 $\times 10^{-01}$ (4.6377 $\times 10^{-04}$)	1.0100 $\times 10^{01}$ (2.5191 $\times 10^{-05}$)	386.7372	394.7331	395.7087	402.0476	0.0769	0.4980	0.5073	1.0370 $\times 10^{-10}$	3.3521
	γ	δ	λ	θ									
EMW	9.5828 $\times 10^{-01}$ (1.8975 $\times 10^{-01}$)	2.6937 $\times 10^{-01}$ (5.4358 $\times 10^{-02}$)	1.7001 $\times 10^{01}$ (5.0648 $\times 10^{-19}$)	1.0000 $\times 10^{-04}$ (8.6130 $\times 10^{-14}$)	209.9658	217.9658	218.9414	225.2804	0.1441	1.0004	0.1519	0.2385	0.1889

The LR test statistic for testing H_0 : GELLoG against H_a : GELLLoG, H_0 : LLLoG against H_a : GELLLoG and H_0 : ELLLoG against H_a : GELLLoG are 66.8962 (p-value < 0.00001), 14.2598 (p-value < 0.000801) and 2.3195 (p-value=0.1277). We can conclude that there are significant differences between GELLoG and GELLLoG distributions, as well as between LLLoG and GELLLoG distributions, respectively based on the LR tests. There is no significant difference between GELLLoG and ELLLoG distributions based on the LR test. The GELLLoG distribution is significantly better than the sub-models considered above. The values of the statistics: AIC, AICC, and BIC are smallest for the GELLLoG distribution. Also, the goodness-of-fit statistics W^* and A^* are the smallest and definitely points to the GELLLoG distribution as the “best” fit for the Repair lifetimes of an airborne transceiver data when compared to the corresponding values for the sub-models. The goodness-of-fit statistics W^* and A^* are also better for the GELLLoG distribution when compared to the values for the non-nested BGE, MMW, BGL and EMW distributions. Thus, there is indeed convincing evidence that the GELLLoG distribution is the “best” fit for the repair lifetimes of an airborne transceiver data.

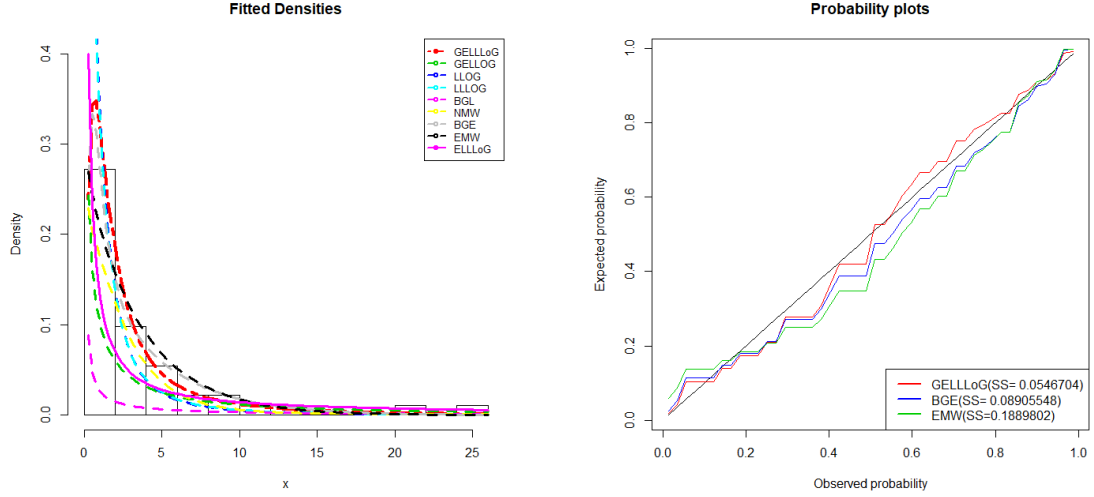


Figure 3.7: Fitted Densities and Probability Plots of the Repair Lifetimes of an Airborne Transceiver Data

3.7 Concluding Remarks

A new generalized distribution called the gamma exponentiated Lindley log-logistic (GELLLoG) distribution is presented. The GELLLoG distribution has several new and known distributions as special cases or sub-models. The density of this new distribution can be expressed as a linear combination of ELLLoG density functions. The GELLLoG distribution possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, mean and median deviations, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. The performance of the GELLLoG distribution was examined by conducting Monte Carlo simulations for different sizes. Finally, the GELLLoG distribution is fitted to real data sets to illustrate the applicability and usefulness of the new generalized distribution.

Chapter 4

Marshall-Olkin

Lindley-Log-logistic Distribution (MOLLLoG)

In this chapter, the Marshall-Olkin transformation is employed to obtain a new distribution involving both the Lindley and log-logistic distributions that is more flexible distribution for describing and modeling reliability and lifetime data. This transformation generalizes the Lindley and log-logistic distributions.

A motivation for developing this model is the advantages presented by this extended distribution with respect to having a hazard function that exhibits increasing, decreasing and bathtub shapes, as well as the versatility and flexibility of the Lindley and log-logistic distributions in modeling lifetime data.

4.1 MOLLLoG Distribution, Expansion of Density Function, Sub-models, Hazard and Quantile Functions

In this section, the MOLLLoG distribution, series expansion of its pdf, some sub-models, quantile function, hazard function as well some graphs are presented. The Marshall-Olkin (Marshall and Olkin [47]) class of distributions

is given by

$$F(x; \delta) = 1 - \frac{\delta \bar{G}(x)}{1 - \delta \bar{G}(x)}, \quad (4.1)$$

for $\delta > 0$ (tilt parameter), $\bar{\delta} = 1 - \delta$ and $\bar{G}(x) = 1 - G(x)$, where $G(x)$ is the baseline cdf. Note that when $\delta = 1$, we have $F(x) = G(x)$, the baseline distribution function. The corresponding pdf is given by

$$f(x; \delta) = \frac{\delta g(x)}{(1 - \delta \bar{G}(x))^2}. \quad (4.2)$$

(Marshall and Olkin [47]) referred to the shape parameter δ as the tilt parameter due to the fact that for all $x > 0$, the hazard functions are ordered based on the values of δ , that is, $h_F(x; \delta) \leq h_G(x)$ when $\delta \geq 1$, and $h_F(x; \delta) \geq h_G(x)$ when $0 < \delta \leq 1$.

The motivation for Marshall-Olkin class of distribution is given in several papers listed above (see references therein), that is, let X_i , $i = 1, 2, \dots$ be independently and identically distributed as $G(x)$ and N a random variable with probability mass function $P(N = n) = \delta(1 - \delta^{n-1})$, $n = 1, 2, \dots$, then the distribution of $X = \min(X_1, \dots, X_N)$ is equivalent to (5.1). That is, the cdf of X is given by

$$P(X \leq x) = 1 - \sum_{n=1}^{\infty} P(X \geq x | N = n) P(N = n) = \frac{G(x)}{G(x) + \delta \bar{G}(x)}. \quad (4.3)$$

Consider the following motivation for the MOLLLoG distribution. Suppose we have a parallel system with N independent components and the random variable N has the geometric distribution with the probability mass function $P(N = n) = \delta^{-1}(1 - \delta^{-1})^{n-1}$, $\delta > 1$, and $n = 1, 2, \dots$. Let $X_1, X_2, \dots, X_N$ represent the lifetimes of each component with each X_i having the Lindley log-logistic (LLLoG) distribution with the cdf given by

$$G(x; \lambda, c) = 1 - \frac{1 + \lambda + \lambda x}{(1 + \lambda)(1 + x^c)} \exp(-\lambda x), \quad (4.4)$$

for $\lambda, c > 0$ and $x > 0$. Then a random variable $X = \min(X_1, \dots, X_N)$ is the lifetime of the system and the distribution of X is given by equation (4.1), where $G(x)$ is the cdf of the Lindley log-logistic (LLLoG) distribution (Oluyede et al. [63]).

The cdf and pdf the MOLLLoG distribution are given by

$$F(x; \lambda, c, \delta) = 1 - \frac{\delta \left(\frac{1 + \lambda + \lambda x}{(1 + \lambda)(1 + x^c)} \right) \exp(-\lambda x)}{1 - \bar{\delta} \left(\frac{1 + \lambda + \lambda x}{(1 + \lambda)(1 + x^c)} \right) \exp(-\lambda x)}, \quad (4.5)$$

and

$$f(x; \lambda, c, \delta) = \frac{\frac{\delta e^{-\lambda x}}{(1+\lambda)(1+x^c)} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \right]}{(1 - \bar{\delta} \frac{1+\lambda+\lambda x}{(1+\lambda)(1+x^c)} e^{-\lambda x})^2}, \quad (4.6)$$

respectively, for $\lambda, c, \delta > 0$. If a random variable X has the Marshall-Olkin Log-logistic (MOLLLoG) density, we write $X \sim MOLLLoG(\lambda, c, \delta)$. Plots of the MOLLLoG pdf shows different shapes including right skewed, left skewed, almost symmetric, increasing and reverse J shape.

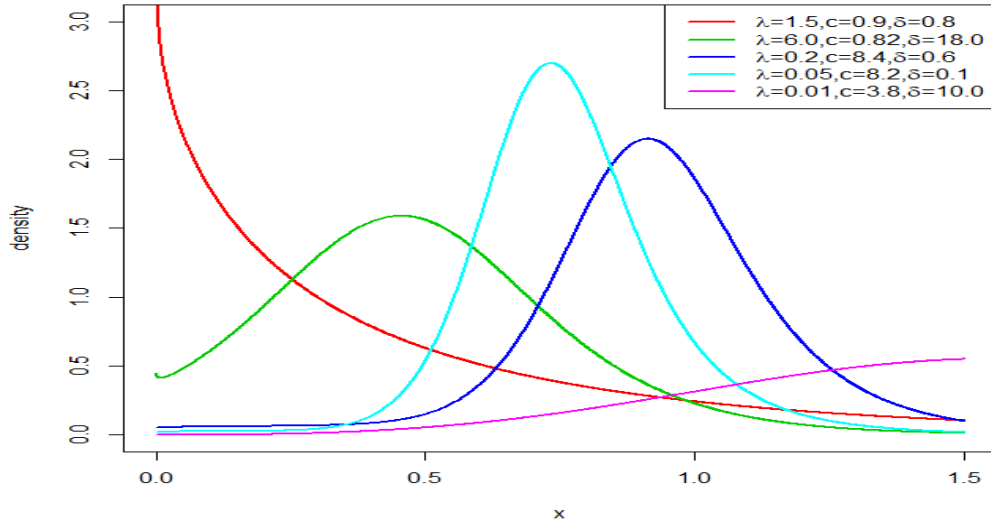


Figure 4.1: Plots of MOLLLoG Density Function

4.1.1 Expansion of Density Function

Presented below is an expansion of the MOLLLoG density function. Note that

$$\begin{aligned} f(x; \lambda, c, \delta) &= \frac{\delta g(x)}{(1 - \bar{\delta} G(x))^2} \\ &= \sum_{j,k=0}^{\infty} \frac{\Gamma(2+j)}{\Gamma(2)j!} \binom{j}{k} (-1)^k \frac{\delta \bar{\delta}^j}{k+1} (k+1) [G(x)]^{k+1-1} g(x) \\ &= \sum_{j,k=0}^{\infty} \frac{\Gamma(2+j)}{\Gamma(2)j!} \binom{j}{k} (-1)^k \frac{\delta \bar{\delta}^j}{k+1} g_{ELLLoG}(x; \lambda, c, k+1), \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} g_{ELLLoG}(x; \lambda, c, k+1) &= \left(1 - \frac{1+\lambda+\lambda x}{(1+\lambda)(1+x^c)} e^{-\lambda x} \right)^{k+1-1} \\ &\quad \times \frac{(k+1)e^{-\lambda x}}{(1+\lambda)(1+x^c)} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c} \right] \end{aligned}$$

is the exponentiated Lindley-Log-logistic (ELLLoG) density function with parameters $\lambda, c, k + 1 > 0$. Thus, the MOLLLoG density function can be written as an infinite linear combination of the ELLLoG density functions. Consequently, the mathematical and statistical properties of the MOLLLoG distribution follows directly from those of the ELLLoG distribution.

4.1.2 Sub-models of MOLLLoG Distribution

In this subsection, some Sub-models of the MOLLLoG distributions are discussed.

- We obtain the parent Lindley-Log-logistic (LLLoG) distribution when $\delta = 1$.
- When $\lambda \rightarrow 0$, we obtain the Marshall-Olkin Log-logistic (MOLLoG) distribution.
- When $\lambda \rightarrow 0$, and $\delta = 1$, we obtain Log-logistic (LLoG) distribution.
- When $c = 1$, the Marshall-Olkin Lindley Log-logistic distribution reduces to the 2 parameter distribution with survival function

$$\bar{F}(x; \lambda, \delta) = \frac{\delta(1 + \lambda + \lambda x)e^{-\lambda x}}{(1 + \lambda)(1 + x) - (1 - \delta)(1 + \lambda + \lambda x)e^{-\lambda x}}, \quad (4.8)$$

for $\lambda, \delta > 0$.

- When $\lambda \rightarrow 0$, and $c=1$, the MOLLLoG distribution reduces to a one parameter distribution with the pdf

$$f(x; \delta) = \frac{\frac{\delta}{(1+x)^{-2}}}{\left(1 - \frac{\delta}{(1+x)}\right)^2}, \quad (4.9)$$

for $\delta > 0$.

- When $c = \delta = 1$, the MOLLLoG distribution reduces to a one parameter distribution with the pdf

$$f(x; \lambda) = \frac{e^{-\lambda x}}{(1 + \lambda)(1 + x)} \left(\lambda^2(1 + x) + \frac{(1 + \lambda + \lambda x)}{1 + x} \right), \quad (4.10)$$

for $\lambda > 0$.

4.1.3 Hazard and Quantile Functions

In this section, we present the hazard and quantile functions of the MOLLLoG distribution. The hazard function is given by

$$h_F(x) = \frac{f(x)}{\bar{F}(x)} = \frac{\frac{\lambda^2(1+x)}{1+\lambda+\lambda x} + (1+x^c)^{-1}cx^{c-1}}{\left(1 - \bar{\delta} \frac{1+\lambda+\lambda x}{(1+\lambda)(1+x^c)} e^{-\lambda x}\right)}, \quad (4.11)$$

for $\lambda, c, \delta > 0$, and $\bar{\delta} = 1 - \delta$. Plots of the hazard function of the MOLLLoG distribution show different shapes including decreasing, increasing, bathtub and upside down bathtub.

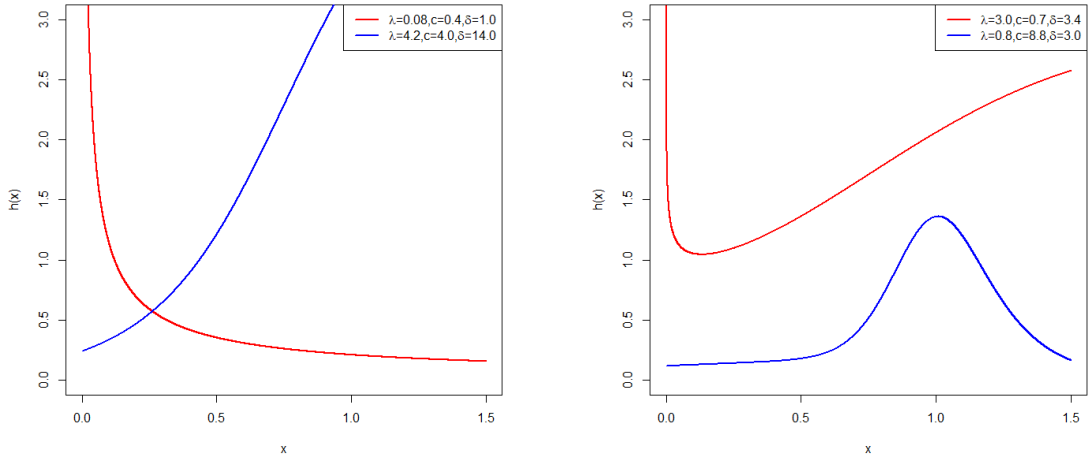


Figure 4.2: Plots of MOLLLoG Hazard Function

The quantile function of the MOLLLoG distribution is obtained by solving the non-linear equation:

$$F(x; \lambda, c, \delta) = 1 - \frac{\delta \left(\frac{1+\lambda+\lambda x}{(1+\lambda)(1+x^c)} \right) \exp(-\lambda x)}{1 - \bar{\delta} \left(\frac{1+\lambda+\lambda x}{(1+\lambda)(1+x^c)} \right) \exp(-\lambda x)} = u, \quad (4.12)$$

$0 \leq u \leq 1$, that is,

$$\lambda x + \ln(1+x^c) - \ln \left(1 + \frac{\lambda x}{1+\lambda} \right) + \ln \left(\frac{1-u}{\delta + (1-u)\bar{\delta}} \right) = 0. \quad (4.13)$$

Therefore, random number can be generated from the MOLLLoG distribution by numerically solving the non-linear equation (4.13). Quantiles of the MOLLLoG distribution for selected parameter values are given in Table 4.1.

Table 4.1: Table of Quantile for MOLLLoG Distribution

u	(λ, c, δ)				
	(0.2, 1.2, 1)	(0.4, 1.9, 0.5)	(0.6, 0.8, 1.6)	(0.8, 1.2, 1)	(0.3, 1.3, 2)
0.1	29.9665	1.2018	4.2442	3.3455	19.1755
0.2	30.8455	1.2011	4.7748	3.6850	20.2345
0.3	31.8308	1.2021	5.3129	4.0393	21.3069
0.4	32.9500	1.1897	5.8717	4.4221	22.4296
0.5	34.2588	1.1895	6.4741	4.8434	23.6380
0.6	35.8316	1.1900	7.1480	5.3342	24.9926
0.7	37.8228	1.2028	7.9477	5.9267	26.5964
0.8	40.5696	1.1998	8.9888	6.7180	28.6778
0.9	45.1362	1.2000	10.6175	7.9861	31.9452

4.2 Moments, Conditional Moments and Mean Deviations

In this section, the r^{th} moment, conditional moments, mean deviations, Lorenz and Bonferroni curves of the MOLLLoG distribution are presented. Let $Y \sim ELLLoG(\lambda, c, k + 1)$. Note that the r^{th} moment of the ELLLoG distribution is given by

$$\begin{aligned}
E(Y^r) &= \int_0^\infty y^r g_{ELLLoG}(y; \lambda, c, k + 1) dy \\
&= \int_0^\infty y^r (k + 1) \left[1 - \frac{1 + \lambda + \lambda y}{1 + \lambda} \frac{e^{-\lambda y}}{(1 + y^c)} \right]^{(k+1)-1} \\
&\quad \times \frac{(1 + y^c)^{-1}}{1 + \lambda} e^{-\lambda y} \left[\lambda^2(1 + y) + \frac{(1 + \lambda + \lambda y)cy^{c-1}}{1 + y^c} \right] dy \\
&= \sum_{s=0}^\infty \binom{(k+1)-1}{s} (-1)^s (k+1) \int_0^\infty y^r \left[\frac{1 + \lambda + \lambda y}{1 + \lambda} \frac{e^{-\lambda y}}{(1 + y^c)} \right]^s \\
&\quad \times \frac{(1 + y^c)^{-1}}{1 + \lambda} e^{-\lambda y} \left[\lambda^2(1 + y) + \frac{(1 + \lambda + \lambda y)cy^{c-1}}{1 + y^c} \right] dy \\
&= \sum_{s,p=0}^\infty \frac{(k+1)(-1)^{s+p} [\lambda(s+1)]^p}{(1 + \lambda)^{s+1} p!} \binom{(k+1)-1}{s} \\
&\quad \times \left[\lambda^2 \sum_{m=0}^\infty \binom{s}{m} \lambda^m (1 + \lambda)^{s-m} \int_0^\infty y^{r+p+m} (1 + y) (1 + y^c)^{-s-1} dy \right. \\
&\quad \left. + c \sum_{m=0}^\infty \binom{s+1}{m} \lambda^m (1 + \lambda)^{s+1-m} \int_0^\infty y^{r+p+c+m-1} (1 + y^c)^{-s-2} dy \right].
\end{aligned}$$

We note that by applying $(1 + \lambda + \lambda y)^{s+1} = \sum_{m=0}^{\infty} \binom{s+1}{m} (\lambda y)^m (1 + \lambda)^{s+1-m}$, $(1 + \lambda + \lambda y)^s = \sum_{m=0}^{\infty} \binom{s}{m} (\lambda y)^m (1 + \lambda)^{s-m}$, and the substitution $t = (1 + y^c)^{-1}$, we have

$$\begin{aligned} E(Y^r) &= \sum_{s,p=0}^{\infty} \frac{(k+1)(-1)^{s+p}[\lambda(s+1)]^p}{(1+\lambda)^{s+1}p!} \binom{(k+1)-1}{s} \\ &\quad \times \left[\sum_{m=0}^{\infty} \binom{s}{m} \lambda^{m+2} \frac{(1+\lambda)^{s-m}}{c} \int_0^1 t^{s+1-\frac{r+p+m+1}{c}-1} (1-t)^{\frac{r+p+m+1}{c}-1} dt \right. \\ &\quad + \int_0^1 t^{s+1-\frac{r+p+m+2}{c}-1} (1-t)^{\frac{r+p+m+2}{c}-1} dt \\ &\quad \left. + c \sum_{m=0}^{\infty} \binom{s+1}{m} \lambda^m (1+\lambda)^{s+1-m} \int_0^1 t^{s+2-\frac{r+p+c+m}{c}-1} (1-t)^{\frac{r+p+c+m}{c}-1} dt \right]. \end{aligned}$$

Thus, the r^{th} moment of the ELLLoG distribution reduces to

$$\begin{aligned} E(Y^r) &= \sum_{s,p=0}^{\infty} \frac{(k+1)(-1)^{s+p}[\lambda(s+1)]^p}{(1+\lambda)^{s+1}p!} \binom{(k+1)-1}{s} \\ &\quad \times \left[\sum_{m=0}^{\infty} \binom{s}{m} \lambda^{m+2} \frac{(1+\lambda)^{s-m}}{c} \left(B\left(s+1 - \frac{r+p+m+1}{c}, \frac{r+p+m+1}{c}\right) \right. \right. \\ &\quad \left. \left. + B\left(s+1 - \frac{r+p+m+2}{c}, \frac{r+p+m+2}{c}\right) \right) \right. \\ &\quad \left. + c \sum_{m=0}^{\infty} \binom{s+1}{m} \lambda^m (1+\lambda)^{s+1-m} \right. \\ &\quad \left. \times B\left(s+2 - \frac{r+p+c+m}{c}, \frac{r+p+c+m}{c}\right) \right]. \end{aligned} \quad (4.14)$$

Consequently, the r^{th} moment of the MOLLLoG distribution is given by

$$E(X^r) = \sum_{j,k=0}^{\infty} \frac{\Gamma(2+j)}{\Gamma(2)j!} \binom{j}{k} (-1)^k \frac{\delta \bar{\delta}^j}{k+1} E(Y^r), \quad (4.15)$$

where $E(Y^r)$ is given by equation (4.14). The moment generating function of the MOLLLoG distribution is given by $E(e^{tX}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k)$, where $E(X^k)$ is given by the equation (4.15). The variance (σ^2), standard deviation (SD= σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$

and

$$CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively. Some moments for selected parameters values are given in Table 4.2 and plots of CS and CK versus shape parameter c and δ are presented in Figure 4.3. Plots of skewness and kurtosis for choices of the model parameters reveal that skewness and kurtosis depend on the shape parameters c and δ .

