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An Inertial Method for Solutions of Split Equality Inclusion Problems

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MSc (Pure and Applied Mathematics)

Botswana International University of Science and Technology (BIUST), Botswana

A Thesis Submitted to the Faculty of Sciences in Partial Fulfilment of the Requirements
for the Award of the Degree of Master of Science in Pure and Applied Mathematics of
BIUST

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February, 2023



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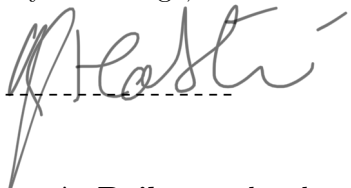
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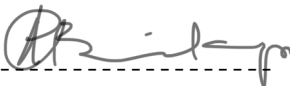
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Abstract

The main objective of this Thesis is to introduce and study the *Split Equality Inclusion Problem* in the framework of real Hilbert spaces. In order to solve this problem, we propose an inertial method for approximating a solution of the problem. Using the method, we prove a strong convergence theorem to a solution of the problem. In addition, we provide some applications of our method and present numerical results to demonstrate the applicability and efficiency of the proposed method. The result obtained in this Thesis generalizes and improves several results in the Literature in this direction.



Dedications

For the love of mathematics.



Acknowledgements

First, I want to give thanks to the Almighty God for the grace and capability He gave generously throughout the research work. I also extend my gratitude to myself for not giving up when it was getting tough, it was not an easy journey. Meanwhile, I owe my advancement and achievements to my supervisors Prof. Habtu Z. Hailu and Prof. Oganeditse A. Boikanyo who gave their full exhortation and didactic guidance on understanding and writing as well as presenting my thesis. I am also grateful to my colleagues for all their assistance. I would like to express my deepest gratitude to Pelonomi M. Kalayakgosi and Keletso T. Thobogang for being a pillar of strength and support throughout the duration of my studies. I would also like to appreciate my mother Lenyatso Thobogang, my family and friends who prayed for me, supported me and believed in me throughout this journey. Lastly, I would not have started and even completed my work without the financial support of BIUST and the Simons Foundation, so I am sincerely grateful.



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1. Introduction

1.1 Background of the Study

This Thesis contributes to the general area of Nonlinear Functional Analysis and its Applications, an area which has been of active research interest in recent years. In particular, we devote our attention to the inertial method for solutions of the Split Equality Inclusion Problem and its special cases and applications.

Throughout this Thesis, unless stated otherwise, H will be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. Let C be a nonempty subset of a real Hilbert space H . A mapping $\mathcal{M} : C \rightarrow H$ is said to be *L-Lipschitz continuous* with Lipschitz constant $L > 0$, if

$$\|\mathcal{M}x - \mathcal{M}y\| \leq L\|x - y\|, \text{ for all } x, y \in C.$$

If $L = 1$, then $\mathcal{M}$ is said to be *nonexpansive* and if $L < 1$, then $\mathcal{M}$ is called a *contraction*. $\mathcal{M}$ is called *firmly nonexpansive* if

$$\|\mathcal{M}x - \mathcal{M}y\|^2 \leq \langle \mathcal{M}x - \mathcal{M}y, x - y \rangle, \text{ for all } x, y \in C.$$

The mapping $\mathcal{M}$ is called *monotone* if

$$\langle \mathcal{M}x - \mathcal{M}y, x - y \rangle \geq 0, \text{ for all } x, y \in C.$$

$\mathcal{M}$ is said to be *co-coercive* if there exists a constant $\alpha > 0$ such that

$$\langle \mathcal{M}x - \mathcal{M}y, x - y \rangle \geq \alpha \|\mathcal{M}x - \mathcal{M}y\|^2, \text{ for all } x, y \in C.$$

Note that if $\mathcal{M}$ is co-coercive, then $\mathcal{M}$ is $\frac{1}{\alpha}$ -Lipschitz continuous and also monotone.

A multi-valued mapping $B : C \rightarrow 2^H$ is called *monotone* if, for all $x, y \in C$, we have

$$\langle u - v, x - y \rangle \geq 0, \text{ for all } u \in Bx \text{ and } v \in By.$$

B is called *maximal monotone* if it is monotone and for any $(x, u) \in H \times H$ such that $\langle u - v, x - y \rangle \geq 0$ for every $(y, v) \in \text{Graph}(B)$ (the graph of the mapping B) implies that $u \in Bx$. B is said to be *co-coercive*, if there exists a constant $k_1 > 0$ such that for all $x, y \in C$,

$$\langle u - v, x - y \rangle \geq k_1 \|u - v\|^2, \text{ for all } u \in Bx \text{ and } v \in By.$$

Examples of monotone mappings

- A function $f : \mathbb{R} \rightarrow \mathbb{R}$ defines a monotone mapping if and only if f is a monotone increasing

function in the usual sense: that is,

$$\langle f(x_2) - f(x_1), x_2 - x_1 \rangle \geq 0, \forall x_2, x_1 \in \mathbb{R}, \text{ iff } x_1 \leq x_2 \implies f(x_1) \leq f(x_2).$$

- Let $f : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ be a multivalued mapping defined by

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 0, \\ [0, 1], & \text{if } x = 0. \end{cases}$$

Then f is a multivalued monotone mapping.

The *resolvent* of a mapping $B : H \rightarrow 2^H$ is a mapping $J_\lambda^B : H \rightarrow C$ defined by

$$J_\lambda^B(x) := (I + \lambda B)^{-1}(x), \forall x \in H,$$

for $\lambda > 0$, where I stands for the identity mapping on H . We note that if $B : H \rightarrow 2^H$ is a multivalued maximal monotone mapping and $\lambda > 0$, then the $\text{Dom}(J_\lambda^B) = H$ and J_λ^B is a single-valued and firmly nonexpansive mapping. Moreover, the set of fixed points of J_λ^B coincides with the set of zeros of B . That is,

$$\text{Fix}(J_\lambda^B) = B^{-1}(0) = \{x \in D(B) : 0 \in Bx\}.$$

A fundamental and classical problem is to find a zero of a monotone mapping $\mathcal{M} : H \rightarrow H$ in a real Hilbert space H , that is,

$$\text{find a point } x^* \in H \text{ such that } x^* \in \mathcal{M}^{-1}(0). \quad (1.1.1)$$

The set of solutions of problem (1.1.1) is denoted by $\mathcal{M}^{-1}(0) := \{x^* \in H : \mathcal{M}x^* = 0\}$.

This problem is connected with many problems in nonlinear analysis and optimization such as convex minimization problems, split feasibility problems, variational inequality problems and equilibrium problems. Due to its diverse applications in economics, medicine, and engineering, the techniques and methods for solving (1.1.1) have received much attention in the optimization community, see [17, 21, 54, 59, 77]. For example, consider the classical convex optimization problem: let $h : H \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper and convex function. The subdifferential of h at $x \in H$ is the mapping $\partial h : H \rightarrow 2^H$ defined by

$$\partial h(x) = \{x^* \in H : h(y) - h(x) \geq \langle y - x, x^* \rangle, \forall y \in H\}.$$

If $x^* \in \partial h(x)$, $y^* \in \partial h(y)$ then $\langle x^*, y - x \rangle \leq h(y) - h(x)$ and $\langle y^*, x - y \rangle \leq h(x) - h(y)$. By adding these two inequalities,

$$\langle x^* - y^*, y - x \rangle \leq 0.$$

Clearly, $\partial h : H \rightarrow 2^H$ is a monotone mapping on H , and $0 \in \partial h(x_0)$ if and only if x_0 is a minimizer of h . In this case, setting $\partial h = \mathcal{M}$, solving the inclusion $0 \in \mathcal{M}x$ is equivalent to solving for minimizers of h .

One simple, yet common generalization of problem (1.1.1), (see, for example, [26, 91, 111]), occurs when the mapping $\mathcal{M}$ is written as the sum of two mappings; that is, $\mathcal{M} = A + B$, such that the resolvent mapping $(I + \lambda B)^{-1}$ (implicit, backward step) and the evaluation of $(I - \lambda A)$ (explicit, forward step) are much easier to compute than the full resolvent $(I + \lambda \mathcal{M})^{-1}$. As a consequence, we consider the following *Monotone Inclusion Problem* (MIP): find a zero of an additive structured mapping $A + B$, acting on the space H , namely,

$$\text{find } x^* \in H \text{ such that } x^* \in (A + B)^{-1}(0), \quad (1.1.2)$$

where $A : H \rightarrow H$ is a monotone mapping for which we can use direct evaluations, and $B : H \rightarrow 2^H$ is a maximal monotone mapping for which we can compute the resolvent. The set of solutions of problem (1.1.2) is denoted by $(A + B)^{-1}(0) := \{x^* \in H : Ax^* + Bx^* \ni 0\}$ and is referred to as the set of zero points of $A + B$. If in (1.1.2), we have $B = 0$, then problem (1.1.2) reduces to problem (1.1.1). Note that some concrete problems in machine learning, image processing and linear inverse problems can be modeled mathematically as MIP (1.1.2) [31, 42, 50, 93].

In the year 2011, Moudafi [68] introduced and investigated a generalization of the problem (1.1.2) known as the *Split Quasi-Inclusion Problem* (SQIP) in the framework of Hilbert spaces. The SQIP is given as follows:

$$\text{find } x^* \in H_1 \text{ such that } x^* \in (A + B)^{-1}(0) \text{ and } Tx^* \in (G + K)^{-1}(0), \quad (1.1.3)$$

where H_1 and H_2 are two real Hilbert spaces, $A : H_1 \rightarrow H_1$ and $G : H_2 \rightarrow H_2$ are nonlinear mappings, $B : H_1 \rightarrow 2^{H_1}$ and $K : H_2 \rightarrow 2^{H_2}$ are set-valued mappings, and $T : H_1 \rightarrow H_2$ is a bounded linear mapping. If in (1.1.3), we have $H_1 = H_2$, $G = K = 0$, and $T = I$ (the identity mapping on H_1), then the problem (1.1.3) reduces to the inclusion problem (1.1.2). Subsequently, similar problems have been studied by a number of authors in Hilbert spaces (see, for instance, [20, 28]).

Moreover, in [67], Moudafi proposed and considered the following *Split Equality Null-Point Problem* (SENPP): Let H_1 , H_2 and H_3 be real Hilbert spaces. Now, given two set-valued maximal monotone mappings $A : H_1 \rightarrow 2^{H_1}$ and $B : H_2 \rightarrow 2^{H_2}$, and two bounded linear mappings $T : H_1 \rightarrow H_3$ and $S : H_2 \rightarrow H_3$, the SENPP is defined as follows:

$$\text{find } x^* \in A^{-1}(0) \text{ and } y^* \in B^{-1}(0), \text{ such that } Tx^* = Sy^*, \quad (1.1.4)$$

where $A^{-1}(0)$ and $B^{-1}(0)$ are nonempty, closed and convex. Note that whenever $S = I$ and $H_2 = H_3$, (1.1.4) reduces to the split common null-point problem (SCNP) which was introduced and studied by Byrne [18]. Let $A : H_1 \rightarrow 2^{H_1}$ and $B : H_2 \rightarrow 2^{H_2}$ be two set-valued maximal monotone mappings, $T : H_1 \rightarrow H_2$ be a bounded linear mapping with adjoint T^* . Then the

SCNP is to,

$$\text{find } x^* \in A^{-1}(0) \text{ such that } Tx^* \in B^{-1}(0). \quad (1.1.5)$$

The SCNP (1.1.5) has received much attention both theoretically and practically as it includes variational inequalities and many constrained optimization problems as special cases, (see for example, [18, 43, 70]).

In this Thesis, it is our main concern to introduce and study a generalization of the above problems called the *Split Equality Inclusion Problem* (SEIP) (see subsection 1.3 below for more details) and develop iterative methods for approximating solutions of the problem.



1.2 Motivation

There have been fruitful works on approximating zero points of a monotone mapping $\mathcal{M} : H \rightarrow H$ in Hilbert spaces for problem (1.1.1), for instance, see [17, 27, 90, 96]. The *proximal point algorithm* (PPA) is the first introduced algorithm for solving problem (1.1.1). This algorithm is recognized as effective and proficient in finding an arithmetical solution of problem (1.1.1) which was introduced by Martinet [65] and studied further by Rockafellar [86] and a swarm of other authors. They have shown that the sequence $\{x_n\}$ generated by the PPA converges weakly to a point x^* satisfying $0 \in \mathcal{M}x^*$, where $\mathcal{M}$ is a maximal monotone mapping (see, other related papers, [11, 73, 85]).

Rockafellar [86] only obtained the weak convergence of the PPA for problem (1.1.1), consequently he proposed two open questions for obtaining the strong convergence of the proximal point algorithm: (1) Does the proximal point algorithm always converge weakly? (2) Can the proximal point algorithm be modified to guarantee strong convergence? In studying the strong convergence, many authors have modified the proximal point algorithm to guarantee strong convergence under different settings, see e.g., [23, 58, 84, 95] and the references therein. However, a drawback of the above mentioned algorithms is the need to compute $(I + \lambda\mathcal{M})^{-1}$ at each step of the iteration process. This means that even if the proximal point algorithm converges strongly, it is not always convenient to use in some applications.

Inspired by the above mentioned algorithm, Passty [78] introduced the *forward-backward splitting method*, which includes, in particular, the proximal point algorithm mentioned above and the gradient method [59]. This method converges weakly to a solution of problem (1.1.2) provided that the mapping $A : H \rightarrow H$ is co-coercive with certain restrictions on the parameters involved. Co-coercivity of a mapping is a strictly strong property than Lipschitz continuity and hence, can be difficult to satisfy for general monotone inclusions.

To overcome this obstacle, Tseng [106] introduced a modified forward–backward splitting method, also known as *Tseng’s Splitting Method*. Tseng proved that the sequence $\{x_n\}$ generated by the Tseng’s splitting method converges weakly to an element of problem (1.1.2) in real Hilbert spaces provided that $A : H \rightarrow H$ is a Lipschitz continuous monotone mapping and $B : H \rightarrow 2^H$ is a maximal monotone mapping.

In recent years, the forward-backward splitting method has received great attention by many authors, who improved it in various ways, (see [8, 11, 14, 30, 31, 32, 40, 41, 44, 48, 57, 59, 61, 69, 76, 83, 99, 100, 101, 102, 103, 104, 106, 113], and the references therein). However, the above methods only obtain weak convergence to a solution of problem (1.1.2). Therefore, one can ask this question; can the forward-backward splitting method be modified to obtain strong convergence?

To answer this question, Thong and Chalamjiak [98] proposed the *Halpern-type forward-backward splitting method*. This method is a natural modification to the forward-backward splitting method mentioned above and can be easily implemented to obtain strong convergence theorems for problem (1.1.2). To get strong convergence, they took $A : H \rightarrow H$ to be L -Lipschitz continuous monotone mapping and $B : H \rightarrow 2^H$ to be a maximal monotone mapping. In

addition, this scheme does not require the knowledge of the Lipschitz constant L of the mapping $A : H \rightarrow H$; thus making the proposed algorithm more practical for computations. Note that in this algorithm, each step of iterates involves only $A : H \rightarrow H$ as the forward step and $B : H \rightarrow 2^H$ as the backward step, but not the sum of the mappings.

In the year 2011, Moudafi [68] proposed an iterative algorithm for solving problem (1.1.3), which yielded weak convergence theorems of the SQIP in the setting of real Hilbert spaces. Recently, Petrot et al. [79] proposed two iterative algorithms to solve the SQIP for the set of zero points of the problem (1.1.3) over the set of fixed points of nonexpansive mappings and also proved weak convergence theorems through a parallel algorithm. Strong convergence results proved difficult to obtain in this problem (1.1.3).

Various authors proposed several other problems such as SENPP, however, the proposed algorithm for solving SENPP also proved a weak convergence theorem to a solution of problem (1.1.4), for example, see [67].

Recently, convergent methods which are rapidly fast for solving optimization problems have become of great interest and got a lot of interest from many researchers. Two very important ways for improving the convergence speed of iterative methods in terms of reduced number of iterations and central processing unit (CPU) time contributed are the inertial technique and the relaxation technique. The inertial technique is founded on a discrete analogue of a second order dissipative dynamical system, which is known for its efficiency in improving the convergence speed of iterative methods (see, for example [2, 3, 12, 37, 38, 53, 60, 72]). Subsequently, Polyak [81] introduced an inertial algorithm which is an iterative procedure in which subsequent terms of the sequence are obtained from the preceding two terms. The relaxation technique is also known to be an essential ingredient in the solution of optimization problems. This is mainly due to the improved convergence rate that it contributes to certain iterative schemes (see [7]). Both techniques arise naturally from an explicit time discretization of a dynamical system (see, for example, [6, 113]). Their influence on the numerical performance of iterative methods was studied in [49]. Moreover, these two techniques have been incorporated into known methods in order to achieve high convergence speed of the resulting modified methods (see [7, 6]). More recently, algorithms with inertial extrapolation steps have been of interest to many authors mainly due to the improved convergence speed in terms of reduced number of iterations and CPU time contributed by the presence of the inertial step (see, for example, [4, 45, 74, 107]).

Motivated and inspired by the above works, it is therefore natural to introduce the generalization of the above problems and propose methods for approximating the solutions in Hilbert spaces.

1.3 Statement of the Problem

One important generalization of all the above mentioned problems is called the *Split Equality Inclusion Problem* (SEIP) which is our area of interest and will be fully studied in this dissertation. The SEIP is formulated as finding a point $(x^*, u^*) \in H_1 \times H_2$ such that

$$x^* \in (A_1 + B_1)^{-1}(0) \text{ and } u^* \in (A_2 + B_2)^{-1}(0) \text{ such that } Tx^* = Su^*, \quad (1.3.1)$$

where $A_1 : H_1 \rightarrow H_1$ and $A_2 : H_2 \rightarrow H_2$ are Lipschitz continuous monotone mappings, $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ are multi-valued maximal monotone mappings, $T : H_1 \rightarrow H_3$ and $S : H_2 \rightarrow H_3$ are bounded linear mappings with adjoints T^* and S^* , respectively.

The inclusion problem (1.3.1) involves, as special cases, convex programming, split feasibility problems, variational inequalities and minimization problems. More precisely, some concrete problems in machine learning, image processing and linear inverse problems can be modeled mathematically as problem (1.1.2) (see, for example, [31, 42, 93]).

