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Department of Mathematics & Statistical Sciences

Some New Transformed and Generalized Families of Distributions with Applications

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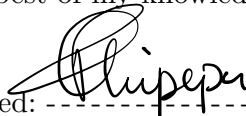
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Abstract

This thesis presents several new generalized classes and families of distributions, namely, the log-logistic modified Weibull power series (LLoGMWPS) class of distributions, type II Topp-Leone exponentiated half logistic-G power series (TII-TL-EHL-GPS) class of distributions, Harris-odd Burr type X-G (Harris-OBX-G) family of distributions, Marshall-Olkin Exponentiated Half Logistic-Type II Topp-Leone-G (MO-EHL-TII-TL-G) family of distributions, and Topp-Leone Harris-G (TL-Harris-G) family of distributions. These classes and families of distributions are developed using different combinations of generators and existing distributions. The mathematical and statistical properties of these new classes and families of distributions, including moments and moment generating functions, distribution of order statistics, probability weighted moments, and Rényi entropy, are derived and thoroughly examined. Additionally, the identifiability issue, concerning whether the parameters of the proposed models can be uniquely determined from observed data is investigated using mathematical functions for each class of distributions and family of distributions.

Finally, the flexibility and applicability of these new classes and families of distributions are demonstrated by fitting them to various real datasets. Several goodness-of-fit statistics and graphical plots are utilized to assess the adequacy of the models in capturing the complex characteristics of datasets across different domains.

Dedications

This dissertation is dedicated to my late father, Mr. Gosiamemang Ranko, who taught me never to give up until I achieve what I started.



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Published, Accepted and Submitted Papers

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- (i) N. Dingalo, B. Oluyede and F. Chipepa. **The Topp-Leone Gompertz-Exponentiated Half Logistic-G Family of Distributions with Applications.** Revista Colombiana de Estadística Journal, 46(2), 55-92, (2023). <https://doi.org/10.15446/rce.v46n2.105209>, doi: 10.15446/rce.v46n2.105209.
- (ii) N. Dingalo, B. Oluyede and F. Chipepa. **The Type II Exponentiated Half Logistic Odd Burr X-G Power Series Class of Distributions with Applications.** Heliyon, 9(1), e22401, (2023). <https://pubmed.ncbi.nlm.nih.gov/38058612/>, doi: 10.1016/j.heliyon.2023.e22402.
- (iii) B. Oluyede, N. Dingalo and F. Chipepa. **The Topp-Leone Harris-G Family of Distributions with Applications.** International Journal of Mathematics in Operational Research, 24(4), 554-582, (2023). <https://www.researchgate.net/publication/368322158>, doi:10.1504/IJMOR.2022.10054027.
- (iv) B. Oluyede, N. Dingalo and F. Chipepa. **A New and Generalized Class of Log-logistic Modified Weibull Power Series Distributions with Applications.** Thailand Statistician Journal, 22(2), 237-273, (2024). <https://statassoc.or.th>.

2. Accepted Papers

- (i) N. Dingalo, B. Oluyede and F. Chipepa. **The Type II Topp-Leone Exponentiated Half Logistic-G Power Series Class of Distributions with Applications.** Pakistan Journal of Statistics and Operation Research, (2022).
- (ii) B. Oluyede, N. Dingalo and F. Chipepa. **The Marshall-Olkin Exponentiated Half Logistic Type II Topp-Leone-G Family of Distributions with Applications.** International Journal of Mathematics in Operational Research, (2023).
- (iii) B. Oluyede, N. Dingalo and F. Chipepa. **The Harris Odd Burr Type X-G Family of Distributions with Applications.** International Journal of Operational Research, (2023).

3. Submitted and under Review

- (i) N. Dingalo, B. Oluyede and F. Chipepa. **The Marshall-Olkin Type II Exponentiated Half-logistic Topp-Leone-G Family of Distributions with Applications.** Pakistan Journal of Statistics and Operation Research, (2023).
- (ii) B. Oluyede, N. Dingalo and F. Chipepa. **The Marshall-Olkin Odd Burr X-Topp-Leone-G Family of Distributions with Applications.** Statistics in Transition New Series, (2023).

List of Abbreviations

Abbreviation	Meaning
ABias	Average Bias
AD	Anderson-Darling
AIC	Akaike Information Criterion
BT	Bathtub
BT-UBT	Bathtub followed by Upside Down Bathtub
BIC	Bayesian Information Criterion
BWLLoG	Beta Weibull Log-Logistic
BrWLn	Burr XII-Weibull logarithmic
CoD	Class of Distributions
CI	Confidence Interval
CAIC	Consistent Akaike Information Criterion
CV	Coefficient of Variation
CVM	Cramér-von Mises
cdf	Cumulative Distribution Function
ECDF	Empirical Cumulative Distribution Function
EBXIIP	Exponentiated Burr-XII Poisson
Exp-G	Exponentiated-G
EHL	Exponentiated Half Logistic
EHLow-TL-LLoG	Exponentiated Half Logistic Odd Weibull-Topp-Leone-Log-Logistic
EHL-TII-TL-G	Exponentiated Half Logistic Type II Topp-Leone-G
EKD	Exponentiated Kumaraswamy Dagum
EKW	Exponentiated Kumaraswamy Weibull
EMW	Exponentiated Modified Weibull
FoD	Family of Distributions
FIM	Fisher Information Matrix
GD	Gamma-Dagum

GoF	Goodness-of-Fit
GGP	Generalized Gompertz-Poisson
Harris-EL	Harris-Extended Lomax
HL-G	Half Logistic-G
HL-TII-TL-G	Half Logistic Type II-Topp-Leone-G
Harris-OBX-G	Harris-Odd Burr Type X-G
Harris-OBX-LLoG	Harris-Odd Burr Type X-Log-Logistic
Harris-OBX-W	Harris-Odd Burr Type X-Weibull
Harris-OBX-U	Harris-Odd Burr Type X-Uniform
hrf	Hazard Rate Function
iid	Independent Identically Distributed
K-M	Kaplan-Meier
K-S	Kolmogorov-Smirnov
KwW	Kumaraswamy Weibull
LRT	Likelihood Ratio Test
LLoG	Log-Logistic
LLoGEP	Log-Logistic Exponential Poisson
LLoGEPs	Log-Logistic Exponential Power Series
LLoGMEPS	Log-Logistic Modified Exponential Power Series
LLoGMW	Log-Logistic Modified Weibull
LLoGMWB	Log-Logistic Modified Weibull Binomial
LLoGMWG	Log-Logistic Modified Weibull Geometric
LLoGMWL	Log-Logistic Modified Weibull Logarithmic
LLoGMWP	Log-Logistic Modified Weibull Poisson
LLoGMWPS	Log-Logistic Modified Weibull Power Series
LLoGMRPS	Log-Logistic Modified Rayleigh Power Series
LLoGP	Log-Logistic Poisson
LLoGPS	Log-Logistic Power Series
LLoGW	Log-Logistic Weibull
LLoGWP	Log-Logistic Weibull Poisson
LLoGWPS	Log-Logistic Weibull Power Series

LLoGR	Log-Logistic Rayleigh
LLoGRPS	Log-Logistic Rayleigh Power Series
ML	Maximum likelihood
MO-EHL-TII-TL-G	Marshall-Olkin-Exponentiated Half Logistic-Type II Topp-Leone-G
MO-EHL-TII-TL-L	Marshall-Olkin-Exponentiated Half Logistic-Type II Topp-Leone-Lindley
MO-EHL-TII-TL-LLoG	Marshall-Olkin-Exponentiated Half Logistic-Type II Topp-Leone-Log-Logistic
MO-EHL-TII-TL-W	Marshall-Olkin-Exponentiated Half Logistic-Type II Topp-Leone-Weibull
MO-KA	Marshall-Olkin Kappa
MOMW	Marshall-Olkin Modified Weibull
MW	Modified Weibull
MGF	Moment Generating Function
MCS	Monte Carlo Simulation
OBX-G	Odd Burr Type X-G
OEHLBXII	Odd Exponentiated Half Logistic Burr XII
OPGWWP	Odd Power Generalised Weibull-Weibull Poisson
pdf	Probability Density Function
pmf	Probability Mass Function
PP	Probability Plot
PV	P-Value
QF	Quantile Function
RE	Rényi Entropy
RMSE	Root Mean Square Error
SD	Standard Deviation
SE	Standard Error
SS	Sum of Squares
TTT	Total Time on Test
TL-Harris-G	Topp-Leone Harris-G
TL-G	Topp-Leone-G
THL-G	Type I Half Logistic-G
TIIHL-G	Type II Half Logistic-G
TII-TL-EHL-G	Type II Topp-Leone Exponentiated Half Logistic-G

TII-TL-EHL-GPS	Type II Topp-Leone Exponentiated Half Logistic-G Power Series
TII-TL-EHL-LLoGL	Type II Topp-Leone Exponentiated Half Logistic-Log-Logistic Logarithmic
TII-TL-EHL-LLoGP	Type II Topp-Leone Exponentiated Half Logistic-Log-Logistic Poisson
TII-TL-EHL-WL	Type II Topp-Leone Exponentiated Half Logistic-Weibull Logarithmic
TII-TL-EHL-WP	Type II Topp-Leone Exponentiated Half Logistic-Weibull Poisson
TII-TL-G	Type II Topp-Leone-G
TIIEHLW	Type II Exponentiated Half Logistic Weibull
TIIGIE-BIII	Type II Generalised Inverse Exponential Burr III
TL-Harris-G	Topp-Leone-Harris-G
TL-Harris-LLoG	Topp-Leone-Harris-Log-Logistic
TL-Harris-U	Topp-Leone-Harris-Uniform
TL-Harris-W	Topp-Leone-Harris-Weibull
TL-MO-G	Topp-Leone Marshall-Olkin-G
TL-MO-W	Topp-Leone Marshall-Olkin-Weibull
TL-GW	Topp-Leone Generated Weibull
UBT	Upside Down Bathtub
UBT-BT	Upside Down Bathtub followed by Bathtub
Var-Cov	Variance-Covariance
$-2\ln(L)$	$-2\log$ -likelihood

Contents

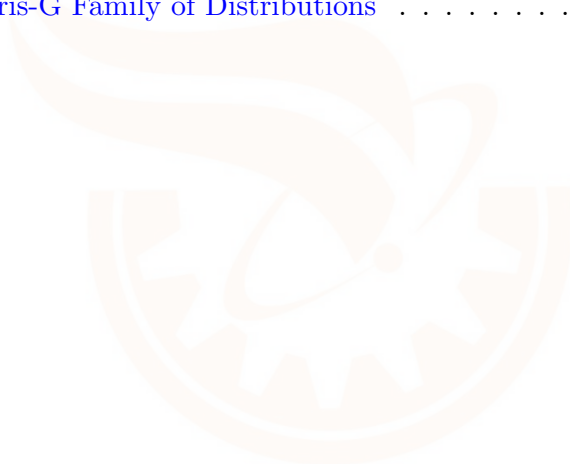
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1. Introduction and Literature Review

1.1 Introduction

The development of new families of distributions has been driven by the need to overcome the limitations for continuous families of distributions and accurately model complex real-life datasets from various fields of study, including economics, engineering, finance, insurance, demographics, biology, environmental, and medical sciences. Many existing classical distributions exhibit complex behavior, which makes them unsuitable for capturing the characteristics of real-world datasets. In many of these fields, the data typically exhibit complex behavior and distinct shapes associated with a non-monotonic hrf, varying degrees of skewness (SK), and kurtosis (KS). In response to these challenges, researchers have proposed several techniques to enhance the flexibility of classical distributions and develop new families of distributions. Some of these techniques include the method of generating skewed distributions, the transformed-transformer method, compounding method, TL-G, Half-Logistic-G, Exp-G, Marshall-Olkin-G, Burr-X-G, type II transformation and Harris-G.

For example, to address the issue of non-monotonic hrf, modifications of classical distributions have been proposed. Murthy et al. [82] discussed that distributions with bathtub-shaped failure rate are sufficiently complex and, therefore, difficult to model. The distribution proposed by Hjorth [53] is such an example. Rajarshi and Rajarshi [112] presented a revision of these distributions, and Haupt and Schäbe [52] developed a new lifetime model with bathtub-shaped failure rates. Unfortunately, these models are insufficient to address various practical situations, so new families of distributions based on modifications of the Weibull distribution have been discussed. For a comprehensive review of these models, please refer to Mudholkar and Srivastava [80] and Pham and Lai [108], where the authors summarized some generalizations of the Weibull distribution. The development and applications of generalized distributions have led to important and useful contributions to the statistical literature in recent times. For instance, Oluyede and Yang [103] introduced the beta generalized Lindley distribution, Oluyede et al. [98] discussed the gamma-Dagum distribution, as well as a general family of univariate distributions generated from Weibull distribution that was introduced by Gurvich et al. [49]. There are other generalizations that include the MW distribution by Lai et al. [65] and the beta exponential distribution presented by Nadarajah and Kotz [85]. Some more recent extensions are the generalized MW distribution studied by Carrasco et al. [23], the beta MW distribution reported by Silva et al. [124], Nadarajah et al. [83], and the gamma-exponentiated Weibull distribution by Pinho et al. [109].

Compounding methods have also been employed to generate new distributions by combining well known continuous distributions such as the exponential, Weibull, Burr XII and exponentiated exponential distributions with the power series distributions, which includes the Poisson, logarithmic, geometric and binomial distributions as particular cases. This approach has led to

the development of generalized distributions, such as the Weibull-power series CoD by Morais and Barreto-Souza [79]. Silva et al. [125] examined the extended Weibull power series CoD, which includes the exponential power series and Weibull power series distributions as special models. Silva and Cordeiro [126] introduced a new family of Burr XII power series distributions, and Oluyede et al. [92] further proposed the Burr-Weibull power series CoD. Oluyede et al. [102] proposed the LLoG-Weibull Poisson distribution which has applications in several areas including lifetime data analysis, reliability and economics. A number of generalized power series distributions have been proposed in the literature, including: Burr XII-Weibull-logarithmic distribution by Oluyede et al. [99], exponentiated generalized power series CoD proposed by Oluyede et al. [100], exponentiated extended Weibull power series CoD proposed by Tahmasebi and Jafari [130], exponentiated odd exponential half logistic-G power series CoD introduced by Rannona et al. [113], odd power generalized Weibull-G power series CoD proposed by Oluyede et al. [94], odd Lindley-G power series CoD by Chipepa et al. [29], Dagum-Poisson distribution by Oluyede et al. [101], exponentiated power Lindley-Poisson distribution by Pararai et al. [105], Ristic and Balakrishnan Lindley-Poisson distribution by Fagbamigbe et al. [43], generalized Burr XII power series CoD by Elbatal et al. [40], TR Y power series CoD by Osatohanmwen et al. [104], and inverse Lindley power series CoD by Shakhatreh et al. [121].

Additionally, techniques like the Marshall-Olkin transformation have been utilized to extend existing classical distributions by introducing additional shape parameters. The Marshall-Olkin FoD (Marshall and Olkin [74]) has some appealing properties that can add flexibility to any existing classical distribution. The Marshall-Olkin technique has been applied to develop the following generalized distributions: Marshall-Olkin Lehmann Lomax distribution by Aboraya [3], alpha power Marshall-Olkin-G FoD by Eghwerido et al. [38], Marshall-Olkin Kumaraswamy exponential distribution by Almarashi et al. [11], Marshall-Olkin Gompertz distribution by Eghwerido et al. [37], Marshall-Olkin half logistic-G FoD by Makubate et al. [69], Marshall-Olkin logistic-exponential distribution by Mansoor et al. [73], Marshall-Olkin alpha power FoD by Nassar et al. [86], extended Marshall-Olkin Burr III distribution by ul Haq et al. [135], Marshall-Olkin extended inverted Kumaraswamy distribution by Usman and ul Haq [136], Marshall-Olkin-Odd Weibull-G FoD by Makubate et al. [68], exponentiated Marshall-Olkin exponential distribution by Tharu et al. [132], Marshall-Olkin Burr X FoD by Jamal et al. [58] and Marshall-Olkin Lindley log-logistic distribution by Moakofi et al. [78].

Furthermore, the exponentiation method (Mudholkar and Srivastava [80]) has been employed to generalize distributions such as the Weibull distribution. Gupta et al. [46], Gupta and Kundu [47], Gupta and Kundu [48], and Kang and Seo [60] applied the exponentiated method. Generalizations via the EHL distribution are of tremendous practical importance, resulting in the development of distributions like the EHL-odd Burr III-G FoD by Oluyede et al. [96], EHL-power generalized Weibull-G FoD by Oluyede et al. [90], EHL-odd Lindley-G FoD by Segweni et al. [120], and EHL-logistic Weibull distribution by Chamunorwa et al. [25].

Other approaches, such as type II transformations, Harris-G and TL-G distributions, have also been explored to extend existing distributions and develop new generalized families of distributions with improved flexibility. The type II transformations including the type II-EHL-

TL-Marshall-Olkin-G FoD by Moakofi et al. [76]. Additionally, Alyami et al. [15] developed the type II half logistic odd Fréchet CoD, ZeinEldin et al. [142] introduced the type II-TL-inverted Kumaraswamy distribution, ZeinEldin et al. [141] introduced the type II half logistic Kumaraswamy distribution, Bantan et al. [19] introduced the type II power-TL-generated FoD, Oluyede and Moakofi [93] introduced the type II-EHL-Gompertz-TL-G FoD, Al-Jadaani and Al-Saiary [5] proposed a type II-TL-generalized inverted exponential distribution, Al-Mofleh et al. [6] developed the type II-EHL-generated FoD. TIIHL-G FoD was proposed by Soliman et al. [128], type II half logistic Exp-G FoD by Bello et al. [22], new TIIHL-G FoD by Altun et al. [13] and TIIGIE FoD by Jamal et al. [56].

Pinho et al. [110] presented the Harris extended CoD. This CoD was generated using the generic construction method described by Aly and Benkherouf [14]. Aly and Benkherouf [14] extended the Marshall-Olkin FoD by considering the pdf of the Harris distribution (Harris [51]). They considered the special case of the Harris extended FoD, then Batsidis and Lemonte [21] considered Harris extended FoD as a proportional hazard rate distribution. Sandhya et al. [118] discussed the Harris family of discrete distributions. Additional work on the Harris distribution include: Harris generalized linear exponential distribution by Paula and Joseb [106], Sivadasa and Joseb [127] introduced Harris extended Burr XII distribution, Harris extended two parameter Lindley distribution by Thomas et al. [133], and Harris extended power Lomax distribution by Ogunde et al. [88].

The TL distribution by Topp and Leone [134] has a closed-form cdf, bathtub-shaped hrf, and J-shaped pdf. It is, however, not a very adaptable distribution due to its domain constraint, (0,1). As a result, numerous researchers amended the TL distribution to increase its flexibility. Al-Shomrani et al. [8] developed a new FoD using the cdf of the TL distribution. Moments of J-shaped distributions were obtained by Nadarajah and Kotz [84]. Tahir et al. [129] obtained a bounded FoD, namely, the TL-G FoD. Rasheed [114] developed the TL Dagum distribution. Chipepa et al. [28] introduced the TL-Marshall-Olkin-G FoD, Chipepa and Oluyede [27] presented the TL-odd exponential half logistic-G FoD, Burr III TL-G FoD by Chipepa et al. [30], EHL-odd Weibull-TL-G FoD by Chipepa et al. [31], and TL-Gompertz-G FoD by Oluyede et al. [89].

In addition to the works listed above, researchers developed and presented new families of distributions using various techniques to extend existing distributions such as Weibull, LLoG, Burr type X, uniform, Lomax, and so on. Some generalizations of the Burr type X distribution include: odd-LLoG Burr type X FoD by Karamikabir et al. [61], Weibull Burr type X distribution by Ibrahim et al. [55], Burr type X exponentiated Weibull distribution by Khalil et al. [62], Burr type X generator distribution by Yousof et al. [140], Burr type X exponential distribution by Pelumi et al. [107], Burr X-Topp-Leone distribution by Usman et al. [137], Poisson Burr type X inverse Rayleigh distribution by Abdelkhalek [2], generalized Burr type X FoD by Jamal and Nasir [57], Burr type X exponentiated exponential distribution by Refaie [115], Marshall-Olkin Burr type X FoD by Jamal et al. [58], and type I half-logistic Burr type X distribution by Shrahili et al. [123], type I half logistic Burr X-G FoD by Algarni et al. [10], Weibull Burr X distribution by Ibrahim et al. [55] and beta Kumaraswamy Burr type X distribution by Madaki

et al. [67].

1.2 Some Generators

In this section, we will discuss the various generators used to develop new generalized classes and families of distributions in this thesis.

1.2.1 Competing Risk Method

The competing risk method arises when individuals or systems face multiple causes of failure, with the occurrence of one type of failure potentially precluding others. In real-world applications such as medical studies or engineering systems, multiple factors can lead to failure (Mukhopadhyay and Basu [81]). By using competing risk methods as one of the generators in our study, we aim to improve the reliability and applicability of our results by adding parameters that account for the shape of the hazard function. Extensive research has been conducted into statistical inference for competing risks models, with contributions from a variety of researchers. For example, Kundu et al. [64] introduced new methodologies and applications for competing risks models, focusing on areas such as survival analysis and event history modeling. Freels et al. [44] discussed recent developments and applications of statistical inference for competing risk method. From these studies, researchers gain a more comprehensive understanding of the competing risk method. This comprehensive perspective not only emphasises the importance of these contributions, but also clarifies their relevance in guiding future research efforts and practical applications.

We assume a series system with the components' lifetimes as follows: $F_1(t)$ and $F_2(t)$ distributions with survival functions $R_1(t) = 1 - F_1(x)$ and $R_2(t) = 1 - F_2(t)$, respectively. The survival function $R(t) = P(T > t)$ of the system is given by

$$R(t) = \prod_{i=1}^2 R_i(t). \quad (1.1)$$

The resulting cdf is given by $F(t) = 1 - R(t)$. In terms of reliability, a series model is referred to as a competing risk model and has the pdf obtained by differentiating $F(t)$.

1.2.2 Compounding Method

The compounding method combines discrete distributions, such as the power series distributions which includes binomial, Poisson, geometric and logarithmic distributions (Noack [87]), with continuous distributions. This method is used in many applications, including income modeling, finance, and lifetime analysis. Compound distributions accurately model real-world data that exhibit complex behavior and unique shapes.

Let X_i , $i=1,2,\dots,N$ be iid random variables with cdf G . Assuming, N be a discrete random

variable, which is zero truncated with a power series distribution, then the pmf is given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, n = 1, 2, 3, \dots, \quad (1.2)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n \geq 1}$ a sequence of positive real numbers. Table 1.1 shows some useful quantities for some power series distributions.

Table 1.1: Useful Quantities for Some Power Series Distributions

Distribution	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	a_n	Parameter Space
Binomial	$(1 + \theta)^m - 1$	$m(1 + \theta)^{m-1}$	$m(m-1)(1 + \theta)^{m-2}$	$(\theta - 1)^{1/m} - 1$	$\binom{m}{n}$	$(0, 1)$
Poisson	$e^\theta - 1$	e^θ	e^θ	$\log(1 + \theta)$	$(n!)^{-1}$	$(0, \infty)$
Geometric	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(1 + \theta)^{-1}$	1	$(0, 1)$
Logarithmic	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	n^{-1}	$(0, 1)$

1.2.2.1 Case 1

Let $X_{(1)} = \min(X_1, X_2, \dots, X_N)$, then the cdf of $X_{(1)}|N = n$ is given by

$$F_{X_{(1)}|N=n}(x) = 1 - (1 - G(x))^n, n \geq 1. \quad (1.3)$$

The marginal distribution of $X_{(1)}$, is given by,

$$\begin{aligned} F_{X_{(1)}}(x) &= \sum_{n=1}^{\infty} P(N = n)P(X_n \leq x|N = n) \\ &= \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} (1 - G(x))^n \\ &= \frac{C(\theta(1 - G(x)))}{C(\theta)}, x > 0. \end{aligned} \quad (1.4)$$

1.2.2.2 Case 2

Let $X_{(1)} = \max(X_1, X_2, \dots, X_N)$, then the cdf of $X_{(1)}|N = n$ is given by

$$F_{X_{(1)}|N=n}(x) = (G(x))^n, n \geq 1. \quad (1.5)$$

The marginal distribution of $X_{(1)}$, is given by,

$$\begin{aligned} F_{X_{(1)}}(x) &= \sum_{n=1}^{\infty} P(N = n)P(X_n \leq x|N = n) \\ &= \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} (G(x))^n \\ &= \frac{C(\theta G(x))}{C(\theta)}, x > 0. \end{aligned} \quad (1.6)$$

1.2.3 Exponentiated-G Generator

The exponentiation method is one of the most popular method that has been widely considered when extending existing classical distributions in-order to improve their flexibility and applicability. Lehmann [66] first introduced the Exp-G generator with the cdf and pdf given by

$$F_{Exp-G}(x; \lambda, \psi) = G^\lambda(x; \psi), \quad (1.7)$$

and

$$f_{Exp-G}(x; \lambda, \psi) = \lambda g(x; \psi) (G(x; \psi))^{\lambda-1}, \quad (1.8)$$

respectively, for $x \geq 0$, $\lambda > 0$ and parameter vector ψ .