Table 4.2: Table of Moments for Selected Parameters for MOLLLoG Distribution

	(0.4,0.8,0.2)	(0.1,0.7,1.2)	(1.0,0.5,1.2)	(0.3,1.0,0.9)	(0.4,1.0,0.5)
$E(X)$	0.1579	0.2718	0.1831	0.2327	0.2222
$E(X^2)$	0.0721	0.1629	0.1048	0.1260	0.1208
$E(X^3)$	0.0453	0.1143	0.0731	0.0842	0.0810
$E(X^4)$	0.0327	0.0874	0.0559	0.0628	0.0604
$E(X^5)$	0.0256	0.0705	0.0452	0.0499	0.0481
$E(X^6)$	0.0210	0.0590	0.0379	0.0413	0.0398
SD	0.0178	0.0507	0.0326	0.0352	0.0340
CV	0.0154	0.0444	0.0286	0.0307	0.0296
CS	0.0136	0.0395	0.0255	0.0272	0.0262
CK	0.0122	0.0355	0.0230	0.0244	0.0236

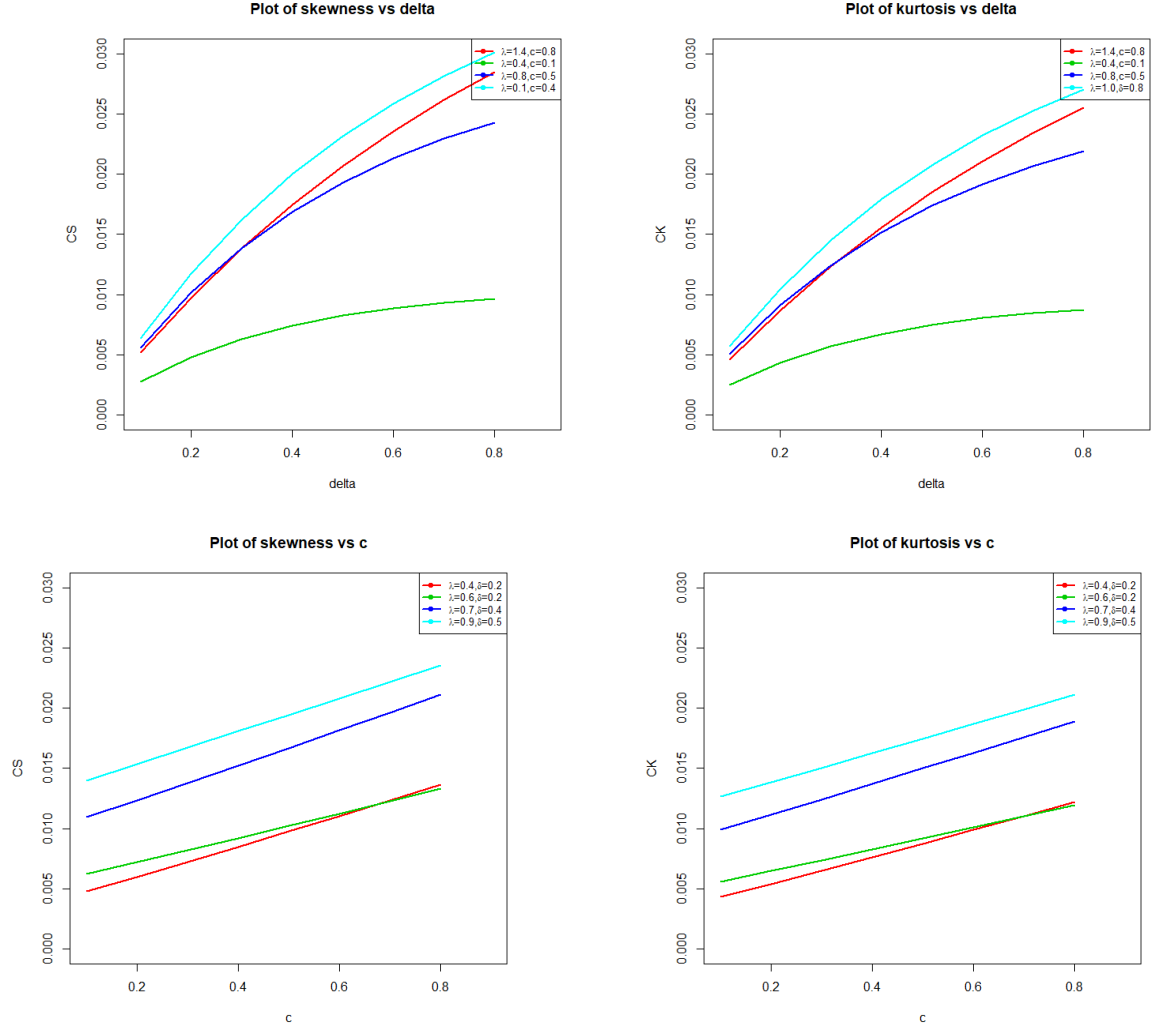


Figure 4.3: Plots of Skewness and Kurtosis for parameter δ and c

4.2.1 Conditional Moments

For lifetime models, it is also of interest to obtain the r^{th} conditional moments and the mean residual lifetime function. The r^{th} conditional moment for the MOLLoG distribution can be readily obtained from those of the ELLLoG distribution. The conditional r^{th} moment for the ELLLoG distribution is given

by

$$\begin{aligned}
E(Y^r|Y > t) &= \frac{1}{\bar{G}_{ELLLoG}(t)} \int_t^\infty x^r g_{ELLLoG}(y; \lambda, c, (k+1)) dy \\
&= \frac{1}{\bar{G}_{ELLLoG}(t)} \sum_{s,p=0}^\infty \frac{(k+1)(-1)^{s+p}[\lambda(s+1)]^p}{(1+\lambda)^{s+1}p!} \binom{(k+1)-1}{s} \\
&\quad \times \left[\sum_{m=0}^\infty \binom{s}{m} \lambda^{m+2} \frac{(1+\lambda)^{s-m}}{c} \right. \\
&\quad \times \left(B_{(1+t^c)^{-1}} \left(s+1 - \frac{r+p+m+1}{c}, \frac{r+p+m+1}{c} \right) \right. \\
&\quad \left. + B_{(1+t^c)^{-1}} \left(s+1 - \frac{r+p+m+2}{c}, \frac{r+p+m+2}{c} \right) \right) \\
&\quad + c \sum_{m=0}^\infty \binom{s+1}{m} \lambda^m (1+\lambda)^{s+1-m} \\
&\quad \left. (1+t^c)^{-1} \left(s+2 - \frac{r+p+c+m}{c}, \frac{r+p+c+m}{c} \right) \right], \quad (4.16)
\end{aligned}$$

where $B_{(1+t^c)^{-1}}(a, b)$ is the incomplete beta function. Consequently, the r^{th} conditional moment of the MOLLLoG distribution is given by

$$\begin{aligned}
E(X^r|X > t) &= \sum_{j,k=0}^\infty \frac{\Gamma(2+j)}{\Gamma(2)j!} \binom{j}{k} (-1)^k \frac{\delta \bar{\delta}^j}{k+1} E(Y^r|Y > t) \\
&= \frac{1}{\bar{G}_{ELLLoG}(t)} \sum_{j,k=0}^\infty \frac{\Gamma(2+j)}{\Gamma(2)j!} \binom{j}{k} (-1)^k \frac{\delta \bar{\delta}^j}{k+1} \\
&\quad \times \sum_{s,p=0}^\infty \frac{(k+1)(-1)^{s+p}[\lambda(s+1)]^p}{(1+\lambda)^{s+1}p!} \binom{(k+1)-1}{s} \\
&\quad \times \left[\sum_{m=0}^\infty \binom{s}{m} \lambda^{m+2} \frac{(1+\lambda)^{s-m}}{c} \right. \\
&\quad \times \left(B_{(1+t^c)^{-1}} \left(s+1 - \frac{r+p+m+1}{c}, \frac{r+p+m+1}{c} \right) \right. \\
&\quad \left. + B_{(1+t^c)^{-1}} \left(s+1 - \frac{r+p+m+2}{c}, \frac{r+p+m+2}{c} \right) \right) \\
&\quad + c \sum_{m=0}^\infty \binom{s+1}{m} \lambda^m (1+\lambda)^{s+1-m} \\
&\quad \left. (1+t^c)^{-1} \left(s+2 - \frac{r+p+c+m}{c}, \frac{r+p+c+m}{c} \right) \right]. \quad (4.17)
\end{aligned}$$

4.2.2 Mean Deviation, Lorenz and Bonferroni Curves

Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the MOLLLoG distribution are presented

in this subsection. The application of Lorenz and Bonferroni curves is not limited to economics for the study of income and poverty, but they can also be used to other fields such as reliability, demography, insurance and medicine. These measures can be readily obtained from those of the ELLLoG distribution. Bonferroni and Lorenz curves can be readily obtained from the incomplete moments.

4.2.3 Mean Deviations

The amount of scatter in a population can be measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median. If Y has the ELLLoG distribution, we can derive the mean deviation about the mean μ by

$$\delta_1 = \int_0^\infty |y - \mu| g_{\text{ELLLoG}}(y) dy = 2\mu G_{\text{ELLLoG}}(\mu) - 2\mu + 2T(\mu),$$

and the median deviation about the median M by

$$\delta_2 = \int_0^\infty |y - M| g_{\text{ELLLoG}}(y) dy = 2T(M) - \mu,$$

where $\mu = E(Y)$ is given in equation (4.15) with $r = 1$, M the median of $G_{\text{ELLLoG}}(x)$ and $T(a) = \int_a^\infty x \cdot g_{\text{ELLLoG}}(y) dy$. Note that

$$\begin{aligned} T(a) &= \sum_{s,p=0}^{\infty} \frac{\alpha(-1)^{s+p} [\lambda(s+1)]^p}{(1+\lambda)^{s+1} p!} \binom{\alpha-1}{s} \\ &\times \left[\sum_{m=0}^{\infty} \binom{s}{m} \lambda^{m+2} \frac{(1+\lambda)^{s-m}}{c} \right. \\ &\times \left(B_{(1+a^c)^{-1}} \left(s+1 - \frac{p+m+2}{c}, \frac{p+m+2}{c} \right) \right. \\ &\left. + B_{(1+a^c)^{-1}} \left(s+1 - \frac{p+m+3}{c}, \frac{p+m+3}{c} \right) \right) \\ &+ c \sum_{m=0}^{\infty} \binom{s+1}{m} \lambda^m (1+\lambda)^{s+1-m} \\ &\left. (1+a^c)^{-1} \left(s+2 - \frac{1+p+c+m}{c}, \frac{1+p+c+m}{c} \right) \right]. \quad (4.18) \end{aligned}$$

Consequently, the mean deviations for MOLLoG can be readily obtained from those of the ELLLoG distribution.

4.2.4 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves for the ELLLoG are given as

$$B(p) = \frac{1}{p\mu} \int_0^q yg_{\text{ELLLoG}}(y)dy = \frac{1}{p\mu}[\mu - T(q)],$$

and

$$L(p) = \frac{1}{\mu} \int_0^q yg_{\text{ELLLoG}}(y)dy = \frac{1}{\mu}[\mu - T(q)],$$

respectively, where $T(q) = \int_q^\infty yg_{\text{ELLLoG}}(y)dy$, $q = G_{\text{ELLLoG}}^{-1}(p)$, $0 \leq p \leq 1$. It follows therefore that Bonferroni and Lorenz curves for the MOLLLoG distribution can be readily obtained from those of the ELLLoG distribution.

4.3 Order Statistics and Rényi Entropy

The concept of entropy plays a very important role in information theory. In this section, we present the distribution of the order statistic and Rényi entropy for the MOLLLoG distribution.

4.3.1 Order Statistics

Order statistics play an important role in probability and statistics. The pdf of the i^{th} order statistic from the MOLLLoG pdf $f_{\text{MOLLLoG}}(x)$ is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!f_{\text{MOLLLoG}}(x)}{(i-1)!(n-i)!} [F_{\text{MOLLLoG}}(x)]^{i-1} [1 - F_{\text{MOLLLoG}}(x)]^{n-i} \\ &= \frac{n!f_{\text{MOLLLoG}}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F_{\text{MOLLLoG}}(x)]^{j+i-1} \end{aligned}$$

by using the binomial expansion

$$[1 - F_{\text{MOLLLoG}}(x)]^{n-i} = \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F_{\text{MOLLLoG}}(x)]^m,$$

so that

$$\begin{aligned} f_{i:n}(x) &= \frac{f_{\text{MOLLLoG}}(x)}{B(i, n-i+1)} \sum_{m=0}^{n-i} \binom{n-i}{m} \frac{(-1)^m}{m+i} (m+i) [F_{\text{MOLLLoG}}(x)]^{m+i-1} \\ &= \sum_{m=0}^{n-i} w_{i,m} g_{m+i}(x), \end{aligned}$$

where $f_{m+i}(x)$ is the pdf of the exponentiated MOLLLoG distribution with parameters c, λ, δ and $(m+i)$, $B(.,.)$ is the beta function and the weights $w_{i,m}$ are given by

$$w_{i,m} = \frac{1}{B(i, n-i+1)} \frac{(-1)^m}{m+i} \binom{n-i}{m} = (-1)^m \binom{m+i-1}{m} \binom{n}{m+i}.$$

4.3.2 Rényi Entropy

In this section, we present Rényi entropy for the MOLLLoG distribution. Rényi entropy (Rényi [70]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f_{\text{MOLLLoG}}(x; \lambda, c, \delta)]^v dx \right), v \neq 1, v > 0. \quad (4.19)$$

Rényi entropy tends to Shannon entropy as $v \rightarrow 1$. Note that

$$\begin{aligned} f_{\text{MOLLLoG}}^v(x) &= \frac{\delta^v e^{-\lambda v x}}{(1+\lambda)^v (1+x^c)^v} \left(\lambda^2 (1+x) + \frac{(1+\lambda+\lambda x) c x^{c-1}}{1+x^c} \right)^v \\ &\quad \times \left(1 - \bar{\delta} \frac{1+\lambda+\lambda x}{(1+\lambda)(1+x^c)} e^{-\lambda x} \right)^{-2v} \\ &= \frac{\delta^v e^{-\lambda v x}}{(1+\lambda)^v (1+x^c)^v} \sum_{p=0}^{\infty} \binom{v}{p} \lambda^{2(v-p)} (1+x)^{v-p} \left(\frac{1+\lambda+\lambda x}{1+x^c} \right)^p c^p x^{cp-p} \\ &\quad \times \sum_{q=0}^{\infty} \frac{\Gamma(2v+q)}{\Gamma(2v)q!} \bar{\delta}^q \frac{(1+\lambda+\lambda x)^q}{(1+\lambda)^q (1+x^c)^q} e^{-\lambda q x} \\ &= \sum_{s,p,q=0}^{\infty} \frac{\delta^v \bar{\delta}^q (-1)^s [\lambda(v+q)]^s}{(1+\lambda)^{v+q} s!} \binom{v}{p} \lambda^{2(v-p)} \frac{\Gamma(2v+q)}{\Gamma(2v)q!} x^{s+cp-p} \\ &\quad \times (1+x^c)^{-v-p-q} (1+x)^{v-p} (1+\lambda+\lambda x)^{p+q} \\ &= \sum_{s,p,q,t,w=0}^{\infty} \frac{\delta^v \bar{\delta}^q (-1)^s [\lambda(v+q)]^s}{(1+\lambda)^{v+q} s!} \binom{v}{p} \lambda^{2(v-p)} \frac{\Gamma(2v+q)}{\Gamma(2v)q!} \binom{v-p}{t} \binom{p+q}{w} \\ &\quad \times \lambda^w (1+\lambda)^{p+q-w} x^{s+cp-p+t+w} (1+x^c)^{-v-p-q}. \end{aligned}$$

Note that, by applying the substitution $y = (1+y^c)^{-1}$, we have the following

$$\int_0^\infty \frac{x^{s+cp-p+t+w}}{(1+x^c)^{v+p+q}} dx = \frac{1}{c} B(v+p+q-k^*, k^*),$$

where $k^* = \frac{s+cp-p+t+w+1}{c}$, so that Rényi entropy for the MOLLLoG distribution reduces to

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left(\sum_{s,p,q,t,w=0}^{\infty} \frac{\delta^v \bar{\delta}^q (-1)^s [\lambda(v+q)]^s}{(1+\lambda)^{v+q} s!} \binom{v}{p} \lambda^{2(v-p)} \frac{\Gamma(2v+q)}{\Gamma(2v)q!} \right. \\ &\quad \left. \times \binom{v-p}{t} \binom{p+q}{w} \lambda^w (1+\lambda)^{p+q-w} \frac{1}{c} B(v+p+q-k^*, k^*) \right). \end{aligned}$$

4.4 Estimation

Let $X_i \sim MOLLLoG(\lambda, c, \delta)$ and $\Delta = (\lambda, c, \delta)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned} \ell = \ell(\Delta) = & n \ln(\delta) - \lambda \sum_{i=1}^n x_i + \sum_{i=1}^n \ln \left[\lambda^2 (1 + x_i) + \frac{(1 + \lambda + \lambda x_i) c x_i^{c-1}}{1 + x_i^c} \right] \\ & - n \ln(1 + \lambda) - \sum_{i=1}^n \ln(1 + x_i^c) - 2 \sum_{i=1}^n \ln \left[1 - \bar{\delta} \frac{1 + \lambda + \lambda x_i}{(1 + \lambda)(1 + x_i^c)} e^{-\lambda x_i} \right] \end{aligned} \quad (4.20)$$

Elements of the score vector $U = (\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial c}, \frac{\partial \ell}{\partial \delta})$ can be readily obtained.

The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters λ, c, δ must be found via iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial c}, \frac{\partial \ell}{\partial \delta})^T = 0$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $I(\Delta) = [I_{\theta_i, \theta_j}]_{3 \times 3} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3$, can be numerically obtained by MATLAB or NLMIXED in SAS or R software. The total Fisher information matrix $nI(\Delta)$ can be approximated by

$$J_n(\hat{\Delta}) \approx \left[- \frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \Big|_{\Delta = \hat{\Delta}} \right]_{3 \times 3}, \quad i, j = 1, 2, 3. \quad (4.21)$$

For a given set of observations, the matrix given in equation (4.21) is obtained after the convergence of the Newton-Raphson procedure via NLMIXED in SAS or R software.

The multivariate normal distribution $N_3(\underline{0}, J^{-1}(\hat{\Delta}))$, where the mean vector $\underline{0} = (0, 0, 0)^T$ and $J^{-1}(\hat{\Delta})$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1 - \eta)\%$ two-sided confidence intervals for the parameters λ, c , and δ are given by:

$$\hat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\Delta})}, \quad \hat{c} \pm Z_{\frac{\eta}{2}} \sqrt{I_{cc}^{-1}(\hat{\Delta})}, \quad \text{and} \quad \hat{\delta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\delta\delta}^{-1}(\hat{\Delta})},$$

respectively, where $I_{\lambda\lambda}^{-1}(\hat{\Delta})$, $I_{cc}^{-1}(\hat{\Delta})$, and $I_{\delta\delta}^{-1}(\hat{\Delta})$, are the diagonal elements of $I_n^{-1}(\hat{\Delta}) = (nI(\hat{\Delta}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

4.5 Monte Carlo Simulations

In this section, the performance of the maximum likelihood estimates is examined by conducting various simulation studies for different sample sizes. We examine the performance of the MOLLLoG distribution by conducting various simulations for different sizes ($n=25, 50, 100, 200, 400, 800, 1200$) via the R package. We simulate $N = 1000$ samples for the true parameters values given in the Table 4.3 and Table 4.4. The Average Bias and Root Mean Square Error (RMSE) were computed. The average bias and RMSE for the estimated parameter $\hat{\theta}$, say, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \text{ and } RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. The table lists the mean MLEs of the parameters along with the respective average bias and root mean squared errors (RMSEs).