If in (1.3.1), we assume that $H_2 = H_3$ and $S = I$, the identity mapping on H_2 , then the Split Equality Inclusion Problem (SEIP) reduces to the Split Quasi-Inclusion Problem (SQIP) given in (1.1.3). If in (1.3.1), we assume that $B_1 = 0 = B_2$, then the Split Equality Inclusion Problem (SEIP) reduces to the Split Equality Null-Point Problem (SENPP) given in (1.1.4). If we consider $H = H_1 = H_2 = H_3$, $A_2 = B_2 = 0$ and $T = S = I$, the identity mapping on H , in (1.3.1), the Split Equality Inclusion Problem (SEIP) reduces to the Monotone Inclusion Problem (MIP) given in (1.1.2). Hence SEIP is a generalization of the problems discussed above.

1.4 Research Questions

Based on the statement of the problem above, the following questions arise quite naturally. The answers to these questions provide affirmative answers to the research gaps mentioned in the motivation above.

1. Can we establish an inertial iterative algorithm which provides strong convergence theorems for approximating solutions of the Split Equality Inclusion Problem for Lipschitz continuous monotone mappings and maximal monotone mappings?
2. Can we find applications of our algorithm to other problems in Hilbert spaces?
3. Is it possible to present a numerical example to fully illustrate the behaviour of the proposed algorithm?

1.5 Objectives of the Study

The main objectives of this study are to:

1. review some known and useful results on monotone inclusion problems.
2. introduce a more general problem called the Split Equality Inclusion Problem.
3. introduce an inertial iterative algorithm and establish strong convergence theorems for approximating solutions of the Split Equality Inclusion Problem.
4. give various applications of our algorithm.
5. come up with numerical examples which demonstrate the performance of the method.

1.6 Organization of the Thesis

This Thesis is organized in the following manner. In Chapter 1, we present the introduction which consists of the background of the Split Equality Inclusion Problem (SEIP), the motivation for the study, some research questions and objectives of this Thesis. In Chapter 2, we state some basic preliminaries on concepts of particular interest accompanied by some proofs for appreciation. These preliminaries will serve to validate and support our main results. We also present a well detailed literature review on problems related to the SEIP in this chapter. Then, in Chapter 3, we provide an iterative algorithm for solving the SEIP in real Hilbert spaces. A result of strong convergence of the given algorithm is then supplied. Subsequently, in Chapter 4, we present the applications of our results and provide numerical examples in Chapter 5. Lastly, in Chapter 6, our contribution to the knowledge is discussed and some areas of future research are also pointed out.

2. Literature Review

In this chapter, we state some known lemmas and important definitions which are relevant to the work which is deliberated in this study. The results in this chapter will be used to expound and fully discuss the upcoming chapters and we will keep on making references where necessary. Subsequently, we present some useful results and give detailed literature review of concepts that are relevant to our work. Throughout the entire chapter, we use the symbols $\rightarrow$ and $\rightharpoonup$ for the strong and weak convergence, respectively.

2.1 Preliminaries

The written proofs of the following Lemmas are put there with the intention to make this Thesis complete.

Lemma 2.1.1. *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H . For each $x, y, z \in H$ and for all $\alpha \in [0, 1]$, the following hold.*

$$1. \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \quad (2.1.1)$$

$$2. \quad \|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \quad (2.1.2)$$

$$3. \quad 2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2. \quad (2.1.3)$$

$$4. \quad \|x - z\|^2 = \|x - y\|^2 + \|y - z\|^2 + 2\langle y - z, x - y \rangle. \quad (2.1.4)$$

Proof. 1.

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x + y \rangle + \langle y, x + y \rangle \\ &\leq |\langle x, x + y \rangle| + \langle y, x + y \rangle \\ &\leq \|x\|\|x + y\| + \langle y, x + y \rangle \\ &\leq \frac{1}{2}(\|x\|^2 + \|x + y\|^2) + \langle y, x + y \rangle \\ &= \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x + y\|^2 + \langle y, x + y \rangle, \end{aligned}$$

and hence,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

2. Left hand side (LHS),

$$\begin{aligned}\|\alpha x + (1 - \alpha)y\|^2 &= \langle \alpha x + (1 - \alpha)y, \alpha x + (1 - \alpha)y \rangle \\ &= \alpha^2\|x\|^2 + 2\alpha(1 - \alpha)\langle x, y \rangle + (1 - \alpha)^2\|y\|^2.\end{aligned}$$

Right hand side (RHS),

$$\begin{aligned}\alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 &= \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\langle x - y, x - y \rangle \\ &= \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x\|^2 + 2\alpha(1 - \alpha)\langle x, y \rangle \\ &\quad - \alpha(1 - \alpha)\|y\|^2 \\ &= \alpha^2\|x\|^2 + 2\alpha(1 - \alpha)\langle x, y \rangle + (1 - \alpha)^2\|y\|^2.\end{aligned}$$

Since LHS = RHS, then proven. □

The proofs for 3. and 4. are trivial.

Lemma 2.1.2. (see, [98]); Let $B : H \rightarrow 2^H$ be a maximal monotone mapping and $A : H \rightarrow H$ be a monotone mapping. Define $T_\lambda := (I + \lambda B)^{-1}(I - \lambda A)$, $\lambda > 0$. Then, it follows that;

$$Fix(T_\lambda) = (A + B)^{-1}(0), \text{ for all } \lambda > 0,$$

where $Fix(T_\lambda) := \{x \in H : T_\lambda(x) = x\}$.

Proof. Using the definition of T_λ , we have the following;

$$\begin{aligned}x \in Fix(T_\lambda) &\iff x = T_\lambda x \\ &\iff (I + \lambda B)^{-1}(I - \lambda A)x = x \\ &\iff (I + \lambda B)x \ni (I - \lambda A)x \\ &\iff x + \lambda Bx \ni x - \lambda Ax \\ &\iff \lambda Bx + \lambda Ax \ni 0 \\ &\iff \lambda(A + B)x \ni 0, \lambda > 0 \\ &\iff (A + B)x \ni 0 \\ &\iff x \in (A + B)^{-1}(0).\end{aligned}$$

□

The following lemmas will be used in analyzing the convergence.

Lemma 2.1.3. ([16], Lemma 2.4) Let $B : H \rightarrow 2^H$ be a maximal monotone mapping and $A : H \rightarrow H$ be a uniformly continuous monotone mapping. Then, the mapping $A + B$ is a maximal monotone mapping.

Lemma 2.1.4. [110] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n + c_n, \quad (2.1.5)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\}, \{c_n\}$ satisfy the following conditions;

- a) $\sum_{n=1}^{\infty} \alpha_n = \infty$,*
- b) $\limsup_{n \rightarrow \infty} b_n \leq 0$,*
- c) $\sum_{n=1}^{\infty} c_n < \infty$ and $c_n \geq 0$, for all n .*

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. Note that $\{\alpha_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ is equivalent to $\prod_{n=1}^{\infty} (1 - \alpha_n) = 0$. Let $\epsilon > 0$ and choose an integer N such that $b_n < \epsilon$ for all $n \geq N$. By mathematical induction on (2.1.5), we get that,

$$a_{n+1} \leq \left[\prod_{k=N}^n (1 - \alpha_k) \right] a_N + \left[1 - \prod_{k=N}^n (1 - \alpha_k) \right] \epsilon, \quad \text{for all } n \geq N.$$

But from the assumption, $\prod_{k=1}^{\infty} (1 - \alpha_k) = 0$ and therefore we get that $\limsup_{n \rightarrow \infty} a_n \leq \epsilon$. Hence shown. $\square$

Lemma 2.1.5. [63] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that there exists $\{a_{n_i}\}$, a subsequence of $\{a_n\}$, such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \rightarrow \infty} m_k = \infty$ and*

$$\max\{a_{m_k}, a_k\} \leq a_{m_k+1}.$$

Proof. From the assumption, $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then, consider the set of integers,

$$J_n = \{j \in \mathbb{N} : a_j \leq a_{j+1}\}, \quad \text{for all } j \leq n.$$

Note that $J_n \neq \emptyset$, $J_n \subset J_{n+1}$ and $|J_n| \rightarrow \infty$ as $n \rightarrow \infty$, where $|J_n|$ denotes the cardinality of J_n . Now, consider the sequence $\{m_k\}$ of $\mathbb{N}$ defined by,

$$m_k = \max J_n, \quad \text{for all } k \in \mathbb{N}.$$

Since $J_n \subset J_{n+1}$, this implies that m_k is a non-decreasing sequence such that $\lim_{k \rightarrow \infty} m_k = \infty$ and $a_{m_k} \leq a_{m_k+1}$, for all $k \in \mathbb{N}$. Moreover, for all $k \in \mathbb{N}$, either $m_k = k$ or $m_k < k$. If $m_k = k$, then $a_k = a_{m_k} \leq a_{m_k+1}$. Again, if $m_k < k$, then

$$a_{m_k+1} > a_{m_k+2} > \cdots > a_{k-1} > a_k.$$

Hence $a_{m_k} \leq a_{m_k+1}$. Therefore, we have shown that $a_{m_k} \leq a_{m_k+1}$, for all $k \in \mathbb{N}$. $\square$

We then recall certain properties of the projection to be used in this Thesis, see [10] for more details.

Definition 2.1.6. Let H be a real Hilbert space, $x \in H$ and $C \subset H$ be nonempty, closed and convex. Then, a projection mapping $P_C : H \rightarrow C$ is defined by

$$\|P_C x - x\| = \inf_{y \in C} \|x - y\|. \quad (2.1.6)$$

The mapping P_C is called the metric projection of H onto C .

Lemma 2.1.7. Let H be a real Hilbert space and $C \subset H$ be nonempty, closed and convex. Let $x \in H$. Then, the following are equivalent.

$$1. \ y = P_C x \text{ (i.e. } \|x - y\| = d(x, C)). \quad (2.1.7)$$

$$2. \ \langle x - P_C x, P_C x - z \rangle \geq 0, \text{ for all } z \in C. \quad (2.1.8)$$

Proof. Firstly, we show that $y = P_C x$ implies $\langle x - P_C x, P_C x - z \rangle \geq 0$, for all $z \in C$. Let $x \in H$ and $P_C x = y$ be the projection of x onto C . Then for $z \in C$, $\lambda \in (0, 1)$ we have,

$$\|x - y\| \leq \|x - (1 - \lambda)y - \lambda z\|$$

and hence,

$$\begin{aligned} \|x - y\|^2 &\leq \|x - (1 - \lambda)y - \lambda z\|^2 \\ &= \|(x - y) + \lambda(y - z)\|^2 \\ &= \|x - y\|^2 + 2\lambda \langle x - y, y - z \rangle + \lambda^2 \|y - z\|^2. \end{aligned}$$

So, we have $2\langle x - y, y - z \rangle + \lambda \|y - z\|^2 \geq 0$. Then, $\langle x - y, y - z \rangle \geq 0$ as $\lambda \rightarrow 0$. Conversely, let $\langle x - y, y - z \rangle \geq 0$ for all $z \in C$. Then

$$\begin{aligned} \|x - y\|^2 &= \langle x - y, x - y \rangle \\ &= \langle x - y, x - z \rangle + \langle x - y, z - y \rangle \\ &\leq \langle x - y, x - z \rangle \\ &\leq \|x - y\| \|x - z\|. \end{aligned}$$

This implies that $\|x - y\| \leq \|x - z\|$, for all $z \in C$. Thus $\|x - y\| = d(x, C)$. $\square$

Lemma 2.1.8. Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H . Then, the following holds.

$$\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2, \text{ for all } y \in C \text{ and } x \in H. \quad (2.1.9)$$

Proof. The fact that $z = P_C x$ is equivalent to $\langle x - z, y - z \rangle \leq 0$ for all $y \in C$ implies that

$$\begin{aligned} \|x - P_C x\|^2 &= \langle x - P_C x, x - P_C x \rangle \\ &= \langle x - P_C x, x - y \rangle + \langle x - P_C x, y - P_C x \rangle \\ &\leq \langle x - P_C x, x - y \rangle. \end{aligned}$$

Subsequently,

$$\begin{aligned} \|P_C x - y\|^2 &= \|(P_C x - x) + (x - y)\|^2 \\ &= \|P_C x - x\|^2 - 2\langle x - P_C x, x - y \rangle + \|x - y\|^2 \\ &\leq \|P_C x - x\|^2 - 2\|x - P_C x\|^2 + \|x - y\|^2, \end{aligned}$$

which is the required. □

The following Lemma is a consequence of Lemma 2.1.7.

Lemma 2.1.9. (see, [13]) Let H_1 and H_2 be real Hilbert spaces and $H = H_1 \times H_2$. If $C \subset H$ is nonempty, closed and convex, $(u, v) \in H$ and $(u^*, v^*) = P_C(u, v)$, then we get,

$$\langle (u, v) - (u^*, v^*), (x, y) - (u^*, v^*) \rangle \leq 0, \text{ for all } (x, y) \in C.$$

2.2 Literature Review on Split Equality Inclusion Problems and its Special Cases

In this section, we discuss the literature review of some past and recent results or findings related to our study.

2.2.1 Inclusion with One Mapping

Let H be a real Hilbert space and $\mathcal{M} : H \rightarrow H$ be a nonlinear mapping. Recall that the *Inclusion Problem with one Mapping* is defined as to find a zero of a monotone mapping $\mathcal{M}$ in a real Hilbert space H , that is,

$$\text{find a point } x^* \in H \text{ such that } x^* \in \mathcal{M}^{-1}(0). \quad (2.2.1)$$

In 1970, Martinet [65] introduced the following popular iteration method known as the *proximal point algorithm* (PPA) that was used for solving problem (2.2.1): for a given $x_0 \in H$,

$$x_{n+1} = J_{\lambda_n} x_n$$

where the resolvent of the maximal monotone mapping $\mathcal{M}$ is denoted by $J_{\lambda_n} = (I + \lambda_n \mathcal{M})^{-1}$, I is the identity mapping, and $\{\lambda_n\} \subset (0, +\infty)$ is a regularization sequence. Martinet [65] showed that the sequence $\{x_n\}$ generated by the PPA converges weakly to a point x^* satisfying $0 \in \mathcal{M}(x^*)$.

Unfortunately, the evaluation of the resolvent operator in many interesting cases is as challenging as finding the solution to the original problem, thus limiting the applicability of the proximal point algorithm in its plain form. To partly overcome this problem, in 1976, Rockafellar [86], showed that this algorithm still converges when using inexact evaluations of the resolvent operator. Rockafellar [86] presented the following method:

$$x_{n+1} = J_{\lambda_n} x_n + e_n,$$

where $\{e_n\}$ is an error sequence, $J_{\lambda_n} = (I + \lambda_n \mathcal{M})^{-1}$ with I as the identity mapping on H and $\lambda_n > 0$. Rockafellar [86] proved that if $e_n \rightarrow 0$ quickly enough, such that $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, $\liminf_{n \rightarrow \infty} \lambda_n > 0$ and $\mathcal{M}^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a solution of a zero point of $\mathcal{M}$ (see also [17, 73] and their references there in). Observe that, the evaluation errors have to satisfy a summability condition which means that the resolvent operators have to be computed with increasing accuracy. This somehow is still a limitation in this algorithm, because in practice, controlling the errors of the resolvent operator in computation has proved to be difficult.

In 2001, Alvarez and Attouch [5] employed a method which was studied in [80, 81] for maximal monotone mappings by the PPA. This algorithm is called the inertial proximal point algorithm

and it is defined as follows:

$$\begin{cases} y_n = x_n + \lambda_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \tau_n \mathcal{M})^{-1} y_n, \quad n \geq 1. \end{cases} \quad (2.2.2)$$

It was proved that if $\{\tau_n\}$ is non-decreasing and $\{\lambda_n\} \subset [0, 1)$ with

$$\sum_{n=1}^{\infty} \lambda_n \|x_n - x_{n-1}\|^2 < \infty,$$

where λ_n is an extrapolation factor, $\lambda_n(x_n - x_{n-1})$ is the inertia term and this condition is true for $\lambda_n < \frac{1}{3}$. Then the algorithm (2.2.2) converges weakly to a zero of $\mathcal{M}$. Note that the inertial term greatly improves the performance of the algorithm and has a nice convergence properties, which are improved convergence speed in terms of reduced number of iterations and CPU time contributed by the presence of the inertial step [36, 39, 72].

Since all the above mentioned authors (see, [5, 65, 86]) only obtained weak convergence of the PPA for problem (2.2.1), one can ask the following question: Can the PPA be modified to guarantee strong convergence or does it always converge weakly? In studying the strong convergence, many authors have modified the PPA to guarantee strong convergence under different settings, see for example, [23, 58, 84, 95] and the references therein. However, they all came across a disadvantage of the fact that with the above mentioned algorithms there is need to compute $(I + \lambda \mathcal{M})^{-1}$ at each step of the iteration process. This means that even if the PPA converges strongly, it is not always convenient for use in applications.

2.2.2 Monotone Inclusion Problem (MIP)

Monotone Inclusion Problem (MIP) also known as Quasi-variational Inclusion Problem is a common generalization of problem (2.2.1) which occurs when the mapping $\mathcal{M}$ is written as the sum of two mappings; that is, $\mathcal{M} = A + B$. Recall that the MIP is to,

$$\text{find } x^* \in H \text{ such that } x^* \in (A + B)^{-1}(0), \quad (2.2.3)$$

where $A : H \rightarrow H$ is a monotone mapping for which we can use direct evaluations, and $B : H \rightarrow 2^H$ is a maximal monotone mapping for which we can compute the resolvent. Several authors have studied different iterative algorithms for approximating a solution of this problem (2.2.3) in the setting of Hilbert spaces (see, e.g., [33, 35, 71, 78, 82, 106] and the references therein).

Note that some concrete problems in machine learning, image processing and linear inverse problems can be modelled mathematically as MIP (2.2.3) (see, e.g., [31, 42, 50, 93]). Some examples are given below.

Example 2.2.2.1. Consider the nonlinear multivalued evolution equation:

$$0 \in \frac{\partial u}{\partial t} + \mathcal{F}(u), \quad u(0) = u_0,$$

and the associated stationary equation:

$$0 \in \mathcal{F}(\bar{u}),$$

where $\mathcal{F}$ is a monotone mapping on a Hilbert space H , which is multivalued, that is, for $u \in H$, $\mathcal{F}(u)$ is a (possibly nonempty) subset of H . Thus, the inclusion problem (2.2.3) can be rewritten as Example 2.2.2.1 when the mapping $\mathcal{F}$ is of the form $\mathcal{F} = A + B$, A and B are maximal monotone (for more details, see [59] and the references therein.)