Gupta et al. [46] proposed the Exp-G generator as a proportional hazard rate family having the cdf and pdf given by

$$F_{Exp-G}(x; \lambda, \psi) = 1 - [1 - G(x; \psi)]^\lambda, \quad (1.9)$$

and

$$f_{Exp-G}(x; \lambda, \psi) = \lambda g(x; \psi) [1 - G(x; \psi)]^{\lambda-1}, \quad (1.10)$$

respectively, for $x \geq 0$, $\lambda > 0$ and parameter vector ψ .

1.2.4 Half-Logistic-G Generator

The half-logistic distribution (Balakrishnan [18]), with its simple pdf and cdf, is tractable but limited in its ability to capture complex data characteristics. The HL-G generator adds flexibility by accommodating skewed and bimodal data modeling, and different hrf shapes. Cordeiro et al. [34] proposed the TIHL-G FoD with the cdf given as

$$\begin{aligned} F(x; \gamma, \psi) &= \int_0^{-\ln(1-G(x;\psi))} \frac{2\gamma e^{-\gamma t}}{(1 + e^{-\gamma t})^2} dt \\ &= \frac{1 - [1 - G(x; \psi)]^\gamma}{1 + [1 - G(x; \psi)]^\gamma}, \end{aligned} \quad (1.11)$$

where $x \geq 0$, $G(x; \psi)$ is the baseline cdf depending on a parameter vector ψ and an additional shape parameter $\gamma > 0$. If we take $\gamma = 1$, then the TIHL-G reduces to the HL-G FoD with cdf

$$F(x; \psi) = \frac{G(x; \psi)}{1 + \overline{G}(x; \psi)}, \quad (1.12)$$

where $\overline{G}(x; \psi) = 1 - G(x; \psi)$.

The corresponding pdf to equation (1.12) is given by

$$f(x; \psi) = \frac{2g(x; \psi)}{[1 + \overline{G}(x; \psi)]^2}, \quad (1.13)$$

for $x \geq 0$, and parameter vector ψ , $\overline{G}(x; \psi)$ and $g(x; \psi)$ are the survival function and pdf of the baseline distribution, respectively.

1.2.5 Topp-Leone-G Generator

The TL distribution (Topp and Leone [134]) was modified by Al-Shomrani et al. [8]. The TL distribution has domain restriction of (0,1) and a bathtub shaped hrf. The generalizations of the TL distribution enhance its flexibility and allow it to model different non-monotonic hrf. The TL-G FoD (Al-Shomrani et al. [8]) cdf and pdf are given by

$$F_{TL-G}(x; b, \psi) = \left[1 - \overline{G}^2(x; \psi)\right]^b \quad (1.14)$$

and

$$f_{TL-G}(x; b, \psi) = 2b \left[1 - \overline{G}^2(x; \psi)\right]^{b-1} \overline{G}(x; \psi) g(x; \psi), \quad (1.15)$$

respectively, for $x \geq 0$, $b > 0$ and parameter vector ψ .

1.2.6 Burr-X Generator

The Burr-X generator (Yousof et al. [140]) is commonly used to model heavy-tailed datasets. The cdf and pdf of Burr X-G FoD are given as follows:

$$F(x; \theta, \psi) = \left[1 - \exp \left\{ - \left(\frac{G(x; \psi)}{\overline{G}(x; \psi)} \right)^2 \right\}\right]^\theta \quad (1.16)$$

and

$$f(x; \theta, \psi) = \frac{2\theta g(x; \psi) G(x; \psi)}{[\overline{G}(x; \psi)]^3} \exp \left\{ - \left(\frac{G(x; \psi)}{\overline{G}(x; \psi)} \right)^2 \right\} \left[1 - \exp \left\{ - \left(\frac{G(x; \psi)}{\overline{G}(x; \psi)} \right)^2 \right\}\right]^{\theta-1}, \quad (1.17)$$

for $x \geq 0$, $\theta > 0$, and baseline parameter vector ψ .

1.2.7 Gamma-G Type I and Type II Transformations

Zografos and Balakrishnan [143] proposed the Zografos-Balakrishnan-G FoD. This transformations provide flexibility in modeling long-tailed, short-tailed, heavy-tailed data, as well as non-monotonic hrf. The cdf and pdf are given by

$$F_{ZB-G}(x; \delta, \xi) = \frac{1}{\Gamma(\delta)} \int_0^{-\log[1-G(x; \xi)]} t^{\delta-1} e^{-t} dt = \frac{\gamma(-\log[1-G(x; \xi)], \delta)}{\Gamma(\delta)}$$

and

$$f_{ZB-G}(x; \delta, \xi) = \frac{1}{\Gamma(\delta)} (-\log[1-G(x; \xi)])^{\delta-1} g(x; \xi), \quad (1.18)$$

respectively, for $x > 0$, $\delta > 0$, where $\Gamma(\delta) = \int_0^\infty t^{\delta-1} e^{-t} dt$ and $\gamma(\delta, z) = \int_0^z t^{\delta-1} e^{-t} dt$ is the incomplete gamma function.

Ristic and Balakrishnan [117] introduced gamma-G type II family known as the Ristic-Balakrishnan-G FoD with pdf and cdf, respectively given by

$$F_{RB-G}(x; \delta, \xi) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log[G(x; \xi)]} t^{\delta-1} e^{-t} dt = 1 - \frac{\gamma(-\log[G(x; \xi)], \delta)}{\Gamma(\delta)} \quad (1.19)$$

and

$$f_{RB-G}(x; \delta, \xi) = \frac{g(x; \xi)}{\Gamma(\delta)} (-\log[G(x; \xi)])^{\delta-1}, \quad (1.20)$$

for $x \geq 0$, $\delta > 0$ and parameter vector ξ .

1.2.8 Harris-G Generator

The Harris distribution (Harris [51]) has been discussed in various studies. Extending the Harris-G allows it to capture different properties of data, such as monotonic, non-monotonic shapes, SK, and KS. This ability to capture a wide range of data characteristics makes the extended Harris distribution versatile in many different applications. Pinho et al. [110] developed the Harris-G FoD with the cdf given by

$$F_{Harris-G}(x; v, c, \psi) = 1 - \left(\frac{c (\bar{G}(x; \psi))^v}{1 - \bar{c} (\bar{G}(x; \psi))^v} \right)^{\frac{1}{v}}, \quad (1.21)$$

for $x \geq 0$, $v, c > 0$, $\bar{c} = 1 - c$, $\bar{G}(x; \psi) = 1 - G(x; \psi)$, where $G(x; \psi)$ is the baseline cdf with parameter vector ψ . The corresponding pdf is given by

$$f(x; v, c, \psi) = \frac{c^{\frac{1}{v}} g(x; \psi)}{[1 - \bar{c} \bar{G}(x; \psi)^v]^{1 + \frac{1}{v}}}, \quad (1.22)$$

for $x \geq 0$, $v, c > 0$, and $\bar{c} = 1 - c$. When $v = 1$, equation (1.21) reduces to the Marshall-Olkin-G FoD.

1.2.9 Marshall-Olkin-G Generator

Marshall and Olkin [74] proposed a method for adding extra parameters to a FoD. Several authors have extended existing classical distributions using the Marshall-Olkin method. This resulted in increased flexibility in modeling complex behaviors and distinct shapes of data. The cdf and pdf of Marshall-Olkin-G FoD are given by

$$F(x; \alpha) = 1 - \frac{\alpha \bar{G}(x; \xi)}{1 - \bar{\alpha} \bar{G}(x; \xi)} \quad (1.23)$$

and

$$f(x; \alpha) = \frac{\alpha g(x; \xi)}{(1 - \bar{\alpha} \bar{G}(x; \xi))^2}, \quad (1.24)$$

for $x \geq 0$, $\alpha > 0$, $\bar{\alpha} = 1 - \alpha$ and $\bar{G}(x; \xi) = 1 - G(x; \xi)$, where $G(x; \xi)$ is the baseline cdf. Note that when $\alpha = 1$, we have $G(x; \psi) = F(x; \xi)$, the baseline distribution function.

Marshall and Olkin called the additional shape parameter α , the tilt parameter, since for all $x > 0$, the hrf of the new FoD are ordered based on the values of α , that is, $h_G(x) \leq h_F(x)$ when $\alpha \geq 1$, and $h_G(x) \geq h_F(x)$, when $0 < \alpha \leq 1$.

1.3 Mathematical and Statistical Properties

1.3.1 Survival Function

The survival function is pivotal in determining the probability of survival or reliability of systems, and is given by $S(x) = P(X \geq x) = 1 - F(x)$. The survival function is used to estimate the probability of surviving beyond a certain time.

The survival function is particularly useful in fields such as medicine, where it can be used for obtaining the probability of a cancer patient surviving a certain length of time after starting chemotherapy treatment. Other applications of the survival function include engineering, finance, forensic science, and ecology.

1.3.2 Hazard Rate Function

The hrf is a non-negative function of time, x , that represents the instantaneous risk of an event occurring at time x . The hrf is given by $h_F(x) = f(x)/S(x)$. The hrf can take on a variety of forms, including monotonically increasing, monotonically decreasing, and non-monotonic.

In practice, the hrf is applied in a variety of fields, including medicine, finance, engineering, insurance, and actuarial science. In medicine, a monotonically increasing hrf can be used to model the increased risk of mortality as people age. A monotonically decreasing hrf can be used to model the decreasing risk of infant mortality as infants grow. A non-monotonic hrf can be used to model situations where the risk of an event varies over time, such as the 'bathtub curve' in medical research, which shows an initial decrease in event risk followed by a constant hrf, and then an increase over time. For example a population being studied from infancy to old age.

1.3.3 Quantile Function

The QF is the inverse of the cdf. The algorithm for obtaining random samples from the derived continuous distribution using the inverse cdf is:

- Define the random variable X with the cdf $F(x)$.
- Obtain the QF: $F^{-1}(u) = \inf \{x : F(x) \geq u\}$, $0 < u < 1$.

- Solve the non-linear equation to find the QF, which represents the inverse cdf of the distribution.

Thus, random numbers can be generated by applying the inverse cdf to uniform random variables. The QF is used for generation of random numbers during simulation studies.

1.3.4 Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution. These random variables are iid, with pdf $f(x)$ and cdf $F(x)$. The pdf of the h^{th} order statistic $X_{h:n}$, is given by

$$f_{h:n}(x) = \frac{n!f(x)}{(h-1)!(n-h)!} [F(x)]^{h-1} [1-F(x)]^{n-h}.$$

Order statistics have many real-world applications, including engineering, survival analysis, finance, and social science. For example in engineering, order statistics are used to determine the probability that the system will fail under a given load.

1.3.5 Rényi Entropy

In information theory, entropy is a measure of the uncertainty or randomness of a system. RE (Rényi [116]) is defined by

$$I_R(\rho) = (1-\rho)^{-1} \log \left[\int_0^\infty f^\rho(x) dx \right], \quad (1.25)$$

where $\rho > 0$ and $\rho \neq 1$, and Shannon entropy (Shannon [122]) is given by $E\{-\log[f(X)]\}$. Shannon entropy is a special case of RE.

RE has been applied in many different areas, including finance, geography, machine learning, physics, medicine and game theory. For example in physics, RE is used to measure the amount of information that is lost when a system undergoes a transformation.

1.3.6 Probability Weighted Moments

In probability theory, probability weighted moments (PWMs) are a generalization of the classical moments, which are defined as the expected values of powers of a random variable. The PWMs are defined as follows:

$$Z_{\eta,i} = E(X^\eta [F(X)]^i) = \int_0^\infty x^\eta [F(x)]^i f(x) dx.$$

The PWMs have a wide range of applications in statistics and engineering. The PWMs approach is a useful alternative to traditional moment-based approaches when estimating distribution parameters, especially when the inverse form of the distribution is unknown. The PWMs method is more accurate than traditional moment-based approach and is often used in cases where ML estimates are not possible or are computationally difficult to obtain.

1.4 Maximum Likelihood Estimation Method

In statistics, ML estimation is a popular method for estimating the parameters of a distribution. ML estimation is based on the assumption that the data follows a given probability distribution. By finding the values of the parameters that maximize the likelihood function, which is the probability of observing the given data, we can obtain the estimate of the parameters. The total likelihood function for a given pdf is $L(\theta; x) = \prod_{i=1}^n f(x_i; \theta)$. The elements of the vector of parameters θ are optimized by setting the partial derivatives of the log-likelihood function equal to zero.

The total FIM $n\mathbf{I}(\Delta)$ can be approximated by

$$\mathbf{J}_n(\hat{\Delta}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \Big|_{\Delta=\hat{\Delta}} \right]_{q \times q}, \quad i, j = 1, 2, 3, 4, \dots, q. \quad (1.26)$$

The total FIM (1.26) can be obtained numerically using MATLAB, SAS, or R. This matrix is used to estimate the asymptotic covariance matrix of the ML estimates. Under the usual regularity conditions and assuming that the parameters are located within the interior of the parameter space, though not on the boundary, we observe the following asymptotic result:

$$\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} \mathcal{N}_q(\mathbf{0}, I(\Delta)^{-1}),$$

where $I(\Delta)$ denotes the expected FIM. This asymptotic behavior remains valid even when $I(\Delta)$ is replaced by the observed information matrix evaluated at $\hat{\Delta}$, denoted as $J(\hat{\Delta})$. Consequently, we can utilize the multivariate normal distribution $\mathcal{N}_q(\mathbf{0}, J^{-1}(\hat{\Delta}))$ to construct CI and confidence regions for the individual model parameters as well as for the survival function and hrf.

The likelihood function is maximized using NLmixed in SAS, as well as the function nlm in Team [131]. The application of these functions to different initial values frequently results in multiple maxima, and if none are found, additional initial values are used to ensure a maximum is obtained. We use profile log-likelihood plots to determine the existence and uniqueness of ML estimates. In these plots, a clear peak at a specific parameter value indicates the presence of an ML estimation, whereas "plateaus" where the ML is achieved at multiple parameter values indicate the absence of a single ML estimation. As profile log-likelihood plots rely on the shape of the likelihood function, using them to identify the uniqueness and existence of ML estimation could potentially offer a more efficient and reliable alternative.

Although ML estimation is a widely used method, it has some limitations that should be taken into consideration such as the potential for not obtaining global optima and difficulties in dealing with constraints in the parameter space. However, despite these limitations, ML estimation remains a popular method of estimation because of its desirable properties such as the consistency and efficiency of the ML estimates.

1.5 Methods for Assessing GoF

The new generalized classes and families of distributions are compared to the nested and non-nested models using several GoF statistics. We use the following GoF statistics: $-2\ln(L)$, $AIC = 2p - 2\ln(L)$, $BIC = p\ln(n) - 2\ln(L)$ and $CAIC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the total number of observations, and p is the number of estimated parameter. Some additional useful GoF statistics include CVM, AD, K-S and its corresponding PV, and SS from the PPs. Chen and Balakrishnan [26] discussed CVM and AD GoF statistics. The SS from the PPs is defined as

$$SS = \sum_{i=1}^n \left[F(x_{(i)}) - \left(\frac{i - 0.375}{n + 0.25} \right) \right]^2,$$

where $i = 1, 2, \dots, n$, and $x_{(i)}$ are the observed data with ordered values (Chambers et al. [24]). The model with the smallest GoF statistics value and the highest PV for the K-S statistics is regarded as the best fitting model.

1.6 Likelihood Ratio Test

The LRT statistic is commonly used to compare the fit of proposed distributions with its sub-models. The LRT statistic is given by:

$$\Lambda = 2 \left[\ln \left(L(\hat{\theta}_1) \right) - \ln \left(L(\hat{\theta}_0) \right) \right],$$

where $\hat{\theta}_1$ represents the parameter estimates under the unrestricted model, and $\hat{\theta}_0$ represents the parameter estimates under the restricted model.

The LRT statistic rejects the null hypothesis if $\Lambda > \chi_{\alpha;p}^2$, where $\chi_{\alpha;p}^2$ denotes the upper $100_{\alpha}\%$ point of the χ^2 distribution with p degrees of freedom, where p is the difference in the number of parameters between the unrestricted and restricted models.

1.7 Aims and Objectives

1.7.1 Main Objectives

The main goal of this work is to use several new combinations of generators in the literature for the development of new, generalized and flexible classes and families of distributions that are suitable for modeling data in different application domains.

1.7.2 Specific Objectives

1. To develop the following generalized classes and families of distributions:

- LLoGMWPS CoD.
- TII-TL-EHL-GPS CoD.

- Harris-OBX-G FoD.
 - MO-EHL-TII-TL-G FoD.
 - TL-Harris-G FoD.
2. To derive some of the statistical and mathematical characteristics of these new generalized classes and families of distributions.
 3. To estimate the parameters of these classes and families of distributions via the ML estimation technique.
 4. To illustrate the consistency of ML estimates through simulation experiments.
 5. To demonstrate the utility and applicability of these new generalized classes and families of distributions by fitting them to real-life datasets from a variety of fields.

1.8 Motivations

The development of generalized classes and families of distributions that are adaptable to both monotonic and non-monotonic hrfs are of tremendous practical importance in probability and statistics. Our main motivation in developing new classes and families of distributions lies in the ability of the hrfs to exhibit different shapes such as increasing, decreasing, UBT, BT, BT-UBT and UBT-BT. This flexibility allows for more accurate representation of real-life situations where the hrf may exhibit complex shapes.

The limitations of classical distributions and existing families of distributions in modeling different hrf emphasize the need to continue developing new classes and families of distributions that can adequately represent the complexities of real-world data. These proposed classes and families of distributions are designed to overcome the limitations of classical and existing distributions, with the aim of providing improved modeling capabilities and increased reliability. By accurately representing real-life situations, these new classes and families of distributions can lead to more effective decision-making across variety of lifetime situations.

Unlike classical and some existing distributions, which are typically restricted to modeling either right-skewed or left-skewed data, the new generalized classes and families of distributions can accommodate wide range of shapes including almost symmetric, negative-skewed or positive-skewed, J and reverse-J. This flexibility of the proposed distributions overcome that of classical and existing distributions, making the new generalized classes and families of distributions more flexible in capturing extreme events and rare occurrences. Such adaptability enhances their applicability and suitability in many applications. These new classes and families of distributions were developed through different generators that introduced additional parameters serving as scale and shape parameters to introduce SK for symmetrical distributions and modulate the weight of the tails of any extended classical distribution.

The applicability of the new classes and families of distributions through real datasets showcase their usefulness in solving practical problems. This bridges the gap between theoretical developments making the research relevant and needful to the current societal needs. These new classes

and families of distributions provide researchers with the ability to better capture lifetime data from different domains. The special cases of the new generalized classes and families of distributions demonstrate superior fit compared to parent distributions and other existing models available in the literature.

1.9 Outline of Thesis

The structure of this thesis is as follows: Chapter 2 presents development of the LLoGMWPS CoD. Chapter 3 introduces the CoD known as TII-TL-EHL-GPS distribution. The Harris-OBX-G FoD is presented in Chapter 4. In Chapter 5, we present the MO-EHL-TII-TL-G FoD. Chapter 6, presents TL-Harris-G FoD. We finally give conclusions and areas of further study under Chapter 7.



2. A New and Generalized Class of Log-logistic Modified Weibull Power Series Distributions

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2.1 Introduction

Compound CoD are important because of their flexibility in modeling distributions with both monotonic and non-monotonic hrf which are encountered in real-life data.

A new generalized CoD is obtained by combining the LLoGMW distribution (Oluyede et al. [97]) with the power series distribution (Noack [87]). The new CoD has added dispersion parameter, depending on the overall form that accounts for the scale of the underlying random variable and has exponential dumping in the upper tail making the distribution suitable for modeling samples that display power behaviour for intermediate observations and a decrease in tail probability for large observations or beyond a certain threshold or specified value. The proposed CoD offer superior flexibility in comparison to existing distributions, such as the Weibull, log-normal, gamma, Rayleigh, and exponential distributions, due to the inclusion of an additional dispersion parameter. This additional parameter allows for greater control over the shape of the distribution and the ability to capture a wider range of data shapes and behaviors.

The mathematical properties of the LLoGMWPS CoD, along with its special cases (LLoGMWP and LLoGMWL distributions), are comprehensively explored. The LLoGMWPS CoD generalizes several existing distributions, encompassing variations such as the LLoGEPS, LLoGWPS, LLoGRPS, LLoGPS CoD, as well as other related distributions like LLoGMW, LLoGW, and LLoG distributions. These generalizations enhance the modeling capabilities of the new CoD, enabling it to effectively represent a wide range of data encountered across various fields. To illustrate the flexibility of LLoGMWP and LLoGMWL distributions, we present plots of the pdf and hrf, as well as the 3D plots of SK and KS. We apply the method of ML estimation to estimate the parameters of this new distribution.

Real-world applications of the LLoGMWP and LLoGMWL distributions are demonstrated through case studies drawn from fields such as reliability and biology. These applications showcase the practical utility and effectiveness of the proposed distributions in modeling complex data scenarios. The results of the LLoGMWP and LLoGMWL distributions are compared with nested and non-nested models. Comparative analyses with nested and non-nested models, including five-parameter distributions and those generated via similar transformation techniques, further highlight the superiority and uniqueness of the LLoGMWP and LLoGMWL distributions

in capturing the underlying characteristics of diverse datasets.

2.2 The Log-Logistic Modified Weibull Power Series Class of Distributions

In this section, the LLoGMWPS CoD and some of its statistical properties are presented. We first of all present the LLoG and MW distributions. The cdf and pdf of the LLoG distribution are given by $F_{LLoG}(x) = 1 - (1 + x^c)^{-1}$ and $f_{LLoG}(x) = cx^{c-1}(1 + x^c)^{-2}$, for $c > 0$ and $x > 0$. The MW distribution by Lai et al. [65] has pdf given by

$$f_{MW}(x; \alpha, \beta, \lambda) = \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x) \exp(-\alpha x^{\beta} e^{\lambda x}),$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, and $\lambda \geq 0$. Note that the parameter α control the scale of the distribution, β controls the shape, whereas λ can be considered to be an accelerating factor in the imperfection time and a factor of fragility in the survival of the individual as time increases. When $\lambda = 0$, we obtain the Weibull distribution. The Weibull distribution is well known and has been extensively used for modeling data in several areas including reliability. Weibull distribution is particularly useful for modeling monotone failure rates.

Consider a sequence of N iid random variable, say X_i , $i = 1, \dots, N$, from the LLoGMW distribution developed by Oluyede et al. [97] with cdf given by

$$F_{LLoGMW}(x; c, \alpha, \beta, \lambda) = 1 - (1 + x^c)^{-1} \exp(-\alpha x^{\beta} e^{\lambda x}),$$

and the corresponding pdf given by

$$f_{LLoGMW}(x; c, \alpha, \beta, \lambda) = e^{-\alpha x^{\beta} e^{\lambda x}} \left(1 + x^c\right)^{-1} \left\{ \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x) + \frac{cx^{c-1}}{(1 + x^c)} \right\},$$

for $c, \alpha, \beta > 0$, $\lambda \geq 0$, and $x > 0$. The survival function and hrf are given by

$$S_{LLoGMW}(x; c, \alpha, \beta, \lambda) = 1 - F_{LLoGMW}(x; c, \alpha, \beta, \lambda) = (1 + x^c)^{-1} \exp(-\alpha x^{\beta} e^{\lambda x})$$

and

$$h_{LLoGMW}(x; c, \alpha, \beta, \lambda) = \frac{f_{LLoGMW}(x; c, \alpha, \beta, \lambda)}{S_{LLoGMW}(x; c, \alpha, \beta, \lambda)} = \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x) + \frac{cx^{c-1}}{(1 + x^c)},$$

respectively.