Table 4.3: Monte Carlo Simulation Results

parameter	Sample Size	(1.0,1.0,1.0)			(1.0,1.2,1.5)			(2.0,3.2,1.5)		
		Mean	RMSE	ABias	Mean	RMSE	ABias	Mean	RMSE	ABias
λ	25	1.0766	0.6101	0.0766	1.0348	0.5757	0.0348	2.5266	1.3586	0.5266
	50	1.0414	0.4337	0.0414	1.0228	0.4498	0.0228	2.3756	1.0222	0.3756
	100	1.0376	0.3278	0.0376	1.0157	0.3239	0.0157	2.1156	0.7910	0.1156
	200	1.0140	0.2228	0.0140	1.0101	0.2358	0.0101	2.0678	0.6616	0.0678
	400	1.0031	0.1521	0.0031	1.0020	0.1690	0.0020	2.0291	0.5215	0.0291
	800	1.0028	0.1072	0.0028	1.0006	0.1228	0.0006	1.9699	0.4179	-0.0300
	1200	0.9984	0.0905	-0.0015	1.0005	0.0994	0.0005	1.9801	0.3367	-0.0199
c	25	1.1680	0.7669	0.1680	1.3245	0.6674	0.1245	5.3011	18.5945	2.1011
	50	1.0502	0.3219	0.0502	1.2474	0.4178	0.0474	3.5220	2.4227	0.3220
	100	1.0150	0.2254	0.0150	1.2316	0.3017	0.0316	3.2662	1.3816	0.0662
	200	1.0106	0.1644	0.0106	1.2154	0.2277	0.0154	3.1533	1.0200	-0.0466
	400	1.0074	0.1113	0.0074	1.2122	0.1636	0.0122	3.1621	0.7508	-0.0378
	800	1.0034	0.0759	0.0034	1.2044	0.1171	0.0044	3.2287	0.5298	0.0287
	1200	1.0024	0.0635	0.0025	1.2023	0.0939	0.0023	3.2191	0.4272	0.0191
δ	25	1.2690	0.9725	0.2690	2.0117	1.7843	0.5117	3.7002	5.0555	2.2002
	50	1.1799	0.7225	0.1799	1.8110	1.1942	0.3111	2.8878	3.3815	1.3878
	100	1.1012	0.4908	0.1012	1.6641	0.7608	0.1641	2.1079	1.7208	0.6079
	200	1.0485	0.3142	0.0485	1.5854	0.5255	0.0854	1.9004	1.2687	0.4004
	400	1.0174	0.2037	0.0174	1.5377	0.3601	0.0377	1.7204	0.9424	0.2204
	800	1.0134	0.1420	0.0134	1.5163	0.2497	0.0163	1.5506	0.6103	0.0506
	1200	1.0045	0.1218	0.0045	1.5107	0.2047	0.0107	1.5296	0.4664	0.0296

Table 4.4: Monte Carlo Simulation Results

parameter	Sample Size	(1.5,1.5,3.0)			(2.0,1.9,2.0)			(3.0,3.8,2.5)		
		Mean	RMSE	ABias	Mean	RMSE	ABias	Mean	RMSE	ABias
λ	25	1.4469	0.8265	-0.0530	2.1867	1.1901	0.1867	3.3764	1.5179	0.3764
	50	1.4604	0.6533	-0.0395	2.0558	0.8827	0.0558	3.1933	1.0682	0.1933
	100	1.4633	0.5259	-0.0366	1.9472	0.7375	-0.0527	3.1867	0.7648	0.1867
	200	1.4765	0.4208	-0.0234	1.9480	0.6539	-0.0519	3.0772	0.5331	0.0772
	400	1.4823	0.3331	-0.0176	1.9496	0.5600	-0.0503	3.0284	0.3461	0.0284
	800	1.4971	0.2467	-0.0028	1.9667	0.4790	-0.0332	3.0055	0.2330	0.0055
	1200	1.4976	0.2394	-0.0023	1.9851	0.4244	-0.0148	2.9992	0.1850	-0.0007
c	25	1.7866	1.0795	0.2866	2.7623	5.1120	0.8623	10.2803	29.9856	6.4802
	50	1.6102	0.7275	0.1102	2.1810	1.3695	0.2810	5.1854	4.7053	1.3854
	100	1.5499	0.6086	0.0499	2.0568	0.9766	0.1568	4.2017	2.3329	0.4017
	200	1.5274	0.4872	0.0274	1.9875	0.7772	0.0875	3.9712	1.4162	0.1712
	400	1.5153	0.4000	0.0153	1.9449	0.6701	0.0449	3.9040	0.8758	0.1040
	800	1.4965	0.2988	-0.0034	1.9218	0.5846	0.0218	3.8627	0.6144	0.0627
	1200	1.5029	0.2923	0.0029	1.8974	0.5423	-0.0025	3.8339	0.4871	0.0339
δ	25	4.0802	4.2275	1.0802	3.6278	5.3864	1.6278	5.2600	7.6445	2.7600
	50	3.6778	2.8050	0.6778	2.7205	2.3209	0.7205	3.7366	3.4053	1.2366
	100	3.3948	2.1131	0.3948	2.3251	1.5538	0.3251	3.3379	2.1097	0.8379
	200	3.2788	1.6519	0.2788	2.2429	1.3451	0.2429	2.6340	0.7107	0.1340
	400	3.1512	1.2450	0.1512	2.1743	1.1072	0.1743	1.7204	0.9424	0.2204
	800	3.1128	0.9293	0.1128	2.1500	0.9724	0.1500	2.5540	0.4755	0.0540
	1200	3.1087	0.8986	0.1087	2.1494	0.9005	0.1494	2.5231	0.3546	0.0231

The result in Tables 4.3 and 4.4 show that the mean MLE_s converges to the true values. Also, the average bias decreases as the sample size n increases, and the $RMSE_s$ decay with increasing sample size n for all the parametric values that were considered.

4.6 Applications

The flexibility and usefulness of the MOLLLoG distribution and its sub-models for data modeling is illustrated in this section. The new MOLLLoG distribution is compared with the generalized Lindley (GL) distribution (Zakerzadeh and Dolati [87]), the Lindley-Weibull (LW) distribution (Asgharzadeh et al. [5]), and a three parameter Lindley (ATPLD) distribution (Shanker et al. [80]).

The pdf of the generalized Lindley (GL) distribution (Zakerzadeh and Dolati [87]) is given by

$$g_{GL}(x; \alpha, \beta, \theta) = \frac{\theta^{\alpha+1}}{(\beta + \theta)\Gamma(\alpha + 1)}(\alpha + \beta x)e^{-\theta x}, \quad (4.22)$$

for $\alpha, \beta, \theta > 0$ and $x > 0$. The LW distribution (see Asgharzadeh et al. [5] for

details) has the pdf given by

$$g_{LW}(x; \lambda, \alpha, \beta) = \frac{e^{-\lambda x - \alpha x^\beta}}{1 + \lambda} [\lambda^2(1 + x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1}], \quad (4.23)$$

for $\lambda, \alpha, \beta > 0$ and $x > 0$. The pdf of a three parameter Lindley (ATPLD) distribution is given by

$$g_{ATPLD}(x; \theta, \alpha, \beta) = \frac{\theta^2}{\theta x + \beta} (\alpha + \beta x) e^{-\theta x}, \quad (4.24)$$

for $\theta, \beta, \alpha > 0$, and $x > 0$. The maximum likelihood estimates (MLEs) of the MOLLLoG parameters and its sub-models are computed by maximizing the objective function via the subroutine NLMIXED in SAS as well as the function nlm in R. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), and Consistent Akaike Information Criterion ($CAIC = AIC + 2\frac{p(p+1)}{n-p-1}$), where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. Tables 4.5 and 4.6 shows results for the data set for MOLLLoG distribution, its sub-models and several non-nested models.

We can use the likelihood ratio (LR) test to compare the fit of the MOLLLoG distribution with its sub-models for a given data set. For example, to test $\delta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\lambda}, \hat{c}, \hat{\delta})) - \ln(L(\tilde{\lambda}, \tilde{c}, 1))]$, where $\hat{\lambda}$, $\hat{c}$, and $\hat{\delta}$ are the unrestricted estimates, and $\tilde{\lambda}$, and $\tilde{c}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi_\epsilon^2$, where χ_ϵ^2 denote the upper 100 $\epsilon\%$ point of the χ^2 distribution with 1 degrees of freedom.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [14]) are given in Figure 4.4 and Figure 4.5. For the probability plot, we plotted $F_{MOLLLoG}(x_{(j)}; \hat{\lambda}, \hat{c}, \hat{\delta})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^n \left[F_{MOLLLoG}(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The goodness-of-fit statistics W^* and A^* , described by (Chen and Balakrishnan [16]) are also presented in the tables. Also presented is the

Kolmogorov-Smirnov (KS) statistics and its P-value. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^* , the better the fit.

4.6.1 Data on halfway house (Failure times in days (49 Cases))

The data for halfway-house parolees failures obtained from (Stollmack and Harris [81]) are given as

13, 16, 20, 22, 22, 25, 32, 45, 49, 59, 64, 70, 88, 8, 89, 93, 95, 10, 112, 116, 122, 147, 150, 151, 177, 179, 190, 204, 207, 221, 233, 240, 245, 247, 264, 267, 272, 283, 291, 301, 307, 320, 337, 343, 352, 362, 367, 396, 421.

Estimates of the parameters of MOLLLoG distribution and its related sub-models (standard error in parentheses), AIC, BIC, and the goodness-of-fit statistics W^* , A^* , KS and its P-value as well as SS are given in Table 4.5. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 4.4.

Table 4.5: Estimates of Models for Data on Halfway House

Model	Estimates			Statistics								
	λ	c	δ	$-2 \log L$	AIC	$CAIC$	BIC	W^*	A^*	KS	P-value	SS
MOLLLoG	1.0515×10^{-02} (2.0166×10^{-03})	8.3765×10^{-01} (7.5631×10^{-02})	1.4434×10^{02} (1.1363×10^{-04})	597.5378	603.5378	604.0712	609.2133	0.0980	0.6551	0.0820	0.8964	0.0760
MOLLoG	0	1.6231	2.8599×10^{03}	614.5057	618.5057	618.7665	622.2893	0.3039	1.8808	0.1359	0.3254	0.21284
	-	(4.7660×10^{-02})	(3.0921×10^{-06})									
LLLoG	0.0101	0.0452	1	671.5517	675.5517	675.8126	679.3354	0.1599	1.0792	0.5253	3.594×10^{-12}	4.2567
	(0.0012)	(0.0313)										
LLoG	0	0.3199	1	768.1772	770.1772	770.2623	772.069	0.3592	2.1621	0.6604	2.2×10^{-16}	7.6245
	-	(0.0354)										
GL	α	β	θ									
	3.6022×10^{-01} (2.4830×10^{-01})	6.6143×10^{05} (2.2713×10^{-14})	7.7106×10^{-03} (1.6950×10^{-03})	602.3196	608.3196	608.853	613.9951	0.2534	1.8066	0.3880	7.778e-07	1.2668
LW	λ	α	β									
	0.0012	1.2649	0.0020	600.4114	606.4114	606.9447	612.0868	0.1445	0.9432	0.1162	0.5217	0.1633
	(0.0012)	(0.2209)	(0.0046)									
ATPLD	θ	α	β									
	0.0093	1.5043	0.0255	601.0656	607.0656	607.599	612.7411	0.1476	0.9643	0.1166	0.5175	0.1516
	(0.0014)	(0.0004)	(0.0239)									

The Likelihood ratio (LR) test statistic for testing H_0 : MOLLoG against H_a : MOLLLoG, H_0 : LLLoG against H_a : MOLLLoG and H_0 : LLoG against H_a : MOLLLoG are 16.9679 (p-value = 0.000038), 74.0139 (p-value < 0.00001) and 170.6394 (p-value < 0.00001). We can conclude that there are significant differences between MOLLoG and MOLLLoG distributions, LLLoG and MOLLLoG distributions as well between LLoG and MOLLLoG distributions, respectively based on the LR tests at 5% level. The values of AIC, CAIC and BIC are smallest for the MOLLLoG distribution, when compared to the

corresponding values for the non-nested GL, LW and ATPLD distributions. The values of the goodness-of-fit-statistics W^* , A^* , KS and its p-value show that the MOLLLoG distribution is the “best” fit for the data on halfway house.

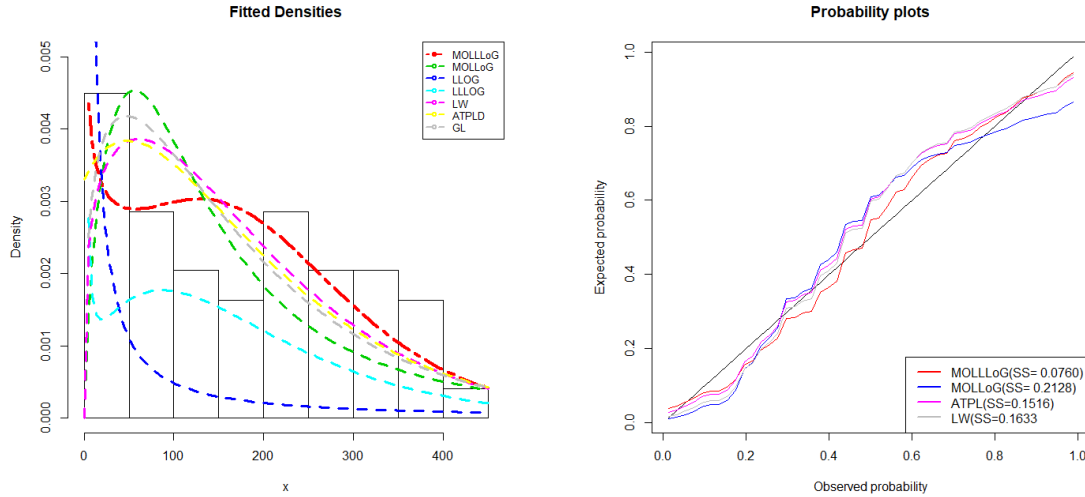


Figure 4.4: Fitted Densities and Probability Plots of the Data on Halfway House

4.6.2 Survival data set

Survival data set that was analyzed by (Feigl and Zelen [25]). The data represent the survival times, in weeks, of 33 patients suffering from acute myelogeneous Leukaemia. The data, that can also be found at library SMIR of the R program (<http://cran.r-project.org>),

65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

Estimates of the parameters of MOLLLoG distribution and its related sub-models (standard error in parentheses), AIC, BIC, and the goodness-of-fit statistics W^* , A^* , KS and its P-value as well as SS are given in Table 4.6. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 4.5.

Table 4.6: Estimates of Models for Survival Data

Model	Estimates			Statistics								
	λ	c	δ	$-2 \log L$	AIC	$CAIC$	BIC	W^*	A^*	KS	P-value	SS
MOLLLoG	0.0184 (0.0070)	0.8540 (0.1741)	14.3933 (7.7615)	305.2288	311.2288	312.0564	315.7183	0.0813	0.5622	0.1172	0.7545	0.0696
MOLLoG	0 -	1.0849 (0.1517)	22.6957 (12.3774)	310.7448	314.7448	315.1448	317.7379	0.1308	0.8319	0.1375	0.5601	0.1123
LLLoG	0.0232 (0.0058)	0.3192 (0.0778)	1	342.7644	346.7644	347.1644	349.7574	0.1327	0.8840	0.5003	1.33×10^{-07}	2.7352
LLoG	0 -	0.5073 (0.0707)	1	360.7053	362.7053	362.8343	364.2018	0.1255	0.7769	0.5449	6.155×10^{-09}	3.5582
GL	α 0.6783 (0.1576)	β 0.0026 (0.0121)	θ 0.0195 (0.0102)	307.3194	313.3194	314.147	317.809	0.4883	2.8500	0.3836	0.0001	0.6478
LW	λ 0.0698 (0.0363)	α 0.7270 (0.1523)	β 0.0085 (0.0121)	306.8979	312.8979	313.7254	317.3874	0.0932	0.6447	0.130	0.6244	0.0883
ATPLD	θ 2.4463×10^{-02} (9.2906×10^{-03})	α 1.9344×10^{02} (6.5274×10^{-07})	β 9.4785×10^{-06} (1.5973)	310.9003	316.9003	317.7279	321.3899	0.0972	0.6729	0.2182	0.0863	0.3295

The Likelihood ratio (LR) test statistic for testing H_0 : MOLLoG against H_a : MOLLLoG, H_0 : LLLoG against H_a : MOLLLoG and H_0 : LLoG against H_a : MOLLLoG are 5.5160 (p-value = 0.018843), 37.5356 (p-value < 0.00001) and 55.4765 (p-value < 0.00001). We can conclude that there are significant differences between MOLLoG and MOLLLoG distributions, LLLoG and MOLLLoG distributions as well between LLoG and MOLLLoG distributions, respectively based on the LR tests at 5% level. The MOLLLoG distribution is significantly better than the sub-models considered above. The values of the statistics: AIC, CAIC, and BIC are smallest for the MOLLLoG distribution. Also, the goodness-of-fit statistics W^* and A^* are the smallest and definitely points to the MOLLLoG distribution as the “best” fit for the Survival data when compared to the corresponding values for the sub-models. The goodness-of-fit statistics W^* and A^* are also better for the MOLLLoG distribution when compared to the values for the non-nested GL, LW and ATPLD distributions. Thus, there is indeed convincing evidence that the MOLLLoG distribution is the “best” fit for the Survival times data.

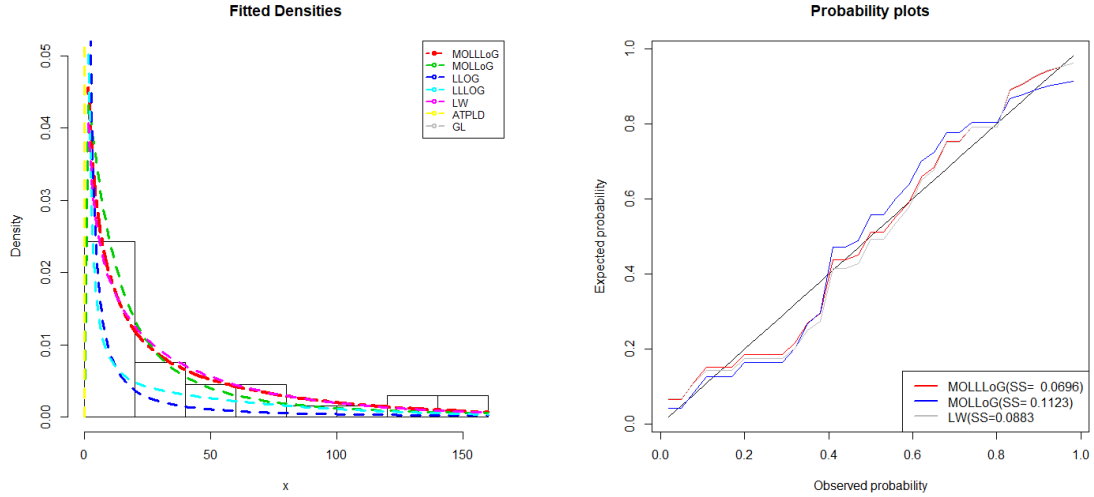


Figure 4.5: Fitted Densities and Probability Plots of the Survival Times Data

4.7 Concluding Remarks

In this chapter, a new generalized distribution called the Marshall-Olkin Lindley-Log-logistic (MOLLLoG) distribution is studied. Several structural properties of the distribution such as hazard function, quantile function, moments, conditional moments, mean and median deviations, Bonferroni and Lorenz curves, distribution of the order statistics and Rényi entropy are investigated. Maximum likelihood estimation technique is used to estimate the model parameters. Monte Carlo simulation study was conducted to examine the performance of MOLLLoG distribution. Examples of two real life datasets proves the importance and potentiality of the MOLLLoG distribution.

Chapter 5

The Half Logistic Log-logistic Weibull Distribution (HLLLoGW)

In this chapter, we study a new three parameter model by combining the half logistic and the log-logistic Weibull distributions called Half-Logistic Log-Logistic Weibull (HLLLoGW) distribution. Half-Logistic transformation was applied to several well known distributions. (Anwar and Zahoor [4]) introduced the half-logistic Lomax distribution for lifetime modeling, (Anwar and Bibi [3]) developed the half-logistic generalized Weibull distribution. (Muhammad and Yahaya [50]) established the half logistic-Poisson distribution.

(Cordeiro et al. [18]) define the cumulative distribution function (cdf) of the new type I half-logistic-G(TIHL-G) family of distributions by

$$\begin{aligned} F(x; \lambda, \underline{\psi}) &= \int_0^{-\ln(1-G(x; \underline{\psi}))} \frac{2\lambda e^{-\lambda t}}{(1 + e^{-\lambda t})^2} dt \\ &= \frac{1 - [1 - G(x; \underline{\psi})]^\lambda}{1 + [1 - G(x; \underline{\psi})]^\lambda}, \end{aligned} \quad (5.1)$$

where $G(x; \underline{\psi})$ is the baseline cdf depending on a parameter vector $\underline{\psi}$ and $\lambda > 0$ is an additional shape parameter. If we take $\lambda = 1$, then the TIHL-G reduces to the half logistic-G (HL-G) distribution with cdf

$$F(x; \underline{\psi}) = \frac{G(x; \underline{\psi})}{1 + \overline{G}(x; \underline{\psi})}, \quad (5.2)$$

where $\overline{G}(x; \underline{\psi}) = 1 - G(x; \underline{\psi})$.

The corresponding probability density function (pdf) to (5.2) is given by

$$f(x; \underline{\psi}) = \frac{2g(x; \underline{\psi})}{[1 + \overline{G}(x; \underline{\psi})]^2}, \quad (5.3)$$

where $g(x; \underline{\psi}) = \frac{dG(x; \underline{\psi})}{dx}$ is the baseline pdf.

The three parameter cdf of the log-logistic Weibull (LLoGW) distribution (Oluyede et al. [66]) is given by

$$G_{LLoGW}(x; \alpha, \beta, c) = 1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}, \quad \text{for } x > 0, \text{ and } \alpha, \beta, c > 0. \quad (5.4)$$

The corresponding pdf is given by

$$g_{LLoGW}(x; \alpha, \beta, c) = e^{-\alpha x^\beta} (1 + x^c)^{-1} [(1 + x^c)^{-1} c x^{c-1} + \alpha \beta x^{\beta-1}],$$

for $x > 0$, and $\alpha, \beta, c > 0$.

The primary motivation of Half-Logistic Log-Logistic Weibull (HLLLoGW) distribution is the applications of log-logistic and Weibull distributions in different areas of sciences and the ability to depict more complex hazard rates and act as a good alternate to both the log-logistic and Weibull distributions.

5.1 HLLLoGW Distribution, Expansion of Density Function, Sub-models, Hazard and Quantile Functions

In this section, the HLLLoGW distribution, series expansion of its pdf, some sub-models, quantile function, hazard function as well some graphs are presented.

Substituting equation (5.4) into equation (5.2) we obtain the cdf of the half logistic log-logistic Weibull (HLLLoGW) distribution as

$$F(x; \alpha, \beta, c) = \frac{1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}}{1 + (1 + x^c)^{-1} e^{-\alpha x^\beta}}. \quad (5.5)$$

The corresponding pdf follows from inserting (5.4) and (5.5) into (5.3) and is given by

$$f(x; \alpha, \beta, c) = 2e^{-\alpha x^\beta} (1 + x^c)^{-1} \left[(1 + x^c)^{-1} c x^{c-1} + \alpha \beta x^{\beta-1} \right] \times \left[1 + (1 + x^c)^{-1} e^{-\alpha x^\beta} \right]^{-2} \quad (5.6)$$

for $\alpha, \beta, c > 0$. If a random variable X has the half logistic log-logistic Weibull (HLLLoGW) density, we write $X \sim HLLLoGW(\alpha, \beta, c)$. Plots of the HLLLoGW pdf shows different shapes including right skewed, left skewed, almost symmetric and reverse J-shapes.

