

ON SOME EXPONENTIATED GENERALISED FAMILIES AND
COMPOUND CLASS OF DISTRIBUTIONS WITH APPLICATIONS

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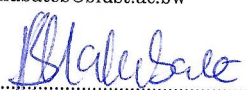
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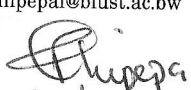
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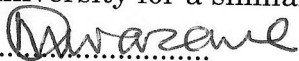
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Dedication

I dedicate this work to my family especially Timukudzeishe Princess for their unwavering support and encouragement.

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Abstract

We introduce the exponentiated odd Nadarajah-Haghighi-G (EO-NH-G) family of distributions and the exponentiated odd exponential Topp-Leone-G (EOE-TL-G) family of distributions as well as the exponentiated odd exponential Topp-Leone-G power series class of distributions. We obtain the series expansions of the probability density functions, hazard rate functions, quantile functions, moments, conditional moments, Bonferroni and Lorenz curves, Rényi entropy and distribution of order statistics. The model parameters are estimated using the maximum likelihood estimation technique and real datasets are used to compare the new distributions to other known distributions with the same number of parameters.

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Chapter 1

Introduction

1.1 Introduction

In the last twenty years or so there has been a lot of changes in the way people live, changes in the technologies, new diseases and climate, just to name a few. These changes have brought in need for development of statistical models that will adequately describe such phenomena. This has necessitated the need for generalizations or modifications of the well-known classical distributions so as to accommodate and provide more flexibility in the modelling of different real life phenomena. This so far has been achieved by adding extra shape, location and scale parameters on the already existing models so as to improve their versatility and flexibility.

One of such models is the Topp-Leone distribution which was developed by Al-Shomrani et al. [4]. The Topp–Leone distribution is a simple bounded J-shaped distribution that has support over a unit interval and most importantly it is an alternative to the Beta generator [4]. It has a hazard rate function (hrf) that possess various shapes top among them being the bathtub which makes it more useful in reliability studies. The Topp–Leone distribution has been studied by several authors which include Genc [27], Feroze and Aslam

[26], MirMostafaei [47], Bayoud [10], MirMostafaei et al. [48], Sultan and Ahmad [63], among others. According to Sangsanit and Bodhisuwan [62], Topp–Leone distribution has a very significant impact in life time data analysis.

Several distributions have been proposed in the literature to model lifetime data by compounding some useful lifetime distributions. These include work by Adamidis and Loukas [1], Kundu and Raqab [39], Kus [40], Chahkandi and Ganjali [12] as well as work by Roozegar and Nadarajah [61].

1.2 Methods of Generating Families of Continuous Distributions

In this section, we look at generators used to formulate continuous distributions.

1.2.1 T-X Family of Distributions

Alzaatreh et al. [5] developed the $T - X$ family of distributions. Let $r(t)$ be the probability density function (pdf) of a random variable $T \in [a, b]$, for $-\infty \leq a < b \leq \infty$ and $W(G(x))$ is a function of the cumulative distribution function (cdf) of a random variable X such that $W[G(x)]$ satisfies the following conditions

- $W[G(x)] \in [a, b]$
- $W[G(x)]$ is differentiable and monotonically non-decreasing, and
- $W[G(x)] \rightarrow a$ as $x \rightarrow -\infty$ and $W[G(x)] \rightarrow b$ as $x \rightarrow \infty$.

According to Alzaatreh et al. [5] the cdf and pdf of the $T - X$ family of distributions are given by

$$F(x) = \int_a^{W(G(x))} r(t) dt \quad (1.1)$$

and

$$f(x) = \left\{ \frac{d}{dx} W[G(x)] \right\} r\{W[G(x)]\}, \quad (1.2)$$

respectively.

1.2.2 Topp-Leone-G Generator

The Topp-Leone-G distribution was developed by Al-Shomrani et al. [4]. It is a one parameter model that exhibits a bathtub shaped hazard rate function. The cdf and pdf of the Topp-Leone-G distribution is given by

$$F_{TL-G}(x; b, \xi) = [1 - \bar{G}^2(x; \xi)]^b \quad (1.3)$$

and

$$f_{TL-G}(x; b, \xi) = 2bg(x; \xi)\bar{G}(x; \xi)[1 - \bar{G}^2(x; \xi)]^{b-1}, \quad (1.4)$$

respectively, where $b > 0$, $G(x; \xi)$ is a baseline distribution with parameter vector ξ and the survival function $\bar{G}(x; \xi) = 1 - G(x; \xi)$.

1.2.3 Beta-G Generator

The beta-G generator pioneered by Eugene et al. [23] is applied on continuous distributions. According to Eugene et al. [23], the cdf and pdf are given by

$$F(x) = I_{G(x; \xi)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x; \xi)} t^{a-1} (1-t)^{b-1} dt \quad (1.5)$$

and

$$f(x) = \frac{1}{B(a, b)} [G(x; \xi)]^{a-1} [1 - G(x; \xi)]^{b-1} g(x; \xi), \quad (1.6)$$

respectively, for $a > 0, b > 0$, where $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta ratio, $B_y(a, b) = \int_0^y t^{a-1} (1-t)^{b-1} dt$ is the incomplete beta function, $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(\cdot)$ is the gamma function and $g(x)$ and $G(x; \xi)$ are the pdf and the cdf of a continuous random variable X respectively.

1.2.4 Compound Class of Distributions

Let N be a discrete random variable following a power series distribution assumed to be truncated at zero, whose probability mass function (pmf) is given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad (1.7)$$

Table 1.1: Useful Quantities for Some Power Series Distributions

Distribution	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	a_n	Parameter Space
Binomial	$(1 + \theta)^m - 1$	$m(1 + \theta)^{m-1}$	$m(m-1)(1 + \theta)^{m-2}$	$(\theta - 1)^{1/m} - 1$	$\binom{m}{n}$	$(0, 1)$
Poisson	$e^\theta - 1$	e^θ	e^θ	$\log(1 + \theta)$	$(n!)^{-1}$	$(0, \infty)$
Geometric	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(1 + \theta)^{-1}$	1	$(0, 1)$
Logarithmic	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	n^{-1}	$(0, 1)$

for $n = 1, 2, 3, \dots$, and $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is a finite series while a_n is a sequence of positive real numbers. Table 1.1 shows values for $C(\theta)$, $C'(\theta)$, $C''(\theta)$, $C^{-1}(\theta)$ and a_n for some usual power series distributions.

Let Y_i , for $i = 1, 2, 3, \dots, N$ be independent and identically distributed random variables. Let $X_1 = \min(Y_1, Y_2, Y_3, \dots, Y_N)$. Then the conditional distribution of X_1 given that $N = n$ is given by

$$\begin{aligned} G_{X_1|N=n}(x) &= 1 - \prod_{i=1}^n (1 - G(x; \xi)) \\ &= 1 - S^n(x), \end{aligned}$$

where $S(x)$ is the survival function given by $1 - G(x; \xi)$ and $G(x; \xi)$ is the baseline cdf. The cdf of the life length of the whole system, X , $F(x)$ will be given by

$$F(x) = 1 - \frac{C(\theta S(x))}{C(\theta)}, \quad (1.8)$$

and the pdf will be given by

$$f(x) = \frac{dF(x)}{dx} = \frac{\theta g(x; \xi) C'(\theta S(x))}{C(\theta)}. \quad (1.9)$$

1.2.5 Exponentiated Generalized Class of Distributions

The Exponentiated Generalized class of distributions was introduced by Gupta et al. [33] when they came up with the exponentiated exponential (EE) distribution. They introduced the exponentiation parameter δ to the exponential distribution. Let $G(x; \xi)$ and $g(x; \xi)$ be the cdf and pdf of a known continuous distribution, then the cdf $F(x)$ of the exponentiated generalised (EG) class of distributions is given by

$$F(x) = [G(x; \xi)]^\delta, \quad (1.10)$$

and the pdf is given by

$$f(x; \xi) = \delta g(x; \xi) [G(x; \xi)]^{\delta-1}, \quad (1.11)$$

for $\delta > 0$. The extra shape parameter brings in more flexibility in the original distribution making it more applicable in areas of life sciences, biology and engineering, just to name a few.

1.3 Some Properties of Distributions

1.3.1 Survival Function, Hazard Function and Quantile Function

For a continuous random variable X with cdf $F(x)$ and pdf $f(x) = dF(x)/dx$, the survival function or reliability function is given by

$$S(x) = P(X > x) = \int_x^\infty f(t)dt = 1 - F(x). \quad (1.12)$$

The survival functions are monotonic, non-increasing functions equal to one at zero and zero as the time approaches infinity. The hazard and the reverse hazard functions are given by

$$h(x) = \frac{f(x)}{S(x)} \quad (1.13)$$

and

$$r(x) = \frac{f(x)}{F(x)}, \quad (1.14)$$

respectively. The quantile function $Q(p)$, where $p \in (0, 1)$ is given by the solution of $F(Q(p)) = p$.

1.3.2 Moments

The r^{th} moment about the origin of a continuous random variable X with pdf $f(x)$ is given by

$$E(X^r) = \int_{-\infty}^\infty x^r f(x)dx. \quad (1.15)$$

Moments are very useful in studying important characteristics of a distribution such as dispersion, central tendency, kurtosis and skewness.

1.3.3 Conditional Moments

The r^{th} conditional moment for a continuous distribution with pdf $f(x)$ is given by

$$E(X^r | X > t) = \frac{1}{S(t)} \int_t^\infty x^r f(x) dx, \quad (1.16)$$

where $S(t)$ is the survival function.

1.3.4 Bonferroni and Lorenz Curves

The Bonferroni curve was introduced by Bonferroni [44]. Let X be a non-negative continuous random variable with mean μ and a cdf $F(x)$. Then the Bonferroni curve is defined as

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx = \frac{1}{p\mu} [\mu - T(q)], \quad (1.17)$$

where $T(q) = \int_q^\infty x f(x) dx$. The Lorenz curve is given by

$$L(p) = \mu^{-1} \int_0^q x f(x) dx = \frac{1}{\mu} [\mu - T(q)],$$

where μ is the mean of the underlying income distribution, $q = F^{-1}(p)$ and $0 \leq p \leq 1$. Bonferroni and Lorenz curves are widely used as a graphical representation of wealth and income distributions. The Lorenz curve was introduced by Lorenz [44]. It is used for analyzing data from information, computer and cognitive sciences and economics.

1.3.5 Entropy

There are two popular measures of entropy, namely, Shannon entropy and Rényi entropy. Rényi entropy by Rényi [67] is an extension of Shannon entropy [69]. Rényi entropy is given by

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f(x)]^v dx \right), v \neq 1, v > 0. \quad (1.18)$$

Rényi entropy tends to Shannon entropy as $v \rightarrow 1$.

1.3.6 Order Statistics

The order statistics of a random sample $X_1, \dots, X_n$ are the sample values placed in ascending order, denoted as $X_{(1)}, \dots, X_{(n)}$ and they are the random variables that satisfy $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \dots \leq X_{(n)}$. The cdf and the pdf of the k^{th} order statistic, for $k = 1, \dots, n$ are defined by

$$F_k(x) = \sum_{i=k}^n \binom{n}{i} F(x)^i [1 - F(x)]^{n-i} \quad (1.19)$$

and

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1} [1 - F(x)]^{n-k}, \quad (1.20)$$

respectively.

1.3.7 Maximum Likelihood Estimation

Suppose we have a random sample $X_1, X_2, \dots, X_n$ with a pdf that depends on one or more unknown parameters $(\theta_1, \theta_2, \theta_3, \dots, \theta_m)$ then the joint pdf of $X_1, X_2, X_3, \dots, X_n$, denoted by $L(\theta_1, \theta_2, \theta_3, \dots, \theta_m)$ is given by

$$L(\theta_1, \theta_2, \theta_3, \dots, \theta_m) = \prod_{i=1}^n (f(x_i; \theta_1, \theta_2, \theta_3, \dots, \theta_m)).$$

The maximum likelihood estimates (MLEs) of the parameter θ_i is the value of θ_i that maximizes $L(\theta_1, \theta_2, \theta_3, \dots, \theta_m)$ if it exists. Since the logarithm is an increasing function, maximizing $L(\theta)$ is equivalent to maximizing its log function denoted as

$$\begin{aligned} \ell(\theta) &= \log(L(\theta)) \\ &= \sum_{i=1}^n \log(f(x_i; \theta_1, \theta_2, \theta_3, \dots, \theta_m)). \end{aligned} \quad (1.21)$$

The maximum likelihood estimates, for large sample sizes, possesses the following desirable limiting properties

- Consistency: the sequence of MLEs converges in probability to the value being estimated ie $\hat{\theta} \xrightarrow{P} \theta$ as $n \rightarrow \infty$.

- **Asymptotic Normality of MLE:** we say that $\hat{\theta}$ is asymptotically normal if $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \sigma_{\theta_0}^2)$, where $\sigma_{\theta_0}^2$ is called the asymptotic variance of the estimate $\hat{\theta}$ (see Ferguson [25]), where θ_0 is the true unknown parameter of the distribution of the sample.

1.3.8 Goodness-of-fit Statistics

The following goodness-of-fit statistics: Akaike information criterion (AIC)

$$AIC = -2p \log(L) + 2p,$$

Corrected Akaike Information Criterion (CAIC)

$$CAIC = AIC + \frac{2p(p+1)}{n-p-1},$$

Bayesian information criterion (BIC)

$$BIC = p \times \log(n) - 2 \log(L),$$

Kolmogorov-Smirnov test (K-S) statistics for a given cdf $F(x)$

$$K - S = \sup_x | F_n(x) - F(x) |,$$

where $F_n(x)$ is the empirical cdf, Anderson-Darling

$$A^* = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\ln(F(x_{(i)})) + \ln(1 - F(x_{(n-i+1)}))],$$

and Cramér-von-Mises statistics

$$W^* = \frac{1}{12n} + \sum_{i=1}^n \left[\frac{2i-1}{2n} - F(x_{(i)}) \right]^2,$$

are used in model selection in this thesis.

1.4 Aims and Objectives

The main aim is to develop new generalized distributions and derive their structural properties. The specific objectives are:

1. To propose and develop three new families of distributions.

2. To obtain mathematical properties of the proposed distributions.
3. To estimate the parameters of the new proposed distributions using maximum likelihood method.
4. To illustrate the performance of the proposed models by means of simulation studies and applications to real data.

1.5 Outline of Thesis

The outline of this thesis is as follows: In Chapter 2, we introduce the exponentiated odd Nadarajah-Haghighi-G (EO-NH-G) family of distributions. We look at the hazard rate function and quantile function, various sub-models, special cases of this new family, Rényi entropy and order statistics. We also look at the estimates of model parameters and assess their performance using the Monte Carlo simulation method. The exponentiated odd Nadarajah-Haghighi-Weibull (EO-NH-W) is compared to other known models with four parameters. Chapter 3 covers the exponentiated odd exponential Topp-Leone-G (EOE-TL-G) family of distributions and its properties including the hazard and quantile function and sub-models, moments, Rényi entropy and distribution of order statistics. In Chapter 4, we introduce the exponentiated odd exponential Topp-Leone-G power series class of distributions and its properties including the hazard and reverse hazard functions, quantile function and sub-models, moments, conditional moments, special cases, Rényi entropy and distribution of order statistics. Estimation of parameters, simulation study and some real data examples to illustrate the usefulness and applicability of the power series class of distributions are also given. Summary and areas of further studies are presented in Chapter 5.

Chapter 2

Exponentiated Odd Nadarajah Haghighi-G Family of Distributions

2.1 Introduction

A lot of work has been done on the Nadarajah-Haghighi (NH) distribution. These include the following distributions: the exponentiated Nadarajah-Haghighi by Lemonte [42], gamma Nadarajah-Haghighi by Bourguignon et al. [11] and the Poisson Nadarajah-Haghighi by Muhammad et al. [51]. The NH distribution is a lifetime model whose density function exhibits some desirable qualities like the reverse-J and uni-modal shapes while the hazard rate has increasing, decreasing and the constant shape. This makes it limited in terms of flexibility to model other types of data which exhibits other forms of the hazard rate function like the bathtub and it is because of this limitation that we propose the new family of distributions called the exponentiated odd Nadarajah-Haghighi-G family of distributions. This proposed family of distributions is more flexible since it applies to almost symmetric, left and right skewed data sets which makes it more useful in the

field of medicine, actuarial sciences, engineering science and environmental science. Furthermore, the proposed generalization applies to both monotonic and non-monotonic hazard rates.

Work done on the Exp-G distributions include exponentiated Weibull distribution by Mudholkar et al. [50], exponentiated Pareto by Gupta et al. [32], exponentiated exponential by Gupta and Kundu [33], exponentiated gamma distribution by Nadarajah and Gupta [54] and many others. This family contains many exponentiated distributions such as exponentiated Weibull, exponentiated exponential, exponentiated Pareto and many others.

The Nadarajah-Haghighi distribution, [55] has its cdf and pdf defined by

$$F_{NH}(x; \beta, \lambda) = 1 - \exp \left(1 - [1 + \lambda x]^\beta \right) \quad (2.1)$$

and

$$f_{NH}(x; \beta, \lambda) = \beta \lambda (1 + \lambda x)^{\beta-1} \exp \left(1 - [1 + \lambda x]^\beta \right), \quad (2.2)$$

respectively for x, λ and $\beta > 0$. Nascimento et al. [56] developed the odd Nadarajah-Haghighi-G (O-NH-G) family of distributions whose cdf and pdf are given by

$$F(x) = 1 - \exp \left(1 - \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]^\beta \right) \quad (2.3)$$

and

$$f(x) = \beta g(x; \xi) \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]^{\beta-1} \exp \left(1 - \left[1 + \left(\frac{G(x; \xi)}{\bar{G}(x; \xi)} \right) \right]^\beta \right), \quad (2.4)$$

respectively for $\beta > 0$, $G(x; \xi)$ is a baseline distribution with parameter vector ξ , $g(x; \xi)$ is the pdf of the baseline distribution and the survival function $\bar{G}(x; \xi) = 1 - G(x; \xi)$. We propose the exponentiated odd Nadarajah-Haghighi-G (EO-NH-G) family of distributions. Special cases from the proposed new family exhibit decreasing, increasing, upside-down bathtub, bathtub and and bathtub followed by upside down bathtub hazard rate shapes. This makes the new

family of distributions more competitive than the O-NH-G distribution.

This chapter is organized as follows: Series expansion of the pdf of the new family is derived in Section 2. In Section 3, we present special cases of the EO-NH-G family of distributions. Main mathematical properties of the EO-NH-G family are studied in Section 4. The parameters of the new distribution are estimated by using the method of maximum likelihood and the observed information matrix are also obtained in Section 5. In Section 6, two real data applications of the EO-NH-G are presented. Concluding remarks are given in Section 7.

2.2 Exponentiated Odd Nadarajah-Haghighi Family of Distributions

In this section, we derive a new family of distributions, namely, the exponentiated odd Nadarajah Haghighi-G (EO-NH-G) family of distributions. Series representation of this new family of distributions is also presented. The EO-NH-G family of distributions has cdf and pdf given by

$$F(x; \delta, \lambda, \beta, \xi) = \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right) \right)^\delta \quad (2.5)$$

and

$$\begin{aligned} f(x; \delta, \lambda, \beta, \xi) &= \frac{\delta \lambda \beta g(x, \xi)}{(1 - G(x; \xi))^2} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta-1} \\ &\times \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right) \\ &\times \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right) \right)^{\delta-1}, \end{aligned} \quad (2.6)$$

respectively, for $\delta, \lambda, \beta > 0$ and parameter vector ξ .

2.2.1 Expansion of Density

Series expansion of the density function of the EO-NH-G family of distributions is presented under this subsection. From equation (2.6) and using the series expansion $(1-x)^{\delta-1} = \sum_{n=1}^{\infty} \binom{\delta-1}{n} (-1)^n x^n$, we obtain,

$$\begin{aligned} \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right) \right)^{\delta-1} &= \sum_{n=0}^{\infty} \binom{\delta-1}{n} (-1)^n \\ &\times \left(\exp \left[1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right] \right)^n, \end{aligned}$$

so that

$$\begin{aligned} f(x; \delta, \lambda, \beta, \xi) &= \frac{\delta \lambda \beta g(x, \xi)}{(1 - G(x; \xi))^2} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta-1} \\ &\times \sum_{n=0}^{\infty} \binom{\delta-1}{n} (-1)^n \left(\exp(n+1) \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right) \right). \end{aligned}$$

But

$$\begin{aligned} \exp \left((n+1) \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right) \right) &= \sum_{m=0}^{\infty} \frac{(n+1)^m}{m!} \\ &\times \left[1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right]^m, \end{aligned}$$

which leads to

$$\begin{aligned} f(x; \delta, \lambda, \beta, \xi) &= \frac{\delta \lambda \beta g(x, \xi)}{(1 - G(x; \xi))^2} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta-1} \\ &\times \sum_{n,m=0}^{\infty} \binom{\delta-1}{n} \frac{(-1)^n (n+1)^m}{m!} \left[1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right]^m. \end{aligned}$$

Also,

$$\left[1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right]^m = \sum_{p=0}^{\infty} (-1)^p \binom{m}{p} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta p},$$

so that

$$\begin{aligned} f(x; \delta, \lambda, \beta, \xi) &= \frac{\delta \lambda \beta g(x, \xi)}{(1 - G(x; \xi))^2} \sum_{n,m,p=0}^{\infty} \binom{\delta-1}{n} \frac{(-1)^{n+p} (n+1)^m}{m!} \binom{m}{p} \\ &\times \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta(p+1)-1}. \end{aligned}$$

Considering the series expansion

$$\left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)\right]^{\beta(p+1)-1} = \sum_{q=0}^{\infty} \binom{\beta(p+1)-1}{q} \left(\lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)\right)^q$$

we get

$$\begin{aligned} f(x; \delta, \lambda, \beta, \xi) &= \delta \lambda^{q+1} \beta g(x, \xi) \sum_{n, m, p, q=0}^{\infty} \binom{\delta-1}{n} \frac{(-1)^{n+p} (n+1)^m}{m!} \binom{m}{p} \\ &\times \binom{\beta(p+1)-1}{q} G^q(x; \xi) (1 - G(x; \xi))^{-(q+2)}. \end{aligned}$$

Now, applying the series expansion

$$[1 - G(x; \xi)]^{-(q+2)} = \sum_{r=0}^{\infty} \binom{-(q+2)}{r} G^r(x; \xi),$$

we have

$$\begin{aligned} f(x; \delta, \beta, \lambda, \xi) &= \sum_{n, m, p, q, r=0}^{\infty} \delta \lambda^{q+1} \beta \binom{\delta-1}{n} \frac{(-1)^{n+p} (n+1)^m}{m!} \binom{m}{p} \\ &\times \binom{\beta(p+1)-1}{q} \binom{-(q+2)}{r} \binom{q+r+1}{q+r+1} g(x, \xi) G^{q+r}(x; \xi) \\ &= \sum_{q, r=0}^{\infty} v_{q, r} g_{q+r+1}(x; \xi), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} v_{q, r} &= \sum_{n, m, p=0}^{\infty} \delta \lambda^{q+1} \beta \binom{\delta-1}{n} \frac{(-1)^{n+p} (n+1)^m}{m!} \binom{m}{p} \\ &\times \binom{\beta(p+1)-1}{q} \binom{-(q+2)}{r} \frac{1}{q+r+1}, \end{aligned} \tag{2.8}$$

and $g_{q+r+1}(x; \xi) = (q+r+1)g(x; \xi)[G(x; \xi)]^{q+r}$ is an Exponentiated-G (Exp-G) distribution with power parameter $(q+r+1)$. The EO-NH-G distribution is an infinite linear combination of Exp-G densities and the mathematical properties of the EO-NH-G family of distributions can be readily obtained from those of the Exp-G family of distributions.

2.2.2 Some Sub-Families

Some sub-families of the EO-NH-G distribution are presented in this subsection.

- When $\delta = \lambda = 1$, we obtain the odd Nadarajah Haghghi-G (O-NH-G) family of distributions (Nascimento et al. [56]).

- When $\beta = 1$, we obtain the exponentiated odd exponential-G (EOE-G) family of distributions with cdf given by

$$F(x; \lambda, \delta, \xi) = \left(1 - \exp \left[-\lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right] \right)^\delta.$$

- When $\beta = 2$, we obtain the new family of distributions with cdf given by

$$F(x; \lambda, \delta, \xi) = \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^2 \right) \right)^\delta.$$

- When $\beta = \delta = 1$, we obtain Odd Exponential-G (OE-G) family of distributions with cdf given by

$$F(x; \lambda, \xi) = 1 - \exp \left[-\lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right].$$

2.3 Some Special Cases

In this section, we present four special cases of the EO-NH-G family of distributions and plots of their pdfs and hazard rate functions (hrf). The baseline distribution is restricted to at most two parameter models to avoid the problem of over-parametrization.

2.3.1 Exponentiated Odd Nadarajah

Haghighi-Uniform (EO-NH-U) Distribution

When we consider the uniform distribution as the baseline distribution with pdf and cdf given by $g(x; \theta) = 1/\theta$ and $G(x; \theta) = x/\theta$, for $0 < x < \theta$, we obtain the EO – NH – U distribution with cdf and pdf given by

$$F(x; \delta, \lambda, \beta, \theta) = \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{x/\theta}{1 - x/\theta} \right) \right]^\beta \right) \right)^\delta$$

and

$$\begin{aligned} f(x; \delta, \lambda, \beta, \theta) &= \frac{\delta \lambda \beta (1/\theta)}{(1 - (x/\theta))^2} \left[1 + \lambda \left(\frac{x/\theta}{1 - x/\theta} \right) \right]^{\beta-1} \\ &\times \left(\exp \left(1 - \left[1 + \lambda \left(\frac{x/\theta}{1 - x/\theta} \right) \right]^\beta \right) \right) \\ &\times \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{x/\theta}{1 - x/\theta} \right) \right]^\beta \right) \right)^{\delta-1}, \end{aligned}$$

respectively for $\delta, \lambda, \beta > 0$ and $0 < x < \theta$.

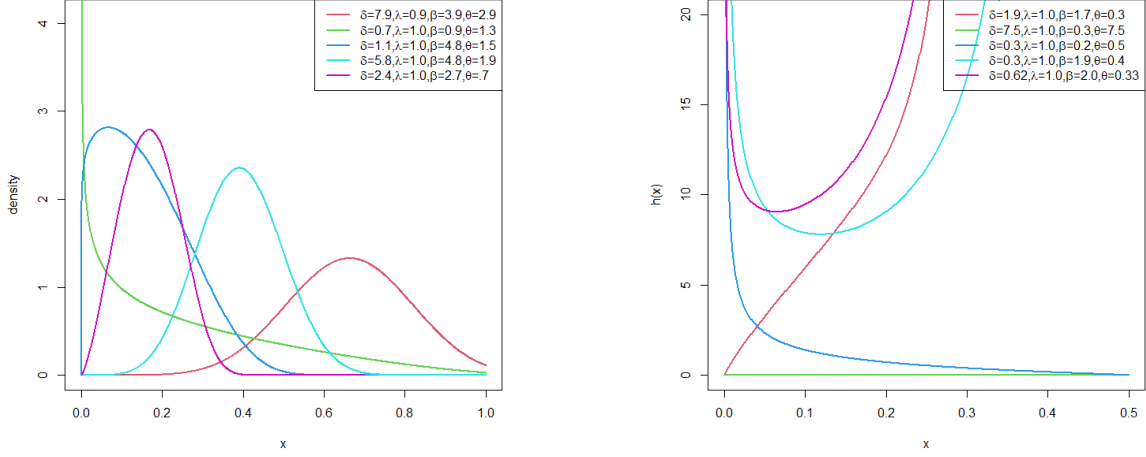


Figure 2.1: Plots of the pdf (left picture) and hrf (right picture) for the EO-NH-U distribution

The pdfs of the EO-NH-U distribution can take various shapes that include reverse-J, uni-modal, left or right skewed. Furthermore, the EO-NH-U distribution exhibit decreasing, increasing, reverse-J and bathtub shapes for the hrf.

2.3.2 The Exponentiated Odd Nadarajah Haghghi-Log Logistic (EO-NH-LLoG) Distribution

By taking the log-logistic distribution as the baseline distribution with the pdf and cdf given by $g(x; c) = cx^{c-1}(1+x^c)^{-2}$ and $G(x; c) = 1 - (1+x^c)^{-1}$, for $c > 0$, we obtain the EO-NH-LLoG distribution with cdf and pdf given by

$$F(x; \delta, \lambda, \beta, c) = \left(1 - \exp \left(1 - \left(1 + \lambda \left(\frac{1 - (1+x^c)^{-1}}{(1+x^c)^{-1}} \right) \right)^\beta \right) \right)^\delta$$

and

$$\begin{aligned}
f(x; \delta, \lambda, \beta, c) &= \delta \lambda \beta c x^{c-1} \left[1 + \lambda \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right) \right]^{\beta-1} \\
&\times \exp \left(1 - \left[1 + \lambda \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right) \right]^{\beta} \right) \\
&\times \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right) \right]^{\beta} \right) \right)^{\delta-1},
\end{aligned}$$

respectively, for $\delta, \lambda, \beta, c > 0$.

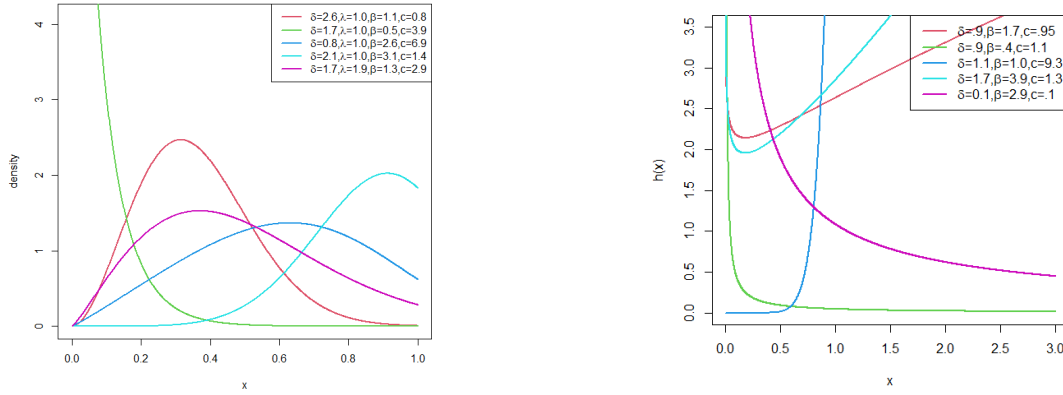


Figure 2.2: Plots of the pdf (left picture) and hrf (right picture) for the EO-NH-LLoG distribution

The pdfs of the EO-NH-LLoG distribution can take various shapes that include reverse-J, left or right skewed shapes as shown in Figure 2.2. Furthermore, the EO-NH-LLoG distribution exhibit reverse-J, decreasing and bathtub shapes for the hrf.

2.3.3 The Exponentiated Odd Nadarajah-

Haghighi-Kumaraswamy (EO-NH-K) Distribution

Considering the Kumaraswamy distribution as the baseline distribution with pdf and cdf given by $g(x; a, b) = abx^{a-1}(1 - x^a)^{b-1}$ and $G(x; a, b) = 1 - (1 - x^a)^b$, for $a, b > 0$, we obtain the EO-NH-K distribution with cdf and pdf given by

$$F(x; \delta, \beta, \lambda, a, b) = \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{1 - (1 - x^a)^b}{(1 - x^a)^b} \right) \right]^{\beta} \right) \right)^{\delta}$$

and

$$\begin{aligned}
f(x; \delta, \lambda, a, \beta, b) &= \delta \lambda a \beta b x^{a-1} (1 - x^a)^{-\beta-1} \\
&\times \left[1 + \lambda \left(\frac{1 - (1 - x^a)^b}{(1 - x^a)^b} \right) \right]^{\beta-1} \\
&\times \left(\exp \left(1 - \left[1 + \lambda \left(\frac{1 - (1 - x^a)^b}{(1 - x^a)^b} \right) \right]^\beta \right) \right) \\
&\times \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{1 - (1 - x^a)^b}{(1 - x^a)^b} \right) \right]^\beta \right) \right)^{\delta-1},
\end{aligned}$$

respectively, for $\delta, \beta, \lambda, a, b > 0$.

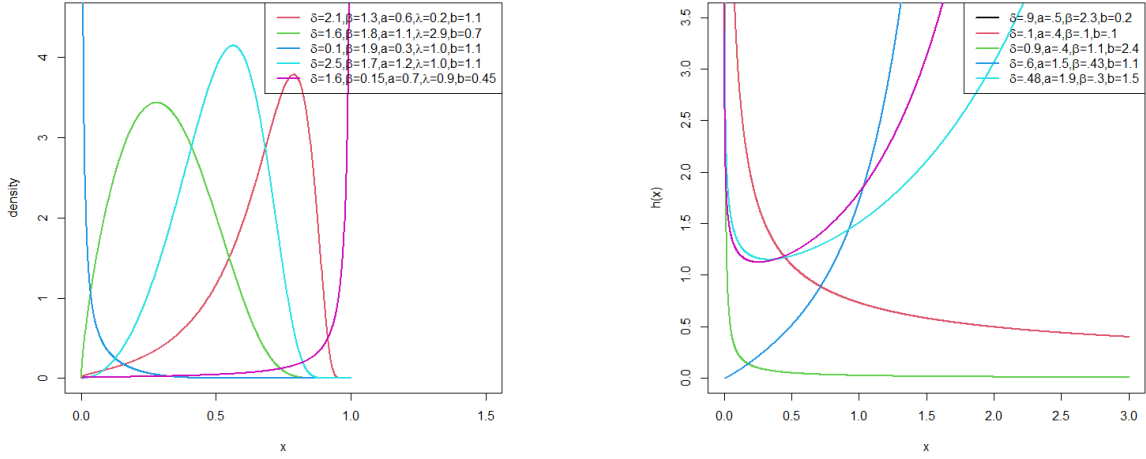


Figure 2.3: Plots of the pdf (left picture) and hrf (right picture) for the EONH-K distribution

The pdfs of the EO-NH-K distribution exhibits various shapes that include reverse-J, uni-modal, left or right skewed shapes. The shapes of EO-NH-K hrf include decreasing, increasing and bathtub.