Now, let N be a discrete random variable that follows a power series distribution assumed to be truncated at zero, whose pmf is given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \dots,$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$, and $\{a_n\}_{n \geq 1}$ a sequence of positive real numbers. Considering $X_{(1)} = \min(X_1, \dots, X_N)$ and conditioning upon $N = n$, the conditional distribution

of $X_{(1)}$ given $N = n$ is obtained as

$$G_{X_{(1)}|N=n}(x) = 1 - \prod_{i=1}^n (1 - F_{LLoGMW}(x)) = 1 - S_{LLoGMW}^n(x) = 1 - (1 + x^c)^{-n} \exp(-n\alpha x^\beta e^{\lambda x}).$$

Theorem 1. The cdf of $X_{(1)}$, say $F_{LLoGMWPS}$, is given by

$$F_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) = 1 - \frac{C\left(\theta(1 + x^c)^{-1} \exp(-\alpha x^\beta e^{\lambda x})\right)}{C(\theta)}.$$

Proof. Note that

$$F_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) = \sum_{n=1}^{\infty} G_{X_{(1)}|N=n}(x) P(N = n) \quad (2.1)$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \left[1 - (1 + x^c)^{-n} \exp(-n\alpha x^\beta e^{\lambda x}) \right] \frac{a_n \theta^n}{C(\theta)} \\ &= \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} - \sum_{n=1}^{\infty} \frac{a_n [\theta(1 + x^c)^{-1} \exp(-\alpha x^\beta e^{\lambda x})]^n}{C(\theta)} \\ &= 1 - \frac{C\left(\theta(1 + x^c)^{-1} \exp(-\alpha x^\beta e^{\lambda x})\right)}{C(\theta)}. \end{aligned} \quad (2.2)$$

□

Theorem 2. $\lim_{\theta \rightarrow 0^+} F_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) = F_{LLoGMW}(x; c, \alpha, \beta, \lambda, \theta).$

Proof. Recall that $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$. Then,

$$\begin{aligned} F_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) &= 1 - \frac{\sum_{n=1}^{\infty} a_n [\theta(1 + x^c)^{-1} \exp(-\alpha x^\beta e^{\lambda x})]^n}{\sum_{n=1}^{\infty} a_n \theta^n} \\ &= 1 - \frac{\sum_{n=1}^{\infty} a_n \theta^n S_{LLoGMW}^n(x)}{\sum_{n=1}^{\infty} a_n \theta^n}. \end{aligned}$$

Letting $\theta \rightarrow 0^+$, the second term to the right of this equation is undetermined. Thus, applying l'Hopital's rule, we have

$$\begin{aligned} \lim_{\theta \rightarrow 0^+} F_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) &= 1 - \lim_{\theta \rightarrow 0^+} \frac{\sum_{n=1}^{\infty} n a_n \theta^{n-1} S_{LLoGMW}^n(x)}{\sum_{n=1}^{\infty} n a_n \theta^{n-1}} \\ &= 1 - \lim_{\theta \rightarrow 0^+} \frac{a_1 S_{LLoGMW}(x) + \sum_{n=2}^{\infty} n a_n \theta^{n-1} S_{LLoGMW}^n(x)}{a_1 + \sum_{n=2}^{\infty} n a_n \theta^{n-1}} \\ &= 1 - S_{LLoGMW}(x; c, \alpha, \beta, \lambda) \\ &= F_{LLoGMW}(x; c, \alpha, \beta, \lambda). \end{aligned}$$

□

Remark 1. Let $C'(\theta)$ be the derivative of $C(\theta)$, that is, $C'(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1}$. Then the

density of $F_{LLoGMWPS}$, say $f_{LLoGMWPS}$, is given by

$$\begin{aligned} f_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) &= \frac{dF_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta)}{dx} \\ &= \frac{\theta f_{LLoGMW}(x; c, \alpha, \beta, \lambda) C'(\theta S_{LLoGMW}(x; c, \alpha, \beta, \lambda))}{C(\theta)}. \end{aligned}$$

2.3 Some Generalized Sub-Classes of Distributions

In this section, we present several new and known sub-classes of distributions including the LLoGPS, LLoGMEPS, LLoGRPS, and other sub CoD.

- If $\lambda = 0$, we obtain the LLoGWPS CoD.
- If $\lambda = 0$, and $\beta = 1$, we obtain the LLoGEPS CoD.
- When $\alpha \rightarrow 0^+$, we have the LLoGPS CoD.
- If $\lambda = 0$ and $\beta = 2$, we have the LLoGRPS CoD.
- If $\beta = 1$, we obtain the LLoGMEPS CoD.
- If $\beta = 2$, we have the LLoGMRPS CoD.
- Several new classes of distributions can be readily obtained by setting $c=1$; $c=1, \beta=1$; $c=1, \beta=2$; $c=1, \beta=1, \lambda=0$; $c=1, \lambda=0$; $\alpha \rightarrow 0^+$, $c=1$ and $c=\alpha=1$, respectively.

2.4 Hazard Rate and Reverse Hazard Rate Functions

We present the hrf and reverse hrf of the proposed LLoGMWPS CoD in this section. The hrf is given by

$$h_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) = \frac{f_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta)}{S_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta)}, \quad (2.3)$$

where $S_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) = 1 - F_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta)$. Explicitly,

$$h_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) = \theta f(x) \frac{C'(\theta S(x))}{C(\theta S(x))},$$

where $f_{LLoGMW}(x) = f(x)$ is the LLoGMW pdf. On the other hand, the reverse hrf, $\tau_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) = f_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) / F_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta)$, is given by

$$\tau_{LLoGMWPS}(x; c, \alpha, \beta, \lambda, \theta) = \theta f(x) \frac{C'(\theta S(x))}{C(\theta) - C(\theta S(x))}.$$

2.5 Identifiability

To assess the identifiability of the LLoGMWPS CoD, we considered two parameter sets $\Delta_1 = (c_1, \alpha_1, \beta_1, \lambda_1, \theta_1)$ and $\Delta_2 = (c_2, \alpha_2, \beta_2, \lambda_2, \theta_2)$. The likelihood ratio is expressed as follows:

$$\begin{aligned} L &= \frac{f_{(\Delta_1)}}{f_{(\Delta_2)}} \\ &= \frac{\theta_1}{\theta_2} \left(1 + x^{c_1}\right)^{-2} \left(1 + x^{c_2}\right)^2 e^{-\alpha_1 x^{\beta_1} e^{\lambda_1 x} + \alpha_2 x^{\beta_2} e^{\lambda_2 x}} \\ &\times \left(\alpha_1 x^{\beta_1-1} e^{\lambda_1 x} (\beta_1 + \lambda_1 x) (1 + x^{c_1}) + c_1 x^{c_1-1}\right) \left(\alpha_2 x^{\beta_2-1} e^{\lambda_2 x} (\beta_2 + \lambda_2 x) (1 + x^{c_2}) + c_2 x^{c_2-1}\right)^{-1} \\ &\times \left(C'(\theta_1 (1 + x^{c_1})^{-1} \exp(-\alpha_1 x^{\beta_1} e^{\lambda_1 x})) C(\theta_1)\right) \left(C'(\theta_2 (1 + x^{c_2})^{-1} \exp(-\alpha_2 x^{\beta_2} e^{\lambda_2 x})) C(\theta_2)\right)^{-1}. \end{aligned}$$

Taking logarithm of likelihood, we get

$$\begin{aligned} \text{Log}(L) &= \log\left(\frac{\theta_1}{\theta_2}\right) + \alpha_2 x^{\beta_2} e^{\lambda_2 x} - \alpha_1 x^{\beta_1} e^{\lambda_1 x} - 2 \log(1 + x^{c_1}) + 2 \log(1 + x^{c_2}) \\ &+ \log\left(\alpha_1 x^{\beta_1-1} e^{\lambda_1 x} (\beta_1 + \lambda_1 x) (1 + x^{c_1}) + c_1 x^{c_1-1}\right) \\ &- \log\left(\alpha_2 x^{\beta_2-1} e^{\lambda_2 x} (\beta_2 + \lambda_2 x) (1 + x^{c_2}) + c_2 x^{c_2-1}\right) \\ &+ \log(C'(\theta_1 (1 + x^{c_1})^{-1} \exp(-\alpha_1 x^{\beta_1} e^{\lambda_1 x}))) + \log(C(\theta_1)) \\ &- \log(C'(\theta_2 (1 + x^{c_2})^{-1} \exp(-\alpha_2 x^{\beta_2} e^{\lambda_2 x}))) - \log(C(\theta_2)). \end{aligned}$$

Now, the derivative of the log-likelihood with respect to x is

$$\begin{aligned} \frac{d \log(L)}{dx} &= \alpha_2 e^{\lambda_2 x} (\beta_2 x^{\beta_2-1} + x^{\beta_2} \lambda_2) - \alpha_1 e^{\lambda_1 x} (\beta_1 x^{\beta_1-1} + x^{\beta_1} \lambda_1) - \frac{2c_1 x^{c_1-1}}{(1 + x^{c_1})} + \frac{2c_2 x^{c_2-1}}{(1 + x^{c_2})} \\ &+ \frac{x^{\beta_1-1} e^{\lambda_1 x} (\beta_1 + \lambda_1 x) [\lambda_1 (1 + x^{c_1}) + c_1 x^{c_1-1}] + (\beta_1 - 1) x^{\beta_1-2} e^{\lambda_1 x} + \lambda_1 (1 + x^{c_1})}{(\alpha_1 x^{\beta_1-1} e^{\lambda_1 x} (\beta_1 + \lambda_1 x) (1 + x^{c_1}) + c_1 x^{c_1-1})} \\ &- \frac{x^{\beta_2-1} e^{\lambda_2 x} (\beta_2 + \lambda_2 x) [\lambda_2 (1 + x^{c_2}) + c_2 x^{c_2-1}] + (\beta_2 - 1) x^{\beta_2-2} e^{\lambda_2 x} + \lambda_2 (1 + x^{c_2})}{(\alpha_2 x^{\beta_2-1} e^{\lambda_2 x} (\beta_2 + \lambda_2 x) (1 + x^{c_2}) + c_2 x^{c_2-1})} \\ &- \frac{\theta_1 C'(\theta_1 S_{LLoGMW}(x; \Delta_1)) f_{LLoGMWP}(x; \Delta_1)}{C(\theta_1 S_{LLoGMW}(x; \Delta_1))} + \frac{\theta_2 C'(\theta_2 S_{LLoGMW}(x; \Delta_2)) f_{LLoGMWP}(x; \Delta_2)}{C(\theta_2 S_{LLoGMW}(x; \Delta_2))}. \end{aligned}$$

Therefore, $\frac{d \log(L)}{dx} = 0$, implying that the parameters uniquely determines the LLoGMWPS CoD if $\Delta_1 = \Delta_2$.

2.6 LLoGMW Poisson, Geometric, Logarithmic and Binomial Distributions

The cdfs, pdfs and hrfs of the LLoGMWP, LLoGMWG, LLoGMWL and LLoGMWB distributions are presented in this section.

2.6.1 Log-Logistic Modified Weibull Poisson (LLoGMWP) Distribution

The LLoGMWP distribution is a special case of the LLoGMWPS CoD with $C(\theta) = e^\theta - 1$ and $a_n = \frac{1}{n!}$. The cdf is given by

$$F_{LLoGMWP}(x; c, \alpha, \beta, \lambda, \theta) = 1 - \frac{e^{\theta(1+x^c)^{-1} \exp(-\alpha x^\beta e^{\lambda x})} - 1}{e^\theta - 1}, \quad (2.4)$$

for $x > 0$, $c, \alpha, \beta, \theta > 0$, $\lambda \geq 0$. The associated pdf is

$$\begin{aligned} f_{LLoGMWP}(x; c, \alpha, \beta, \lambda, \theta) &= \frac{\theta e^{-\alpha x^\beta e^{\lambda x}} (1+x^c)^{-2} e^{\theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}}}{e^\theta - 1} \\ &\times \left(c x^{c-1} + \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x) (1+x^c) \right), \end{aligned} \quad (2.5)$$

and the hrf is given by

$$\begin{aligned} h_{LLoGMWP}(x; c, \alpha, \beta, \lambda, \theta) &= \frac{\theta e^{-\alpha x^\beta e^{\lambda x}} (1+x^c)^{-2} e^{\theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}}}{e^{\theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}} - 1} \\ &\times \left(c x^{c-1} + \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x) (1+x^c) \right), \end{aligned} \quad (2.6)$$

respectively.

2.6.2 Log-Logistic Modified Weibull Geometric (LLoGMWG) Distribution

The LLoGMWG distribution is obtained from the LLoGMWPS CoD with $C(\theta) = \theta(1-\theta)^{-1}$ and $a_n = 1$. The cdf of the LLoGMWG distribution is given by

$$F_{LLoGMWG}(x; c, \alpha, \beta, \lambda, \theta) = 1 - \frac{(1-\theta)(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}}{1 - \theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}},$$

for $x > 0$, $c, \alpha, \beta, \theta > 0$ and $\lambda \geq 0$. The corresponding pdf is given by

$$\begin{aligned} f_{LLoGMWG}(x; c, \alpha, \beta, \lambda, \theta) &= \frac{(1-\theta)(e^{-\alpha x^\beta e^{\lambda x}} (1+x^c)^{-2})}{(1 - \theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}})^2} \\ &\times \left(c x^{c-1} + \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x) (1+x^c) \right), \end{aligned}$$

and the hrf is given by

$$h_{LLoGMWG}(x; c, \alpha, \beta, \lambda, \theta) = \frac{(1+x^c)^{-1} (c x^{c-1} + \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x) (1+x^c))}{1 - \theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}},$$

respectively.

2.6.3 Log-Logistic Modified Weibull Logarithmic (LLoGMWL) Distribution

The LLoGMWL distribution is obtained from the the LLoGMWPS CoD with $C(\theta) = -\log(1-\theta)$ and $a_n = \frac{1}{n}$. The cdf is given by

$$F_{LLoGMWL}(x; c, \alpha, \beta, \lambda, \theta) = 1 - \frac{\log(1 - \theta(1 + x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}})}{\log(1 - \theta)}, \quad (2.7)$$

for $x > 0$, $c, \alpha, \beta, \theta > 0$ and $\lambda \geq 0$. The corresponding pdf is given by

$$f_{LLoGMWL}(x; c, \alpha, \beta, \lambda, \theta) = \frac{\theta e^{-\alpha x^\beta e^{\lambda x}} (1 + x^c)^{-2} (cx^{c-1} + \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x)(1 + x^c))}{-(1 - \theta(1 + x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}) \log(1 - \theta)}, \quad (2.8)$$

and the hrf is

$$h_{LLoGMWL}(x; c, \alpha, \beta, \lambda, \theta) = \frac{\theta e^{-\alpha x^\beta e^{\lambda x}} (1 + x^c)^{-2} (cx^{c-1} + \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x)(1 + x^c))}{-(1 - \theta(1 + x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}) \log(1 - \theta(1 + x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}})}, \quad (2.9)$$

respectively.

2.6.4 Log-Logistic Modified Weibull Binomial (LLoGMWB) Distribution

The LLoGMWB distribution is a special case of the LLoGMWPS CoD with $C(\theta) = (1 + \theta)^m - 1$, ($n \leq m$), and $a_n = \binom{m}{n}$. The cdf is given by

$$F_{LLoGMWB}(x; c, \alpha, \beta, \lambda, \theta) = 1 - \frac{(1 + \theta(1 + x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}})^m - 1}{(1 + \theta)^m - 1},$$

for $x > 0$, $c, \alpha, \beta, \theta > 0$ and $\lambda \geq 0$. The corresponding pdf and hrf are given by

$$f_{LLoGMWB}(x; c, \alpha, \beta, \lambda, \theta) = e^{-\alpha x^\beta} (1 + x^c)^{-2} (cx^{c-1} + \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x)(1 + x^c)) \\ \times \frac{m(1 + \theta e^{-\alpha x^\beta} (1 + x^c)^{-1})^{m-1}}{(1 + \theta)^m - 1},$$

and

$$h_{LLoGMWB}(x; c, \alpha, \beta, \lambda, \theta) = e^{-\alpha x^\beta} (1 + x^c)^{-2} (cx^{c-1} + \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x)(1 + x^c)) \\ \times \frac{m(1 + \theta e^{-\alpha x^\beta e^{\lambda x}} (1 + x^c)^{-1})^{m-1}}{(1 + \theta e^{-\alpha x^\beta e^{\lambda x}} (1 + x^c)^{-1})^m - 1},$$

respectively.

2.7 Quantile Function

Let X be a random variable with cdf as in Theorem 1. The QF is obtained by solving non-linear equation

$F_{LLogMWPS}(x; c, \alpha, \beta, \lambda, \theta) = u$, $0 \leq u \leq 1$, Thus, we solve the equation

$$C\left(\theta\left(1+(Q_{X_{(1)}}(u))^c\right)^{-1}e^{-\alpha(Q_{X_{(1)}}(u))^\beta}e^{\lambda(Q_{X_{(1)}}(u))}\right)-C(\theta)(1-u)=0,$$

equivalently

$$-\ln\left[1+(Q_{X_{(1)}}(u))^c\right]-\alpha(Q_{X_{(1)}}(u))^\beta e^{\lambda(Q_{X_{(1)}}(u))}-\ln\left[\frac{C^{-1}(C(\theta)(1-u))}{\theta}\right]=0, \quad (2.10)$$

which can be done using numerical methods. Consequently, random number can be generated based on Equation (2.10).

2.8 Moments, Order Statistics and Entropy

In this section, moments, order statistics and RE from LLoGMWPS CoD are presented. We note that using the result of a power series distribution raised to a positive integer, (Gradshteyn and Ryzhik [45]), that is,

$$\left(\sum_{j=0}^{\infty} a_j y^j\right)^s = \sum_{j=0}^{\infty} b_{s,j} y^j,$$

where the coefficients $b_{s,j}$ for $(j = 1, 2, \dots)$ are determined by the recurrence equations $b_{s,j} = (ja_0)^{-1} \sum_{m=1}^j [m(s+1) - j] a_m b_{s,j-m}$, and $b_{s,0} = a_0^s$.

2.8.1 Moments

Moments of the LLoGMWPS CoD are presented in this subsection. The r^{th} raw moment, μ'_r of the LLoGMWPS CoD is given by

$$\mu'_r = E(X^r) = \int_0^\infty x^r \frac{\theta f(x) C'(\theta S(x))}{C(\theta)} dx.$$

Note that

$$\begin{aligned} C'(\theta(1+x^c)^{-1}e^{-\alpha x^\beta}e^{\lambda x}) &= \sum_{n=1}^{\infty} n a_n [\theta(1+x^c)^{-1}e^{-\alpha x^\beta}e^{\lambda x}]^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} [\theta(1+x^c)^{-1}e^{-\alpha x^\beta}e^{\lambda x}]^n \\ &= \sum_{n=0}^{\infty} b_n [\theta(1+x^c)^{-1}e^{-\alpha x^\beta}e^{\lambda x}]^n, \end{aligned}$$

where $b_n = (n+1)a_{n+1}$, and $(\sum_{n=0}^{\infty} b_n y^n)^v = \sum_{n=0}^{\infty} d_{v,n} y^n$, where $y = \theta(1+x^c)^{-1}e^{-\alpha x^\beta}$, $d_{v,n} = b_0^n$, and $d_{v,m} = \frac{1}{mb_0} \sum_{j=0}^m (jv - m + j) b_j d_{v,m-j}$. So, when using transformation $y = (1+x^c)^{-1}$,

$x = \left(\frac{1-y}{y}\right)^{\frac{1}{c}}$ and $dx = \frac{dy}{-(1+x^c)^{-2}cx^{c-1}}$. Therefore,

$$\begin{aligned}
 \mu_r' &= E(X^r) \\
 &= \frac{\theta^{n+1}}{C(\theta)} \sum_{n=0}^{\infty} b_n \left\{ \frac{\alpha\beta}{c} \sum_{j,m=0}^{\infty} \frac{(-1)^j \alpha^j (n+1)^j x^m (j+1)^m}{j!m!} \right. \\
 &\quad \times B\left(n+c-\frac{1}{c}(m+r+\beta(j+1)-1), \frac{1}{c}(m+r+\beta(j+1)-1)-c+2\right) \\
 &\quad + \frac{\alpha\lambda}{c} \sum_{k,p=0}^{\infty} \frac{(-1)^k \alpha^k (n+1)^k (\lambda(k+1))^p}{k!p!} \\
 &\quad \times B\left(n+c-\frac{1}{c}(p+r+\beta(k+1)), \frac{1}{c}(p+r+\beta(k+1))-c+2\right) \\
 &\quad + \sum_{k,q=0}^{\infty} \frac{(-1)^k (n+1)^k (\lambda(k))^q}{k!q!} \\
 &\quad \times \left. B\left(n+c-\frac{1}{c}(q+r+\beta(k)+c-1)-c+2, \frac{1}{c}(p+r+\beta(k)-c+2)\right) \right\},
 \end{aligned}$$

where $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the complete beta function.