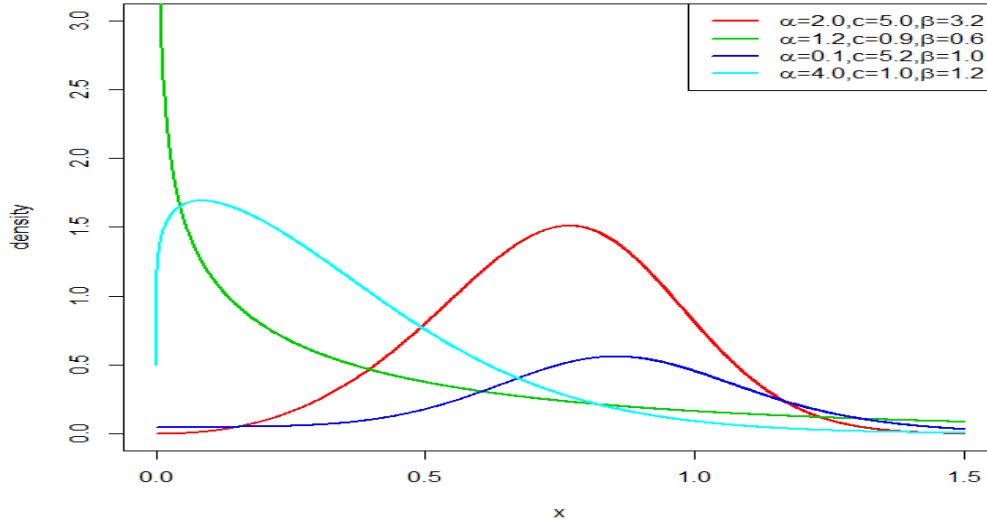


Figure 5.1: Plots of HLWLLoG Density Function

5.1.1 Expansion of Density Function

In this subsection, an expansion of the HLLLoGW density function is presented.

Note that

$$\begin{aligned}
 f(x; \alpha, \beta, c) &= \frac{2g_{LLoGW}(x)}{[1 + \overline{G}_{LLoGW}(x)]^2} \\
 &= 2 \sum_{p=0}^{\infty} \frac{\Gamma(2+p)}{\Gamma(2)p!} (-1)^p [1 - G_{LLoGW}(x)]^p g_{LLoGW}(x) \\
 &= 2 \sum_{p,k=0}^{\infty} \frac{\Gamma(2+p)}{\Gamma(2)p!} \binom{p}{k} (-1)^{p+k} \frac{k+1}{k+1} [G_{LLoGW}(x)]^{k+1-1} g_{LLoGW}(x) \\
 &= \sum_{p,k=0}^{\infty} \frac{\Gamma(2+p)}{\Gamma(2)p!} \binom{p}{k} (-1)^{p+k} \frac{2}{k+1} g_{ELLoGW}(x; \alpha, \beta, c, k+1), \quad (5.7)
 \end{aligned}$$

where

$$\begin{aligned}
 g_{ELLoGW}(x; \alpha, \beta, c, k+1) &= \left(1 - (1+x^c)^{-1} e^{-\alpha x^\beta}\right)^{k+1-1} \\
 &\quad \times (k+1) e^{-\alpha x^\beta} (1+x^c)^{-1} [(1+x^c)^{-1} c x^{c-1} + \alpha \beta x^{\beta-1}]
 \end{aligned}$$

is the exponentiated log-logistic Weibull (ELLoGW) density function with parameters $\alpha, \beta, c, k+1 > 0$. Thus, the HLLLoGW density function can be written as an infinite linear combination of the ELLoGW density functions. Consequently, the mathematical and statistical properties of the HLLLoGW distribution follows directly from those of the ELLoGW distribution.

5.1.2 Sub-models of HLLLoGW Distribution

In this subsection, some special models of the HLLLoGW distributions are discussed.

- If $\alpha \rightarrow 0$, the HLLLoGW model reduces to the Half logistic log-logistic model (HLLLoG) with pdf

$$f(x; c) = \frac{2(1+x^c)^{-2} c x^{c-1}}{[1 + (1+x^c)^{-1}]^2}, \quad (5.8)$$

for $c > 0$.

- If $\beta = 1$ we obtain the Half logistic log-logistic Exponential (HLLLoGE) with the pdf

$$f(x; \alpha, c) = \frac{2e^{-\alpha x} (1+x^c)^{-1} [(1+x^c)^{-1} c x^{c-1} + \alpha]}{[1 + (1+x^c)^{-1} e^{-\alpha x}]^2} \quad (5.9)$$

for $\alpha, c > 0$.

- If $\beta = 2$, we obtain the Half logistic log-logistic Rayleigh (HLLLoGR) with the pdf

$$\begin{aligned} f(x; \alpha, c) &= 2e^{-\alpha x^2} (1 + x^c)^{-1} [(1 + x^c)^{-1} c x^{c-1} + 2\alpha x] \\ &\times [1 + (1 + x^c)^{-1} e^{-\alpha x^2}]^{-2} \end{aligned} \quad (5.10)$$

for $\alpha, c > 0$.

- If $c = 1$, the HLLLoGW model reduces to a two parameter distribution with the pdf

$$\begin{aligned} f(x; \alpha, \beta) &= 2e^{-\alpha x^\beta} (1 + x)^{-1} [(1 + x)^{-1} + \alpha \beta x^{\beta-1}] \\ &\times [1 + (1 + x)^{-1} e^{-\alpha x^\beta}]^{-2} \end{aligned} \quad (5.11)$$

for $\alpha, \beta > 0$.

- If $c = \beta = 1$, the HLLLoGW model reduces to a one parameter distribution with the pdf,

$$\begin{aligned} f(x; \alpha) &= 2e^{-\alpha x} (1 + x)^{-1} [(1 + x)^{-1} + \alpha] \\ &\times [1 + (1 + x)^{-1} e^{-\alpha x}]^{-2} \end{aligned} \quad (5.12)$$

for $\alpha > 0$.

- If $\alpha = 1$, the HLLLoGW model reduces to a two parameter distribution with the pdf,

$$\begin{aligned} f(x; \beta, c) &= 2e^{-x^\beta} (1 + x^c)^{-1} [(1 + x^c)^{-1} c x^{c-1} + \beta x^{\beta-1}] \\ &\times [1 + (1 + x^c)^{-1} e^{-x^\beta}]^{-2} \end{aligned} \quad (5.13)$$

for $\beta, c > 0$.

- If $\alpha = c = 1$, the HLLLoGW model reduces to a one parameter distribution with the pdf,

$$f(x; \beta) = 2e^{-x^\beta}(1+x)^{-1}[(1+x)^{-1} + \beta x^{\beta-1}] \times [1 + (1+x)^{-1}e^{-x^\beta}]^{-2} \quad (5.14)$$

for $\beta > 0$.

5.1.3 Hazard and Quantile Functions

In this section, the hazard and quantile functions of the HLLLoGW distribution are presented. The hazard function is given by

$$h_F(x; \alpha, \beta, c) = \frac{f_{HLLLoGW}(x; \alpha, \beta, c)}{F_{HLLLoGW}(x; \alpha, \beta, c)} = \frac{2e^{-\alpha x^\beta}(1+x^c)^{-1}[(1+x^c)^{-1}cx^{c-1} + \alpha\beta x^{\beta-1}]}{\left(1 - \frac{1-(1+x^c)^{-1}e^{-\alpha x^\beta}}{1+(1+x^c)^{-1}e^{-\alpha x^\beta}}\right) [1 + (1+x^c)^{-1}e^{-\alpha x^\beta}]^2}, \quad (5.15)$$

for $\alpha, \beta, c > 0$. Plots of the hazard function of the HLLLoGW distribution show different shapes including decreasing, increasing, bathtub followed by upside down, as well as bathtub shapes.

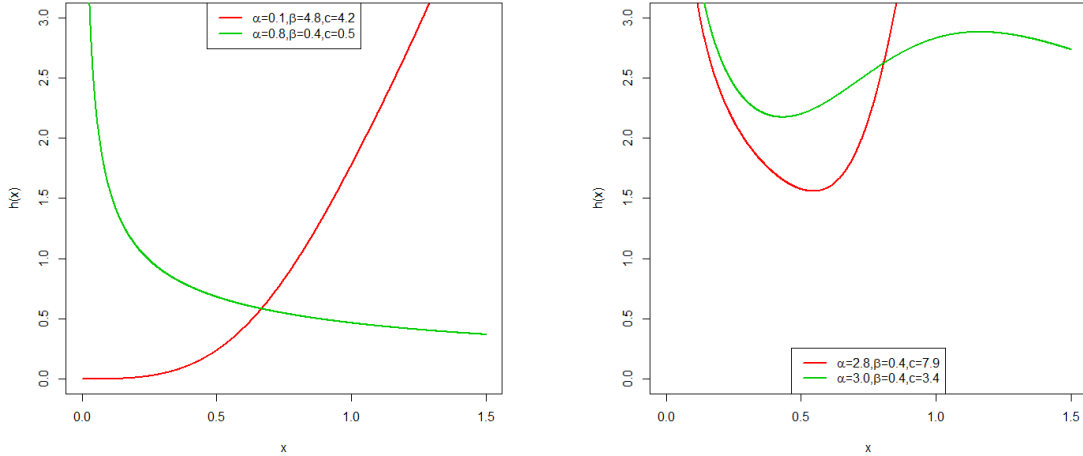


Figure 5.2: Plots of HLLLoGW Hazard Function

The quantile function of the HLLLoGW distribution is obtained by solving the non-linear equation:

$$F(x; \alpha, \beta, c) = \frac{1 - (1+x^c)^{-1}e^{-\alpha x^\beta}}{1 + (1+x^c)^{-1}e^{-\alpha x^\beta}} = u, \quad (5.16)$$

$0 \leq u \leq 1$, that is,

$$\alpha x^\beta + \ln(1 + x^c) + \ln\left(\frac{1 - u}{1 + u}\right) = 0. \quad (5.17)$$

Therefore, random numbers can be generated from the HLLLoGW distribution by numerically solving the non-linear equation (5.17).

Table 5.1: Table of Quantile for HLLLoGW Distribution

(α, β, c)					
u	(0.1, 0.5, 4.02)	(0.2, 0.3, 3.8)	(0.02, 0.004, 2.1)	(0.3, 0.2, 1.4)	(0.2, 0.1, 0.9)
0.1	0.7024	0.7091	0.4949	0.4154	0.2328
0.2	0.8641	0.8868	0.7289	0.7869	0.6025
0.3	0.9950	1.0333	0.9413	1.2242	1.1523
0.4	1.1185	1.1765	1.1649	1.8002	1.9866
0.5	1.2450	1.3252	1.4131	2.6226	3.3435
0.6	1.3898	1.5035	1.7190	3.9421	5.7227
0.7	1.5698	1.7270	2.1227	6.4557	10.5563
0.8	1.8300	2.0675	2.7542	13.1099	23.2730
0.9	2.3281	2.7459	4.0818	68.6328	85.6170

5.2 Moments, Conditional Moments and Mean Deviations

In this section, the r^{th} moment, conditional moments, mean deviations, Lorenz and Bonferroni curves of the HLLLoGW distribution are presented. Let $Y \sim$

$ELLoGW(\alpha, \beta, c, k+1)$. Note that

$$\begin{aligned}
E(Y^r) &= \int_0^\infty y^r g_{ELLoGW}(y; \alpha, \beta, c, k+1) dy \\
&= \int_0^\infty y^r (k+1) \left(1 - (1+y^c)^{-1} e^{-\alpha y^\beta}\right)^{(k+1)-1} e^{-\alpha y^\beta} (1+y^c)^{-1} \\
&\quad \times [(1+y^c)^{-1} c y^{c-1} + \alpha \beta y^{\beta-1}] dy \\
&= \sum_{i=0}^\infty \binom{(k+1)-1}{i} (-1)^i (k+1) \int_0^\infty y^r (1+y^c)^{-i-1} e^{-\alpha y^\beta (i+1)} \\
&\quad \times [(1+y^c)^{-1} c y^{c-1} + \alpha \beta y^{\beta-1}] dy \\
&= \sum_{i,j=0}^\infty \frac{(k+1)(-1)^{i+j} [\alpha(i+1)]^j}{j!} \binom{(k+1)-1}{i} \left[c \int_0^\infty y^{r+\beta j+c-1} \right. \\
&\quad \left. \times (1+y^c)^{-i-2} dy + \alpha \beta \int_0^\infty y^{r+\beta(j+1)-1} (1+y^c)^{-i-1} dy \right].
\end{aligned}$$

We note that by applying the substitution $t = (1+y^c)^{-1}$, we have

$$\begin{aligned}
E(Y^r) &= \sum_{i,j=0}^\infty \frac{(k+1)(-1)^{i+j} [\alpha(i+1)]^j}{j!} \binom{(k+1)-1}{i} \\
&\quad \times \left[\int_0^1 t^{i-\frac{r+\beta j+c}{c}-1} (1-t)^{\frac{r+\beta j+c}{c}-1} dt \right. \\
&\quad \left. + \alpha \beta \int_0^1 t^{i-1-\frac{r+\beta(j+1)}{c}-1} (1-t)^{\frac{r+\beta(j+1)}{c}-1} dt \right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
E(Y^r) &= \sum_{i,j=0}^\infty \frac{(k+1)(-1)^{i+j} [\alpha(i+1)]^j}{j!} \binom{(k+1)-1}{i} \\
&\quad \times \left[B\left(i - \frac{r+\beta j+c}{c}, \frac{r+\beta j+c}{c}\right) \right. \\
&\quad \left. + B\left(i-1 - \frac{r+\beta(j+1)}{c}, \frac{r+\beta(j+1)}{c}\right) \right]. \tag{5.18}
\end{aligned}$$

Consequently, the r^{th} moment of the HLLLoGW distribution is given by

$$E(X^r) = \sum_{p,k=0}^\infty \frac{\Gamma(2+p)}{\Gamma(2)p!} \binom{p}{k} (-1)^{p+k} \frac{2}{k+1} E(Y^r), \tag{5.19}$$

where $E(Y^r)$ is given by equation (5.18). The moment generating function of the ELLoGW distribution is given by $E(e^{tY}) = \sum_{r=0}^\infty \frac{t^r}{r!} E(Y^r)$, where $E(Y^r)$ is given by the equation (5.18). The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD= σ), CV, CS and CK are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$

and

$$CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively. Some moments for selected parameters values are given in Table 5.2 and plots are presented in Figure 5.3. Plots of skewness and kurtosis for choices of the model parameters reveal that skewness and kurtosis depend on the shape parameters β and c .

Table 5.2: Table of Moments for Selected Parameters for HLLLoGW Distribution

	(α, β, c)				
	(0.04,0.8,0.2)	(0.2,1.5,0.7)	(0.9,0.2,0.5)	(0.6,0.9,1.0)	(0.9,0.8,0.4)
E(X)	0.0146	0.0634	0.0604	0.1181	0.0890
E(X ²)	0.0073	0.0351	0.0292	0.0659	0.0502
E(X ³)	0.0049	0.0241	0.0190	0.0448	0.0349
E(X ⁴)	0.0036	0.0183	0.0140	0.0336	0.0267
E(X ⁵)	0.0029	0.0148	0.0111	0.0268	0.0216
E(X ⁶)	0.0024	0.0124	0.0092	0.0223	0.0182
SD	0.0021	0.0106	0.0078	0.0190	0.0157
CV	0.0018	0.0093	0.0068	0.0166	0.0138
CS	0.0016	0.0083	0.0060	0.0147	0.0123
CK	0.0014	0.0075	0.0054	0.0132	0.0111

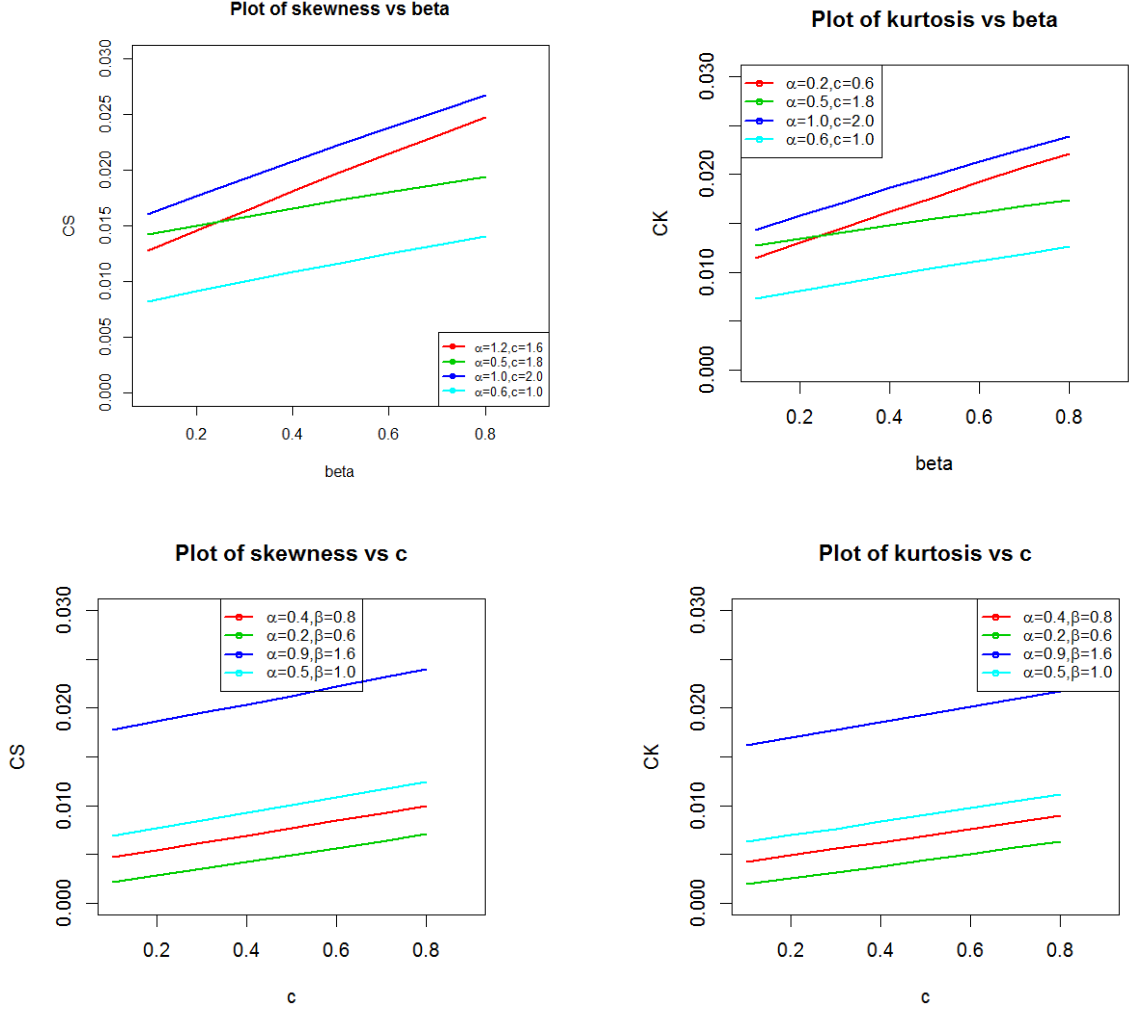


Figure 5.3: Plots of Skewness and Kurtosis for shape parameters β and c

5.2.1 Conditional Moments

For lifetime models, it is also of interest to obtain the r^{th} conditional moments and the mean residual lifetime function. The r^{th} conditional moment for the HLLoGW distribution can be readily obtained from those of the ELLoGW distribution. The conditional r^{th} moment for the ELLoGW distribution is given

by

$$\begin{aligned}
E(Y^r|Y > t) &= \frac{1}{\overline{G}_{ELLoGW}(t)} \int_t^\infty x^r g_{ELLoGW}(y; \alpha, \beta, c, k+1) dy \\
&= \frac{1}{\overline{G}_{ELLoGW}(t)} \sum_{i,j=0}^\infty \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!} \binom{(k+1)-1}{i} \\
&\quad \times \left[B_{(1+t^c)-1} \left(i - \frac{r+\beta j+c}{c}, \frac{r+\beta j+c}{c} \right) \right. \\
&\quad \left. + B_{(1+t^c)-1} \left(i-1 - \frac{r+\beta(j+1)}{c}, \frac{r+\beta(j+1)}{c} \right) \right], \quad (5.20)
\end{aligned}$$

where $B_{(1+t^c)-1}(a, b)$ is the incomplete beta function. Consequently, the r^{th} conditional moment of the HLLLoGW distribution is given by

$$\begin{aligned}
E(X^r|X > t) &= \sum_{p,k=0}^\infty \frac{\Gamma(2+p)}{\Gamma(2)p!} \binom{p}{k} (-1)^{p+k} \frac{2}{k+1} E(Y^r|Y > t) \\
&= \frac{1}{\overline{G}_{ELLoGW}(t)} \sum_{p,k=0}^\infty \frac{\Gamma(2+p)}{\Gamma(2)p!} \binom{p}{k} (-1)^{p+k} \frac{2}{k+1} \\
&\quad \times \sum_{i,j=0}^\infty \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!} \binom{(k+1)-1}{i} \\
&\quad \times \left[B_{(1+t^c)-1} \left(i - \frac{r+\beta j+c}{c}, \frac{r+\beta j+c}{c} \right) \right. \\
&\quad \left. + B_{(1+t^c)-1} \left(i-1 - \frac{r+\beta(j+1)}{c}, \frac{r+\beta(j+1)}{c} \right) \right]. \quad (5.21)
\end{aligned}$$

5.2.2 Mean Deviation, Lorenz and Bonferroni Curves

In this subsection, we present Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the HLLLoGW distribution. Bonferroni and Lorenz curves are widely used tool for analyzing and visualizing income inequality.

5.2.3 Mean Deviations

If Y has the ELLoGW distribution, we can derive the mean deviation about the mean μ by

$$\delta_1 = \int_0^\infty |y - \mu| f_{ELLoGW}(y) dy = 2\mu F_{ELLoGW}(\mu) - 2\mu + 2T(\mu),$$

and the median deviation about the median M by

$$\delta_2 = \int_0^\infty |y - M| f_{ELLoGW}(y) dy = 2T(M) - \mu,$$

where $\mu = E(Y)$ is given in equation (5.18) with $r = 1$, M the median of $F_{ELLoGW}(x)$ and $T(a) = \int_a^\infty x \cdot f_{ELLoGW}(y)dy$. Note that

$$\begin{aligned} T(a) = & \sum_{i,j=0}^{\infty} \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!} \binom{(k+1)-1}{i} \\ & \times \left[B_{(1+a^c)-1} \left(i - \frac{1+\beta j+c}{c}, \frac{1+\beta j+c}{c} \right) \right. \\ & \left. + B_{(1+a^c)-1} \left(i-1 - \frac{1+\beta(j+1)}{c}, \frac{1+\beta(j+1)}{c} \right) \right]. \end{aligned} \quad (5.22)$$

Consequently, the mean deviations for HLLLoGW can be readily obtained from those of the ELLoGW distribution.