Example 2.2.2.2. Let H be a real Hilbert space. Let $f : H \rightarrow \mathbb{R}$ be a proper, convex and smooth function and $g : H \rightarrow \mathbb{R}$ be a proper, convex and lower semicontinuous function. We consider the following minimization problem of finding $\bar{x} \in H$ such that,

$$f(\bar{x}) + g(\bar{x}) = \min_{x \in H} \{f(x) + g(x)\}. \quad (2.2.4)$$

Many optimization problems from image processing [15], statistical regression, machine learning (see, e.g., [108] and the references contained therein), can be adapted into the form of (2.2.4). In view of the Fermat's rule, the problem (2.2.4) is equivalent to the following problem of finding $\bar{x} \in H$ such that,

$$0 \in \nabla f(\bar{x}) + \partial g(\bar{x}), \quad (2.2.5)$$

where the subdifferential ∂g is a maximal monotone mapping and the gradient ∇f is a Lipschitz continuous monotone mapping [10, 87]. Then problem (2.2.5) is a special case of the inclusion problem (2.2.3) with $A := \nabla f$ and $B := \partial g$.

Several authors studied algorithms to solve problem (2.2.3) with co-coerciveness as a property of one of the mappings involved. The following are the sampled algorithms that included this property.

In 1979, Passty [78] was the first to present the *forward-backward splitting method* which incorporates, the proximal point algorithm which was mentioned in problem (2.2.1) (see, [22, 47, 64, 86] and the references therein) and the gradient method [59]. The forward-backward splitting method is a classical method to solve the Quasi-variational Inclusion Problem (2.2.3) and it is defined as follows; $x_1 \in H$ and

$$x_{n+1} = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n, \quad n \geq 1, \quad (2.2.6)$$

where x_1 is chosen arbitrarily in H , $\{\lambda_n\}$ is a sequence of positive numbers, A and B are maximal monotone mappings with $Dom(A) \subset Dom(B)$. In general, this method converges weakly

to a solution of problem (2.2.3) provided that $A : H \rightarrow H$ is $\frac{1}{L}$ -co-coercive, where L is Lipschitz constant of A .

In 2003, Moudafi and Oliny [69] proposed the following *inertial proximal point algorithm*, for finding zeros of the sum of two monotone mappings in a real Hilbert space H : For any $x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= (I + \gamma_n B)^{-1}(y_n - \gamma_n A x_n), \quad n \geq 1, \end{aligned}$$

where $A : H \rightarrow H$ is a single valued, co-coercive and monotone mapping while $B : H \rightarrow 2^H$ is a maximal monotone mapping. They obtained weak convergence theorem with the conditions that $\gamma_n < 2/L$ where, L is the Lipschitz constant of A and $\{\theta_n\} \subset [0, 1)$ with $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$.

Subsequently, in 2015, Lorenz and Pock [61] proposed the following *inertial forward-backward algorithm* for monotone mappings which is basically a modification of the above mentioned forward-backward splitting algorithm in Hilbert spaces: For any $x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= (I + \gamma_n B)^{-1}(y_n - \gamma_n A y_n), \quad n \geq 1, \end{aligned}$$

where $\{\gamma_n\}$ is a positive real sequence, $A : H \rightarrow H$ and $B : H \rightarrow 2^H$ are co-coercive and maximal monotone mappings, respectively. They obtained weak convergence of the sequence $\{x_n\}$ to a point in $(A + B)^{-1}(0)$.

Recently, Dadashi and Postolache [34] studied the following modified forward-backward splitting algorithm in a Hilbert space setting: For any $x_1 \in H$,

$$\begin{aligned} w_n &= J_{\gamma_n}^B(x_n - \gamma_n A x_n), \\ x_{n+1} &= \alpha_n x_n + \beta_n w_n + \gamma_n e_n, \quad n \geq 1, \end{aligned}$$

where $J_{\gamma_n}^B = (I + \gamma_n B)^{-1}$, with A is a co-coercive monotone and single valued mapping on H , B is a maximal monotone mapping of H into 2^H , e_n is an error vector, $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$. They proved that the sequence $\{x_n\}$ given by the algorithm converges strongly to a point $x^* \in (A + B)^{-1}(0)$ provided that the control sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

However, co-coercivity of a mapping is a strictly strong property compared to Lipschitz continuity and, therefore, unlikely to satisfy for general monotone inclusions.

With the intention to overcome this drawback, Tseng [106] introduced a modified forward-backward

splitting method, also known as *Tseng's splitting method* and is defined as follows: Let $C \subset H$ be a closed and convex set. Choose an arbitrary initial point $x_0 \in C$; given the current iterate x_n , compute the next iterate as follows:

$$y_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n,$$

where λ_n is the largest element λ of the set $\{\delta, \delta l, \delta l^2, \delta l^3, \dots, \delta l^j, \dots\}$ satisfying,

$$\lambda \langle Ax_n - Ay_n, x_n - y_n \rangle \leq \mu \|x_n - y_n\|^2,$$

where $\delta > 0$, $l \in (0, 1)$ and $\mu \in (0, 1)$ are constants.

Set

$$x_{n+1} = P_C(y_n - \lambda_n(Ay_n - Ax_n)). \quad (2.2.7)$$

Tseng proved that the sequence $\{x_n\}$ generated by (2.2.7) converges weakly to an element of the solution set of problem (2.2.3) in real Hilbert spaces provided that A is a Lipschitz continuous monotone mapping and B is a maximal monotone mapping.

In recent years, the forward-backward splitting method has received great attention by many authors, who improved it in various ways, (see [8, 11, 14, 30, 31, 32, 40, 41, 44, 48, 57, 59, 61, 69, 76, 89, 83, 99, 100, 101, 102, 103, 104, 106, 113], and the references therein). However, all the above authors only obtained weak convergence theorems to a solution of problem (2.2.3). Therefore, the following question comes naturally; How can we modify the forward-backward splitting method to obtain strong convergence?

Consequently, in the year 2019, Thong and Choleamjiak [98] came up with a modification of the forward-backward splitting method in (2.2.7) to obtain the strong convergence to a solution of the monotone inclusion problem (2.2.3) known as the *viscosity or Halpern-type forward-backward splitting method*. They constructed the following algorithm with a Lipschitz continuous monotone mapping, $A : H \rightarrow H$, and a maximal monotone mapping, $B : H \rightarrow 2^H$, such that the solution set is denoted by $\Omega_* := (A + B)^{-1}(0) \neq \emptyset$ and a contraction mapping $f : H \rightarrow H$ with contraction parameter $\rho \in [0, 1)$. In addition, this scheme does not require the knowledge of the Lipschitz constant L of the mapping A ; thus making the proposed algorithm more practical for computations.

Algorithm 2.2.1.

Initialization: Let $\delta > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$ and $\gamma \in (0, 2)$.

Iterative Steps: Choose $x_0 \in H$ arbitrary. Given x_n , calculate x_{n+1} as follows;

Step 1. Compute

$$y_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n,$$

where λ_n is the largest element λ of the set $\{\delta, \delta l, \delta l^2, \delta l^3, \dots, \delta l^j, \dots\}$ satisfying,

$$\lambda \langle Ax_n - Ay_n, x_n - y_n \rangle \leq \mu \|x_n - y_n\|^2.$$

If $x_n = y_n$, then stop and y_n is an element of the solution set Ω_* . Otherwise go to **Step 2**.

Step 2. Compute

$$z_n = x_n - \gamma \eta_n d_n,$$

where

$$d_n = x_n - y_n - \lambda_n(Ax_n - Ay_n) \text{ and } \eta_n = (1 - \mu) \frac{\|x_n - y_n\|^2}{\|d_n\|^2}.$$

Step 3. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)z_n.$$

Step 4. Set $n := n + 1$ and go to **Step 1**.

Thong and Chalamjiak [98] proved the strong convergence of Algorithm 2.2.1 to the solution of problem (2.2.3), that is, they approximated an element $p \in \Omega_*$, where $p = P_{\Omega_*} f(p)$. Note that in this algorithm, each step of iterates involves only with A as the forward step and B as the backward step, but not the sum of the mappings.

Very recently, Chalamjiak et al. [25] proposed the following modified inertial forward-backward algorithm for monotone mappings: Let H be a real Hilbert space. Let $A : H \rightarrow H$ and $B : H \rightarrow 2^H$ be maximal monotone mappings such that $(A + B)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_0, x_1 \in H$ and

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= \alpha_n u + \beta_n y_n + \gamma_n J_{\lambda_n}^B(y_n - \lambda_n A y_n), \quad n \geq 1. \end{aligned}$$

where $J_{\lambda_n}^B = (I + \lambda_n B)^{-1}$, $0 < \lambda_n \leq 2L$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$, $\theta_n < \frac{1}{3}$ is an extrapolation factor and $\theta_n(x_n - x_{n-1})$ is the inertia term. They proved that the sequence $\{x_n\}$ generated by the above algorithm strongly converges to a solution of the inclusion problem (2.2.3) in Hilbert spaces.

From the above discussions, it is natural to ask the following general questions. Can we have strong convergence results for approximating zeros of the sum of two maximal monotone mappings without the assumption that one of the mappings is single valued and co-coercive (also known as inverse-strongly monotone)? We continue to explore more general problems to pursue answers to the above questions.

2.2.3 Split Quasi-Inclusion Problem (SQIP)

In 2011, Moudafi [68] was the first author who proposed and explored the *Split Quasi-Inclusion Problem* (SQIP) in the setting of Hilbert spaces. Recall that, the SQIP is to;

$$\text{find } x^* \in H_1 \text{ such that } 0 \in Ax^* + Bx^* \text{ and } 0 \in G(Tx^*) + K(Tx^*), \quad (2.2.8)$$

where $A : H_1 \rightarrow H_1$ and $G : H_2 \rightarrow H_2$ are nonlinear mappings, $B : H_1 \rightarrow 2^{H_1}$ and $K : H_2 \rightarrow 2^{H_2}$ are set-valued mappings, and $T : H_1 \rightarrow H_2$ is a bounded linear mapping. If in (2.2.8), we have $H_1 = H_2$, $G = K = 0$, and $T = I$ (identity mapping on H_1), then the problem (2.2.8) reduces to the Quasi-variational Inclusion Problem (2.2.3).

Moudafi [68] proposed the following iterative algorithm to solve the problem (2.2.8), which is defined by the following manner: let $x_0 \in H_1$, and

$$x_{n+1} = U(x_n + \gamma T^*(V - I)Tx_n), \quad n \geq 1, \quad (2.2.9)$$

where $\gamma > 0$, $V := J_\lambda^K(I - \lambda G)$, $U := J_\lambda^B(I - \lambda A)$, for some $\lambda > 0$, B and K are maximal monotone, and A and G are co-coercive (also known as inverse-strongly monotone). Moudafi [68] proved that the sequence $\{x_n\}$ generated by the above Algorithm (2.2.9) converges weakly to a solution of the SQIP, that is problem (2.2.8) provided that the solution set of that problem is nonempty. Subsequently, other methods for solving this kind of problems in the framework of Hilbert spaces have been considered, proposed and studied (see, for example, [1, 20, 28, 46, 51, 56, 75, 88, 112] and the references there in). Unfortunately, like Moudafi [68], all these authors use the co-coercive condition for the mappings A and G , and this might prevent or limit the potential applications of these methods.

In 2021, Izuchukwu et al. [52] combined the inertial and relaxation techniques with the proximal contraction algorithm, that is, they introduced and analysed new relaxed inertial proximal contraction-type method given below for solving the SQIP (2.2.8) in real Hilbert spaces. Let the set of solutions of problem (2.2.8) be denoted by $\Omega = \{z \in (A + B)^{-1}(0) : Tz \in (G + K)^{-1}(0)\}$ is nonempty. Let $A : H_1 \rightarrow H_1$ and $G : H_2 \rightarrow H_2$ be monotone and Lipschitz continuous with Lipschitz constants L_1 and L_2 , respectively, and let $B : H_1 \rightarrow 2^{H_1}$ and $K : H_2 \rightarrow 2^{H_2}$ be maximal monotone mappings, with $T : H_1 \rightarrow H_2$ as a bounded linear mapping. Let L_1 and L_2 be known.

Izuchukwu et al. [52] presented the following Algorithm for solving problem (2.2.8).

Algorithm 2.2.2. Relaxed inertial proximal contraction-type method with fixed stepsize.

Initialization: Let $\alpha \in (0, 2)$, $\theta_n \in [0, 1]$, $\rho_n \in (0, 1]$ for all $n \geq 1$. Let $\eta_n > 0$, $0 < \bar{\alpha} < \alpha < 2$, $\mu \in (0, \frac{1}{L_1})$, $\lambda \in (0, \frac{1}{L_2})$.

Iterative Steps: Choose $x_0, x_1 \in H_1$ arbitrarily. Set $n := 1$,

Step 1. Set:

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then compute

$$\begin{aligned} y_n &= J_\lambda^K(Tw_n - \lambda GTw_n), \\ \bar{t}_n &= Tw_n - \bar{\alpha}\gamma_n v_n, \end{aligned}$$

where $v_n = Tw_n - y_n - \lambda(GTw_n - Gy_n)$ and $\gamma_n = \frac{\langle Tw_n - y_n, v_n \rangle}{\|v_n\|^2}$, if $v_n \neq 0$; otherwise $\gamma_n = 0$.

Step 2. Compute

$$b_n = w_n + \eta_n T^*(\bar{t}_n - Tw_n),$$

where the stepsize η_n is chosen such that for small $\bar{\epsilon} > 0$, $\eta_n \in \left[\bar{\epsilon}, \frac{\|Tw_n - \bar{t}_n\|^2}{\|T^*(Tw_n - \bar{t}_n)\|^2} - \bar{\epsilon}\right]$, if $\bar{t}_n \neq Tw_n$; otherwise $\eta_n = \eta$.

Step 3. Compute

$$\begin{aligned} u_n &= J_\lambda^B(b_n - \mu Ab_n Tw_n), \\ z_n &= b_n - \alpha\beta_n r_n, \end{aligned}$$

where $r_n = b_n - u_n - \mu(Ab_n - Au_n)$ and $\beta_n = \frac{\langle b_n - u_n, r_n \rangle}{\|r_n\|^2}$, if $r_n \neq 0$; otherwise $\beta_n = 0$.

Step 4. Compute

$$x_{n+1} = (1 - \rho_n)w_n + \rho_n z_n.$$

Step 7. Set $n := n + 1$ and go to **Step 1**.

The above Algorithm 2.2.2 has a few major advantages as compared to other mentioned methods. First, it does not require the restrictive co-coerciveness assumption on A and G , instead

it requires A and G to be monotone and Lipschitz continuous rather than co-coercive. Note that for general monotone mappings, Lipschitz continuity is a much weaker assumption than co-coercivity thus easier to satisfy for general monotone inclusion problems. The other advantage is that this method combines the inertial and relaxation techniques in order to improve convergence speed and it is analyzed numerically. Lastly, the assumptions on the inertial parameter does not depend on the coefficient of the co-coercivity which is sometimes very difficult to estimate in practice.

Izuchukwu et al. [52] proved that the sequence $\{x_n\}$ generated by Algorithm 2.2.2 converges weakly to an element of Ω without assuming a co-coerciveness condition on one of the mappings. From all these findings, one can ask the following question: Can strong convergence theorems be obtained for solving problem (2.2.8)?

2.2.4 Split Equality Null-Point Problem (SENPP)

The *Split Equality Null-Point Problem* (SENPP) was proposed and explored by Moudafi [66]. Let $A : H_1 \rightarrow 2^{H_1}$ and $B : H_2 \rightarrow 2^{H_2}$ be two set-valued maximal monotone mappings, $T : H_1 \rightarrow H_3$ and $S : H_2 \rightarrow H_3$ be bounded linear mappings with adjoints T^* and S^* , respectively. Then, recall that the SENPP is to,

$$\text{find } x^* \in A^{-1}(0), u^* \in B^{-1}(0) \text{ such that } Tx^* = Su^*, \quad (2.2.10)$$

where $A^{-1}(0) = \{x^* \in H_1 : 0 \in Ax^*\}$ and $B^{-1}(0) = \{u^* \in H_2 : 0 \in Bu^*\}$ are closed and convex sets.

Note that if $S = I$, (2.2.10) reduces to the split common null-point problem (SCNP) which was cordially introduced and deliberated by Byrne [18] given below:

$$\text{find } x^* \in A^{-1}(0) \text{ such that } Tx^* \in B^{-1}(0). \quad (2.2.11)$$

where $A : H_1 \rightarrow 2^{H_1}$ and $B : H_2 \rightarrow 2^{H_2}$ are two set-valued maximal monotone mappings, $T : H_1 \rightarrow H_2$ a bounded linear mapping with adjoint T^* . Theoretically and practically, the SCNP (2.2.11) received considerably much attention as it includes variational inequalities and many constrained optimization problems as special cases, (see for example, [18, 43, 70]). Byrne [18] introduced an algorithm for approximating the solution of problem (2.2.11) in the framework of real Hilbert spaces. The algorithm is as follows:

Algorithm 2.2.3.

Initialization: Given $\lambda > 0$ and choose $x_0 \in H_1$ arbitrarily.

Iterative Steps: Given the current iterate x_n , calculate x_{n+1} as follows;

$$x_{n+1} = J_\lambda^A[x_n - \gamma T^*(I - J_\lambda^B)Tx_n], \quad (2.2.12)$$

where the step-size $\gamma \in (0, \frac{2}{L})$ with $L = \|T^*T\|$, and $J_\lambda^A = (I + \lambda A)^{-1}$ and $J_\lambda^B = (I + \lambda B)^{-1}$ are the resolvents of A and B , respectively.

Byrne [18] showed weak convergence theorems for Algorithm (2.2.3) to the solution of (2.2.11) in real Hilbert spaces.

In the attempt to solve the SENPP (2.2.10), Moudafi [66] introduced the following algorithm and he obtained a weak convergence theorem for maximal monotone mappings A and B . Let $\lambda > 0$. Given $(x_0, u_0) \in H_1 \times H_2$ and the current iterate (x_n, u_n) , calculate (x_{n+1}, u_{n+1}) as follows:

$$\begin{aligned} x_{n+1} &= J_r^A(x_n - \gamma_n T^*(Tx_n - Sy_n)), \\ y_{n+1} &= J_r^B(y_n - \gamma_n S^*(Tx_n - Sy_n)), \quad \text{for all } n \geq 0, \end{aligned} \quad (2.2.13)$$

where $\{\gamma_n\} \subset (\epsilon, \min\{\frac{1}{\|T\|^2}, \frac{1}{\|S\|^2}\} - \epsilon)$, for small enough $\epsilon > 0$, and T^* and S^* denote the adjoint of T and S , respectively. Note that in the above algorithm, the step-size, γ_n depends on the norms $\|T\|$ and $\|S\|$. There is a difficulty encountered when trying to implement the alternating algorithm for solving SENPP (2.2.13) as it requires calculation of the step-size γ_n , where we need to compute $\|T\|$ and $\|S\|$, which is generally not an easy task in practice. Hence the following question comes naturally: How can one obtain strong convergence theorems for problem (2.2.10) with a condition that when calculating the step-size, one does not require computation of the norms?