2.8.2 Order Statistics

The pdf of the i^{th} order statistic from the LLoGMWPS CoD is given by

$$\begin{aligned}
 f_{i:n}(x) &= \frac{1}{B(i, n-i+1)} f_{\theta}(x) F_{\theta}^{i-1}(x) (1-F_{\theta}(x))^{n-i} \\
 &= \frac{1}{B(i, n-i+1)} f_{\theta}(x) \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j F_{\theta}^{j+i-1}(x) \\
 &= \frac{f_{\theta}(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \\
 &\quad \times \left(1 - \frac{C(\theta(1+x^c)^{-1} \exp(-\alpha x^{\beta} e^{\lambda x}))}{C(\theta)}\right)^{j+i-1}.
 \end{aligned}$$

Using binomial expansion

$$\begin{aligned}
 \left(1 - \frac{C(\theta(1+x^c)^{-1} \exp(-\alpha x^{\beta} e^{\lambda x}))}{C(\theta)}\right)^{j+i-1} &= \sum_{s=0}^{j+i-1} (-1)^s \binom{j+i-1}{s} \\
 &\quad \times \left(\frac{C(\theta(1+x^c)^{-1} \exp(-\alpha x^{\beta} e^{\lambda x}))}{C(\theta)}\right)^s,
 \end{aligned}$$

the pdf of the i^{th} order statistics can be written as

$$\begin{aligned}
 f_{i:n}(x) &= \frac{f_{\theta}(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \binom{n-i}{j} \binom{j+i-1}{s} (-1)^{j+s} \\
 &\quad \times \left(\frac{C(\theta(1+x^c)^{-1} \exp(-\alpha x^{\beta} e^{\lambda x}))}{C(\theta)}\right)^s.
 \end{aligned}$$

Note that,

$$\begin{aligned} (C(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x})))^s &= \left(\sum_{w=0}^{\infty} a_w \theta(1+x^c)^{-1} \exp(-\alpha x^\beta e^{\lambda x}) \right)^s \\ &= \sum_{w=0}^{\infty} e_{s,w} \left(\theta(1+x^c)^{-1} \exp(-\alpha x^\beta e^{\lambda x}) \right)^w, \end{aligned}$$

where If $e_{s,w} = (wa_0)^{-1} \sum_{m=1}^w [m(s+1) - w] a_m e_{s,w-m}$, and $e_{s,0} = a_0^s$, so that the pdf of the i^{th} order statistic can be written as

$$\begin{aligned} f_{i:n}(x) &= \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \sum_{w=0}^{\infty} \binom{n-i}{j} \binom{j+i-1}{s} \frac{(-1)^{j+s} e_{s,w} \theta^w}{(w+1)B(i, n-i+1)(C(\theta))^s} \\ &\times (w+1)[(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}]^w f_\theta(x) \\ &= \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \sum_{w=0}^{\infty} \binom{n-i}{j} \binom{j+i-1}{s} \frac{(-1)^{j+s} e_{s,w} \theta^w}{(w+1)B(i, n-i+1)(C(\theta))^s} f^*(x), \end{aligned}$$

where

$$\begin{aligned} f^*(x) &= (w+1)[(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}]^w f_\theta(x) \\ &= (w+1)[(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}]^w \\ &\times [e^{-\alpha x^\beta e^{\lambda x}} (1+x^c)^{-2} (cx^{c-1} + (1+x^c)\alpha\beta x^{\beta-1} e^{\lambda x} (\beta + \lambda x))]. \end{aligned}$$

The corresponding cdf is given by

$$\begin{aligned} F_{i:n}(x) &= \sum_{k=i}^n \binom{n}{k} F_\theta^k(x) (1 - F_\theta(x))^{n-k} \\ &= \sum_{k=i}^n \sum_{j=0}^{n-k} \binom{n-k}{j} \binom{n}{k} (-1)^j [F_\theta(x)]^{j+k} \\ &= \sum_{k=i}^n \sum_{j=0}^{n-k} \sum_{m=0}^{j+k} \binom{j+k}{m} \binom{n-k}{j} \binom{n}{k} (-1)^{j+m} \\ &\times \left(\frac{C(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x}))}{C(\theta)} \right)^m, \end{aligned}$$

where $F_\theta(x) = 1 - \frac{C(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x}))}{C(\theta)}$, and

$$C(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x})) = \sum_{j=0}^{\infty} a_j \left[\theta(1+x^c)^{-1} \exp(-\alpha x^\beta e^{\lambda x}) \right]^j.$$

2.8.3 Rényi Entropy

The most widely used information measures are RE (Rényi [116]) and Shannon entropy (Shannon [122]). Statistically, RE is an expansion of Shannon entropy and is given as

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f(x; c, \alpha, \beta, \lambda, \theta)]^v dx \right), v \neq 1, v > 0.$$

RE tends to Shannon entropy as $v \rightarrow 1$. With

$$\int_0^\infty f^v(x; c, \alpha, \beta, \lambda, \theta) dx = \int_0^\infty f_\theta^v(x) dx$$

written as

$$\int_0^\infty f_\theta^v(x) dx = \left(\frac{\theta}{C(\theta)} \right)^v \int_0^\infty (f(x) C'(\theta S(x)))^v dx,$$

and

$$\begin{aligned} C'(\theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}) &= \sum_{n=1}^{\infty} n a_n [\theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}]^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} [\theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}]^n \\ &= \sum_{n=0}^{\infty} b_n [\theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}]^n, \end{aligned}$$

where $b_n = (n+1)a_{n+1}$, and $(\sum_{n=0}^{\infty} b_n y^n)^v = \sum_{n=0}^{\infty} d_{v,n} y^n$, where $y = \theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}$, $d_{v,n} = b_0^n$, and $d_{v,m} = \frac{1}{mb_0} \sum_{j=0}^m (jv - m + j) b_j d_{v,m-j}$, we have

$$\begin{aligned} \int_0^\infty f_\theta^v(x) dx &= \left(\frac{\theta}{C(\theta)} \right)^v \sum_{n=0}^{\infty} d_{v,n} \int_0^\infty [\theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}]^n \\ &\quad \times \left[(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}} \left(\alpha \beta x^{\beta-1} e^{\lambda x} (\beta + \lambda x) + \frac{cx^{c-1}}{1+x^c} \right) \right]^v dx \\ &= \left(\frac{\theta}{C(\theta)} \right)^v \sum_{n,p,k,l,s=0}^{\infty} \frac{(-1)^p [\alpha(n+1)]^p [\lambda(p+v-k)]^l}{p! l!} \binom{v}{k} \binom{v-k}{s} \\ &\quad \times \theta^n d_{v,n} c^k \alpha^{v-k} \lambda^s \beta^{v-k-s} \int_0^\infty x^{ck+v\beta-k\beta-v+s+l} (1+x^c)^{-n+v-k-2v} dx. \end{aligned}$$

Now, let $y = (1+x^c)^{-1}$, and set $\delta^* = ck + v\beta - k\beta - v + s + l + 1$, then RE for LLoGMWPS CoD is given by

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left(\left(\frac{\theta}{C(\theta)} \right)^v \sum_{n,p,k,l,s=0}^{\infty} \frac{(-1)^p [\alpha(n+1)]^p [\lambda(p+v-k)]^l}{p! l!} \right. \\ &\quad \times \left. \binom{v}{k} \binom{v-k}{s} \theta^n d_{v,n} c^k \alpha^{v-k} \lambda^s \beta^{v-k-s} B \left(n+v+k - \frac{\delta^*+1}{c}, \frac{\delta^*+1}{c} \right) \right), \end{aligned}$$

for $v \neq 1$, and $v > 0$.

2.9 Estimation

Let $X \sim LLoGMWPS(c, \alpha, \beta, \lambda, \theta)$ and $\Delta = (c, \alpha, \beta, \lambda, \theta)^T$ be the parameter vector. The log-likelihood function $\ell = \ell(\Delta)$ based on a random sample of size, n is given by

$$\begin{aligned} \ell = \ell_n(\Delta) &= n \ln(\theta) - n \ln(C(\theta)) + \sum_{i=1}^n \ln(f(x_i)) + \sum_{i=1}^n \ln(C'(\theta S(x_i))) \\ &= n \ln(\theta) - n \ln(C(\theta)) + \sum_{i=1}^n (-\alpha x_i^\beta e^{\lambda x_i}) - 2 \sum_{i=1}^n \ln(1 + x_i^c) \\ &\quad + \sum_{i=1}^n \ln[cx_i^{c-1} + (1 + x_i^c)\alpha x_i^{\beta-1} e^{\lambda x_i}(\beta + \lambda x_i)] \\ &\quad + \sum_{i=1}^n \ln(C'(\theta e^{-\alpha x_i^\beta e^{\lambda x_i}}(1 + x_i^c)^{-1})). \end{aligned}$$

The equations obtained by setting the partial derivatives to zero are not in closed form and the values of the parameters $c, \alpha, \beta, \lambda$ and θ must be found via iterative methods. The ML estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial c}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \theta})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The FIM given by $\mathbf{I}(\Delta) = [\mathbf{I}_{\theta_i, \theta_j}]_{5 \times 5} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4, 5$, can be numerically obtained by MATLAB or NLMIXED in SAS or R software. The total FIM $n\mathbf{I}(\Delta)$ can be approximated by

$$\mathbf{J}_n(\hat{\Delta}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \Big|_{\Delta=\hat{\Delta}} \right]_{5 \times 5}, \quad i, j = 1, 2, 3, 4, 5. \quad (2.11)$$

For a given set of observations, the matrix given in Equation (2.11) is obtained after the convergence of the Newton-Raphson procedure via NLMIXED in SAS or R software. Note that the expectations in the FIM can be obtained numerically. Let $\hat{\Delta} = (\hat{c}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$ be the ML estimate of $\Delta = (c, \alpha, \beta, \lambda, \theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} N_5(\mathbf{0}, I(\Delta)^{-1})$, where $I(\Delta)$ is the expected FIM. The asymptotic behaviour is still valid if $I(\Delta)$ is replaced by the observed information matrix evaluated at $\hat{\Delta}$, that is $J(\hat{\Delta})$. The multivariate normal distribution $N_5(\mathbf{0}, J^{-1}(\hat{\Delta}))$, where the mean vector $\mathbf{0} = (0, 0, 0, 0, 0)^T$, can be used to construct CI and confidence regions for the individual model parameters and for the survival function and hrf. That is, the approximate $100(1 - \eta)\%$ CI for $c, \lambda, \alpha, \beta$ and θ are given by:

$$\begin{aligned} \hat{c} \pm Z_{\frac{\eta}{2}} \sqrt{\mathbf{I}_{cc}^{-1}(\hat{\Delta})}, \quad \hat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{\mathbf{I}_{\alpha\alpha}^{-1}(\hat{\Delta})}, \quad \hat{\beta} \pm Z_{\frac{\eta}{2}} \sqrt{\mathbf{I}_{\beta\beta}^{-1}(\hat{\Delta})}, \quad \hat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{\mathbf{I}_{\lambda\lambda}^{-1}(\hat{\Delta})}, \\ \text{and } \hat{\theta} \pm Z_{\frac{\eta}{2}} \sqrt{\mathbf{I}_{\theta\theta}^{-1}(\hat{\Delta})}, \end{aligned}$$

respectively, where $\mathbf{I}_{cc}^{-1}(\hat{\Delta}), \mathbf{I}_{\alpha\alpha}^{-1}(\hat{\Delta}), \mathbf{I}_{\beta\beta}^{-1}(\hat{\Delta}), \mathbf{I}_{\lambda\lambda}^{-1}(\hat{\Delta})$ and $\mathbf{I}_{\theta\theta}^{-1}(\hat{\Delta})$ are the diagonal elements of $\mathbf{I}_n^{-1}(\hat{\Delta}) = (n\mathbf{I}(\hat{\Delta}))^{-1}$ and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

2.10 Some Special Cases of the LLoGMWPS CoD

Two special cases of the LLoGMWPS CoD, namely, the LLoGMWP and LLoGMWL distributions are presented. We will examine their properties, including pdfs, cdfs, hrfs, and QFs. Plots of the pdf and hrf for selected values of the model parameters are given. The method of ML estimation is used for estimating parameters. A simulation study is conducted to illustrate the performance of the two special cases of the LLoGMWPS CoD. Real-life datasets are applied to demonstrate the flexibility and applicability of the two special cases.

2.11 Log-Logistic Modified Weibull Poisson (LLoGMWP) Distribution

The cdf and pdf of the LLoGMWP distribution are given by Equations (2.4) and (2.5). Plots of the LLoGMWP pdf for selected values of the model parameters are given in Figure 2.1.

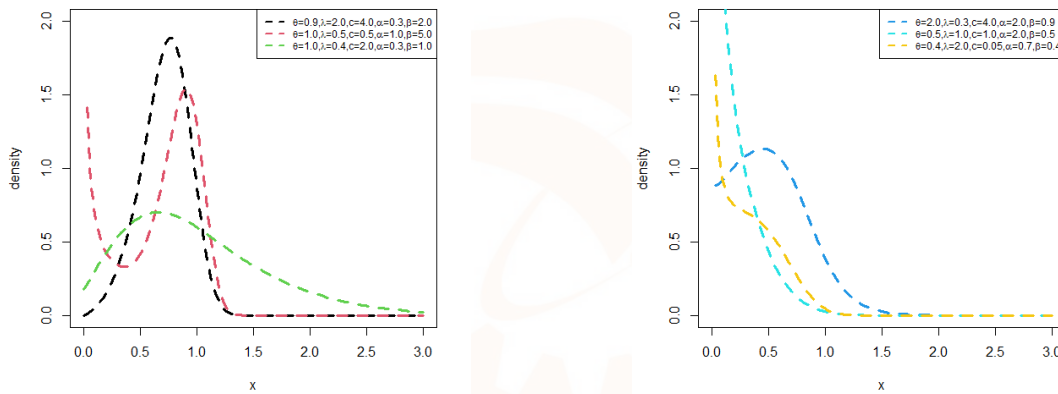


Figure 2.1: Pdf Plots of LLoGMWP Distribution

Figure 2.1 shows that the plots of the pdf can be positive-skewed, reverse-J and decreasing among many potential shapes. The density can exhibit different shapes depending on the values of the parameters as shown in Figure 2.1.

2.11.1 Hazard Rate Function

The hrf of the LLoGMWP distribution is given in Equation (2.6). Plots of the hrf for selected values of the model parameters are also given in Figure 2.2.

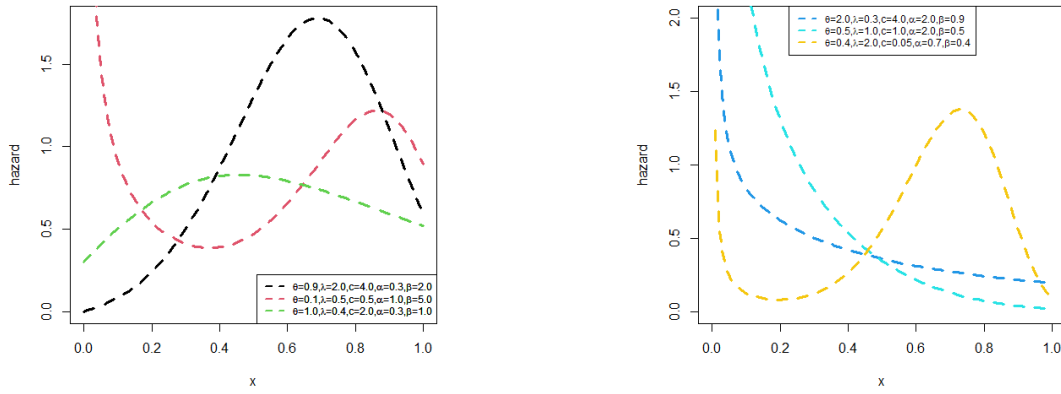


Figure 2.2: Hrf Plots of the LLoGMWP Distribution

The graphs in Figure 2.2 exhibit decreasing, UBT, UBT-BT and BT-UBT shapes for the selected values of the model parameters. This very attractive flexibility makes the LLoGMWP hrf useful and suitable for non-monotonic empirical hazard behaviours that are commonly observed in real-world applications.

2.11.2 Quantile Function

The QF of the LLoGMWP distribution can be obtained by inverting $F_{LLoGMWP}(x) = u$, $0 \leq u \leq 1$, where

$$F_{LLoGMWP}(x) = 1 - \frac{e^{\theta((1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}})} - 1}{e^\theta - 1}.$$

The QF of the LLoGMWP distribution is obtained by the solving non-linear equation

$$-\alpha(Q_{X_{(1)}}(u))^\beta e^{\lambda Q_{X_{(1)}}(u)} - \log(1 + (Q_{X_{(1)}}(u))^c) - \log\left(\left(\frac{1}{\theta} \log\left(1 - (1 - u)(1 - e^\theta)\right)\right)\right) = 0, \quad (2.12)$$

using numerical methods. Consequently, random numbers can be generated based on Equation (2.12). Table of quantiles from the LLoGMWP distribution for selected values of model parameters are given in Table 2.1.

Table 2.1: LLoGMWP Quantile for Selected Values

u	$(c, \lambda, \theta, \alpha, \beta)$				
	(0.1,0.2,2.8,0.4,0.5)	(2.8,3.5,4.0,3.0,1.0)	(2.0,1.0,1.0,1.0,0.3)	(1.0,0.9,1.0,1.0,1.0)	(1.0,1.0,1.0,0.6,1.0)
0.1	0.0732	0.0964	0.1241	0.0846	0.1067
0.2	0.2539	0.1417	0.0642	0.1677	0.2123
0.3	0.4259	0.1740	0.1390	0.2505	0.3174
0.4	0.5736	0.2007	0.2258	0.3344	0.4231
0.5	0.7138	0.2249	0.3222	0.4215	0.5310
0.6	0.8611	0.2484	0.4296	0.5145	0.6439
0.7	1.0327	0.2730	0.5532	0.6181	0.7667
0.8	1.2619	0.3011	0.7045	0.7418	0.9093
0.9	1.6561	0.3391	0.9191	0.9137	1.1009

The first five moments, SD, CV, SK and KS for selected values of the parameters of the LLoGMWP distribution are listed in Table 2.2.

Table 2.2: LLoGMWP Moments for Selected Values

$Moments$	$(c, \theta, \lambda, \alpha, \beta)$				
	(1.0,3.0,2.0,3.0,1.0)	(1.0,3.0,1.0,1.5,2.0)	(1.0,3.0,0.9,1.5,1.0)	(0.9,2,0.8,0.5,2.0)	(0.4,1.0,1.0,3.0,1.0)
μ'_1	4.3759	1.5883	1.4355	1.0340	0.5515
μ'_2	6.1566	2.8842	2.1994	0.4162	0.4804
μ'_3	2.7520	6.1940	3.6307	0.5738	0.4916
μ'_4	3.0311	6.4748	6.5323	0.9791	0.5564
μ'_5	6.4121	7.0603	12.9794	1.9405	0.6783
SD	6.5129	0.6011	0.3723	0.2541	0.8774
CV	1.4883	0.3784	0.2593	0.1212	0.4197
CS	7.6433	2.1399	1.4564	1.3556	0.7610
CK	1.4500	12.9121	7.2475	6.4151	0.2858



Figure 2.3: SK and KS Plots of the LLoGMWP Distribution, when $\alpha = 0.5, \beta = 0.2$ and $\theta = 0.5$.



Figure 2.4: SK and KS of LLoGMWP Distribution, when $c = 1.0, \beta = 1.2$ and $\theta = 0.35$.

The 3D plots of SK and KS for the LLoGMWP distribution are given in Figure 2.3 and 2.4. We observe that

- When we fix the parameters α, λ and θ , the SK and KS of LLoGMWP distribution decrease then increase as c and β increase.
- When we fix the parameters c, β and θ , the SK and KS of LLoGMWP distribution increase as α and λ increase.

2.11.3 Estimation

Let $X \sim LLoGMWP(c, \alpha, \beta, \lambda, \theta)$ and $\Delta = (c, \alpha, \beta, \lambda, \theta)^T$ be the parameter vector. The log-likelihood function $\ell = \ell(\Delta)$ based on a random sample of size, n is given by

$$\begin{aligned}
 \ell &= n \ln(\theta) - n \ln(C(\theta)) + \sum_{i=1}^n \ln(g(x_i)) + \sum_{i=1}^n \ln(C'(\theta S(x_i))) \\
 &= n \ln(\theta) - n \ln(e^\theta - 1) + \sum_{i=1}^n (-\alpha x_i^\beta e^{\lambda x_i}) - 2 \sum_{i=1}^n \ln(1 + x_i^c) \\
 &\quad + \sum_{i=1}^n \ln[cx_i^{c-1} + (1 + x_i^c)\alpha\beta x_i^{\beta-1} e^{\lambda x_i}(\beta + \lambda x_i)] \\
 &\quad + \theta \sum_{i=1}^n [1 - e^{-\alpha x_i^\beta e^{\lambda x_i}} (1 + x_i^c)^{-1}].
 \end{aligned}$$

We obtain the ML estimates of parameters denoted by $\hat{\Delta}$ by solving the nonlinear equation $(\frac{\partial \ell}{\partial c}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \theta})^T = \mathbf{0}$.

(See elements of Score vectors in the appendix)

2.11.4 Simulation Study

In this section, the performance of the ML estimates is examined by conducting simulation studies for different sample sizes, n . We examine the performance of the LLoGMWP distribution by conducting various simulations for different sample sizes ($n=25, 50, 100, 200, 400, 1600$) via

the R package. We simulate $N = 1000$ samples for the true parameters values $I : \theta = 0.5, c = 0.5, \alpha = 0.1, \beta = 0.5, \lambda = 1.2$, $II : \theta = 1.2, c = 1.2, \alpha = 1.2, \beta = 0.5, \lambda = 0.9$. Additionally, to generate these samples, we employed an algorithm based on the inverse cdf of the LLoGMWP distribution. The random numbers were generated based on the derived QF, enabling the simulation of samples from the LLoGMWP distribution. The mean, ABias and RMSE were computed to assess the performance of ML estimates under different sample sizes and sets of parameters. Table 2.3 lists the mean ML estimates of the parameters along with the respective ABias and RMSE.

The results in Table 2.3 shows that the mean ML estimates converge to the true values as sample sizes, n increases. The ABias and the RMSE decay towards zero as the sample sizes, n increases for all the sets of parameter values. Consequently, the simulations affirm the consistency of the ML estimation method and support the suitability of the LLoGMWP distribution for modeling real-world data.

Table 2.3: MCS Results: Mean, ABias and RMSE

		I			II		
Parameter	n	Mean	RMSE	ABias	Mean	RMSE	ABias
θ	25	0.9575	1.5802	0.4523	2.3700	1.6334	1.1700
	50	0.7539	0.5933	0.2539	2.3089	1.5417	1.1089
	100	0.6183	0.4247	0.1183	1.9789	1.2970	0.7789
	200	0.5425	0.3424	0.0425	1.6310	1.0121	0.4310
	400	0.4845	0.2698	-0.0154	1.4045	0.8311	0.2045
	1600	0.4983	0.0528	-0.0016	1.1603	0.5396	-0.0496
c	25	0.5686	0.2964	0.0686	1.7323	0.7665	0.5323
	50	0.5410	0.1724	0.0410	1.6534	0.6235	0.4534
	100	0.5288	0.0994	0.0288	1.5430	0.4700	0.3430
	200	0.5211	0.0627	0.0211	1.4458	0.3353	0.2458
	400	0.5208	0.0467	0.0208	1.3948	0.2519	0.1948
	1600	0.5167	0.0168	0.0167	1.3351	0.1788	0.1351
α	25	0.1792	0.2468	0.0792	2.0741	4.6995	0.8741
	50	0.1642	0.1543	0.0642	2.0009	1.6220	0.8009
	100	0.1338	0.0955	0.0338	1.7908	1.2870	0.5908
	200	0.1187	0.0634	0.0187	1.6558	1.2655	0.4558
	400	0.1050	0.0420	0.0050	1.3432	1.2107	0.1432
	1600	0.1037	0.0041	0.0037	1.2828	1.1935	0.0828
β	25	1.2602	1.2992	0.7602	5.1485	5.2093	4.6485
	50	1.1215	1.1941	0.6215	4.7931	4.4139	4.2931
	100	0.9277	1.0101	0.4277	4.5573	4.1493	4.0573
	200	0.7493	0.7232	0.2493	4.2198	3.7370	3.7198
	400	0.7333	0.7094	0.2333	2.1840	1.6840	1.6840
	1600	0.5132	0.0521	0.0132	0.5530	0.0530	0.0530
λ	25	0.9752	0.8024	-0.1195	1.2360	0.5183	0.3360
	50	0.0804	0.6738	-0.2247	1.1751	0.4138	0.2751
	100	1.0338	0.5376	-0.1661	1.0995	0.3055	0.1995
	200	1.0936	0.3791	-0.1063	1.0235	0.1883	0.1235
	400	1.1255	0.3376	-0.0744	0.9859	0.1311	0.0859
	1600	1.2005	0.0140	0.0005	0.9419	0.0652	0.0419

2.11.5 Applications

In this section, we present applications of LLoGMWP distribution to real data.

The ML estimates of the LLoGMWP parameters $\Delta = (c, \alpha, \beta, \lambda, \theta)$ are computed by maximizing the objective function via the subroutine NLMIXED in SAS as well as the function `nlm` in R Team [131]. The estimated values of the parameters (SE in parenthesis), $(-2\ln(L))$, AIC, BIC, and CAIC, are presented. Additionally, the CVM and AD GoF statistics, as described by Chen and Balakrishnan [26], are also presented. These statistics serve to assess the GoF of the distribution to the data, with smaller values indicating a better fit.

Furthermore, the SS (Chambers et al. [24]) from the PPs and K-S statistics, along with their associated PV, are used for model comparison. It is worth noting that a model with the smallest value of GoF statistics and highest PV for the K-S statistic is considered the best fitting model.

We use the LRT statistic to compare the fit of the LLoGMWP distribution with its sub-models for a given dataset. For example, to test $\lambda = 1, \alpha = 1, \beta = 1$, the LRT statistic is $\omega = 2[\ln(L(\hat{c}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})) - \ln(L(\tilde{c}, 1, 1, 1, \tilde{\theta}))]$, where $\hat{c}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}$, and $\hat{\theta}$ are the unrestricted estimates, and $\tilde{c}$, and $\tilde{\theta}$ are the restricted estimates. The LRT rejects the null hypothesis if $\omega > \chi_{\epsilon}^2$, where χ_{ϵ}^2 denote the upper 100 ϵ % point of the χ^2 distribution with 3 df.