5.2.4 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves for the ELLoGW are given as

$$B(p) = \frac{1}{p\mu} \int_0^q y f_{ELLoGW}(y) dy = \frac{1}{p\mu} [\mu - T(q)],$$

and

$$L(p) = \frac{1}{\mu} \int_0^q y f_{ELLoGW}(y) dy = \frac{1}{\mu} [\mu - T(q)],$$

respectively, where $T(q) = \int_q^\infty y f_{ELLoGW}(y) dy$, $q = F_{ELLoGW}^{-1}(p)$, $0 \leq p \leq 1$. It follows therefore that Bonferroni and Lorenz curves for the HLLLoGW distribution can be readily obtained from those of the ELLoGW distribution. Note that

$$\begin{aligned} T(q) = & \sum_{i,j=0}^{\infty} \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!} \binom{(k+1)-1}{i} \\ & \times \left[B_{(1+q^c)-1} \left(i - \frac{1+\beta j+c}{c}, \frac{1+\beta j+c}{c} \right) \right. \\ & \left. + B_{(1+q^c)-1} \left(i-1 - \frac{1+\beta(j+1)}{c}, \frac{1+\beta(j+1)}{c} \right) \right]. \end{aligned} \quad (5.23)$$

Consequently, the mean deviations for HLLLoGW can be readily obtained from those of the ELLoGW distribution.

5.3 Order Statistics and Rényi Entropy

The concept of entropy plays a very important role in information theory. In this section, we present the distribution of the order statistic and Rényi entropy for the HLLLoGW distribution.

5.3.1 Distribution of Order Statistics

Order statistics play an important role in probability and statistics. Let $X_1, X_2, \dots, X_n$ be a random sample from the HLLLoGW distribution and suppose $X_{1:n} < X_{2:n}, \dots < X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l f(x) [F(x)]^{k+l-1}. \quad (5.24)$$

Note, $f(x) [F(x)]^{k+l-1} = \frac{1}{k+l} \frac{d}{dx} [F(x)]^{k+l}$. The corresponding pdf of $f_{k:n}(x)$ is given by

$$\begin{aligned} f_{k:n}(x) &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \frac{\binom{n-k}{l} (-1)^l}{k+l} \frac{d}{dx} [F(x)]^{k+l} \\ &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \frac{\binom{n-k}{l} (-1)^l}{k+l} \frac{d}{dx} \left[\frac{1 - (1+x^c)^{-1} e^{-\alpha x^\beta}}{1 + (1+x^c)^{-1} e^{-\alpha x^\beta}} \right]^{k+l} \\ &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \frac{\binom{n-k}{l} (-1)^l}{k+l} \frac{d}{dx} G_{EHLLLoGW}(x; \lambda, c, \gamma, k+l), \end{aligned}$$

where $G_{EHLLLoGW}(x; \alpha, \beta, c, k+l)$ is exponentiated half logistic log-logistic Weibull (EHLLLoGW) distribution function with parameters α, β, c and $k+l > 0$. Thus, the pdf of the k^{th} order statistic can be expressed as a linear combination of the pdf of the EHLLLoGW distribution.

5.3.2 Rényi Entropy

In this section, we present Rényi entropy for the HLLLoGW distribution. Rényi entropy (Rényi [70]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f(x; \alpha, \beta, c)]^v dx \right), v \neq 1, v > 0. \quad (5.25)$$

Rényi entropy tends to Shannon entropy as $v \rightarrow 1$. Note that

$$\begin{aligned}
f_{HLLLoGW}^v(x) &= 2^v e^{-\alpha v x^\beta} (1+x^c)^{-v} [(1+x^c)^{-1} c x^{c-1} + \alpha \beta x^{\beta-1}]^v \\
&\quad \times [1 + (1+x^c)^{-1} e^{-\alpha x^\beta}]^{-2v} \\
&= \sum_{p=0}^{\infty} 2^v e^{-\alpha v x^\beta} (1+x^c)^{-v} \binom{v}{p} (1+x^c)^{-p} c^p x^{pc-p} x^{(\beta-1)(v-p)} \\
&\quad \times (\alpha \beta)^{v-p} \sum_{z=0}^{\infty} \frac{\Gamma(2v+z)}{\Gamma(2v)z!} (-1)^z (1+x^c)^{-z} e^{-\alpha z x^\beta} \\
&= \sum_{p,z=0}^{\infty} 2^v (\alpha \beta)^{v-p} c^p \binom{v}{p} \frac{\Gamma(2v+z)}{\Gamma(2v)z!} (-1)^z (1+x^c)^{-z-p-v} \\
&\quad e^{pc-p+(\beta-1)(v-p)} e^{-\alpha x^\beta(z+v)} \\
&= \sum_{p,z,s=0}^{\infty} 2^v (\alpha \beta)^{v-p} c^p [\alpha(z+v)]^s \binom{v}{p} \frac{\Gamma(2v+z)}{\Gamma(2v)z!} (-1)^{z+s} \\
&\quad e^{pc+\beta s-p+(\beta-1)(v-p)} (1+x^c)^{-z-p-v}.
\end{aligned}$$

Note that, by applying the substitution $y = (1+x^c)^{-1}$, we have the following

$$\begin{aligned}
\int_0^\infty \frac{x^{pc+\beta s-p+(\beta-1)(v-p)}}{(1+x^c)^{z+p+v}} dx &= \frac{1}{c} B\left(z+p+v-2 - \frac{pc+\beta s-p+(\beta-1)(v-p)+1}{c}\right. \\
&\quad \left., \frac{pc+\beta s-p+(\beta-1)(v-p)+1}{c}\right).
\end{aligned}$$

Consequently, Rényi entropy for the HLLLoGW distribution reduces to

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v} \log \left(\sum_{p,z,s=0}^{\infty} 2^v (\alpha \beta)^{v-p} c^p [\alpha(z+v)]^s \binom{v}{p} \frac{\Gamma(2v+z)}{\Gamma(2v)z!} (-1)^{z+s} \right. \\
&\quad \times \frac{1}{c} B\left(z+p+v-2 - \frac{pc+\beta s-p+(\beta-1)(v-p)+1}{c}\right. \\
&\quad \left., \left. \frac{pc+\beta s-p+(\beta-1)(v-p)+1}{c}\right) \right),
\end{aligned}$$

for $v \neq 1$ and $v > 0$.

5.4 Estimation

Let $X_i \sim HLLLoGW(\alpha, \beta, c)$ and $\Delta = (\alpha, \beta, c)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned}
\ell &= n \ln(2) - \alpha \sum_{i=1}^n x_i^\beta - \sum_{i=1}^n \ln(1+x_i^c) + \sum_{i=1}^n \ln [(1+x_i^c)^{-1} c x_i^{c-1} + \alpha \beta x_i^{\beta-1}] \\
&\quad - 2 \sum_{i=1}^n \ln [1 + (1+x_i^c)^{-1} e^{-\alpha x_i^\beta}].
\end{aligned} \tag{5.26}$$

Elements of the score vector $U = (\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial c})$ can be readily obtained. The equations obtained by setting the partial derivatives to zero are not in closed form and the values of the parameters α, c, β must be found via iterative methods. The maximum likelihood estimates (MLE) of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial c})^T = 0$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $I(\Delta) = [I_{\theta_i, \theta_j}]_{3 \times 3} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3$, can be numerically obtained by MATLAB or NLMIXED in SAS or R software. The total Fisher information matrix $nI(\Delta)$ can be approximated by

$$J_n(\hat{\Delta}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \Big|_{\Delta=\hat{\Delta}} \right]_{3 \times 3}, \quad i, j = 1, 2, 3. \quad (5.27)$$

For a given set of observations, the matrix given in equation (5.27) is obtained after the convergence of the Newton-Raphson procedure via NLMIXED in SAS or R software.

The multivariate normal distribution $N_3(\underline{0}, J^{-1}(\hat{\Delta}))$, where the mean vector $\underline{0} = (0, 0, 0)^T$ and $J^{-1}(\hat{\Delta})$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1 - \eta)\%$ two-sided confidence intervals for λ , c , and δ are given by:

$$\hat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\Delta})}, \quad \hat{c} \pm Z_{\frac{\eta}{2}} \sqrt{I_{cc}^{-1}(\hat{\Delta})}, \quad \text{and} \quad \hat{\delta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\delta\delta}^{-1}(\hat{\Delta})},$$

respectively, where $I_{\lambda\lambda}^{-1}(\hat{\Delta})$, $I_{cc}^{-1}(\hat{\Delta})$, and $I_{\delta\delta}^{-1}(\hat{\Delta})$, are the diagonal elements of $I_n^{-1}(\hat{\Delta}) = (nI(\hat{\Delta}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

5.5 Monte Carlo Simulations

In this section, the performance of the maximum likelihood estimates is examined by conducting simulation studies for different sample sizes. We examine the performance of the HLLLoGW distribution by conducting various simulations for different sizes ($n=25, 50, 100, 200, 400, 800, 1200$) via the R package. We simulate $N = 1000$ samples for the true parameters values

given in the Table 5.3 and Table 5.4. The Average Bias and Root Mean Square Error (RMSE) were computed. The average bias and RMSE for the estimated parameter $\hat{\theta}$, say, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. The table lists the mean MLEs of the parameters along with the respective root mean squared errors (RMSEs).

Table 5.3: Monte Carlo Simulation Results

parameter	Sample Size	(0.4,0.9,0.8)			(1.4,1.5,1.8)			(2.0,1.8,0.5)		
		Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	0.3770	0.2238	-0.0229	1.4484	0.4283	0.0484	2.2140	0.8278	0.2140
	50	0.3778	0.1647	-0.0221	1.4140	0.2665	0.0140	2.0932	0.3781	0.0932
	100	0.3889	0.1113	-0.0110	1.4095	0.1932	0.0095	2.0461	0.2593	0.0461
	200	0.3952	0.0788	-0.0047	1.3945	0.1301	-0.0054	2.0124	0.1662	0.0124
	400	0.3982	0.0557	-0.0017	1.3959	0.0918	-0.0040	2.0036	0.1182	0.0036
	800	0.3986	0.0411	-0.0013	1.3963	0.0640	-0.0036	1.9981	0.0872	-0.0018
	1200	0.3992	0.0307	-0.0007	1.3969	0.0505	-0.0030	2.0004	0.0666	0.0004
β	25	1.1515	0.6706	0.2515	1.7665	0.6636	0.2665	1.9857	0.7112	0.1857
	50	1.0041	0.3799	0.1041	1.6411	0.4169	0.141	1.8706	0.3721	0.0706
	100	0.9392	0.2029	0.0392	1.5879	0.2907	0.0879	1.8417	0.2574	0.0417
	200	0.9114	0.1297	0.0114	1.5541	0.2105	0.0541	1.8204	0.1685	0.0204
	400	0.9042	0.0905	0.0042	1.5327	0.1615	0.0327	1.8136	0.1149	0.0136
	800	0.9033	0.0613	0.0033	1.5205	0.1313	0.0205	1.8052	0.0777	0.0052
	1200	0.9011	0.0482	0.0011	1.5150	0.1145	0.0150	1.8050	0.0666	0.0050
c	25	0.8826	0.3218	0.0826	1.9035	0.9145	0.1035	0.6457	0.3520	0.1457
	50	0.8632	0.2515	0.0632	1.9477	0.9020	0.1477	0.5795	0.2586	0.0795
	100	0.8390	0.1721	0.0390	1.8668	0.6362	0.0668	0.54749	0.1730	0.0474
	200	0.8304	0.1384	0.0304	1.8438	0.4899	0.0438	0.5265	0.0688	0.0265
	400	0.8158	0.0857	0.0158	1.8388	0.4158	0.0388	0.5232	0.0495	0.0232
	800	0.8079	0.0554	0.0079	1.8248	0.3487	0.0248	0.5206	0.0365	0.0206
	1200	0.8052	0.0445	0.0052	1.8231	0.3025	0.0231	0.5197	0.0312	0.0197

Table 5.4: Monte Carlo Simulation Results

parameter	Sample Size	(2.0,2.8,2.0)			(3.0,2.5,2.5)			(3.0,3.2,3.4)		
		Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	2.2207	0.7338	0.2207	3.4622	1.2786	0.4622	3.4705	1.3094	0.4705
	50	2.0876	0.3748	0.0876	3.2513	0.7018	0.2513	3.2182	0.7654	0.2182
	100	2.0431	0.2376	0.0431	3.1385	0.4579	0.1385	3.1235	0.4219	0.1235
	200	2.0198	0.1620	0.0198	3.0656	0.2854	0.0656	3.0672	0.2715	0.0672
	400	2.0141	0.1205	0.0141	3.0332	0.1829	0.0332	3.0343	0.1835	0.0343
	800	2.0087	0.0849	0.0087	3.0290	0.1321	0.0290	3.0314	0.1332	0.0314
	1200	2.0022	0.0649	0.0022	3.0185	0.1086	0.0185	3.0152	0.1037	0.0152
β	25	3.2089	1.0516	0.4089	2.9459	0.9958	0.4459	3.7442	1.1905	0.5442
	50	2.9372	0.5824	0.1372	2.7485	0.5654	0.2485	3.4776	0.7698	0.2776
	100	2.8377	0.4105	0.0377	2.6677	0.4040	0.1677	3.3758	0.5001	0.1758
	200	2.7816	0.3171	-0.0183	2.6009	0.2851	0.1009	3.3013	0.3739	0.1013
	400	2.7870	0.2418	-0.0129	2.5612	0.2069	0.0612	3.2674	0.2756	0.0674
	800	2.7904	0.1766	-0.0095	2.5454	0.1665	0.0454	3.2526	0.2110	0.0526
	1200	2.7916	0.1449	-0.0083	2.5345	0.1433	0.0345	3.2431	0.2005	0.0431
c	25	2.2358	0.8174	0.2358	3.3568	17.4269	0.8568	4.0698	5.9043	0.6698
	50	2.2474	0.7931	0.2474	2.6406	3.3831	0.1406	4.0007	6.2780	0.6007
	100	2.2319	0.7259	0.2319	2.4733	0.7425	-0.0266	3.6606	2.3400	0.2606
	200	2.2086	0.6263	0.2086	2.4779	0.6413	-0.0220	3.5826	1.6641	0.1820
	400	2.1423	0.4936	0.1423	2.4783	0.5246	-0.0216	3.4765	0.9394	0.0765
	800	2.0935	0.3592	0.0935	2.5126	0.4851	0.0126	3.4605	0.7565	0.0605
	1200	2.0588	0.2875	0.0588	2.5002	0.4322	0.0002	3.4442	0.7111	0.0442

From Table 5.3 and Table 5.4 above we conclude that estimation method is adequate as the simulated estimates are closed to the true values of parameters. We also observed that estimated root mean square errors ($RMSE_s$) consistently decreases with increasing sample size and the average bias decreases as the sample size n increases.

5.6 Applications

The flexibility and usefulness of the HLLLoGW distribution and its sub-models for data modeling is illustrated in this section. We compare the HLLLoGW distribution with the generalized Lindley (GL) distribution (Zakerzadeh and Dolati [87]), Lindley-Weibull (LW) distribution (Asgharzadeh et al. [5]), Log-Logistic Weibull (LLoGW) distribution (Oluyede et al. [66]), Marshall-Olkin Log-logistic distribution (MOLLD) (Wenhao [84]), the exponentiated Fréchet (EF) distribution (Nadarajah and Kotz [55]) and the Odd Lindley Fréchet (OLiFr) distribution (Mansour et al. [46]). The pdf of the

generalized Lindley (GL) distribution is given by

$$g_{GL}(x; \alpha, \beta, \theta) = \frac{\theta^{\alpha+1}}{(\beta + \theta)\Gamma(\alpha + 1)}(\alpha + \beta x)e^{-\theta x}, \quad (5.28)$$

for $\alpha, \beta, \theta > 0$ and $x > 0$. The LW distribution (see Asgharzadeh et al. [5] for details) has the pdf given by

$$g_{LW}(x; \lambda, \alpha, \beta) = \frac{e^{-\lambda x - \alpha x^\beta}}{1 + \lambda} [\lambda^2(1 + x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1}], \quad (5.29)$$

for $\lambda, \alpha, \beta > 0$ and $x > 0$. The pdf of the Log-logistic Weibull (LLoGW) distribution is given by

$$g_{LLoGW}(x; c, \alpha, \beta) = (1 + x^c)^{-1} e^{-\alpha x^\beta} [(1 + x^c)^{-1} c x^{c-1} + \alpha \beta x^{\beta-1}], \quad (5.30)$$

for $c, \alpha, \beta > 0$, and $x \geq 0$. The MOLLD pdf is given by

$$g_{MOLLD}(x; \alpha, \beta, \gamma) = \frac{\alpha^\beta \beta \gamma x^{\beta-1}}{(x^\beta + \alpha^\beta \gamma)^2} \quad (5.31)$$

for $\alpha, \beta, \gamma > 0$, and $x > 0$. The pdf of the exponentiated Fréchet (EF) distribution is given by

$$g_{EF}(x; \alpha, \beta, \theta) = \theta \beta \alpha^\beta x^{-\beta-1} \exp(-(\alpha/x)^\beta) (1 - \exp(-(\alpha/x)^\beta))^{\theta-1} \quad (5.32)$$

for $\alpha, \beta, \theta > 0$, and $x > 0$.

The pdf of the Odd Lindley Fréchet (OLiFr) distribution is given by

$$g_{OLiFr}(x; \theta, \alpha, \beta) = \frac{\theta \alpha^\beta \theta^2 x^{-\beta-1} \exp[-(\frac{\alpha}{x})^\beta]}{(1 + \theta) \left(1 - \exp[-(\frac{\alpha}{x})^\beta]\right)^3} \exp\left(\frac{-\theta \exp[-(\frac{\alpha}{x})^\beta]}{1 - \exp[-(\frac{\alpha}{x})^\beta]}\right) \quad (5.33)$$

for $\theta, \alpha, \beta > 0$, and $x > 0$.

The maximum likelihood estimates (MLEs) of the HLLLoGW parameters and its sub-models are computed by maximizing the objective function via the subroutine NLMIXED in SAS as well as the function nlm in R. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic ($-2 \ln(L)$), Akaike Information Criterion ($AIC = 2p - 2 \ln(L)$), Bayesian Information Criterion ($BIC = p \ln(n) - 2 \ln(L)$), and Consistent Akaike Information Criterion ($AICC = AIC + 2 \frac{p(p+1)}{n-p-1}$), where $L = L(\hat{\Delta})$ is the value

of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. Tables 5.5 and 5.6 shows results for the data set for HLLLoGW distribution, its sub-models and several non-nested models.

The likelihood ratio (LR) test can be used to compare the fit of the HLLLoGW distribution with its sub-models for a given data set. For example, to test $\beta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\alpha}, \hat{c}, \hat{\beta})) - \ln(L(\tilde{\alpha}, \tilde{c}, 1))]$, where $\hat{\alpha}$, $\hat{c}$, and $\hat{\beta}$ are the unrestricted estimates, and $\tilde{\alpha}$, and $\tilde{c}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_{\epsilon}$, where χ^2_{ϵ} denote the upper 100 ϵ % point of the χ^2 distribution with 1 degrees of freedom.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [14]) are given in Figure 5.4 and Figure 5.5. For the probability plot, we plotted $G_{HLLLoGW}(x_{(j)}; \hat{\alpha}, \hat{c}, \hat{\beta})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^n \left[G_{HLLLoGW}(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The goodness-of-fit statistics W^* and A^* , described by (Chen and Balakrishnan [16]) are presented in the tables. The Kolmogorov-Smirnov (KS) and its P-value are also presented in Tables 5.5 and 5.6. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of KS, W^* and A^* , the better the fit.

5.6.1 Time to failure of kevlar 49/epoxy strands tested at various stress level data

The data consists of 101 data points representing the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90% stress level until all had failed, so that we have complete data with exact times of failure. The failure times in hours from (Barlow et al. [6]) are shown below

0.01,0.01,0.02,0.02,0.02,0.03,0.03,0.04,0.05,0.06,0.07,0.07,0.08,0.09,0.09,
0.10,0.10,0.11,0.11,0.12,0.13,0.18,0.19,0.20,0.23,0.24,0.24,0.29,0.34,0.35,

0.36,0.38,0.40,0.42,0.43,0.52,0.54,0.56,0.60,0.60,0.63,0.65,0.67,0.68,0.72,
0.72,0.72,0.73,0.79,0.79,0.80,0.80,0.83,0.85,0.90,0.92,0.95,0.99,1.00,1.01,
1.02,1.03,1.05,1.10,1.10,1.11,1.15,1.18,1.20,1.29,1.31,1.33,1.34,1.40,1.43,
1.45,1.50,1.51,1.52,1.53,1.54,1.54,1.55,1.58,1.60,1.63,1.64,1.80,1.80,1.81,
2.02,2.05,2.14,2.17,2.33,3.03,3.03,3.34,4.20,4.69,7.89.

Estimates of the parameters of HLLLoGW distribution and its related sub-models (standard error in parentheses), AIC, BIC, and the goodness-of-fit statistics W^* , A^* , Kolmogorov-Smirnov (KS) and its P-value as well as SS are given in Table 5.5. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 5.4.