In the next Chapter, we introduce a more general problem called the Split Equality Inclusion Problem (SEIP) and provide a method for approximating its solution. Our results provides positive answers to the questioned raised above.

3. Split Equality Inclusion Problem

Let H_1 , H_2 and H_3 be real Hilbert spaces. The *Split Equality Inclusion Problem* (SEIP) is defined as finding a point $(x, u) \in H_1 \times H_2$ such that

$$x^* \in (A_1 + B_1)^{-1}(0) \text{ and } u^* \in (A_2 + B_2)^{-1}(0) \text{ such that } Tx^* = Su^*, \quad (3.0.1)$$

where $A_1 : H_1 \rightarrow H_1$ and $A_2 : H_2 \rightarrow H_2$ are Lipschitz continuous monotone mappings and $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ are multi-valued maximal monotone mappings and $T : H_1 \rightarrow H_3$ and $S : H_2 \rightarrow H_3$ are bounded linear mappings with adjoints T^* and S^* , respectively. The solution set of the SEIP is denoted by Ω .

We then develop an iterative algorithm and show that the resulting method converges strongly to a solution of (3.0.1). As a consequence, we obtain a minimum norm solution of the SEIP.

3.1 Main Results

In this section, we present a detailed description of our projection method and discuss its convergence analysis. We assume the following conditions.

Conditions

- C1. Let H_1 , H_2 and H_3 be real Hilbert spaces.
- C2. Let $A_1 : H_1 \rightarrow H_1$ and $A_2 : H_2 \rightarrow H_2$ be L_1 -Lipschitz continuous monotone mapping and L_2 -Lipschitz continuous monotone mapping, respectively. Let $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ be multi-valued maximal monotone mappings.
- C3. Let $T : H_1 \rightarrow H_3$ and $S : H_2 \rightarrow H_3$ be bounded linear operators with adjoints T^* and S^* , respectively.
- C4. Let $\{\alpha_n\} \subset (0, 1)$ be a real sequence such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.
- C5. Let $\{\tau_n\}$ be a positive sequence such that $\tau_n \in \left(0, \frac{1}{2}\right)$ and $\frac{\tau_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$.
- C6. Let $\Omega := \{p \in (A_1 + B_1)^{-1}(0), q \in (A_2 + B_2)^{-1}(0) : Tp = Sq\}$ be nonempty.

Next, we introduce an inertial iterative algorithm to establish strong convergence theorems for approximating solutions of the SEIP (3.0.1).

Algorithm 3.1.1.

Initialization: Let $x_0, x_1 \in H_1$, $u_0, u_1 \in H_2$, $\delta, e, \tau > 0$, $l, \mu \in (0, 1)$, $\gamma \in (0, 2)$.

Choose $(x, u) \in H_1 \times H_2$.

Iterative Steps: Given $x_{n-1}, x_n \in H_1$, $u_{n-1}, u_n \in H_2$, calculate x_{n+1} and u_{n+1} as follows;

Step 1. Compute

$$a_n = x_n + \xi_n(x_n - x_{n-1}) \text{ and } b_n = u_n + \xi_n(u_n - u_{n-1}),$$

where $0 \leq \xi_n \leq \bar{\xi}_n$ and

$$\bar{\xi}_n = \begin{cases} \min \left\{ \frac{\tau_n}{\|x_n - x_{n-1}\| + \|u_n - u_{n-1}\|}, \tau \right\}, & \text{if } x_n \neq x_{n-1} \text{ and } u_n \neq u_{n-1}, \\ \tau, & \text{Otherwise.} \end{cases}$$

Step 2. Compute $\{\gamma_n\}$ satisfying the following:

$$0 < \gamma_0 \leq \gamma_n \leq \frac{\|Ta_n - Sb_n\|^2}{\|T^*(Ta_n - Sb_n)\|^2 + \|S^*(Ta_n - Sb_n)\|^2},$$

for $n \in \Omega_n = \{n \in \mathbb{N} : Ta_n - Sb_n \neq 0\}$.

Otherwise, let $\gamma_n = \gamma > 0$.

Step 3. Compute

$$t_n = a_n - \gamma_n T^*(Ta_n - Sb_n) \text{ and } w_n = b_n + \gamma_n S^*(Ta_n - Sb_n).$$

Step 4. Compute

$$y_n = (I + \lambda_n B_1)^{-1}(I - \lambda_n A_1)t_n \text{ and } v_n = (I + \theta_n B_2)^{-1}(I - \theta_n A_2)w_n,$$

where λ_n is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, \delta l^3, \dots, \delta l^j, \dots\}$, $j \in \mathbb{N}$ satisfying,

$$\lambda \|A_1 t_n - A_1 y_n\| \leq \mu \|y_n - t_n\|, \quad (3.1.1)$$

and θ_n is chosen to be the largest $\theta \in \{e, el, el^2, el^3, \dots, el^k, \dots\}$, $k \in \mathbb{N}$ satisfying,

$$\theta \|A_2 w_n - A_2 v_n\| \leq \mu \|v_n - w_n\|. \quad (3.1.2)$$

Step 5. Compute

$$m_n = t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \text{ and } e_n = w_n - v_n - \theta_n(A_2 w_n - A_2 v_n).$$

Step 6. Compute

$$z_n = t_n - \gamma\eta_n m_n \text{ and } h_n = w_n - \gamma\beta_n e_n,$$

where

$$\eta_n = \begin{cases} (1 - \mu) \frac{\|t_n - y_n\|^2}{\|m_n\|^2}, & m_n \neq 0, \\ 0, & m_n = 0, \end{cases}$$

$$\beta_n = \begin{cases} (1 - \mu) \frac{\|w_n - v_n\|^2}{\|e_n\|^2}, & e_n \neq 0, \\ 0, & e_n = 0. \end{cases}$$

Step 7. Compute

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) z_n \text{ and } u_{n+1} = \alpha_n u + (1 - \alpha_n) h_n.$$

Step 8. Set $n := n + 1$ and go to **Step 1**.

We now prove some lemmas that will be used to determine the convergence of Algorithm 3.1.1.

Lemma 3.1.2. *Assume that conditions C1 – C6 hold. Then, the inequalities in (3.1.1) and (3.1.2) are well defined. Furthermore, $\forall n \geq 0$, we have that $\delta \geq \lambda_n \geq \min\{\delta, \frac{\mu l}{L_1}\}$ and $e \geq \theta_n \geq \min\{\delta, \frac{\mu l}{L_2}\}$.*

Proof. For fixed n , $\lim_{j \rightarrow \infty} \delta l^j \|A_1 t_n - A_1 y_n\| = 0$. Therefore, there exists $j \in \mathbb{N}$ such that $\delta l^j \|A_1 t_n - A_1 y_n\| \leq \mu \|t_n - y_n\|$. This shows that the inequality (3.1.1) holds.

If $\lambda_n = \delta$ for all $n \geq 1$, then we have $\{\lambda_n\} = \{\delta\} \subset (0, 1)$.

Observe that inequality (3.1.1) fails to hold for $\lambda = \frac{\lambda_n}{l}$, which means (3.1.1) becomes

$$\frac{\lambda_n}{l} \|A_1 t_n - A_1 y_n\| > \mu \|t_n - y_n\|.$$

By Lipschitz continuity of A_1 , we have

$$\frac{\lambda_n L_1}{l} \|t_n - y_n\| > \mu \|t_n - y_n\|.$$

That is $\frac{\lambda_n}{l} > \frac{\mu}{L_1}$, which follows that $\delta \geq \lambda_n \geq \min\{\delta, \frac{\mu l}{L_1}\}$. Similarly, for A_2 which is L_2 -Lipschitz with constant $L_2 > 0$, the inequality (3.1.2) holds and $e \geq \theta_n \geq \min\{\delta, \frac{\mu l}{L_2}\}$. $\square$

Lemma 3.1.3. *Assume that conditions C1 – C6 hold and the sequences $\{t_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{h_n\}$ are generated by Algorithm 3.1.1. Let $(p, q) \in \Omega$. Then, the following inequalities hold.*

$$\mathbf{a)} \ \|z_n - p\|^2 \leq \|t_n - p\|^2 - \left(\frac{2 - \gamma}{\gamma}\right) \|t_n - z_n\|^2, \quad \forall n \geq 0,$$

$$\mathbf{b}) \|h_n - q\|^2 \leq \|w_n - q\|^2 - \left(\frac{2-\gamma}{\gamma}\right) \|w_n - h_n\|^2, \quad \forall n \geq 0.$$

Proof. **a)** By the definition of m_n and inequality (3.1.1), we get

$$\begin{aligned} \langle t_n - p, m_n \rangle &= \langle t_n - y_n + y_n - p, m_n \rangle \\ &= \langle t_n - y_n, m_n \rangle + \langle y_n - p, m_n \rangle \\ &= \langle t_n - y_n, t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \rangle + \langle y_n - p, t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \rangle \\ &= \|t_n - y_n\|^2 - \lambda_n \langle t_n - y_n, A_1 t_n - A_1 y_n \rangle + \langle y_n - p, t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \rangle \\ &\geq \|t_n - y_n\|^2 - \lambda_n \|t_n - y_n\| \|A_1 t_n - A_1 y_n\| + \langle y_n - p, t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \rangle \\ &\geq \|t_n - y_n\|^2 - \mu \|t_n - y_n\|^2 + \langle y_n - p, t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \rangle \\ &= (1 - \mu) \|t_n - y_n\|^2 + \langle y_n - p, t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \rangle. \end{aligned} \quad (3.1.3)$$

Since $y_n = (I + \lambda_n B_1)^{-1}(I - \lambda_n A_1)t_n$, then $(I - \lambda_n A_1)t_n \in (I + \lambda_n B_1)y_n$, which implies that $t_n - \lambda_n A_1 t_n \in y_n + \lambda_n B_1 y_n$. Now, clearly there exists $j_n \in B_1 y_n$ such that $t_n - \lambda_n A_1 t_n = y_n + \lambda_n j_n$. Hence

$$j_n = \frac{1}{\lambda_n} (t_n - y_n - \lambda_n A_1 t_n). \quad (3.1.4)$$

On the other hand, we note that $A_1 p + B_1 p \ni 0$ and $A_1 y_n + j_n \in A_1 y_n + B_1 y_n$. Using (3.1.4) and the fact that $A_1 + B_1$ is monotone, we have $\langle A_1 y_n + j_n, y_n - p \rangle \geq 0$ and hence $\langle A_1 y_n + \frac{1}{\lambda_n} (t_n - y_n - \lambda_n A_1 t_n), y_n - p \rangle \geq 0$. This implies that $\frac{1}{\lambda_n} \langle t_n - y_n - \lambda_n A_1 t_n + \lambda_n A_1 y_n, y_n - p \rangle \geq 0$. Thus, we have

$$\langle t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n), y_n - p \rangle \geq 0. \quad (3.1.5)$$

Combining (3.1.3) and (3.1.5), we get

$$\langle t_n - p, m_n \rangle \geq (1 - \mu) \|t_n - y_n\|^2. \quad (3.1.6)$$

On the other hand, applying the definition of z_n , we have

$$\begin{aligned} \|z_n - p\|^2 &= \|t_n - \gamma \eta_n m_n - p\|^2 \\ &= \|t_n - p - \gamma \eta_n m_n\|^2 \\ &= \langle t_n - p - \gamma \eta_n m_n, t_n - p - \gamma \eta_n m_n \rangle \\ &= \|t_n - p\|^2 - 2\gamma \eta_n \langle t_n - p, m_n \rangle + \gamma^2 \eta_n^2 \|m_n\|^2 \\ &\leq \|t_n - p\|^2 - 2\gamma \eta_n (1 - \mu) \|t_n - y_n\|^2 + \gamma^2 \eta_n^2 \|m_n\|^2. \end{aligned} \quad (3.1.7)$$

Note that if $m_n = 0$, then **a)** holds. Now, assume that $m_n \neq 0$. Then, $\eta_n = (1 - \mu) \frac{\|t_n - y_n\|^2}{\|m_n\|^2}$ implies that $\eta_n \|m_n\|^2 = (1 - \mu) \|t_n - y_n\|^2$. Hence

$$-2\gamma \eta_n^2 \|m_n\|^2 = -2\gamma \eta_n (1 - \mu) \|t_n - y_n\|^2. \quad (3.1.8)$$

Combining (3.1.7) and (3.1.8), we obtain

$$\begin{aligned}
 \|z_n - p\|^2 &\leq \|t_n - p\|^2 - 2\gamma\eta_n^2\|m_n\|^2 + \gamma^2\eta_n^2\|m_n\|^2 \\
 &= \|t_n - p\|^2 - (2\gamma\eta_n^2 - \gamma^2\eta_n^2)\|m_n\|^2 \\
 &= \|t_n - p\|^2 - \gamma^2\eta_n^2\left(\frac{2}{\gamma} - 1\right)\|m_n\|^2 \\
 &= \|t_n - p\|^2 - \left(\frac{2-\gamma}{\gamma}\right)\|\gamma\eta_n m_n\|^2 \\
 &= \|t_n - p\|^2 - \left(\frac{2-\gamma}{\gamma}\right)\|t_n - z_n\|^2, \quad \forall n \geq 0.
 \end{aligned}$$

Similarly, following the methods in **a)**, we obtain inequality **b)**,

$$\|h_n - q\|^2 \leq \|w_n - q\|^2 - \left(\frac{2-\gamma}{\gamma}\right)\|w_n - h_n\|^2, \quad \forall n \geq 0.$$

□

Lemma 3.1.4. *Assume that conditions C1–C6 hold and the sequences $\{t_n\}$, $\{w_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{h_n\}$ are generated by Algorithm 3.1.1. Then, the following hold.*

$$\begin{aligned}
 \mathbf{a)} \quad \|t_n - y_n\|^2 &\leq \frac{1 + \mu^2}{[(1 - \mu)\gamma]^2} \|t_n - z_n\|^2, \quad \forall n \geq 0, \\
 \mathbf{b)} \quad \|w_n - v_n\|^2 &\leq \frac{1 + \mu^2}{[(1 - \mu)\gamma]^2} \|w_n - h_n\|^2, \quad \forall n \geq 0.
 \end{aligned}$$

Proof. **a)** Note that if $m_n = 0$, from (3.1.6), we have $t_n = y_n$. Hence the inequality in **a)** holds. Now, assume that $m_n \neq 0$. Using (3.1.1) and the monotonicity of A_1 , we obtain

$$\begin{aligned}
 \|m_n\|^2 &= \|t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n)\|^2 \\
 &= \langle t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n), t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \rangle \\
 &= \|t_n - y_n\|^2 + \lambda_n^2 \|A_1 t_n - A_1 y_n\|^2 - 2\lambda_n \langle t_n - y_n, A_1 t_n - A_1 y_n \rangle \\
 &\leq \|t_n - y_n\|^2 + \lambda_n^2 \|A_1 t_n - A_1 y_n\|^2 \\
 &\leq \|t_n - y_n\|^2 + \mu^2 \|t_n - y_n\|^2 \\
 &= (1 + \mu^2) \|t_n - y_n\|^2.
 \end{aligned} \tag{3.1.9}$$

Equivalently,

$$\frac{1}{\|m_n\|^2} \geq \frac{1}{(1 + \mu^2) \|t_n - y_n\|^2}. \tag{3.1.10}$$

Thus, from the definition of η_n and (3.1.10), we obtain that

$$\eta_n = (1 - \mu) \frac{\|t_n - y_n\|^2}{\|m_n\|^2} \geq \frac{(1 - \mu) \|t_n - y_n\|^2}{(1 + \mu^2) \|t_n - y_n\|^2} = \frac{1 - \mu}{1 + \mu^2}. \tag{3.1.11}$$

On the other hand, from the definition of η_n and (3.1.11), we have

$$\begin{aligned}
 \|t_n - y_n\|^2 &= \frac{1}{(1-\mu)} \eta_n \|m_n\|^2 \\
 &= \frac{1}{1-\mu} \left(\frac{1}{\gamma^2 \eta_n} \right) \|\gamma \eta_n m_n\|^2 \\
 &= \frac{1}{1-\mu} \left(\frac{1}{\gamma^2 \eta_n} \right) \|t_n - z_n\|^2 \\
 &= \frac{1}{(1-\mu)\gamma^2} \|t_n - z_n\|^2 \left(\frac{1}{\eta_n} \right) \\
 &\leq \frac{1}{(1-\mu)\gamma^2} \|t_n - z_n\|^2 \left(\frac{1+\mu^2}{1-\mu} \right) \\
 &= \frac{1+\mu^2}{[(1-\mu)\gamma]^2} \|t_n - z_n\|^2, \quad \forall n \geq 0.
 \end{aligned} \tag{3.1.12}$$

Similarly, following the methods in **a)**, we obtain that

$$\|w_n - v_n\|^2 \leq \frac{1+\mu^2}{[(1-\mu)\gamma]^2} \|w_n - h_n\|^2, \quad \forall n \geq 0.$$

□

Lemma 3.1.5. Suppose C1–C6 hold and the sequences $\{t_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{v_n\}$ are generated by Algorithm 3.1.1.

- a)** If $\lim_{k \rightarrow \infty} \|t_{n_k} - y_{n_k}\| = 0$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ converges weakly to some $z \in H_1$, then $z \in \Omega_1 = \{z \in H_1 | 0 \in (A_1 + B_1)z\}$.
- b)** If $\lim_{k \rightarrow \infty} \|w_{n_k} - v_{n_k}\| = 0$ and a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ converges weakly to some $h \in H_2$, then $h \in \Omega_2 = \{h \in H_2 | 0 \in (A_2 + B_2)h\}$.