We compare the LLoGMWP distribution with the GD distribution by Oluyede et al. [98], EKD distribution by Huang and Oluyede [54], BWLLoG distribution by Makubate et al. [71], EKW distribution by Eissa and Abdulaziz [39], LLoGWP distribution by Oluyede et al. [102], and BrWLn distribution by Oluyede et al. [99]. The pdfs of the GD, EKD, BWLLoG, EKW, LLoGWP, and BrWLn distributions are given by

$$f_{GD}(x) = \frac{\lambda\beta\delta x^{-\delta-1}}{\Gamma(\alpha)\theta^{\alpha}}(1+\lambda x^{-\delta})^{-\beta-1} \left(-\log[1-(1+\lambda x^{-\delta})^{-\beta}] \right)^{\alpha-1} \\ \times [1-(1+\lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1},$$

for $\alpha, \lambda, \delta, \beta, \theta > 0, x > 0$,

$$f_{EKD}(x) = \alpha\lambda\delta\phi\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\alpha-1} \left[1-(1+\lambda x^{-\delta})^{-\alpha} \right]^{\phi-1} \\ \times \left[1 - \left[1-(1+\lambda x^{-\delta})^{-\alpha} \right]^{\theta-1} \right],$$

for $\alpha, \lambda, \delta, \phi, \theta > 0, x > 0$,

$$f_{BWLLoG}(x) = \frac{\alpha\beta c}{B(a,b)} x^{c-1} (1+x^c)^{-2} \frac{[1-(1+x^c)^{-1}]^{\beta-1}}{[(1+x^c)^{-1}]^{\beta+1}} \\ \times \exp \left\{ -\alpha b [(1+x^c) - 1]^{\beta} \right\} \\ \times \left[1 - \exp \left\{ -\alpha [(1+x^c) - 1]^{\beta} \right\} \right]^{a-1},$$

for $a, b, c, \alpha, \beta > 0, x > 0$,

$$f_{EKW}(x) = \theta abc \lambda^c x^{c-1} \exp - (\lambda x)^c [1 - \exp - (\lambda x)^c]^{a-1} \\ \times \left[1 - [1 - \exp - (\lambda x)^c]^a \right]^{b-1} \left[1 - [1 - [1 - \exp - (\lambda x)^c]^a]^b \right]^{\theta-1},$$

for $a, b, c, \lambda, \theta > 0, x > 0$,

$$f_{LLoGWP}(x) = \frac{\theta e^{\theta \left(1 - \left(1 + \left(\frac{x}{s}\right)^c\right)^{-1} e^{-\alpha x^\beta}\right)}}{e^{\theta-1}} \left(1 + \left(1 + \left(\frac{x}{s}\right)^c\right)^{-1}\right) e^{-\alpha x^\beta} \\ \times \left[\left(1 + \left(\frac{x}{s}\right)^c\right)^c \frac{c}{s} \left(\frac{x}{s}\right)^{c-1} + \alpha \beta x^{\beta-1} \right],$$

for $s, c, \alpha, \beta, \theta > 0$, $x > 0$, and

$$f_{BrWLn}(x) = \frac{\theta e^{-\alpha x^\beta} (1 + x^c)^{-k-1} (k c x^{c-1} + \alpha \beta x^{\beta-1} (1 + x^c))}{- \left(1 - \theta (1 + x^c)^{-k} e^{-\alpha x^\beta}\right) \log(1 - \theta)},$$

for $k, c, \alpha, \beta, \theta > 0$, $x > 0$.

2.11.5.1 Failure Rates of Components Data

The first set of data represent failure rates of fifty (50) components from Chipepa et al. [33]. Figure 2.5 shows the boxplot and violin plots of failure rates of components data. The violin plot for the data has a wider base and narrows at the top, with few outliers, indicating that the data is skewed and has light tails.

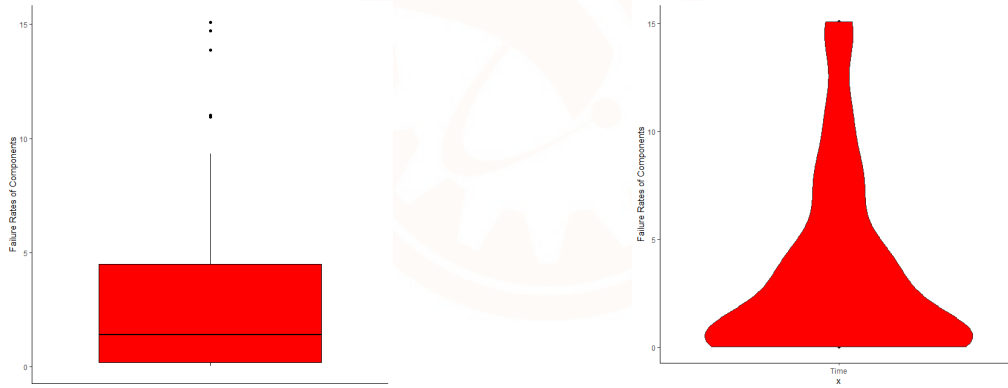


Figure 2.5: Boxplot and Violin Plots of Failure Rates of Components Data

The asymptotic Var-Cov matrix of the ML estimates for the LLoGMWP distribution is given by

$$\begin{pmatrix} 0.0739 & 0.0134 & 0.0004 & -0.4383 & -0.0170 \\ 0.0134 & 0.0157 & -0.0011 & -0.0156 & -0.0144 \\ 0.0004 & -0.0011 & 0.0007 & -0.0418 & 0.0054 \\ -0.4383 & -0.0156 & -0.0418 & 4.7653 & -0.1960 \\ -0.0170 & -0.0144 & 0.00542 & -0.1960 & 0.2903 \end{pmatrix},$$

and the approximate 95% CI for $\lambda, c, \alpha, \beta$ and θ are given by 0.2478 ± 0.5329 , 0.7776 ± 0.2459 , 0.0066 ± 0.0523 , 0.7740 ± 4.2786 , and 0.3708 ± 1.0560 , respectively.

The Table 2.4 provides estimates of the parameters, along with their SE in parenthesis.

Table 2.4: Estimates of Models for Failure Rates of Components Data

Model	Estimates				
	λ	c	α	β	θ
LLoGMWP	0.2478 (0.2719)	0.7776 (0.1246)	0.0066 (0.0267)	0.7740 (2.1832)	0.3708 (0.5388)
LLoGP	-	0.9098 (0.1029)	-	-	0.9558 (0.4911)
LLoGEP	-	0.7372 (0.1238)	0.7800 (0.3954)	-	3.6898×10^{-08} (0.2131)
EKD	α	λ	δ	ϕ	θ
	1.0279 (0.3081)	1.846 (0.0000014)	1486.00 (0.7378)	0.7485 (0.000002)	9250.1000 (1.5469)
GD	λ	β	δ	α	θ
	0.0004 (0.0018)	0.5934 (0.2332)	1.2483 (0.6123)	12.5790 (0.0391)	0.6682 (0.3247)
EKW	c	a	b	θ	λ
	0.5110 (0.3118)	10.0989 (9.7929)	7.1749 (11.3697)	0.1173 (0.1166)	0.2359 (0.2870)
BWLLoG	c	a	b	α	β
	0.4256 (2.4626)	1.4584 (3.2680)	13.3910 (56.1346)	0.0709 (0.2913)	1.2299 (7.1157)
LLoGWP	s	c	α	β	θ
	1.6389 (0.8462)	0.6939 (0.1230)	0.0268 (0.0457)	1.5685 (0.6521)	5.5051×10^{-06} (0.2441)
BrWLn	k	c	α	β	θ
	1.051×10^{-10} (1.0310×10^{-06})	1.2728 (0.6154)	0.2679 (0.04024)	0.6168 (0.0721)	5.3097×10^{-09} (0.0107)

The GoF statistics AIC, CAIC, BIC, CVM, AD, K-S and its PV in Table 2.5 shows that the LLoGMWP distribution fits the data better than the sub-models, and the non-nested models.

Table 2.5: GoF Statistics of Models for Failure Rates of Components Data

Model	Statistics								
	$-2\log(L)$	AIC	$CAIC$	BIC	CVM	AD	SS	$K-S$	PV
LLoGMWP	197.6134	207.6134	208.9771	217.1735	0.0941	0.6137	0.0853	0.0994	0.6696
LLoGP	210.9849	214.9849	215.2402	218.8089	0.2140	1.3296	0.5448	0.1372	0.2774
LLoGEP	206.2124	212.2124	212.7342	217.9485	0.1523	0.9451	0.3066	0.1839	0.0592
EKD	204.7135	214.7135	216.0771	224.2736	0.1499	0.9460	3.4806	0.1331	0.3101
GD	208.2542	218.2543	219.6179	227.8144	0.2148	1.3219	0.2162	0.1606	0.1359
EKW	202.8997	212.8998	214.2634	222.4599	0.1318	0.8531	0.1353	0.1363	0.2845
BWLLoG	204.6975	214.6975	216.0612	224.2577	0.1549	0.9658	0.1616	0.1187	0.4475
LLoGWP	200.4967	210.4967	211.8603	220.0568	0.1123	0.7305	0.0970	0.1075	0.5723
BrWLn	228.4620	238.4625	239.8261	248.0226	0.1524	0.9498	1.8593	0.2750	0.0007

The LRT for the following hypothesis testing $H_0 : \text{LLogP}$ against $H_a : \text{LLogMWP}$ and $H_0 : \text{LLogGEP}$ against $H_a : \text{LLogMWP}$ are 13.3715 (PV=0.0038) and 8.599 (PV= 0.0136). We conclude that there are significant differences between the nested models and LLogMWP distribution.

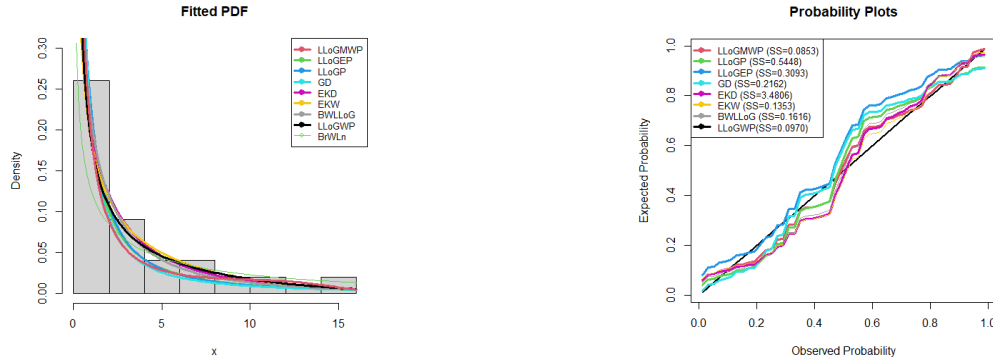


Figure 2.6: Fitted PDF and PPs for Failure Rates of Components Data

Plots of the fitted densities and PPs are given in Figure 2.6. For PPs figure, the plot with the smallest SS value corresponds to the model with points that are closer to the diagonal line. See Table 2.5 for the SS values. Figure 2.7 shows the TTT scaled plot with a BT hrf.

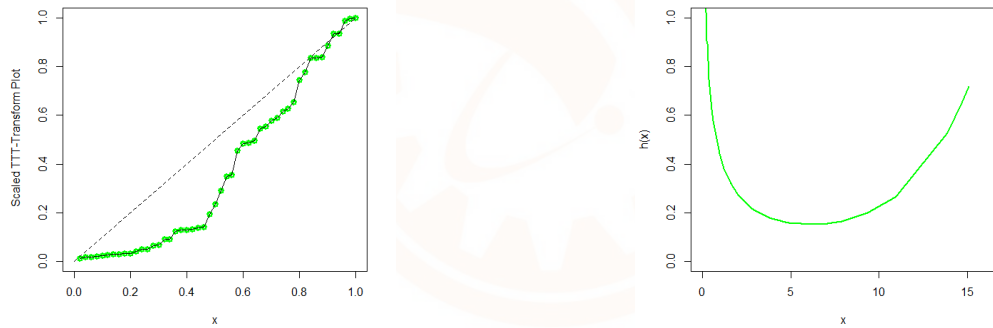


Figure 2.7: Fitted TTT and Hrf Plots for Failure Rates of Components Data

The K-M survival and ECDF curves for the failure rates of components data are shown in Figure 2.8. The fitted ECDF curve and K-M survival curve are closer to the empirical curve, indicating a better fit of the data.

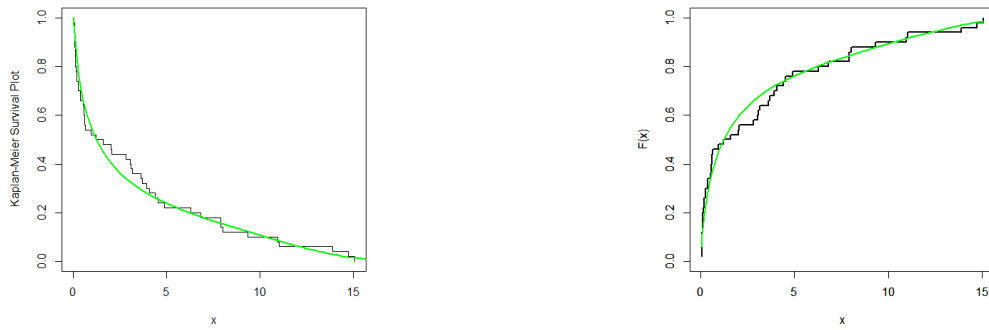


Figure 2.8: Fitted K-M Survival and ECDF Plots for Failure Rates of Components Data

We present the profile log-likelihood functions of the ML estimates of the LLoGMWP distribution for failure rates of components data in Figure 2.9. The ML estimates for the LLoGMWP distribution from failure rates of components data exist and can be obtained uniquely.

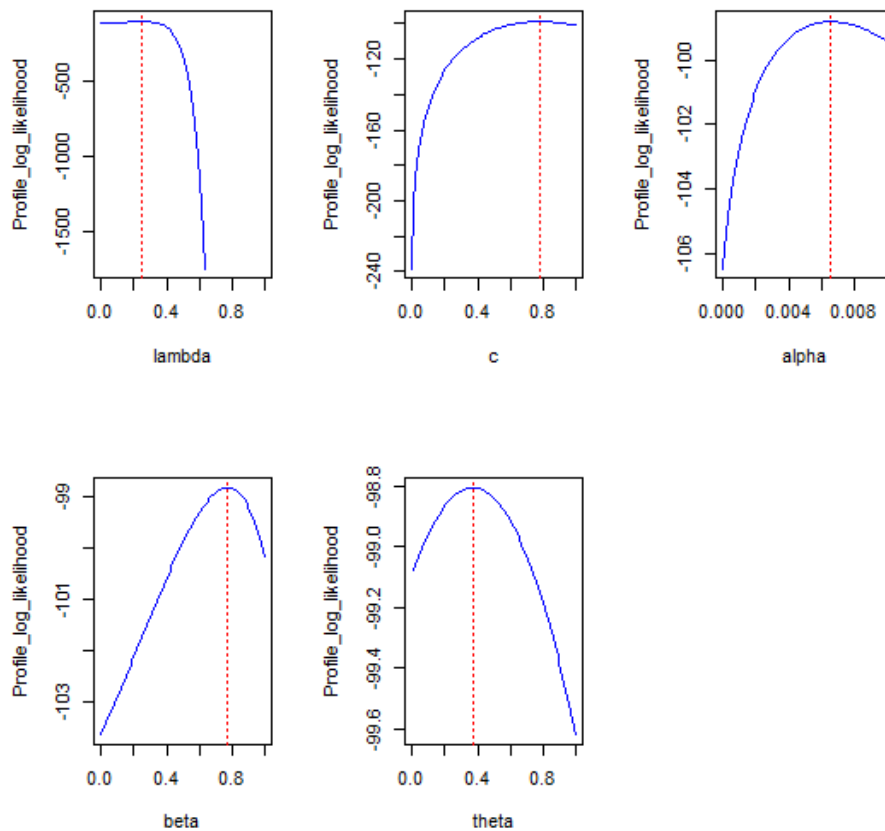


Figure 2.9: Plots of the Profile Log-Likelihood Function of the Parameters of the LLoGMWP Distribution on Failure Rates of Components Data

2.11.5.2 Aircraft Windshield Failure Data

The dataset analyzed in Tables 2.6 and 2.7 represents the dataset for a particular aircraft windshield reported by Anabike et al. [16]. The dataset is divided into two different sets of

data being the 88 observations that are classified as failed windshields and the 65 observations which are service times of windshields that had not failed at the time of observation. The measurement unit is 1000 hours. The violin shape indicates that the data has outliers at the edges and a concentration of observations in the middle range. The box plot in this data have have whiskers that are long at the top and bottom.

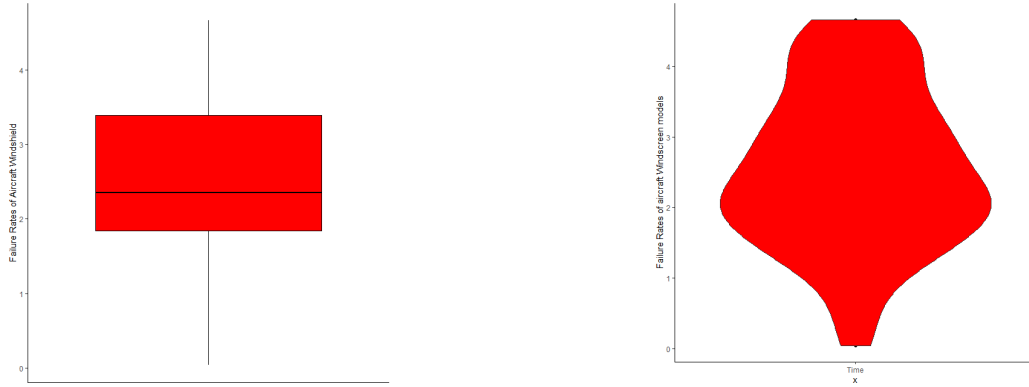


Figure 2.10: Boxplot and Violin Plots of Aircraft Windshield Failure Data

The estimated Var-Cov matrix for the LLoGMWP distribution is given by

$$\begin{pmatrix} 0.0152 & 0.03261 & -0.1009 & -0.0025 & -2.9859 \\ 0.03261 & 0.1696 & -0.1149 & -0.06028 & -2.8573 \\ -0.1009 & -0.1149 & 0.9210 & -0.0891 & 28.9171 \\ -0.0025 & -0.0603 & -0.0891 & 0.0489 & -3.282 \\ -2.9858 & -2.8573 & 28.9171 & -3.2824 & 924.5021 \end{pmatrix}, \quad (2.13)$$

and the approximate 95% CI for $\lambda, c, \alpha, \beta$ and θ are given by 0.3222 ± 0.2416 , 0.1024 ± 0.8072 , 1.1776 ± 1.8810 , 0.0691 ± 0.4337 , and 25.0918 ± 59.5950 , respectively.

Table 2.6 provides the parameter estimates for various models used in aircraft windshield failure data.

Table 2.6: Estimates of Models for Aircraft Windshield Failure Data

Model	Estimates				
	λ	c	α	β	θ
LLoGMWP	0.3222	0.1024	0.1776	0.06971	25.0918
	(0.1233)	(0.4118)	(0.9597)	(0.2213)	(30.4056)
LLoGP	-	2.3891	-	-	5.6505
	-	(0.1737)	-	-	(0.76491)
LLoGEP	-	0.4780	0.2590	-	1.2934×10^{-08}
	-	(0.5083)	(0.1490)	-	(0.0090)
GD	λ	β	δ	α	θ
	5.0919	6.6830	0.7827	0.4698	0.0010
	(0.0101)	(0.0038)	(0.2214)	(0.1492)	(0.0016)
EKD	α	λ	δ	ϕ	θ
	2.8909	16.1328	1.9388	10.5093	0.4057
	(1.4745)	(12.3388)	(0.4457)	(5.4744)	(0.2462)
EKW	c	a	b	θ	λ
	1.8858	3.6003	1.53512	0.3442	0.34900
	(0.5653)	(2.2276)	(1.1677)	(0.2446)	(0.0721)
BWLLoG	c	a	b	α	β
	5.8439	0.2885	5.3215	0.0005	2.5840
	(0.0018)	(0.0307)	(2.0250×10^{-6})	(1.1249×10^{-6})	(0.0113)
LLoGWP	s	c	α	β	θ
	2.5559	4.3028	0.0714	0.8998	0.3062
	(0.6394)	(0.5457)	(0.1189)	(0.6295)	(2.8061)
BrWLn	k	c	α	β	θ
	1.7919×10^{-06}	1.9719	0.5418	0.4345	1.8402×10^{-09}
	(1.0310×10^{-06})	(2.3539×10^{-06})	(0.0250)	(0.0459)	(0.0052)

Table 2.7 shows that the LLoGMWP distribution fits the data better than the non-nested models and the nested models.

Table 2.7: GoF Statistics of Models for Aircraft Windshield Failure Data

Model	Statistics								
	$-2\log(L)$	AIC	$CAIC$	BIC	CVM	AD	$K-S$	PV	SS
LLoGMWP	253.6165	263.6165	264.3857	275.7706	0.0469	0.5376	0.0642	0.9689	0.0728
LLoGP	286.4005	290.4005	290.5487	295.5622	0.3113	2.2440	0.1304	0.1147	0.2087
LLoGEP	419.3331	425.3331	425.6333	432.6257	0.1512	1.3009	0.1785	0.5743	0.3222
GD	260.4438	270.4440	271.2132	282.5981	0.0658	0.6310	0.0775	0.6934	0.5995
EKD	261.9840	271.9840	272.7532	284.1381	0.0649	0.6581	0.0756	0.7224	0.0881
EKW	261.1730	271.1730	271.9422	283.3271	0.0643	0.6435	0.0753	0.7278	0.0843
BWLLoG	255.4797	265.4797	266.2490	277.6338	0.1185	0.7838	0.0680	0.9421	7.8727
LLoGWP	258.8826	268.8826	269.6519	281.0367	0.0770	0.5822	0.0770	0.5822	0.0877
BrWLn	450.7534	460.7547	461.5239	472.9088	0.3257	2.4118	0.3257	2.591×10^{-10}	1.8593

The following hypothesis were tested using the LRT. The values for LRT statistic for testing $H_0 : \text{LLogP}$ against $H_a : \text{LLogMWP}$ and $H_0 : \text{LLogEP}$ against $H_a : \text{LLogMWP}$ are 32.7840 (PV<0.00001) and 165.7166 (PV<0.00001). We conclude that there are significant differences between LLogP, LLogEP and LLogMWP distributions. The PPs and fitted densities are given in Figure 2.11.

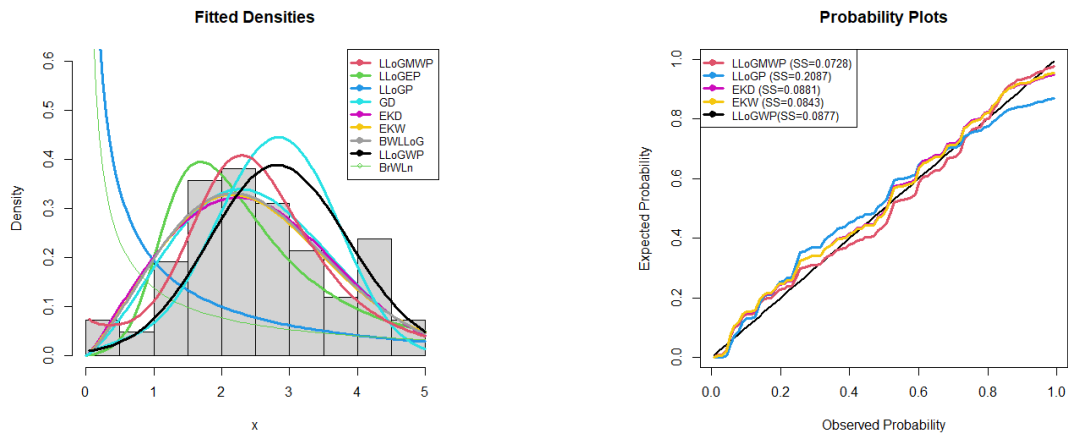


Figure 2.11: Fitted Densities Plots and PPs for Aircraft Windshield Failure data

Figure 2.12 shows the TTT scaled plot with an increasing hrf.