Table 5.5: MLEs of the parameters, SEs in parenthesis and the goodness-of-fit statistics for kevlar 49/epoxy failure time data

Model	Estimates			Statistics								
	α	β	c	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
HLLLoGW	0.7797 (0.1249)	0.5256 (0.0941)	2.2184 (0.4272)	203.5713	209.5713	209.8187	217.4166	0.0822	0.5948	0.0700	0.7049	0.1038
HLLLoG	0 (-)	0 (-)	1.2229 (0.1086)	282.8573	284.8573	284.8977	287.4725	0.3966	2.1424	0.3834	2.5410×10^{-13}	5.7580
HLLLoGR	0.2311 (0.0435)	2 (-)	0.7482 (0.1053)	233.7703	237.7703	237.8928	243.0006	0.2067	1.5106	0.2211	0.0001	1.6510
HLLLoGW(1, β , c)	1 (0.0944)	0.8914 (0.0944)	0.7852 (-)	207.1673	211.1673	211.2898	216.3976	0.1718	0.9811	0.1422	0.0335	0.5198
GL	0.7447 (0.1739)	0.9518 (1.0959)	1.1653 (0.1847)	205.0725	211.0725	211.3199	218.9178	0.6847	4.5739	0.3043	1.493e-08	0.8912
LW	λ 0.8890 (0.5527)	α 0.4716 (0.4809)	β 0.7481 (0.2543)	205.2548	211.2548	211.5023	219.1002	0.1479	0.8797	0.0785	0.5617	0.1441
LLoGW	c 2.1997 (0.3761)	α 0.4113 (0.0919)	β 0.5412 (0.1021)	207.4965	213.4965	213.7439	221.3419	0.1343	0.8644	0.0998	0.2664	0.2699
MoLLD	α 0.6380 (0.0684)	β 1.2704 (0.1069)	γ 0.9721 (0.0353)	210.4262	231.3724	231.6198	239.2178	0.5653	3.0709	0.1113	0.1637	0.3892
EF	α 2.2673e+08 (1.6273e-14)	β 1.0818e-01 (6.5876e-04)	θ 3.2881e+03 (1.2979e-09)	210.4262	216.4264	216.6738	224.2717	0.3373	1.8018	0.1218	0.0998	0.9359
OLiFr	α 0.03768 (0.0353)	θ 0.2649 (0.1626)	β 0.6348 (0.04799)	207.2066	213.2066	213.4540	221.0519	0.2858	1.5263	0.1065	0.2021	1.1409

The Likelihood ratio (LR) test statistic for testing the hypothesis H_0 : $HLLLoG$ against H_a : $HLLLoGW$, H_0 : $HLLLoGR$ against H_a : $HLLLoGW$ and H_0 : $HLLLoG(1, \beta, c)$ against H_a : $HLLLoGW$ are 78.2860 (p-value < 0.00001), 30.1990 (p-value < 0.00001) and 3.596 (p-value = 0.0579). We conclude that there are significant differences between HLLLoG and HLLLoGW distributions, HLLLoGR and HLLLoGW distributions as well as between $HLLLoG(1, \beta, c)$ and HLLLoGW distributions at the 10% level of significance based on the LR tests. The values of the goodness-of-fit statistics: W^* , A^* , KS and its p-value clearly

show that the HLLLoGW distribution is by far the better fit for the kevlar 49/epoxy failure time data. The value of sum of squares (SS=0.1038) from the probability plots in Figure 5.4 is smaller for HLLLoGW distribution. Also, the goodness-of-fit statistics KS, W^* and A^* are smaller for HLLLoGW distribution as compared to nested and non-nested distributions for the kevlar 49/epoxy failure time data, hence we can conclude that the HLLLoGW distribution is by far better fit.

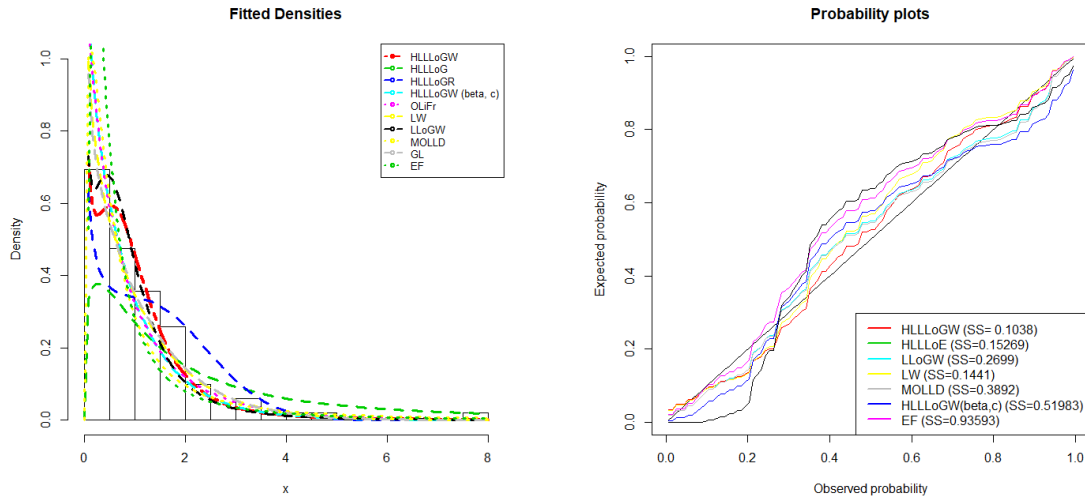


Figure 5.4: Fitted Densities and Probability Plots for Time to Failure of kevlar 49/epoxy strands tested at various stress level data

5.6.2 Electronics Data

The data for lifetimes of 20 electronic components obtained from (Murthy et al. [51]) are given as

0.03,0.12,0.22,0.35,0.73,0.79,1.25,1.41,1.52,1.79,1.80,1.94,2.38,2.40,2.87,2.99, 3.14,3.17,4.72,5.09.

Estimates of the parameters of HLLLoGW distribution and its related sub-models (standard error in parentheses), AIC, BIC, and the goodness-of-fit statistics W^* , A^* , Kolmogorov-Smirnov (KS) and its P-value as well as SS are given in Table 5.6. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 5.5.

Table 5.6: Estimates of Models for Lifetimes of 20 Electronic Components Data

Model	Estimates			Statistics								
	α	β	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
HLLLoGW	0.0613 (0.0827)	2.4057 (0.8385)	0.8223 (0.3242)	62.9432	68.9432	70.44325	71.93044	0.0288	0.2022	0.1194	0.9058	0.0368
HLLLoG	0	0	1.5334 (0.2692)	71.9342	73.9342	74.1564	74.9299	0.1572	0.9015	0.1542	0.6723	0.0865
HLLLoGE	0.3056 (0.1736)	1 -	0.9868 (0.4452)	68.1241	72.1241	72.8300	74.11559	0.0997	0.5790	0.2339	0.1907	0.2397
HLLLoGW(1, β , c)	1 -	0.7358 (0.1740)	0.5908 (0.3861)	79.5829	83.5829	84.2887	85.5743	0.0904	0.5281	0.4486	0.0003	1.3743
	α	β	θ									
GL	0.8162 (0.4566)	1.8070 (2.8134)	0.7823 (0.2144)	64.9147	70.9147	72.4147	73.9019	0.0767	0.6479	0.2118	0.2882	0.0626
	λ	α	β									
LW	0.7740 (0.2021)	0.0346 (0.1779)	0.5474 (0.9914)	65.0536	71.0536	72.5536	74.0408	0.0610	0.3644	0.1352	0.8108	0.0628
	c	α	β									
LLoGW	0.9210 (0.2984)	0.0145 (0.0287)	3.1704 (1.2036)	66.3973	72.3973	73.8973	75.3845	0.0539	0.3194	0.2642	0.101	0.3366
	α	β	γ									
MoLLD	0.7952 (0.2248)	1.4979 (0.2889)	2.4902 (0.0479)	71.8445	77.8445	79.3445	80.8317	0.1628	0.9318	0.1415	0.7669	0.0971
	α	β	θ									
EF	3.5757×10^{08} (2.2441×10^{-14})	1.1679×10^{-01} (1.3058×10^{-03})	1.0381×10^{04} (7.2556×10^{-10})	67.8969	73.8966	75.3966	76.8838	0.1105	0.6387	0.1642	0.5961	0.1116
	α	θ	β									
OLiFr	0.0635 (0.1042)	0.1428 (0.1863)	0.7798 (0.1355)	66.8895	72.8895	74.3895	75.8767	0.0975	0.5664	0.1475	0.7232	0.0875

The Likelihood ratio (LR) test statistic for testing the hypothesis H_0 : $HLLLoG$ against H_a : $HLLLoGW$, H_0 : $HLLLoGE$ against H_a : $HLLLoGW$ and H_0 : $HLLLoG(1, \beta, c)$ against H_a : $HLLLoGW$ are 8.991 (p-value= 0.0111), 5.1809 (p-value= 0.0143) and 16.6397 (p-value= 0.00004). We conclude that there are significant differences between HLLLoGW and its Sub-models at the 5% level of significance. The values in table 5.6 show that the HLLLoGW distribution gives the smallest values for the goodness-of-fit statistics and the greatest p-value of KS-test. Thus, the HLLLoGW distribution provides better fit than the rest of the distributions for the lifetimes of 20 electronic components data. Additionally, the value of sum of squares (SS=0.0368) from the probability plots in Figure 5.5 is smaller for HLLLoGW distribution confirming that the HLLLoGW distribution is a better fit for the lifetimes of 20 electronic components data compared to the nested and non-nested distributions in Table 5.6.

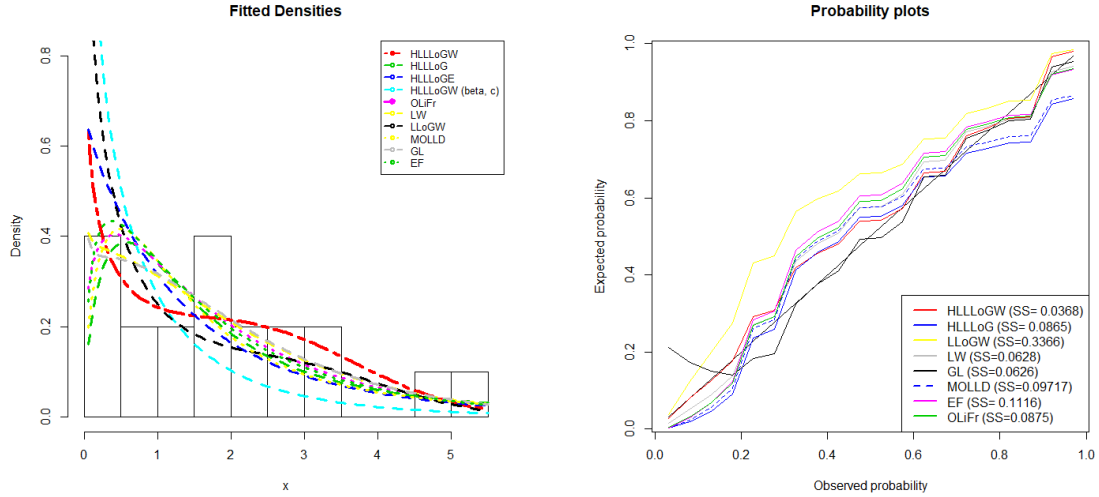


Figure 5.5: Fitted Densities and Probability Plots of Lifetimes of 20 Electronic Components Data

5.7 Concluding Remarks

We have proposed and studied a new three parameter distribution called the Half Logistic Log-Logistic Weibull (HLLLoGW) distribution. Distributional properties of this model are derived. Maximum likelihood estimation technique is used to estimate the model parameters. Monte Carlo simulation study is carried out to examine the accuracy of the maximum likelihood estimates. The importance of the HLLLoGW is exemplified by two real life datasets.

Chapter 6

Conclusion and Suggestions For Further Studies

6.1 Conclusion

Lifetime modeling and reliability analysis are very important in areas such as computer science, engineering, social sciences. We developed and studied new models namely, Lindley-Weibull Power Series (LWPS) class of distributions and their special case called Lindley-Weibull logarithmic (LWL) distribution, a new gamma generalized Lindley log-logistic (GELLLoG) distribution, Marshall-Olkin Lindley log-logistic (MOLLLoG) distribution and half logistic log-logistic Weibull (HLLLoGW) distribution. These distributions exhibit monotonic and non-monotonic hazard rate behavior. Distributional properties of these models were derived. Monte Carlo simulation study at different sample sizes was carried out to examine the accuracy of the maximum likelihood estimates. The importance of the proposed distributions was exemplified by real life datasets. We conclude that the proposed models are useful since they performed better than the nested models and other non-nested models on the selected datasets.

6.2 Suggestions for Further Studies

Future research may include the use of other methods of estimation such as Weighted Least Square estimation and Bayesian estimation. Also, further research on proposed distributions can be study on mixtures and multivariate extensions.

Bibliography

- [1] Afify, A. Z., Altun, E., Alizadeh, M., Ozel, G., and Hamedani, G. G., (2017) *The Odd Exponentiated Half-Logistic-G Family: Properties, Characterizations and Applications*, Chilean Journal of Statistics , 8 (2), 65-91.
- [2] Afify, A. Z., Cordeiro, G. M., Yousof, H. M. , Saboor, A., and Ortega, E. M.M., (2018). *The Marshall-Olkin Additive Weibull Distribution with Variable Shapes for the Hazard Rate*, Hacettepe Journal of Mathematics and Statistics, 47(2), 365-381.
- [3] Anwar, A., and Bibi, A., (2018). *The Half-Logistic Generalized Weibull Distribution*, Journal of Probability and Statistics, Volume 2018, Article ID 8767826, 12 pages, <https://doi.org/10.1155/2018/8767826>
- [4] Anwar, M., and Zahoor, J., (2018). *The Half-Logistic Lomax Distribution for Lifetime Modeling*, Journal of Probability and Statistics , Volume 2018, Article ID 3152807, 12 pages, <https://doi.org/10.1155/2018/3152807>.
- [5] Asgharzadeh, A., Nadarajah, S., and Sharafi, F. (2016). *Weibull Lindley Distribution*, Revstat-Statistical Journal, Vol. 16, No 1, 87-113.
- [6] Barlow, R. E., Towland, R. H., and Freeman, T., (1984). *A Bayesian Analysis of Stress-Rupture Life of Kevlar 49/epoxy Spherical Pressure Vessels*, Proceedings of the Canadian Conference on Applied Statistics.
- [7] Barreto-Souza, W., Santos, A. H. S. and Cordeiro, G. M. (2010). *The Beta Generalized Exponential Distribution*, Journal of Statistical Computation and Simulation, 80(2), 159-172.

- [8] Barreto-Souza, W., Lemonte, A. J., and Cordeiro, G. M. (2013). *General Results for the Marshall and Olkin Family of Distributions*, Annals of the Brazilian Academy of Sciences, 85(1), 3-21.
- [9] Bebbington M., Lai C. D., and Zitikis R. (2007). *A Flexible Weibull Extension*, Reliability Engineering System Safety, 92, 719-726.
- [10] Bourguignon, M., Silva, R. B., and Cordeiro, G. M. (2014). *A New Class of Fatigue Life Distributions*, Journal of Statistical Computation and Simulation, 84(12), 2619-2635.
- [11] Bidram, H. and Nekoukhou, V. (2013). *Double Bounded Kumaraswamy-Power Series Class of Distributions*, Statistics and Operations Research Transactions, 37(2), 211-230.
- [12] Chahkandi, M. and Ganjali, M. (2009). *On Some Lifetime Distributions with Decreasing Failure Rate*, Computational Statistics and Data Analysis, 53(12), 4433-4440.
- [13] Chakraborty, S., and Handique, L., (2017). *The Generalized Marshall-Olkin-Kumaraswamy-G Family of Distributions*, Journal of Data Science, 15(3), 391-422.
- [14] Chambers, J., Cleveland, W., Kleiner, B. and Tukey, P. (1983). *Graphical Methods of Data Analysis*, Chapman and Hall.
- [15] Chhikara R.S., and Folks J. I., (1977). *The Inverse Gaussian Distribution as a Lifetime Model*, Technometrics, No 19, 461-8.
- [16] Chen, G. and Balakrishnan, N. (1985). *A General Purpose Approximate Goodness-of-fit Test*, Journal of Quality Technology, 27(2), 154-161.
- [17] Cordeiro, G. M., Afify, A. Z., Yousof, H. M., Cakmakyapan, S., and Ozel, G.,(2018). *The Lindley Weibull Distribution: Properties and Applications*, Annals of the Brazilian Academy of Sciences, 90(3): 2579-2598.

- [18] Cordeiro, G. M., Alizadeh, M., and Marinho, P. R. D., (2016). *The Type I Half-Logistic Family of Distributions*, Journal of Statistical Computation and Simulation, vol.86, no.4, pp.707-728.
- [19] Cordeiro, G. M., and Lemonte A. J. A. (2013). *On the Marshall-Olkin Extended Weibull Distribution*, Statistical Papers, 54, 333-353.
- [20] Cordeiro, G. M., Ortega E. M., and Nadarajah S., (2010). *The Kumaraswamy Weibull Distribution with Applications to Failure Data*, Journal of Franklin Institute, 347, 1399-1429.
- [21] Cordeiro, G. M. and Silva, R. B. (2014). *The Complementary Extended Weibull Power Series Class of Distributions*, Ciência e Natura, 36(3).
- [22] Doostmoradi, A., Zadkarami, M. R. and Roshani Sheykhabad, A. (2014). *A New Modified Weibull Distribution and Its Application*, Journal of Statistical Research and Training Center, Iran 11, 97-118.
- [23] Elbatal, I. (2011). *Exponentiated Modified Weibull Distribution*. Economic Quality Control. 26, 189-200.
- [24] Famoye, F., Lee, C., and Olumolade, O., (2005). *The Beta Weibull Distribution*, Journal of Statistical Theory and Applications, 4 (2), 121-136.
- [25] Feigl, P., and Zelen, M., (1965). *Estimation of Exponential Survival Probabilities with Concomitant Information*, Biometrics, 21 (4), 826-838.
- [26] Flores, J., Borges, P., Cancho, V. G., and Louzada, F. (2013). *The Complementary Exponential Power Series Distribution*, Brazilian Journal of Probability and Statistics, 27(4), 565-584.
- [27] Ghitany, M. E, Al-Hussaini, E. K., and Al-Jarallah, R. A., (2005). *Marshall-Olkin Extended Weibull Distribution and Its Application to Censored Data*, Journal of Applied Statistics, 32(10), 1025-1034.
- [28] Ghitany, M.E., Al-Mutairi, D.K., Balakrishnan, N., and Al-Enezi, L.J. (2013). *Power Lindley Distribution and Associated Inference*, Comput. Stat. Data Anal. 64:20-33.

- [29] Ghitany, M.E., Atieh, B., and Nadarajah, S. (2008). *Lindley Distribution and Its Applications*, Math. Comput. Simul. 78(4):493-506.
- [30] Gradshteyn, I. S., and Ryzhik, I. M. (2000). Tables of Integrals, Series and Products, sixth edition, Academic Press, San Diego.
- [31] Gross, A.J., and Clark, V.A.,(1975). *Survival distributions: reliability applications in the biomedical sciences*, NewYork, JohnWiley and Sons.
- [32] Javed, M., Nawaz, T., and Irfan M., (2018) *The Marshall-Olkin Kappa Distribution*, Journal of King Saud University-Science, 684-691.
- [33] Krishna, E., Jose, K. K., and Ristic, M. M., (2013). *Applications of Marshall-Olkin Fréchet Distribution*, Communications in Statistics-Simulation and Computation, 42:1, 76-89.
- [34] Kumar, D., (2016). *Ratio and Inverse Moments of Marshall-Olkin Extended Burr Type III Distribution Based on Lower Generalized Order Statistics*, Journal of Data Science, 14(1), 53-66.
- [35] Lai, C. D., Xie, M., and Murthy, D. N. P., (2003). *A Modified Weibull Distribution*, IEEE Transactions on Reliability, 52(1), 33-37.
- [36] Lai, C. D., Xie, M., and Murthy, D. N. P., (2005). *Reply to On Some Recent Modifications of Weibull distribution*, IEEE Transactions on Reliability, 54(4), 563.
- [37] Lazhar, B., (2017). *Marshall-Olkin Extended Generalized Gompertz Distribution*, Journal of Data Science, 15(2), 239-266.
- [38] Leiva V., Barros M., and Paula G. A., (2009). *Generalized Birnbaum-Sanders Models using R*, Brazil, Brazilian Statistical Association, No 53, 1511-1528.
- [39] Lemonte, A. J., (2014). *The Beta Log-Logistic Distribution*, Brazilian Journal of Probability and Statistics, 28, 313-332. <https://doi.org/10.1214/12-BJPS209>.

- [40] Lepetu, L., Oluyede, B. O, Makubate, B., Foya, S., and Mdlongwa, P., (2017). *Marshall-Olkin Log-Logistic Extended Weibull Distribution*, Journal of Data Science, 691-722.
- [41] Lindley, D. V., (1958). *Fiducial Distributions and Bayes Theorem*, J. Royal Stat. Soc. Ser. B. 20:102-107.
- [42] Louzada, F., Marchi, V., and Roman, M. (2014). *The Exponentiated Exponential-Geometric Distribution: a Distribution with Decreasing, Increasing and Unimodal Failure Rate*, Statistics, 48(1), 167-181.
- [43] Lu, W. and Shi, D. (2011). *A New Compounding Life Distribution: The Weibull-Poisson Distribution*, Journal of Applied Statistics, 39(1), 21-38.
- [44] Mahmoudi, E. and Jafari, A. A. (2012). *Generalized Exponential-Power Series Distributions*, Computational Statistics and Data Analysis, 56(12), 4047-4066.
- [45] Mahmoudi, E and Jafari, A. A. (2017). *The Compound Class of Linear Failure Rate-Power Series Distributions: Model, Properties and Applications*, Communications in Statistics - Simulation and Computation, 46(2), 1414-1440.
- [46] Mansour, M., M., Elrazik¹, E. M. A., Altun, E., Afify, A., Z., and Iqbal, Z. (2018). *A New Three-Parameter Fréchet Distribution: Properties and Applications*, Pakistan Journal of Statistics, 34(6), 441-458.
- [47] Marshall A. W., and Olkin I., (1997). *A New Method for Adding a Parameter to a Family of Distributions with Application to the Exponential and Weibull Families*, Biometrika 84: 641-652.
- [48] Morais, A. L. and Barreto-Souza, W. (2011). *A Compound Class of Weibull and Power Series Distributions*, Computational Statistics and Data Analysis, 55(3), 1410-1425.
- [49] Mudholkar, G. S., and Srivastava D. K., (1993). *Exponentiated Weibull Family for Analyzing Bathtub Failure data*, IEEE Transactions on Reliability, 43, 299-302.