Proof. **a)** Let $(v, u) \in \text{Graph}(A_1 + B_1)$, that is $u - A_1 v \in B_1 v$. By definition of y_n , we have $y_{n_k} = (I + \lambda_{n_k} B_1)^{-1} (I - \lambda_{n_k} A_1) t_{n_k}$. Thus, we get $(I - \lambda_{n_k} A_1) t_{n_k} \in (I + \lambda_{n_k} B_1) y_{n_k}$, which implies that

$$\frac{1}{\lambda_{n_k}} (t_{n_k} - y_{n_k} - \lambda_{n_k} A_1 t_{n_k}) \in B_1 y_{n_k}. \tag{3.1.13}$$

By the monotonicity of B_1 , it follows that $\langle v - y_{n_k}, u - A_1 v - \frac{1}{\lambda_{n_k}} (t_{n_k} - y_{n_k} - \lambda_{n_k} A_1 t_{n_k}) \rangle \geq 0$. Thus

$$\begin{aligned}
 \langle v - y_{n_k}, u \rangle &\geq \langle v - y_{n_k}, A_1 v - A_1 t_{n_k} + \frac{1}{\lambda_{n_k}} (t_{n_k} - y_{n_k}) \rangle \\
 &= \langle v - y_{n_k}, A_1 v - A_1 t_{n_k} \rangle + \langle v - y_{n_k}, \frac{1}{\lambda_{n_k}} (t_{n_k} - y_{n_k}) \rangle \\
 &= \langle v - y_{n_k}, A_1 v - A_1 y_{n_k} + A_1 y_{n_k} - A_1 t_{n_k} \rangle + \langle v - y_{n_k}, \frac{1}{\lambda_{n_k}} (t_{n_k} - y_{n_k}) \rangle \\
 &= \langle v - y_{n_k}, A_1 v - A_1 y_{n_k} \rangle + \langle v - y_{n_k}, A_1 y_{n_k} - A_1 t_{n_k} \rangle + \langle v - y_{n_k}, \frac{1}{\lambda_{n_k}} (t_{n_k} - y_{n_k}) \rangle \\
 &\geq \langle v - y_{n_k}, A_1 y_{n_k} - A_1 t_{n_k} \rangle + \langle v - y_{n_k}, \frac{1}{\lambda_{n_k}} (t_{n_k} - y_{n_k}) \rangle.
 \end{aligned}$$

Since by assumption, $\|t_{n_k} - y_{n_k}\| \rightarrow 0$ and A_1 is a Lipschitz continuous mapping, we have $\|A_1 y_{n_k} - A_1 t_{n_k}\| \rightarrow 0$. Hence

$$\langle v - z, u \rangle = \lim_{k \rightarrow \infty} \langle v - y_{n_k}, u \rangle \geq 0. \quad (3.1.14)$$

Now, the fact that $A_1 + B_1$ is a maximal monotone mapping and inequality (3.1.14) imply that $z \in \Omega_1 = \{z \in H_1 | 0 \in (A_1 + B_1)z\}$. Similarly, following the methods in **a)**, one can also show that $h \in \Omega_2 = \{h \in H_2 | 0 \in (A_2 + B_2)h\}$. $\square$

Lemma 3.1.6. *Assume that C1 – C6 hold and the sequences $\{t_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{u_n\}$ are generated by Algorithm 3.1.1. Let $(p, q) \in \Omega$. Then, for some $\epsilon \in (0, \frac{1}{2})$ and $N > 0$,*

$$\|t_n - p\|^2 + \|w_n - q\|^2 \leq (1 + 2\alpha_n \epsilon)(\|x_n - p\|^2 + \|u_n - q\|^2) + 2\alpha_n \epsilon, \quad \forall n \geq N.$$

Proof. From Algorithm 3.1.1, we obtain that

$$\begin{aligned} \|t_n - p\|^2 &= \|a_n - \gamma_n T^*(Ta_n - Sb_n) - p\|^2 \\ &= \|a_n - p - \gamma_n T^*(Ta_n - Sb_n)\|^2 \\ &= \|a_n - p\|^2 - 2\gamma_n \langle a_n - p, T^*(Ta_n - Sb_n) \rangle + \gamma_n^2 \|T^*(Ta_n - Sb_n)\|^2. \end{aligned} \quad (3.1.15)$$

Using the equality (2.1.3), we have

$$\begin{aligned} -2\langle a_n - p, T^*(Ta_n - Sb_n) \rangle &= -2\langle Ta_n - Tp, Ta_n - Sb_n \rangle \\ &= -\|Ta_n - Tp\|^2 - \|Ta_n - Sb_n\|^2 + \|Ta_n - Tp - (Ta_n - Sb_n)\|^2 \\ &= -\|Ta_n - Tp\|^2 - \|Ta_n - Sb_n\|^2 + \|Sb_n - Tp\|^2. \end{aligned} \quad (3.1.16)$$

Thus, from (3.1.15) and (3.1.16), we have

$$\begin{aligned} \|t_n - p\|^2 &= \|a_n - p\|^2 - \gamma_n \|Ta_n - Tp\|^2 - \gamma_n \|Ta_n - Sb_n\|^2 \\ &\quad + \gamma_n \|Sb_n - Tp\|^2 + \gamma_n^2 \|T^*(Ta_n - Sb_n)\|^2. \end{aligned} \quad (3.1.17)$$

Moreover, (2.1.4) gives

$$\|a_n - p\|^2 = \|x_n - p\|^2 - \|a_n - x_n\|^2 + 2\langle p - a_n, x_n - a_n \rangle, \quad (3.1.18)$$

and

$$\begin{aligned} \langle p - a_n, x_n - a_n \rangle &\leq \|p - a_n\| \|x_n - a_n\| \\ &= \xi_n \|p - a_n\| \|x_n - x_{n-1}\| \\ &\leq \frac{\xi_n}{2} \|x_n - x_{n-1}\| (\|p - a_n\|^2 + 1) \\ &\leq \frac{\xi_n}{2} \|x_n - x_{n-1}\| (\|p - x_n\| + \|x_n - a_n\|)^2 + 1) \\ &\leq \xi_n \|x_n - x_{n-1}\| (\|x_n - p\|^2 + \|x_n - a_n\|^2) + \frac{\xi_n}{2} \|x_n - x_{n-1}\| \\ &\leq \tau_n \|x_n - p\|^2 + \tau_n \|x_n - a_n\|^2 + \frac{\tau_n}{2}. \end{aligned} \quad (3.1.19)$$

From (3.1.18) and (3.1.19),

$$\|a_n - p\|^2 \leq (1 + 2\tau_n)\|x_n - p\|^2 - (1 - 2\tau_n)\|x_n - a_n\|^2 + \tau_n. \quad (3.1.20)$$

Thus (3.1.20) with (3.1.17) gives,

$$\begin{aligned} \|t_n - p\|^2 &\leq (1 + 2\tau_n)\|x_n - p\|^2 - (1 - 2\tau_n)\|x_n - a_n\|^2 + \tau_n - \gamma_n\|Ta_n - Tp\|^2 \\ &\quad - \gamma_n\|Ta_n - Sb_n\|^2 + \gamma_n\|Sb_n - Tp\|^2 + \gamma_n^2\|T^*(Ta_n - Sb_n)\|^2. \end{aligned} \quad (3.1.21)$$

Similarly,

$$\begin{aligned} \|w_n - q\|^2 &\leq (1 + 2\tau_n)\|u_n - q\|^2 - (1 - 2\tau_n)\|u_n - b_n\|^2 + \tau_n - \gamma_n\|Sb_n - Sq\|^2 \\ &\quad - \gamma_n\|Ta_n - Sb_n\|^2 + \gamma_n\|Ta_n - Sq\|^2 + \gamma_n^2\|S^*(Ta_n - Sb_n)\|^2. \end{aligned} \quad (3.1.22)$$

By adding (3.1.21) and (3.1.22), taking into account that $Tp = Sq$, we obtain

$$\begin{aligned} &\|t_n - p\|^2 + \|w_n - q\|^2 \\ &\leq (1 + 2\tau_n)(\|x_n - p\|^2 + \|u_n - q\|^2) - (1 - 2\tau_n)(\|x_n - a_n\|^2 + \|u_n - b_n\|^2) \\ &\quad + 2\tau_n - \gamma_n(2\|Ta_n - Sb_n\|^2 - \gamma_n[\|S^*(Ta_n - Sb_n)\|^2 + \|T^*(Ta_n - Sb_n)\|^2]) \\ &= (1 + 2\tau_n)(\|x_n - p\|^2 + \|u_n - q\|^2) - (1 - 2\tau_n)(\|x_n - a_n\|^2 + \|u_n - b_n\|^2) \\ &\quad + 2\tau_n - \gamma_n\|Ta_n - Sb_n\|^2 - \gamma_n(\|Ta_n - Sb_n\|^2 - \gamma_n[\|S^*(Ta_n - Sb_n)\|^2 \\ &\quad + \|T^*(Ta_n - Sb_n)\|^2]) \\ &\leq (1 + 2\tau_n)(\|x_n - p\|^2 + \|u_n - q\|^2) - (1 - 2\tau_n)(\|x_n - a_n\|^2 + \|u_n - b_n\|^2) \\ &\quad + 2\tau_n - \gamma_0\|Ta_n - Sb_n\|^2 \end{aligned} \quad (3.1.23)$$

$$\leq (1 + 2\tau_n)(\|x_n - p\|^2 + \|u_n - q\|^2) + 2\tau_n. \quad (3.1.24)$$

Since $\frac{\tau_n}{\alpha_n} \rightarrow 0$, for $\epsilon \in (0, \frac{1}{2})$, there exists $N \in \mathbb{N}$ such that $\tau_n < \alpha_n \epsilon$ for all $n \geq N$. This yields the following

$$\|t_n - p\|^2 + \|w_n - q\|^2 \leq (1 + 2\alpha_n \epsilon)(\|x_n - p\|^2 + \|u_n - q\|^2) + 2\alpha_n \epsilon, \text{ for all } n \geq N.$$

□

Theorem 3.1.7. *Assume that conditions C1–C6 hold. Then, the sequence $\{(x_n, u_n)\}$ generated by Algorithm 3.1.1 converges strongly to some element $(p, q) \in \Omega$, where $(p, q) = P_\Omega(x, u)$.*

Proof. We divided the proof into four (4) steps.

Claim 1. The following sequences are bounded.

- i. $\{(x_n, u_n)\}$
- ii. $\{(y_n, v_n)\}$

iii. $\{(z_n, h_n)\}$

i. Let $(p, q) \in \Omega$, where $(p, q) = P_\Omega(x, u)$, and $\Gamma_n(p, q) = \|x_n - p\|^2 + \|u_n - q\|^2$. Then, from Lemma 3.1.3, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n x + (1 - \alpha_n)z_n - p\|^2 \\ &= \|\alpha_n(x - p) + (1 - \alpha_n)(z_n - p)\|^2 \\ &\leq \alpha_n \|x - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 \\ &\leq \alpha_n \|x - p\|^2 + (1 - \alpha_n) \|t_n - p\|^2. \end{aligned} \quad (3.1.25)$$

Similarly,

$$\|u_{n+1} - q\|^2 \leq \alpha_n \|u - q\|^2 + (1 - \alpha_n) \|w_n - q\|^2. \quad (3.1.26)$$

Combining (3.1.25) with (3.1.26), and Lemma 3.1.6, we obtain, for all $n \geq N$

$$\begin{aligned} \Gamma_{n+1}(p, q) &= \|x_{n+1} - p\|^2 + \|u_{n+1} - q\|^2 \\ &\leq \alpha_n (\|x - p\|^2 + \|u - q\|^2) + (1 - \alpha_n) (\|t_n - p\|^2 + \|w_n - q\|^2) \\ &\leq \alpha_n (\|x - p\|^2 + \|u - q\|^2) + (1 - \alpha_n) [(1 + 2\alpha_n \epsilon) \Gamma_n(p, q) + 2\alpha_n \epsilon] \\ &\leq \alpha_n (\|x - p\|^2 + \|u - q\|^2) + (1 - \alpha_n) \Gamma_n(p, q) + 2\alpha_n \epsilon \Gamma_n(p, q) + 2\alpha_n \epsilon \\ &= \alpha_n (\|x - p\|^2 + \|u - q\|^2) + [(1 - \alpha_n) + 2\alpha_n \epsilon] \Gamma_n(p, q) + 2\alpha_n \epsilon \\ &= \alpha_n (\|x - p\|^2 + \|u - q\|^2) + [1 - \alpha_n(1 - 2\epsilon)] \Gamma_n(p, q) + \alpha_n(1 - 2\epsilon) \frac{2\epsilon}{1 - 2\epsilon} \\ &\leq \max \left\{ \Gamma_n(p, q), \frac{1}{1 - 2\epsilon} (\|x - p\|^2 + \|u - q\|^2 + 2\epsilon) \right\}. \end{aligned}$$

By induction, for all $n \geq N$, we have

$$\Gamma_n(p, q) \leq \max \left\{ \Gamma_N(p, q), \frac{1}{1 - 2\epsilon} (\|x - p\|^2 + \|u - q\|^2 + 2\epsilon) \right\},$$

and hence

$$\Gamma_n(p, q) \leq \max \left\{ M, \Gamma_N(p, q), \frac{1}{1 - 2\epsilon} (\|x - p\|^2 + \|u - q\|^2 + 2\epsilon) \right\}, \quad \forall n \geq 0,$$

where $M = \max\{\Gamma_0(p, q), \Gamma_1(p, q), \Gamma_2(p, q), \dots, \Gamma_{N-1}(p, q)\}$. This implies that the sequence $\{(x_n, u_n)\}$ is bounded.

ii. From the definition of $\{(y_n, v_n)\}$ and the nonexpansive of the resolvents $(I + \lambda_n B_1)^{-1}$ and $(I + \theta_n B_2)^{-1}$, we get

$$\begin{aligned} &\|y_n - p\|^2 + \|v_n - q\|^2 \\ &= \|(I + \lambda_n B_1)^{-1}(I - \lambda_n A_1)t_n - p\|^2 + \|(I + \theta_n B_2)^{-1}(I - \theta_n A_2)w_n - q\|^2 \\ &= \|(I + \lambda_n B_1)^{-1}(I - \lambda_n A_1)t_n - (I + \lambda_n B_1)^{-1}(I - \lambda_n A_1)p\|^2 \\ &\quad + \|(I + \theta_n B_2)^{-1}(I - \theta_n A_2)w_n - (I + \theta_n B_2)^{-1}(I - \theta_n A_2)q\|^2 \\ &\leq \|t_n - \lambda_n A_1 t_n - p + \lambda_n A_1 p\|^2 + \|w_n - \theta_n A_2 w_n - q + \theta_n A_2 q\|^2 \end{aligned}$$

$$\begin{aligned}
 &\leq \left(\|t_n - p\| + \lambda_n \|A_1 t_n - A_1 p\| \right)^2 + \left(\|w_n - q\| + \theta_n \|A_2 w_n - A_2 q\| \right)^2 \\
 &\leq \left(\|t_n - p\| + \lambda_n L \|t_n - p\| \right)^2 + \left(\|w_n - q\| + \theta_n L \|w_n - q\| \right)^2 \\
 &= (1 + \lambda_n L)^2 \|t_n - p\|^2 + (1 + \theta_n L)^2 \|w_n - q\|^2.
 \end{aligned}$$

We note that from Lemma 3.1.6 and the fact that $\{(x_n, u_n)\}$ is bounded, there exists $M > 0$ such that

$$\|t_n - p\|^2 + \|w_n - q\|^2 \leq M, \quad \text{for all } n \geq 0. \quad (3.1.27)$$

This implies that $\|t_n - p\|^2$ and $\|w_n - q\|^2$ are bounded. Moreover, since λ_n and θ_n are bounded, then there exists $M' > 0$ such that

$$\|y_n - p\|^2 + \|v_n - q\|^2 \leq M', \quad \text{for all } n \geq 0.$$

Hence $\{(y_n, v_n)\}$ is bounded.

iii. From Lemma 3.1.3 and (3.1.27),

$$\begin{aligned}
 \|z_n - p\|^2 + \|h_n - q\|^2 &\leq \|t_n - p\|^2 + \|w_n - q\|^2 - \left(\frac{2-\gamma}{\gamma} \right) \|t_n - z_n\|^2 \\
 &\quad - \left(\frac{2-\gamma}{\gamma} \right) \|w_n - h_n\|^2 \\
 &\leq \|t_n - p\|^2 + \|w_n - q\|^2 \\
 &\leq M, \quad \text{for all } n \geq 0.
 \end{aligned}$$

Hence $\{(z_n, h_n)\}$ is bounded.