2.3.4 The Exponentiated Odd Nadarajah Haghighi-Weibull (EO-NH-W) Distribution

By considering the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x; \omega) = \omega x^{\omega-1} e^{-x^\omega}$ and $G(x; \omega) = 1 - e^{-x^\omega}$, for $\omega > 0$, we obtain

the EO-NH-W distribution given by

$$F(x; \delta, \lambda, \beta, \omega) = \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{1 - e^{-x^\omega}}{e^{-x^\omega}} \right) \right]^\beta \right) \right)^\delta$$

and

$$\begin{aligned} f(x; \delta, \lambda, \beta, \omega) &= \frac{\delta \lambda \beta \omega x^{\omega-1}}{e^{-x^\omega}} \left[1 + \lambda \left(\frac{1 - e^{-x^\omega}}{e^{-x^\omega}} \right) \right]^{\beta-1} \\ &\times \left(\exp \left(1 - \left[1 + \lambda \left(\frac{1 - e^{-x^\omega}}{e^{-x^\omega}} \right) \right]^\beta \right) \right) \\ &\times \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{1 - e^{-x^\omega}}{e^{-x^\omega}} \right) \right]^\beta \right) \right)^{\delta-1}, \end{aligned}$$

for $\delta, \beta, \omega > 0$.

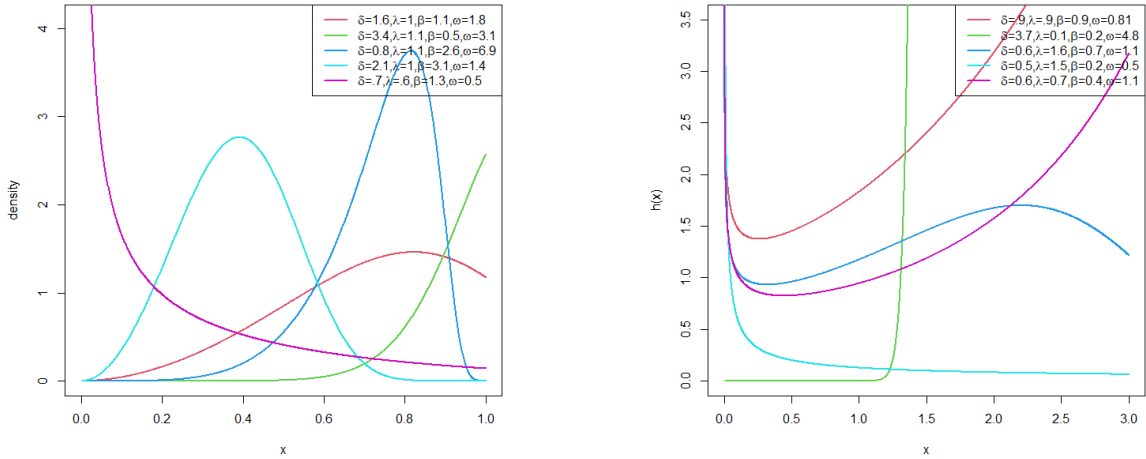


Figure 2.4: Plots of the pdf (left picture) and hrf (right picture) for the EO-NH-W distribution

The pdfs of the EO-NH-W distribution exhibits various shapes that include uni-modal, left or right-skewed shapes. The shapes of EO-NH-W hrf are decreasing, increasing, bathtub, and bathtub followed by upside down bathtub.

2.4 Mathematical and Statistical Properties

This section presents some statistical properties of the EO-NH-G family of distributions, namely; distribution of order statistics, entropy, moments,

incomplete moments, probability weighted moments (PWMs), quantile and generating functions.

2.4.1 Distribution of Order Statistics

The pdf of the i^{th} order statistics is given by

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} F(x)^{j+i-1}, \quad (2.9)$$

where $B(.,.)$ is the beta function. Substituting equations (2.5) and (2.6) into equation (2.9), we get

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \frac{\delta\lambda\beta g(x, \xi)}{(1-G(x; \xi))^2} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta-1} \\ &\times \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \\ &\times \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right)^{\delta(j+i)-1}. \end{aligned}$$

Using the generalized binomial series expansion

$$\begin{aligned} \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right)^{\delta(j+i)-1} &= \sum_{m=0}^{\infty} (-1)^m \binom{\delta(j+i)-1}{m} \\ &\times \exp \left(m \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right), \end{aligned}$$

and substituting into equation (2.9), we obtain

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{m=0}^{\infty} \frac{g(x; \xi)}{(1-G(x; \xi))^2} (-1)^m \binom{\delta(j+i)-1}{m} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta-1} \\ &\times \delta\beta\lambda \exp \left((m+1) \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right). \end{aligned}$$

Now, using the series expansion

$$\begin{aligned}
\exp \left((m+1) \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right) \right) &= \sum_{n=0}^{\infty} \frac{(m+1)^n}{n!} \\
&\times \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^\beta \right)^n \\
&= \sum_{n,p=0}^{\infty} \frac{(m+1)^n (-1)^p}{n!} \binom{n}{p} \\
&\times \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta p},
\end{aligned}$$

we get

$$\begin{aligned}
f(x)F(x)^{j+i-1} &= \delta\beta\lambda \sum_{m,n,p=0}^{\infty} \frac{(m+1)^p (-1)^{m+p}}{n!} \binom{n}{p} \binom{\delta(j+i)-1}{m} \\
&\times \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta(p+1)-1} \frac{g(x; \xi)}{(1 - G(x; \xi))^2}.
\end{aligned} \tag{2.10}$$

Considering the following generalized binomial expansions

$$\left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta(p+1)-1} = \sum_{k=0}^{\infty} \binom{\beta(p+1)-1}{k} \lambda^k \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)^k$$

and

$$(1 - G(x; \xi))^{-(k+2)} = \sum_{q=0}^{\infty} \binom{-(k+2)}{q} (G(x; \xi))^q,$$

we have

$$\begin{aligned}
f(x)F(x)^{j+i-1} &= \delta\beta \sum_{m,n,p,k,q=0}^{\infty} \lambda^{k+1} \binom{-(k+2)}{q} \binom{\beta(p+1)-1}{k} \\
&\times \frac{(m+1)^p (-1)^{k+m}}{p!} \binom{n}{p} \binom{\delta(j+i)-1}{m} \\
&\times (G(x; \xi))^{k+q} g(x; \xi).
\end{aligned}$$

We can therefore, write the pdf of the i^{th} order statistic from the EO-NH-G family of distributions as

$$\begin{aligned}
f_{i:n}(x) &= \frac{\delta\beta}{B(i, n-i+1)} \binom{n-i}{j} \sum_{m,n,p,k,q=0}^{\infty} \lambda^{k+1} \binom{-(k+2)}{q} \binom{\beta(p+1)-1}{k} \\
&\times \frac{(m+1)^p (-1)^{k+m}}{p!} \binom{n}{p} \binom{\delta(j+i)-1}{m} (G(x; \xi))^{k+q} g(x; \xi) \left(\frac{k+q+1}{k+q+1} \right) \\
&= \sum_{k,q=0}^{\infty} v_{k,q}^* g_{k+q+1}(x; \xi),
\end{aligned} \tag{2.11}$$

where $g_{k+q+1}(x; \xi) = (k + q + 1)g(x; \xi)[G(x; \xi)]^{k+q}$ is an Exp-G distribution with power parameter $(k + q + 1)$ and

$$\begin{aligned} v_{k,q}^* &= \frac{\delta\beta}{B(i, n-i+1)} \binom{n-i}{j} \sum_{m,n,p,k,q=0}^{\infty} \lambda^{k+1} \binom{\beta(n+1)-1}{k} \binom{-(k+2)}{q} \\ &\times \frac{(m+1)^p (-1)^{k+m}}{p!} \binom{n}{p} \binom{\delta(j+i)-1}{m} \frac{1}{k+q+1}. \end{aligned}$$

Therefore, the pdf of the i^{th} order statistics from the EO-NH-G family of distributions can be expressed as an infinite linear combination of Exp-G densities.

2.4.2 Entropy

Entropy measures variation of uncertainty for a random variable X having the probability distribution $f(x)$. There are two common types of entropy, namely, Rényi entropy [67] and Shannon entropy [69]. Shannon entropy is a special case of Rényi entropy. We derive the Rényi entropy ($I_R(\nu)$) of the EO-NH-G family of distributions as follows

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\int_0^{\infty} f^{\nu}(x) dx \right], \nu \neq 1, \nu > 0. \quad (2.12)$$

Using equation (2.6), $f^{\nu}(x)$, can be written as

$$\begin{aligned} f^{\nu}(x) &= (\delta\beta\lambda)^{\nu} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\nu(\beta-1)} \exp \left(\nu \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right) \\ &\times \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right)^{\nu(\delta-1)} \left(\frac{g(x; \xi)}{(1 - G(x; \xi))^2} \right)^{\nu}. \end{aligned}$$

Let $t = \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right)$, then

$$\begin{aligned} f^{\nu}(x) &= (\delta\beta\lambda)^{\nu} [1 + \lambda t]^{\nu(\beta-1)} \exp \left(\nu \left(1 - [1 + \lambda t]^{\beta} \right) \right) \\ &\times \left(1 - \exp \left(1 - [1 + \lambda t]^{\beta} \right) \right)^{\nu(\delta-1)} \left(\frac{g(x; \xi)}{(1 - G(x; \xi))^2} \right)^{\nu}. \end{aligned}$$

Consider

$$\left(1 - \exp \left(1 - [1 + \lambda t]^{\beta} \right) \right)^{\nu(\delta-1)} = \sum_{j=0}^{\infty} (-1)^j \binom{\nu(\delta-1)}{j} \exp \left(j \left(1 - [1 + \lambda t]^{\beta} \right) \right),$$

then we can write,

$$f^\nu(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\nu(\delta-1)}{j} (\delta\beta\lambda)^\nu [1+\lambda t]^{\nu(\beta-1)} \\ \times \exp\left((j+1)\left(1-[1+\lambda t]^\beta\right)\right) \left(\frac{g(x;\xi)}{(1-G(x;\xi))^2}\right)^\nu.$$

By considering the series expansions

$$\exp\left((j+1)\left(1-[1+\lambda t]^\beta\right)\right) = \sum_{i=0}^{\infty} \frac{(j+1)^i}{i!} \left(1-[1+\lambda t]^\beta\right)^i$$

and

$$\left(1-[1+\lambda t]^\beta\right)^i = \sum_{k=0}^{\infty} \binom{i}{k} (-1)^k [1+\lambda t]^{\beta k},$$

we get

$$f^\nu(x) = \sum_{j=0}^{\infty} (-1)^{j+k} \binom{\nu(\delta-1)}{j} (\delta\beta\lambda)^\nu \frac{(j+1)^i}{i!} \binom{i}{k} [1+\lambda t]^{\nu(\beta-1)+\beta k} \\ \times \left(\frac{g(x;\xi)}{(1-G(x;\xi))^2}\right)^\nu. \quad (2.13)$$

By applying the binomial expansion of

$$[1+\lambda t]^{\nu(\beta-1)+\beta k} = \sum_{l=0}^{\infty} \binom{\nu(\beta-1)+\beta k}{l} (\lambda t)^l$$

where $t = \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)$, and substituting into (2.13), we obtain

$$f^\nu(x) = \sum_{j,i,k,l=0}^{\infty} (-1)^{j+k} \binom{\nu(\delta-1)}{j} (\delta\beta\lambda)^\nu \frac{(j+1)^i}{i!} \binom{i}{k} \binom{\nu(\beta-1)+\beta k}{l} (\lambda t)^l \\ \times \left(\frac{g(x;\xi)}{(1-G(x;\xi))^2}\right)^\nu.$$

and applying the generalized binomial series expansion of

$$(1-G(x;\xi))^{-(l+2\nu)} = \sum_{q=0}^{\infty} \binom{-(l+2\nu)}{q} (-1)^q (G(x;\xi))^q,$$

we have

$$f^\nu(x) = \sum_{j,i,k=0}^{\infty} \binom{-(l+2\nu)}{q} \frac{(\nu+j)^i}{i!} \binom{i}{k} \frac{(j+1)^i}{i!} \binom{\nu(\beta-1)+\beta k}{l} \\ \times (-1)^{i+k} \binom{\nu(\delta-1)}{j} (\delta\beta)^\nu \lambda^{\nu+l} [G(x;\xi)]^{q+l} g(x;\xi)^\nu, \quad (2.14)$$

and by considering

$$\begin{aligned}
[G(x; \xi)]^{q+l} &= [1 - (1 - G(x; \xi))]^{q+l} \\
&= \sum_{n=0}^{\infty} \binom{q+l}{n} (-1)^n [1 - G(x; \xi)]^n \\
&= \sum_{n,p=0}^{\infty} \binom{q+l}{n} \binom{n}{p} (-1)^{n+p} G(x; \xi)^p,
\end{aligned}$$

we can write equation (2.14) as

$$\begin{aligned}
f^\nu(x) &= \sum_{j,i,k=0}^{\infty} \binom{-(l+2\nu)}{q} \frac{(\nu+j)^i}{i!} \binom{i}{k} \frac{(j+1)^i}{j!} \binom{\nu(\beta-1)+\beta k}{l} \\
&\times (-1)^{i+k} \binom{\nu(\delta-1)}{j} (\delta\beta)^\nu \lambda^{\nu+k} [G(x; \xi)]^{q+l} g(x; \xi)^\nu, \\
f^\nu(x) &= \sum_{j,i,k,n,p=0}^{\infty} \lambda^{\nu+k} \binom{-(k+2\nu)}{l} \frac{(\nu+j)^i}{i!} \binom{i}{k} \frac{(j+1)^i}{i!} \binom{k+l}{n} \\
&\times (-1)^{i+k+n+p} \binom{\nu(\delta-1)}{j} (\delta\beta)^\nu G(x; \xi)^p g(x; \xi)^\nu. \tag{2.15}
\end{aligned}$$

Rényi entropy for the EO-NH-G family of distributions can therefore be written as

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\sum_{m=0}^{\infty} v_m \exp(1-\nu) I_{REG} \right], \nu \neq 1, \nu > 0, \tag{2.16}$$

where

$$\begin{aligned}
v_m &= \sum_{j,i,k,n,p=0}^{\infty} \binom{-(\beta(j+\nu)-\nu)}{l} \frac{(\nu+j)^i}{i!} \binom{i}{k} \frac{(j+1)^i}{i!} \\
&\times (-1)^{i+k+n+p} \binom{\nu(\delta-1)}{j} (\delta\beta)^\nu \lambda^{\nu+k} \left(\frac{p}{\nu} + 1\right)^{-\nu} \tag{2.17}
\end{aligned}$$

and $I_{REG} = (1-\nu)^{-1} \log \int_0^\infty \left[\left(\frac{p}{\nu} + 1\right) g(x; \xi) [G(x; \xi)]^{\frac{p}{\nu}} \right]^\nu dx$ is Rényi entropy of Exp-G distribution with parameter $(\frac{p}{\nu} + 1)$. Therefore, Rényi entropy of the EO-NH-G family of distributions can be derived directly from Rényi entropy of Exp-G distribution.

2.4.3 Moments and Generating Function

The r^{th} ordinary moment can be derived using equation (2.7) as follows:

$$\mu'_r = E(X^r) = \sum_{q,r=0}^{\infty} v_{q,r} E(Y_{q,r}^r), \quad (2.18)$$

where $Y_{q,r}$ follows an Exp-G distribution with power parameter $(q + r + 1)$ and $v_{q,r}$ is given in equation (2.8). The s^{th} central moment of X is given by

$$\mu_s = \sum_{t=0}^s \binom{s}{t} (-\mu'_1)^{s-t} E(X^t) = \sum_{t=0}^s \sum_{q,t=0}^{\infty} v_{q,r} \binom{s}{t} (-\mu'_1)^{s-t} E(Y_{q,r}^t).$$

The cumulants of X follow recursively from

$$k_s = \mu'_s - \sum_{t=0}^{s-1} \binom{s-1}{t-1} k_t \mu'_{s-t},$$

where $k_1 = \mu'_1$, $k_2 = \mu'_2 - \mu_1'^2$, $k_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu_1'^3$, etc. Ordinary moments may also be used to calculate the measures of dispersion, namely, variance, skewness and kurtosis.

The t^{th} incomplete moment of X is given by

$$\phi_r(z) = \int_{-\infty}^z x^t f(x) dx = \sum_{q,r=0}^{\infty} v_{q,r} \int_{-\infty}^z x^t g_{q+r+1}(x; \xi) dx. \quad (2.19)$$

The incomplete moment is very useful and can be used to estimate some important quantities such as Lorenz and Bonferroni curves. These quantities have a wide application in demography, economics, insurance, medicine and reliability. Mathematically, the Lorenz and Bonferroni curves for a given probability p are given by $L(p) = \phi_1(q)/\mu'_1$ and $B(p) = \phi_1(q)/(p\mu'_1)$, respectively, where μ'_1 is given by equation (2.18), with $r = 1$ and $q = Q(p)$ is the quantile function of X at p . The incomplete moment (equation (2.19)) can be expressed as

$$\phi_r(z) = \sum_{q,r=0}^{\infty} v_{q,r} H_{q,r}(z), \quad (2.20)$$

where $H_{q,r}(z) = \int_{-\infty}^z x^t g_{q+r+1}(x; \xi) dx$ is the t^{th} incomplete moment of the Exp-G distribution.

The variance (σ^2), Standard deviation ($SD=\sigma$), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by $\sigma^2 =$

$$\mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1}, \quad CS = \frac{E[(X-\mu)^3]}{[E(X-\mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}}, \quad \text{and} \\ CK = \frac{E[(X-\mu)^4]}{[E(X-\mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2}, \quad \text{respectively.}$$

Note that the r^{th} cumulant of the random variable X can be readily obtained from the recursive relationship: $\kappa_r = \mu'_r - \sum_{s=1}^{r-1} \binom{r-1}{s-1} \mu'_{r-s} \kappa_s$, where $\mu'_r = E(X - \mu_1)^r$, so that the CS and CK are given by $\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}$ and $\gamma_2 = \frac{\kappa_4}{\kappa_2^2}$.

We present the first five moments together with the standard deviation (SD or σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the EO-NH-LLoG distribution for selected values of the parameters (see Table 2.1, for details).

Table 2.1: Moments of the EO-NH-LLoG distribution for some parameter values

	$(\delta, \lambda, \beta, c)$				
	(0.8,0.3,1.3,1.1)	(0.4,1.1,0.7,0.1)	(2.1,1.6,2.7,1.6)	(1.1,0.2,2.5,1.4)	(1.1,1.6,1.7,0.7)
$E(X)$	0.3052	0.3160	0.3483	0.19	0.0899
$E(X^2)$	0.1875	0.1737	0.1942	0.0885	0.0509
$E(X^3)$	0.1341	0.1153	0.1286	0.0543	0.0354
$E(X^4)$	0.1038	0.0849	0.0936	0.0383	0.0272
$E(X^5)$	0.0845	0.0666	0.0724	0.0292	0.0221
$E(X^6)$	0.0712	0.0546	0.0584	0.0235	0.0185
SD	0.0461	0.0486	0.0195	0.016	0.0160
CV	0.0399	0.0414	0.0167	0.0141	0.0141
CS	0.0351	0.0359	0.0146	0.0126	0.0126
CK	0.0313	0.0317	0.0129	0.0113	0.0113

The moment generating function (mgf) of EO-NH-G family of distributions is given by

$$M_x(t) = E(e^{tX}) = \sum_{p=0}^{\infty} v_{q,r} M_{Y_{q,r}}(t),$$

where $M_{Y_{q,r}}(t)$ is the mgf of Exp-G with power parameter $q + r + 1$. The mgf of EO-NH-G family of distributions can be obtained directly from that of the Exp-G distribution.

2.4.4 Probability Weighted Moments

Probability weighted moments (PWMs) are very useful in estimating parameters of distributions which are not in closed form. The $(j, i)^{th}$ PWM, say $\eta_{j,i}$ of $X \sim \text{EO-NH-G}(\delta, \beta; \xi)$ distribution is defined by

$$\eta_{j,i} = E(X^j F(X)^i) = \int_{-\infty}^{\infty} x^j f(x) F(x)^i dx.$$

Using equation (2.10), we can write

$$\begin{aligned} f(x)F(x)^i &= \frac{\delta\lambda\beta g(x, \xi)}{(1 - G(x; \xi))^2} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta-1} \\ &\times \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \\ &\times \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right)^{\delta-1} \\ &\times \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right)^{\delta i}. \end{aligned}$$

$$\begin{aligned} \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right)^{\delta(i+1)-1} &= \sum_{n=0}^{\infty} \binom{\delta(i+1)-1}{n} (-1)^n \\ &\times \left(\exp \left[1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right] \right)^n, \end{aligned}$$

so that

$$\begin{aligned} f(x; \delta, \lambda, \beta, \xi) &= \frac{\delta\lambda\beta g(x, \xi)}{(1 - G(x; \xi))^2} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta-1} \\ &\times \sum_{n=0}^{\infty} \binom{\delta(i+1)-1}{n} (-1)^n \left(\exp(n+1) \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right). \end{aligned}$$

But

$$\begin{aligned} \exp \left((n+1) \left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right) \right) &= \sum_{m=0}^{\infty} \frac{(n+1)^m}{m!} \\ &\times \left[1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right]^m, \end{aligned}$$

which leads to

$$\begin{aligned} f(x; \delta, \lambda, \beta, \xi) &= \frac{\delta\lambda\beta g(x, \xi)}{(1 - G(x; \xi))^2} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta-1} \\ &\times \sum_{n,m=0}^{\infty} \binom{\delta(i+1)-1}{n} \frac{(-1)^n (n+1)^m}{m!} \left[1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right) \right]^{\beta} \right]^m. \end{aligned}$$

Also,

$$\left[1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)\right]^\beta\right]^m = \sum_{p=0}^{\infty} (-1)^p \binom{m}{p} \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)\right]^{\beta p},$$

so that

$$\begin{aligned} f(x; \delta, \lambda, \beta, \xi) &= \frac{\delta \lambda \beta g(x, \xi)}{(1 - G(x; \xi))^2} \sum_{n, m, p=0}^{\infty} \binom{\delta(i+1) - 1}{n} \frac{(-1)^{n+p} (n+1)^m}{m!} \binom{m}{p} \\ &\times \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)\right]^{\beta(p+1)-1}. \end{aligned}$$

Considering the series expansion

$$\left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)\right]^{\beta(p+1)-1} = \sum_{q=0}^{\infty} \binom{\beta(p+1) - 1}{q} \left(\lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)\right)^q$$

we get

$$\begin{aligned} f(x; \delta, \lambda, \beta, \xi) &= \delta \lambda^{q+1} \beta g(x, \xi) \sum_{n, m, p, q=0}^{\infty} \binom{\delta(i+1) - 1}{n} \frac{(-1)^{n+p} (n+1)^m}{m!} \binom{m}{p} \\ &\times \binom{\beta(p+1) - 1}{q} G^q(x; \xi) (1 - G(x; \xi))^{-(q+2)}. \end{aligned}$$

Now, applying the series expansion

$$[1 - G(x; \xi)]^{-(q+2)} = \sum_{r=0}^{\infty} \binom{-(q+2)}{r} G^r(x; \xi),$$

we have

$$\begin{aligned} f(x) F(x)^i &= \sum_{m, n, p, k, q=0}^{\infty} \binom{\beta(k+1) - 1}{q} \frac{(m+1)^n (-1)^{k+n}}{p!} \binom{p}{k} \binom{\delta(i+1) - 1}{m} \\ &\times (G(x; \xi))^{k+q} g(x; \xi), \end{aligned}$$

which simplifies to

$$f(x) F(x)^i = \sum_{k, q=0}^{\infty} v_{k, q}^* g_{k+q+1}(x; \xi),$$

where

$$v_{k, q}^* = \sum_{m, n, p, k, q=0}^{\infty} \binom{\beta(k+1) - 1}{q} \frac{(m+1)^p (-1)^{k+n}}{p! (k+q+1)} \binom{p}{k} \binom{\delta(j+i) - 1}{m}$$

and $g_{k+q+1}(x; \xi) = (k+q)g(x; \xi)[G(x; \xi)]^{k+q}$ is an Exp-G distribution with power parameter $(k+q+1)$. The PWM is therefore given by

$$\eta_{j,i} = \sum_{k,q=0}^{\infty} v_{k,q}^* \int_{-\infty}^{\infty} x^j g_{k,q}(x; \xi) dx = \sum_{k,q=0}^{\infty} v_{k,q}^* E(T_{k+q+1}^j),$$

where $T_{k,q}^j$ is j^{th} power of an Exp-G distributed random variable with power parameter $(k+q+1)$.

2.4.5 Quantile Function

We present in this sub-section the quantile function for the EO-NH-G family of distributions by inverting the cdf given by equation (2.5). We invert the function

$$\left(1 - \exp\left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)\right]^\beta\right)\right)^\delta = u,$$

for $0 \leq u \leq 1$. Note that

$$1 - u^{\frac{1}{\delta}} = \exp\left(1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)\right]^\beta\right)$$

so that

$$[1 - \ln(1 - u^{\frac{1}{\delta}})]^{1/\beta} - 1 = \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right),$$

which simplifies to

$$G(x; \xi) = \frac{[1 - \ln(1 - u^{\frac{1}{\delta}})]^{1/\beta} - 1}{[1 - \ln(1 - u^{\frac{1}{\delta}})]^{1/\beta} + \lambda - 1}.$$

The quantile values of the EO-NH-G family of distributions are obtained by solving the non-linear equation

$$x(u) = G^{-1}\left[\frac{[1 - \ln(1 - u^{\frac{1}{\delta}})]^{1/\beta} - 1}{[1 - \ln(1 - u^{\frac{1}{\delta}})]^{1/\beta} + \lambda - 1}\right], \quad (2.21)$$

using iterative methods.

Quantiles for selected parameter values for the EO-NH-LLoG distribution are shown in Table 2.2.

Table 2.2: Table of Quantiles for Selected Parameters of the EO-NH-LLoG Distribution

u	$(\delta, \lambda, \beta, c)$				
	(0.8,0.3,1.3,1.1)	(0.4,1.1,0.7,0.1)	(2.1,1.6,2.7,1.6)	(0.1,0.2,2.5,1.4)	(1.1,1.6,1.7,0.7)
0.1	0.1647	0.0000	0.2076	0.3455	0.0123
0.2	0.3441	0.0000	0.2603	0.5186	0.0311
0.3	0.5195	0.0000	0.2998	0.6521	0.0544
0.4	0.6899	0.0001	0.3340	0.7654	0.0822
0.5	0.8572	0.0001	0.3659	0.8674	0.1153
0.6	1.0254	0.0001	0.3975	0.9637	0.1553
0.7	1.2007	0.0032	0.4309	1.0595	0.2052
0.8	1.3951	0.1442	0.4691	1.1618	0.2720
0.9	1.6413	7.2765	0.5202	1.2868	0.3770

2.5 Maximum Likelihood Estimation

This section covers the maximum likelihood estimation procedures for the EO-NH-G parameters: δ, λ and β , as well as ξ the baseline parameter vector. Let $X_i \sim EO - NH - G(\delta, \lambda, \beta; \xi)$ with the parameter vector $\Delta = (\delta, \lambda, \beta; \xi)^T$, then the log-likelihood for $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\begin{aligned}
\ell &= n \log(\delta \lambda \beta) - 2 \sum_{i=1}^n \log(1 - G(x_i; \xi)) + (\beta - 1) \sum_{i=1}^n \log \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right] \\
&+ \sum_{i=1}^n \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^\beta \right) + \sum_{i=1}^n \log(g(x_i; \xi)) \\
&+ (\delta - 1) \sum_{i=1}^n \log \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^\beta \right) \right).
\end{aligned}$$

The score vector $U = (\frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \xi_k})$ has elements given by

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^n \log \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^\beta \right) \right),$$

$$\begin{aligned}\frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} + (\beta - 1) \sum_{i=1}^n \left(\frac{\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)}}{\left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]} \right) - \beta \sum_{i=1}^n \frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta-1} \\ &+ \beta(\delta - 1) \sum_{i=1}^n \frac{\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta-1} \exp \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \right)}{\left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \right) \right)},\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n \log \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right] \\ &- \sum_{i=1}^n \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \log \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right] \\ &- \sum_{i=1}^n \frac{\left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \log \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right] \exp \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \right)}{\left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \right) \right)}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \xi_k} &= -2 \sum_{i=1}^n \frac{1}{1 - G(x_i; \xi)} \frac{\partial(1 - G(x_i; \xi))}{\partial \xi_k} \\ &+ (\beta - 1) \sum_{i=1}^n \frac{\frac{\partial \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]}{\partial \xi_k}}{1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right)} + \sum_{i=1}^n \frac{1}{g(x_i; \xi)} \frac{\partial g(x_i; \xi)}{\partial \xi_k} \\ &- \sum_{i=1}^n \frac{\frac{\partial \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \right)}{\partial \xi_k}}{\left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \right)} \\ &+ (\delta - 1) \sum_{i=1}^n \frac{\frac{\partial \left(1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \right) \right)}{\partial \xi_k}}{1 - \exp \left(1 - \left[1 + \lambda \left(\frac{G(x_i; \xi)}{\bar{G}(x_i; \xi)} \right) \right]^{\beta} \right)},\end{aligned}$$

respectively.

The elements of the score vector are not in closed-form, so numerical methods will be used to obtain solution to the equations.

2.6 Simulation Study

In this section, we conducted a simulation study to evaluate the consistency of the maximum likelihood estimators. We simulated for N=1000 times with sample size n=35,

70, 140, 280, 560 and 1120 from the EO-NH-W distribution. Simulation results are shown in Table 2.3 and 2.4. The results shows that as the sample size increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs and average bias decays toward zero for all the parameters. For the simulations, we set $\lambda = 1$. The bias and RMSE of $\hat{\theta}$ relative to θ are given by: $Bias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta$, and $RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}}$, respectively.

From 2.3 and 2.4 it can be seen that the means approximates the true parameters values as the sample size n increases. We also observe that as the sample size n increases, the RMSEs decay toward zero and that for all the parameters values, the biases decreases as the sample size n increases.

2.7 Applications

We applied the EO-NH-W model to two real data examples to demonstrate usefulness of the new distribution compared to several other known non-nested distributions. Model performance was assessed using the following goodness-of-fit statistics: -2loglikelihood ($-2 \log L$), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér von Mises (W^*) and Andersen-Darling (A^*) statistic as described by Chen and Balakrishnan [14]. The best model has smaller values of these statistics.

We used **R** software to estimate the model parameters via the `nlm` package. Model parameter estimates (standard errors in parenthesis) and the goodness-of-fit statistics for the two data sets are shown in Tables 2.5 and 2.6. We also present plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [13]) to show how well our model fits the observed data sets. The plots are shown in Figures 2.5 and 2.7.