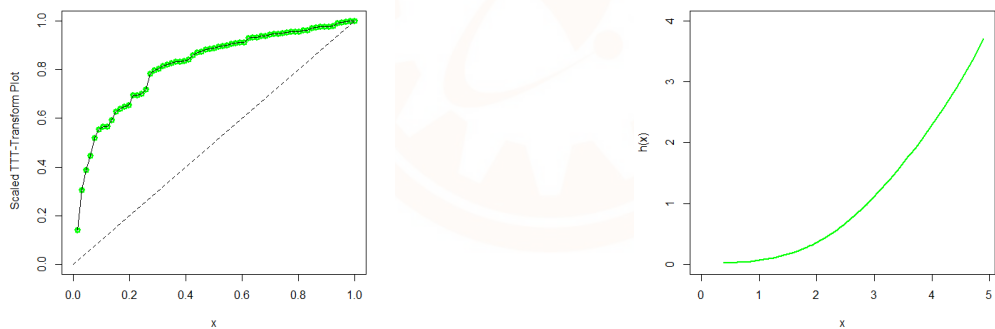


Figure 2.12: Fitted TTT and Hrf Plots for Aircraft Windshield Failure Data

In Figure 2.13, the K-M survival and ECDF curves for the aircraft windshield failure data are shown. The fitted ECDF curve and K-M curve are closer to the empirical curve, indicating a better fit of the aircraft windshield failure data.

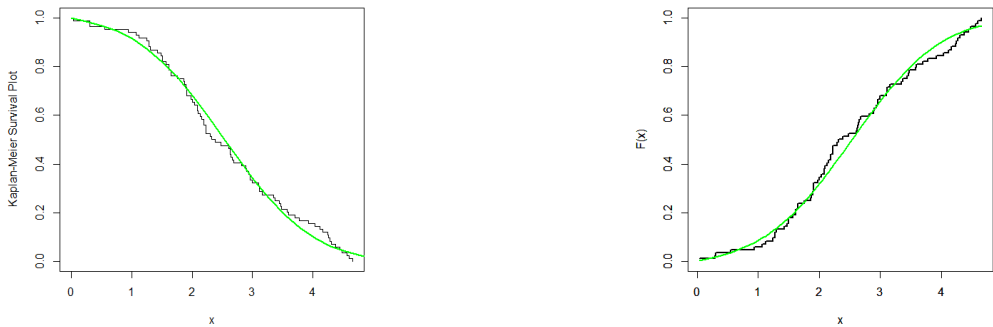


Figure 2.13: Fitted K-M Survival and ECDF Plots for Aircraft Windshield Failure Data

We present the profile log-likelihood functions of the parameter of the LLoGMWP distribution on aircraft windshield failure data in Figure 2.14. Figure 2.14 shows that the ML estimates of the LLoGMWP distribution can be achieved uniquely.

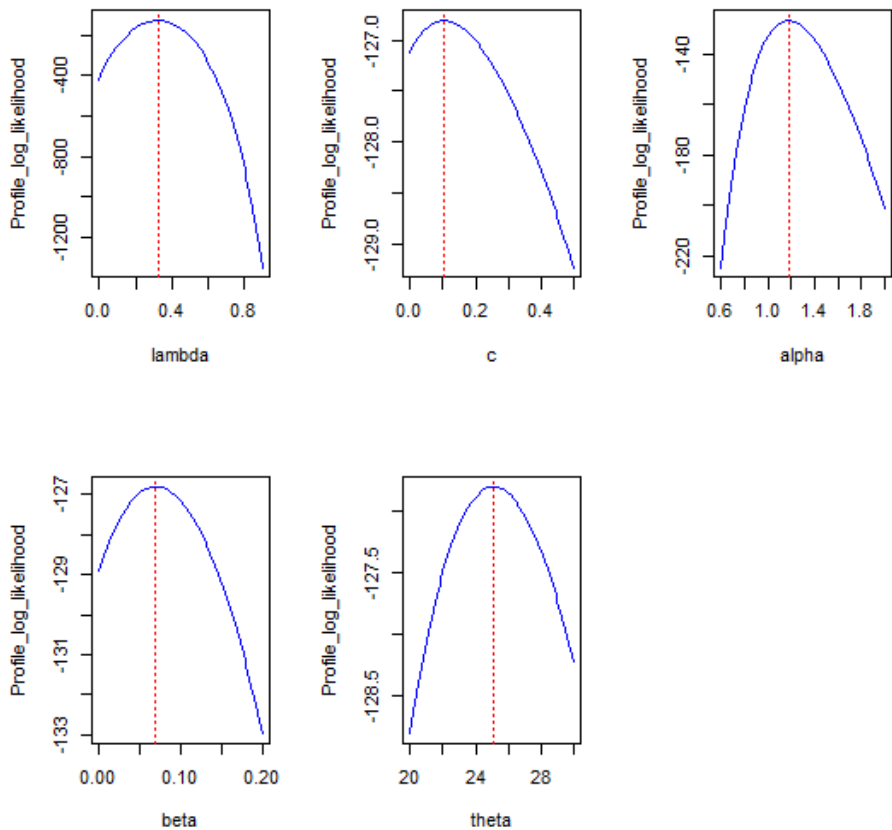


Figure 2.14: Plots of the Profile Log-Likelihood Function of the Parameters of the LLoGMWP Distribution on Aircraft Windshield Failure Data

2.12 Log-Logistic Modified Weibull Logarithmic (LLoGMWL) Distribution

The cdf and pdf of the LLoGMWL distribution are given by Equations (2.7) and (2.8), respectively. Plots of the pdf of the LLoGMWL distribution are given in the Figure 2.15.

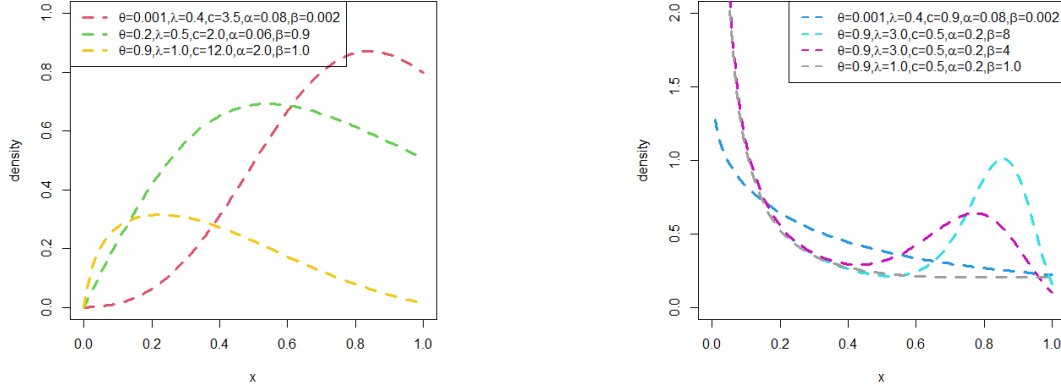


Figure 2.15: LLoGMWL Density Function Plots

Figure 2.15 shows that plots of the the LLoGMWL pdf can be decreasing, increasing-decreasing, right-skewed, uni-modal and reverse-J shapes for different selected values of the parameters.

2.12.1 Hazard Rate Function

The hrf of the LLoGMWL distribution is given in Equation (2.9). Figure 2.16 show the hrf of LLoGMWL distribution in different shapes.

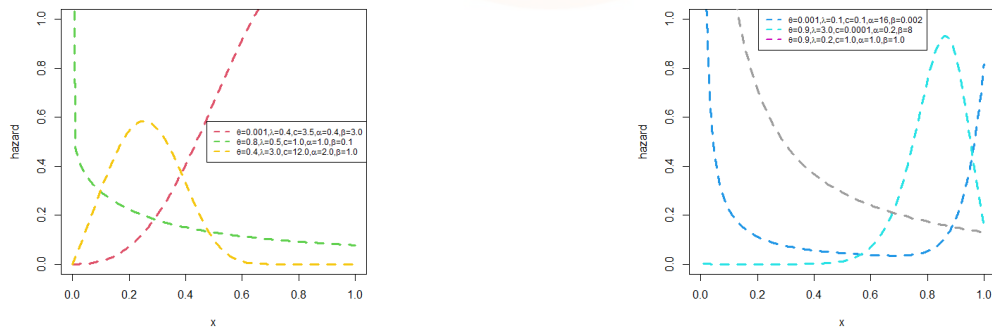


Figure 2.16: LLoGMWL Hrf Plots

Figure 2.16 exhibit increasing, decreasing, BT and unimodal shapes of the hrf for the selected values of the model parameters.

2.12.2 Quantile Function

The QF of the LLoGMWL distribution can be obtained by inverting $F_{LLoGMWL}(x) = u$, $0 \leq u \leq 1$, where

$$F_{LLoGMWL}(x) = 1 - \frac{\log(1 - \theta(1 + x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}})}{\log(1 - \theta)}.$$

The QF of the LLoGMWL distribution is obtained by solving the non-linear equation

$$-\log(1 + (Q_{X_1}(u))^c) - \alpha(Q_{X_1}(u))^\beta e^{\lambda(Q_{X_1}(u))} - \log\left[\frac{1 - e^{(1-u)\log(1-\theta)}}{\theta}\right] = 0.$$

Quantiles of the LLoGMWL distribution are presented in Table 2.8 for selected values of the model parameters.

Table 2.8: LLoGMWL Quantiles for Selected Parameter Values

u	$(c, \lambda, \theta, \alpha, \beta)$				
	(0.1,0.2,2.8,0.4,0.5)	(0.1,3.5,4.0,3.0,0.8)	(0.2,1.0,1.0,1.0,0.3)	(0.1,0.9,1.0,1.0,0.5)	(0.9,1.0,1.0,0.6,1.0)
0.1	0.0607	0.0134	0.0003	0.0083	0.0182
0.2	0.2236	0.0318	0.0043	0.0312	0.0418
0.3	0.3921	0.0524	0.0182	0.0674	0.0727
0.4	0.5388	0.0748	0.0476	0.1167	0.1135
0.5	0.6777	0.0989	0.0967	0.1810	0.1681
0.6	0.8230	0.1254	0.1691	0.2636	0.2423
0.7	0.9918	0.1555	0.2716	0.3715	0.3457
0.8	1.2166	0.1917	0.4194	0.5191	0.4959
0.9	1.6030	0.2422	0.6565	0.7484	0.7369

The first six moments, SD, CV, SK and KS for selected values of the parameters of the LLoGMWL distribution are listed in Table 2.9.

Table 2.9: LLoGMWL Moments for Selected Parameter Values

$Moments$	$(c, \theta, \lambda, \alpha, \beta)$				
	(1.0,1.0,1.0,1.0,1.0)	(0.2,0.0,0.1,1.0,0.2)	(1.0,1.0,1.0,0.9,1.0)	(0.6,0.6,0.5,0.5,0.05)	(0.3,1.0,1.6,1.0,0.1)
μ'_1	0.0802	0.0530	0.0288	0.0319	0.0996
μ'_2	0.0567	0.0360	0.0194	0.0213	0.0707
μ'_3	0.0443	0.0275	0.0147	0.0160	0.0560
μ'_4	0.0365	0.0222	0.0120	0.0128	0.0451
μ'_5	0.0310	0.0188	0.0102	0.0107	0.0382
μ'_6	0.0271	0.0162	0.0088	0.0092	0.0332
SD	0.2243	0.1821	0.1363	0.1425	0.2465
CV	2.7982	3.4367	4.7233	4.4695	2.4747
CS	2.8062	3.6496	5.2023	4.8617	2.3936
CK	9.6071	15.5148	30.1957	25.5557	7.3354



Figure 2.17: SK and KS Plots of the LLoGMWL Distribution, when $\beta = 1.5, \lambda = 8.8$ and $\theta = 0.9$.



Figure 2.18: SK and KS Plots of LLoGMWL Distribution, when $c = 5.5, \alpha = 0.05$ and $\theta = 0.1$.

The 3D plots of SK and KS for the LLoGMWL distribution are given in Figure 2.17 and 2.18. We observe that

- When the parameters β, λ and θ are fixed, the SK and KS of LLoGMWL distribution increase as c and α increase.
- When the parameters c, α and θ are fixed, the SK and KS of LLoGMWL distribution decrease as β and λ increase.

2.12.3 Estimation

Let $X \sim LLoGMWL(c, \alpha, \beta, \lambda, \theta)$ and $\Delta = (c, \alpha, \beta, \lambda, \theta)^T$ be the parameter vector. The log-likelihood function $\ell = \ell(\Delta)$ for a single observation x of X is given by

$$\begin{aligned} \ell = \log(L) &= \ln(\theta) - 2 \ln(1 + x^c) - \alpha x^\beta e^{\lambda x} + \ln \left(c x^{c-1} + \alpha x^{\beta-1} e^{\lambda x} (\beta + \lambda x) \right) \\ &- \ln \left(- \left(1 - \theta (1 + x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}} \right) \right) - \ln(\ln(1 - \theta)). \end{aligned}$$

(See Elements of the score vector are given in appendix).

2.12.4 Simulation Study

A simulation study was conducted to check the performance of the ML estimates. The simulation study was repeated $N = 1000$ times each with $n = 25, 50, 100, 200, 400, 800$ and with the true parameters values $I : \theta = 0.6, \lambda = 0.5, c = 0.8, \alpha = 4.0, \beta = 1.8$, and $II : \theta = 0.5, \lambda = 0.5, c = 0.5, \alpha = 1.2, \beta = 3.5$, The Table 2.10 shows the mean ML estimates with their respective ABias and RMSE. The simulation study generated random samples from derived distributions using the QF. The ML estimates were computed for each simulated dataset, mean, ABias, and RMSE of these estimates were calculated across all iterations.

The results in Table 2.10 show that the mean ML estimates converge to the true value whereas the RMSE and ABias decay towards zero as the sample size, n increases. The simulations results demonstrate that the ML estimation method is consistent. Therefore, the LLoGMWL distribution is good for modeling real-world data.

Table 2.10: MCS Results: Mean, ABias and RMSE

I					II		
Parameter	n	Mean	ABias	RMSE	Mean	RMSE	ABias
θ	25	0.6977	0.0977	0.3026	1.7553	1.6827	1.2553
	50	0.6474	0.0474	0.2564	1.1745	0.9477	0.6745
	100	0.6337	0.0337	0.2398	0.9082	0.7045	0.4082
	200	0.6242	0.0244	0.2333	0.6353	0.3810	0.1353
	400	0.6127	0.0127	0.2285	0.5025	0.3038	0.0025
	800	0.6079	0.0079	0.2252	0.4950	0.2933	-0.0150
λ	25	1.2407	0.7407	2.5277	0.8747	0.5599	0.3747
	50	0.9818	0.4818	1.7046	0.7523	0.5599	0.2523
	100	0.7942	0.2942	0.8581	0.6944	0.2434	0.1944
	200	0.7169	0.2169	0.8580	0.6519	0.1749	0.1519
	400	0.6039	0.1039	0.6147	0.6318	0.1441	0.1318
	800	0.4540	-0.0454	0.4992	0.5591	0.0638	0.0591
c	25	0.9122	0.1122	0.4366	7.6618	16.0246	7.1618
	50	0.8915	0.0915	0.2559	7.4830	13.4181	6.9830
	100	0.8635	0.0635	0.1838	5.3290	8.9145	4.8290
	200	0.8630	0.0630	0.1410	4.9502	7.5622	4.4502
	400	0.8382	0.0382	0.1032	3.5074	5.5076	3.0074
	800	0.8309	0.0309	0.0882	0.5938	0.0999	0.0938
α	25	7.3775	3.3776	13.5132	3.3944	3.2655	2.8944
	50	5.3212	1.3212	4.8617	2.8797	2.4745	1.6797
	100	4.6752	0.6752	3.0319	2.5477	2.0223	1.3477
	200	4.3856	0.3856	2.4560	2.4353	1.9064	1.2353
	400	4.1856	0.1856	2.1026	2.4097	1.6373	1.2097
	800	4.0351	0.0350	1.7837	1.2587	0.2387	0.0587
β	25	2.0542	0.2542	3.2785	4.6865	4.5777	1.1865
	50	1.9266	0.1266	0.6707	4.4325	4.2237	0.9325
	100	1.9225	0.1225	0.5261	4.4027	3.9167	0.9027
	200	1.9138	0.1138	0.4689	3.7454	2.4708	0.2454
	400	1.9116	0.1115	0.4215	3.6065	2.1832	0.1065
	800	1.8273	0.0270	0.3901	3.5868	1.4612	0.0868

2.12.5 Applications

In this subsection, some datasets are used to illustrate the usefulness and applicability of the LLoGMWL distribution. The LLoGMWL distribution is compared with its sub-models: LLoGW, LLoGR and LLoG distributions. We also compare the LLoGMWL distribution with GD distribution by Oluyede et al. [98], EKD distribution by Huang and Oluyede [54], EKW distribution by Eissa and Abdulaziz [39], LLoGWP distribution by Oluyede et al. [102], and BrWLn distribution by Oluyede et al. [99].

2.12.5.1 Caterpillar Body Mass Data

The first dataset comes from R software package "Stat2Data" Caterpillars data, consisting of 267 observations corresponding to body mass (in grams) of *Manduca Sexta* caterpillars. The violin and boxplot in Figure 2.19 suggests that the data is right-skewed and have outliers

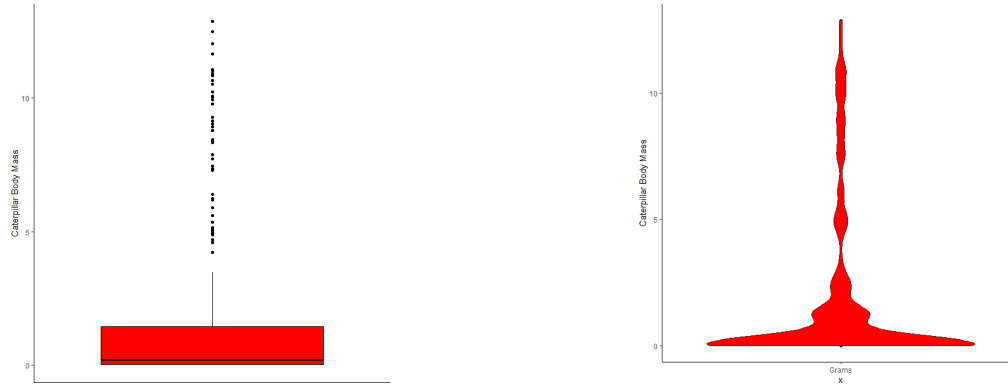


Figure 2.19: Boxplot and Violin Plots of Components Data

The estimated Var-Cov matrix of the LLoGMWL distribution is given by

$$\begin{pmatrix} 0.01584 & 0.00108 & -0.00016 & 0.00014 & 0.00096 \\ 0.00108 & 0.00140 & -0.00001 & 0.00009 & 0.00106 \\ -0.00016 & -0.00001 & 0.000001 & -0.00001 & -0.00001 \\ 0.00145 & 0.00009 & -0.00001 & 0.00013 & 0.00008 \\ 0.00096 & 0.00106 & -0.00001 & 0.00008 & 0.00169 \end{pmatrix},$$

and the approximate 95% CI for $\lambda, c, \alpha, \beta$ and θ are given by 0.5371 ± 0.2467 , 0.5722 ± 0.0735 , 0.0008 ± 0.0025 , 0.5926 ± 0.0227 , and 0.8651 ± 0.0806 , respectively.

We present the parameter estimates of the models and their SE in Table 2.11.

Table 2.11: Estimates of Models for Caterpillar Body Mass Data

Model	Estimates				
	λ	c	α	β	θ
LLoGMWL	0.5371 (0.1259)	0.5723 (0.0375)	0.0008 (0.0013)	0.5926 (0.0116)	0.8651 (0.0412)
LLoGMWL($\lambda, c, 1, \beta, \theta$)	0.0147 (0.0102)	0.4505 (0.0452)	- -	0.3880 (0.0326)	9.1888×10^{-09} (0.0085)
LLoGMWL($\lambda, 1, \alpha, \beta, \theta$)	0.1803 (0.0245)	- -	0.1797 (0.0351)	0.1688 (0.0216)	8.4552×10^{-09} (0.0057)
LLoGMWL($\lambda, 1, \alpha, 1, \theta$)	0.0100 (0.0651)	- -	0.0644 (0.0302)	- -	5.3589×10^{-09} (0.0046)
GD	λ	β	δ	α	θ
	3.3889 (0.000009)	3.0748 (0.00005)	0.0612 (0.0027)	3.0667 (0.00001)	0.0015 (0.00001)
EKD	α	λ	δ	ϕ	θ
	0.0050 (0.3081)	312.5400 (0.0000014)	2.3520 (0.7378)	2.0050 (0.000002)	22.2950 (1.5469)
EKW	c	a	b	θ	λ
	0.8727 (0.7512)	3.7969 (3.4702)	0.9770 (0.8459)	0.0891 (0.0807)	0.2629 (0.2582)
LLoGWP	s	c	α	β	θ
	0.2276 (0.0553)	0.4769 (0.3135)	0.0975 (0.0385)	1.0561 (0.1771)	1.7582×10^{-09} (0.0020)
BrWLn	k	c	α	β	θ
	1.6843 (0.0553)	0.4602 (0.3135)	1.8759×10^{-05} (0.0385)	$4.67796.4784 \times 10^{-03}$ (0.1771)	 (0.0020)

Table 2.12 shows that the LLoGMWL distribution outperforms the nested and non-nested models in-terms of the CVM, AD and SS values. Also, AIC, CAIC and BIC all indicate that the LLoGMWL distribution provides a better fit than the non-nested distributions. Thus, the LLoGMWL distribution can be considered as the best-fitting model for the caterpillar body mass dataset.

Table 2.12: GoF Statistics of Models for Caterpillar Body Mass Data

Model	Statistics								
	$-2\log(L)$	AIC	$CAIC$	BIC	CVM	AD	$K-S$	PV	SS
LLoGMWL	281.2231	291.2231	291.4530	309.1594	0.3005	1.9010	0.0606	0.9779	0.2747
LLoGMWL($\lambda, c, 1, \beta, \theta$)	373.4067	381.4067	381.5594	395.7557	0.5787	0.3932	0.1920	5.5510×10^{-09}	2.3202
LLoGMWL($\lambda, 1, \alpha, \beta, \theta$)	472.6697	480.6691	480.8218	495.0181	1.6186	9.9862	0.2552	1.5540×10^{-15}	5.0552
LLoGMWL($\lambda, 1, \alpha, 1, \theta$)	649.9628	655.9627	656.0540	666.7245	0.7415	4.8965	0.3544	2.2000×10^{-16}	10.9931
GD	341.9914	351.9913	352.2211	369.9275	0.5309	3.6239	0.1064	0.0046	0.5396
EKD	324.4124	334.4124	334.3423	352.3486	0.4682	3.2383	0.0912	0.0235	0.5282
EKW	357.6407	447.8232	448.1462	469.3467	1.1149	7.2879	0.1594	2.574×10^{-06}	0.3250
LLoGWP	322.8534	332.8395	333.0694	350.7757	0.4386	3.0787	0.0732	0.1136	0.2969
BrWLn	292.0598	302.0598	302.2897	319.9961	0.3701	2.4162	0.0814	0.5770	0.3097

The LRT for the following hypothesis tests are: H_0 : LLoGMWL($\lambda, c, 1, \beta, \theta$) against H_a : LLoGMWL, H_0 : LLoGMWL($\lambda, 1, \alpha, \beta, \theta$) against H_a : LLoGMWL, and H_0 : LLoGMWL($\lambda, 1, \alpha, 1, \theta$) against H_a : LLoGMWL yield test statistics of 92.1836 ($PV < 0.00001$), 191.4466 ($PV < 0.00001$), and 368.7397 ($PV < 0.00001$), respectively. We conclude that there are significant differences between the selected nested distributions and LLoGMWL distribution. The fitted densities plots and PPs are given in Figure 2.20.