Claim 2. The following inequalities hold:

$$\begin{aligned}
 \text{a)} \quad &(1 - \alpha_n) \left(\frac{2-\gamma}{\gamma} \right) \|t_n - z_n\|^2 \leq \|t_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \|x - p\|^2, \quad \forall n \geq 0, \\
 \text{b)} \quad &(1 - \alpha_n) \left(\frac{2-\gamma}{\gamma} \right) \|w_n - h_n\|^2 \leq \|w_n - q\|^2 - \|u_{n+1} - q\|^2 + \alpha_n \|u - q\|^2, \quad \forall n \geq 0.
 \end{aligned}$$

a) From the definition of x_n and using Lemma 3.1.3, we have

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|\alpha_n(x - p) + (1 - \alpha_n)(z_n - p)\|^2 \\
 &\leq \alpha_n \|x - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 \\
 &\leq \alpha_n \|x - p\|^2 + (1 - \alpha_n) \left[\|t_n - p\|^2 - \left(\frac{2-\gamma}{\gamma} \right) \|t_n - z_n\|^2 \right] \\
 &\leq \alpha_n \|x - p\|^2 + \|t_n - p\|^2 - (1 - \alpha_n) \left(\frac{2-\gamma}{\gamma} \right) \|t_n - z_n\|^2.
 \end{aligned}$$

Rearranging

$$(1 - \alpha_n) \left(\frac{2 - \gamma}{\gamma} \right) \|t_n - z_n\|^2 \leq \|t_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \|x - p\|^2, \quad \forall n \geq 0. \quad (3.1.28)$$

b) Similarly, following the same process as **a)**, we obtain that

$$(1 - \alpha_n) \left(\frac{2 - \gamma}{\gamma} \right) \|w_n - h_n\|^2 \leq \|w_n - q\|^2 - \|u_{n+1} - q\|^2 + \alpha_n \|u - q\|^2, \quad \forall n \geq 0. \quad (3.1.29)$$

Claim 3. The following inequality hold for all $n \geq N$.

$$\begin{aligned} \Gamma_{n+1}(p, q) &\leq [1 - \alpha_n(1 - 2\epsilon)]\Gamma_n(p, q) + 2\alpha_n(\langle x - p, x_{n+1} - p \rangle + \langle u - q, u_{n+1} - q \rangle) \\ &\quad + 2\tau_n. \end{aligned} \quad (3.1.30)$$

Using (2.1.1) and Lemma 3.1.3, we have the following

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(x - p) + (1 - \alpha_n)(z_n - p)\|^2 \\ &\leq (1 - \alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle x - p, x_{n+1} - p \rangle \\ &\leq (1 - \alpha_n)\|t_n - p\|^2 - (1 - \alpha_n) \left(\frac{2 - \gamma}{\gamma} \right) \|t_n - z_n\|^2 \\ &\quad + 2\alpha_n\langle x - p, x_{n+1} - p \rangle. \end{aligned} \quad (3.1.31)$$

Similarly,

$$\begin{aligned} \|u_{n+1} - q\|^2 &\leq (1 - \alpha_n)\|w_n - q\|^2 - (1 - \alpha_n) \left(\frac{2 - \gamma}{\gamma} \right) \|w_n - h_n\|^2 \\ &\quad + 2\alpha_n\langle u - q, u_{n+1} - q \rangle. \end{aligned} \quad (3.1.32)$$

From (3.1.23), (3.1.31) and (3.1.32), we have, for all $n \geq N$

$$\begin{aligned} \Gamma_{n+1}(p, q) &= \|x_{n+1} - p\|^2 + \|u_{n+1} - q\|^2 \\ &\leq (1 - \alpha_n)[\|t_n - p\|^2 + \|w_n - q\|^2] - (1 - \alpha_n) \left(\frac{2 - \gamma}{\gamma} \right) [\|t_n - z_n\|^2 + \|w_n - h_n\|^2] \\ &\quad + 2\alpha_n(\langle x - p, x_{n+1} - p \rangle + \langle u - q, u_{n+1} - q \rangle) \\ &\leq (1 - \alpha_n) \left[(1 + 2\tau_n)(\|x_n - p\|^2 + \|u_n - q\|^2) - (1 - 2\tau_n)(\|x_n - a_n\|^2 + \|u_n - b_n\|^2) \right. \\ &\quad \left. + 2\tau_n - \gamma_0 \|Ta_n - Sb_n\|^2 \right] - (1 - \alpha_n) \left(\frac{2 - \gamma}{\gamma} \right) [\|t_n - z_n\|^2 + \|w_n - h_n\|^2] \\ &\quad + 2\alpha_n(\langle x - p, x_{n+1} - p \rangle + \langle u - q, u_{n+1} - q \rangle) \\ &= (1 - \alpha_n) \left[(1 + 2\tau_n)\Gamma_n(p, q) - (1 - 2\tau_n)(\|x_n - a_n\|^2 + \|u_n - b_n\|^2) \right. \\ &\quad \left. + 2\tau_n - \gamma_0 \|Ta_n - Sb_n\|^2 \right] - (1 - \alpha_n) \left(\frac{2 - \gamma}{\gamma} \right) [\|t_n - z_n\|^2 + \|w_n - h_n\|^2] \\ &\quad + 2\alpha_n(\langle x - p, x_{n+1} - p \rangle + \langle u - q, u_{n+1} - q \rangle) \\ &\leq (1 - \alpha_n)(1 + 2\tau_n)\Gamma_n(p, q) + 2\alpha_n(\langle x - p, x_{n+1} - p \rangle + \langle u - q, u_{n+1} - q \rangle) + 2\tau_n \\ &\leq [1 - \alpha_n(1 - 2\epsilon)]\Gamma_n(p, q) + 2\alpha_n(\langle x - p, x_{n+1} - p \rangle + \langle u - q, u_{n+1} - q \rangle) + 2\tau_n. \end{aligned} \quad (3.1.33)$$

The above inequality is the same as (3.1.30).

Claim 4. The $\{\Gamma_n(p, q)\}$ sequence converges to zero as $n \rightarrow \infty$. We demonstrate this by examining two possible cases.

Case 1 Let $N_1 \in \mathbb{N}$ such that $\Gamma_{n+1}(p, q) \leq \Gamma_n(p, q)$ for all $n \geq N_1$. Then, $\lim_{n \rightarrow \infty} \Gamma_n(p, q)$ exists. Since $\lim_{n \rightarrow \infty} \alpha_n = 0$ and using (3.1.33), we get

$$\lim_{n \rightarrow \infty} \|Ta_n - Sb_n\| = 0, \quad (3.1.34)$$

which implies that,

$$\lim_{n \rightarrow \infty} \|T^*(Ta_n - Sb_n)\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|S^*(Ta_n - Sb_n)\| = 0. \quad (3.1.35)$$

Inequality (3.1.33) also implies that

$$\lim_{n \rightarrow \infty} \|x_n - a_n\| = 0, \quad \lim_{n \rightarrow \infty} \|u_n - b_n\| = 0, \quad (3.1.36)$$

$$\lim_{n \rightarrow \infty} \|t_n - z_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|w_n - h_n\| = 0. \quad (3.1.37)$$

Using Lemma 3.1.4 and (3.1.37), we obtain

$$\lim_{n \rightarrow \infty} \|t_n - y_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|w_n - v_n\| = 0.$$

Moreover, from **Step 3.** of Algorithm 3.1.1 and (3.1.35) we have

$$\lim_{n \rightarrow \infty} \|t_n - a_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|w_n - b_n\| = 0,$$

which with (3.1.36) imply that

$$\lim_{n \rightarrow \infty} \|t_n - x_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|w_n - u_n\| = 0. \quad (3.1.38)$$

Combining (3.1.37) and (3.1.38), we obtain

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|h_n - u_n\| = 0. \quad (3.1.39)$$

By definition of x_{n+1} and using (3.1.39), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n x + (1 - \alpha_n)z_n - x_n\| \\ &\leq \alpha_n \|x - x_n\| + (1 - \alpha_n) \|z_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.1.40)$$

Similarly, one can show that

$$\|u_{n+1} - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.1.41)$$

Since the sequence $\{(x_n, u_n)\}$ is bounded, there exists a subsequence $\{(x_{n_k}, u_{n_k})\}$ of $\{(x_n, u_n)\}$ that converges weakly to some $(z, h) \in H_1 \times H_2$, such that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} (\langle x - p, x_n - p \rangle + \langle u - q, u_n - q \rangle) \\ &= \lim_{k \rightarrow \infty} (\langle x - p, x_{n_k} - p \rangle + \langle u - q, u_{n_k} - q \rangle) \\ &= \langle x - p, z - p \rangle + \langle u - q, h - q \rangle \\ &= \langle (x, u) - (p, q), (z, h) - (p, q) \rangle. \end{aligned} \quad (3.1.42)$$

From Lemma 3.1.5, we have $z \in \Omega_1 := \{z \in H_1 | 0 \in (A_1 + B_1)z\}$ and $h \in \Omega_2 := \{h \in H_2 | 0 \in (A_2 + B_2)h\}$. Next, we show that $Tz = Sh$. Note that using (3.1.36), we have $a_{n_k} \rightharpoonup z$ and $b_{n_k} \rightharpoonup h$. Thus, using (2.1.1), we have

$$\begin{aligned} \|Tz - Sh\|^2 &= \|(Tz - Ta_{n_k} + Sb_{n_k} - Sh) + (Ta_{n_k} - Sb_{n_k})\|^2 \\ &\leq \|Ta_{n_k} - Sb_{n_k}\|^2 + 2\langle Tz - Sh, Tz - Ta_{n_k} + Sb_{n_k} - Sh \rangle, \end{aligned}$$

and using the fact that $Ta_{n_k} \rightharpoonup Tz$ and $Sb_{n_k} \rightharpoonup Sh$ as $k \rightarrow \infty$ and taking lim sup on both sides and using (3.1.34), we get

$$\|Tz - Sh\|^2 \leq \limsup_{k \rightarrow \infty} \|Ta_{n_k} - Sb_{n_k}\|^2 + 2 \limsup_{k \rightarrow \infty} \langle Tz - Sh, Tz - Ta_{n_k} + Sb_{n_k} - Sh \rangle = 0.$$

Hence, $Tz = Sh$ and so $(z, h) \in \Omega$. Using Lemma 2.1.9 and the fact that $(p, q) = P_\Omega(x, u)$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} (\langle x - p, x_n - p \rangle + \langle u - q, u_n - q \rangle) &= \langle x - p, z - p \rangle + \langle u - q, h - q \rangle \\ &= \langle (x, u) - (p, q), (z, h) - (p, q) \rangle \\ &\leq 0. \end{aligned} \quad (3.1.43)$$

Moreover, from (3.1.40), (3.1.41) and (3.1.43), we get

$$\begin{aligned} & \limsup_{n \rightarrow \infty} (\langle x - p, x_{n+1} - p \rangle + \langle u - q, u_{n+1} - q \rangle) \\ &= \limsup_{n \rightarrow \infty} \langle x - p, x_{n+1} - x_n \rangle + \limsup_{n \rightarrow \infty} \langle x - p, x_n - p \rangle \\ &\quad + \limsup_{n \rightarrow \infty} \langle u - q, u_{n+1} - u_n \rangle + \limsup_{n \rightarrow \infty} \langle u - q, u_n - q \rangle \\ &= \limsup_{n \rightarrow \infty} \langle x - p, x_n - p \rangle + \limsup_{n \rightarrow \infty} \langle u - q, u_n - q \rangle \\ &= \langle x - p, z - p \rangle + \langle u - q, h - q \rangle \\ &= \langle (x, u) - (p, q), (z, h) - (p, q) \rangle \leq 0. \end{aligned} \quad (3.1.44)$$

Therefore, using (3.1.44) and (3.1.30) with reference to Lemma 2.1.4, we have

$$\lim_{n \rightarrow \infty} \Gamma_n(p, q) = 0.$$

That is, $\lim_{n \rightarrow \infty} (\|x_n - p\|^2 + \|u_n - q\|^2) = 0$ as $n \rightarrow \infty$. Hence $x_n \rightarrow p$ and $u_n \rightarrow q$ as $n \rightarrow \infty$.

Case 2 Let $\{\Gamma_{n_j}(p, q)\}$ be a subsequence of $\{\Gamma_n(p, q)\}$ such that $\Gamma_{n_j}(p, q) < \Gamma_{n_j+1}(p, q)$ for all $j \in \mathbb{N}$. Then, from Lemma 2.1.5, there exists a non-decreasing sequence m_k of $\mathbb{N}$, such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following inequalities holds for all $k \in \mathbb{N}$.

$$\Gamma_{m_k}(p, q) \leq \Gamma_{m_k+1}(p, q) \text{ and } \Gamma_k(p, q) \leq \Gamma_{m_k+1}(p, q). \quad (3.1.45)$$

Using (3.1.33) and $\lim_{k \rightarrow \infty} \alpha_{m_k} = 0$, we get the limit

$$\lim_{k \rightarrow \infty} \|Ta_{m_k} - Sb_{m_k}\| = 0,$$

which implies that

$$\lim_{k \rightarrow \infty} \|S^*(Ta_{m_k} - Sb_{m_k})\| = 0 \text{ and } \lim_{k \rightarrow \infty} \|T^*(Ta_{m_k} - Sb_{m_k})\| = 0. \quad (3.1.46)$$

Inequality (3.1.33) also implies that

$$\lim_{k \rightarrow \infty} \|x_{m_k} - a_{m_k}\| = 0, \quad \lim_{k \rightarrow \infty} \|u_{m_k} - b_{m_k}\| = 0, \quad (3.1.47)$$

$$\lim_{k \rightarrow \infty} \|t_{m_k} - z_{m_k}\| = 0 \text{ and } \lim_{k \rightarrow \infty} \|w_{m_k} - h_{m_k}\| = 0. \quad (3.1.48)$$

Using Lemma 3.1.4 and (3.1.48), we obtain

$$\lim_{k \rightarrow \infty} \|t_{m_k} - y_{m_k}\| = 0 \text{ and } \lim_{k \rightarrow \infty} \|w_{m_k} - v_{m_k}\| = 0.$$

Moreover, from **Step 3.** of Algorithm 3.1.1 and (3.1.46) we have

$$\lim_{k \rightarrow \infty} \|t_{m_k} - a_{m_k}\| = 0 \text{ and } \lim_{k \rightarrow \infty} \|w_{m_k} - b_{m_k}\| = 0,$$

which with (3.1.47) imply that

$$\lim_{k \rightarrow \infty} \|t_{m_k} - x_{m_k}\| = 0 \text{ and } \lim_{k \rightarrow \infty} \|w_{m_k} - u_{m_k}\| = 0. \quad (3.1.49)$$

Combining (3.1.48) and (3.1.49), we obtain

$$\lim_{k \rightarrow \infty} \|z_{m_k} - x_{m_k}\| = 0 \text{ and } \lim_{k \rightarrow \infty} \|h_{m_k} - u_{m_k}\| = 0. \quad (3.1.50)$$

Using (3.1.50) and the definition of x_{n+1} , we have

$$\begin{aligned} \|x_{m_k+1} - x_{m_k}\| &= \|\alpha_{m_k}x + (1 - \alpha_{m_k})z_{m_k} - x_{m_k}\| \\ &\leq \alpha_{m_k}\|x - x_{m_k}\| + (1 - \alpha_{m_k})\|z_{m_k} - x_{m_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned} \quad (3.1.51)$$

Similarly, we can show that

$$\|u_{m_k+1} - u_{m_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (3.1.52)$$

Using the same methods as in the proof of **Case 1**, we obtain that

$$\limsup_{k \rightarrow \infty} (\langle x - p, x_{m_k+1} - p \rangle + \langle u - q, u_{m_k+1} - q \rangle) \leq 0. \quad (3.1.53)$$

Now, from (3.1.30) and (3.1.45), we have

$$\begin{aligned} \Gamma_{m_k+1}(p, q) &\leq [1 - \alpha_{m_k}(1 - 2\epsilon)]\Gamma_{m_k+1}(p, q) + 2\tau_{m_k} \\ &\quad + 2\alpha_{m_k}(\langle x - p, x_{m_k+1} - p \rangle + \langle u - q, u_{m_k+1} - q \rangle), \end{aligned} \quad (3.1.54)$$

which implies that

$$\alpha_{m_k}(1 - 2\epsilon)\Gamma_{m_k+1}(p, q) \leq 2\alpha_{m_k}(\langle x - p, x_{m_k+1} - p \rangle + \langle u - q, u_{m_k+1} - q \rangle) + 2\tau_{m_k}.$$

This together with (3.1.45) yields that

$$\begin{aligned} \Gamma_k(p, q) \leq \Gamma_{m_k+1}(p, q) &\leq \frac{2}{1 - 2\epsilon}(\langle x - p, x_{m_k+1} - p \rangle + \langle u - q, u_{m_k+1} - q \rangle) \\ &\quad + \frac{2\tau_{m_k}}{\alpha_{m_k}(1 - 2\epsilon)}. \end{aligned} \quad (3.1.55)$$

Hence, from (3.1.55), (3.1.53) and the property of $\{\tau_n\}$, we obtain that $\limsup_{k \rightarrow \infty} \Gamma_k(p, q) \leq 0$, and hence

$$\lim_{k \rightarrow \infty} (\|x_k - p\|^2 + \|u_k - q\|^2) = 0, \text{ which implies that } x_k \rightarrow p \text{ and } u_k \rightarrow q \text{ as } k \rightarrow \infty.$$

□

If, in Theorem 3.1.7, we assume that $x = 0 \in H_1$ and $u = 0 \in H_2$ in Algorithm 3.1.1, then Algorithm 3.1.1 reduces to the following algorithm which can be used to locate an element of the solution of (3.0.1) with the minimum norm.

Algorithm 3.1.8.

Initialization: Let $x_0, x_1 \in H_1$, $u_0, u_1 \in H_2$, $\delta, e, \tau > 0$, $l, \mu \in (0, 1)$, $\gamma \in (0, 2)$.

Choose $(x, u) \in H_1 \times H_2$.

Iterative Steps: Given $x_{n-1}, x_n \in H_1$, $u_{n-1}, u_n \in H_2$, calculate x_{n+1} and u_{n+1} as follows;

Step 1. Compute

$$a_n = x_n + \xi_n(x_n - x_{n-1}) \text{ and } b_n = u_n + \xi_n(u_n - u_{n-1}),$$

where $0 \leq \xi_n \leq \bar{\xi}_n$ and

$$\bar{\xi}_n = \begin{cases} \min \left\{ \frac{\tau_n}{\|x_n - x_{n-1}\| + \|u_n - u_{n-1}\|}, \tau \right\}, & \text{if } x_n \neq x_{n-1} \text{ and } u_n \neq u_{n-1}, \\ \tau, & \text{Otherwise.} \end{cases}$$

Step 2. Compute $\{\gamma_n\}$ satisfying the following:

$$0 < \gamma_0 \leq \gamma_n \leq \frac{\|Ta_n - Sb_n\|^2}{\|T^*(Ta_n - Sb_n)\|^2 + \|S^*(Ta_n - Sb_n)\|^2},$$

for $n \in \Omega_n = \{n \in \mathbb{N} : Ta_n - Sb_n \neq 0\}$.

Otherwise, let $\gamma_n = \gamma > 0$.

Step 3. Compute

$$t_n = a_n - \gamma_n T^*(Ta_n - Sb_n) \text{ and } w_n = b_n + \gamma_n S^*(Ta_n - Sb_n).$$

Step 4. Compute

$$y_n = (I + \lambda_n B_1)^{-1}(I - \lambda_n A_1)t_n \text{ and } v_n = (I + \theta_n B_2)^{-1}(I - \theta_n A_2)w_n,$$

where λ_n is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, \delta l^3, \dots, \delta l^j, \dots\}$, $j \in \mathbb{N}$ satisfying,

$$\lambda \|A_1 t_n - A_1 y_n\| \leq \mu \|y_n - t_n\|,$$

and θ_n is chosen to be the largest $\theta \in \{e, el, el^2, el^3, \dots, el^k, \dots\}$, $k \in \mathbb{N}$ satisfying,

$$\theta \|A_2 w_n - A_2 v_n\| \leq \mu \|v_n - w_n\|.$$

Step 5. Compute

$$m_n = t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \text{ and } e_n = w_n - v_n - \theta_n(A_2 w_n - A_2 v_n).$$

Step 6. Compute

$$z_n = t_n - \gamma \eta_n m_n \text{ and } h_n = w_n - \gamma \beta_n e_n,$$

where

$$\eta_n = \begin{cases} (1 - \mu) \frac{\|t_n - y_n\|^2}{\|m_n\|^2}, & m_n \neq 0, \\ 0, & m_n = 0, \end{cases}$$

$$\beta_n = \begin{cases} (1 - \mu) \frac{\|w_n - v_n\|^2}{\|e_n\|^2}, & e_n \neq 0, \\ 0, & e_n = 0. \end{cases}$$

Step 7. Compute

$$x_{n+1} = (1 - \alpha_n)z_n \text{ and } u_{n+1} = (1 - \alpha_n)h_n.$$

Step 8. Set $n := n + 1$ and go to **Step 1**.

We now state the approximation of the solution of problem (3.0.1) with minimum norm, which follows from Theorem 3.1.7.