We compared the EO-NH-W distribution with other competing four parameter non-nested models, namely Gamma-Burr III (GB) distribution by Oluyede et al. [57]), beta-Weibull (BW) by Cordeiro et al. [19], the Kumaraswamy Odd Lindley log-logistic (KOL-LLoG) by Chipepa et al. [16], beta odd Lindley-uniform (BOL-E) by Chipepa et al. [16], the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. [36], Weibull Lomax (WL) by Tahir et al. [64], and beta generalized Lindley (BGL) by

Table 2.3: Monte Carlo Simulation Results for EO-NH-W Distribution: Mean, RMSE and Average Bias

		$\delta = 1.10, \beta = 0.05, \omega = 1.10$			$\delta = 1.10, \beta = 0.05, \omega = 0.50$		
parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
δ	35	2.6584	3.5749	1.5584	2.5245	3.1912	1.4245
	70	1.4623	0.8670	0.3623	1.5100	0.8962	0.4100
	140	1.2944	0.5597	0.1944	1.2979	0.5762	0.1979
	280	1.1771	0.2845	0.0771	1.1660	0.2965	0.0660
	560	1.1177	0.1869	0.0177	1.1372	0.1945	0.0372
	1120	1.1140	0.1398	0.0140	1.1079	0.1411	0.0079
β	35	0.1694	0.2071	0.1194	0.1623	0.2023	0.1123
	70	0.0875	0.0899	0.0375	0.0919	0.0935	0.0419
	140	0.0694	0.0584	0.0194	0.0688	0.0586	0.0188
	280	0.0589	0.0329	0.0089	0.0580	0.0346	0.0080
	560	0.0517	0.0207	0.0017	0.0540	0.0223	0.0040
	1120	0.0516	0.0163	0.0016	0.0509	0.0164	0.0009
ω	35	0.8841	0.4126	-0.2159	0.4107	0.1839	-0.0893
	70	1.0389	0.2984	-0.0611	0.4632	0.1331	-0.0368
	140	1.0756	0.2365	-0.0244	0.4902	0.1077	-0.0098
	280	1.0884	0.1809	-0.0116	0.5001	0.0877	0.0001
	560	1.1096	0.1301	0.0096	0.4987	0.0607	-0.0013
	1120	1.1018	0.0983	0.0018	0.5030	0.0460	0.0030
		$\delta = 1.10, \beta = 0.05, \omega = 0.80$			$\delta = 0.50, \beta = 1.10, \omega = 1.10$		
δ	35	2.6702	3.7498	1.5702	0.9722	2.6932	0.4722
	70	1.5218	0.9312	0.4218	0.6176	0.4106	0.1176
	140	1.3157	0.5703	0.2157	0.5386	0.2406	0.0386
	280	1.1711	0.2760	0.0711	0.5206	0.1509	0.0206
	560	1.1178	0.1916	0.0178	0.5353	0.1140	0.0353
	1120	1.1061	0.1333	0.0061	0.4960	0.0614	-0.0040
β	35	0.1655	0.2081	0.1155	1.1882	0.3156	0.0882
	70	0.0939	0.0975	0.0439	1.1283	0.2499	0.0283
	140	0.0715	0.0595	0.0215	1.1037	0.1699	0.0037
	280	0.0582	0.0324	0.0082	1.0962	0.1239	-0.0038
	560	0.0519	0.0216	0.0019	1.1167	0.0853	0.0167
	1120	0.0507	0.0154	0.0007	1.0934	0.0527	-0.0066
ω	35	0.6619	0.3051	-0.1381	1.3561	0.9969	0.2561
	70	0.7390	0.2181	-0.0610	1.2713	0.7067	0.1713
	140	0.7709	0.1671	-0.0291	1.2305	0.4810	0.1305
	280	0.7927	0.1256	-0.0073	1.1661	0.3224	0.0661
	560	0.8079	0.0991	0.0079	1.0947	0.1836	-0.0053
	1120	0.8043	0.0692	0.0043	1.1336	0.1304	0.0336

Table 2.4: Monte Carlo Simulation Results for EO-NH-W Distribution: Mean, RMSE and Average Bias

parameter	n	$\delta = 0.50, \beta = 0.50, \omega = 1.10$			$\delta = 0.80, \beta = 0.50, \omega = 1.10$		
		Mean	RMSE	Bias	Mean	RMSE	Bias
δ	35	0.9334	2.5344	0.4334	1.4448	2.8755	0.6448
	70	0.6134	0.4057	0.1134	0.9148	0.6359	0.1148
	140	0.5198	0.2377	0.0198	0.8018	0.3470	0.0018
	280	0.5133	0.1550	0.0133	0.7960	0.2240	-0.0040
	560	0.5268	0.1092	0.0268	0.7999	0.1483	-0.0001
	1120	0.5093	0.0725	0.0093	0.7990	0.1040	-0.0010
β	35	0.5843	0.3489	0.0843	0.5626	0.3360	0.0626
	70	0.5287	0.2572	0.0287	0.4922	0.2401	-0.0078
	140	0.4760	0.1943	-0.0240	0.4641	0.1801	-0.0359
	280	0.4818	0.1442	-0.0182	0.4747	0.1278	-0.0253
	560	0.5069	0.0991	0.0069	0.4886	0.0849	-0.0114
	1120	0.4958	0.0679	-0.0042	0.4929	0.0605	-0.0071
ω	35	1.2918	0.8847	0.1918	1.3107	0.9000	0.2107
	70	1.2476	0.6448	0.1476	1.2986	0.6764	0.1986
	140	1.2685	0.5240	0.1685	1.2609	0.4973	0.1609
	280	1.1796	0.3485	0.0796	1.1714	0.2817	0.0714
	560	1.1011	0.1950	0.0011	1.1247	0.1648	0.0247
	1120	1.1094	0.1288	0.0094	1.1111	0.1100	0.0111

Oluyede and Yang [58]. The pdfs of these models are

$$f_{GB}(x; \alpha, \beta, \theta, \delta) = \frac{\beta \delta x^{-\delta-1}}{\Gamma(\alpha) \theta^\alpha} (1 + x^{-\delta})^{-\beta-1} (-\log(1 - (1 + x^{-\delta})^{-\beta}))^{\alpha-1} \\ \times (1 - (1 + x^{-\delta})^{-\beta})^{(1/\theta-1)},$$

for $\alpha, \beta, \theta, \delta > 0$ and $x > 0$,

$$f_{BW}(x; a, b, \alpha, \beta) = \frac{\beta \alpha^\beta}{B(a, b)} x^{\beta-1} e^{-b(\alpha x)^\beta} (1 - e^{-(\alpha x)^\beta})^{a-1},$$

for $a, b, \alpha, \beta > 0$ and $x > 0$,

$$f_{KOL-LLoG}(x; a, b, \lambda, c) = ab \left[\frac{\lambda^2}{(1 + \lambda)} \frac{cx^{c-1}}{(1 + x^c)^{-1}} \exp(-\lambda z) \right] \\ \times \left[1 - \frac{\lambda + ((1 + x^c)^{-1})}{(1 + \lambda)((1 + x^c)^{-1})} \exp(-\lambda z) \right]^{a-1} \\ \times \left(1 - \left[1 - \frac{\lambda + ((1 + x^c)^{-1})}{(1 + \lambda)((1 + x^c)^{-1})} \exp(-\lambda z) \right]^a \right)^{b-1},$$

where $z = \frac{(1 - (1 + x^c)^{-1})}{((1 + x^c)^{-1})}$, $a, b, \lambda, c > 0$,

$$f_{BOL-E}(x; a, b, \lambda, \theta) = \frac{1}{B(a, b)} \left[1 - \frac{\lambda + e^{-\theta x}}{(1 + \lambda)e^{-\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\} \right]^{a-1} \\ \times \left[\frac{\lambda + e^{-\theta x}}{(1 + \lambda)e^{-\theta x}} \exp \left\{ -\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}} \right\} \right]^{b-1} \\ \times \frac{\lambda^2}{(1 + \lambda)} \frac{(\theta e^{-\theta x})}{e^{-3\theta x}} \exp \left\{ -\lambda \frac{1 - e^{-\theta x}}{e^{-\theta x}} \right\},$$

for $a, b, \lambda, \theta > 0$ and $x > 0$,

$$f_{ELOLLW}(x; \beta, \gamma, \theta, \lambda) = \frac{\theta^2 \gamma \lambda^\gamma x^{\gamma-1} e^{-(\lambda x)^\gamma} (e^{-(\lambda x)^\gamma})^{\theta-1} (1 - e^{-(\lambda x)^\gamma})^{\alpha-1}}{(\theta + \beta)((1 - e^{-(\lambda x)^\gamma}) + e^{-(\lambda x)^\gamma})^{\theta-1}} \\ \times \left(1 - \beta \log \left[\frac{e^{-(\lambda x)^\gamma}}{(1 - e^{-(\lambda x)^\gamma}) + e^{-(\lambda x)^\gamma}} \right] \right)$$

for $\beta, \gamma, \theta, \lambda > 0$ and $x > 0$,

$$f_{WL}(x; \alpha, \beta, a, b) = \frac{ab\alpha}{\beta} \left[1 + \left(\frac{x}{\beta} \right) \right]^{b\alpha-1} \left\{ 1 - \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha} \right\}^{b-1} \\ \times \exp \left\{ -a \left\{ \left[1 + \left(\frac{x}{\beta} \right) \right]^\alpha - 1 \right\} \right\}$$

for $\alpha, \beta, a, b > 0$ and $x > 0$, and

$$f_{BGL}(x; a, b, \alpha, \lambda) = \frac{\lambda^2 \alpha (1 + x) \exp(-\lambda x)}{B(a, b)(1 + \lambda)} \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \exp(-\lambda x) \right)^{\alpha(\alpha-1)} \\ \times \left(1 - \left(1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\alpha \right)^{(b-1)},$$

for $a, b, \alpha, \lambda > 0$ and $x > 0$.

2.7.1 Time to Failure of Turbocharger Data

The first data set represents failure times (10^3h) of turbocharger of one type of engine as report by Xu et al [71]. The data are: 1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0 ,8.4, 2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0, 8.7, 8.8, 9.0.

The estimated variance-covariance matrix for the EO-NH-W model based on the turbocharger data set is given by

$$\begin{bmatrix} 1.4010 \times 10^{-4} & -4.4840 \times 10^{-7} & 4.3183 \times 10^{-8} & 4.8804 \times 10^{-4} \\ -4.4840 \times 10^{-7} & 1.5049 \times 10^{-9} & -1.3821 \times 10^{-10} & -1.5621 \times 10^{-6} \\ 4.3183 \times 10^{-8} & -1.3821 \times 10^{-10} & 1.3311 \times 10^{-11} & 1.5043 \times 10^{-7} \\ 4.8804 \times 10^{-4} & -1.5621 \times 10^{-6} & 1.5043 \times 10^{-7} & 1.7001 \times 10^{-3} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$\delta \in [0.8135 \pm 0.0240]$, $\lambda \in [7.7565 \times 10^{-5} \pm 7.6035 \times 10^{-5}]$, $\beta \in [52.384 \pm 7.1508 \times 10^{-6}]$ and $\omega \in [0.8128 \pm 0.0808]$.

Table 2.5: Parameter estimates and goodness-of-fit statistics for various models fitted for time to failure of turbocharger data set

Model	Estimates				Statistics							
	δ	λ	β	ω	$-2 \log L$	AIC	AICC	BIC	W^*	A^*	$K - S$	$p - value$
EO – NH – W	0.8135 (1.1836×10^{-2})	7.7565×10^{-5} (3.8793×10^{-5})	52.384 (3.6484×10^{-6})	0.8128 (4.1232×10^{-2})	158.1015	166.0941	167.2370	172.8497	0.0296	0.1855	0.0703	0.9889
BOLE	a	b	λ	θ	158.9558	166.9558	168.0987	173.7114	0.0343	0.2175	0.0831	0.9450
	0.2105 (0.0371)	0.3251 (0.2604)	0.0003 (0.0002)	1.1911 (0.0088)								
WL	a	b	α	β	163.5453	171.5453	172.6881	178.3008	0.0629	0.4777	0.1036	0.7842
	16.406 (5.9494)	3.3430 (0.44667)	1.2177×10^4 (3.6108×10^{-2})	2.3526×10^5 (1.8688×10^{-3})								
ELOLLW	β	α	θ	γ	163.7752	171.7752	172.9181	178.5308	0.0636	0.4815	0.1017	0.8028
	1.9944 (1.0760)	0.1393 (0.0351)	1.7917 (1.1969)	3.4997 (0.6303)								
KOL-LLoG	a	b	λ	c	164.9051	172.9051	174.0480	179.6607	0.0759	0.5657	0.1061	0.7591
	0.4752 (1.3006×10^{-10})	29.694 (5.3436×10^{-13})	1.5939×10^{-5} (1.7835×10^{-6})	4.0409 (3.0193×10^{-11})								
BGL	a	b	λ	α	456.9146	484.9459	486.0888	491.7014	0.2478	1.6045	0.8324	$< 2.2 \times 10^{-16}$
	1.8000 (2.0586×10^{-6})	1.9000 (6.2739×10^{-6})	4.8030×10^{-7} (3.8617×10^{-6})	1.0936×10^{-2} (1.2564×10^{-3})								
GB	α	β	δ	θ	165.2263	173.2263	174.3692	179.9818	0.3575	2.2146	0.5956	9.5020×10^{-13}
	0.3825 (7.0975×10^{-2})	45.925 (1.1786×10^{-3})	0.7215 (0.2444)	1.3095×10^{-4} (5.6382×10^{-4})								

The EO – NH – W model has the smallest values of $-2 \log L$, W^* and A^* as well as the largest p-value of the $K - S$ statistic compared to the other models presented in Table 7.1. Therefore, we can conclude that the EO – NH – W model provides a better fit for the time to failure of turbocharger data set. Presented in Figure 2.5 is the

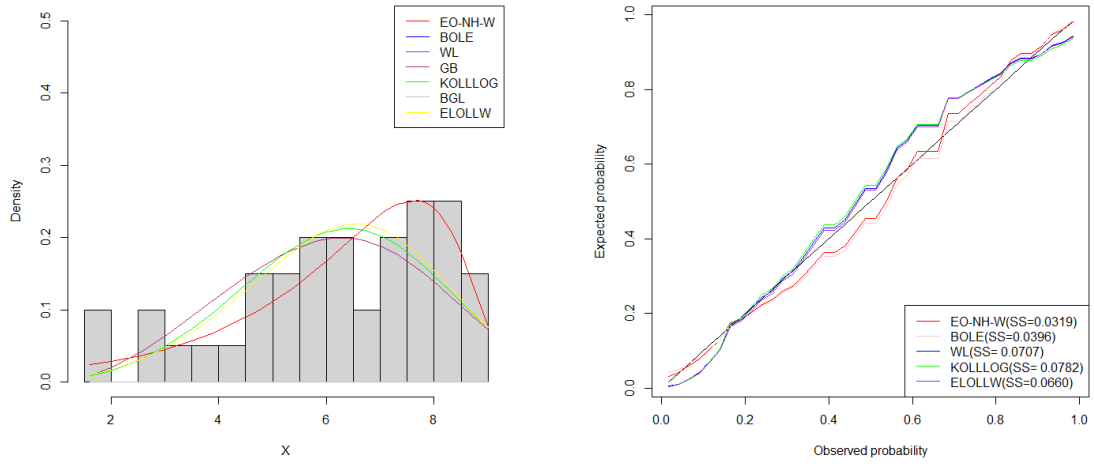


Figure 2.5: Fitted pdfs and probability plots for time to failure of turbocharger data set

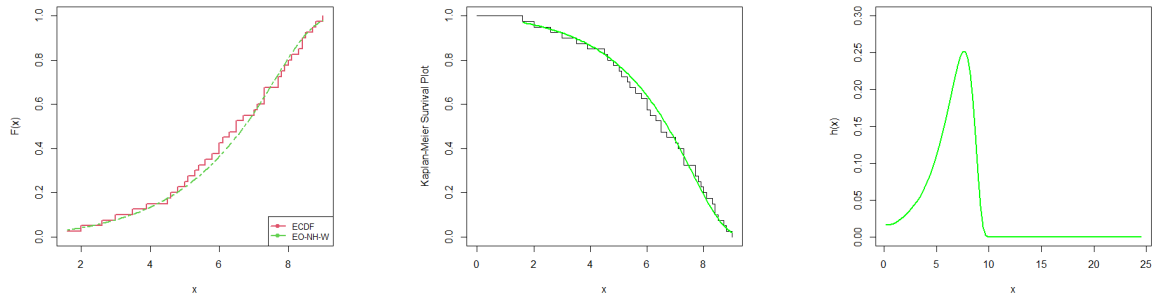


Figure 2.6: Plots for empirical cdf, Kaplan–Meier curve and the hrf of the EO-NH-W model for the time to failure of turbocharger data set

histogram and fitted densities as well as the probability plots. We can conclude that the EO-NH-G distribution fit the time to failure of turbocharger data set better than the models considered in this paper.

2.7.2 Repair Lifetimes of an Airborne Transceiver Data

The second data corresponds to maintenance data with 46 observations based on active repair times (in hours) for an airborne communication transceiver cited by Leiva et al.[41] and Chhikara and Folks [15]. The data is as follows: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 3.3861 \times 10^{-15} & -4.6214 \times 10^{-15} & -1.0155 \times 10^{-11} & 1.3827 \times 10^{-9} \\ -4.6214 \times 10^{-15} & 6.3238 \times 10^{-15} & 2.0177 \times 10^{-11} & -1.8848 \times 10^{-9} \\ -1.0155 \times 10^{-11} & 2.0177 \times 10^{-11} & 2.454 \times 10^{-6} & -3.2629 \times 10^{-6} \\ 1.3827 \times 10^{-9} & -1.8848 \times 10^{-9} & -3.2629 \times 10^{-6} & 5.6490 \times 10^{-4} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$\delta \in [0.1997 \pm 0.2748]$, $\lambda \in [0.0540 \pm 0.3349]$, $\beta \in [0.0157 \pm 0.0863]$ and $\omega \in [2.6925 \pm 3.1048]$.

Table 2.6: Parameter estimates and goodness-of-fit statistics for various models fitted for repair lifetimes of an airborne transceiver data set

Model	Estimates				Statistics							
	δ	λ	β	ω	$-2\log L$	AIC	AICC	BIC	W^*	A^*	$K - S$	$p - value$
EO - NH - W	1.1886×10^4 (5.8190×10^{-8})	6.9153×10^3 (7.9522×10^{-8})	0.2472 (1.5666×10^{-3})	0.2344 (0.0238)	198.9639	206.9639	207.9395	214.2785	0.0421	0.2616	0.0812	0.9224
BOLE	a	b	λ	θ	209.8693	217.8693	218.8449	225.1839	0.1434	0.9948	0.1456	0.2836
	0.9320 (6.0917×10^{-10})	67.7680 (8.4355×10^{-12})	63.2570 (9.1731×10^{-12})	6.1204×10^{-5} (9.3447×10^{-6})								
WL	a	b	α	β	200.6187	208.6187	209.5943	215.9333	0.0565	0.3653	0.0939	0.8121
	1.1881×10^3 (0.0050×10^{-5})	2.6116 (1.1891)	0.0232 (0.0141)	0.1914 (0.3409)								
ELOLLW	β	α	θ	γ	205.3792	213.3792	214.3548	220.6938	0.0983	0.6768	0.1166	0.5592
	1.9588×10^3 (8.4202×10^{-10})	50.924 (3.3971×10^{-5})	0.0865 (3.1982×10^{-2})	0.6301 (6.5918×10^{-2})								
KOL-LLoG	a	b	λ	c	198.8821	206.8821	207.8577	214.1967	0.0386	0.2629	0.0884	0.8651
	11.8395 (10.1581)	0.2140 (0.2319)	4.5046 (2.3042)	0.5529 (0.1909)								
BGL	a	b	λ	α	456.9146	468.4403	469.4160	475.7549	0.6385	0.2665	0.0776	$< 2.2 \times 10^{-16}$
	1.8000 (2.3505×10^{-6})	1.9000 (5.8951×10^{-6})	1.3382×10^{-2} (3.6170×10^{-6})	1.1117×10^{-2} (1.1918×10^{-3})								
GB	α	β	δ	θ	200.0809	208.0809	209.0565	215.3955	0.2284	1.4706	1.0000	$< 2.2 \times 10^{-16}$
	36.9507 (0.0219)	0.9893 (1.5650)	0.2164 (0.2119)	0.0212 (0.0333)								

Furthermore, the results shown in Table 2.6 shows that the EO-NH-W distribution fit the repair lifetimes of an airborne transceiver data set better than the equi-parameter models considered in this paper. The EO-NH-W distribution has smallest values of the the goodness-of-fit statistics A^* and W^* . From the fitted densities plots (Figure 2.5), we can notice that the EO-NH-W model fit the repair lifetimes of an airborne transceiver data set better than the selected models.

2.8 Concluding Remarks

We propose a new family of distributions referred to as the exponentiated odd Nadarajah Haghighi (EO-NH-G) family of distributions. This new distribution is an infinite linear combination of the exponentiated-G distribution. We use the maximum

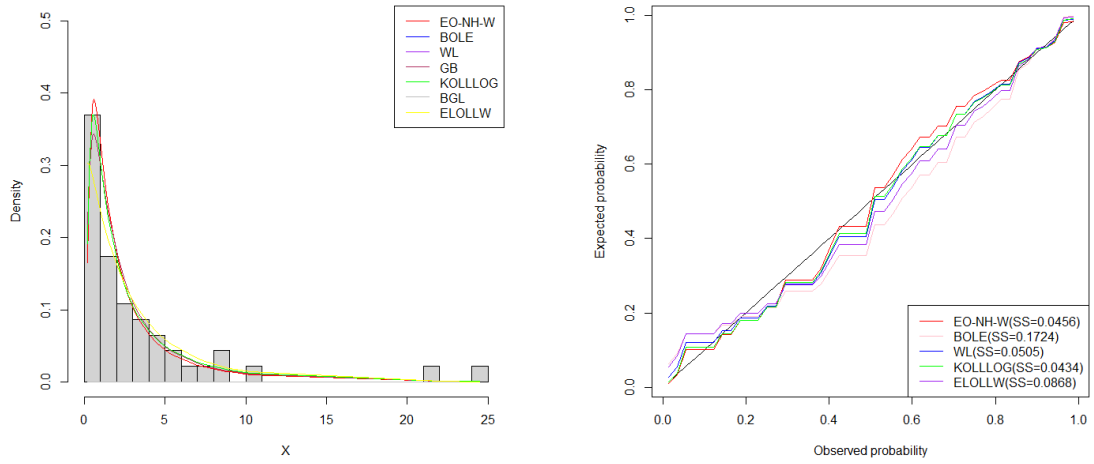


Figure 2.7: Fitted pdfs and probability plots for repair lifetimes of an airborne transceiver data set

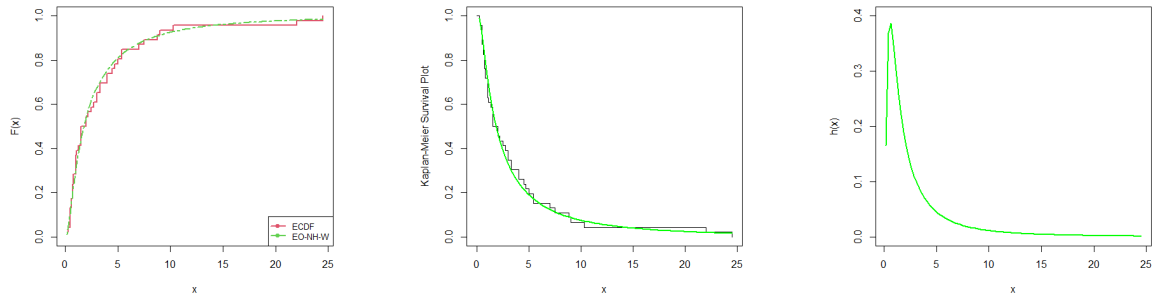


Figure 2.8: Plots for empirical cdf, Kaplan–Meier curve and the hrf of the EO-NH-W model for the repair lifetimes of an airborne transceiver data set

likelihood estimation technique to estimate the model parameters. We applied the special case, the EO-NH-W distribution to two real data sets. Our proposed model performs better than the selected equal-parameter models as shown in the applications that were presented.

Chapter 3

The Exponentiated Odd Exponential Topp-Leone-G Family of Distributions

3.1 Introduction

The Topp-Leone (TL) distribution was originally proposed by Topp and Leone [65] as an alternative to the beta distribution. The TL distribution is well known to be an attractive model for survival analysis and reliability studies since it has a bathtub shaped hazard function which is characteristic of most data from engineering and related areas (Lienig and Bruemmer [43]). Many electronic consumer product life cycles are believed to strongly exhibit the bathtub curve which is a property of the TL distribution. Some generalisations of the TL distribution available in the literature include results given by Nadarajah and Kotz [53], Ghitany et al. [28], Kotz and Nadarajah [37], Kotz and Seier [38], Vicari et al. [72], Zghoul [74], Genç [27] and Bayoud [9], Chipepa et al. [18], just to name a few.

The goal of this study is to develop a new flexible family of distributions called the exponentiated odd exponential Topp-Leone-G (EOE-TL-G) family of distributions using the generalised Weibull model introduced by Gurvich et al. [31] as the generator and study its mathematical properties. Al-Shomrani et al. [4] proposed the Topp

Leone-G (TL-G) family of distributions by generalising the Topp-Leone distribution. The EOE-TL-G has a lot of desirable properties which include various shapes of the probability density function (pdf), unrestricted domain unlike the TL distribution and also it has various shapes of the hazard rate function other than the bathtub. The TL-G family has the cumulative distribution function (cdf) and probability density function, (pdf) given by

$$F_{TL-G}(x; b, \xi) = [1 - \bar{G}^2(x; \xi)]^b \quad (3.1)$$

and

$$f_{TL-G}(x; b, \xi) = 2bg(x; \xi)\bar{G}(x; \xi)[1 - \bar{G}^2(x; \xi)]^{b-1}, \quad (3.2)$$

respectively, where $b > 0$ and $G(x; \xi)$ is a baseline distribution with parameter vector ξ and the survival function $\bar{G}(x; \xi) = 1 - G(x; \xi)$.

This chapter is organized in the following manner. The model and its properties are provided in Section 2. The special cases, order statistics, Rényi entropy and moments generating functions are presented in Section 3. Section 4 covers the Maximum Likelihood estimates, Section 5 contain Monte Carlo simulations and finally Section 6 covers applications.

3.2 The Model and Properties

In this section, we present the proposed model, sub-families and some statistical properties. The statistical properties covered include expansion of the density, order statistics, hazard rate function, quantile function and moments generating functions.

3.2.1 The Model

Several families of univariate distributions have been generated from the Weibull family of distributions introduced by Gurvich et al. [31]. This was achieved by extending the classical Weibull model. The cdf of the generalized Weibull distribution is given by

$$F(x; \alpha, \xi) = 1 - \exp(-\alpha H(x; \xi)), \quad (3.3)$$

where $x \in D \subseteq \mathbb{R}$, $\alpha > 0$. From equation (3.3), we set $\alpha = 1$ and let

$H(x; \xi) = \frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b}$, then the odd exponential Topp-Leone-G (OE-TL-G) family of distributions is given by

$$F_{OE-TL-G}(x; b, \xi) = 1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right]. \quad (3.4)$$

The cdf and pdf of the proposed Exponentiated Odd Exponential Topp-Leone-G (EOE-TL-G) family of distributions are given by

$$F_{EOE-TL-G}(x; b, \delta, \xi) = \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^\delta \quad (3.5)$$

and

$$\begin{aligned} f_{EOE-TL-G}(x; b, \delta, \xi) &= \frac{2\delta b g(x; \xi) \bar{G}(x; \xi) (1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \\ &\times \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\delta-1} \\ &\times \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right], \end{aligned} \quad (3.6)$$

for $b, \delta > 0$ and parameter vector, ξ .

3.2.2 Quantile and Hazard Rate Functions

The quantile function is obtained by solving the non-linear equation

$$\left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^\delta = u$$

for $0 < u < 1$, that is

$$-\ln(1 - u^{\frac{1}{\delta}}) = \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right)$$

so that

$$1 - G(x; \xi) = \sqrt[1 - \left(\frac{\ln(1 - u^{\frac{1}{\delta}})}{\ln(1 - u^{\frac{1}{\delta}}) - 1} \right)^{\frac{1}{b}}]{1 - \left(\frac{\ln(1 - u^{\frac{1}{\delta}})}{\ln(1 - u^{\frac{1}{\delta}}) - 1} \right)^{\frac{1}{b}}}.$$

Consequently, the quantile function or the EOE-TL-G family of distributions is given by

$$Q(u) = G^{-1} \left(1 - \sqrt[1 - \left(\frac{\ln(1 - u^{\frac{1}{\delta}})}{\ln(1 - u^{\frac{1}{\delta}}) - 1} \right)^{\frac{1}{b}}]{1 - \left(\frac{\ln(1 - u^{\frac{1}{\delta}})}{\ln(1 - u^{\frac{1}{\delta}}) - 1} \right)^{\frac{1}{b}}} \right).$$

The hazard rate function (hrf) of the EOE-TL-G family of distributions is given

$$\begin{aligned} h(x; b, \delta, \xi) &= \frac{f_{EOE-TL-G}(x; b, \delta, \xi)}{1 - F_{EOE-TL-G}(x; b, \delta, \xi)} \\ &= \frac{\frac{2\delta b g(x; \xi) \bar{G}(x; \xi) (1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2}}{1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^\delta} \\ &\times \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\delta-1} \\ &\times \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right], \end{aligned}$$

for $b, \delta > 0$ and parameter vector, ξ .

3.2.3 Expansion of Density Function

In this section, we provide a linear representation of the EOE-TL-G family of distributions. Using the series expansion

$$\begin{aligned} \left(1 - \exp \left\{ - \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} \right)^{\delta-1} &= \sum_{a=0}^{\infty} (-1)^a \binom{\delta-1}{a} \\ &\times \exp \left\{ -a \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\}, \end{aligned}$$

the EOE-TL-G pdf can be written as

$$\begin{aligned} f_{EOE-TL-G}(x; b, \delta, \xi) &= \frac{2\delta b g(x; \xi) \bar{G}(x; \xi) (1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \\ &\times \sum_{a=0}^{\infty} (-1)^a \binom{\delta-1}{a} \\ &\times \exp \left\{ - (a+1) \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\}. \end{aligned} \quad (3.7)$$

Using the following series expansion

$$\begin{aligned} \exp \left\{ - (a+1) \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} &= \sum_{m=0}^{\infty} \frac{(-1)^m (a+1)^m}{m!} \\ &\times \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right]^m \end{aligned}$$

and substituting into equation (3.7), we obtain

$$\begin{aligned} f_{EOE-TL-G}(x; b, \delta, \xi) &= \sum_{a,m=0}^{\infty} (-1)^{a+m} \binom{\delta-1}{a} 2\delta b g(x, \xi) \bar{G}(x, \xi) \\ &\times \frac{(1 - \bar{G}^2(x, \xi))^{b-1+bm}}{(1 - (1 - \bar{G}^2(x, \xi))^b)^{2+m}} \frac{(a+1)^m}{m!}. \end{aligned}$$

Applying the generalized binomial expansion

$$[1 - (1 - \bar{G}^2(x; \xi))^b]^{-(m+2)} = \sum_{n=0}^{\infty} (-1)^n \binom{-(m+2)}{n} (1 - \bar{G}^2(x; \xi))^{bn},$$

we can write the EOE-TL-G pdf as

$$\begin{aligned} f_{EOE-TL-G}(x; b, \delta, \xi) &= 2\delta b g(x; \xi) \bar{G}(x; \xi) \sum_{a,m,n=0}^{\infty} (-1)^{a+m+n} \binom{\delta-1}{a} \\ &\times \frac{(a+1)^m}{m!} \binom{-(m+2)}{n} (1 - \bar{G}^2(x; \xi))^{b-1+bm+bn}. \end{aligned}$$

Also, considering the expansion

$$(1 - \bar{G}^2(x; \xi))^{b-1+bm+bn} = \sum_{p=0}^{\infty} (-1)^p \binom{b-1+bm+bn}{p} \bar{G}^{2p}(x; \xi),$$

we can write the pdf of the EOE-TL-G as

$$\begin{aligned} f_{EOE-TL-G}(x; b, \delta; \xi) &= 2\delta b g(x; \xi) \sum_{a,m,n,p=0}^{\infty} (-1)^{a+m+n+p} \binom{\delta-1}{a} \frac{(a+1)^m}{m!} \\ &\times \binom{-(m+2)}{n} \binom{b(m+n+1)-1}{p} \bar{G}^{2p+1}(x; \xi). \end{aligned}$$

Now, with the expansion

$$\bar{G}^{2p+1}(x; \xi) = (1 - G(x; \xi))^{2p+1} = \sum_{q=0}^{\infty} (-1)^q \binom{2p+1}{q} G^q(x; \xi),$$

we have

$$\begin{aligned} f_{EOE-TL-G}(x; b, \delta; \xi) &= 2\delta b g(x; \xi) \sum_{a,m,n,p,q=0}^{\infty} (-1)^{a+m+n+p+q} \binom{\delta-1}{a} \frac{(a+1)^m}{m!} \\ &\times \binom{-(m+2)}{n} \binom{b-1+bm+bn}{p} \binom{2p+1}{q} \\ &\times \left(\frac{q+1}{q+1} \right) G^q(x; \xi) \\ &= \sum_{q=0}^{\infty} v_{q+1} g_{q+1}(x; \xi), \end{aligned}$$

where

$$\begin{aligned} v_{q+1} &= \sum_{a,m,n,p=0}^{\infty} (-1)^{a+m+n+p} \binom{\delta-1}{a} \frac{(a+1)^m}{m!(q+1)} \\ &\times \binom{-(m+2)}{n} \binom{b-1+bm+bn}{p} \binom{2p+1}{q}, \end{aligned} \quad (3.8)$$

and $g_q(x; \xi) = (q+1)g(x; \xi)G^q(x; \xi)$ is an exponentiated-G (Exp-G) distribution with power parameter $(q+1)$. The EOE-TL-G distribution can be written as an infinite linear combination of Exp-G densities and the mathematical properties of the EOE-TL-G family of distributions can be readily obtained from those of the Exp-G family of distributions.

3.2.4 Some Sub-Families of the Distributions

Some sub-families of the EOE-TL-G family of distributions are presented in this subsection.

- When $\delta = 1$, we obtain the Odd Exponential Topp-Leone-G (OE-TL-G) family of distributions.

- When $b = 1$, we obtain a new family of distributions with cdf given by

$$F_{OE-TL-G}(x; \delta, \xi) = \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))}{\bar{G}^2(x; \xi)} \right) \right] \right)^\delta.$$

- If $\delta = b = 1$, we obtain a new family of distributions with the cdf given by

$$F_{OE-TL-G}(x; \xi) = 1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))}{\bar{G}^2(x; \xi)} \right) \right].$$

3.3 Some Special Cases

Some special cases of the EOE-TL-G family of distributions with the baseline distributions being the uniform, Bur XII, Lomax, log-logistic and Weibull distributions are presented in this section.