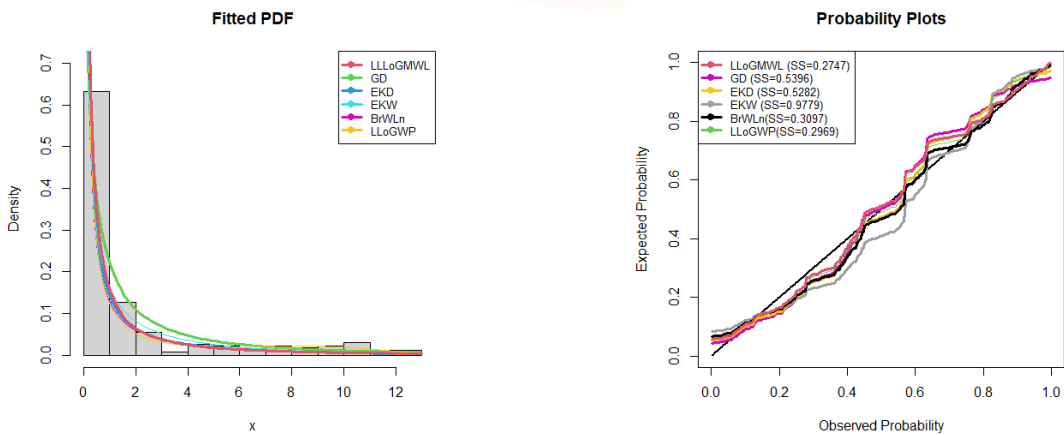


Figure 2.20: Fitted Pdf Plots and PPs for Caterpillar Body Mass Data

Figure 2.21 shows the TTT scaled plot with a UBT-BT hrf.

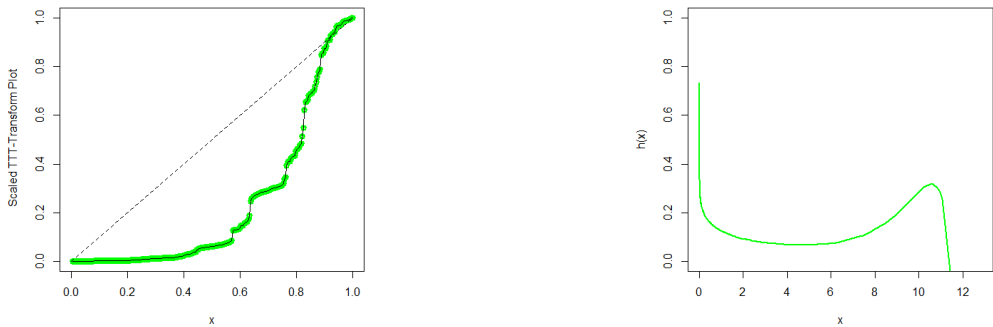


Figure 2.21: TTT Plots and Hrf for Caterpillar Body Mass Patients Data

The closeness between the fitted K-M survival and ECDF curves and the empirical curve, as shown in Figure 2.22, suggest a good fit of the model to the caterpillar body mass data.

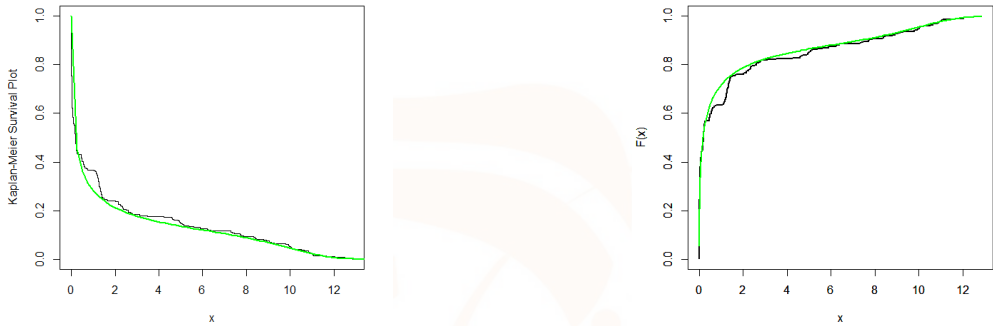


Figure 2.22: Fitted K-M Survival and ECDF Plots for Caterpillar Body Mass Data

We present the profile log-likelihood function of the parameters of the LLoGMWL distribution on caterpillar body mass data in Figure 2.23. Figure 2.23 shows that the ML estimates of the LLoGMWL distribution can be achieved uniquely.

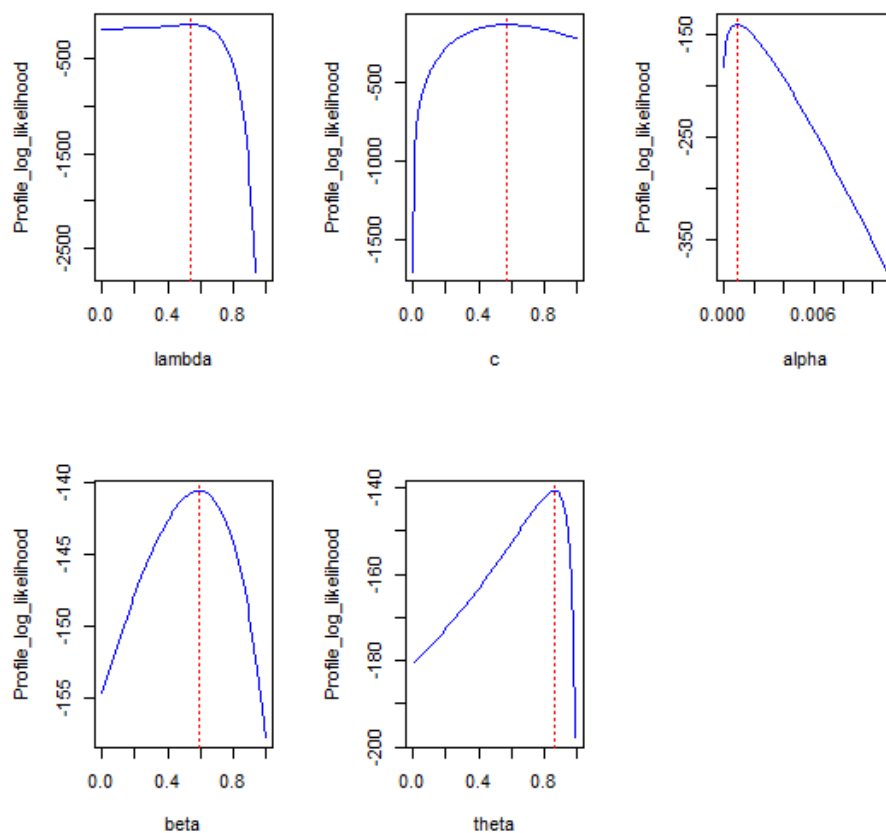


Figure 2.23: Plots of the Profile Log-Likelihood Function of the Parameters of the LLoGMWL Distribution on Caterpillar Body Mass Data

2.12.5.2 Survival Years of Chemotherapy Patients Data

The dataset is a part of the data reported by Moakofi et al. [77], and it relates to the survival years of a group of patients who received chemotherapy alone. The data include the survival times for 46 patients. The data observations are: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033. The data is right skewed with outliers.

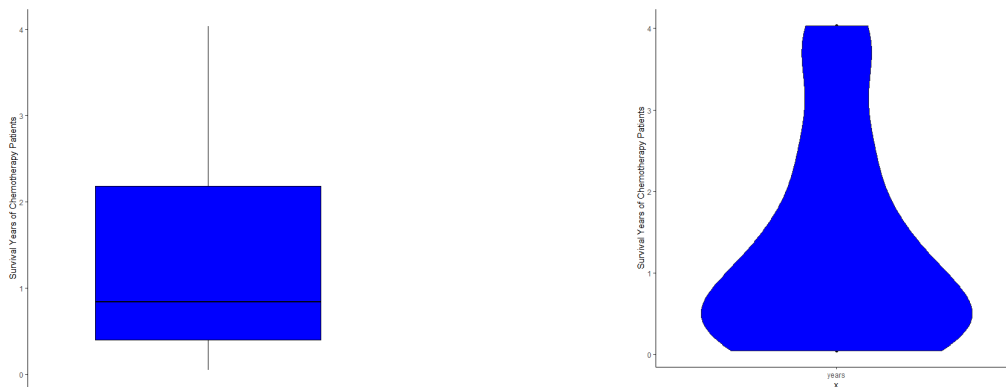


Figure 2.24: Boxplot and Violin Plots of Survival Years of Chemotherapy Patients Data

The estimated Var-Cov matrix of the LLoGMWL distribution is given by

$$\begin{pmatrix} 0.25478 & 0.00105 & -0.01485 & -0.22856 & 1.25722 \times 10^{-04} \\ 0.00105 & 0.03562 & -5.07652 \times 10^{-04} & -0.00462 & 5.98563 \times 10^{-05} \\ -0.01485 & -5.07652 \times 10^{-04} & 9.84519 \times 10^{-04} & 0.01202 & -1.3771 \times 10^{-05} \\ -0.22855 & -4.62249 \times 10^{-03} & 0.01203 & 0.28076 & -1.39966 \times 10^{-04} \\ 0.00012 & 5.98563 \times 10^{-05} & -1.37706 \times 10^{-05} & -0.00013 & 2.99234 \times 10^{-04} \end{pmatrix},$$

and the approximate 95% CI for $\lambda, c, \alpha, \beta$ and θ are given by 0.9694 ± 0.9893 , 1.2068 ± 0.3699 , 0.0221 ± 0.0614 , 0.1463 ± 1.03854 , and $9.8566 \times 10^{-09} \pm 0.0339$, respectively.

Table 2.13 gives the parameter estimates and their SE in parenthesis.

Table 2.13: Estimates of Models for Survival Years of Chemotherapy Patients Data

Model	Estimates				
	λ	c	α	β	θ
LLoGMWL	0.9694 (0.5047)	1.2068 (0.1887)	0.0221 (0.0313)	0.1463 (0.5298)	9.8566×10^{-09} (0.1729)
LLoGMWL($\lambda, c, 1, \beta, \theta$)	0.1389 (0.0570)	0.2295 (0.1245)	- -	0.0298 (0.0834)	2.8004×10^{-09} (0.0082)
LLoGMWL($\lambda, 1, \alpha, \beta, \theta$)	0.2347 (0.7503)	- -	0.0752 (0.1797)	0.1285 (0.7691)	9.3559×10^{-09} (0.0239)
LLoGMWL($\lambda, 1, \alpha, 1, \theta$)	0.1385 (0.1384)	- -	0.1122 (0.1223)	1 -	1.4395×10^{-08} (1.4395×10^{-08})
GD	λ	β	δ	α	θ
	23.8726 (0.0136)	1.8234 (0.6666)	0.3988 (0.1641)	1.9918 (0.1211)	0.0015 (0.0035))
EKD	α	λ	δ	ϕ	θ
	0.8742 (0.2161)	0.0035 (1.0938×10^{-05})	0.9229 (0.4773)	1419.3000 (2.8032×10^{-05})	1.6138 (1.9805)
EKW	c	a	b	θ	λ
	5.2827 (5.4515)	0.0354 (0.0663)	2.0262 (1.1080)	8.8587 (15.6819)	0.1795 (0.0811)
LLoGWP	s	c	α	β	θ
	1.0942 (0.7657)	1.1595 (0.2863)	0.1031 (0.3325)	1.8544 (2.1447)	6.3799×10^{-08} (0.0325)
BrWLn	k	c	α	β	θ
	1.1598 (0.2242)	1.5476 (0.2477)	0.0236 (0.0590)	2.5852 (2.4566)	9.0726×10^{-09} (1.5184)

According to the values of CVM, AD, SS, AIC, CAIC, and BIC in Table 2.14, the LLoGMWL distribution outperforms the nested and non-nested models. Therefore, it can be concluded that the LLoGMWL distribution is the best-fitting model for the survival years of chemotherapy patients dataset.

Table 2.14: GoF Statistics of Models for Survival Years of Chemotherapy Patients Data

Model	Statistics								
	$-2 \log(L)$	AIC	$CAIC$	BIC	CVM	AD	$K-S$	PV	SS
LLoGMWL	111.3002	121.3002	122.8386	130.3335	0.0549	0.3834	0.0772	0.9319	0.0364
LLoGMWL($\lambda, c, 1, \beta, \theta$)	197.9248	205.9248	206.9248	213.1515	0.0931	0.6160	0.62563	0.2300	6.2472
LLoGMWL($\lambda, 1, \alpha, \beta, \theta$)	131.3158	139.3158	140.3158	146.5424	0.0694	0.4807	0.1575	0.1922	0.1936
LLoGMWL($\lambda, 1, \alpha, 1, \theta$)	119.0507	125.0507	125.6362	130.4708	0.0553	0.3881	0.0926	0.8013	0.0714
GD	116.0515	126.0515	127.5900	135.0848	0.0657	0.4510	0.0959	0.7661	0.0697
EKD	116.0872	126.0872	127.6256	135.1205	0.0691	0.4705	0.1003	0.7176	0.0722
EKW	113.5734	123.5734	125.1119	132.6067	0.0737	0.4961	0.1058	0.6563	0.0793
LLoGWP	114.5491	124.5490	126.0875	133.5823	0.0652	0.4466	0.0937	0.7892	1.1895
BrWLn	116.3818	126.3818	127.9197	135.4145	0.0625	0.4279	0.1123	0.5822	0.1434

The LRT conducted for the following hypothesis comparisons: H_0 : LLoGMWL($\lambda, c, 1, \beta, \theta$) versus H_a : LLoGMWL, H_0 : LLoGMWL($\lambda, 1, \alpha, \beta, \theta$) versus H_a : LLoGMWL, and H_0 : LLoGMWL($\lambda, 1, \alpha, 1, \theta$) versus H_a : LLoGMWL, resulted in test statistics of 86.5246 ($PV < 0.00001$), 20.0156 ($PV < 0.00001$), and 7.7505 ($PV = 0.020749$), respectively. These result indicate that there are significant differences between the non-nested models and the LLoGMWP distribution.

To support the results in Table 2.14, we provide fitted density plots and PPs. These plots demonstrate that the LLoGMWL distribution offers a superior fit compared to other models considered for comparison.

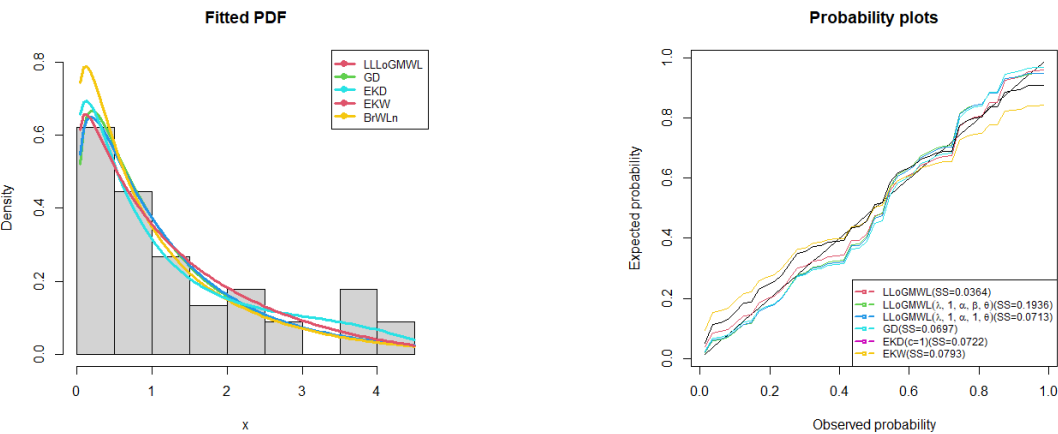


Figure 2.25: Fitted Pdf Plots and PPs for Survival Years of Chemotherapy Patients Data

Figure 2.26 shows the TTT scaled plot with a bathtub hrf.

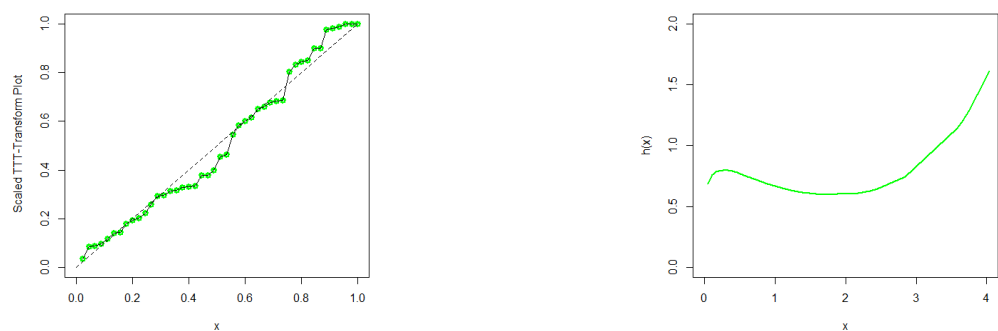


Figure 2.26: TTT Plots and Hrf for Survival Years of Chemotherapy Patients Data

The empirical curve, as displayed in Figure 2.27 is closer to the fitted K-M survival and ECDF curves, suggest a good fit of the model to the survival years of chemotherapy data.

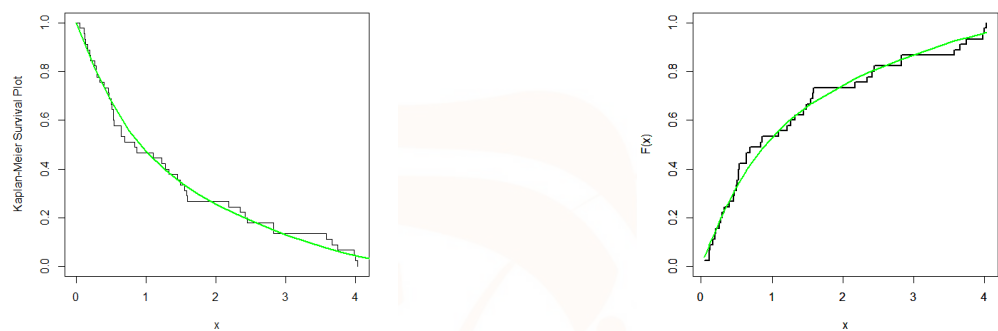


Figure 2.27: Fitted K-M Survival and ECDF Plots for Survival Years of Chemotherapy Patients Data

The profile log-likelihood function of the LLoGMWL distribution on survival years of chemotherapy patients data, as shown in Figure 2.28, demonstrates the existence and uniqueness of the ML estimates of the distribution.

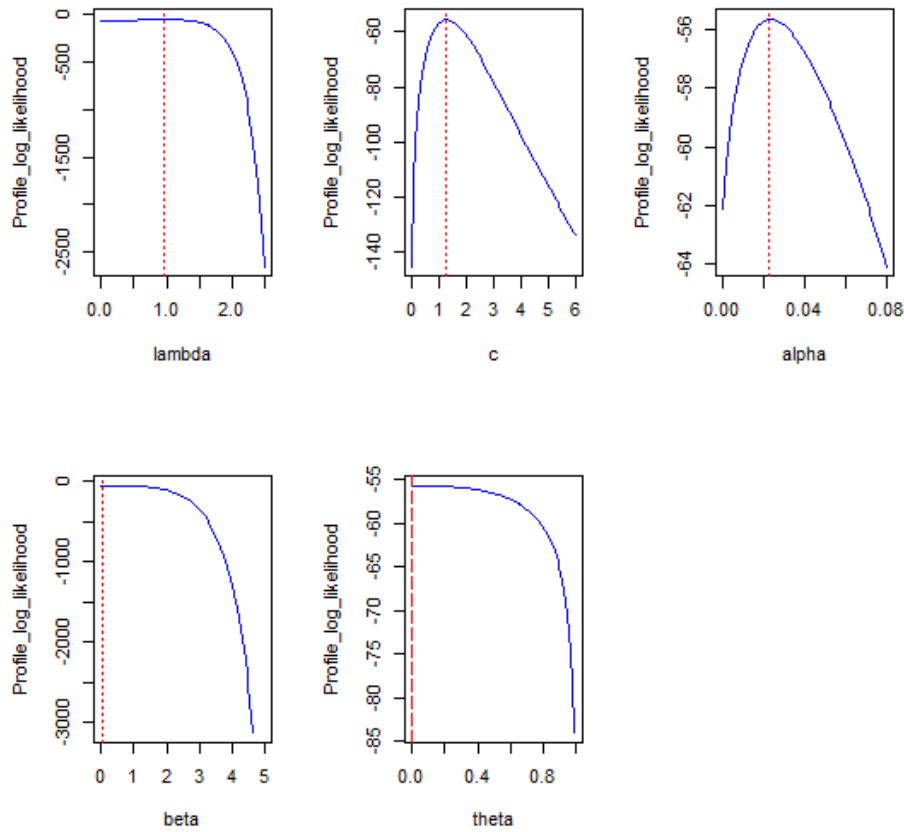


Figure 2.28: Plots of the Profile Log-Likelihood Function of the Parameters of the LLoGMWL Distribution on Survival Years of Chemotherapy Patients Data

2.13 Concluding Remarks

We have presented a new class of generalized distributions called the LLoGMWPS distribution that is suitable for applications in various areas including reliability, survival analysis, just to mention a few areas. The structural properties including hrf, QF, moments, RE, distribution of order statistics, ML estimates, and asymptotic CI are presented. Two special cases of the LLoGMWPS CoD, namely, LLoGMWP and LLoGMWL distributions are considered in details. Applications of these two special cases to real datasets are given in order to illustrate the applicability and usefulness of the proposed CoD.

The LLoGMWPS CoD's flexibility and ability to model non-monotonic hrf significantly improve reliability and survival analysis in critical infrastructure sectors like transportation, energy, and healthcare. It enhances decision-making, safety measures, and risk assessments, enabling accurate modeling of component failure rates and effective maintenance planning. By accurately modelling the shape of the hrf, we can identify critical failure points and plan maintenance activities accordingly.

3. The Type II Topp-Leone Exponentiated Half Logistic-G Power Series Class of Distributions with Applications

This paper is accepted by the *Scopus Indexed Pakistan Journal of Statistics and Operation Research*.

3.1 Introduction

The introduction of compounded distributions represents a significant advancement in statistical theory, particularly with the development of power series CoD. These classes combine established generalized families of distributions with discrete distributions, offering greater flexibility and applicability compared to existing distributions.

Despite the flexibility of existing distributions like TII-TL-G (Elgarhy et al. [41]) and EHL-G (Cordeiro et al. [35]) families of distributions, they often struggle to capture all the characteristics of real-world datasets. This limitations becomes apparent when the data deviates significantly from the assumptions of these distributions. In such cases, alternative models might be more appropriate. The new CoD offers an adequate parametric fit to skewed and heavy-tailed data than the non-nested distributions used for comparisons. Moreover, it outperforms the nested and non-nested models in terms of fit.

To illustrate the usefulness of the new CoD, we provide recent examples of compounded distributions such as the TII-EHL-TL-GPS CoD by Moakofi et al. [75], EHL-TL-G power series CoD by Chipepa et al. [32], EHL power series CoD by Esmaeili and Niaparast [42], TL-G power series CoD by [50], type II TL-G power series CoD by Makubate et al. [70], odd Weibull-TL-G power series CoD by Vasileva et al. [138], new class of TL-G power series CoD by Volodin [139], odd Weibull-TL-G power series CoD by Oluyede et al. [91], and TL-G power series CoD by Makubate et al. [72].

Motivated by the limitations of existing distributions, we propose and present a new CoD referred to as the TII-TL-EHL-GPS distribution. The TII-TL-EHL-GPS CoD can be obtained by compounding TII-TL-EHL-G FoD with power series distributions. The special cases of the new CoD such as TII-TL-EHL-LLoGL distribution, TII-TL-EHL-LLoGP distribution, TII-TL-EHL-WL distribution, and TII-TL-EHL-WP distribution, are discussed. Therefore, we show plots of the pdf, hrf, SK, and KS to demonstrate the effect of the extra parameters. The model parameters are estimated using the ML estimation technique and the consistency of ML estimators are evaluated by conducting a simulation study. To demonstrate the flexibility and versatility of the TII-TL-EHL-GPS CoD, we apply one of its special cases (TII-TL-EHL-WL distribution) to reliability and health sciences datasets. Also, in applications, we use the GoF statistics to

compare the TII-TL-EHL-WL distribution with the non-nested models and LRT to compare the TII-TL-EHL-WL distribution with the nested models.

3.2 The Model, Sub-Classes and Some Properties

In this subsection, we develop the TII-TL-EHL-GPS CoD. Some sub-classes and statistical properties for this new CoD are also presented.

3.2.1 The Model

Elgarhy et al. [41] introduced the TII-TL-G FoD. The cdf and pdf of the TII-TL-G FoD are given by

$$F(x; b, \xi) = 1 - [1 - G^2(x; \xi)]^b \quad (3.1)$$

and

$$f(x; b, \xi) = 2b [1 - G^2(x; \xi)]^{b-1} G(x; \xi)g(x; \xi),$$

respectively, for $b > 0$, $x \geq 0$, and parameter vector ξ .