Corollary 3.1.9. *Suppose that conditions C1 – C6 hold. Then, the sequence $\{(x_n, u_n)\}$ generated by Algorithm 3.1.8 converges strongly to the minimum norm point $(x^*, u^*) \in \Omega$, that is, $(x^*, u^*) = P_\Omega(0, 0)$.*

4. Applications

In this chapter, we provide a few applications of Theorem 3.1.7 to demonstrate that our method is applicable.

4.1 Application to Split Equality Null-Point Problem

Let $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ be maximal monotone mappings, $T : H_1 \rightarrow H_3$ and $S : H_2 \rightarrow H_3$ be bounded linear mappings with adjoints T^* and S^* , respectively. The *Split Equality Null-Point Problem* (SENPP) is to find $x^* \in H_1, u^* \in H_2$ such that

$$x^* \in B_1^{-1}(0), u^* \in B_2^{-1}(0) \text{ and } Tx^* = Su^*. \quad (4.1.1)$$

If, in Algorithm 3.1.1, we assume $A_1 = 0$ and $A_2 = 0$, then the algorithm reduces to the following.

Algorithm 4.1.1.

Initialization: Let $x_0, x_1 \in H_1$, $u_0, u_1 \in H_2$, $\delta, e, \tau > 0$, $l, \mu \in (0, 1)$, $\gamma \in (0, 2)$.

Choose $(x, u) \in H_1 \times H_2$.

Iterative Steps: Given $x_{n-1}, x_n \in H_1$, $u_{n-1}, u_n \in H_2$, calculate x_{n+1} and u_{n+1} as follows;

Step 1. Compute

$$a_n = x_n + \xi_n(x_n - x_{n-1}) \text{ and } b_n = u_n + \xi_n(u_n - u_{n-1}),$$

where $0 \leq \xi_n \leq \bar{\xi}_n$ and

$$\bar{\xi}_n = \begin{cases} \min \left\{ \frac{\tau_n}{\|x_n - x_{n-1}\| + \|u_n - u_{n-1}\|}, \tau \right\}, & \text{if } x_n \neq x_{n-1} \text{ and } u_n \neq u_{n-1}, \\ \tau, & \text{Otherwise.} \end{cases}$$

Step 2. Compute $\{\gamma_n\}$ satisfying the following:

$$0 < \gamma_0 \leq \gamma_n \leq \frac{\|Ta_n - Sb_n\|^2}{\|T^*(Ta_n - Sb_n)\|^2 + \|S^*(Ta_n - Sb_n)\|^2},$$

for $n \in \Omega_n = \{n \in \mathbb{N} : Ta_n - Sb_n \neq 0\}$.

Otherwise, let $\gamma_n = \gamma > 0$.

Step 3. Compute

$$t_n = a_n - \gamma_n T^*(Ta_n - Sb_n) \text{ and } w_n = b_n + \gamma_n S^*(Ta_n - Sb_n).$$

Step 4. Compute

$$y_n = (I + \lambda_n B_1)^{-1} t_n \text{ and } v_n = (I + \theta_n B_2)^{-1} w_n,$$

where $\lambda_n = \delta$ and $\theta_n = e$.

Step 5. Compute

$$m_n = t_n - y_n \text{ and } e_n = w_n - v_n.$$

Step 6. Compute

$$z_n = t_n - \gamma \eta_n m_n \text{ and } h_n = w_n - \gamma \beta_n e_n,$$

where

$$\eta_n = \begin{cases} (1 - \mu) \frac{\|t_n - y_n\|^2}{\|m_n\|^2}, & m_n \neq 0, \\ 0, & m_n = 0, \end{cases}$$

$$\beta_n = \begin{cases} (1 - \mu) \frac{\|w_n - v_n\|^2}{\|e_n\|^2}, & e_n \neq 0, \\ 0, & e_n = 0. \end{cases}$$

Step 7. Compute

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) z_n \text{ and } u_{n+1} = \alpha_n u + (1 - \alpha_n) h_n.$$

Step 8. Set $n := n + 1$ and go to **Step 1**.

We apply Theorem 3.1.7 and we get the following corollary for the approximation of the SENP solution.

Corollary 4.1.2. *Assume that C1 – C5 with $A_1 = A_2 = 0$ hold. Let $\Omega_1 := \{x^* \in B_1^{-1}(0), u^* \in B_2^{-1}(0) \text{ and } Tx^* = Su^*\}$ be nonempty. Then, the sequence $\{(x_n, u_n)\}$ generated by Algorithm 4.1.1 strongly converges to an element $(x^*, u^*) = P_{\Omega_1}(x, u)$.*

4.2 Application to Split Inclusion Problem

Let H_1 and H_2 be real Hilbert spaces. Let $A_1 : H_1 \rightarrow H_1$, $A_2 : H_2 \rightarrow H_2$ be L_1 -Lipschitz continuous monotone mapping and L_2 -Lipschitz continuous monotone mapping, respectively

and $B_1 : H_1 \rightarrow 2^{H_1}$, $B_2 : H_2 \rightarrow 2^{H_2}$ be multi-valued maximal monotone mappings. Let $T : H_1 \rightarrow H_2$ be a bounded linear mapping with T^* as its adjoint. The *Split Inclusion Problem* (SIP) (see, e.g., Censor et al.[19]) is to find $x^* \in H_1, u^* \in H_2$ such that;

$$x^* \in (A_1 + B_1)^{-1}(0), u^* \in (A_2 + B_2)^{-1}(0) \text{ and } u^* = Tx^*. \quad (4.2.1)$$

The SIP was introduced by Censor et al. [19] in 2012. This problem is known to have wide applications in phase retrieval, image and signal processing, data compression, among others (see, [54, 94, 97, 109] and the references therein).

If, in Algorithm 3.1.1, we assume $H_2 = H_3$ and $S = I$, the identity mapping on H_2 , then Algorithm 3.1.1 reduces to the following.

Algorithm 4.2.1.

Initialization: Let $x_0, x_1 \in H_1$, $u_0, u_1 \in H_2$, $\delta, e, \tau > 0$, $l, \mu \in (0, 1)$, $\gamma \in (0, 2)$.

Choose $(x, u) \in H_1 \times H_2$.

Iterative Steps: Given $x_{n-1}, x_n \in H_1$, $u_{n-1}, u_n \in H_2$, calculate x_{n+1} and u_{n+1} as follows;

Step 1. Compute

$$a_n = x_n + \xi_n(x_n - x_{n-1}) \text{ and } b_n = u_n + \xi_n(u_n - u_{n-1}),$$

where $0 \leq \xi_n \leq \bar{\xi}_n$ and

$$\bar{\xi}_n = \begin{cases} \min \left\{ \frac{\tau_n}{\|x_n - x_{n-1}\| + \|u_n - u_{n-1}\|}, \tau \right\}, & \text{if } x_n \neq x_{n-1} \text{ and } u_n \neq u_{n-1}, \\ \tau, & \text{Otherwise.} \end{cases}$$

Step 2. Compute $\{\gamma_n\}$ satisfying the following:

$$0 < \gamma_0 \leq \gamma_n \leq \frac{\|Ta_n - Sb_n\|^2}{\|T^*(Ta_n - Sb_n)\|^2 + \|S^*(Ta_n - Sb_n)\|^2},$$

for $n \in \Omega_n = \{n \in \mathbb{N} : Ta_n - Sb_n \neq 0\}$.

Otherwise, let $\gamma_n = \gamma > 0$.

Step 3. Compute

$$t_n = a_n - \gamma_n T^*(Ta_n - b_n) \text{ and } w_n = b_n + \gamma_n(Ta_n - b_n).$$

Step 4. Compute

$$y_n = (I + \lambda_n B_1)^{-1}(I - \lambda_n A_1)t_n \text{ and } v_n = (I + \theta_n B_2)^{-1}(I - \theta_n A_2)w_n,$$

where λ_n is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, \delta l^3, \dots, \delta l^j, \dots\}$, $j \in \mathbb{N}$ satisfying,

$$\lambda \|A_1 t_n - A_1 y_n\| \leq \mu \|y_n - t_n\|,$$

and θ_n is chosen to be the largest $\theta \in \{e, el, el^2, el^3, \dots, el^k, \dots\}$, $k \in \mathbb{N}$ satisfying,

$$\theta \|A_2 w_n - A_2 v_n\| \leq \mu \|v_n - w_n\|.$$

Step 5. Compute

$$m_n = t_n - y_n - \lambda_n(A_1 t_n - A_1 y_n) \text{ and } e_n = w_n - v_n - \theta_n(A_2 w_n - A_2 v_n).$$

Step 6. Compute

$$z_n = t_n - \gamma \eta_n m_n \text{ and } h_n = w_n - \gamma \beta_n e_n,$$

where

$$\eta_n = \begin{cases} (1 - \mu) \frac{\|t_n - y_n\|^2}{\|m_n\|^2}, & m_n \neq 0, \\ 0, & m_n = 0, \end{cases}$$

$$\beta_n = \begin{cases} (1 - \mu) \frac{\|w_n - v_n\|^2}{\|e_n\|^2}, & e_n \neq 0, \\ 0, & e_n = 0. \end{cases}$$

Step 7. Compute

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) z_n \text{ and } u_{n+1} = \alpha_n u + (1 - \alpha_n) h_n.$$

Step 8. Set $n := n + 1$ and go to **Step 1**.

Applying Theorem 3.1.7, we get the following corollary.

Corollary 4.2.2. *Assume that C1 – C5 hold with $H_2 = H_3$ and $S = I$, the identity mapping on H_2 . Suppose that $\Omega_2 = \{x^* \in (A_1 + B_1)^{-1}(0), u^* \in (A_2 + B_2)^{-1}(0) \text{ such that } u^* = T x^*\}$ is nonempty. Then, the sequence $\{(x_n, u_n)\}$ generated by Algorithm 4.2.1 strongly converges to a point $(x^*, u^*) = P_{\Omega_2}(x, u)$.*

4.3 Application to Convex Minimization Problem

Let H_i , for $i = 1, 2, 3$, be real Hilbert spaces. Let $f_i : H_i \rightarrow \mathbb{R}$ be proper, convex and smooth functions and $g_i : H_i \rightarrow \mathbb{R}$ be proper, convex and lower semicontinuous functions, $i = 1, 2$. We consider the following minimization problem: Find $(p, q) \in H_1 \times H_2$ such that

$$f_1(p) + g_1(p) = \min_{x \in H_1} \{f_1(x) + g_1(x)\}, \quad (4.3.1)$$

$$f_2(q) + g_2(q) = \min_{y \in H_2} \{f_2(y) + g_2(y)\} \quad (4.3.2)$$

and

$$Tp = Sq, \quad (4.3.3)$$

where $T : H_1 \rightarrow H_3$ and $S : H_2 \rightarrow H_3$ are bounded linear mappings.

Denote $\Gamma = \{(p, q) \in H_1 \times H_2 : p \text{ solves (4.3.1), } q \text{ solves (4.3.2) and } Tp = Sq\}$.

This problem is equivalent, by Fermat's rule, to the problem of finding $(p, q) \in H_1 \times H_2$ such that

$$(p, q) \in (\nabla f_1 + \partial g_1)^{-1}(0) \times (\nabla f_2 + \partial g_2)^{-1}(0) \text{ and } T(p) = S(q), \quad (4.3.4)$$

where ∇f_i are gradients of f_i and ∂g_i are subdifferentials of g_i , $i = 1, 2$. It is known that ∇f_i and ∂g_i are monotone and maximal monotone mappings, respectively.

If we set $A_1 = \nabla f_1$, $A_2 = \nabla f_2$, $B_1 = \partial g_1$ and $B_2 = \partial g_2$ in Algorithm 3.1.1, then it reduces to the following Algorithm.

Algorithm 4.3.1.

Initialization: Let $x_0, x_1 \in H_1$, $u_0, u_1 \in H_2$, $\delta, e, \tau > 0$, $l, \mu \in (0, 1)$, $\gamma \in (0, 2)$.

Choose $(x, u) \in H_1 \times H_2$.

Iterative Steps: Given $x_{n-1}, x_n \in H_1$, $u_{n-1}, u_n \in H_2$, calculate x_{n+1} and u_{n+1} as follows;

Step 1. Compute

$$a_n = x_n + \xi_n(x_n - x_{n-1}) \text{ and } b_n = u_n + \xi_n(u_n - u_{n-1}),$$

where $0 \leq \xi_n \leq \bar{\xi}_n$ and

$$\bar{\xi}_n = \begin{cases} \min \left\{ \frac{\tau_n}{\|x_n - x_{n-1}\| + \|u_n - u_{n-1}\|}, \tau \right\}, & \text{if } x_n \neq x_{n-1} \text{ and } u_n \neq u_{n-1}, \\ \tau, & \text{Otherwise.} \end{cases}$$

Step 2. Compute $\{\gamma_n\}$ satisfying the following:

$$0 < \gamma_0 \leq \gamma_n \leq \frac{\|Ta_n - Sb_n\|^2}{\|T^*(Ta_n - Sb_n)\|^2 + \|S^*(Ta_n - Sb_n)\|^2},$$

for $n \in \Omega_n = \{n \in \mathbb{N} : Ta_n - Sb_n \neq 0\}$.

Otherwise, let $\gamma_n = \gamma > 0$.

Step 3. Compute

$$t_n = a_n - \gamma_n T^*(Ta_n - Sb_n) \text{ and } w_n = b_n + \gamma_n S^*(Ta_n - Sb_n).$$

Step 4. Compute

$$y_n = (I + \lambda_n \partial g_1)^{-1}(I - \lambda_n \nabla f_1)t_n \text{ and } v_n = (I + \theta_n \partial g_2)^{-1}(I - \theta_n \nabla f_2)w_n,$$

where λ_n is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, \delta l^3, \dots, \delta l^j, \dots\}$, $j \in \mathbb{N}$ satisfying,

$$\lambda \|\nabla f_1 t_n - \nabla f_1 y_n\| \leq \mu \|y_n - t_n\|,$$

and θ_n is chosen to be the largest $\theta \in \{e, el, el^2, el^3, \dots, el^k, \dots\}$, $k \in \mathbb{N}$ satisfying,

$$\theta \|\nabla f_2 w_n - \nabla f_2 v_n\| \leq \mu \|v_n - w_n\|.$$

Step 5. Compute

$$m_n = t_n - y_n - \lambda_n (\nabla f_1 t_n - \nabla f_1 y_n) \text{ and } e_n = w_n - v_n - \theta_n (\nabla f_2 w_n - \nabla f_2 v_n).$$

Step 6. Compute

$$z_n = t_n - \gamma \eta_n m_n \text{ and } h_n = w_n - \gamma \beta_n e_n,$$

where

$$\eta_n = \begin{cases} (1 - \mu) \frac{\|t_n - y_n\|^2}{\|m_n\|^2}, & m_n \neq 0, \\ 0, & m_n = 0, \end{cases}$$

$$\beta_n = \begin{cases} (1 - \mu) \frac{\|w_n - v_n\|^2}{\|e_n\|^2}, & e_n \neq 0, \\ 0, & e_n = 0. \end{cases}$$

Step 7. Compute

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) z_n \text{ and } u_{n+1} = \alpha_n u + (1 - \alpha_n) h_n.$$

Step 8. Set $n := n + 1$ and go to **Step 1**.

The corollary for approximating a solution of the convex minimization problem below follows from Theorem 3.1.7.

Corollary 4.3.2. *Let H_1, H_2 and H_3 be real Hilbert spaces. Let $f_i : H_i \rightarrow \mathbb{R}$ be proper, convex and smooth functions such that ∇f_i are Lipschitz continuous and $g_i : H_i \rightarrow \mathbb{R}$ be proper, convex and lower semicontinuous functions, $i = 1, 2$. Assume that the conditions C3 – C5 are satisfied. If $\Gamma \neq \emptyset$, then the sequence $\{(x_n, u_n)\}$ generated by Algorithm 4.3.1 converges strongly to (p, q) in Γ , where $(p, q) = P_\Gamma(x, u)$.*



5. Numerical Examples

In this chapter, we present two numerical examples which show the implementability of the proposed algorithm.

Example 5.1. Let $H_1 = H_2 = H_3 = H = L_2([0, 1])$ with the norm $\|x\|_{L_2} = \left(\int_0^1 |x(t)|^2 dt \right)^{\frac{1}{2}}$. Let $A_1, A_2 : H \rightarrow H$ and $B_1, B_2 : H \rightarrow H$ be defined by $A_1x(t) = 2x(t)$, $B_1x(t) = 3x(t) + 1$, $A_2u(t) = 2u(t) - 1$ and $B_2u(t) = 4u(t)$. Note that one can easily show that A_1 and A_2 are Lipschitz continuous monotone mappings, and B_1 and B_2 are monotone mappings. In addition, we observe that $x^*(t) = \frac{-1}{5}$ and $u^*(t) = \frac{1}{6}$ are solutions of the equations $A_1x(t) + B_1x(t) = 0$ and $A_2u(t) + B_2u(t) = 0$, respectively. Now, define $T : H_1 \rightarrow H_3$ and $S : H_2 \rightarrow H_3$ by

$$Tx(t) = 2x(t) \text{ and } Su(t) = \frac{-12}{5}u(t).$$

Clearly, T and S are bounded linear operators with

$$T\left(\frac{-1}{5}\right) = \frac{-2}{5} = S\left(\frac{1}{6}\right).$$

Therefore, we have

$$X^* = (x^*(t), u^*(t)) = \left(\frac{-1}{5}, \frac{1}{6}\right) \in \Omega \quad := \quad \left\{ (x^*(t), u^*(t)) \in H \times H : \begin{aligned} &A_1x(t) + B_1x(t) = 0, \\ &A_2u(t) + B_2u(t) = 0 \text{ and } Tx^*(t) = Su^*(t) \end{aligned} \right\}. \quad (5.1.1)$$

Consider $x(t) = 0.5$, $u(t) = 0.1$, $\delta = e = 0.5$, $l = 0.7$, $\gamma = 0.5$ with fixed initial points $(x_0(t), u_0(t)) = (0.5, 0.9)$ and $(x_1(t), u_1(t)) = (1, 0.5)$. If $\alpha_n = \frac{1}{n+100}$ and $\tau_n = \frac{1}{n^2+1000}$ for all $n \geq 1$, then conditions $C1 - C6$ are satisfied and the numerical MATLAB (Mathworks online using a Lenovo Intel(R) Core(TM) i5) experiment is provided below to illustrate the behavior of the convergence of the sequence $\{d_n(t)\} = \{(x_n(t), u_n(t))\}$ generated by Algorithm 3.1.1 to a solution for different values of μ and τ .