3.3.1 The Exponentiated Odd Exponential Topp-Leone-Uniform Distribution

Consider the uniform distribution as the baseline distribution. The uniform distribution has pdf and cdf given by $g(x) = 1/\theta$ and $G(x; \theta) = x/\theta$, respectively, for $0 < x < \theta$. Therefore, the exponentiated odd exponential Topp-Leone-Uniform distribution (EOE-TL-U) distribution has the cdf and pdf given by

$$F_{EOE-TL-U}(x; b, \delta, \theta) = \left(1 - \exp \left[- \left(\frac{(1 - (1 - x/\theta)^2)^b}{1 - (1 - (1 - x/\theta)^2)^b} \right) \right] \right)^\delta$$

and

$$\begin{aligned} f_{EOE-TL-U}(x; b, \delta; \theta) &= \frac{2\delta(b/\theta)(1 - x/\theta)(1 - (1 - x/\theta)^2)^{b-1}}{(1 - (1 - (1 - x/\theta)^2)^b)^2} \\ &\times \left(1 - \exp \left[- \left(\frac{(1 - (1 - x/\theta)^2)^b}{1 - (1 - (1 - x/\theta)^2)^b} \right) \right] \right)^{\delta-1} \\ &\times \exp \left[- \left(\frac{(1 - (1 - x/\theta)^2)^b}{1 - (1 - (1 - x/\theta)^2)^b} \right) \right], \end{aligned}$$

respectively, for $\delta, b > 0$ and $0 < x < \theta$. Plots of the density function and hazard rate function (hrf) are given in Figures 3.1(a) and 3.1(b).

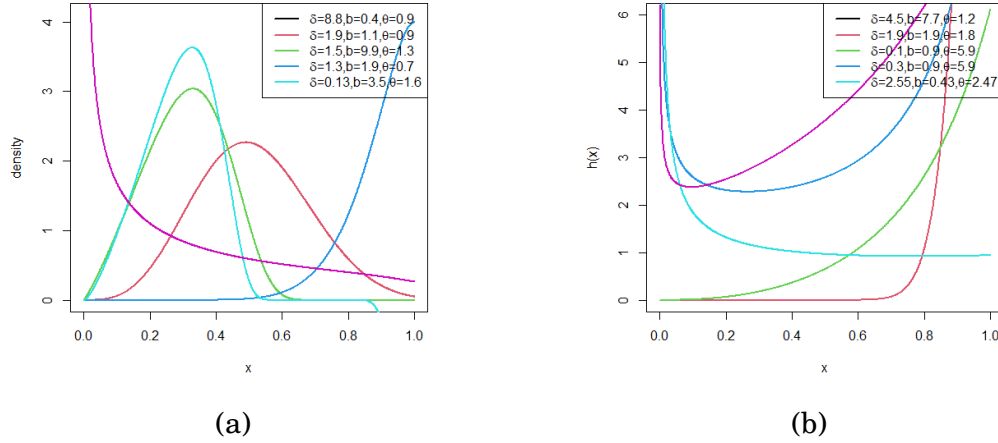


Figure 3.1: Plots of the pdf and hrf for the EOE-TL-U distribution

Figures 3.1 (a) and 3.1 (b) show the plots of the pdfs and hrfs of the EOE-TL-U distribution for selected parameters values. The pdf can take various shapes including uni-modal, almost symmetric, left and right skewed. The hrf of the EOE-TL-U distribution exhibits both monotonic and non-monotonic shapes including increasing, decreasing and bathtub shapes.



Figure 3.2: Plots of skewness and kurtosis of EOE-TL-U distribution
The plot for skewness shows that skewness increases as δ is increasing and b is decreasing for $\theta = 6.9$ while that of Kurtosis shows that kurtosis increases as both δ and b are both increasing for $\theta = 7.8$.

3.3.2 The Exponentiated Odd Exponential Topp-Leone-Burr-XII Distribution

In this subsection, we take the baseline cdf $G(x; \xi)$ to be Burr-XII distribution with cdf $G(x; c, \theta) = 1 - (1 + x^c)^{-\theta}$ and pdf $g(x; \xi) = c\theta x^{(c-1)}(1 + x^c)^{-(\theta+1)}$. Therefore, the exponentiated odd exponential Topp-Leone-Burr-XII distribution (EOE-TL-Burr-XII) has the cdf and pdf given by

$$F_{OE-TL-B}(x; b, \delta, \theta, c) = \left(1 - \exp \left[- \left(\frac{(1 - (1 + x^c)^{-2\theta})^b}{1 - (1 - (1 + x^c)^{-2\theta})^b} \right) \right] \right)^\delta \quad (3.9)$$

and

$$\begin{aligned}
f_{EOE-TL-B}(x; b, \delta, \theta, c) &= \frac{2\delta b c \theta x^{(c-1)} (1+x^c)^{-(2\theta+1)} (1 - (1+x^c)^{-2\theta})^{b-1}}{(1 - (1 - (1+x^c)^{-2\theta})^b)^2} \\
&\times \left(1 - \exp \left[- \left(\frac{(1 - (1+x^c)^{-2\theta})^b}{1 - (1 - (1+x^c)^{-2\theta})^b} \right) \right] \right)^{\delta-1} \\
&\times \exp \left[- \left(\frac{(1 - (1+x^c)^{-2\theta})^b}{1 - (1 - (1+x^c)^{-2\theta})^b} \right) \right], \quad (3.10)
\end{aligned}$$

for $b, c, \delta, \theta > 0$. Plots of the pdf and hrf are given in Figures 3.2(a) and 3.2(b).

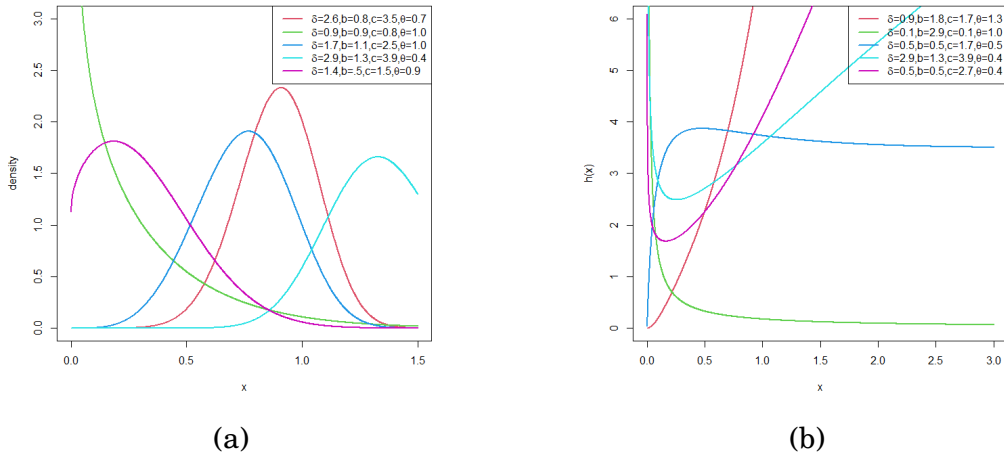


Figure 3.3: Plots of the pdf and hrf for the EOE-TL-Burr-XII distribution

Figures 3.3 (a) and 3.3 (b) show the plots of the pdfs and hrfs of the EOE-TL-Burr-XII distribution for selected parameters values. The pdf can take various shapes including uni-modal, almost symmetric, left and right skewed. The EOE-TL-Burr-XII distribution exhibits bathtub, upside down bathtub, monotonic and non-monotonic hazard rate functions. For $c = 1.1$ and



Figure 3.4: Plots of skewness and kurtosis of EOE-TL-Burr-XII distribution

$\theta = 6.9$, skewness increases as both δ and b increase while for $c = 0.1$ and $\theta = 6.9$, kurtosis increases as δ is increasing and b is decreasing.

The Exponentiated Odd Exponential Topp-Leone-Lomax Distribution

If $c = 1$ in (3.11), the exponentiated odd exponential Topp-Leone-Lomax distribution (EOE-TL-L) has a cdf and pdf given by

$$F_{OE-TL-L}(x; b, \delta, \theta) = \left(1 - \exp \left[- \left(\frac{(1 - (1 + x)^{-2\theta})^b}{1 - (1 - (1 + x)^{-2\theta})^b} \right) \right] \right)^\delta \quad (3.11)$$

and

$$\begin{aligned} f_{OE-TL-L}(x; b, \delta, \theta) &= \frac{2\delta b\theta(1+x)^{-(2\theta+1)}(1 - (1+x)^{-2\theta})^{b-1}}{(1 - (1 - (1+x)^{-2\theta})^b)^2} \\ &\times \left(1 - \exp \left[- \left(\frac{(1 - (1+x)^{-2\theta})^b}{1 - (1 - (1+x)^{-2\theta})^b} \right) \right] \right)^{\delta-1} \\ &\times \exp \left[- \left(\frac{(1 - (1+x)^{-2\theta})^b}{1 - (1 - (1+x)^{-2\theta})^b} \right) \right], \end{aligned}$$

respectively, for b, δ and $\theta > 0$.

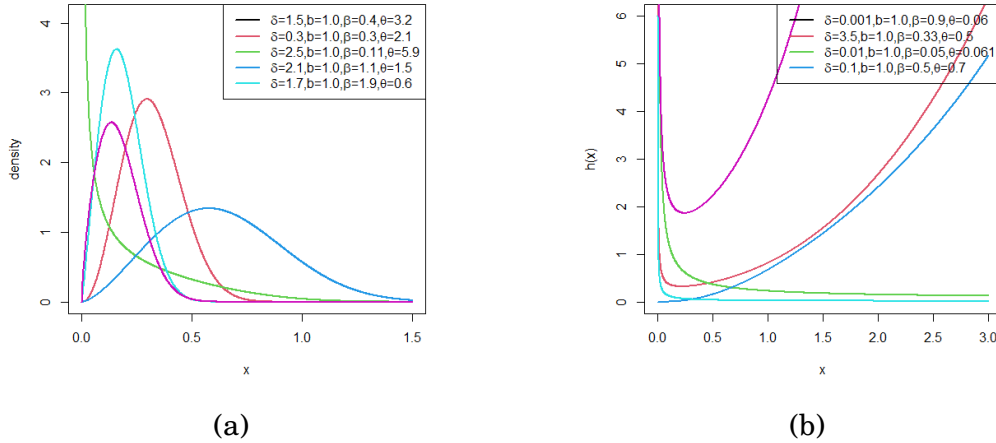


Figure 3.5: Plots of the pdf and hrf for the EOE-TL-L distribution

Figures 3.5 (a) and 3.5 (b) show the plots of the pdfs and hrfs of the EOE-TL-L distribution for selected parameters values. The pdf can take various shapes including reverse-J, uni-modal, almost symmetric and right skewed. The hrf of the EOE-TL-L distribution can exhibit shapes that are both monotonic and non-monotonic including decreasing, increasing and bathtub shapes.

The Exponentiated Odd Exponential Topp-Leone-Log-Logistic Distribution

Consider the log-logistic distribution which is a special case of the Burr-XII distribution with pdf and cdf given by $g(x) = cx^{c-1}(1+x^c)^{-2}$ and $G(x) = 1 - (1+x^c)^{-1}$, for $c > 0$, respectively. Therefore, the exponentiated odd exponential Topp-Leone-log logistic (EOE-TL-LLoG) distribution have the cdf and pdf given by

$$F_{EOE-TL-LLoG}(x; b, \delta, c) = \left(1 - \exp \left[- \left(\frac{(1 - (1 + x^c)^{-2})^b}{1 - (1 - (1 + x^c)^{-2})^b} \right) \right] \right)^\delta$$

and

$$\begin{aligned} f_{EOE-TL-LLoG}(x; b, \delta, c) &= \frac{2\delta b c x^{(c-1)} (1 + x^c)^{-3} (1 - (1 + x^c)^{-2})^{b-1}}{(1 - (1 - (1 + x^c)^{-2})^b)^2} \\ &\times \left(1 - \exp \left[- \left(\frac{(1 - (1 + x^c)^{-2})^b}{1 - (1 - (1 + x^c)^{-2})^b} \right) \right] \right)^{\delta-1} \\ &\times \exp \left[- \left(\frac{(1 - (1 + x^c)^{-2})^b}{1 - (1 - (1 + x^c)^{-2})^b} \right) \right], \end{aligned}$$

respectively, for δ, b and $c > 0$.

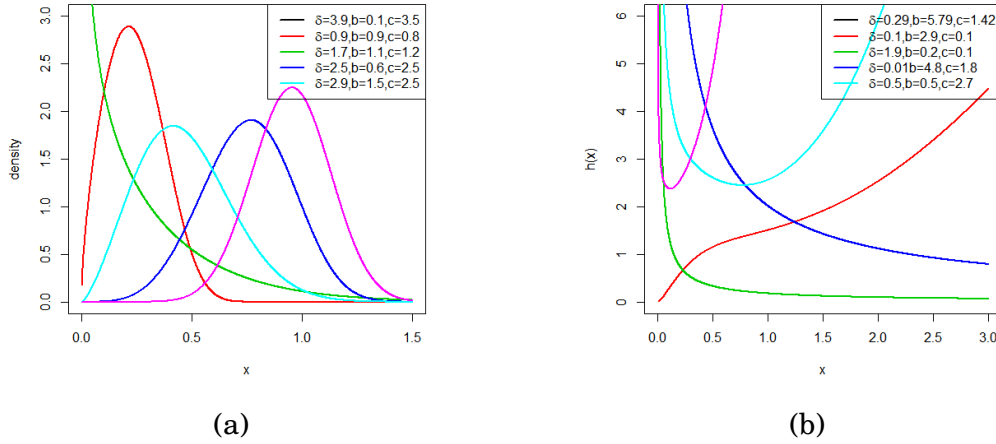


Figure 3.6: Plots of the pdf and hrf for the EOE-TL-LLoG distribution

Figures 3.6 (a) and 3.6 (b) show the plots of the pdfs and hrfs of the EOE-TL-LLoG distribution for selected parameters values. The pdf can take various shapes including reverse-J, uni-modal, left and right skewed. The EOE-TL-LLoG distribution exhibits both monotonic and non-monotonic shapes for the hrf.

3.4 Distribution of Order Statistics and Rényi Entropy

In this section the distribution of the i^{th} order statistics and Rényi entropy of the EOE-TL-G family of distributions are presented. The pdf of the i^{th} order statistics is given by

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} F(x)^{j+i-1}, \quad (3.12)$$

where $B(.,.)$ is the beta function. Substituting equations for the cdf and pdf of the EOE-TL-G distribution into equation (3.12), we have

$$\begin{aligned} f_{i:n}(x) &= \frac{2\delta b g(x, \xi) \bar{G}(x, \xi) (1 - \bar{G}^2(x, \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x, \xi))^b)^2} \\ &\times \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} \exp \left[- \left(\frac{(1 - \bar{G}^2(x, \xi))^b}{1 - (1 - \bar{G}^2(x, \xi))^b} \right) \right] \\ &\times \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x, \xi))^b}{1 - (1 - \bar{G}^2(x, \xi))^b} \right) \right] \right)^{\delta(j+i)-1}. \end{aligned}$$

Using the series expansion

$$\begin{aligned} \left(1 - \exp \left\{ - \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} \right)^{\delta(j+i)-1} &= \sum_{s=0}^{\infty} (-1)^s \binom{\delta(j+i)-1}{s} \\ &\times \exp \left\{ -s \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\}, \end{aligned}$$

we can write

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{j=0}^{n-i} \sum_{s=0}^{\infty} \frac{1}{B(i, n-i+1)} (-1)^s \binom{\delta(j+i)-1}{s} \\ &\times \frac{2b\delta g(x; \xi) \bar{G}(x; \xi) [1 - \bar{G}^2(x; \xi)]^{b-1}}{[1 - (1 - \bar{G}^2(x; \xi))^b]^2} \\ &\times \exp \left\{ - (s+1) \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\}. \end{aligned}$$

Using the expansion

$$\exp \left\{ - (s+1) \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} = \sum_{t=0}^{\infty} \frac{(-1)^t (s+1)^t}{t!} \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right]^t,$$

we have

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{j=0}^{n-i} \sum_{s,t=0}^{\infty} \frac{1}{B(i, n-i+1)} \sum_{s,t=0}^{\infty} \frac{(-1)^{s+t} (s+1)^t}{t!} \binom{\delta(j+i)-1}{s} \\ &\times \frac{2b\delta g(x; \xi) \bar{G}(x; \xi) [1 - \bar{G}^2(x; \xi)]^{bt+b-1}}{[1 - (1 - \bar{G}^2(x; \xi))^b]^{t+2}}. \end{aligned}$$

Considering the generalized binomial series expansion

$$[1 - (1 - \bar{G}^2(x; \xi))^b]^{-(t+2)} = \sum_{r=0}^{\infty} (-1)^r \binom{-(t+2)}{r} (1 - \bar{G}^2(x; \xi))^{br},$$

leads to

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{j=0}^{n-i} \sum_{s,t,r=0}^{\infty} \frac{(-1)^{s+t+r} (s+1)^t}{t!} \binom{\delta(j+i)-1}{s} \binom{-(t+2)}{r} \\ &\times \frac{1}{B(i, n-i+1)} 2b\delta g(x; \xi) \bar{G}(x; \xi) [1 - \bar{G}^2(x; \xi)]^{b(t+r+1)-1}, \end{aligned}$$

and by considering

$$[1 - \bar{G}^2(x; \xi)]^{bt+br+b-1} = \sum_{u=0}^{\infty} (-1)^u \binom{b(t+r+1)-1}{u} \bar{G}^{2u}(x; \xi),$$

yields

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{j=0}^{n-i} \sum_{s,t,r,u=0}^{\infty} \frac{1}{B(i, n-i+1)} \frac{(-1)^{s+t+r+u}(s+1)^t}{t!} \binom{\delta(j+i)-1}{s} \\ &\times \binom{-(t+2)}{r} \binom{b(t+r+1)-1}{u} 2b\delta g(x; \xi) \bar{G}^{2u+1}(x; \xi). \end{aligned}$$

Considering the generalized binomial series expansion

$$\bar{G}^{2u+1}(x; \xi) = (1 - G(x; \xi))^{2u+1} = \sum_{v=0}^{\infty} (-1)^v \binom{2u+1}{v} G^v(x; \xi),$$

we have

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{j=0}^{n-i} \sum_{s,t,r,u,v=0}^{\infty} \frac{(-1)^{s+t+r+u+v}(s+1)^t}{t!} \binom{\delta(j+i)-1}{s} \binom{-(t+2)}{r} \\ &\times \frac{1}{B(i, n-i+1)} \binom{b(t+r+1)-1}{u} \\ &\times 2b\delta \binom{2u+1}{v} \left(\frac{w+1}{w+1} \right) g(x; \xi) G^v(x; \xi). \end{aligned}$$

We can therefore, write the pdf of the i^{th} order statistic from the EOE-TL-G family of distributions as

$$\begin{aligned} f_{i:n}(x) &= \sum_{j=0}^{n-i} \sum_{s,t,r,u,v=0}^{\infty} \frac{(-1)^{s+t+r+u+v}(s+1)^t}{t!} \binom{\delta(j+i)-1}{s} \binom{-(t+2)}{r} \\ &\times \frac{1}{B(i, n-i+1)} \binom{b(t+r+1)-1}{u} 2b\delta \binom{2u+1}{v} \frac{(v+1)}{(v+1)} g(x; \xi) G^v(x; \xi) \\ &= \sum_{v=0}^{\infty} m_{v+1}^* g_{v+1}(x; \xi), \end{aligned}$$

where $g_{v+1}(x; \xi) = (v+1)g(x; \xi)[G(x; \xi)]^v$ is an Exp-G distribution with power parameter $(v+1)$ and

$$\begin{aligned} m_{v+1}^* &= \sum_{j=0}^{n-i} \sum_{s,t,r,u=0}^{\infty} \frac{1}{B(i, n-i+1)} \frac{(-1)^{s+t+r+u+v}(s+1)^t}{t!} \binom{\delta(j+i)-1}{s} \binom{-(t+2)}{r} \\ &\times \binom{b(t+r+1)-1}{u} \binom{2u+1}{v} \frac{2b\delta}{(v+1)}. \end{aligned} \quad (3.13)$$

Therefore, the pdf of the i^{th} order statistics from the EOE-TL-G family of distributions can be expressed as an infinite linear combination of Exp-G densities.

3.4.1 Rényi Entropy

We derive Rényi entropy ($I_R(\nu)$) of the EOE-TL-G family of distributions as follows

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\int_0^\infty f^\nu(x) dx \right], \nu \neq 1, \nu > 0. \quad (3.14)$$

Using equation (4.2), $f^\nu(x; b, \delta; \xi)$, can be written as

$$\begin{aligned} f^\nu(x; b, \delta, \xi) &= \frac{(2b\delta)^\nu g^\nu(x; \xi) \bar{G}(x; \xi)^\nu [1 - \bar{G}^2(x; \xi)]^{\nu(b-1)}}{[1 - (1 - \bar{G}^2(x; \xi))^b]^{2\nu}} \\ &\times \exp \left(-\nu \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right) \\ &\times \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\nu(\delta-1)}. \end{aligned}$$

Using the series expansion

$$\begin{aligned} \left(1 - \exp \left\{ - \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} \right)^{\nu(\delta-1)} &= \sum_{d=0}^{\infty} (-1)^d \binom{\nu(\delta-1)}{d} \\ &\times \exp \left\{ -d \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\}, \end{aligned}$$

we can write

$$\begin{aligned} f^\nu(x; b, \delta, \xi) &= \sum_{d=0}^{\infty} \frac{(2b\delta)^\nu g^\nu(x; \xi) \bar{G}(x; \xi)^\nu [1 - \bar{G}^2(x; \xi)]^{\nu(b-1)}}{[1 - (1 - \bar{G}^2(x; \xi))^b]^{2\nu}} \\ &\times (-1)^d \binom{\nu(\delta-1)}{d} \underbrace{\left(\exp \left[-(d+1) \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)}_A, \end{aligned}$$

where

$$A = \sum_{e=0}^{\infty} \frac{(-1)^e (d+1)^e}{e!} \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right]^e,$$

and substituting into our expansion we obtain

$$\begin{aligned} f^\nu(x; b, \delta, \xi) &= \sum_{d,e=0}^{\infty} \frac{(2b\delta)^\nu g^\nu(x; \xi) \bar{G}(x; \xi)^\nu [1 - \bar{G}^2(x; \xi)]^{\nu(b-1)+be}}{[1 - (1 - \bar{G}^2(x; \xi))^b]^{2\nu+e}} \\ &\times (-1)^{d+e} \binom{\nu(\delta-1)}{d} \frac{(d+1)^e}{e!}. \end{aligned}$$

Applying the generalized binomial series expansion

$$[1 - (1 - \bar{G}^2(x; \xi))^b]^{-(2\nu+e)} = \sum_{f=0}^{\infty} \binom{-(2\nu+e)}{f} (1 - \bar{G}^2(x; \xi))^{bf},$$

we can write

$$f^\nu(x; b, \delta, \xi) = \sum_{d,e,f=0}^{\infty} (2b\delta)^\nu g^\nu(x; \xi) \bar{G}(x; \xi)^\nu \binom{\nu(\delta-1)}{d} \times \binom{-(2\nu+e)}{f} \frac{(-1)^{d+e} (d+1)^e}{e!} [1 - \bar{G}^2(x; \xi)]^{\nu(b-1)+be+bf}.$$

Also, considering the expansion

$$(1 - \bar{G}^2(x; \xi))^{\nu(b-1)+be+bf} = \sum_{g=0}^{\infty} (-1)^g \binom{\nu(b-1)+be+bf}{g} \bar{G}^{2g}(x; \xi),$$

$$f^\nu(x; b, \delta, \xi) = (2b\delta)^\nu g^\nu(x; \xi) \sum_{d,e,f,g=0}^{\infty} (-1)^{d+e+g} \binom{\nu(\delta-1)}{d} \frac{(d+1)^e}{e!} \times \binom{\nu(b-1)+be+bf}{g} \bar{G}^{2g+\nu}(x; \xi).$$

Now, from the generalized binomial series expansion

$$\bar{G}^{2g+\nu}(x; \xi) = (1 - G(x; \xi))^{2g+\nu} = \sum_{h=0}^{\infty} (-1)^h \binom{2g+\nu}{h} G(x; \xi)^h,$$

we have

$$f^\nu(x; b, \delta, \xi) = (2b\delta)^\nu \sum_{d,e,f,g,h=0}^{\infty} \binom{\nu(\delta-1)}{d} \frac{(-1)^{d+e+g+h} (d+1)^e}{e!} \binom{-(2\nu+e)}{f} \times \binom{\nu(b-1)+be+bf}{g} \binom{2g+\nu}{h} \left(\frac{1}{\frac{h}{\nu} + 1} \right)^\nu \times \left(\left(\frac{h}{\nu} + 1 \right) g(x; \xi) G^{\frac{h}{\nu}}(x; \xi) \right)^\nu.$$

Therefore, the Rényi entropy of the EOE-TL-G family of distributions can be written as

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\sum_{h=0}^{\infty} r_h e^{(1-\nu)I_{REG}} \right], \nu \neq 1, \nu > 0,$$

where

$$r_h = \left(\frac{2b\delta}{\frac{h}{\nu} + 1} \right)^\nu \sum_{d,e,f,g=0}^{\infty} \binom{\nu(\delta-1)}{d} \binom{-(2\nu+e)}{f} \times \frac{(-1)^{d+e+g+h} (d+1)^e}{e!} \binom{\nu(b-1)+be+bf}{g} \binom{2g+\nu}{h},$$

and $I_{REG} = (1 - \nu)^{-1} \int_0^\infty \left[\left(\frac{h}{\nu} + 1 \right) g(x; \xi) [G(x; \xi)]^{\frac{h}{\nu}} \right]^\nu dx$ is Rényi entropy of Exp-G distribution with parameter $\left(\frac{h}{\nu} + 1 \right)$. Therefore, the Rényi entropy of the EOE-TL-G family of distributions can be derived directly from Rényi entropy of Exp-G distribution.

3.4.2 Moments and Generating Function

Moments are useful in statistics and probability. The first few moments are useful in describing a probability distribution. We therefore, derive moments of the EOE-TL-G distribution and present some results of the first five moments of the exponentiated odd exponential Topp-Leone-log-logistic (EOE-TL-LLoG) distribution together with the standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of Kurtosis (CK).

From the expansion of the EOE-TL-G density, we conclude that the distribution can be expressed as an infinite linear combination of the Exp-G distribution. The moments of the Exp-G densities are well documented in literature and we will refer to this distribution in the derivations of moments of the EOE-TL-G family of distributions. The EOE-TL-G family of distributions is an infinite linear combination of Y_{q+1} , which has an Exp-G distribution with power parameter $q + 1$. Therefore the r^{th} ordinary moment can be derived using equation (3.15) as follows:

$$\mu'_r = E(X^r) = \sum_{q=0}^{\infty} v_{q+1} E(Y_{q+1}^r), \quad (3.15)$$

where v_{q+1} is given in equation (3.8). Incomplete moments are very useful and can be used to obtain some important quantities such as Lorenz and Bonferroni curves. These quantities have a wide application in demography, economics, insurance, medicine and reliability. The r^{th} incomplete moment of X is given by

$$\phi_r(z) = \int_{-\infty}^z x^r f_{EOE-TL-G}(x; b, \delta; \xi) dx = \sum_{q=0}^{\infty} v_q \int_{-\infty}^z x^r g_{q+1}(x; \xi) dx. \quad (3.16)$$

Quantiles for selected parameter values for the EOE-TL-LLoG distribution are shown in Table 3.1.

Table 3.1: Table of Quantiles for Selected Parameters of the EOE-TL-LLoG Distribution

u	(δ, b, c)				
	(0.1,2.6,1.5)	(0.1,1.2,2.7)	(0.1,0.9,0.7)	(1.1,1.5,2.5)	(0.1,2.5,1.1)
0.1	0.1904	0.1951	0.0304	0.5809	0.1367
0.2	0.0938	0.0953	0.0114	0.4355	0.0649
0.3	0.0601	0.0594	0.0064	0.3426	0.0413
0.4	0.0437	0.0417	0.0043	0.2792	0.0300
0.5	0.0341	0.0314	0.0032	0.2339	0.0234
0.6	0.0279	0.0249	0.0025	0.2001	0.0192
0.7	0.0235	0.0205	0.0021	0.1742	0.0162
0.8	0.0203	0.0172	0.0018	0.1538	0.0140
0.9	0.0179	0.0148	0.0015	0.1374	0.0124



Figure 3.7: Plots of skewness and kurtosis of EOE-TL-LLoG distribution
For $c = 6.9$, skewness increases as both b and δ increase and if $c = 7.8$ kurtosis also increases as both δ and b increase.

3.4.3 Exponentiated Odd Exponential Topp-Leone -Weibull Distribution

Consider the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x; \alpha, \omega) = \alpha \omega x^{\omega-1} e^{-\alpha x^\omega}$ and $G(x; \alpha, \omega) = 1 - e^{-\alpha x^\omega}$, respectively, for $\alpha, \omega > 0$. The cdf and pdf of the exponentiated odd exponential Topp Leone -Weibull (EOE-TL-W) distribution are given by

$$F_{EOE-TL-W}(x; \delta, \alpha, b, \omega) = \left(1 - \exp \left[- \left(\frac{(1 - \exp(-2\alpha x^\omega))^b}{1 - (1 - \exp(-2\alpha x^\omega))^b} \right) \right] \right)^\delta$$

and

$$\begin{aligned} f_{EOE-TL-W}(x; \delta, \alpha, b, \omega) &= \frac{2\delta b \alpha \omega x^{\omega-1} \exp(-\alpha x^\omega) (1 - \exp(-2\alpha x^\omega))^{b-1}}{(1 - (1 - \exp(-2\alpha x^\omega))^b)^2} \\ &\times \left(1 - \exp \left(\left[- \left(\frac{(1 - \exp(-2\alpha x^\omega))^b}{1 - (1 - \exp(-2\alpha x^\omega))^b} \right) \right] \right) \right)^{\delta-1} \\ &\times \exp \left[- \left(\frac{(1 - \exp(-2\alpha x^\omega))^b}{1 - (1 - \exp(-2\alpha x^\omega))^b} \right) \right], \end{aligned}$$

respectively, for δ, α, b and $\omega > 0$.

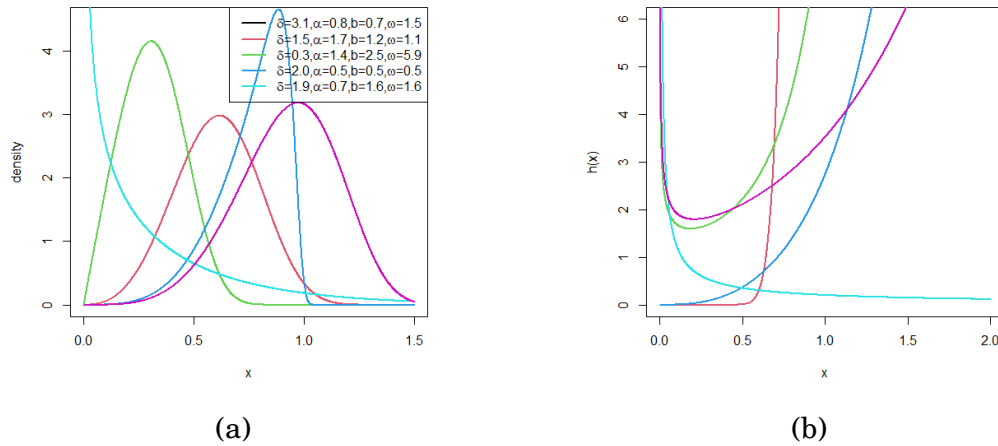


Figure 3.8: Plots of the pdf and hrf for the EOE-TL-W distribution

Figures 3.8 (a) and 3.8 (b) show the plots of the pdfs and hrfs of the EOE-TL-W distribution for selected parameters values. The pdf can take various shapes including reverse-J, uni-modal, left and right skewed. The EOE-TL-W distribution exhibits both monotonic and non-monotonic hazard rate functions.



Figure 3.9: Plots of skewness and kurtosis of EOE-TL-W distribution
For $\omega = 1.27$, both skewness and kurtosis increase as both b and δ increase.

3.5 Maximum Likelihood Estimation

If $X_i \sim EOE - TL - G(\delta, b, \xi)$ with the parameter vector $\Delta = (\delta, b, \xi)^T$, then the total log-likelihood function $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\begin{aligned} \ell &= n \log(2\delta) + n \log(b) + \sum_{i=1}^n \log[g(x_i; \xi)] + \sum_{i=1}^n \log[\bar{G}(x_i; \xi)] \\ &+ (b-1) \sum_{i=1}^n \log[1 - \bar{G}^2(x_i; \xi)] - 2 \sum_{i=1}^n \log[1 - (1 - \bar{G}^2(x_i; \xi))^b] \\ &- \sum_{i=1}^n \left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right] \\ &+ (\delta-1) \sum_{i=1}^n \log \left(1 - \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right] \right) \right). \end{aligned}$$

The score vector $U = (\frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \xi_k})$ has elements given by:

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^n \log \left(1 - \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right] \right) \right),$$

$$\begin{aligned}
\frac{\partial \ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \log[1 - \bar{G}^2(x_i; \xi)] \\
&+ 2 \sum_{i=0}^n \frac{(1 - \bar{G}^2(x_i; \xi))^b \log(1 - \bar{G}^2(x_i; \xi))}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \\
&+ \sum_{i=1}^n \frac{(1 - \bar{G}^2(x_i; \xi))^b \log(1 - \bar{G}^2(x_i; \xi))}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)^2} \\
&- (\delta - 1) \sum_{i=0}^n \frac{(1 - \bar{G}^2(x_i; \xi))^b \log(1 - \bar{G}^2(x_i; \xi))}{((1 - (1 - \bar{G}^2(x_i; \xi))^b))^2} \\
&\times \frac{\exp\left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)}\right]\right)}{\left(1 - \exp\left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)}\right]\right)\right)},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell}{\partial \xi_k} &= \sum_{n=1}^n \frac{1}{g(x_i; \xi)} \frac{\partial g(x_i; \xi)}{\partial \xi_k} + \sum_{i=1}^n \frac{1}{\bar{G}(x_i; \xi)} \frac{\partial \bar{G}(x_i; \xi)}{\partial \xi_k} \\
&- 2(b-1) \sum_{i=1}^n \frac{1}{1 - \bar{G}^2(x_i; \xi)} \frac{\partial}{\partial \xi_k} (1 - \bar{G}^2(x_i; \xi)) \\
&- 2 \sum_{i=1}^n \frac{1}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)^2} \frac{\partial}{\partial \xi_k} (1 - (1 - \bar{G}^2(x_i; \xi))^b) \\
&- \sum_{i=0}^n \frac{(1 - (1 - \bar{G}^2(x_i; \xi))^b)}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)^2} \\
&\times \left[\frac{\partial}{\partial \xi_k} ((1 - \bar{G}^2(x_i; \xi))^b) - (1 - \bar{G}^2(x_i; \xi))^b \frac{\partial}{\partial \xi_k} (1 - (1 - \bar{G}^2(x_i; \xi))^b) \right] \\
&- (\delta - 1) \sum_{i=0}^n \frac{1}{\left(1 - \exp\left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)}\right]\right)\right)} \\
&\times \frac{\partial}{\partial \xi_k} \left(1 - \exp\left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)}\right]\right)\right),
\end{aligned}$$

respectively. These functions are not in closed form and can only be solved using iterative methods using R, MATLAB or SAS software.