- [50] Muhammad, M., and Yahaya, M. A. (2017). *The Half Logistic-Poisson Distribution*, Asian Journal of Mathematics and Applications, 1-15.
- [51] Murthy, D. N. P., Xie, M., and Jiang, R. (2004). *Weibull Models*, John Wiley, New York, doi:10.1002/047147326X.
- [52] Nadarajah, S., Bakouch, H. S., and Tahmasbi, R. (2011). *A Generalized Lindley Distribution*, Sankhya B, 73:331-359.
- [53] Nadarajah, S., Cordeiro, G. M., and Ortega, E. M. M. (2015). *The Exponentiated G Geometric Family of Distributions*, Journal of Statistical Computation and Simulation, 85(8), 1634-1650.
- [54] Nadarajah, S. and Kotz, S. (2008). *Strength Modeling using Weibull Distributions* Journal of Mechanical Science and Technology, 22(7), 1247-1254.
- [55] Nadarajah, S. and Kotz, S. (2003). *The Exponentiated Fréchet Distribution* Interstat Electron. J., 1-7.
- [56] Nadarajah, S., Popović, B. V., and Ristić, M. M. (2013). *Compounding: An R Package for Computing Continuous Distributions obtained by Compounding a Continuous and a Discrete Distribution*, Computational Statistics, 28(3), 977-992.
- [57] Najar zadegan, H., and Alamatsaz, M. H. (2017). *A new generalization of weighted geometric distribution*, Journal of Statistical Theory and Applications, Vol. 16, No. 4, 522-654.
- [58] Pal, M., Ali, M. M., Woo, J., (2006). *Exponentiated Weibull Distribution*, STATISTICA, anno LXVI, n. 2.
- [59] Nichols, M. D., and Padgett, W. J. (2006). *A Bootstrap Control Chart for Weibull Percentiles*, Quality and Reliability Engineering International, 22, 141-151.
- [60] Nekoukhou, V., Alamatsaz, M. H., and Aghajani, A. H. (2013). *A Flexible Skew Generalized Normal Distribution*, Communications

- in Statistics - Theory and Methods, 42:13, 2324-2334, DOI: 10.1080/03610926.2011.599003.
- [61] Nekoukhou, V., Alamatsaz, M. H., and Bidram, H. (2015). *On an extension of the exponentiated Weibull distribution*, Communications in Statistics - Simulation and Computation, 44,1389-1404. DOI:10.1080/03610918.2013.819918.
- [62] Oluyede, B. O., Raheem, S., and Wanduku, D. (2020). *The beta Lindley Log-logistic Distribution with Applications to Lifetime Data*, submitted.
- [63] Oluyede, B. O., Kenkwo, E. and Wandaku, D., (2020). *Lindley-Log Logistic Distribution with Applications* (in preparation).
- [64] Oluyede, B.O, and Yang, T. (2015). *A New Class of Generalized Lindley Distributions with Applications*, Journal of Statistical Computation and Simulation, 85:10, 2072-2100, DOI: 10.1080/00949655.2014.917308
- [65] Oluyede, B. O., Yang, T., and Omolo, B. (2015). *A Generalized Class of Kumaraswamy Lindley Distribution with Applications to Lifetime Data*, Journal of Computations & Modelling, vol.5, no.1, 27-70.
- [66] Oluyede, B. O., Foya, S., Warahena-Liyanage, G. and Huang, S. (2016). *The Log-logistic Weibull Distribution with Applications to Income and Lifetime Data*, Austrian Journal of Statistics, 45, 43-69.
- [67] Oluyede, B.O., Makubate, B., Fagbamigbe, A. F. and Mdlongwa, P. (2018). *A New Burr XII-Weibull-Logarithmic Distribution for Survival and Lifetime Data Analysis: Model, Theory and Applications*, Stats 2018, 1(1), 77-91; <https://doi.org/10.3390/stats1010006>
- [68] Roman, M., Louzada, F., Cancho, V. G., and Leite, J. G. (2012). *A New Long-Term Survival Distribution for Cancer Data*, Journal of Data Science, 10(2), 241-258.
- [69] R Development Core Team (2014). *A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria

- [70] Rényi, A., (1960). *On Measures of Entropy and Information*, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, **1**, 547 - 561.
- [71] Santos-Neo, M., Bourguignon, M., Zea, L. M., and Nascimento, A. D. C., (2014). *The Marshall-Olkin Extended Weibull Family of Distributions*, Journal of Statistical Distributions and Applications, 1-9.
- [72] Santos Silva, J. M. C., and Tenreyro, S., (2010). *On the Existence of Maximum Likelihood Estimates in Poisson Regression*, Econ. Lett., **107**, 310-312.
- [73] Seregin, A., (2010). *Uniqueness of the Maximum Likelihood Estimator for k -Monotone Densities*, Proc. Amer. Math. Soc. **138**, 4511-4515.
- [74] Silva, R. B., Bourguignon, M., Dias, C. R. B. and Cordeiro, G. M. (2013). *The compound class of extended Weibull power series distributions* Computational Statistics and Data Analysis, 58, 352–367.
- [75] Silva, R. B. and Cordeiro, G. M. (2015). *The Burr XII Power Series Distributions: A New Compounding Family*, Brazilian Journal of Probability and Statistics, 29(3), 565-589.
- [76] Silva, G. O., Ortega, E. M. M., Cordeiro, G. M., (2010). *The Beta Modified Weibull Distribution*, Lifetime Data Analysis, 16, 409-430.
- [77] Shams Harandi, S., and Alamatsaz, M. H. (2013). *Alpha-skew Laplace distribution*, Statistics and Probability Letters, 83, 774–782, DOI.10.1016/j.spl.2012.11.024.
- [78] Shams Harandi, S., and Alamatsaz, M. H. (2017). *A complementary generalized linear failure rate-geometric distribution*, Communications in Statistics: Simulation and Computation, 46 (1), 679-703. DOI: 10.1080/03610918.2014.950744.
- [79] Shams Harandi, S., Alamatsaz, M. H. (2016). *Generalized Linear Failure Rate Power Series Distribution*, Communications in Statistics - Theory and Methods, 45(8), 2204-2227.

- [80] Shanker, R., Shukla, K. K., Shanker, Ravi., Leonida, T. A., (2017). *A Three-Parameter Lindley Distribution*, American Journal of Mathematics and Statistics, 7(1): 15-26.
- [81] Stollmack, S., and Harris, C. M., (1974). *Failure-Rate Analysis Applied to Recidivism Data*, INFORMS, 22, No. 6, 1192-1205.
- [82] Usman, R. M., and Haq, M. A. A. U, (2018). *The Marshall-Olkin Extended Inverted Kumaraswamy Distribution*, Journal of King Saud University-Science, 356-365.
- [83] Warahena-Liyanage, G., and Pararai, M. (2015). *The Lindley Power Series Class of Distributions: Model, Properties and Applications*, Journal of Computations and Modelling, Vol. 5, No. 3, 35-80.
- [84] Wenhao, G., (2013). *Marshall-Olkin Extended Log-Logistic Distribution and Its Application in Minification Processes*, Applied Mathematical Sciences, 7(80), 3947-3961.
- [85] Xia, J., Mi, J., and Zhou, Y. Y., (2009). *On the Existence and Uniqueness of the Maximum Likelihood Estimators of Normal and Log-normal Population Parameters with Grouped Data*, J. Probab. Statist., Article id 310575, 16 pages.
- [86] Xie, M., Tang, y., and T. N. Goh., (2002). *A modified Weibull Extension with Bathtub-Shaped Failure Rate Function*, Reliability Engineering and System Safety, 76(3), 279-285.
- [87] Zakerzadeh, H., and Dolati, A., (2009). *Generalized Lindley Distribution*, Journal of Mathematical Extension, 3, No 2, 13-25.
- [88] Zhang T., and Xie, M., (2007). *Failure Data Analysis with Extended Weibull Distribution*, Communications in Statistics-Simulation and Computation, 36:579-592.
- [89] Zhou, C., (2009). *Existence and Consistency of the Maximum Likelihood Estimator for the Extreme Index*, J. Multivariate Analysis, **100**, 794-815.

-
- [90] Zografos, K. and Balakrishnan, N., (2009). *On Families of Beta and Generalized Gamma-Generated Distributions and Associated Inference*, Statistical Methodology, 6, 344-362.

Appendix

Chapter 2 Appendix

The elements of the score vector are given by

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= -\alpha \sum_{i=1}^n x_i^{\beta} \ln x_i + \sum_{i=1}^n \frac{\alpha(1+\lambda+\lambda x_i) [x_i^{\beta-1}(1+\beta \ln x_i)]}{\lambda^2(1+x_i) + (1+\lambda+\lambda x_i)\alpha\beta x_i^{\beta-1}} \\ &+ \sum_{i=1}^n \frac{C'' \left(\theta \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) \right)}{C' \left(\theta \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) \right)} \theta \left(\frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) \alpha x_i^{\beta} \ln x_i, \\ \frac{\partial \ell}{\partial \theta} &= \frac{n}{\theta} - \frac{n C'(\theta)}{C(\theta)} + \sum_{i=1}^n \frac{C'' \left(\theta \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) \right)}{C' \left(\theta \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) \right)} \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right),\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= -\sum_{i=1}^n x_i^{\beta} + \sum_{i=1}^n \frac{(1+\lambda+\lambda x_i)(\beta x_i^{\beta-1})}{\lambda^2(1+x_i) + (1+\lambda+\lambda x_i)\alpha\beta x_i^{\beta-1}} \\ &+ \sum_{i=1}^n \frac{C'' \left(\theta \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) \right)}{C' \left(\theta \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) \right)} \theta \left(\frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) x_i^{\beta},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \lambda} &= -n x_i + \sum_{i=1}^n \frac{2\lambda(1+x_i) + (1+x_i)\alpha\beta x_i^{\beta-1}}{\lambda^2(1+x_i) + (1+\lambda+\lambda x_i)\alpha\beta x_i^{\beta-1}} \\ &+ \left(\sum_{i=1}^n \frac{C'' \left(\theta \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) \right)}{C' \left(\theta \left(1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} e^{-\lambda x_i - \alpha x_i^{\beta}} \right) \right)} \right) \\ &\times \theta e^{-\lambda x_i - \alpha x_i^{\beta}} \left[x_i \left(\frac{1+\lambda+\lambda x_i}{1+\lambda} \right) - \frac{(1+\lambda)(1+x_i) - (1+\lambda+\lambda x_i)}{(1+\lambda)^2} \right].\end{aligned}$$

LWL R Algorithms

pdf plot of LWL Distribution

```
f1 = function(x, lambda, alpha, beta, theta){
y = (theta * exp(-lambda * x - alpha * x^beta))/(1 + lambda)) * ((lambda^2) * (1 + x)
+ (1 + lambda + lambda * x) * alpha * beta * x^(beta-1)) *
((1 - theta * (1 - (((1 + lambda + lambda * x)/(1 + lambda))
* exp(-lambda * x - alpha * x^beta))))^( - 1)/(-log(1 - theta)))
return(y)}
```

```

x = seq(0, 1.5, by = 0.001)
y1 = f1(x, 1.5, 1, 8, 0.99)
plot(x, y1, ylim = c(0, 3), col = 2, lwd = 2, xlab = "x", ylab = "density")
y2 = f1(x, 1.5, 1, 8, 0.49)
lines(x, y2, col = 3, lwd = 2)
y3 = f1(x, 1.5, 1, 8, 0.0001)
lines(x, y3, col = 4, lwd = 2)
y4 = f1(x, 1.5, 1, 8, 0.82)
lines(x, y4, col = 5, lwd = 2)
y5 = f1(x, 1.5, 1, 8, 0.45)
lines(x, y5, col = 6, lwd = 2)
legend("topright", c(expression(paste(lambda,' = 1.5',' , alpha,' = 1',' , beta,' = 8',' ,
theta,' = 0.99')), expression(paste(lambda,' = 1.5',' , alpha,' = 1',' , beta,' = 8',' ,
theta,' = 0.49')), expression(paste(lambda,' = 1.5',' , alpha,' = 1',' , beta,' = 8',' ,
theta,' = 0.0001')), expression(paste(lambda,' = 1.5',' , alpha,' = 1',' , beta,' = 8',' ,
theta,' = 0.82'))),
expression(paste(lambda,' = 1.5',' , alpha,' = 1',' , beta,' = 8',' ,
theta,' = 0.45'))), col = c(2, 3, 4, 5, 6), lwd = c(2, 2, 2, 2, 2))

```

hazard plot of LWL Distribution

```

f1 = function(x, alpha, lambda, beta, theta){
y = ((theta * exp((-lambda * x) - alpha * x^beta))/(1 + lambda)) * ((lambda^2)
* (1 + x) + (1 + lambda + lambda * x) * (alpha * beta * x^(beta-1))) *
(1 - theta * (1 - ((1 + lambda + lambda * x)/(1 + lambda)))
* (exp((-lambda * x) - alpha * x^beta)))^(-1)/(log(1 - theta *
(1 - ((1 + lambda + lambda * x)/(1 + lambda)))
* (exp((-lambda * x) - alpha * x^beta)))) - log(1 - theta))
return(y)}
x = seq(0, 1.5, by = 0.001)
y1 = f1(x, 1.0, 1.8, 1.4, 0.9)
plot(x, y1, ylim = c(0, 3), col = 2, lwd = 2, xlab = "x", ylab = "h(x)")
y5 = f1(x, 0.2, 1.4, 3.8, 0.99)
lines(x, y5, col = 6, lwd = 2)

```

```

legend("topright", c(expression(paste(alpha,' = 1.0,' , lambda,' = 1.8,' , beta,' = 1.4,' ,
theta,' = 0.9')),
expression(paste(alpha,' = 0.2,' , lambda,' = 1.4,' , beta,' = 3.8,' ,
theta,' = 0.99'))), col = c(2, 6), lwd = c(2, 2))

```

Quantile numbers and moments of LWL Distribution

```

install.packages("stats4")
install.packages("bbmle")
install.packages("stats")
install.packages("numDeriv")
install.packages("Matrix")
install.packages("zipfR")
install.packages("rootSolve")
library(rootSolve)
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library(Matrix)
library(zipfR)
define pdf LWL
LWLpdf <- function(alpha, lambda, beta, theta, x){
y = (theta * exp(-lambda * x - alpha * xbeta)/(1 + lambda)) * ((lambda2) *
(1 + x) + (1 + lambda + lambda * x) * alpha * beta * x(beta-1)) * ((1 - theta * (1 - (((1 +
lambda + lambda * x)
/(1 + lambda)) * exp(-lambda * x - alpha * xbeta))))(-1)/(-log(1 - theta))))}

```

define cdf LWL Distribution

```

LWLcdf <- function(alpha, lambda, beta, theta, x){
y = log(1 - theta * (1 - (((1 + lambda + lambda * x)/(1 + lambda)) *
exp(-lambda * x - alpha * xbeta))))/log(1 - theta)}

```

define hazard LWL

```

LWLhazard <- function(alpha, lambda, beta, theta, x){
y = (theta * exp(-lambda * x - alpha * xbeta)/(1 + lambda)) * ((lambda2)

```

```

* (1 + x) + (1 + lambda + lambda * x) * alpha * beta * x(beta-1)) *
((1 - theta * (1 - (((1 + lambda + lambda * x)/(1 + lambda))
* exp(-lambda * x - alpha * xbeta))))(-1)/(log(1 - theta *
(1 - (((1 + lambda + lambda * x)/(1 + lambda))
* exp(-lambda * x - alpha * xbeta)))) - log(1 - theta)))
}

```

define LWL quantile

```

LWLQuantile = function(parameter, u){
  alpha = parameter[1]
  lambda = parameter[2]
  beta = parameter[3]
  theta = parameter[4]
  f = function(x){
    LWLcdf(alpha, lambda, beta, theta, x) - u
  }
  x = min(uniroot.all(f, lower = 0, upper = 100, tol = 0.01))
  return(x)
}

```

```

LWLQuantileTable = function(parametermatrix){u = seq(0.1, 0.9, 0.1)
size = dim(parametermatrix)[1]
TableQuantile = matrix(NA, nrow = length(u), ncol = size)
row.names(TableQuantile) = u
colnames(TableQuantile) = apply(parametermatrix, 1, function(x)
paste0("'", paste0(x, collapse = ','), ")"))
TableQuantile
for(iter in 1 : size){
  parameter = parametermatrix[iter, ]
  for(i in 1 : length(u)){
    TableQuantile[i, iter] = LWLQuantile(parameter, u[i])
  }
}
return(TableQuantile)

```

}

table of quantile

```
parameter_matrix = as.matrix(rbind(par1 = c(0.1, 0.5, 2.5, 0.5),  
par2 = c(0.4, 1.0, 1.2, 0.2),  
par3 = c(1.5, 1.0, 2.5, 0.5),  
par4 = c(3.0, 2.5, 3.5, 0.8),  
par5 = c(1.6, 4.5, 6.5, 0.6)  
))
```

```
LWLQuantileTable(parameter_matrix)  
print(parameter_matrix)  
print(LWLQuantileTable(parameter_matrix))
```

End of Quantile

```
LWLpdf <- function(alpha, lambda, beta, theta, x){  
y = (theta * exp(-lambda * x - alpha * x^beta)/(1 + lambda)) *  
((lambda^2) * (1 + x) + (1 + lambda + lambda * x) * alpha * beta * x^(beta-1))  
* ((1 - theta * (1 - (((1 + lambda + lambda * x)/(1 + lambda)) *  
exp(-lambda * x - alpha * x^beta)))) - 1)/(-log(1 - theta)))  
return(y)  
}  
  
LWLmoments = function(alpha, lambda, beta, theta, n){  
f = function(alpha, lambda, beta, theta, n, x){(x^n) * (LWLpdf(alpha, lambda, beta, theta, x))  
}y = integrate(f, lower = 0, upper = 1, alpha = alpha, lambda = lambda,  
beta = beta, theta = theta, n = n)  
return(y$value)}  
  
table of moments  
  
parameter_matrix_Moments = matrix(c(0.1, 0.2, 0.2, 0.5, 0.7, 1.0, 1.5, 0.5, 0.2, 2.0, 2.0,  
0.2, 3.4, 4.0, 0.5, 0.9, 0.2, 2.0, 1.5, 0.5)  
, ncol = 4, byrow = T)  
  
parameter_matrix_Moments  
  
list_Moments = c('EX', 'EX2', 'EX3', 'EX4', 'EX5', 'EX6', 'SD', 'CV', 'CS', 'CK')
```

```

TableMoments = matrix(0, nrow = length(listMoments),
ncol = dim(parametermatrixMoments)[1])
row.names(TableMoments) = listMoments
TableMoments
for(iter in 1 : dim(parametermatrixMoments)[1]){
parameter = parametermatrixMoments[iter, ]
for(i in 1 : 10){
TableMoments[i, iter] = EOLWmoments(parameter[1], parameter[2], parameter[3]
, parameter[4], i)
}
}

```

APPLICATION FOR LWL Distribution

```

rm(list = ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library('bbmle')
x <- c(5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70,
3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08,
2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20,
4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80,
5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30,
5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60,
3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70,
4.40, 4.90, 4.90, 5.00)
LWL <- function(theta, lambda, alpha, beta){
- sum(log(theta) - lambda * x - alpha * xbeta + log(((lambda)2) * (1 + x) +
(1 + lambda + lambda * x) * alpha * beta * x(beta-1))
+ log((1 - (theta * (1 - ((1 + lambda + lambda * x)
* exp((-lambda * x) - (alpha * xbeta))))/(1 + lambda))))(-1)) - log(1 + lambda)
- log(-log(1 - theta)))}

```

```
LWL.result <- mle2(LWL, hessian = NULL, start = list(theta = 0.991, lambda = 0.114,
alpha = 0.8, beta = 0.008), optimizer = "nlminb", lower = 0)
summary(LWL.result)
```

GOODNESS OF FIT FOR LWL Distribution

```
rm(list = ls())
install.packages("stats4")
install.packages("AdequacyModel")
install.packages("bbmle")
install.packages("stats")
library(stats4)
library(AdequacyModel)
library(bbmle)
library(stats)

DATA <- -c(5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01,
4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08,
2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50,
4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80,
5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20,
5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60,
3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40,
3.70, 4.40, 4.90, 4.90, 5.00)

define LWL pdf
LWLpdf <- function(parameter, x){alpha = parameter[1]
lambda = parameter[2]
beta = parameter[3]
theta = parameter[4]
(theta * exp(-lambda * x - alpha * xbeta)/(1 + lambda)) * ((lambda2) * (1 + x) +
(1 + lambda + lambda * x) * alpha * beta * x(beta-1)) *
((1 - theta * (1 - (((1 + lambda + lambda * x)/(1 + lambda)) *
exp(-lambda * x - alpha * xbeta))))(-1)/(-log(1 - theta)))
}

define LWLcdf
LWLcdf <- function(parameter, x){
```

```

alpha = parameter[1]
lambda = parameter[2]
beta = parameter[3]
theta = parameter[4]
log(1 - theta * (1 - (((1 + lambda + lambda * x)/(1 + lambda)) *
exp((-lambda * x) - alpha * xbeta))))/log(1 - theta)
}
goodness.fit(pdf = LWLpdf, cdf = LWLcdf,
mle = c(0.0018821, 0.0023050, 4.3660237, 0.8486558),
data = DATA, method = "BFGS", domain = c(0, 100),
liminf = c(0, 0, 0, 0), limsup = c(100, 100, 100, 100))

```

Chapter 3 Appendix

Elements of Score Vector

Elements of the score vector are given by

$$\begin{aligned}
\frac{\partial \ell_n}{\partial c} &= (\delta - 1) \sum_{i=1}^n \frac{\left(1 - \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^\alpha\right)^{-1}}{\ln \left(1 - \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^\alpha\right)} \alpha \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha-1} \\
&\times \left[\frac{(1+\lambda+\lambda x_i)e^{-\lambda x_i}}{1+\lambda} + \frac{x_i^c \ln x_i}{(1+x_i^c)^2} \right] + (\alpha - 1) \sum_{i=1}^n \frac{\left[\frac{(1+\lambda+\lambda x_i)e^{-\lambda x_i}}{1+\lambda} + \frac{x_i^c \ln x_i}{(1+x_i^c)^2} \right]}{\left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]} \\
&- \sum_{i=1}^n \frac{x_i^c \ln x_i}{(1+x_i^c)} + \sum_{i=1}^n \frac{\frac{(1+\lambda+\lambda x_i)}{(1+x_i^c)^2} \left((x_i^{c-1} + c x_i^{c-1} \ln x_i)(1+x_i^c) - c x_i^{c-1} x_i^c \ln x_i \right)}{\left[\lambda^2(1+x_i) + \frac{(1+\lambda+\lambda x_i)c x_i^{c-1}}{(1+x_i^c)} \right]},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \alpha} &= (\delta - 1) \sum_{i=1}^n \frac{\left(1 - \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^\alpha\right)^{-1}}{\ln \left(1 - \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^\alpha\right)} \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^\alpha \\
&\times \ln \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right] + \frac{n}{\alpha} + \sum_{i=1}^n \ln \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \lambda} &= (\delta - 1) \sum_{i=1}^n \frac{\left(1 - \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^\alpha\right)^{-1}}{\ln \left(1 - \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^\alpha\right)} \alpha \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha-1} \\
&\times \frac{-e^{-\lambda x_i}}{(1+x_i^c)(1+\lambda)} \left(\frac{(1+x_i)(1+\lambda) - (1+\lambda+\lambda x_i)}{1+\lambda} + (1+\lambda+\lambda x_i)(x_i) \right) \\
&+ (\alpha - 1) \sum_{i=1}^n \frac{\frac{e^{-\lambda x_i}}{(1+x_i^c)} \left[\frac{(1+x_i)(1+\lambda) - (1+\lambda+\lambda x_i)}{(1+\lambda)^2} + \frac{(1+\lambda+\lambda x_i)x_i}{1+\lambda} \right]}{\left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]} - \frac{n}{1+\lambda} - \sum_{i=1}^n x_i \\
&+ \sum_{i=1}^n \frac{2\lambda x_i + \frac{c x_i^{c-1}(1+x_i)}{(1+x_i^c)}}{\left[\lambda^2(1+x_i) + \frac{(1+\lambda+\lambda x_i)c x_i^{c-1}}{(1+x_i^c)} \right]},
\end{aligned}$$

and

$$\frac{\partial \ell_n}{\partial \delta} = -\frac{n\Gamma'(\delta)}{\Gamma(\delta)} + \sum_{i=1}^n \ln \left[-\ln \left(1 - \left[1 - \frac{1+\lambda+\lambda x_i}{1+\lambda} \frac{e^{-\lambda x_i}}{(1+x_i^c)} \right]^\alpha \right) \right].$$