To implement Algorithm 3.1.1, we chose $\tau = 0.5$ with values of μ in $\{0.1, 0.3, 0.5, 0.9\}$ and obtain Fig. 5.1 and Fig. 5.2 below.

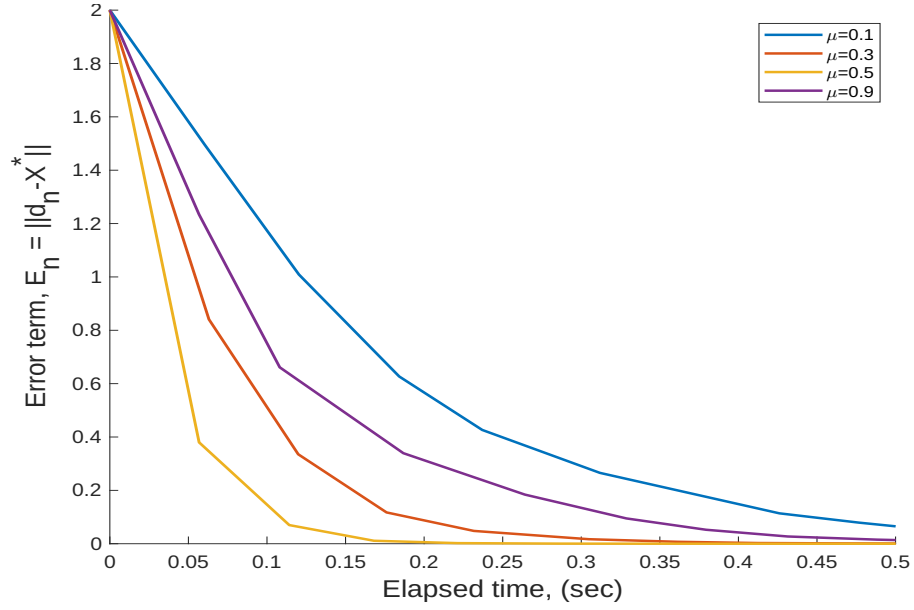


Fig. 5.1: Convergence of the sequence $\{d_n(t)\}$ to the solution with different values of μ with respect to elapsed time.

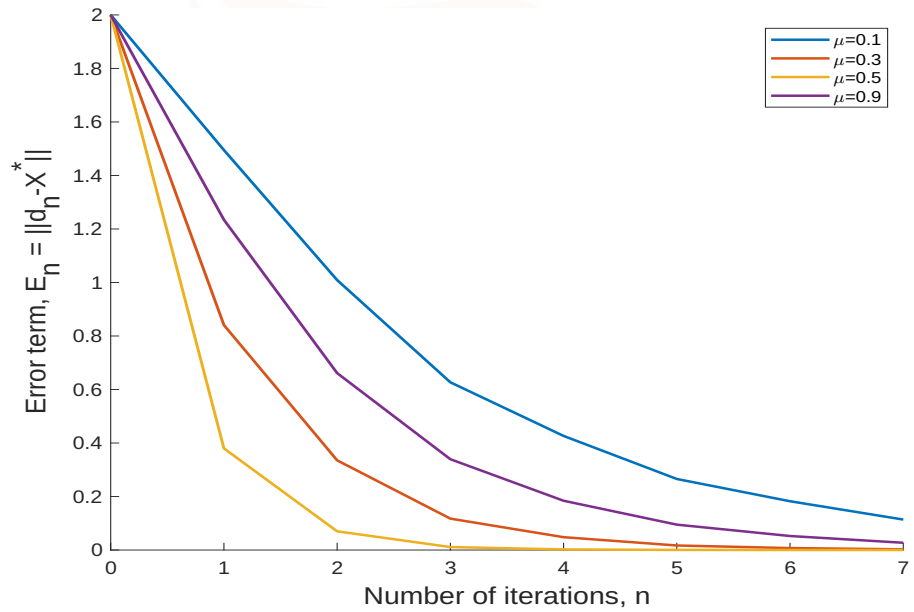


Fig. 5.2: Convergence of the sequence $\{d_n(t)\}$ to the solution with different values of μ with respect to number of iterations.

Furthermore, with the same initial points used in the figures above and $\mu = 0.5$ and varying $\tau \in \{0.0, 0.5, 0.8, 10.0\}$, we get the convergence behavior of the Algorithm 3.1.1 shown in Fig. 5.3 below.

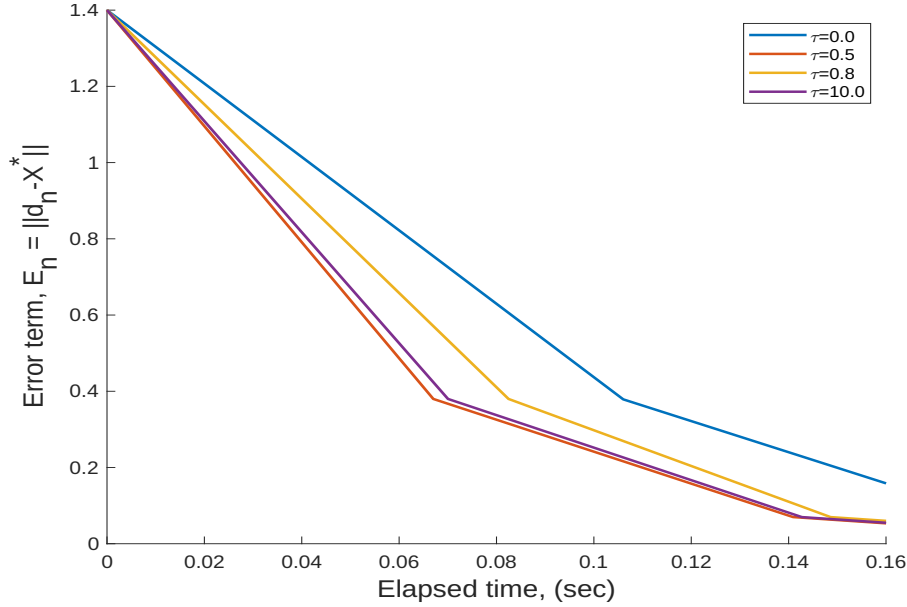


Fig. 5.3: Convergence of the sequence $\{d_n(t)\}$ to the solution with different values of τ with respect to elapsed time.

Remark 5.1.1. In Fig. 5.1, we obtain strong convergence of the sequence $\{d_n(t)\} = \{(x_n(t), u_n(t))\}$ to the solution $(x^*(t), u^*(t)) = \left(\frac{-1}{5}, \frac{1}{6}\right)$ of problem (5.1.1) for different values of μ and different values of τ . From Fig. 5.1, we observe that the rate of convergence of Algorithm 3.1.1 is fast when $\mu = 0.5$ compared to other values of μ near 0 and 1. Now, looking at Fig. 5.2, we have a similar case to the observation in Fig. 5.1. That is, at $\mu = 0.5$, the number of iterations needed for convergence of Algorithm 3.1.1 are fewer which implies faster convergence as compared to other values of μ near 0 and 1. Furthermore, Fig. 5.3 shows that the rate of convergence of Algorithm 3.1.1 is slow when it is non-inertial version (that is, $\tau = 0$) than those with inertial version ($\tau \neq 0$).

Example 5.2. Let $H_1 = \mathbb{R}$, $H_2 = \mathbb{R}^2$ and $H_3 = \mathbb{R}^3$ with the Euclidean norm. Let $A_1, B_1 : H_1 \rightarrow H_1$ and $A_2, B_2 : H_2 \rightarrow H_2$ be mappings defined by $A_1(x) = x - 1$, $B_1(x) = 3x$, $A_2(x, y) = (3x, y - 2)$ and $B_2(x, y) = (x - 1, 3y)$. Then, note that A_1, A_2 are Lipschitz continuous monotone mappings, and B_1, B_2 are monotone mappings. In addition, we observe that $x^* = \frac{1}{4}$ and $u^* = (\tilde{x}, \tilde{y}) = (\frac{1}{4}, \frac{1}{2})$ are solutions of the equations $A_1x + B_1x = 0$ and $A_2(x, y) + B_2(x, y) = (0, 0)$, respectively. Now, define $T : H_1 \rightarrow H_3$ and $S : H_2 \rightarrow H_3$ by

$$T(x) = (x, 2x, 0) \text{ and } S(x, y) = (x, y, 0).$$

Note that T and S are bounded linear operators with

$$T\left(\frac{1}{4}\right) = \left(\frac{1}{4}, \frac{1}{2}, 0\right) = S\left(\frac{1}{4}, \frac{1}{2}\right).$$

Therefore, we have

$$X^* = (x^*, u^*) = \left(\frac{1}{4}, \left(\frac{1}{4}, \frac{1}{2} \right) \right) \in \Omega := \left\{ (x^*, u^*) \in H_1 \times H_2 : A_1 x^* + B_1 u^* = 0, \right. \\ \left. A_2 u^* + B_2 u^* = (0, 0) \text{ and } T x^* = S u^* \right\}.$$

Consider $x = 0.5$, $u = (0.1, 0.1)$, $\delta = e = 0.5$, $l = 0.7$ and $\gamma = 0.5$ with fixed initial points $(x_0, u_0) = (0.5, (0.9, 0.2))$ and $(x_1, u_1) = (0.3, (0.8, 1))$. If $\alpha_n = \frac{1}{n+100}$ and $\tau_n = \frac{1}{n^2+1000}$, for $n \geq 1$, then conditions $C1 - C6$ are satisfied and the numerical MATLAB (Mathworks online using a Lenovo Intel(R) Core(TM) i5) experiment provided below illustrates that the scheme $\{d_n\} = \{(x_n, u_n)\}$ generated by Algorithm 3.1.1 converges strongly to a solution for different values of μ and τ .

Taking $\tau = 0.5$, the numerical experiment in Fig. 5.4 and Fig. 5.5 shows that the sequence $\{d_n\} = \{(x_n, u_n)\}$ generated by Algorithm 3.1.1 converges strongly to the solution (x^*, u^*) with different values of μ in $\{0.1, 0.2, 0.5, 0.8\}$.

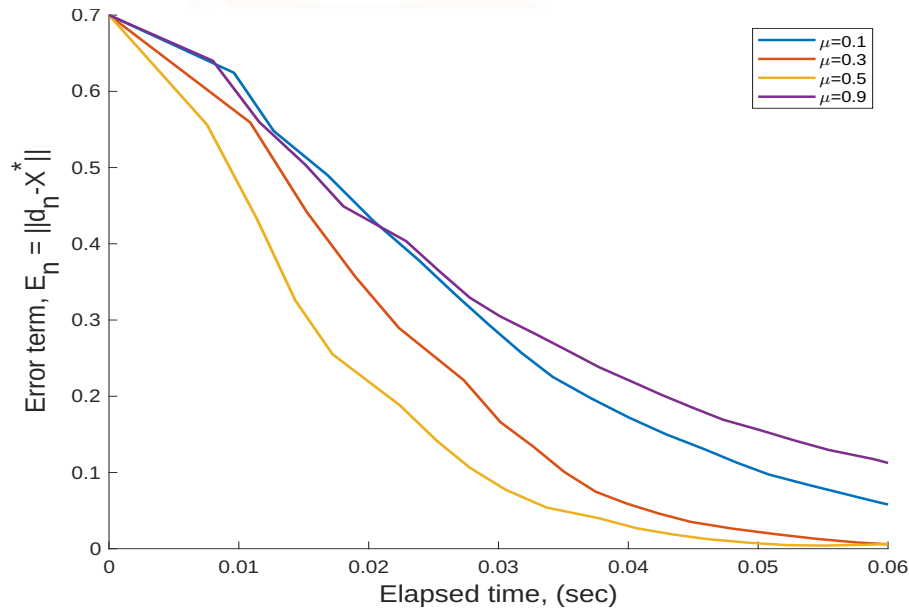


Fig. 5.4: Convergence of the sequence $\{d_n\}$ for different values of μ with respect to elapsed time.

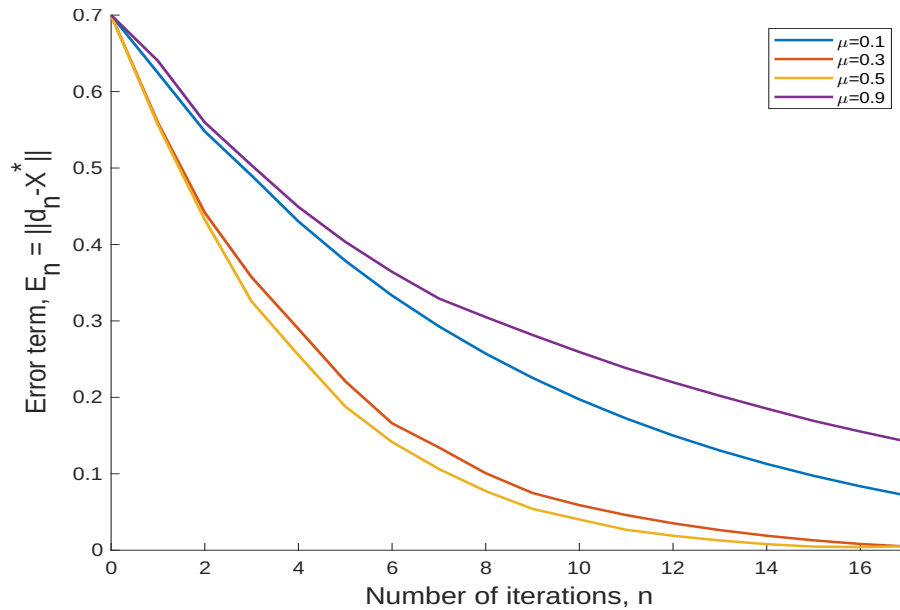


Fig. 5.5: Convergence of the sequence $\{d_n\}$ for different values of μ with respect to number of iterations.

In addition, to implement Algorithm 3.1.1, taking the same initial points and $\mu = 0.5$ with different values of τ in $\{0.0, 0.3, 0.5, 0.8, 10.0\}$, we obtain Fig. 5.6 below.

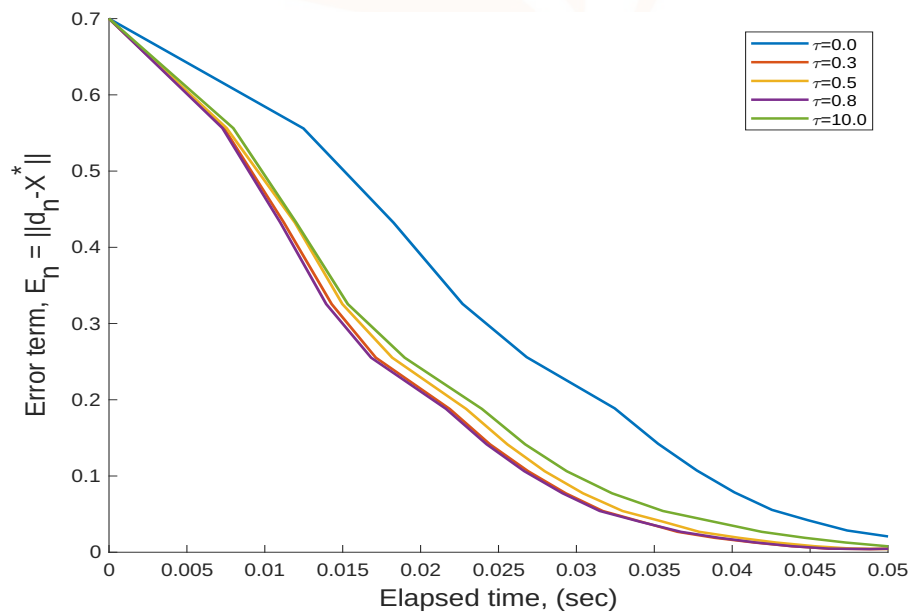


Fig. 5.6: Convergence of the sequence $\{d_n\}$ for different values of τ with respect to elapsed time.

Remark 5.2.1. From Fig. 5.4, we observe that the rate of convergence is faster when $\mu = 0.5$ relative to other $\mu \in (0, 1)$. In Fig. 5.5, we notice that $\mu = 0.5$ has the lowest number of iterations hence faster convergence as compared to those values of μ approaching 0 or 1. Furthermore, Fig. 5.6 shows that the convergence of Algorithm 3.1.1 is slower for the non-inertial version (that is, $\tau = 0$) than for the inertial version.

6. Conclusion and Future Research Problems

6.1 Conclusion

In this Thesis, we introduced a more general problem called Split Equality Inclusion Problem. Moreover, we proposed and studied an inertial iterative method for finding solutions of the problem in real Hilbert spaces. In Chapter 1, we presented a brief background of our study, that is, we defined a few keywords, subject matters and reviewed most of the essential papers published in relation to our study. In chapter 2, we recalled some preliminaries which are of great importance to our study. Then, Chapter 3 was devoted to the main results of our study. The Split Equality Inclusion Problem was introduced. Then the inertial-based algorithm is devised for approximating the solutions of this problem and some strong convergence theorems were established under the assumption that B_1 and B_2 are maximal monotone and A_1 and A_2 are Lipschitz continuous monotone mappings. The prior knowledge of the Lipschitz constants is not required. We then experienced an improved convergence speed in terms of reduced number of iterations and CPU time contributed by the presence of the inertial step. Some applications were provided in Chapter 4 to clearly establish the purpose of the method and numerical examples were given in Chapter 5 to clarify the convergence behavior of the suggested method. The major outcomes in this thesis expands the results of several authors, see Byrne [18], Cholakjiak et. al. [25], Izuchukwu [52], Moudafi [66, 68], and Thong and Cholakjiak [98] in the sense that our result provides strong convergence theorems for the solution of the class of Split Equality Inclusion Problems in real Hilbert spaces which is more general than the problems considered by these authors. We also gave positive and affirmative answers to questions in Chapter 2.

The results of this Thesis have been published as follows:

Thobogang, O.T., Zegeye, H., Boikanyo, O.A.: An Inertial method for solutions of split equality inclusion problems. *Rendiconti del Circolo Matematico di Palermo Series 2*, 1-23. (2023).

6.2 Future Research Problems

The following questions will be at the core of future research work:

1. Can the results obtained in Chapter 3 be extended to Banach spaces which is more general than Hilbert spaces?
2. Can we alter some of the assumptions of the main result, Theorem 3.1.7, and still obtain strong convergence theorems for the solution of the class of Split Equality Inclusion Problems?

3. Can we produce algorithms which are simpler and more efficient in the sense of rate of convergence?
4. Can we have strong convergence results for approximating zeros of the sum of a finite family of maximal monotone mappings?

These questions will surely stipulate further research problems in this direction.



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