For interval estimation for model parameters (δ, b, ξ) and the hypothesis tests on these parameters, we obtained the observed information matrix since its expectation requires numerical intergration. The $(2+p) \times (2+p)$ observed information matrix $J(\Delta)$ is given by

$$J(\Delta) = \begin{pmatrix} J_{\delta\delta}(\Delta) & J_{\delta b}(\Delta) & J_{\delta\xi}(\Delta) \\ J_{b\delta}(\Delta) & J_{bb}(\Delta) & J_{b\xi}(\Delta) \\ J_{\xi\delta}(\Delta) & J_{\xi b}(\Delta) & J_{\xi\xi}(\Delta) \end{pmatrix}, \quad (3.17)$$

where $J_{i,j} = -\frac{\partial^2 \ell(\Delta)}{\partial \delta_i \partial \delta_j}$, for $i, j = \delta, b, \xi$, where ξ is a $(p \times 1)$ component vector. Under the usual regularity conditions $\hat{\Delta}$ is asymptotically normal distributed, that is $\hat{\Delta} \sim N(\underline{0}, I^{-1}(\Delta))$ as $n \rightarrow \infty$, where $I(\Delta)$ is the expected information matrix. The asymptotic behaviour remains valid if $I(\Delta)$ is replaced by $J(\hat{\Delta})$, the information matrix evaluated at $\hat{\Delta}$.

3.6 Simulation Study

In this section, we conducted a simulation study to evaluate the consistency of the maximum likelihood estimators. We simulated $N=1000$ times with sample size $n=35, 70, 140, 280, 560$ and 1120 from the EOE-TL-LLoG distribution. Simulation results are shown in Table 3.2. The bias and Root Mean Square Error (RMSE) for the estimated parameter, say, $\hat{\theta}$, say, are given by

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. The results shows that as the sample size increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs and average bias decays toward zero for all the parameters.

Table 3.2: Monte Carlo Simulation Results for EOE-TL-LLoG Distribution: Mean, RMSE and Average Bias

		I: $\delta = 0.498, b = 1.140, c = 1.140$			II: $\delta = 1.060, b = 0.500, c = 1.060$		
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
δ	35	1.4483	4.7915	0.9503	1.8527	3.3032	0.8577
	70	0.7764	0.7098	0.2784	1.2748	0.6835	0.2798
	140	0.6081	0.3593	0.1101	1.1452	0.3840	0.1502
	280	0.5484	0.2159	0.0504	1.0904	0.2370	0.0954
	560	0.5404	0.1638	0.0424	1.0997	0.1919	0.1047
	1120	0.5097	0.1069	0.0117	1.0609	0.1085	0.0659
b	35	1.0144	0.4428	-0.1256	0.4443	0.1636	-0.0557
	70	1.0754	0.3469	-0.0646	0.4719	0.1232	-0.0281
	140	1.1212	0.2552	-0.0188	0.4772	0.0924	-0.0228
	280	1.1462	0.1951	0.0062	0.4981	0.0612	-0.0019
	560	1.1389	0.1534	-0.0011	0.4986	0.0383	-0.0014
	1120	1.1466	0.1092	0.0066	0.5002	0.0387	0.0002
c	35	1.2909	0.8390	0.1509	1.3325	0.9036	0.2325
	70	1.2166	0.5541	0.0766	1.2211	0.5962	0.1211
	140	1.2052	0.3851	0.0652	1.1650	0.3811	0.0650
	280	1.1712	0.2535	0.0312	1.0940	0.2059	-0.0060
	560	1.1443	0.1913	0.0043	1.0649	0.1328	-0.0351
	1120	1.1555	0.1386	0.0155	1.0714	0.0875	-0.0286
		III: $\delta = 1.050, b = 0.488, c = 1.050$			IV: $\delta = 0.498, b = 1.140, c = 1.140$		
δ	35	1.9631	3.4854	0.9131	1.4373	4.7708	0.9393
	70	1.3781	0.7561	0.3281	0.7836	0.7133	0.2856
	140	1.2186	0.4154	0.1686	0.6081	0.3593	0.1101
	280	1.1739	0.2644	0.1239	0.5478	0.2162	0.0498
	560	1.1890	0.2223	0.1390	0.5403	0.1638	0.0423
	1120	1.1374	0.1260	0.0874	0.5097	0.1069	0.0117
b	35	0.4351	0.1607	-0.0529	1.0161	0.4367	-0.1239
	70	0.4633	0.1180	-0.0247	1.0707	0.3430	-0.0693
	140	0.4675	0.0893	-0.0205	1.1184	0.2530	-0.0216
	280	0.4874	0.0601	-0.0006	1.1466	0.1948	0.0066
	560	0.4951	0.0369	0.0071	1.1396	0.1571	-0.0004
	1120	0.4854	0.0395	-0.0026	1.1466	0.1092	0.0066
c	35	1.2815	0.9066	0.2315	1.2974	0.8189	0.1574
	70	1.1423	0.5599	0.0923	1.2031	0.5340	0.0631
	140	1.1026	0.3481	0.0526	1.2114	0.4158	0.0714
	280	1.0331	0.1972	-0.0169	1.1728	0.2562	0.0328
	560	0.9863	0.1270	-0.0637	1.1440	0.1901	0.0040
	1120	1.0180	0.0907	-0.0320	1.1555	0.1386	0.0155
		V: $\delta = 1.140, b = 0.498, c = 1.140$			VI: $\delta = 1.030, b = 0.528, c = 1.030$		
δ	35	2.1521	4.6649	1.0121	1.9815	4.1107	0.9515
	70	1.4264	0.7893	0.2864	1.3305	0.7616	0.3005
	140	1.2679	0.4246	0.1279	1.1818	0.4082	0.1518
	280	1.2082	0.2520	0.0682	1.1190	0.2483	0.0890
	560	1.2063	0.1845	0.0663	1.1361	0.1952	0.1061
	1120	1.1692	0.1176	0.0292	1.0815	0.1003	0.0515
b	35	0.4455	0.1578	-0.0525	0.4675	0.1657	-0.0605
	70	0.4706	0.1185	-0.0274	0.4997	0.1218	-0.0283
	140	0.4741	0.0894	-0.0239	0.5049	0.0900	-0.0231
	280	0.4897	0.0605	-0.0083	0.5235	0.0614	-0.0045
	560	0.4964	0.0386	-0.0016	0.5252	0.0385	-0.0028
	1120	0.4961	0.0293	-0.0019	0.5324	0.0350	0.0044
c	35	1.3972	0.9888	0.2572	1.2569	0.8694	0.2269
	70	1.2608	0.5593	0.1208	1.1250	0.5140	0.0950
	140	1.2301	0.4003	0.0901	1.0836	0.3295	0.0536
	280	1.1680	0.2232	0.0280	1.0329	0.1894	0.0029
	560	1.1276	0.1355	-0.0124	0.9994	0.1193	-0.0306
	1120	1.1412	0.1038	0.0012	1.0040	0.0830	-0.0260

3.7 Applications

In this section, two real data sets are used to illustrate the usefulness of the proposed new model compared to other competing known models. We use the following goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér von Mises (W^*) and Andersen-Darling (A^*) as described by Chen and Balakrishnan [14], Kolmogorov-Sminov (K-S) and its p-value to assess the best fitting model. The model with the smallest values of these statistics is regarded as the best.

We used the `nlm` function in **R** software to estimate the model parameters and the package **AdequacyModel** in **R** for goodness-of-fit test. Model parameters estimates (standard errors in parenthesis) and the goodness-of-fit statistics for the two data sets are shown in tables 3.3 and 3.4. Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [13]) are also presented to show how well our model fits the observed data sets compared to the selected models. The models considered are type II generalized Topp-Leone-Rayleigh (TIGTLR) and type II generalized Topp-Leone-uniform (TIGTLU) both by Hassan et al. [34]), Marshall-Olkin-log-logistic (MO-LLoG) by Wenhao et al. [73], Marshall-Olkin extended generalized exponential (MOEGE) distribution by Barreto-Souza et al. [8], Topp-Leone-Weibull (TL-W) by Rezaei et al. [66] and the Dagum (D) distribution by Dagum [20]. The pdfs of the non-nested models are as follows:

$$f_{TIGTLR}(x; \alpha, \beta, \theta) = 4\alpha\beta\theta x e^{\theta x^2} [1 - e^{-\theta x^2}]^{2\beta-1} (1 - (1 - e^{-\theta x^2})^{2\beta})^{\alpha-1},$$

for $\alpha, \beta, \theta > 0$,

$$f_D(x; \delta, \beta, \lambda) = \beta\delta\lambda x^{-\delta-1} (1 + \lambda x^{-\beta-1}),$$

for $\delta, \beta, \lambda > 0$,

$$f_{TIGTLU}(x; \alpha, \beta, \theta) = \frac{2\alpha\beta}{\theta} \left(\frac{x}{\theta}\right)^{2\beta-1} \left(1 - \frac{x}{\theta}\right)^{\alpha-1},$$

for $\alpha, \beta, \theta > 0$,

$$f_{TL-W}(x; \alpha, a, b) = 2\alpha b a^b x^{b-1} \exp(-2(ax)^b) (1 - \exp(-2(ax)^b))^{\alpha-1},$$

for $\alpha, a, b > 0$,

$$f_{MO-LLoG}(x; \alpha, \beta, \gamma) = \alpha^\beta \beta \gamma \frac{x^{\beta-1}}{(x^\beta + \gamma \alpha^\beta)^2},$$

for $\alpha, \beta, \gamma > 0$, and

$$f_{MOEGE}(x; \alpha, \gamma, \lambda) = \frac{\alpha \gamma \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\gamma-1}}{(1 - \bar{\alpha} [1 - e^{-\lambda x}]^\gamma)^2},$$

for $\alpha, \gamma, \lambda > 0$.

3.7.1 Failure Times Data

The first data set represent failure times (per 1000h) of 50 components. The data is from Murthy et al. [52]. The data are 0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183, 0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904, 0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228, 3.931, 9.337, 0.086, 0.379, 1.600, 4.073, 10.940, 0.102, 0.381, 2.006, 4.393, 11.020, 0.103, 0.538, 2.054, 4.534, 13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116, 0.574, 3.058, 6.274, 15.08.

Table 3.3: Parameters estimates and goodness-of-fit statistics for various models fitted for failure times data set

Model	Estimates			Statistics							
	δ	b	c	$-2 \log L$	AIC	AICC	BIC	W^*	A^*	$K-S$	$p-value$
<i>EOE-TL-LLoG</i>	0.3555 (0.2284)	7.7748 (4.0922)	0.4790 (0.0786)	201.1584	207.1584	207.6801	212.8945	0.1314	0.8180	0.1271	0.3641
THGTLR	b 2.6390 (2.7737)	δ 0.1396 (0.0295)	c 1.9655×10^{-3} (0.00429)	203.4759	209.4759	209.9976	215.2119	0.1370	0.8857	0.1369	0.2799
D	β 0.0910 (0.0129)	λ 6.7694×10^4 (1.4197×10^{-6})	δ 4.5698 (0.2715)	206.5926	212.5926	213.1144	218.3287	0.1483	1.0121	0.1572	0.1517
<i>TL-W</i>	a 0.0.1180 (0.1420)	b 0.7845 (1.5463)	α 0.7845 (0.9925)	204.7125	210.717	211.2388	216.4531	0.1554	0.9892	0.2930	2.689×10^{-4}
THGTLU	a 1.6795 (6.9473)	b 0.2431 (0.1264)	α 17.2032 (8.7858)	202.0347	208.0347	208.5565	213.7708	17.2664	99.7861	0.9906	8.882×10^{-16}
<i>MO-LLoG</i>	α 0.3938 (0.1182)	β 0.9108 (0.1033)	λ 2.6172 (0.0195)	210.9703	216.9703	217.492	222.7063	0.2138	1.3281	0.1355	0.291
MOEGE	α 2.0188 (1.4968)	γ 0.6664 (0.1573)	λ 0.1521 (0.0608)	203.8013	209.8013	210.3231	215.5374	18.0662	100.2202	0.9964	8.882×10^{-16}

The estimated variance-covariance matrix for EOE-TL-LLoG model on failure times data set is given by

$$\begin{bmatrix} 0.0522 & -0.9097 & -0.0154 \\ -0.9097 & 16.7465 & 0.2717 \\ -0.0154 & 0.2717 & 0.0062 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [0.3555 \pm 0.4476]$, $b \in [7.7748 \pm 8.0208]$ and $c \in [0.4790 \pm 0.1540]$.

Results in Table 3.3 and plots in Figures 3.10 (a) and 3.10 (b) show that the EOE-TL-LLoG distribution performs better than the various non-nested models on failure times.

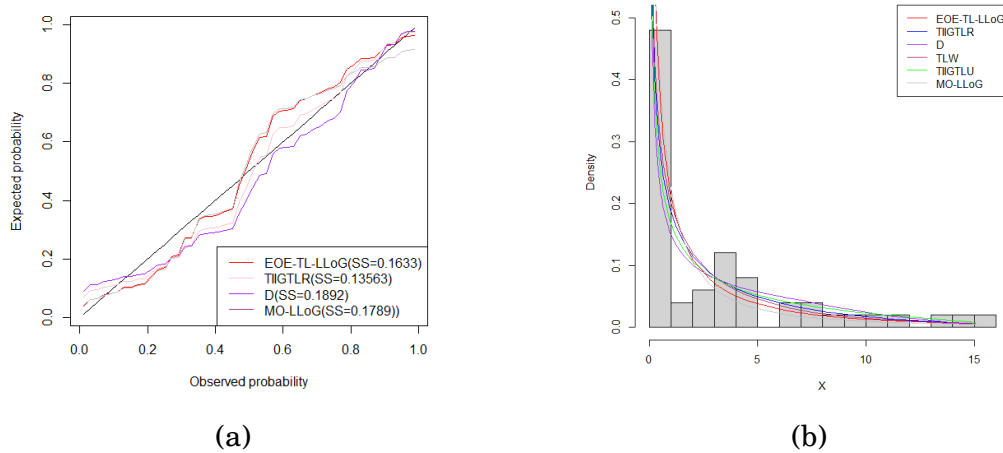


Figure 3.10: Fitted densities and probability plots for Failure times data

The *EOE-TL-LLoG* has the lowest values of $-2 \log(L)$, W^* , A^* , AIC, AICC, KS and the highest p-value therefore we can conclude that the EOE-TL-LLoG model performs better than the six selected non-nested models for failure times data set.

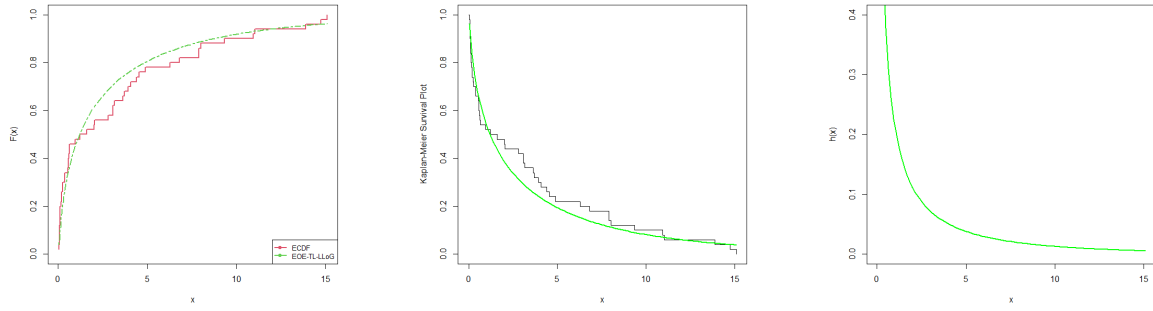


Figure 3.11: Plots for empirical cdf, Kaplan–Meier curve and the hrf of the EOE-TL-LLoG model for the failure times data set

3.7.2 Growth Hormone Data

The data consists of the estimated time since growth hormone medication until the children reached the targeted age. This data was also used by Alizadeh et al.[2] and are as follows: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

Table 3.4: Parameters estimates and goodness of fit statistics for various models fitted for growth hormone data set

Model	Estimates			Statistics							
	δ	b	c	$-2 \log L$	AIC	AICC	BIC	W^*	A^*	$K-S$	$p-value$
<i>EOETL-LoG</i>	2.8414×10^4 (3.7091×10^{-8})	0.4289 (0.0153)	0.1755 (0.02064)	155.0985	161.0985	161.8727	165.7645	0.0344	0.2454	0.0859	0.9585
THGTLR	b 0.2382 (0.3243)	δ 0.9223 (0.7778)	c 0.130 (0.1769)	162.584	168.584	169.3582	173.2501	0.1276	0.8177	0.1707	0.2598
D	β 354.01 (8.4601×10^{-6})	λ 0.06927 (0.02795)	δ 2.4380 (0.3226)	155.4559	161.456	162.2301	166.122	0.0434	0.2881	0.0975	0.8934
<i>TL-W</i>	a 0.0547 (1.1421×10^{-07})	b 0.2198 (2.1537×10^{-3})	α 4.5137×10^4 (5.9602×10^{-10})	155.0936	161.0936	161.8678	165.7596	0.1485	1.0015	0.9997	$< 2.2 \times 10^{-16}$
THGTLU	α 1.5439×10^4 (1.2554×10^{-7})	β 0.9963 (0.01820)	θ 762.40 (5.0664×10^{-6})	164.9781	170.9781	171.7523	170.9781	11.3443	69.1721	0.9941	$< 2.2 \times 10^{-16}$
<i>MO-LLoG</i>	α 3.3212 (0.2844)	β 3.5219 (0.4909)	λ 3.0700 (0.0874)	158.5828	164.5828	165.357	169.2488	0.0560	0.4061	0.0978	0.8911
MOEGE	a 4.4408 (5.8569)	b 6.5817 (2.0173)	α 0.3289 (0.1501)	156.473	162.473	163.2471	167.139	11.6054	69.5788	0.1330	$< 2.2 \times 10^{-16}$

The *EOE-TL-LLoG* has the lowest values of $-2 \log(L)$, W^* , A^* , AIC, AICC, KS and the highest p -value therefore we can conclude that the EOE-TL-LLoG model performs better than the six selected non-nested models on growth hormone data set.

The estimated variance-covariance matrix for EOE-TL-LLoG model on growth hormone data set is given by

$$\begin{bmatrix} 1.3758 \times 10^{-15} & 5.6397 \times 10^{-10} & 7.2131 \times 10^{-10} \\ 5.6397 \times 10^{-10} & 2.3517 \times 10^{-04} & 2.8189 \times 10^{-04} \\ 7.2131 \times 10^{-10} & 2.8189 \times 10^{-04} & 4.2592 \times 10^{-04} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$$\delta \in [2.8414 \times 10^4 \pm 7.2699 \times 10^{-08}], \quad b \in [4.2894 \times 10^{-01} \pm 3.0057 \times 10^{-02}] \quad \text{and} \quad c \in [1.7554 \times 10^{-01} \pm 4.0450 \times 10^{-02}].$$

Results in Table 3.4 and plots in Figures 3.12 (a) and 3.12 (b) show that the EOE-TL-LLoG distribution performs better than the various selected models on growth hormone data.

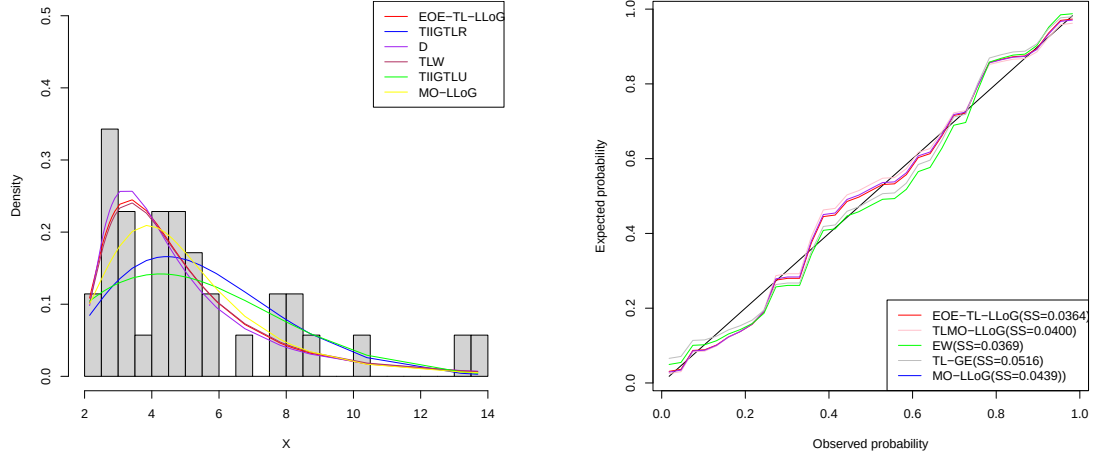


Figure 3.12: Fitted densities and probability plots for growth hormone data

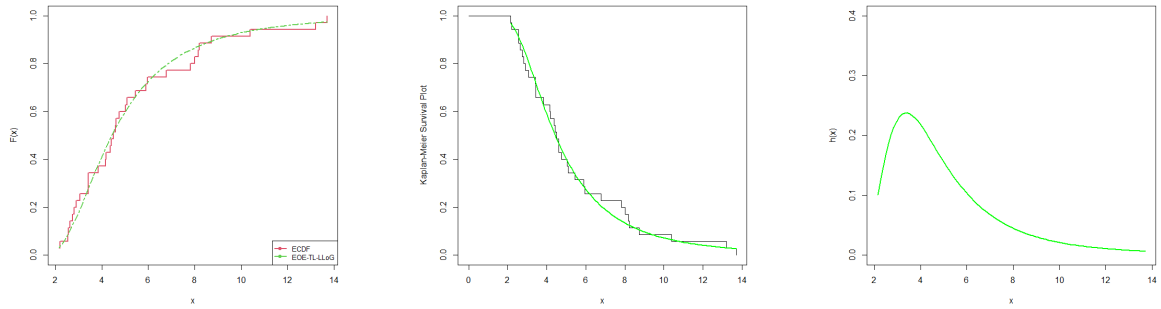


Figure 3.13: Plots for the empirical cdf, Kaplan–Meier curve and the hrf of the EOE-TL-LLoG model for the growth hormone data set

3.8 Concluding Remarks

In this chapter, we have introduced a new family of distributions called the EOE-TL-G distribution. The statistical properties for the EOE-TL-G family of distributions are addressed. The performance of the maximum likelihood estimates is quantified using Monte Carlo simulations. Finally, applications of the model to real data are presented and according to goodness-of-fit statistics, the results indicate that the EOE-TL-G model provides a better fit as compared to the selected models.

Chapter 4

The Exponentiated Odd Exponential Topp-Leone-G Power Series Class of Distributions

4.1 Introduction

The power series distributions are discrete and come from power series and the examples of such distributions are the geometric, logarithmic, binomial and poisson distributions, just to name a few. These are mainly used to model count data. Their inclusion in the classical theories is intended to bring in more flexibility as researchers try to meet the demands of the ever changing data needs in the world. Work done in this respect includes the work by Lu and Shi [45] who proposed the Weibull geometric (WG) and Weibull Poisson (WP) distributions. Furthermore, Rodrigues et al. [68] defined the Weibull negative binomial (WNB) distribution, which includes sub-models such as the WG and WP distributions. Adamidis and Loukas [1] introduced a two-parameter distribution known as exponential geometric (EG) distribution by compounding an exponential distribution with a geometric distribution, Chahkandi and Ganjali [12] proposed the exponential power series (EPS) distribution, the exponentiated exponential Poisson (EEP) distribution by Barreto [6], the Weibull power series (WPS) distribution by Barreto et al. [7] and the complementary exponential-power series distributions by Flores et al. [24], and the odd Weibull Topp-Leone-G power series distributions by Oluyede et al. [59].

We propose the exponentiated odd exponential Topp-Leone-G power series (EOE-TL-GPS) class of distributions. The EOE-TL-GPS has an upside down bathtub which makes it more applicable than the exponentiated odd exponential Topp-Leone-G (EOE-TL-G) family of distributions which has a bathtub only and because of this, the EOE-TL-G distribution does not provide a good parametric fit for modeling phenomenon with failure rates which have an upside bathtub shape. This makes it less popular in reliability and biological studies which exhibits hazard rates which include upside down bathtub. Dimitrakopoulou et al. [21] states that lifetime models that present upside-down bathtub failure rates curves can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

The EOE-TL-GPS has an upside bathtub and this makes it most suited to model lifetime data sets.

4.2 The Model and Properties

In this section, we develop the model EOE-TL-GPS and explore its properties by looking at the hazard rate function, linear representation, special cases, moments, order statistics and Rényi entropy. The cumulative distribution function (cdf) and probability distribution function (pdf) of the exponentiated odd exponential Topp-Leone-G (EOE-TL-G) family of distributions by Chipepa et al. [17] are given by

$$F_{EOE-TL-G}(x; b, \delta, \xi) = \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^\delta \quad (4.1)$$

and

$$\begin{aligned} f_{EOE-TL-G}(x; b, \delta, \xi) &= \frac{2\delta b g(x; \xi) \bar{G}(x; \xi) (1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \\ &\times \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\delta-1} \\ &\times \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right], \end{aligned} \quad (4.2)$$

for $b, \delta > 0$ and vector ξ , $G(x; \xi)$ and $g(x; \xi)$ are the baseline cdf and pdf, respectively. We compound the EOE-TL-G distribution and power series distribution to obtain a new class of distributions, namely, EOE-TL-GPS distribution. Given N , let $X_1, X_2, \dots, X_N$ be identically and independently distributed (iid) random variable following EOE-TL-G distribution. Let $X = X_{(1)} = \min(X_1, \dots, X_N)$. The conditional distribution of X given $N = n$ is given by

$$\begin{aligned} G_{X|N=n}(x; \delta, b, \xi) &= 1 - \prod_{i=1}^n (1 - G(x; \delta, b, \xi)) \\ &= 1 - S^n(x; \delta, b, \xi) \\ &= 1 - \left[1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \delta, b, \xi))^b}{1 - (1 - \bar{G}^2(x; \delta, b, \xi))^b} \right) \right] \right)^\delta \right]^n, \end{aligned}$$

where $S(x; \delta, b, \xi)$ is the survival function given by $1 - F(x; \delta, b, \xi)$. Thus, the cdf of the life length of the whole system, X , say $F(x; \theta, b, \delta, \xi)$, is given by

$$\begin{aligned} F(x; \theta, b, \delta, \xi) &= 1 - \frac{C(\theta S(x; \delta, b, \xi))}{C(\theta)} \\ &= 1 - \frac{C\left(\theta \left[1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \delta, b, \xi))^b}{1 - (1 - \bar{G}^2(x; \delta, b, \xi))^b} \right) \right] \right)^\delta \right]}{C(\theta)}, \end{aligned} \quad (4.3)$$

for $b, \delta > 0$ and parameter vector ξ . Several sub-classes of the EOE-TL-GPS distribution can be obtained by varying the baseline distribution function $G(x; \xi)$ and the power series distribution as shown in Table 4.1.

Table 4.1: Special Cases of the EOE-TL-GPS

Distribution	a_n	$C(\theta)$	cdf
EOE-TL-G Poisson	$(n!)^{-1}$	$e^\theta - 1$	$1 - \frac{\exp\left(\theta\left(1 - \left(1 - \exp\left[-\left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b}\right)\right]\right)^\delta\right)\right)^{-1}}{e^\theta - 1}$
EOE-TL-G Geometric	1	$\theta(1 - \theta)^{-1}$	$1 - \frac{(1-\theta)\left(1 - \left(1 - \exp\left[-\left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b}\right)\right]\right)^\delta\right)}{1-\theta\left(1 - \left(1 - \exp\left[-\left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b}\right)\right]\right)^\delta\right)}$
EOE-TL-G Logarithmic	n^{-1}	$-\log(1 - \theta)$	$1 - \frac{\log\left(1 - \theta\left(1 - \left(1 - \exp\left[-\left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b}\right)\right]\right)^\delta\right)\right)}{\log(1-\theta)}$
EOE-TL-G Binomial	$\binom{m}{n}$	$(1 + \theta)^m - 1$	$1 - \frac{\left(1 + \theta\left(1 - \left(1 - \exp\left[-\left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b}\right)\right]\right)^\delta\right)\right)^m}{(1+\theta)^m - 1}$

Table 4.1 shows some special cases of the EOE-TL-GPS distribution. Let $C'(\theta)$ be the derivative of $C(\theta)$, that is, $C'(\theta) = \sum_{n=1}^{\infty} na_n\theta^{n-1}$. Then the pdf of the *EOE-TL-GPS* distribution is given by

$$f(x; \theta, b, \delta, \xi) = \frac{dF(x; \theta, b, \delta, \xi)}{dx} = \frac{\theta g(x; \xi) C'(\theta S(x))}{C(\theta)}$$

so that

$$\begin{aligned} f(x; \theta, b, \delta, \xi) &= \frac{2\theta\delta bg(x; \xi)\bar{G}(x; \xi)(1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \\ &\times \left(1 - \exp\left[-\left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b}\right)\right]\right)^{\delta-1} \\ &\times \exp\left[-\left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b}\right)\right] \\ &\times \frac{C'\left(\theta\left(1 - \left(1 - \exp\left[-\left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b}\right)\right]\right)^\delta\right)\right)}{C(\theta)}. \end{aligned} \quad (4.4)$$

for $b, \delta > 0$ and parameter vector ξ .

4.2.1 Hazard Rate Function and Quantile Function

The hrf is given by

$$\begin{aligned} h(x; \theta, b, \delta, \xi) &= \frac{2\delta bg(x; \xi)\bar{G}(x; \xi)(1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \\ &\times \left(1 - \exp\left[-\left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b}\right)\right]\right)^{\delta-1} \\ &\times \exp\left[-\left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b}\right)\right] \\ &\times \frac{C'\left(\theta\left(1 - \exp\left[-\left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b}\right)\right]\right)^\delta\right)}{C\left(\theta\left(1 - \left(1 - \exp\left[-\left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b}\right)\right]\right)^\delta\right)\right)}. \end{aligned} \quad (4.5)$$

The quantile function of the EOE-TL-GPS distribution is obtained by inverting $F(x; \theta, b, \delta, \xi) = u$, $0 \leq u \leq 1$. This is equivalent to solving the equation

$$1 - \frac{C\left(\theta \left[1 - \left(1 - \exp\left[-\left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b}\right)\right]^\delta\right)\right]\right)}{C(\theta)} = u.$$

Note that

$$\exp\left[-\left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b}\right)\right] = 1 - \left(1 - \frac{C^{-1}(C(\theta)(1-u))}{\theta}\right)^{\frac{1}{\delta}},$$

so that

$$\bar{G}(x;\xi) = \left(1 - \left(\frac{\ln\left(1 - \left(1 - \frac{C^{-1}(C(\theta)(1-u))}{\theta}\right)^{\frac{1}{\delta}}\right)}{\ln\left(1 - \left(1 - \frac{C^{-1}(C(\theta)(1-u))}{\theta}\right)^{\frac{1}{\delta}}\right) + 1}\right)^{\frac{1}{b}}\right)^{1/2}.$$

Consequently, the quantile function of the EOE-TL-GPS class of distributions is given by

$$Q(u) = G^{-1}\left(\left[1 - \left(\frac{\ln\left(1 - \left(1 - \frac{C^{-1}(C(\theta)(1-u))}{\theta}\right)^{\frac{1}{\delta}}\right)}{\ln\left(1 - \left(1 - \frac{C^{-1}(C(\theta)(1-u))}{\theta}\right)^{\frac{1}{\delta}}\right) + 1}\right)^{\frac{1}{b}}\right]^{1/2}\right). \quad (4.6)$$

Therefore, random numbers for the parameters of the model can be generated using equation (4.6), when the function $G(x;\xi)$ is specified. Consider a special case where $G(x;\xi)$ is Log-Logistic distribution. Table 4.2 shows the quantiles for selected parameter values of the exponentiated odd exponential Topp-Leone-log logistic Poisson (EOE-TL-LLoGP) distribution.