R Code: Pdf plot of GELLoG

```

f1=function(x,lambda,c,alpha,delta){
  y=(1/gamma(delta))*((-log(1-(1-(((1+lambda+lambda*x)
*(exp(-lambda*x))))^(alpha))^(delta-1))*alpha*
((1-(((1+lambda+lambda*x)
*(exp(-lambda*x)))/((1+lambda)*(1+x^c))))^(alpha-1))*
(((1+x^c)^-1)
*(exp(-lambda*x))/(1+lambda))*((lambda^2)*
(1+x)+((1+lambda+lambda*x)
*(c*x^(c-1))/(1+x^c))))
  return(y)
}
x=seq(0,1.5,by=0.001)
y1=f1(x,4.0,4.0,3.8,0.8)
plot(x,y1,ylim=c(0,3),col=2,'l',lwd=2,xlab="x",ylab="density")
y2=f1(x,2.4,8.0,3.0,2.0)
lines(x,y2,col=3,lwd=2)
y3=f1(x,4.0,1.0,0.2,2.0)
lines(x,y3,col=4,lwd=2)

```

```

y4=f1(x,8.0,5.8,1.2,5.4)
lines(x,y4,col=5,lwd=2)
y5=f1(x,1.5,1,9.0,1.0)
lines(x,y5,col=6,lwd=2)
legend("topright",c(
  expression(paste(lambda,'=4.0',',alpha','=4.0',',beta','=3.8',',
  theta','=0.8')),
  expression(paste(lambda,'=2.4',',alpha','=8.0',',beta','=3.0',',
  theta','=2.0')),
  expression(paste(lambda,'=4.0',',alpha','=1.0',',beta','=0.2',',
  theta','=2.0')),
  expression(paste(lambda,'=8.0',',alpha','=5.8',',beta','=1.2',',
  theta','=5.4')),
  expression(paste(lambda,'=1.5',',alpha','=1',',beta','=9.0',',
  theta','=1.0'))),col=c(2,3,4,5,6),lwd=c(2,2,2,2,2))

```

R Code: Hazard plot of GELLLoG

```

f1=function(x,lambda,c,alpha,delta){
  y((((-log(1-(1-(((1+lambda+lambda*x)*(exp(-lambda*x)))/
  ((1+lambda)*
  (1+x^c))))^alpha))^(delta-1))*alpha*((1-(((1+lambda+lambda*x)*
  (exp(-lambda*x)))/((1+lambda)*(1+x^c))))^(alpha-1))*
  (((1+x^c)^-1)*
  (exp(-lambda*x))/(1+lambda))*((lambda^2)*(1+x)+((1+lambda+lambda*x)*
  (c*x^(c-1))/(1+x^c)))*(gamma(delta)-
  pgamma((-log(1-(1-(((1+lambda+lambda*x)
  *(exp(-lambda*x)))/((1+lambda)*(1+x^c))))^alpha)),delta))^( -1)
  return(y)
}
x=seq(0,1.5,by=0.001)
y1=f1(x,0.4,3.0,1.2,0.6)
plot(x,y1,ylim=c(0,3),col=2,'l',lwd=2,xlab="x",ylab="h(x)")
y2=f1(x,2.4,8.0,3.0,2.0)
lines(x,y2,col=3,lwd=2)

```

```

y3=f1(x,2.0,3.8,0.2,2.0)
lines(x,y3,col=4,lwd=2)
y4=f1(x,8.0,5.8,1.2,5.4)
lines(x,y4,col=5,lwd=2)
y5=f1(x,2.0,2.0,1.8,0.2)
lines(x,y5,col=6,lwd=2)
legend("topright",c(
  expression(paste(lambda,'=0.4',',alpha','=3.0',',beta','=1.2',',
  theta','=0.6')),
  expression(paste(lambda,'=2.4',',alpha','=8.0',',beta','=3.0',',
  theta','=2.0')),
  expression(paste(lambda,'=2.0',',alpha','=3.8',',beta','=0.2',',
  theta','=2.0')),
  expression(paste(lambda,'=8.0',',alpha','=5.8',',beta','=1.2',',
  theta','=5.4')),
  expression(paste(lambda,'=2.0',',alpha','=2.0',',beta','=1.8',',
  theta','=0.2'))),col=c(2,3,4,5,6),lwd=c(2,2,2,2,2))

```

R Code: Moments and Quantiles of GELLLoG

```

install.packages("stats4")
install.packages("bbmle")
install.packages("stats")
install.packages("numDeriv")
install.packages("Matrix")
install.packages("zipfR")
install.packages("rootSolve")
library(rootSolve)
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library(Matrix)
library(zipfR)
##define pdf GELLLoG

```

```
GELLLoG_pdf=function(lambda,c,alpha,delta,x){
  y=(1/gamma(delta))*((-log(1-(1-(((1+lambda+lambda*x)*
  (exp(-lambda*x)))/((1+lambda)*(1+x^c))))^alpha))^(delta-1))*
  alpha*((1-(((1+lambda+lambda*x)*
  (exp(-lambda*x)))/((1+lambda)*(1+x^c))))^(alpha-1))*
  (((1+x^c)^-1)*
  (exp(-lambda*x))/(1+lambda))*((lambda^2)*
  (1+x)+((1+lambda+lambda*x)*
  (c*x^(c-1))/(1+x^c)))
  return(y)
}
```

```
##define cdf GELLLoG
```

```
GELLLoG_cdf=function(lambda,c,alpha,delta,x){
  y=pgamma((-log(1-(1-(((1+lambda+lambda*x)*
  (exp(-lambda*x)))/((1+lambda)*
  (1+x^c))))^alpha)),delta)/gamma(delta)
}
```

```
##define hazard GELLLoG
```

```
GELLLoG_hazard=function(lambda,c,alpha,delta){
  y=(((-log(1-(1-(((1+lambda+lambda*x)*(exp(-lambda*x)))/
  ((1+lambda)*(1+x^c))))^alpha))^(delta-1))*
  alpha*((1-(((1+lambda+lambda*x)*(exp(-lambda*x)))/
  ((1+lambda)*(1+x^c))))^(alpha-1))*
  (((1+x^c)^-1)*(exp(-lambda*x))/(1+lambda))*((lambda^2)*
  (1+x)+((1+lambda+lambda*x)*
  (c*x^(c-1))/(1+x^c))))*(gamma(delta)-
  pgamma((-log(1-(1-(((1+lambda+lambda*x)*
  (exp(-lambda*x)))/((1+lambda)*(1+x^c))))^alpha)),delta))^(-1)
}
```

```
#-----
```

```

## define GELLoG quantile
GELLLoG_quantile =function(parameter,u){
  lambda=parameter[1]
  c=parameter[2]
  alpha=parameter[3]
  delta=parameter[4]
  f=function(x){
    GELLLoG_cdf(lambda,c,alpha,delta,x)-u
  }
  x=min(uniroot.all(f,lower=0,upper=100,tol=0.01))
  return(x)
}

GELLLoG_QuantileTable=function(parameter_matrix){
  u=seq(0.1,0.9,0.1)
  size=dim(parameter_matrix)[1]
  Table_Quantile=matrix(NA,nrow=length(u),ncol=size)
  row.names(Table_Quantile)=u
  colnames(Table_Quantile)=apply(parameter_matrix,1,function(x)
  {paste0('(',paste0(x,collapse=','),')')})
  Table_Quantile
  for(iter in 1:size){
    parameter=parameter_matrix[iter,]
    for(i in 1:length(u)){
      Table_Quantile[i,iter]=GELLLoG_quantile(parameter,u[i])
    }
  }
  return(Table_Quantile)
}

#-----
## table of quantile
parameter_matrix=as.matrix(rbind(
  par1=c(1.2,1.2,1.0,1.8),
  par2=c(0.2,1.5,2.2,2.0),

```

```

par3=c(0.8,1.2,2.0,1.0),
par4=c(2.0,1.0,1.0,2.2),
par5=c(1.0,1.8,2.6,2.0)

))

GELLLoG_QuantileTable(parameter_matrix)
print(parameter_matrix)
print(GELLLoG_QuantileTable(parameter_matrix))

##End of Quantile
##End of Quantile
##define pdf GELLLoG

GELLLoG_pdf=function(lambda,c,alpha,delta,x){
  y=(1/gamma(delta))*((-log(1-(1-(((1+lambda+lambda*x)*
    (exp(-lambda*x)))/((1+lambda)*(1+x^c))))^alpha))^(delta-1))*
    alpha*(((1-(((1+lambda+lambda*x)*
    (exp(-lambda*x)))/((1+lambda)*(1+x^c))))^(alpha-1))*
    (((1+x^c)^-1)*
    (exp(-lambda*x))/(1+lambda))*((lambda^2)*(1+x)+
    ((1+lambda+lambda*x)*
    (c*x^(c-1))/(1+x^c)))
  return(y)
}

GELLLoG_moments=function(lambda,c,alpha,delta,n){
  f=function(lambda,c,alpha,delta,n,x){
    (x^n)*(GELLLoG_pdf(lambda,c,alpha,delta,x))
  }
  y=integrate(f,lower = 0, upper =1,lambda=lambda,c=c,
  alpha=alpha,

```

```

    delta=delta ,n=n)
    return(y$value)
}
### table of moments
parameter_matrix_Moments=matrix(c(0.1,0.2,0.2,0.5,0.7,
1.0,1.5,0.5,
0.2,2.0,2.0,0.2,
3.4,4.0,0.5,0.9,0.2,2.0,1.5,0.5),ncol=4,byrow=T)
parameter_matrix_Moments
list_Moments=c('EX','EX2','EX3','EX4','EX5','EX6','SD',
'CV','CS','CK')
Table_Moments=matrix(0,nrow=length(list_Moments),
ncol=dim(parameter_matrix_Moments)[1])
row.names(Table_Moments)=list_Moments
Table_Moments
for(iter in 1:dim(parameter_matrix_Moments)[1]){
    parameter=parameter_matrix_Moments[iter,]
    for(i in 1:10){
        Table_Moments[i,iter]=GELLLoG_moments(parameter[1],
        parameter[2],parameter[3],parameter[4],i)
    }
}

```

Chapter 4 Appendix

Elements of Score Vector

Elements of the score vector are given by

$$\begin{aligned}
 \frac{\partial \ell_n}{\partial \lambda} = & - \sum_{i=1}^n x_i - \frac{n}{(1+\lambda)} + \sum_{i=1}^n \frac{2\lambda(1+x_i) + \frac{(1+x_i)cx_i^{c-1}}{1+x_i^c}}{\left[\lambda^2(1+x_i) + \frac{(1+\lambda+\lambda x_i)cx_i^{c-1}}{1+x_i^c} \right]} \\
 & + 2 \sum_{i=1}^n \frac{\frac{\bar{\delta} e^{-\lambda x_i}}{(1+x_i^c)(1+\lambda)^2} [(1+x_i)(1+\lambda) - (1+\lambda+\lambda x_i)]}{\left[1 - \bar{\delta} \frac{1+\lambda+\lambda x_i}{(1+\lambda)(1+x_i^c)} e^{-\lambda x_i} \right]}, \tag{6.1}
 \end{aligned}$$

$$\begin{aligned} \frac{\partial \ell_n}{\partial c} = & - \sum_{i=1}^n \frac{x_i \ln(x_i)}{(1+x_i^c)} + \sum_{i=1}^n \frac{\frac{c(1+\lambda+\lambda x_i)x_i^{c-1}}{(1+x_i^c)^2} [\ln(x_i)(1+x_i^c) - x_i^c \ln(x_i)]}{\left[\lambda^2(1+x_i) + \frac{(1+\lambda+\lambda x_i)cx_i^{c-1}}{1+x_i^c} \right]} \\ & - 2 \sum_{i=1}^n \frac{\bar{\delta} \frac{(1+\lambda+\lambda x_i)e^{-\lambda x_i}}{1+\lambda(1+x_i^c)^2} [x_i^c \ln(x_i)]}{\left[1 - \bar{\delta} \frac{1+\lambda+\lambda x_i}{(1+\lambda)(1+x_i^c)} e^{-\lambda x_i} \right]}, \end{aligned} \quad (6.2)$$

and

$$\frac{\partial \ell_n}{\partial \delta} = \frac{n}{\delta} - 2 \sum_{i=1}^n \frac{\frac{(1+\lambda+\lambda x_i)e^{-\lambda x_i}}{(1+\lambda)(1+x_i^c)}}{\left[1 - \bar{\delta} \frac{1+\lambda+\lambda x_i}{(1+\lambda)(1+x_i^c)} e^{-\lambda x_i} \right]}. \quad (6.3)$$

R Code: Pdf plot of MOLLLoG

```
f1=function(x,lambda,c,delta){
  y=((delta*exp(-lambda*x))/((1+lambda)*(1+x^c)))*((lambda^2
  *(1+x)+((1+lambda+lambda*x)*(c*x^(c-1)))/(1+x^c)))/((1-(1-delta)
  *((1+lambda+lambda*x)*exp(-lambda*x))/((1+x^c)*(1+lambda)))^2)
  return(y)
}
x=seq(0,1.5,by=0.001)
y1=f1(x,1.0,1.0,0.2)
plot(x,y1,ylim=c(0,3),col=2,'l',lwd=2,xlab="x",ylab="density")
y2=f1(x,0.01,3.8,10.0)
lines(x,y2,col=3,lwd=2)
y3=f1(x,0.2,8.4,0.6)
lines(x,y3,col=4,lwd=2)
y4=f1(x,0.2,3.4,0.2)
lines(x,y4,col=5,lwd=2)
y5=f1(x,0.05,8.2,0.1)
lines(x,y5,col=6,lwd=2)
legend("topright",c(
  expression(paste(lambda,'=1.0',',',c,'=1.0',',',delta,'=0.2')),
  expression(paste(lambda,'=0.01',',',c,'=3.8',',',delta,'=10.0')),
  expression(paste(lambda,'=0.2',',',c,'=8.4',',',delta,'=0.6')),
  expression(paste(lambda,'=0.2',',',c,'=3.4',',',delta,'=0.2')),
  expression(past
```

R Code: Hazard plot of MOLLLoG

```
f1=function(x,lambda,c,delta){
  y=((lambda^2)*(1+x)/(1+lambda+lambda*x))+((1+x^c)^(-1))*c*
  (x^(c-1))/(1-((1-delta)*((1+lambda+lambda*x)/(1+lambda)*
  (1+x^c))*exp(-lambda*x)))
  return(y)
}
x=seq(0,1.5,by=0.0001)
y1=f1(x,3.0,0.7,3.4)
plot(x,y1,ylim=c(0,3),col=2,'l',lwd=2,xlab="x",ylab="h(x)")

y3=f1(x,0.8,8.8,3.0)
lines(x,y3,col=4,lwd=2)

legend("topright",c(
  expression(paste(lambda,'=3.0','c','=0.7','delta','=3.4')),
  expression(paste(lambda,'=0.8','c','=8.8','delta','=3.0'))
),col=c(2,4),lwd=c(2,2))

y4=f1(x,2.0,1.0,4.8)
lines(x,y4,col=3,l
```

R Code: Application of MOLLLoG

```
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library('bbmle')
x<-c(
13, 16, 20, 22, 22, 25, 32, 45, 49, 59, 64, 70, 88, 8, 89,
93, 95, 10, 112, 116, 122, 147, 150, 151, 177, 179, 190,
204,207, 221, 233, 240, 245, 247, 264, 267, 272, 283,
```

```

291, 301, 307, 320, 337, 343, 352, 362, 367, 396, 421)
hist(x)

MOLLLoG.LL<- function(lambda,c,delta) {
  -sum(log(((delta*exp(-lambda*x)/((1+lambda)*(1+x^c)))*
    ((lambda^2)
    *(1+x)+((1+lambda+lambda*x)*(c*x^(c-1)))/(1+x^c)))/
    ((1-((1-delta)
    *((1+lambda+lambda*x)*exp(-lambda*x))/((1+x^c)*(1+lambda))))^2))))
}
MOLLLoG.result<-mle2(MOLLLoG.LL,hessian = NULL,start=
list(lambda=1.09,c=1.0 ,delta=1.0),optimizer="nllminb",lower=0)
summary(MOLLLoG.result)

```

Chapter 5 Appendix

Elements of Score Vector

Elements of the score vector are given by

$$\frac{\partial \ell_n}{\partial \alpha} = -\sum_{i=1}^n x_i^\beta + \sum_{i=1}^n \frac{\beta x_i^{\beta-1}}{[(1+x_i^c)^{-1} c x_i^{c-1} + \alpha \beta x_i^{\beta-1}]} + 2 \sum_{i=1}^n \frac{(1+x_i^c)^{-1} x_i^\beta e^{-\alpha x_i^\beta}}{[1 + (1+x_i^c)^{-1} e^{-\alpha x_i^\beta}]}, \quad (6.4)$$

$$\begin{aligned} \frac{\partial \ell_n}{\partial c} &= -\sum_{i=1}^n \frac{x_i^c \ln(x_i)}{(1+x_i^c)} + \sum_{i=1}^n \frac{[x_i^{c-1} + c x_i^{c-1} \ln(x_i)](1+x_i^c) - x_i^c \ln(x_i) c x_i^{c-1}}{(1+x_i^c)^2 [(1+x_i^c)^{-1} c x_i^{c-1} + \alpha \beta x_i^{\beta-1}]} \\ &+ 2 \sum_{i=1}^n \frac{(1+x_i^c)^{-2} x_i^c \ln(x_i) e^{-\alpha x_i^\beta}}{[1 + (1+x_i^c)^{-1} e^{-\alpha x_i^\beta}]}, \end{aligned} \quad (6.5)$$

and

$$\begin{aligned} \frac{\partial \ell_n}{\partial \beta} &= \sum_{i=1}^n \frac{\alpha [x_i^{\beta-1} + \beta x_i^{\beta-1} \ln(x_i)]}{[(1+x_i^c)^{-1} c x_i^{c-1} + \alpha \beta x_i^{\beta-1}]} + 2 \sum_{i=1}^n \frac{(1+x_i^c)^{-1} \alpha x_i^\beta \ln(x_i) e^{-\alpha x_i^\beta}}{[1 + (1+x_i^c)^{-1} e^{-\alpha x_i^\beta}]} \\ &- \alpha \sum_{i=1}^n x_i^\beta \ln(x_i). \end{aligned} \quad (6.6)$$

R Code: Pdf plot of HLLLoGW

```

f1=function(x,alpha,c,beta){
  y=(2*exp(-alpha*x^beta)*((1+x^c)^(-1))*(((1+x^c)^(-1))*

```

```

c*(x^(c-1))+alpha*beta*(x^(beta-1))*(1+(1+x^c)*
exp(-alpha*x^beta))^(-2) )/(1-((1-((1+x^c)^(-1))*
exp(-alpha*x^beta))/(1+((1+x^c)^(-1))*exp(-alpha*x^beta))))
return(y)
}
x=seq(0,1.5,by=0.001)
y1=f1(x,2.0,5.0,3.2)
plot(x,y1,ylim=c(0,3),col=2,'l',lwd=2,xlab="x",ylab="density")
y2=f1(x,1.2,0.9,0.6)
lines(x,y2,col=3,lwd=2)
y3=f1(x,0.1,5.2,1.0)
lines(x,y3,col=4,lwd=2)
y4=f1(x,4.0,1.0,1.2)
lines(x,y4,col=5,lwd=2)
legend("topright",c(
  expression(paste(alpha,'=2.0',',',c,'=5.0',',',beta,'=3.2')),
  expression(paste(alpha,'=1.2',',',c,'=0.9',',',beta,'=0.6')),
  expression(paste(alpha,'=0.1',',',c,'=5.2',',',beta,'=1.0')),
  expression(paste(alpha,'=4.0',',',c,'=1.0',',',beta,'=1.2'))),
  col=c(2,3,4,5),lwd=c(2,2,2,2))

```

R Code: Application of HLLLoGW

```

#####
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library('bbmle')
x<-c(0.03,0.12,0.22,0.35,0.73,0.79,1.25,1.41,1.52,1.79, 1.80,
1.94,2.38,2.40,2.87,2.99,3.14,3.17,4.72,5.09)
hist(x)

HLLLoGWLL<- function(alpha, beta, c) {

```

```

      -sum(log(2*exp(-alpha*x^beta)*((1+x^c)^(-1))*(((1+x^c)^(-1))*c*
      (x^(c-1))+alpha*beta*(x^(beta-1)))*(1+((1+x^c)^(-1))*
      exp(-alpha*x^beta))^(-2))))}
main.result<-mle2(HLLLoGWLl,hessian = NULL,start=
list(alpha=0.082996, beta=0.008799878 ,c=19.899827),
optimizer="nlminb",lower=0)
summary(main.result)

```

R Code: Goodness-of-Fit for HLLLoGW

```

rm(list=ls())
install.packages("stats4")
install.packages("AdequacyModel")
install.packages("bbmle")
install.packages("stats")
library(stats4)
library(AdequacyModel)
library(bbmle)
library(stats)

air<-c(0.03,0.12,0.22,0.35,0.73,0.79,1.25,1.41,1.52,1.79,
1.80,1.94,2.38,2.40,2.87,2.99,3.14,3.17,4.72,5.09)

# define HLLLoGW pdf
HLLLoGW_pdf<-function(parameter,x){
  alpha=parameter[1]
  beta=parameter[2]
  c=parameter[3]
  2*exp(-alpha*x^beta)*((1+x^c)^(-1))*(((1+x^c)^(-1))*c*(x^(c-1))
+alpha*beta*(x^(beta-1)))*(1+((1+x^c)^(-1))*exp(-alpha*
x^beta))^(-2)
}
# define HLLLoGW cdf

```

```

HLLLoGW_cdf<-function(parameter,x){
  alpha=parameter[1]
  beta=parameter[2]
  c=parameter[3]
  (1-((1+x^c)^(-1))*exp(-alpha*x^beta))/(1+((1+x^c)^(-1))*
exp(-alpha*x^beta))
}
goodness.fit(pdf=HLLLoGW_pdf,cdf= HLLLoGW_cdf,mle=c(0.061398 ,
2.405721 ,
0.822303 ),data=air,method = "BFGS",domain = c(0,100)
,lim_inf = c(0,0,0,0),
lim_sup = c(100,100,100,100))

```