Table 4.2: Table of Quantiles for Selected Parameters of the EOE-TL-LLoGP Distribution

(θ, δ, b, c)					
u	(0.8,0.3,1.3,1.1)	(0.4,1.1,0.7,0.1)	(2.1,1.6,2.7,1.6)	(1.1,0.2,2.5,1.4)	(1.1,1.6,1.7,0.7)
0.1	0.3162	0.5311	1.2116	0.4597	1.2007
0.2	0.2237	0.2873	1.1454	0.3811	1.0598
0.3	0.1632	0.1639	1.0964	0.3221	0.9571
0.4	0.1172	0.0917	1.0533	0.2706	0.8688
0.5	0.0801	0.0481	1.0118	0.2222	0.7861
0.6	0.0501	0.0226	0.9687	0.1747	0.7036
0.7	0.0266	0.0087	0.9206	0.1267	0.6166
0.8	0.0102	0.0023	0.8613	0.0778	0.5180
0.9	0.0016	0.0002	0.7740	0.0304	0.3916

4.2.2 Expansion of Density Function

In this sub-section, we present the series expansion of the EOE-TL-GPS distribution. The pdf in (4.4) can be written as

$$\begin{aligned}
 f(x; \theta, b, \delta, \xi) &= \sum_{n=1}^{\infty} \frac{na_n \theta^n}{C(\theta)} \frac{2\delta b g(x; \xi) \bar{G}(x; \xi) (1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \\
 &\times \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\delta-1} \\
 &\times \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \\
 &\times \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\delta} \right)^{n-1}. \quad (4.7)
 \end{aligned}$$

Using the series expansion

$$\left(1 - \left(1 - \exp \left\{ - \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} \right)^{\delta} \right)^{n-1} = \sum_{a=0}^{\infty} (-1)^a \binom{n-1}{a} A, \quad (4.8)$$

where $A = \left(1 - \exp \left\{ - \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} \right)^{a\delta}$.

Substituting equation (4.8) into equation (4.7), we obtain

$$\begin{aligned}
 f(x; \theta, b, \delta, \xi) &= \sum_{a=0}^{\infty} \sum_{n=1}^{\infty} \frac{na_n \theta^n}{C(\theta)} \frac{2\delta b g(x; \xi) \bar{G}(x; \xi) (1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \\
 &\times (-1)^a \binom{n-1}{a} \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \\
 &\times \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\delta(a+1)-1} \\
 &= \sum_{n,a=1}^{\infty} \frac{na_n \theta^n}{C(\theta)} f_{EOE-TL-G}(x; \delta(a+1), b, \xi). \quad (4.9)
 \end{aligned}$$

The various mathematical properties of the EOE-TL-GPS distribution can be readily obtained from equation (4.2) and the corresponding properties of the EOE-TL-G distribution.

4.2.3 Moments

The r^{th} of the EOE-TL-GPS class of distributions is given by

$$E(X^r) = \int_0^{\infty} x^r f(x; \theta, b, \delta, \xi) dx$$

but from equation (4.9) $f(x; \theta, b, \delta, \xi) = \sum_{n,a=1}^{\infty} \frac{na_n \theta^n}{C(\theta)} f_{EOE-TL-G}(x; \delta(a+1), b, \xi)$

$$E(X^r) = \sum_{n,a=1}^{\infty} \frac{na_n \theta^n}{C(\theta)} \int_t^{\infty} x^r f_{EOE-TL-G}(x; \delta(a+1), b, \xi) dx.$$

4.2.4 Conditional Moments

The r^{th} conditional moments for the EOE-TL-GPS class of distributions are given by

$$E(X^r|X > t) = \frac{1}{1 - F_{EOE-TL-GPS}(t)} \int_0^\infty x^r f(x; \theta, b, \delta, \xi) dx.$$

We present the first five moments together with the standard deviation (SD or σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the EOE-TL-GPS distribution for selected values of the parameters (see Table 4.3, for details).

Table 4.3: Moments of the exponentiated odd exponential Topp-Leone-log logistic Poisson distribution for some parameter values

	(δ, b, c, θ)				
	(1.6,1.7,.7,0.9)	(0.7,1.9,0.3,0.9)	(1.9,1.9,1.9,0.8)	(1.5,1.5,1.5,0.9)	(0.9,2.3,2.5,0.9)
$E(X)$	0.3392	0.1041	0.5056	0.5035	0.5499
$E(X^2)$	0.1995	0.0493	0.3989	0.3386	0.4146
$E(X^3)$	0.1371	0.0318	0.3254	0.2470	0.3270
$E(X^4)$	0.1031	0.0234	0.2726	0.1909	0.2668
$E(X^5)$	0.0821	0.0185	0.2332	0.1538	0.2237
$E(X^6)$	0.0680	0.0153	0.2030	0.1279	0.1916
SD	0.0579	0.0130	0.1792	0.1089	0.1669
CV	0.0504	0.0113	0.1600	0.0946	0.1475
CS	0.0446	0.0100	0.1443	0.0834	0.1318
CK	0.0400	0.0090	0.1313	0.0745	0.1190

4.3 Some Special Cases

Some special cases of the EOE-TL-GPS family of distributions with the baseline distributions being the Weibull, uniform and log-logistic distributions are presented in this section.

4.3.1 The Exponentiated Odd Exponential Topp-Leone-Weibull Poisson (EOE-TL-WP) Distribution

Consider the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x; \omega) = \omega x^{\omega-1} e^{-x^\omega}$ and $G(x; \omega) = 1 - e^{-x^\omega}$, respectively, for $\omega > 0$, the cdf and pdf of the EOE-TL-WP distribution are given by

$$F_{EOE-TL-WP}(x; \theta, b, \delta, \omega) = 1 - \frac{\exp\left(\theta \left(1 - \left(1 - \exp\left[-\left(\frac{(1-\exp(-2x^\omega))^b}{1-(1-\exp(-2x^\omega))^b}\right)\right]\right)^\delta\right)\right) - 1}{e^\theta - 1}$$

and

$$\begin{aligned}
 f_{EOE-TL-WP}(x; \theta, b, \delta, \omega) &= \frac{2\delta\theta b\omega x^{\omega-1} \exp(-x^\omega)(1 - \exp(-2x^\omega))^{b-1}}{(1 - (1 - \exp(-2x^\omega))^b)^2} \\
 &\times \left(1 - \exp\left(\left[-\left(\frac{(1 - \exp(-2x^\omega))^b}{1 - (1 - \exp(-2x^\omega))^b}\right)\right]\right)\right)^{\delta-1} \\
 &\times \exp\left[-\left(\frac{(1 - \exp(-2x^\omega))^b}{1 - (1 - \exp(-2x^\omega))^b}\right)\right] \\
 &\times \frac{\exp\left(\theta\left(1 - \left(1 - \exp\left[-\left(\frac{(1 - \exp(-2x^\omega))^b}{1 - (1 - \exp(-2x^\omega))^b}\right)\right]\right)^\delta\right)\right)}{e^\theta - 1},
 \end{aligned}$$

respectively, for $\delta, \theta, b, \omega > 0$.

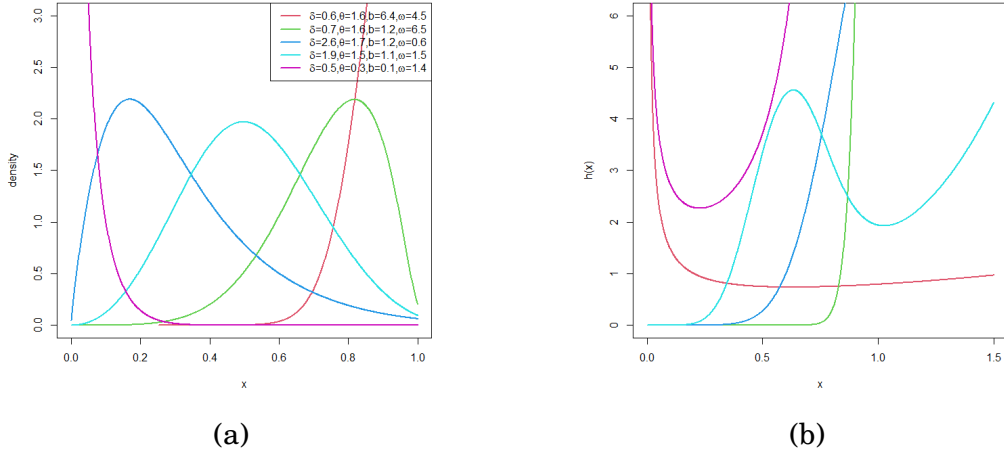


Figure 4.1: Plots of the pdf and hrf for the EOE-TL-WP distribution

Figures 4.1 (a) and 4.1 (b) show the plots of the pdfs and hrfs of the EOE-TL-WP distribution for selected parameters values. The pdf can take various shapes including reverse-J, uni-modal, left and right skewed. While the plot of the EOE-TL-WP hazard function shows that it can be decreasing, increasing, bathtub, upside down bathtub followed by a bathtub for different selected parameter values.

4.3.2 The Exponentiated Odd Exponential Topp-Leone-Weibull Geometric (EOE-TL-WG) Distribution

The cdf and pdf of the EOE-TL-WG distribution are given by

$$F_{EOE-TL-WG}(x; \theta, b, \delta, \omega) = 1 - \frac{(1 - \theta) \left(1 - \exp\left[-\left(\frac{(1 - (1 - e^{-x^\omega})^2)^b}{1 - (1 - (1 - e^{-x^\omega})^2)^b}\right)\right]\right)^\delta}{1 - \theta \left(1 - \exp\left[-\left(\frac{(1 - (1 - e^{-x^\omega})^2)^b}{1 - (1 - (1 - e^{-x^\omega})^2)^b}\right)\right]\right)^\delta}$$

and

$$f(x; \theta, b, \delta, \omega) = \frac{2(1-\theta)\delta b \omega x^{\omega-1} e^{-2x^\omega} (1-e^{-2x^\omega})^{b-1}}{(1-(1-e^{-2x^\omega})^b)^2 \left(1-\theta \left(1-\left(1-\exp\left[-\left(\frac{(1-e^{-2x^\omega})^b}{1-(1-e^{-2x^\omega})^b}\right)\right]\right)^\delta\right)\right)^2} \\ \times \left(1-\exp\left[-\left(\frac{(1-e^{-2x^\omega})^b}{1-(1-e^{-2x^\omega})^b}\right)\right]\right)^{\delta-1} \exp\left[-\left(\frac{(1-e^{-2x^\omega})^b}{1-(1-e^{-2x^\omega})^b}\right)\right],$$

for $\theta, b, \delta, \omega > 0$.

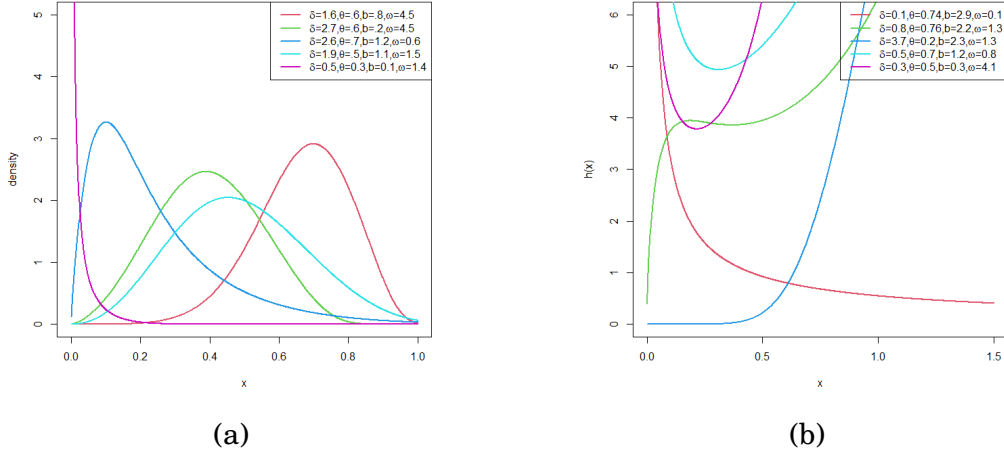


Figure 4.2: Plots of the pdf and hrf for the EOE-TL-WG distribution

Figures 4.2 (a) and 4.2 (b) show the plots of the pdfs and hrfs of the EOE-TL-WG distribution for selected parameters values. The pdf can take various shapes including reverse-J, uni-modal, left and right skewed. While the plot of the EOE-TL-WG hazard rate function shows that it can be decreasing, increasing, upside down bathtub followed by a bathtub for different selected parameter values.

4.3.3 The Exponentiated Odd Exponential Topp-Leone-Log-Logistic Poisson Distribution

If we consider the log-logistic distribution with pdf and cdf given by

$g(x) = cx^{c-1}(1+x^c)^{-2}$ and $G(x) = 1 - (1+x^c)^{-1}$, for $c > 0$, respectively, we obtain the exponentiated odd exponential Topp-Leone-log logistic Poisson (EOE-TL-LLoGP) distribution with cdf and pdf given by

$$F_{EOE-TL-LLoGP}(x; \theta, b, \delta, c) = 1 - \frac{\exp\left(\theta \left(1 - \left(1 - \exp\left[-\left(\frac{(1-(1+x^c)^{-2})^b}{1-(1-(1+x^c)^{-2})^b}\right)\right]\right)^\delta\right)\right) - 1}{e^\theta - 1}$$

and

$$\begin{aligned}
f_{EOE-TL-LLoGP}(x; \theta, b, \delta, c) &= \frac{2\delta\theta bc x^{(c-1)}(1+x^c)^{-3}(1-(1+x^c)^{-2})^{b-1}}{(1-(1-(1+x^c)^{-2})^b)^2} \\
&\times \left(1 - \exp\left[-\left(\frac{(1-(1+x^c)^{-2})^b}{1-(1-(1+x^c)^{-2})^b}\right)\right]\right)^{\delta-1} \\
&\times \exp\left[-\left(\frac{(1-(1+x^c)^{-2})^b}{1-(1-(1+x^c)^{-2})^b}\right)\right] \\
&\times \frac{\exp\left(\theta\left(1 - \left(1 - \exp\left[-\left(\frac{(1-(1+x^c)^{-2})^b}{1-(1-(1+x^c)^{-2})^b}\right)\right]\right)^{\delta}\right)\right)}{e^{\theta} - 1},
\end{aligned}$$

respectively, for $x > 0$ and $\theta, b, c, \delta > 0$.

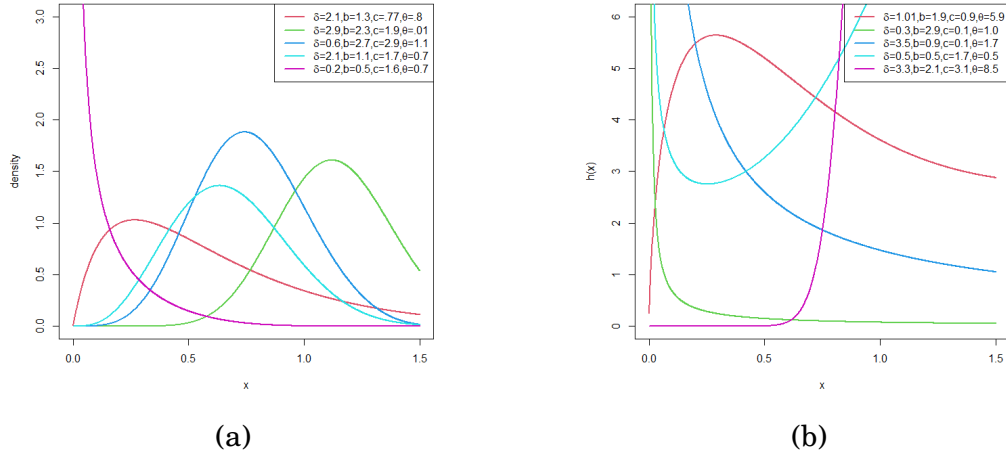


Figure 4.3: Plots of the pdf and hrf for the EOE-TL-LLoGP distribution

Figures 4.3 (a) and 4.3 (b) show the plots of the pdfs and hrfs of the EOE-TL-LLoGP distribution for selected parameters values. The pdf can take various shapes including decreasing, uni-modal, left and right skewed. The EOE-TL-LLoGP distribution exhibits a bathtub, J, monotonic and non-monotonic hazard rate functions.

4.3.4 The Exponentiated Odd Exponential Topp-Leone-Log-Logistic Geometric Distribution (EOE-TL-LLoGG)

The EOE-TL-LLoGG distribution has cdf and pdf given by

$$F_{EOE-TL-LLoGG}(x; \theta, b, \delta, c) = 1 - \frac{(1 - \theta) \left(1 - \exp\left[-\left(\frac{(1-(1+x^c)^{-2})^b}{1-(1-(1+x^c)^{-2})^b}\right)\right]\right)^{\delta}}{1 - \theta \left(\left(1 - \exp\left[-\left(\frac{(1-(1+x^c)^{-2})^b}{1-(1-(1+x^c)^{-2})^b}\right)\right]\right)^{\delta}\right)}$$

and

$$\begin{aligned}
f_{EOE-TL-LLoGG}(x; \theta, b, \delta, c) &= \frac{2(1-\theta)\delta b c x^{c-1} (1+x^c)^{-3} (1-(1+x^c)^{-2})^{b-1}}{(1-(1-(1+x^c)^{-2})^b)^2} \\
&\times \left(1 - \exp \left[- \left(\frac{(1-(1+x^c)^{-2})^b}{1-(1-(1+x^c)^{-2})^b} \right) \right] \right)^{\delta-1} \\
&\times \exp \left[- \left(\frac{(1-(1+x^c)^{-2})^b}{1-(1-(1+x^c)^{-2})^b} \right) \right] \\
&\times \frac{1}{\left(1 - \theta \left(1 - \left(1 - \exp \left[- \left(\frac{(1-(1+x^c)^{-2})^b}{1-(1-(1+x^c)^{-2})^b} \right) \right] \right)^\delta \right) \right)^2},
\end{aligned}$$

for $0 < \theta < 1$, and $b, c, \delta > 0$.

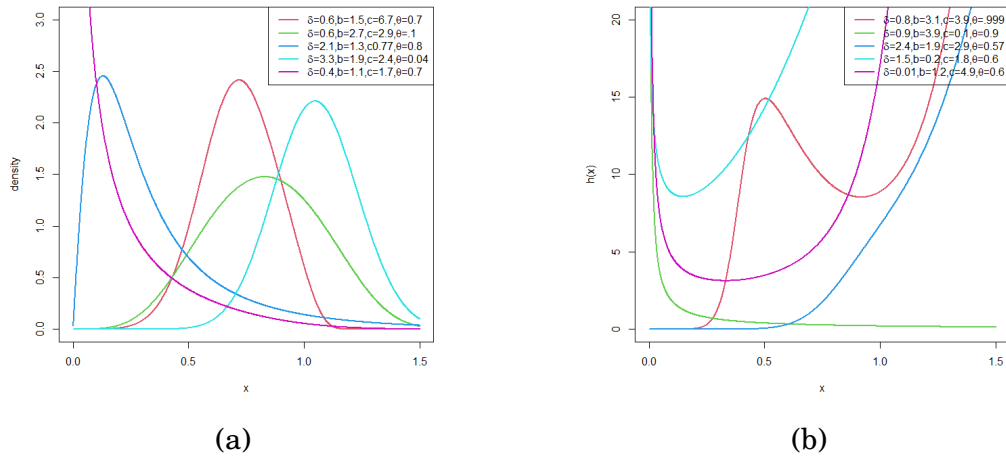


Figure 4.4: Plots of the pdf and hrf for the EOE-TL-LLoGG distribution

Figures 4.4 (a) and 4.4 (b) show the plots of the pdfs and hrfs of the EOE-TL-LLoGG distribution for selected parameters values. The pdf can take various shapes including decreasing, uni-modal, left and right skewed. The EOE-TL-LLoGG distribution exhibits a bathtub, reverse J, increasing and an upside bathtub followed by a bathtub hazard rate functions.

4.4 Order Statistics and Rényi Entropy

In this section, we present the distribution of the order statistic and Rényi entropy for the EOE-TL-GPS family of distributions.

4.4.1 Order Statistics

The pdf of the i^{th} order statistic from the EOE-TL-GPS pdf $f_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)$ is given by

$$\begin{aligned}
g_{i:n}(x) &= \frac{n! f_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)}{(i-1)!(n-i)!} \\
&\times [F_{EOE-TL-GP}(x; \delta, b, \theta, \xi)]^{i-1} [1 - F_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)]^{n-i}.
\end{aligned}$$

Using the generalized binomial expansion

$[1 - F_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)]^{n-i} = \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j [F_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)]^j$,
we get

$$\begin{aligned} g_{i:n}(x) &= \frac{n! f_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \\ &\times [F_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)]^{j+i-1}. \end{aligned} \quad (4.10)$$

Substituting equation (4.3) into equation (4.10), we get

$$\begin{aligned} g_{i:n}(x) &= \frac{n! f_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \\ &\times \left[1 - \frac{C\left(\theta \left[1 - \left(1 - \exp \left[- \left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b} \right) \right] \right)^\delta \right]}{C(\theta)} \right]^{j+i-1}. \end{aligned}$$

Using the generalized binomial expansion

$$\left[1 - \frac{C\left(\theta \left[1 - \left(1 - \exp \left[- \left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b} \right) \right] \right)^\delta \right]}{C(\theta)} \right]^{j+i-1} = \sum_{m=0}^{j+i-1} (-1)^m \binom{j+i-1}{m} B,$$

where $B = \left[\frac{C\left(\theta \left[1 - \left(1 - \exp \left[- \left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b} \right) \right] \right)^\delta \right]}{C(\theta)} \right]^m$, we have

$$\begin{aligned} g_{i:n}(x) &= \frac{n! f_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)}{(i-1)!(n-i)!} \sum_{m=0}^{j+i-1} \sum_{j=0}^{n-i} (-1)^{j+m} \binom{n-i}{j} \binom{j+i-1}{m} \\ &\times \left[\frac{C\left(\theta \left[1 - \left(1 - \exp \left[- \left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b} \right) \right] \right)^\delta \right]}{C(\theta)} \right]^m, \end{aligned} \quad (4.11)$$

by using the power series raised to a positive integer by Gradshteyn and Ryzhik [29],
where $d_{m,w} = (mb_0)^{-1} \sum_{l=1}^m [s(l+1) - m] b_l d_{m-l,w}$ and $d_{0,w} = b_0^w$.

We can write equation (4.11) as

$$\begin{aligned} g_{i:n}(x) &= \frac{n! f_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{j+i-1} \frac{(-1)^{j+m}}{C^m(\theta)} \binom{n-i}{j} \binom{j+i-1}{m} \\ &\times \sum_{w=0}^{\infty} d_{w,m} \theta^w \left[1 - \left(1 - \exp \left[- \left(\frac{(1-\bar{G}^2(x;\xi))^b}{1-(1-\bar{G}^2(x;\xi))^b} \right) \right] \right)^\delta \right]^w. \end{aligned}$$

Using the series expansion

$$\left(1 - \left(1 - \exp \left\{ - \left[\frac{[1 - \bar{G}^2(x;\xi)]^b}{[1 - (1 - \bar{G}^2(x;\xi))^b]} \right] \right\} \right)^\delta \right)^w = \sum_{p=0}^{\infty} (-1)^p \binom{w}{p} A,$$

where $A = \left(1 - \exp \left\{ - \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} \right)^{p\delta}$,
and substituting $f_{EOE-TL-GPS}(x; \delta, b, \theta, \xi)$, we get

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{a,w=0}^{\infty} \sum_{n=1}^{\infty} \sum_{j=0}^{n-i} \sum_{m=0}^{j+i-1} \frac{(-1)^{j+m+p+a}}{C^m(\theta)} \binom{n-i}{j} \binom{j+i-1}{m} \\
&\times \binom{w}{p} \binom{n-1}{a} d_{w,m} \theta^w \frac{n a_n \theta^n}{C(\theta)} \frac{2\delta b g(x; \xi) \bar{G}(x; \xi) (1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \\
&\times \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\delta(a+p+1)-1} \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{a,w=0}^{\infty} \sum_{n=1}^{\infty} \sum_{j=0}^{n-i} \sum_{m=0}^{j+i-1} \frac{(-1)^{j+m+p}}{C^m(\theta)} \binom{n-i}{j} \binom{j+i-1}{m} \\
&\times f_{EOE-TL-G}(x; \delta(a+p+1), b, \xi). \tag{4.12}
\end{aligned}$$

The various mathematical properties of the EOE-TL-GPS distribution can be readily obtained from equation (4.2) and the corresponding properties of the EOE-TL-G distribution.

4.4.2 Rényi Entropy

Rényi entropy is defined by

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\int_0^{\infty} f_{\theta}^{\nu}(x) dx \right], \nu \neq 1, \nu > 0, \tag{4.13}$$

and Shannon entropy is given by $E \{ \log[f(X)] \}$. Shannon entropy is a special case of Rényi entropy, we therefore, derive expressions for Rényi entropy for the EOE-TL-GPS distribution. From equation (4.4), $f^{\nu}(x)$ can be written as

$$\begin{aligned}
f_{EOE-TL-GPS}^{\nu}(x; \theta, \delta, b, \xi) &= \left(\frac{\theta g(x) C'(\theta S(x))}{C(\theta)} \right)^{\nu} \\
&= \left(\frac{2\theta \delta b g(x; \xi) \bar{G}(x; \xi) (1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \right)^{\nu} \\
&\times \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\nu(\delta-1)} \\
&\times \exp \left[-\nu \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \\
&\times \left(\frac{C' \left(\theta \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\delta} \right) \right)}{C(\theta)} \right)^{\nu}.
\end{aligned}$$

Considering the expansion

$$(C'(\theta A))^{\nu} = \sum_{k=0}^{\infty} d_{k,\nu} \theta^k A^k,$$

where $A = \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^\delta \right)$,
 $d_{k,\nu} = (kb_0)^{-1} \sum_{i=1}^k [\nu(l+1) - k] b_l d_{k-1,\nu}$ and $d_{0,\nu} = b_0^\nu$.
Applying the generalized binomial expansion

$$\left(1 - \left(1 - \exp \left\{ - \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} \right)^\delta \right)^\nu = \sum_{t=0}^{\infty} (-1)^t \binom{\nu}{t} B,$$

where $B = \left(1 - \exp \left\{ - \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} \right)^{t\delta}$,
yields

$$\begin{aligned} f_{EOE-TL-GPS}^\nu(x; \theta, \delta, b, \xi) &= \sum_{k,t=0}^{\infty} d_{k,\nu} \theta^{k+1} \left(\frac{2\delta b g(x; \xi) \bar{G}(x; \xi) (1 - \bar{G}^2(x; \xi))^{b-1}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^2} \right)^\nu \\ &\times \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] \right)^{\nu(\delta-1)+t\delta} \\ &\times \exp \left[-\nu \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right] (-1)^t \binom{\nu}{t}. \end{aligned}$$

Furthermore, applying the generalized binomial expansion

$$\begin{aligned} \left(1 - \exp \left(- \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right) \right)^{\delta(\nu+t)-\nu} &= \sum_{s=0}^{\infty} (-1)^s \binom{\delta(\nu+t)-\nu}{s} \\ &\times \exp \left\{ -s \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\}, \end{aligned}$$

we get

$$\begin{aligned} f_{EOE-TL-GPS}^\nu(x; \theta, \delta, b, \xi) &= \sum_{k,t,s=0}^{\infty} (-1)^{t+s} \frac{d_{k,\nu} \theta^{k+1}}{C^\nu(\theta)} (2\delta b) \nu g^\nu(x; \xi) \bar{G}^\nu(x; \xi) \\ &\times \frac{(1 - \bar{G}^2(x; \xi))^{(b-1)\nu}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^{2\nu}} \binom{\nu}{t} \\ &\times \binom{\delta(\nu+t)-\nu}{s} \exp \left[-(\nu+s) \left(\frac{(1 - \bar{G}^2(x; \xi))^b}{1 - (1 - \bar{G}^2(x; \xi))^b} \right) \right]. \end{aligned}$$

Applying series expansion

$$\begin{aligned} \exp \left\{ -(\nu+s) \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right] \right\} &= \sum_{r=0}^{\infty} \frac{(-1)^r (\nu+s)^r}{r!} \\ &\times \left[\frac{[1 - \bar{G}^2(x; \xi)]^b}{[1 - (1 - \bar{G}^2(x; \xi))^b]} \right]^r, \end{aligned}$$

yields

$$\begin{aligned} f_{EOE-TL-GPS}^\nu(x; \theta, \delta, b, \xi) &= \sum_{k,t,s,r=0}^{\infty} (-1)^{t+s+r} \frac{d_{k,\nu} \theta^{k+1}}{C^\nu(\theta)} (2\delta b)^\nu g^\nu(x; \xi) \bar{G}^\nu(x; \xi) \\ &\times \frac{(1 - \bar{G}^2(x; \xi))^{(b-1)\nu+br}}{(1 - (1 - \bar{G}^2(x; \xi))^b)^{2\nu+r}} \binom{\delta(\nu+t)-\nu}{s} \frac{(\nu+s)^r}{r!} \binom{\nu}{t}. \end{aligned}$$

Also applying the generalized binomial expansions

$$[1 - (1 - \bar{G}^2(x; \xi))^b]^{-(2\nu+r)} = \sum_{u=0}^{\infty} (-1)^u \binom{-(2\nu+r)}{u} (1 - \bar{G}^2(x; \xi))^{bu},$$

we can write the expansion as

$$\begin{aligned} f_{EOE-TL-GPS}^{\nu}(x; \theta, \delta, b, \xi) &= \sum_{k,t,s,r,u=0}^{\infty} (-1)^{t+s+r+u} \frac{d_{k,\nu} \theta^k}{C^{\nu}(\theta)} (2\delta b)^{\nu} g^{\nu}(x; \xi) \bar{G}^{\nu}(x; \xi) \\ &\times (1 - \bar{G}^2(x; \xi))^{b(\nu+r+u)-\nu} \binom{\delta(\nu+t)-\nu}{s} \frac{(\nu+s)^r}{r!}. \end{aligned}$$

Also, considering the expansion

$$(1 - \bar{G}^2(x; \xi))^{b(\nu+r+u)-\nu} = \sum_{w=0}^{\infty} (-1)^w \binom{b(\nu+r+u)-\nu}{w} \bar{G}^{2w}(x; \xi),$$

substituting into our expansion

$$\begin{aligned} f_{EOE-TL-GPS}^{\nu}(x; \theta, \delta, b, \xi) &= \sum_{k,t,s,r,u,w=0}^{\infty} (-1)^{t+s+r+u+w} \frac{d_{k,\nu} \theta^k}{C^{\nu}(\theta)} (2\delta b)^{\nu} g^{\nu}(x; \xi) \bar{G}^{\nu+2w}(x; \xi) \\ &\times \binom{\delta(\nu+t)-\nu}{s} \frac{(\nu+s)^r}{r!} \binom{b(\nu+r+u)-\nu}{w}. \end{aligned}$$

Now, from the generalised binomial series expansion

$$\bar{G}^{2w+\nu}(x; \xi) = (1 - G(x; \xi))^{2w+\nu} = \sum_{y=0}^{\infty} (-1)^y \binom{2w+\nu}{y} G(x; \xi)^y,$$

we have

$$\begin{aligned} f_{EOE-TL-GPS}^{\nu}(x; \theta, \delta, b, \xi) &= \sum_{k,t,s,r,u,v,w,y=0}^{\infty} (-1)^{t+s+r+u+w+y} \frac{d_{k,\nu} \theta^{k+1}}{C^{\nu}(\theta)} (2\delta b)^{\nu} \\ &\times \binom{\delta(\nu+t)-\nu}{s} \frac{(\nu+s)^r}{r!} \binom{b(\nu+r+u)-\nu}{w} \binom{2w+\nu}{y} \binom{\nu}{t} \\ &= \sum_{k,t,s,r,u,w,y=0}^{\infty} (-1)^{t+s+r+u+w+y} \frac{(na_n \theta^n)^{\nu}}{C^{\nu}(\theta)} (2\delta b)^{\nu} g^{\nu}(x; \xi) G^y(x; \xi) \\ &\times \binom{\delta(\nu+t)-\nu}{s} \frac{(\nu+s)^r}{r!} \binom{b(\nu+r+u)-\nu}{w} \binom{2w+\nu}{y} \left(\frac{1}{\frac{y}{\nu} + 1} \right)^{\nu} \\ &\times \left(\left(\frac{y}{\nu} + 1 \right) g(x; \xi) G^{\frac{y}{\nu}}(x; \xi) \right)^{\nu}. \end{aligned}$$

Therefore, the Rényi entropy of the EOE-TL-GPS family of distributions can be written as

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\sum_{y=0}^{\infty} \eta_y e^{(1-\nu)I_{REG}} \right], \nu \neq 1, \nu > 0,$$

where

$$\begin{aligned}
\eta_y &= \left(\frac{2b\delta}{\frac{y}{\nu} + 1} \right)^\nu \sum_{k,t,s,r,u,v,w,y=0}^{\infty} \frac{(-1)^{t+s+r+u+w+y} d_{k,\nu} \theta^k}{\left(\frac{y}{\nu} + 1 \right)^\nu C^\nu(\theta)} (2\delta b)^\nu \\
&\times \binom{\delta(\nu+t)-\nu}{s} \frac{(\nu+s)^r}{r!} \binom{b(\nu+r+u)-\nu}{w} \binom{2w+\nu}{y} \binom{\nu}{t} \\
&= \sum_{k,t,s,r,u,v,w,y=0}^{\infty} \frac{(-1)^{t+s+r+u+w+y} (na_n \theta^n)^\nu}{\left(\frac{y}{\nu} + 1 \right)^\nu C^\nu(\theta)} (2\delta b)^\nu g^\nu(x; \xi) G^y(x; \xi) \\
&\times \binom{\delta(\nu+t)-\nu}{s} \frac{(\nu+s)^r}{r!} \binom{b(\nu+r+u)-\nu}{w} \binom{2w+\nu}{y} \binom{\nu}{t}.
\end{aligned}$$

and $I_{REG} = (1 - \nu)^{-1} \int_0^\infty \left[\left(\frac{y}{\nu} + 1 \right) g(x; \xi) [G(x; \xi)]^{\frac{h}{\nu}} \right]^\nu dx$ is Rényi entropy of Exp-G distribution with parameter $\left(\frac{y}{\nu} + 1 \right)$. Therefore, the Rényi entropy of the EOE-TL-GPS family of distributions can be derived directly from Rényi entropy of Exp-G distribution.

4.4.3 Maximum Likelihood Estimation

Let $X_i \sim EOE - TL - GPS(\theta, \delta, b; \xi)$ and $\Delta = (\theta, \delta, b; \xi)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned}
\ell &= n \log(2\delta) + n \log(b) + n \ln \theta + \sum_{i=1}^n \log[g(x_i; \xi)] + \sum_{i=1}^n \log[\bar{G}(x_i; \xi)] \\
&+ (b-1) \sum_{i=1}^n \log[1 - \bar{G}^2(x_i; \xi)] - 2 \sum_{i=1}^n \log[1 - (1 - \bar{G}^2(x_i; \xi))^b] \\
&- \sum_{i=1}^n \left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right] - n \log(C(\theta)) \\
&+ (\delta-1) \sum_{i=1}^n \log \left(1 - \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right] \right) \right) \\
&+ \sum_{i=1}^n \log \left(C' \left(\theta \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right) \right) \right).
\end{aligned}$$

The score vector $U = \left(\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \xi_k} \right)$ has elements given by

$$\begin{aligned}
\frac{\partial \ell}{\partial \theta} &= \frac{n}{\theta} - \frac{n C'(\theta)}{C(\theta)} + \sum_{i=1}^n C'' \left(\theta \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right) \right) \\
&\times \frac{\left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right)}{C' \left(\theta \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right) \right)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \delta} &= \frac{n}{\delta} + \sum_{i=1}^n \log \left(1 - \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \right] \right) \right) \\
&- \delta \theta \sum_{i=1}^n \frac{C'' \left(\left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right) \right)}{C' \left(\theta \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right) \right)} \\
&\times \left(1 - \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \right] \right) \right)^{\delta-1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \log[1 - \bar{G}^2(x_i; \xi)] \\
&+ 2 \sum_{i=0}^n \frac{(1 - \bar{G}^2(x_i; \xi))^b \log(1 - \bar{G}^2(x_i; \xi))}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \\
&+ \sum_{i=1}^n \frac{(1 - \bar{G}^2(x_i; \xi))^b \log(1 - \bar{G}^2(x_i; \xi))}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)^2} \\
&- (\delta - 1) \sum_{i=0}^n \frac{(1 - \bar{G}^2(x_i; \xi))^b \log(1 - \bar{G}^2(x_i; \xi))}{((1 - (1 - \bar{G}^2(x_i; \xi))^b))^2} \\
&\times \frac{\exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \right] \right)}{\left(1 - \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \right] \right) \right)} \\
&+ \delta \theta \sum_{i=1}^n \frac{C'' \left(\left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right) \right)}{C' \left(\theta \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right) \right)} \\
&\times \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right)^{\delta-1} \\
&\times \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \right] \right) \log((1 - \bar{G}^2(x_i; \xi))^b (1 - \bar{G}^2(x_i; \xi))^b),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell}{\partial \xi_k} &= \sum_{n=1}^n \frac{1}{g(x_i; \xi)} \frac{\partial g(x_i; \xi)}{\partial \xi_k} + \sum_{i=1}^n \frac{1}{\bar{G}(x_i; \xi)} \frac{\partial \bar{G}(x_i; \xi)}{\partial \xi_k} \\
&- 2(b-1) \sum_{i=1}^n \frac{1}{1 - \bar{G}^2(x_i; \xi)} \frac{\partial}{\partial \xi_k} (1 - \bar{G}^2(x_i; \xi)) \\
&- 2 \sum_{i=1}^n \frac{1}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)^2} \frac{\partial}{\partial \xi_k} (1 - (1 - \bar{G}^2(x_i; \xi))^b) \\
&- \sum_{i=0}^n \frac{(1 - (1 - \bar{G}^2(x_i; \xi))^b)}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)^2} \\
&\times \left[\frac{\partial}{\partial \xi_k} ((1 - \bar{G}^2(x_i; \xi))^b) - (1 - \bar{G}^2(x_i; \xi))^b \frac{\partial}{\partial \xi_k} (1 - (1 - \bar{G}^2(x_i; \xi))^b) \right] \\
&- (\delta - 1) \sum_{i=0}^n \frac{1}{\left(1 - \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \right] \right) \right)} \\
&\times \frac{\partial}{\partial \xi_k} \left(1 - \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \right] \right) \right) \\
&+ \delta \theta \sum_{i=1}^n \frac{C'' \left(\left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right) \right)}{C' \left(\theta \left(1 - \left(1 - \exp \left[- \left(\frac{(1 - \bar{G}^2(x_i; \xi))^b}{1 - (1 - \bar{G}^2(x_i; \xi))^b} \right) \right] \right)^\delta \right) \right)} \\
&\times \frac{\partial}{\partial \xi_k} \left(1 - \left(1 - \exp \left(\left[\frac{(1 - \bar{G}^2(x_i; \xi))^b}{(1 - (1 - \bar{G}^2(x_i; \xi))^b)} \right] \right) \right)^\delta \right),
\end{aligned}$$

respectively. By letting $U = (\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \xi_k}) = 0$ will result in the maximum likelihood estimates. Solving these equations may require the use of iterative methods such as the Newton-Raphson method or making use of statistical packages such as R.

For interval estimation for model parameters (θ, δ, b, ξ) and the hypothesis tests on these parameters, we obtain the observed information matrix since its expectation requires numerical integration. The $(3+p) \times (3+p)$ observed information matrix $J(\Delta)$ is given by

$$J(\Delta) = \begin{pmatrix} J_{\theta\theta}(\Delta) & J_{\theta\delta}(\Delta) & J_{\theta b}(\Delta) & J_{\theta\xi}(\Delta) \\ J_{\delta\theta}(\Delta) & J_{\delta\delta}(\Delta) & J_{\delta b}(\Delta) & J_{\delta\xi}(\Delta) \\ J_{b\theta}(\Delta) & J_{b\delta}(\Delta) & J_{bb}(\Delta) & J_{b\xi}(\Delta) \\ J_{\xi\theta}(\Delta) & J_{\xi\delta}(\Delta) & J_{\xi b}(\Delta) & J_{\xi\xi}(\Delta) \end{pmatrix}, \quad (4.14)$$

where $J_{i,j} = \frac{-\partial^2 \ell(\Delta)}{\partial i \partial j}$, for $i, j = \delta, b, \xi$, and ξ is a $p \times 1$ component vector. Under the usual regularity conditions $\hat{\Delta}$ is asymptotically normal distributed, that is $\hat{\Delta} \sim N(0, I^{-1}(\Delta))$ as $n \rightarrow \infty$, where $I(\Delta)$ is the expected information matrix. The asymptotic behaviour remains valid if $I(\Delta)$ is replaced by $J(\hat{\Delta})$, the information matrix evaluated at $\hat{\Delta}$.

Table 4.4: Monte Carlo Simulation Results for EOE-TL-LLoGP Distribution: Mean, RMSE and Average Bias

		II: $\delta = 1.3, b = 2.3, c = 2.3, \theta = 1.3$			III: $\delta = 0.4, b = 2.9, c = 2.9, \theta = 0.4$		
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
δ	30	1.1036	1.6971	-0.1964	0.4339	0.3782	0.0339
	60	1.1192	1.2708	-0.1808	0.4734	0.3247	0.0734
	120	1.2336	1.0862	-0.0664	0.4768	0.3030	0.0768
	240	1.2116	0.8769	-0.0884	0.4814	0.2947	0.0814
	480	1.2301	0.6817	-0.0699	0.4870	0.2643	0.0870
	960	1.2540	0.4906	-0.0460	0.4715	0.2133	0.0715
b	30	4.0334	3.0019	1.7334	3.9302	1.9927	1.0302
	60	3.7054	2.6728	1.4054	3.4435	1.3454	0.5435
	120	3.2733	2.2398	0.9733	3.2308	1.1165	0.3308
	240	3.0889	1.8601	0.7889	3.1050	0.9863	0.2050
	480	2.8765	1.5588	0.5765	2.9172	0.7683	0.0172
	960	2.5955	0.9307	0.2955	2.8607	0.6771	-0.0393
c	30	2.9315	1.0683	0.6315	3.2108	0.9283	0.3108
	60	2.7397	0.8723	0.4397	2.9977	0.6570	0.0977
	120	2.5713	0.7354	0.2713	2.9413	0.5327	0.0413
	240	2.5133	0.6009	0.2133	2.8767	0.4737	-0.0233
	480	2.4571	0.5157	0.1571	2.8316	0.4134	-0.0684
	960	2.3903	0.3521	0.0903	2.8322	0.3619	-0.0678
θ	30	2.3316	1.4645	1.0316	1.1992	1.1122	0.7992
	60	2.1969	1.4378	0.8969	0.9028	0.8058	0.5028
	120	2.0723	1.5409	0.7723	0.6971	0.5423	0.2971
	240	1.7014	1.1232	0.4014	0.5540	0.3668	0.1540
	480	1.5813	0.9624	0.2813	0.4613	0.2850	0.0613
	960	1.3937	0.6048	0.0937	0.4100	0.2114	0.0100
		I: $\delta = 0.68, b = 2.70, c = 0.68, \theta = 0.68$			II: $\delta = 0.68, b = 2.70, c = 0.68, \theta = 0.59$		
δ	30	0.4609	0.4116	0.0609	0.5627	0.7765	0.1627
	60	0.4892	0.3415	0.0892	0.5660	0.5100	0.1660
	120	0.4818	0.3031	0.0818	0.5963	0.5106	0.1963
	240	0.5003	0.2949	0.1003	0.5500	0.3736	0.1500
	480	0.4759	0.2378	0.0759	0.5707	0.3691	0.1707
	960	0.4626	0.1882	0.0626	0.5302	0.2637	0.1302
b	30	4.0790	2.2881	1.1790	6.8518	3.8903	2.0518
	60	3.5177	1.4837	0.6177	5.6860	2.5357	0.8860
	120	3.3076	1.1950	0.4076	5.1983	2.1747	0.3983
	240	3.0721	0.9499	0.1721	4.9318	1.7104	0.1318
	480	2.9917	0.7712	0.0917	4.5881	1.4775	-0.2119
	960	2.9027	0.6482	0.0027	4.4531	1.1835	-0.3469
c	30	3.1679	0.8922	0.2679	3.0391	0.6987	0.1391
	60	2.9810	0.6446	0.0810	2.8775	0.5647	-0.0225
	120	2.9588	0.5506	0.0588	2.7957	0.4944	-0.1043
	240	2.8357	0.4607	-0.0643	2.7771	0.4170	-0.1229
	480	2.8395	0.3833	-0.0605	2.7381	0.3964	-0.1619
	960	2.8418	0.3290	-0.0582	2.7537	0.3225	-0.1463
θ	30	1.1647	1.2436	0.9647	1.1571	1.6683	0.9571
	60	0.8294	0.8666	0.6294	0.8759	1.1336	0.6759
	120	0.6217	0.5936	0.4217	0.7272	1.2896	0.5272
	240	0.4627	0.4031	0.2627	0.5351	0.4775	0.3351
	480	0.3675	0.3010	0.1675	0.4256	0.3452	0.2256
	960	0.2924	0.2015	0.0924	0.3577	0.2573	0.1577

4.5 Simulation Study

In this section, we examine the performance of the EOE-TL-LLoGP distribution by conducting various simulations for different sample sizes ($n=30, 60, 120, 240, 480, 960$) via the R package. We simulate 1000 samples for the true parameter values given in Table 4.4. The bias and RMSE of $\hat{\theta}$ relative to θ are given by

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta \text{ and } RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}}.$$

The results shows that as the sample size increases, the mean estimates of the parameters tend to be closer to the true parameter values since RMSEs and average bias decreases for all the parameters.

4.6 Applications

In this section, two real data sets are used to illustrate the usefulness of the proposed new model compared to other competing known models. We use the following goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér von Mises (W^*) and Andersen-Darling (A^*) as described by Chen and Balakrishnan [14], Kolmogorov-Sminov (K-S) and its p-value to assess the best fitting model. The model with the smallest values of these statistics is regarded as the best model.

We used the **nlm** function in **R** software to estimate the model parameters and the package **AdequacyModel** in **R** for goodness-of-fit test. Model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the two data sets are shown in tables 4.5 and 4.6. Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [13]) are also presented to show how well our model fits the observed data sets compared to the selected models. The models considered are the exponentiated power Lindley Poisson (EPLP) distribution by Pararai et al. [60], the Burr XII Poisson (BXIIP) and Burr XII geometric (BXIIG) by Silva and Cordeiro [70], the exponentiated log-logistic geometric distribution by Mendoza et al. [46], Weibull Poisson (WP) by Morais and Barreto-Souza [49], Exponential Pareto Poisson (EPP) by Elbatal et al. [22] and the Gompertz-Poisson (GP) distribution by Jafari and Tahmasebi [35]. The pdfs of the non-nested models are as follows:

$$f_{EPLP}(x; \theta, \alpha, \beta, \omega) = \frac{\alpha\beta^2\omega\theta}{(\beta+1)(\exp(\theta)-1)}(1+x^\alpha)x^{\alpha-1}\exp(-\beta x^\alpha) \\ \times \left(1 - \left(1 + \frac{\beta x^\alpha}{\beta+1}\right)\exp(-\beta x^\alpha)\right)^{\omega-1},$$

for $\alpha, \theta, \omega, \beta > 0$ and $x > 0$,

$$f_{BXIIP}(x; \theta, c, k) = \frac{\theta ck}{\exp(\theta)-1}x^{c-1}(1+x^c)^{-k-1}\exp(\theta(1+x^c)^{-k}),$$

for $c, k, \theta > 0$ and $x > 0$,

$$f_{BXIIG}(x; \theta, c, k) = \frac{ck(1-\theta)x^{-c-1}(1+x^c)^{-k-1}}{(1-\theta(1+x^c)^{-k})^2},$$

for $c, k, \theta > 0$ and $x > 0$,

$$f_{ELLGII}(x; \theta, \alpha, a, \beta) = \frac{(1-\theta)\beta a}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta a-1}\left[1 + \left(\frac{x}{\alpha}\right)^\beta\right]^{-(a+1)}\{1 - \theta[1 + \left(\frac{x}{\alpha}\right)^\beta]^{-a}\}^{-2},$$

for $0 < \theta < 1$, $\alpha, a, \beta > 0$ and $x > 0$,

$$f_{WP}(x; \theta, \alpha, \beta) = \frac{\theta\alpha\beta^\alpha}{\exp(\theta)-1}x^{\alpha-1}\exp(-(\beta x)^\alpha)\exp(\theta\exp(-(\beta x)^\alpha)),$$

for $\theta, \alpha, \beta > 0$ and $x > 0$, and

$$f_{GP}(x; \theta, \alpha, \beta, \gamma) = \frac{\theta\beta\exp(\gamma x)\exp(-(\frac{\beta}{\gamma}\exp(\gamma x)-1))}{\exp(\theta)-1}\exp(\theta\exp(\gamma x)-1),$$

for $\theta, \alpha, \beta, \gamma > 0$ and $x > 0$.

4.6.1 Relief Times Data

The following data set represents relief times (in minutes) of 20 patients receiving an analgesic which was reported by Gross and Clark [30]. The data is as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

Table 4.5: Parameter estimates and goodness-of-fit statistics for various models fitted for relief time data set

Model	Estimates				Statistics							
	θ	δ	b	c	$-2\log L$	AIC	AICC	BIC	W^*	A^*	$K-S$	$p-value$
<i>EOE-TL-LLoGP</i>	2.4478 (2.9703)	1.3884×10^4 (4.8805×10^{-6})	0.4224 (1.2905×10^{-2})	0.2648 (9.1235×10^{-2})	26.7911	34.7911	37.6482	38.5688	0.0308	0.1840	0.0969	0.9941
<i>EOE-TL-LLoGP</i>	4.21387 (2.1044)	1 -	18.04467 (3.8043)	1.38800 (0.2271)	29.0428	35.0428	36.6428	37.8761	0.0658	0.4168	0.1382	0.8610
<i>EOE-TL-LLoGP</i>	9.8986×10^{-9} (0.0171)	1 -	5.9315 (0.9961)	1 -	40.7086	44.7086	45.4586	46.5975	0.1433	0.8838	0.2427	0.2129
<i>EOE-TL-LLoGP</i>	6.3822×10^{-11} (1.9737×10^{-3})	0.8494 (0.1958)	2.3102 (0.2173)	1 (-)	87.9346	93.9334	95.5334	96.7667	0.1879	1.14000	0.7462	1.295×10^{-9}
<i>EOE-TL-LLoGP</i>	3.5114×10^{-8} (0.0333)	1 -	1 -	0.3763 (0.0676)	148.2826	152.2826	153.0326	154.1715	0.1160	0.7240	0.9570	1.554×10^{-15}
<i>EOE-TL-LLoGP</i>	1.3913×10^{-9} (6.4810×10^{-3})	1.9608 (0.4500)	1 -	0.4114 (0.0700)	123.6561	129.6559	131.2559	132.4892	0.1137	0.7098	0.9186	2.365×10^{-14}
<i>EOE-TL-LLoGP</i>	1.8584×10^{-8} (0.0241)	10.304 (2.3639)	1 -	1 -	137.8377	141.8368	142.5868	142.5868	0.3579	2.1227	0.7622	5.173×10^{-10}
BXIIP	1.7402×10^{-8} (0.0384)	134.80 (2.9040×10^{-7})	0.0131 (2.9997×10^{-3})	- -	38.0890	44.0890	45.6890	46.9223	1.4593	7.0928	1.0000	$< 2.2 \times 10^{-16}$
EPLP	4.2803×10^{-8} (5.7815×10^{-6})	1.4503 (2.3229)	1.2015 (0.01997)	4.3934 (0.7393)	32.5840	40.5841	43.4413	44.3619	0.1203	0.7503	0.1586	0.7259
GP	7.5221×10^{-7} (0.2411)	0.1611 (0.0750)	0.8582 (0.2069)	- -	46.7863	63.8191	65.4191	66.6524	0.3529	2.0271	0.4400	0.0013
ELLGH	0.0877 (2.7122)	0.6784 (0.1098)	34.1689 (0.0250)	4.3535 (2.7852)	26.8935	34.8935	37.7506	38.6712	0.0364	0.2099	0.1084	0.9789
WP	4.020×10^{-9} (0.0110)	0.2227 (0.0508)	0.2311 (0.0580)	- -	160.7121	166.7121	168.3121	169.5454	0.0917	0.5767	0.5215	6.49×10^{-5}
BXIIG	0.0000 (1.5592)	164.04 (2.2314×10^{-7})	0.0107 (4.5638×10^{-3})	- -	38.0890	44.0891	45.6891	46.9224	0.0490	0.2795	0.2891	0.0835

$$\begin{bmatrix} 8.8225 & 1.4492 \times 10^{-5} & 0.01296 & -0.2320 \\ 1.4492 \times 10^{-5} & 2.3819 \times 10^{-11} & 2.2655 \times 10^{-8} & -3.7532 \times 10^{-7} \\ 0.01296 & 2.2655 \times 10^{-8} & 1.665 \times 10^{-4} & 1.0446 \times 10^{-4} \\ -0.2320 & -3.7532 \times 10^{-7} & 1.0446 \times 10^{-4} & 8.3238 \times 10^{-3} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$\theta \in [0.0537 \pm 5.8217]$, $\delta \in [3.0819 \pm 9.5658 \times 10^{-6}]$, $b \in [0.3361 \pm 0.0253]$ and $c \in [0.0903 \pm 0.1788]$.

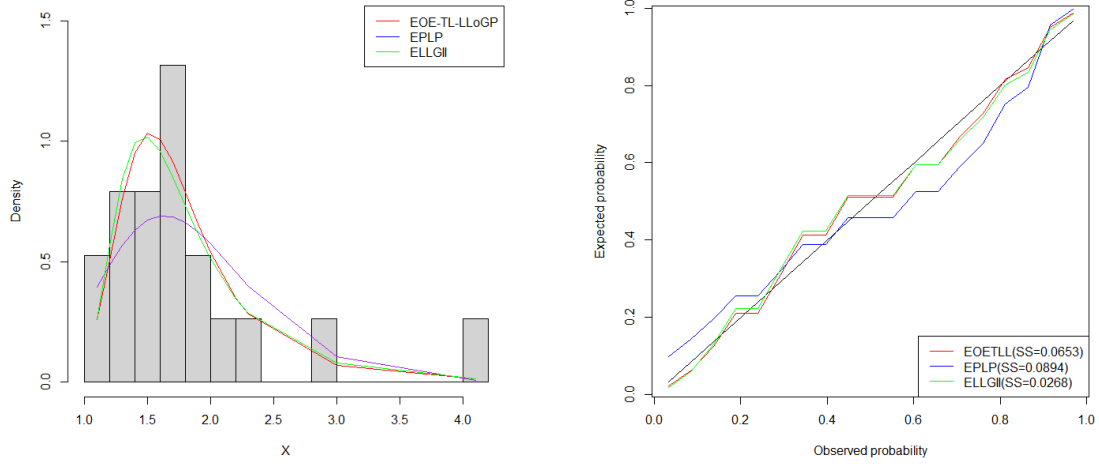


Figure 4.5: Fitted pdfs and probability plots for relief time data set

From the results shown in Table 4.5, we can conclude that the EOE-TL-LLoGP distribution fit the relief time data set better than the nested and non-nested models considered. Furthermore, from the fitted densities plots (Figure 4.5), we can conclude that the proposed model fits well on data that is heavy-tailed compared to the non nested models.

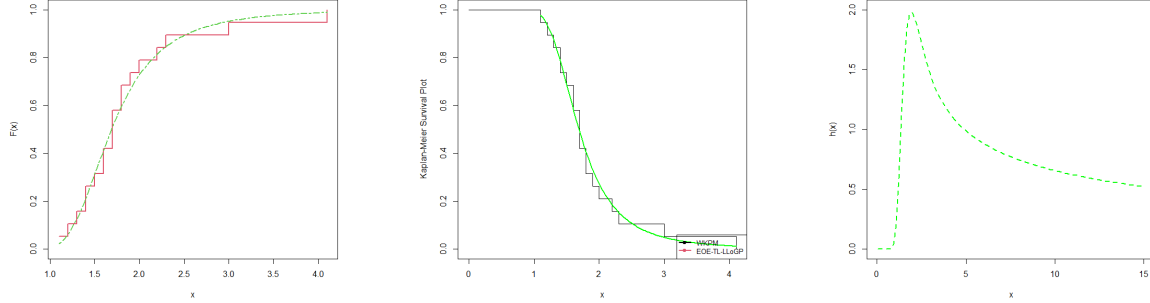


Figure 4.6: Plots for the empirical cdf, Kaplan-Meier curve and the hrf of the EOE-TL-LLoGP model for the relief time data set

4.6.2 Active Repair Times Data

These data correspond to the maintenance data with 46 observations reported on active repair times (hours) for an airborne communication transceiver discussed by Alven [3], Chhikara and Folks [15] and Dimitrakopoulou et al. [21].

The data are as follows: 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

Table 4.6: Parameter estimates and goodness-of-fit statistics for various models fitted for active repair times data set

Model	Estimates				Statistics							
	θ	δ	b	c	$-2\log L$	AIC	AICC	BIC	W^*	A^*	$K-S$	$p-value$
EOE-TL-LLoGP	0.0482 (1.4560)	3.0819×10^4 (2.5293×10^{-6})	0.3362 (0.0197)	0.0905 (0.0140)	178.9258	186.929	188.0718	193.6845	0.0584	0.3801	0.0984	0.8332
EOE-TL-LLoGP	9.9133×10^{-9} (0.0125)	1	3.9368 (0.4645)	0.4279 (0.0411)	194.7365	200.7368	201.4034	205.8034	0.1108	0.8164	0.2402	0.0198
EOE-TL-LLoGP	4.8064×10^{-9} (9.1514×10^{-3})	0.4649 (0.0736)	2.6548 (0.0689)	1	1457.549	1463.526	1464.192	1468.592	0.4849	3.2761	0.5776	$< 2.2 \times 10^{-16}$
EOE-TL-LLoGP	1.2819×10^{-8} (0.0163)	1.3129 (0.2076)	1	0.1978 (0.0230)	359.5079	365.5098	366.1764	370.5764	0.0910	0.6625	0.8941	5.146×10^{-12}
EOE-TL-LLoGP	3.7601×10^{-9} (7.1742×10^{-3})	1	1	0.1979 (0.0231)	380.4944	384.4944	384.8187	387.8722	0.0932	0.6785	0.9182	$< 2.2 \times 10^{-16}$
EOE-TL-LLoGP	1.8657 (0.6239)	1	18.5783 (1.3000)	-	353.3661	357.3661	357.6905	360.7439	0.0892	0.7921	0.3650	4.71×10^{-5}
EOE-TL-LLoGP	3.5417×10^{-9} (9.4951×10^{-3})	10.295 (1.6278)	1	1	3702.443	3702.767	3760.38	3705.821	0.4066	2.8470	0.5973	8.067×10^{-13}
BXIIP	2.3569×10^{-5} (0.0555)	2.9722 (0.6778)	3.3835 (0.0873)	-	181.9732	187.9732	188.6399	193.0399	0.0846	0.5402	0.5097	1.89×10^{-9}
EPLP	0.0135 (5.7815×10^{-6})	1.0435 (2.3229)	11.188 (0.01997)	5.9983×10^4 (0.7393)	178.8315	186.8315	187.9744	193.5871	0.0594	0.3822	0.0976	0.8405
GP	0.0159 (0.9845)	0.2076 (0.0593)	2.3723×10^{-11} (3.2981×10^{-3})	-	192.3193	198.3191	198.9858	203.3858	0.1450	1.0459	0.1809	0.1460
ELLGH	0.4156 (0.9091)	0.1163 (1.9215×10^{-6})	23.0805 (0.0178)	1.3611 (0.5327)	178.8877	186.8877	188.0306	193.6432	0.0624	0.3957	0.0993	0.8256
EPP	1.4597×10^{-8} (0.0140)	0.2182 (0.1597)	0.4551 (0.0789)	0.0843 (0.0909)	220.0751	228.0754	229.2183	234.8309	0.1004	0.3456	25202	$< 2.2 \times 10^{-6}$
WP	1.0736×10^{-8} (1.5592)	0.7585 (2.2314×10^{-7})	0.2857 (0.0832)	-	219	225.0078	225.6745	230.0744	0.1224	0.8891	0.2058	0.0675
BXIIG	2.3569×10^{-5} (0.0555)	2.9722 (0.6778)	3.3835 (0.0873)	-	181.9732	187.9732	188.6399	193.0399	0.0898	0.5623	0.1033	0.7868

$$\begin{bmatrix} 2.1198 & 3.6826 \times 10^{-6} & 0.0157 & -0.0135 \\ 3.6826 \times 10^{-6} & 6.3975 \times 10^{-12} & 2.7370 \times 10^{-8} & -2.3028 \times 10^{-8} \\ 0.0157 & 2.7370 \times 10^{-8} & 1.4322 \times 10^{-4} & -8.8357 \times 10^{-5} \\ -0.0135 & -2.3028 \times 10^{-8} & -8.8357 \times 10^{-5} & 1.9522 \times 10^{-4} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\theta \in [0.0482 \pm 2.8537]$, $\delta \in [3.0819 \times 10^4 \pm 4.9575 \times 10^{-6}]$, $b \in [0.3362 \pm 0.0235]$ and $c \in [0.0905 \pm 0.0274]$.

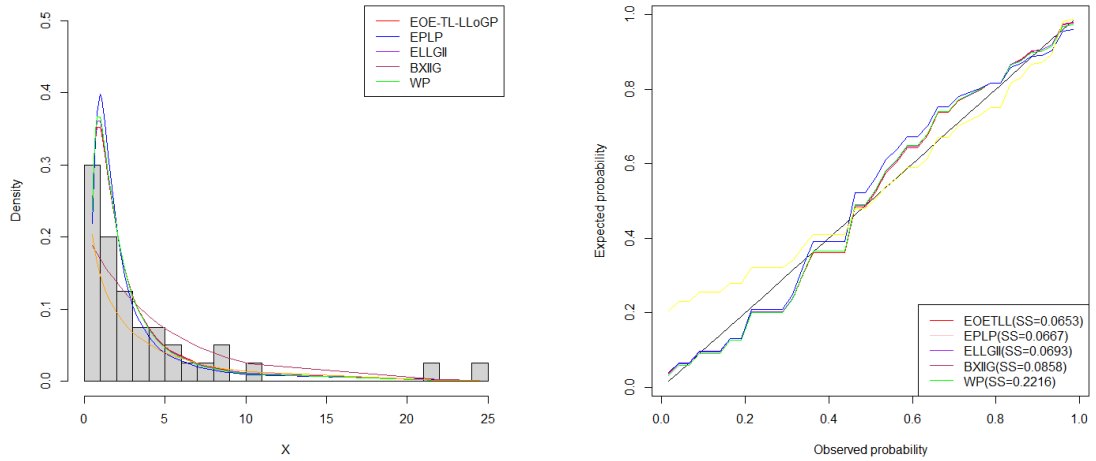


Figure 4.7: Fitted pdfs and probability plots for active repair time data set

From the results shown in Table 4.6, we can conclude that the EOE-TL-LLoGP distribution fit the active repair time data set better than the nested and non-nested models considered. Furthermore, from the fitted densities plots (Figure 4.7)

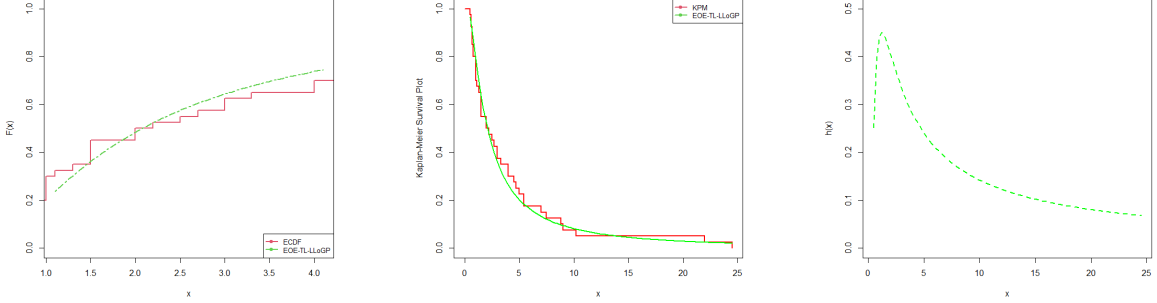


Figure 4.8: Plots for the empirical cdf, Kaplan–Meier curve and the hrf of the EOE-TL-LLoGP model for the active repair time data set

4.6.3 Likelihood Ratio Test

We present results of Likelihood ratio test for testing if the $EOE - TL - LLoGP$ model performs better than its nested models. We conclude from the values of the χ^2 shown in Table 4.7 at 5% level of significance that the $EOE - TL - LLoGP$ model performs better than its nested models on the two data sets considered except for $EOE - TL - LLoGP(\theta, 1, b, c)$ on the relief dataset. Basing on other goodness-of-fit statistics A^* , W^* , $K - S$ and its p-value, we conclude that the main model performs better than the $EOE - TL - LLoGP(\theta, 1, b, c)$.

Table 4.7: Likelihood Ratio Test Results

	Relief Time Data	Active Repair Times Data
Model	χ^2 (p-value)	χ^2 (p-value)
$EOE - TL - LLoGP(\theta, 1, b, c)$	2.2517 (0.1335)	15.8075 (0.00007)
$EOE - TL - LLoGP(\theta, \delta, b, 1)$	61.1435 (< 0.00001)	1278.62 (< 0.00001)
$EOE - TL - LLoGP(\theta, \delta, 1, c)$	96.865 (< 0.00001)	180.5789 (< 0.00001)
$EOE - TL - LLoGP(\theta, 1, 1, c)$	121.4915 (< 0.00001)	201.5654 (< 0.00001)
$EOE - TL - LLoGP(\theta, \delta, 1, 1)$	111.0466 (< 0.00001)	3523.514 (< 0.00001)
$EOE - TL - LLoGP(\theta, 1, b, 1)$	11.6658 (.002931)	174.4371 (< 0.00001)

4.7 Concluding Remarks

A new class of distributions referred to as the exponentiated odd exponential Topp-Leone-G Power Series (EOE-TL-GP) class of distributions is developed. We derive the statistical properties of the proposed class of distributions. The new class of distributions can be expressed as an infinite linear combination of EOE-TL-G distribution. We derive maximum likelihood estimates of the EOE-TL-GP class of distributions. A simulation study on the EOE-TL-LLoGP distribution is also conducted to assess efficiency of the maximum likelihood estimates. The EOE-TL-LLoGP distribution was applied to real data examples and compared to several equi-parameter

non-nested models and nested models. The EOE-TL-LLoGP distribution performed better than the selected non-nested and nested models on the two data sets considered.

Chapter 5

Conclusions and Suggestions for Further Studies

5.1 Conclusions

Three new generalized distributions called the exponentiated odd Nadarjah-Haghighi (EO-NH-G), exponentiated odd exponential Topp-Leone-G family of distributions and the exponentiated odd exponential Topp-Leone-G power series class of distributions were proposed and developed. Properties of these distributions including the series expansion of pdfs, cdfs, moments, hazard functions, reverse hazard functions, quantile functions and sub-models, moments, conditional moments, Rényi entropy and distribution of order statistics are derived. Estimation of the parameters of the models and applications to real dataset are also given.

5.2 Suggestions for Further Studies

We present some suggestions for further studies that may be conducted. In this thesis, we introduced and presented the three generalized models, their properties and applications. Other possible research topic could include the application of the proposed models to regression analysis and also the application of Bayesian techniques to the proposed models, as well as estimation of the model parameters under various censoring schemes. Also of interest is the estimation of the parameters of the proposed models using different estimation techniques or methods.

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Appendix

PART 1A

```
##define EONHW pdf
EONHW_pdf <- function(delta ,lambda ,beta ,omega ,x){
  (delta*lambda*beta*omega*((x^(omega-1))/(exp(-x^omega))
  )*((1+lambda*((1-exp(-x^omega))/exp(-x^omega)))^(beta-1))
  *(exp(1-lambda*(1+((1-exp(-x^omega))/exp(-x^omega)))^beta))*
  ((1-exp(1-(1+lambda*((1-exp(-x^omega))/exp(-x^omega)
  ))^beta))^(delta-1)))
  }
##define HLWL cdf
EONHW_cdf <- function(delta ,lambda ,beta ,omega ,x){
  (1-exp(1-(1+lambda*((1-exp(-x^omega))/
  exp(-x^omega)))^beta))^delta}
## define EONHW hazard
EONHW_hazard <-function(delta ,lambda ,
beta ,omega ,x){
  EONHW_pdf(delta ,lambda ,beta ,omega ,x) /
  (1 -EONHW_cdf(delta ,lambda ,beta ,omega ,x))
  }
x=seq(0,3,by=0.001)
z1=EONHW_hazard(.9 ,.9 ,.9 ,.81 ,x)
plot(x,z1,ylim=range(c(0,3.5)),col=2,'l',
lwd=2,xlab="x",ylab="h(x)")

z2=EONHW_hazard(3.7 ,.1 ,.2 ,4.8 ,x)
z3=EONHW_hazard(.6 ,1.6 ,.7 ,1.1 , x)
z4=EONHW_hazard(.5 ,1.5 ,.2 ,.5 ,x)
z5=EONHW_hazard(.6 ,.7 ,.4 ,1.1 ,x)
lines(x,z2,col=3,lwd=2)
lines(x,z3,col=4,lwd=2)
lines(x,z4,col=5,lwd=2)
lines(x,z5,col=6,lwd=2)
legend("topright",c(
expression(paste(delta ,'= .9 ,',lambda ,'= .9 ,',
beta ,'=0.9 ,',omega ,'=0.81')),
expression(paste(delta ,'=3.7 ,',lambda ,'=0.1 ,',
,beta ,'=0.2 ,',omega ,'=4.8')),
expression(paste(delta ,'=0.6 ,',lambda ,'=1.6 ,',
,beta ,'=0.7 ,',omega ,'=1.1')),
expression(paste(delta ,'=0.5 ,',lambda ,'=1.5 ,',
,beta ,'=0.2 ,',omega ,'=0.5')),
expression(paste(delta ,'=0.6 ,',lambda ,'=0.7 ,',
,beta ,'=0.4 ,',omega ,'=1.1'))),
col=c(2,3,4,5,6),lwd=c(2,2,2,2,2))
```

Applications

```

rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)

# maximum likelihood estimators

x<-c(0.2,0.3,0.5,0.5,0.5,0.5,0.6,0.6,0.7,0.7,
0.7,0.8,0.8,1.0,1.0,1.0,1.0,1.1,1.3,1.5,1.5,
1.5,1.5,2.0,2.0,2.2,2.5,2.7,3.0,3.0,3.3,3.3,
4.0,4.0,4.5,4.7,5.0,5.4,5.4,7.0,7.5,8.8,9.0,
10.3,22.0,24.5)

EONHLW<-function(delta,lambda,beta,omega){-sum(
log(((delta*lambda*beta*omega*((x^(omega-1))/
(exp(-x^omega)))*((1+lambda*((1-exp(-x^omega))
/exp(-x^omega)))^(beta-1))*(exp(1-(1+lambda*
((1-exp(-x^omega))/exp(-x^omega)))^beta))
*((1-exp(1-(1+lambda*((1-exp(-x^omega))/
exp(-x^omega)))^beta))^(delta-1))))))
}
mle.result<-mle2(EONHLW,hessian = NULL,
start=list(delta=0.51048,
lambda=.001 ,beta=.01,omega=1.2),
optimizer="nllminb",lower=0)
summary(mle.result)
vcov(mle.result)
output1<-vcov(mle.result)
1.96*sqrt(diag(output1))
solve(output1)
KOLLLoG_LL<-function(a,b,lambda,c)
{-sum(log(a*b*(lambda^2/(1+lambda))*((c*x^(c-1))/
(1+x^c))^(b-1))*(exp(-lambda*
(1-(1+x^c))^(b-1))/(1+x^c))^(b-1))*((1-((
lambda+(1+x^c))^(b-1))/(1+lambda)
*(1+x^c))^(b-1))*exp(-lambda*(1-(1+x^c))^(b-1))/
(1+x^c))^(b-1))^(a-1))*(1-((1-((lambda+(
1+x^c))^(b-1))/(1+lambda)*(1+x^c))^(b-1))*exp(-lambda
*(1-(1+x^c))^(b-1))/(1+x^c))^(b-1))^(a-1))^(b-1))}
mle.result<-mle2(KOLLLoG_LL,hessian = NULL,
start=list(a=.1,b=1.2,
lambda=.1, c=1.1),optimizer="nllminb",lower=0)
summary(mle.result)
ELOLLW_LL<-function(beta,lambda,theta,gamma)
{-sum(log(((1*(theta^2)*gamma*(lambda^gamma)*
(x^(gamma-1))*(exp(-(lambda*x)^gamma))*((exp(-
(lambda*x)^gamma))^(1*theta-1))*((1-exp(-
(lambda*x)^gamma))^(1-1)))*(1-beta*log(((exp
(-1*(lambda*x)^gamma))/((1-exp(-(lambda*x)^gamma))^1)+
(exp(-1*(lambda*x)^gamma)))))))/((theta+beta)

```

```

(((1 - exp(-(lambda*x)^gamma))^1)+exp(-1*
(lambda*x)^gamma))^(theta-1)))}
mle.result<-mle2(ELOLLWLL,hessian = NULL,
start=list(beta=2,lambda=0.2,theta=1.7418,
gamma=1.5233),optimizer="nlminb",lower=0)
BGL<-function(a,b,lambda,alpha){-sum(log(
(gamma(a+b)/(gamma(a)*gamma(b)))
*((alpha*lambda**2)/(1+lambda))*(1+x)
*exp(-lambda*x)*((1-(1+(lambda*x)/
(lambda+1))*exp(-lambda*x))*((a*alpha-1))
*(1-(1-(1+(lambda*x)/(lambda+1))
*exp(-lambda*x))*alpha)*(b-1)))})
mle.result<-mle2(BGL,hessian = NULL,
start=list(a=10.1,b=.01,lambda=10.2 ,alpha=.1 )
,optimizer="nlminb",lower=0)
summary(mle.result)
ELOLLWLL<-function(beta,lambda,theta,
gamma){-sum(log(((1*(theta^2)*gamma*
(lambda^gamma)*(x^(gamma-1))*(exp(-(
lambda*x)^gamma))*((exp(-(
lambda*x)^gamma))^(1*theta-1))*
((1-exp(-(lambda*x)^gamma))^(1-1)))
*(1-beta*log(((exp(-1*(lambda*x)^gamma))
/(((1-exp(-(lambda*x)^gamma))^1
+(exp(-1*(lambda*x)^gamma)))))))/
((theta+beta)*(((1-exp(-(
lambda*x)^gamma))^1)+exp(-1*
(lambda*x)^gamma))^(theta-1))))})
mle.result4<-mle2(ELOLLWLL,hessian
= NULL,start=list(beta=2,lambda=0.2,
theta=1.7418,gamma=1.5233),optimizer="nlminb",lower=0)
summary(mle.result4)

```

Goodness of fit test

```

library(AdequacyModel)
data=c(0.2,0.3,0.5,0.5,0.5,0.5,0.6,0.6,0.7,0.7,0.7,0.8,
0.8,1.0,1.0,1.0,1.0,1.1,1.3,1.5,1.5,1.5,1.5,2.0,2.0,2.2,
2.5,2.7,3.0,3.0,3.3,3.3,4.0,4.0,4.5,4.7,5.0,5.4,5.4,7.0,
7.5,8.8,9.0,10.3,22.0,24.5)
EONHW.pdf<-function(par,x){
  delta=par[1]
  lambda=par[2]
  beta =par[3]
  omega=par[4]
  (delta*beta*lambda*omega*((x^(omega-1))
/(exp(-x^omega)))*((1+lambda*((1-exp(-x^omega))
/exp(-x^omega)))^(beta-1))*(exp(1-(1+lambda*
((1-exp(-x^omega))/exp(-x^omega)))^beta))*((
(1-exp(1-(1+lambda*((1-exp(-x^omega))/
exp(-x^omega))))^beta))^(delta-1)))
}

```

```

EONHW_cdf<-function(par,x){

    delta=par[1]
    lambda=par[2]
    beta =par[3]
    omega=par[4]

    (1-exp(1-(1+lambda*((1-exp(-x^omega))/
    exp(-x^omega)))^beta))^delta
}

goodness.fit(pdf=EONHW_pdf, cdf=EONHW_cdf,
mle=c(1.1886e+04 ,6.9153e+03,2.4715e-01,
2.3437e-01 ),data = data,method = "BFGS",
domain = c(0,1), lim_inf = c(0,0,0,0),
lim_sup = c(10000000000000000,10000000000000000,
10000000000000000))

KOLLLoG_pdf<-function(par,x){
    delta=1

    a=par[1]
    b=par[2]
    lambda=par[3]
    c=par[4]
a*b*(lambda^2/(1+lambda))*((c*x^(c-1))/(1+x^c)^(-1))*
(exp(-lambda*(1-(1+x^c)^(-1))/(1+x^c)^(-1)))*
((1-((lambda+(1+x^c)^(-1))/(1+lambda)*(1+x^c)^(-1)))
*exp(-lambda*(1-(1+x^c)^(-1))/(1+x^c)^(-1)))^(a-1)*
(1-((1-((lambda+(1+x^c)^(-1))/(1+lambda)*(1+x^c)^(-1)))
*exp(-lambda*(1-(1+x^c)^(-1))/(1+x^c)^(-1))))^(a)))^(b-1)
}

KOLLLoG_cdf<-function(par,x){
    delta=1

    a=par[1]
    b=par[2]
    lambda=par[3]
    c=par[4]

    -(1-((1-((lambda+(1+x^c)^(-1))/(1+lambda)*(1+x^c)^(-1)))*
exp(-lambda*(1-(1+x^c)^(-1))/(1+x^c)^(-1))))^(a)))^b

```

```
}
```

```
goodness.fit(pdf=KOLLLoG_pdf, cdf=KOLLLoG_cdf,
mle=c(0.221756,2.580236,0.008049,
3.042535 ),data = data,method = "BFGS",domain = c(0,1),
lim_inf = c(0,0,0,0),lim_sup = c(10000000000000000,
10000000000000000,100000000000,100000000000))
```

```
BGL_pdf<-function(par,x){
  a=par[1]
  b=par[2]
  lambda=par[3]
  alpha=par[4]
```

```
(gamma(a+b)/(gamma(a)*gamma(b)))*(
(alpha*lambda**2)/(1+lambda))*(1+x)*
exp(-lambda*x)*((1-(1+(lambda*x)/(lambda+1)))*
exp(-lambda*x)**(a*alpha-1))*(1-
(1-(1+(lambda*x)/(lambda+1))*
exp(-lambda*x)**alpha)**(b-1)
}
```

```
BGL_cdf<-function(par,x){
```

```
  a=par[1]
  b=par[2]
  lambda=par[3]
  alpha=par[4]
```

```
pbeta(((1-((1+lambda+lambda*x)/(1+lambda))
*exp(-lambda*x))^alpha),a,b,log=FALSE)
```

```
}
```

```
goodness.fit(pdf=BGL_pdf, cdf=BGL_cdf,
mle=c(0.253815,44.743212,0.185752,
2.693302),data = data,method = "BFGS",
domain = c(0,100), lim_inf = c(0,0,0,0),
lim_sup = c(100,100,100,100))
```

Fitted curves

```
data=c(0.2,0.3,0.5,0.5,0.5,0.5,0.6,0.6,0.7,0.7,
0.7,0.8,0.8,1.0,1.0,1.0,1.0,1.1,1.3,1.5,1.5,1.5,
1.5,2.0,2.0,2.2,2.5,2.7,3.0,3.0,3.3,3.3,4.0,4.0,
4.5,4.7,5.0,5.4,5.4,7.0,7.5,8.8,9.0,10.3,22.0,24.5)
```

```
data<-sort(data)
n=length(data)
```

```
#####
delta1=1.1886e+04
lambda1=6.9153e+03
beta1 = 2.4715e-01
omega1 = 2.3437e-01
```

```
#####
a2=9.3197e-01
b2=6.7768e+01
lambda2=6.3257e+01
theta2=6.1204e-05
```

```
#####
a3 = 1.1881e+03
b3 = 2.6116e+00
alpha3 =2.3172e-02
beta3 = 1.9141e-01
```

```
#####
#####
alpha4=36.950656
beta4=0.989296
delta4=0.216366
theta4=0.021152
```

```
#####
a5=11.83946
b5 =0.21398
lambda5=4.50458
c5= 0.55291
```

```
#####
a6=2.0000e+00
b6=1.7000e+00
lambda6=1.3223e-06
alpha6 = 1.1372e-02
```

```
#####
beta7=1.9588e+03
lambda7=5.0924e+01
theta7= 8.6500e-02
gamma7= 6.3010e-01
```

```
F_observed<-c(rep(0,n))
```

```
m=n+0.25
```

```
for (i in 1:n)
```

```
{
```

```
    F_observed[i]<-(i-0.375)/m
```

```
}
```

```
require(gsl)
```

```

require(stats)

F_observed<-c(rep(0,n))
m=n+0.25
for (i in 1:n)
{
    F_observed[i]<-(i-0.375)/m
}

require(gsl)
require(stats)
EONHW_cdf<- function(data,delta,lambda,beta,omega,n){
    F<-c(rep(0,n))
    for (i in 1:n)
    {
        x<-data[i]
        F[i]<-(1-exp(1-(1+lambda*((1-exp(-x^omega))/
            exp(-x^omega)))^beta))^delta}
    return(F)
}

BOLE_cdf<- function(data,a,b,lambda,theta,n) {
    F<-c(rep(0,n))
    for (i in 1:n)
    {
        x<-data[i]

F[i]<-pbeta(1-((lambda+(exp(-theta*x)))/((1+
lambda)*(exp(-theta*x)))*exp(-lambda*(1-
exp(-theta*x))/(exp(-theta*x))),a,b,log=FALSE)
    }
    return(F)
}

WL_cdf<- function(data,a,b,alpha,beta,n) {
    F<-c(rep(0,n))
    for (i in 1:n)
    {
        x<-data[i]

F[i]<-1-exp((-a)*(((1+(x/beta))^alpha)-1)^b) }
    return(F)
}

#####Non nested EW#####
GB_cdf<-function(data,alpha,beta,delta,theta,n){
    F<-c(rep(0,n))
    for (i in 1:n)
    {
        x<-data[i]

F[i]<-pgamma((-theta^(-1))*log(1-

```

```

        (1+x^(-delta))^(-beta)),alpha)/gamma(alpha)
    }
    return(F)
}
#####Non nested GLLo#####

KOLLLoG_cdf<-function(data,a,b,lambda,c,n){
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]

F[i]<- 1-(1-((1-((lambda+(1+x^c)^(-1))/(1+lambda)
*(1+x^c)^(-1)))*exp(-lambda*(1-(1+x^c)^(-1))/
(1+x^c)^(-1)))^(a)))^b
  }
  return(F)
}
BGL_cdf<-function(data,a,b,lambda,alpha,n){
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]

F[i]<- pbeta(((1-((1+lambda+lambda*x)/(1+lambda))
*exp(-lambda*x))^alpha),a,b,log=FALSE)}
  return(F)
}
ELOLLW_cdf<-function(data,beta,lambda,theta,gamma,n){
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]

F[i]<- 1-((exp(-1*theta*(lambda*x)^gamma))*(theta+
beta-theta*beta*log((((exp(-1*(lambda*x)^gamma))
/((1-exp(-(lambda*x)^gamma))^1+exp(-1*(lambda
*x)^gamma)))))))/(((theta+beta)*((1-exp(-
(lambda*x)^gamma))^1)+exp(-1*(lambda*
x)^gamma))^theta))}
  return(F)
}

#####Non nested EPGW#####
F1<-EONHW_cdf(data,delta1,lambda1,beta1,omega1,n)

```

```

F1
F2<-BOLE_cdf( data ,a2,b2,lambda2,theta2 ,n)
F2
F3<-WL_cdf( data ,a3,b3,alpha3 ,beta3 ,n)
F3
F4<-GB_cdf( data ,alpha4 ,beta4 ,delta4 ,theta4 ,n)
F4
F5<-KOLLLoG_cdf( data ,a5,b5,lambda5 ,c5 ,n)
F5
F6<-BGL_cdf( data ,a6,b6,lambda6 ,alpha6 ,n)
F6
F7<-ELOLLW_cdf( data ,beta7 ,lambda7 ,theta7 ,gamma7,n)
F7

SS<-function(F,F_line ,n){
  SS<-0
  for (i in 1:n)
  {
    SS<-SS+(F[i]-F_line[i])^2
  }
  return(SS)
}

SS1<-SS(F1,F_observed ,n)
SS1
SS2<-SS(F2,F_observed ,n)
SS2
SS3<-SS(F3,F_observed ,n)
SS3
SS4<-SS(F4,F_observed ,n)
SS4
SS5<-SS(F5,F_observed ,n)
SS5
SS6<-SS(F6,F_observed ,n)
SS6
SS7<-SS(F7,F_observed ,n)
SS7
op <- par(mfrow=c(1,1))
plot(F_observed,F_observed ,type='l' ,
xlab="Observed probability",ylab="Expected probability")
lines(F_observed,F1,col='red' ,type='l')
lines(F_observed,F2,col='pink')
lines(F_observed,F3,col='blue' ,type='l')
#lines(F_observed,F4,col='maroon' ,type='l')
lines(F_observed,F5,col='green' ,type='l')
#lines(F_observed,F6,col='gray' ,type='l')
lines(F_observed,F7,col='purple' ,type='l')
#lines(F_observed,F8,col='pink' ,type='l')
#lines(F_observed,F9,col='maroon' ,type='l')

```

```
#lines(F_observed,F10,col=13)
#lines(F_observed,F11,col=1)
#lines(F_observed,F12,col=14)
#lines(F_observed,F13,col=15)
#lines(F_observed,F14,col=16)
#lines(F_observed,F15,col=17)
legend("bottomright",cex=1,c("EO-NH-W(SS=0.0456)",
"BOLE(SS=0.1724)", "WL(SS=0.0505)", "KOLLLOG(SS=0.0434)",
"ELOLLW(SS=0.0868)"),lty=c(1,1,1,1),
col=c("red","pink","blue","green","purple"),
lwd=c(1,1,1,1,1,1))
```

```
#define pdf EOETLLoG.
f1=function(x,delta,b,c){
y= 2*delta*b*(c*x^(c-1))*((1+x^c)^(-3))*
((1-(1+x^c)^(-2))^(b-1))*(exp(-(((1-(1+x^c)^(-2))^b)
/(1-(1-(1+x^c)^(-2))^b))))*
((1-exp(-(((1-(1+x^c)^(-2))^b)/
(1-(1-(1+x^c)^(-2))^b))))^(delta-1))
*(1-(1-(1+x^c)^(-2))^b)^(-2))
return(y)
}
x=seq(0,1.5,by=0.001)
y1=f1(x,3.9,.1,3.5)
plot(x,y1,ylim=c(0,3),col=2,lwd=2,xlab="x",
ylab="density")
```

```
y2=f1(x,0.9,0.9,.8)
lines(x,y2,col=3,lwd=2)
y3=f1(x,1.7,1.1,2.5)
lines(x,y3,col=4,lwd=2)
y4=f1(x,2.5,.6,1.5)
lines(x,y4,col=5,lwd=2)
y5=f1(x,2.9,1.5,2.5)
lines(x,y5,col=6,lwd=2)
legend("topright",c(
expression(paste(delta,'=3.9',b,'=0.1',c,'=3.5')),
expression(paste(delta,'=0.9',b,'=0.9',c,'=0.8')),
expression(paste(delta,'=1.7',b,'=1.1',c,'=1.2')),
expression(paste(delta,'=2.5',b,'=0.6',c,'=2.5')),
expression(paste(delta,'=2.9',b,'=1.5',c,'=2.5'))),
col=c(1,2,3,4,5),lwd=c(2,2,2,2,2))
```

Simulations

The EOETLG

```
rm(list=ls())
library(stats4)
```

```

library(bbmle)
library(stats)
library(numDeriv)
library(rootSolve)

##define EOETLG pdf
EOETLLoG_pdf <- function(delta,b,c,x){
2*delta*b*(c*x^(c-1))*((1+x^c)^(-3))*
((1-(1+x^c)^(-2))^(b-1))*(exp(-(((1-
(1+x^c)^(-2))^b)/(1-(1-(1+x^c)^(-2)
)^b))))*((1-exp(-(((1-(1+x^c)^(-2))^b)
/(1-(1-(1+x^c)^(-2))^b))))^(delta-1))*
(1-(1-(1+x^c)^(-2))^b)^(-2))
}
##define EOETLLoG cdf
EOETLLoG_cdf <- function(delta,b,c,x){
(1-exp(-(((1-(1+x^c)^(-2))^b)/(1-(1-
(1+x^c)^(-2))^b))))^delta}
## define EOETLLoG hazard
EOETLLoG_hazard <-function(delta,b,c,x){
EOETLLoG_pdf(delta,b,c,x)/(1
-EOETLLoG_cdf(delta,b,c,x))
}
EOETLLoG_neglogl=function(delta,b,c,x){
-sum(log(EOETLLoG_pdf(delta,b,c,x)))
}
EOETLLoG_hazard2=function(delta,b,c,x){
f=EOETLLoG_pdf(delta,b,c,x)
F=EOETLLoG_cdf(delta,b,c,x)
y=f/(1-F)
return(y)}

#-----
EOETLLoG_quantile=function(parameter,u){
delta=parameter[1]
b=parameter[2]
c=parameter[3]

f=function(x){
EOETLLoG_cdf(delta,b,c,x)-u
}
x=min(uniroot.all(f,lower=0,upper=1000000000000,
tol=0.0001))
return(x)
}

EOETLLoG_QuantileTable=function(parameter_matrix){
u=seq(0.1,0.9,0.1)
size=dim(parameter_matrix)[1]
Table_Quantile=matrix(NA,nrow=length(u),ncol=size)
row.names(Table_Quantile)=u

```

```

colnames( Table_Quantile)=apply( parameter_matrix ,1 ,
function(x)
{paste0( '( ',paste0(x,collapse = ', ' ),"')" )})
Table_Quantile
for(iter in 1:size){
    parameter=parameter_matrix[ iter ,]
    for(i in 1:length(u)){
        Table_Quantile[i ,iter]=MOHLW_quantile
            (parameter,u[i ])
    }
}
return( Table_Quantile )
}

#-----
## table of quantile
parameter_matrix=as.matrix(rbind(
par1=c(0.09,2.66,1.47) ,
par2=c(.1,1.2,2.7) ,
par3=c(.1,2.9,2.7) ,
par4=c(.1,1.5,2.5) ,
par5=c(.1,2.5,1.1)

))
EOETLLoG.QuantileTable( parameter_matrix)
sink("C:/Users/HpProbook4530s/Desktop
/Biust/EOETLLLOG_quantile_table.txt")
print( parameter_matrix)
print(EOETLLLOG.QuantileTable( parameter_matrix))
sink()

#-----

##define pdf EOETLG
EOETLLoG_pdf=function( delta ,b,c,x){
y= 2*delta*b*(c*x^(c-1))*((1+x^c)^(-3))
*((1-(1+x^c)^(-2))^(b-1))*(exp(-(((1-
(1+x^c)^(-2))^b)/ (1-((1+x^c)^(-2))^b)))
)*((1-exp(-(((1-(1+x^c)^(-2))^b)/(1-(1-
(1+x^c)^(-2))^b))))^(delta-1))*
(1-(1-(1+x^c)^(-2))^b)^(-2))

}

EOETLLoG_moments=function( delta ,b,c,n){
f=function( delta ,b,c,n,x){
    (x^n)*(EOETLLLOG_pdf( delta ,b,c,x))
}
y=integrate( f,lower=0,upper=1,subdivisions=10,

```

```

delta=delta ,b=b ,c=c ,n=n)
return(y$value)
}

### table of moments
parameter_matrix_Moments=matrix(c(0.09,2.66,1.47,
.1,1.2,2.7,.1,2.9,2.7,.1,
1.5,2.5,.1,2.5,1.1),ncol=3,byrow=T)
parameter_matrix_Moments
list_Moments=c('EX','EX2','EX3','EX4','EX5','EX6','SD',
'CV','CS','CK')
Table_Moments=matrix(0,nrow=length(list_Moments),
ncol=dim(parameter_matrix_Moments)[1])
row.names(Table_Moments)=list_Moments
Table_Moments

for(iter in 1:dim(parameter_matrix_Moments)[1]){
parameter=parameter_matrix_Moments[iter,]
for(i in 1:10){
Table_Moments[i,iter]=EOETLLoG_moments(parameter[1],
parameter[2],parameter[3],i)
}
}
print(Table_Moments)
###moments end
sink("C:/Users/HpProbook4530s/Desktop/
Biust/Table_Moments")
sink()

#-----

EOETLLoG_simulation=function(size=c(30,60,120,
240,480,960),
samp,par){
set.seed(1000)
K=length(par)
Mean=vector()
RMSE=vector()
Bias=vector()
samplesize=as.vector(t(mapply(rep,size,K)))
for(iter_size in 1:length(size)){
coef1=matrix(NA,samp,K)
colnames(coef1)=c('delta','b','c')
for(nsamp in 1:samp){
tryCatch(
{
x1 EOETLLoG=NULL
q=runif(size[iter_size],0,1)
x1=sapply(q,EOETLLoG_quantile,
parameter=par)

```

```

# MLE
x1_EOETLLoG=mle2(EOETLLoG_neglogl,
start=list(delta=par[1],
b=par[2],
c=par[3]),
method="L-BFGS-B",data=list(x=x1),
lower=c(delta=0,b=0,c=0),
upper=c(delta=Inf,b=Inf,c=Inf),
use.ginv=TRUE)coef1[nsamp,]=coef(x1_EOETLLoG)
},error=function(e){}
)
}
mean[length(size)*(0:(K-1))+iter_size]=
apply(coef1,2,mean,na.rm=TRUE)
RMSE[length(size)*(0:(K-1))+iter_size]
=apply((coef1-matrix(rep(par,
nsamp),ncol=K,byrow=T))^2,2,function(x)
{sqrt(mean(x,na.rm=TRUE))})
}
Bias=as.vector(sapply(1:K,function(x)
{Bias[(length(size)*(x-1)+1):
(length(size)*x)]=Mean[(length(size)*(x-1)+1):
(length(size)*x)]-par[x]}))
return(cbind(samplesize,Mean,RMSE,Bias))
}

-----

sink("C:/Users/HpProbook4530s/Desktop/Biust
/EOETLLLOG.simulation.1.txt",
append=F,split=F)
par=c(.1,.5)
par_p=c(1)
for(i in 1:length(par)){
for(j in 1:length(par)){
for(p in 1:length(par)){
for(q in 1:length(par_p)){
print(c(par[i],par[j],par[p]))
print(MOHLW_simulation(samp=1000,par=c(par[i],
par[j],par[p])))
}
}
}
}

sink()

## simulation
parameter_matrix=as.matrix(rbind(
par1=c(0.09,2.66,1.47),
par2=c(.1,1.2,2.7),
par3=c(.1,2.9,2.7),

```

```

par4=c(.1,1.5,2.5),
par5=c(.1,2.5,1.1)

}
Applications
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)

# maximum likelihood estimators

x<-c(0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183,
0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904,
0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228,
3.931, 9.337,0.086, 0.379, 1.600, 4.073, 10.940, 0.102,
0.381, 2.006,4.393,11.020, 0.103, 0.538, 2.054, 4.534,
13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116,
0.574, 3.058, 6.274, 15.08

)
####EOETLL
EOETLLLL<-function(delta,b,c)
{-sum(log(2*delta*b*(c*x^(c-1))*
((1+x^c)^(-3))*((1-(1+x^c)^(-2))^(b-1))
*(exp(-(((1-(1+x^c)^(-2))^b)/(1-
(1-(1+x^c)^(-2))^b))))*((1-
exp(-(((1-(1+x^c)^(-2))^b)/
(1-(1-(1+x^c)^(-2))^b))))^(delta-1))*
(1-(1-(1+x^c)^(-2))^b)^(-(2))))}
mle.result<-mle2(EOETLLLL,hessian = NULL,
start=list(delta=.902175,

b=1.5,c=1.178770),optimizer="nlminb",lower=0)
summary(mle.result)
vcov(mle.result)
output1<-vcov(mle.result)
1.96*sqrt(diag(output1))
solve(output1)
####TLMO-LLo
TLMOLLLL<-function(b,delta,c){-sum(log((2*b)*
(((delta*c*x^(c-1))*(1+x^c)^(-2))/
((1-(1-delta)*(1+x^c)^(-1))^2))*(((delta*
(1+x^c)^(-1))/(1-(1-delta)
*(1+x^c)^(-1))))*(1-((delta^2)*((1+x^c)^(-2)))/
(1-(1-delta)*((1+x^c)^(-1))^2)^(b-1))))}
mle.result<-mle2(TLMOLLLL,hessian =
NULL,start=list(b=1.902175,

```

```
delta=0.35190,c= 0.178770),optimizer=
"nllminb",lower=0)
summary(mle.result)
```

```
#####EW
EWLL<-function(alpha,beta,delta){
-sum(log(alpha*beta*delta*(x^(beta-1))
*(exp(-alpha*(x^beta)))*(1-exp(-alpha*
x^beta))^delta))}
mle.result<-mle2(EWLL,hessian = NULL,
start=list(alpha=1.2576,beta=.766,delta=.9),
optimizer="nllminb",lower=0)
summary(mle.result)
```

```
#####TLW
TLWLL<-function(a,b,alpha){-sum(log(2*alpha*b
*(a^b)*(x^(b-1))*(exp(-2*(a*x)^b))*
(1-exp(-2*(a*x)^b))^(alpha-1)))}
mle.result<-mle2(TLWLL,hessian = NULL,start=list(a=1,
b=.6,alpha=.891),
optimizer="nllminb",lower=0)
summary(mle.result)
```

```
#####TLGE
TLGE<-function(alpha,beta,lambda){-sum(log(2*alpha*
beta*lambda*(exp(-lambda*x))*((1-(1-exp(-
lambda*x))^beta))*
((1-exp(-lambda*x))^(beta*alpha-1))*(2-(1-exp(
-lambda*x))^beta)^(alpha-1)))}
mle.result<-mle2(TLGE,hessian = NULL,start=list
(beta=.6,alpha=.891,lambda=0.1),
optimizer="nllminb",lower=0)
summary(mle.result)
```

```
#####WLx
WLx<-function(a,b,alpha){-sum(log(alpha*a*b*((1+b*x)
)^(a*alpha-1))*((1-(1+b*x)^(-a))^(alpha-1))*
exp(-(((1-(1+b*x)^(-a))/(1+b*x)^(-a))^alpha))))}
mle.result<-mle2(WLx,hessian = NULL,
start=list(a=1,b=.6,
alpha=.891),optimizer="nllminb",lower=0)
summary(mle.result)
```

```
#####IWMO
IWMO<-function(alpha,lambda,theta){-sum(log((alpha*lambda
*(theta^(-lambda))*(x^(-lambda-1))*exp(-
(x*theta)^(-lambda)))/(alpha-(alpha-1)
*exp(-(x*theta)^(-lambda))^2))}
mle.result<-mle2(IWMO,hessian = NULL,start=list(alpha=2.5,
mbda=0.1,theta=.5),optimizer="nllminb",lower=0)
```

```
summary(mle.result)
```

```
#####MO-LLoG
```

```
MOLLD<-function(alpha,beta,gamma){-sum(log((
alpha^beta)*beta*gamma*
(x^(beta-1))/(x^beta+gamma*alpha^beta)^2))}
mle.result<-mle2(MOLLD,hessian = NULL,
start=list(alpha=1 ,beta=1.1961,
gamma=2.8644),optimizer="nlminb",lower=0)
summary(mle.result)
```

```
#####EF
```

```
EF<-function(alpha,delta,lambda){-sum(log((
alpha*lambda*delta^lambda)*((1-exp(-
(delta/x)^lambda))* (alpha-1))*(x^(-(lambda+1)))*
exp(-(lambda+1)*(delta/x)^lambda)))}
mle.result<-mle2(EF,hessian = NULL,
start=list(alpha=1.1787 ,
delta=0.01961,lambda= 2.8644),
optimizer="nlminb",lower=0)
summary(mle.result)
```