Cordeiro et al. [35] presented the EHL-G FoD. The cdf and pdf of the EHL-G FoD are given by

$$F(x; \alpha, \xi) = \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^\alpha \quad (3.2)$$

and

$$f(x; b, \xi) = 2\alpha (G(x; \xi))^{\alpha-1} \frac{g(x; \xi)}{[1 + \overline{G}(x; \xi)]^{(\alpha+1)}},$$

respectively, for $\alpha > 0$, $x \geq 0$, and parameter vector ξ .

Note that the TII-TL-EHL-G FoD is developed using the Equation (3.1) and taking Equation (3.2) as the parent distribution. The resulting cdf and pdf of TII-TL-EHL-G FoD are given by

$$F(x; b, \alpha, \xi) = 1 - \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \quad (3.3)$$

and

$$f(x; b, \alpha, \xi) = 4\alpha b \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^{b-1} (G(x; \xi))^{2\alpha-1} \frac{g(x; \xi)}{[1 + \overline{G}(x; \xi)]^{2(\alpha+1)-1}}, \quad (3.4)$$

respectively, for $b, \alpha > 0$, $x \geq 0$, and parameter vector ξ .

Let N be a discrete random variable assumed to be zero truncated with a power series distribution, whose pmf is given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, n = 1, 2, 3, \dots, \quad (3.5)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n \geq 1}$ a sequence of positive real numbers. Noack [87] noted that the power series CoD include binomial, Poisson, geometric and logarithmic distributions.

Suppose we have a sequence of N iid random variable from the TII-TL-EHL-G FoD whose cdf and pdf are given by Equations (3.3) and (3.4), respectively. Let $X_{(1)} = \min(X_1, X_2, \dots, X_N)$, then the cdf of $X_{(1)}|N = n$ is given by

$$F_{X_{(1)}|N=n}(x; b, \alpha, \xi) = 1 - \left(\left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)^n, \quad (3.6)$$

for $b, \alpha > 0$, $x \geq 0$, $n \geq 1$ and parameter vector ξ . The TII-TL-EHL-GPS CoD is defined by the marginal distribution of $X_{(1)}$, that is,

$$F_{X_{(1)}}(x) = 1 - \frac{C \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)}{C(\theta)}, \quad (3.7)$$

for $b, \alpha, \theta > 0$, $x \geq 0$, and parameter vector ξ . Note that

$$\begin{aligned} F_{X_{(1)}}(x) &= \sum_{n=1}^{\infty} F_{X_{(1)}|N=n}(x) P(N = n) \\ &= \sum_{n=1}^{\infty} \left(1 - \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right) \frac{a_n \theta^n}{C(\theta)} \\ &= \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} - \sum_{n=1}^{\infty} \frac{a_n \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)^n}{C(\theta)} \\ &= 1 - \frac{C \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)}{C(\theta)}. \end{aligned}$$

The pdf of the TII-TL-EHL-GPS CoD is given by

$$\begin{aligned} f_{X_{(1)}}(x) &= 4\alpha b \theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^{b-1} (G(x; \xi))^{2\alpha-1} \frac{C' \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)}{C(\theta)} \\ &\quad \times \frac{g(x; \xi)}{[1 + \overline{G}(x; \xi)]^{2\alpha+1}}, \end{aligned} \quad (3.8)$$

for $b, \alpha, \theta > 0$, and parameter vector ξ , where $C'(\theta)$ is the derivative of $C(\theta)$, such that, $C'(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1}$.

The hrf is given by $h_{F_{X_{(1)}}}(x) = f_{X_{(1)}}(x)/S_{X_{(1)}}(x)$, where $S_{X_{(1)}}(x) = 1 - F_{X_{(1)}}(x)$.

Explicitly,

$$h_{F_{X_{(1)}}}(x) = \theta f(x) \frac{C'(\theta S(x))}{C(\theta S(x))},$$

where $f(x) = f_{TII-TL-EHL-G}(x)$ is the TII-TL-EHL-G FoD pdf and $S(x)$ is the survival function of the TII-TL-EHL-G FoD. The TII-TL-EHL-GPS CoD hrf is given by

$$\begin{aligned} h_{F_{X_{(1)}}}(x) &= 4\alpha b\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right)^{2\alpha} \right]^{b-1} \frac{g(x; \xi)}{[1 + \bar{G}(x; \xi)]^{2\alpha+1}} (G(x; \xi))^{2\alpha-1} \\ &\times \frac{C' \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)}{C \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)}, \end{aligned}$$

for $\alpha, b, \theta > 0$ and parameter vector ξ .

Table 3.1 below presents the special classes of TII-TL-EHL-GPS CoD when $C(\theta)$ is specified in Equation (3.7).

Table 3.1: Special Classes of the TII-TL-EHL-GPS CoD

Distribution	$C(\theta)$	$C'(\theta)$	cdf
TII-TL-EHL-G Poisson	$e^\theta - 1$	$(n!)^{-1}$	$1 - \frac{\exp \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right)^{2\alpha} \right]^b \right) - 1}{\exp(\theta) - 1}$
TII-TL-EHL-G Geometric	$\theta(1 - \theta)^{-1}$	1	$1 - \frac{(1 - \theta) \left(\left[1 - \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)}{\left(1 - \theta \left(\left[1 - \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right)^{2\alpha} \right]^b \right) \right)}$
TII-TL-EHL-G Logarithmic	$-\log(1 - \theta)$	n^{-1}	$1 - \frac{\log \left(1 - \theta \left(\left[1 - \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right)^{2\alpha} \right]^b \right) \right)}{\log(1 - \theta)}$
TII-TL-EHL-G Binomial	$(1 + \theta)^m - 1$	$\binom{m}{n}$	$1 - \frac{\left(1 + \theta \left(\left[1 - \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right)^{2\alpha} \right]^b \right) \right)^m - 1}{(1 + \theta)^m - 1}$

3.2.2 Sub-Classes of TII-TL-EHL-GPS Class of Distributions

- When $b = 1$, we obtain a new CoD with the cdf

$$F(x; \alpha, \theta, \xi) = 1 - \frac{C \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right)^{2\alpha} \right] \right)}{C(\theta)},$$

for $x \geq 0$, $\alpha, \theta > 0$, and parameter vector ξ .

- When $\alpha = 1$, we get the TII-TL-HL-GPS CoD with the cdf

$$F(x; b, \theta, \xi) = 1 - \frac{C \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^2 \right]^b \right)}{C(\theta)},$$

for $x \geq 0$, $b, \theta > 0$, and parameter vector ξ .

- When $\alpha = b = 1$, we get a new CoD with the cdf

$$F(x; \theta, \xi) = 1 - \frac{C \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^2 \right] \right)}{C(\theta)},$$

for $x \geq 0$, $\theta > 0$, and parameter vector ξ .

- When $\theta \rightarrow 0^+$, we get a TII-TL-EHL-G FoD with the cdf

$$F(x; b, \alpha, \xi) = 1 - \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b,$$

respectively, for $x \geq 0$, $b, \alpha > 0$ and parameter vector ξ .

- When $b = 1$ and $\theta \rightarrow 0^+$, we obtain a new FoD with the cdf

$$F(x; \alpha, \xi) = 1 - \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right] = \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha},$$

respectively, for $x \geq 0$, $\alpha > 0$ and parameter vector ξ .

- When $\alpha = 1$ and $\theta \rightarrow 0^+$, we get a TII-TL-HL-G FoD with the cdf

$$F(x; b, \xi) = 1 - \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^2 \right]^b,$$

respectively, for $x \geq 0$, $b > 0$ and parameter vector ξ .

- When $\alpha = b = 1$ and $\theta \rightarrow 0^+$, we get a new FoD with the cdf

$$F(x; \xi) = \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^2,$$

for $x \geq 0$ and parameter vector ξ .

3.3 Identifiability

Identifiability ensures model parameters can be determined uniquely based on observed data. In this proof, we demonstrate the identifiability of the TII-TL-EHL-GPS CoD. We consider two sets of parameters for the TII-TL-EHL-GPS CoD. Let

$$\textcircled{\mathbf{H}}_1 = (b_1, \alpha_1, \theta_1, \xi_1) \quad \text{and} \quad \textcircled{\mathbf{H}}_2 = (b_2, \alpha_2, \theta_2, \xi_2).$$

The likelihood ratio is given by:

$$\begin{aligned} \frac{f(\textcircled{\mathbf{H}}_1)}{f(\textcircled{\mathbf{H}}_2)} &= \left(\frac{\alpha_1 b_1 \theta_1}{\alpha_2 b_2 \theta_2} \right) \left[1 - \left(\frac{G(x; \xi_1)}{1 + \overline{G}(x; \xi_1)} \right)^{2\alpha_1} \right]^{b_1-1} \frac{C' \left(\theta_1 \left[1 - \left(\frac{G(x; \xi_1)}{1 + \overline{G}(x; \xi_1)} \right)^{2\alpha_1} \right]^{b_1} \right)}{C(\theta_1)} \\ &\times \left[1 - \left(\frac{G(x; \xi_2)}{1 + \overline{G}(x; \xi_2)} \right)^{2\alpha_2} \right]^{-(b_2-1)} \frac{C(\theta_2)}{C' \left(\theta_2 \left[1 - \left(\frac{G(x; \xi_2)}{1 + \overline{G}(x; \xi_2)} \right)^{2\alpha_2} \right]^{b_2} \right)} \\ &\times [1 + \overline{G}(x; \xi_1)]^{-(2\alpha_1+1)} [1 + \overline{G}(x; \xi_2)]^{2\alpha_2+1} [G(x; \xi_1)]^{2\alpha_1-1} \\ &\times [G(x; \xi_2)]^{-(2\alpha_2-1)} \frac{g(x; \xi_1)}{g(x; \xi_2)}. \end{aligned}$$

The logarithm of the likelihood ratio is

$$\begin{aligned} \log \left(\frac{f(\textcircled{\mathbf{H}}_1)}{f(\textcircled{\mathbf{H}}_2)} \right) &= \log \left(\frac{\alpha_1 b_1 \theta_1}{\alpha_2 b_2 \theta_2} \right) + (b_1 - 1) \log \left[1 - \left(\frac{G(x; \xi_1)}{1 + \overline{G}(x; \xi_1)} \right)^{2\alpha_1} \right] \\ &- (b_2 - 1) \log \left[1 - \left(\frac{G(x; \xi_2)}{1 + \overline{G}(x; \xi_2)} \right)^{2\alpha_2} \right] \\ &+ \log \left(C' \left(\theta_1 \left[1 - \left(\frac{G(x; \xi_1)}{1 + \overline{G}(x; \xi_1)} \right)^{2\alpha_1} \right]^{b_1} \right) \right) \\ &- \log \left(C' \left(\theta_2 \left[1 - \left(\frac{G(x; \xi_2)}{1 + \overline{G}(x; \xi_2)} \right)^{2\alpha_2} \right]^{b_2} \right) \right) \\ &- (2\alpha_1 + 1) \log [1 + \overline{G}(x; \xi_1)] + (2\alpha_2 + 1) \log [1 + \overline{G}(x; \xi_2)] \\ &+ (2\alpha_1 - 1) \log [G(x; \xi_1)] - (2\alpha_2 - 1) \log [G(x; \xi_2)] \\ &+ \log [g(x; \xi_1)] - \log [g(x; \xi_2)]. \end{aligned} \tag{3.9}$$

Now, the derivative of the log-likelihood with respect to x is

$$\begin{aligned}
 \frac{d}{dx} \log \left(\frac{f(\textcircled{\text{H}}_1)}{f(\textcircled{\text{H}}_2)} \right) &= -(b_1 - 1) \frac{A_1}{\left[1 - \left(\frac{G(x; \xi_1)}{1 + \overline{G}(x; \xi_1)} \right)^{2\alpha_1} \right]} + (b_2 - 1) \frac{A_2}{\left[1 - \left(\frac{G(x; \xi_2)}{1 + \overline{G}(x; \xi_2)} \right)^{2\alpha_2} \right]} \\
 &\quad - \frac{C'' \left(\theta_1 \left[1 - \left(\frac{G(x; \xi_1)}{1 + \overline{G}(x; \xi_1)} \right)^{2\alpha_1} \right]^{b_1} \right)}{C' \left(\theta_1 \left[1 - \left(\frac{G(x; \xi_1)}{1 + \overline{G}(x; \xi_1)} \right)^{2\alpha_1} \right]^{b_1} \right)} \theta_1 b_1 \left[1 - \left(\frac{G(x; \xi_1)}{1 + \overline{G}(x; \xi_1)} \right)^{2\alpha_1} \right]^{b_1 - 1} A_1 \\
 &\quad + \frac{C'' \left(\theta_2 \left[1 - \left(\frac{G(x; \xi_2)}{1 + \overline{G}(x; \xi_2)} \right)^{2\alpha_2} \right]^{b_2} \right)}{C' \left(\theta_2 \left[1 - \left(\frac{G(x; \xi_2)}{1 + \overline{G}(x; \xi_2)} \right)^{2\alpha_2} \right]^{b_2} \right)} \theta_2 b_2 \left[1 - \left(\frac{G(x; \xi_2)}{1 + \overline{G}(x; \xi_2)} \right)^{2\alpha_2} \right]^{b_2 - 1} A_2 \\
 &\quad + (2\alpha_1 + 1) \frac{g(x; \xi_1)}{[1 + \overline{G}(x; \xi_1)]} - (2\alpha_2 + 1) \frac{g(x; \xi_2)}{[1 + \overline{G}(x; \xi_2)]} + (2\alpha_1 - 1) \frac{g(x; \xi_1)}{G(x; \xi_1)} \\
 &\quad - (2\alpha_2 - 1) \frac{g(x; \xi_2)}{G(x; \xi_2)} + \frac{g'(x; \xi_1)}{g(x; \xi_1)} - \frac{g'(x; \xi_2)}{g(x; \xi_2)},
 \end{aligned} \tag{3.10}$$

where A_1 and A_2 are defined as: $A_1 = \frac{4\alpha_1[G(x; \xi_1)]^{2\alpha_1-1}}{[1+G(x; \xi_1)]^{2\alpha_1+1}} g(x; \xi_1)$ and $A_2 = \frac{4\alpha_2[G(x; \xi_2)]^{2\alpha_2-1}}{[1+G(x; \xi_2)]^{2\alpha_2+1}} g(x; \xi_2)$. Therefore, when the derivative of the logarithm of the likelihood ratio with respect to x is equal to zero, that is,

$$\frac{d}{dx} \log \left(\frac{f(\textcircled{\text{H}}_1)}{f(\textcircled{\text{H}}_2)} \right) = 0,$$

it suggest that the TII-TL-EHL-GPS CoD is identifiable if and only if $\textcircled{\text{H}}_1 = \textcircled{\text{H}}_2$.

3.4 Statistical Properties

In this section, we present the statistical properties of TII-TL-EHL-GPS CoD.

3.4.1 Quantile Function

Let X be a random variable with cdf defined by Equation (3.7). The QF ($Q_{X(1)}(u)$) is defined by $F_{X(1)}(Q_{X(1)}(u)) = u, 0 \leq u \leq 1$. Note that

$$1 - \frac{C \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)}{C(\theta)} = u,$$

so that

$$C \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right) = C(\theta) (1 - u),$$

which further simplifies to

$$G(x; \xi) = \frac{2 \left([1 - A]^{\frac{1}{2\alpha}} \right)}{\left([1 - A]^{\frac{1}{2\alpha}} + 1 \right)},$$

where $A = \left[\frac{C^{-1}(C(\theta)(1-u))}{\theta} \right]^{\frac{1}{b}}$. Therefore, the QF of the TII-TL-EHL-GPS CoD is given by,

$$Q_{X_{(1)}}(u) = G^{-1} \left[\frac{2 \left([1 - A]^{\frac{1}{2\alpha}} \right)}{\left([1 - A]^{\frac{1}{2\alpha}} + 1 \right)} \right]. \quad (3.11)$$

It follows therefore that random numbers can be generated from the TII-TL-EHL-GPS CoD using Equation (3.11), for specified baseline cdf and function $C(\theta)$.

3.4.2 Expansion of Density

Expansion of the pdf of the TII-TL-EHL-GPS CoD is presented in this sub-section. Equation (3.8) can be written as

$$\begin{aligned} f_{X_{(1)}}(x) &= \sum_{n=1}^{\infty} \frac{na_n\theta^n}{C(\theta)} \left(\left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)^{n-1} \\ &\times 4\alpha b\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^{b-1} (G(x; \xi))^{2\alpha-1} \frac{g(x; \xi)}{[1 + \overline{G}(x; \xi)]^{2\alpha+1}} \\ &= 4\alpha b\theta \sum_{n=1}^{\infty} \frac{na_n\theta^n}{C(\theta)} \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^{bn-1} (G(x; \xi))^{2\alpha-1} \frac{g(x; \xi)}{[1 + \overline{G}(x; \xi)]^{2\alpha+1}}. \end{aligned}$$

Using the generalized binomial series expansion

$$\left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^{bn-1} = \sum_{k=0}^{\infty} \binom{bn-1}{k} (-1)^k \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha k},$$

we write the pdf of the TII-TL-EHL-GPS CoD as

$$f_{X_{(1)}}(x) = 4\alpha b\theta \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{bn-1}{k} (-1)^k \frac{na_n\theta^n}{C(\theta)} (G(x; \xi))^{2\alpha(k+1)-1} \frac{g(x; \xi)}{[1 + \overline{G}(x; \xi)]^{2\alpha(k+1)-1}}.$$

Considering the following series expansions:

$$[1 + \overline{G}(x; \xi)]^{-(2\alpha(k+1)-1)} = \sum_{p=0}^{\infty} \binom{(2\alpha(k+1)+p)}{p} (-1)^p \overline{G}^p(x; \xi)$$

and

$$\overline{G}^p(x; \xi) = \sum_{m=0}^{\infty} (-1)^m \binom{p}{m} G^m(x; \xi),$$

we can write

$$\begin{aligned} f_{X(1)}(x) &= \frac{4\alpha b \theta n a_n \theta^n}{C(\theta)} \sum_{n=1}^{\infty} \sum_{k,p,m=0}^{\infty} \binom{bn-1}{k} \binom{(2\alpha(k+1)+p)}{p} (-1)^{k+m+p} \binom{p}{m} \\ &\times g(x; \xi) (G(x; \xi))^{2\alpha(k+1)+m-1} \\ &= \sum_{k,m=0}^{\infty} \omega_{k,m} g_{2\alpha(k+1)+m}(x; \xi), \end{aligned} \quad (3.12)$$

where $g_{2\alpha(k+1)+m}(x; \xi) = (2\alpha(k+1)+m) (G(x; \xi))^{2\alpha(k+1)+m-1} g(x; \xi)$ is an Exp-G distribution with power parameter $(2\alpha(k+1)+m)$ and

$$\omega_{k,m} = \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \binom{bn-1}{k} \binom{(2\alpha(k+1)+p)}{p} \binom{p}{m} (-1)^{k+m+p} \frac{n a_n \theta^n}{C(\theta)} \left(\frac{4\alpha b \theta}{2\alpha(k+1)+m} \right). \quad (3.13)$$

Thus, the pdf of TII-TL-EHL-GPS CoD can be written as an infinite linear combination of Exp-G distribution.

3.4.3 Moments and Generating Function

If X follows the TII-TL-EHL-GPS CoD and $Y \sim \text{Exp-G}(2\alpha(k+1)+m)$. Then using Equation (3.12), the i^{th} raw moment, μ'_i of the TII-TL-EHL-GPS CoD is obtained as

$$\mu'_i = E(X^i) = \int_{-\infty}^{\infty} x^i f(x) dx = \sum_{k,m=0}^{\infty} \omega_{k,m} E(Y^i),$$

where $E(Y^i)$ is the i^{th} moment of Exp-G distribution and $\omega_{k,m}$ is given by Equation (3.13).

The MGF, $M_X(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{k,m=0}^{\infty} \omega_{k,m} M_Y(t),$$

where $M_Y(t)$ is the MGF of Y and $\omega_{k,m}$ is given by Equation (3.13).

3.4.4 Distribution of Order Statistics

In this subsection, we present the distribution of the i^{th} order statistic of the TII-TL-EHL-GPS CoD. In many areas of statistical theory and application, order statistics are essential. Let $X_1, X_2, \dots, X_n$ be a random sample from TII-TL-EHL-GPS CoD. The pdf of the i^{th} order statistic is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l f_{X_{(1)}}(x) [F_{X_{(1)}}(x)]^{i+l-1}. \quad (3.14)$$

Using Equations (3.7) and (3.8), we note that

$$\begin{aligned} f_{X_{(1)}}(x) [F_{X_{(1)}}(x)]^{i+l-1} &= 4\alpha b\theta \sum_{n=1}^{\infty} \frac{na_n\theta^n}{C(\theta)} \left[1 - \left(\frac{G(x;\xi)}{1+\overline{G}(x;\xi)} \right)^{2\alpha} \right]^{bn-1} (G(x;\xi))^{2\alpha-1} \\ &\quad \times \frac{g(x;\xi)}{[1+\overline{G}(x;\xi)]^{2\alpha+1}} \left(1 - \frac{C\left(\theta \left[1 - \left(\frac{G(x;\xi)}{1+\overline{G}(x;\xi)} \right)^{2\alpha} \right]^b\right)}{C(\theta)} \right)^{i+l-1}. \end{aligned}$$

Using the generalized binomial expansion

$$\left(1 - \frac{C\left(\theta \left[1 - \left(\frac{G(x;\xi)}{1+\overline{G}(x;\xi)} \right)^{2\alpha} \right]^b\right)}{C(\theta)} \right)^{i+l-1} = \sum_{s=0}^{\infty} \binom{i+l-1}{s} (-1)^s \left(\frac{C\left(\theta \left[1 - \left(\frac{G(x;\xi)}{1+\overline{G}(x;\xi)} \right)^{2\alpha} \right]^b\right)}{C(\theta)} \right)^s,$$

and the series expansion

$$\left(\frac{C\left(\theta \left[1 - \left(\frac{G(x;\xi)}{1+\overline{G}(x;\xi)} \right)^{2\alpha} \right]^b\right)}{C(\theta)} \right)^s = \sum_{w=0}^{\infty} d_{w,s} \theta^w \frac{1}{C(\theta)^w} \left[1 - \left(\frac{G(x;\xi)}{1+\overline{G}(x;\xi)} \right)^{2\alpha} \right]^{bw},$$

obtained by applying the power series distribution raised to a non positive integer (Gradshteyn and Ryzhik [45]), where $d_{w,s} = (wb_0)^{-1} \sum_{m=1}^w [s(m+1) - w] b_m d_{w-m,s}$ and $d_{0,s} = b_0^s$, we get

$$\begin{aligned} f_{X_{(1)}}(x) [F_{X_{(1)}}(x)]^{i+l-1} &= \sum_{n=1}^{\infty} \sum_{s,w=0}^{\infty} \binom{i+l-1}{s} (-1)^s \frac{na_n d_{w,s} \theta^{n+w}}{(C(\theta))^{w+1}} (G(x;\xi))^{2\alpha-1} \\ &\quad \times 4\alpha b\theta \left[1 - \left(\frac{G(x;\xi)}{1+\overline{G}(x;\xi)} \right)^{2\alpha} \right]^{b(n+w)-1} \frac{g(x;\xi)}{[1+\overline{G}(x;\xi)]^{2\alpha+1}}. \end{aligned}$$

Using the generalized binomial series expansion

$$\left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)}\right)^{2\alpha}\right]^{b(n+w)-1} = \sum_{k=0}^{\infty} \binom{b(n+w)-1}{k} (-1)^k \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)}\right)^{2\alpha k},$$

we can write

$$\begin{aligned} f_{X_{(1)}}(x) \left[F_{X_{(1)}}(x)\right]^{i+l-1} &= \sum_{n=1}^{\infty} \sum_{k,s,w=0}^{\infty} \binom{b(n+w)-1}{k} \binom{i+l-1}{s} \frac{na_n d_{w,s} \theta^{n+w}}{(C(\theta))^{w+1}} \\ &\times 4\alpha b \theta (-1)^{s+k} (G(x; \xi))^{2\alpha(k+1)-1} \frac{g(x; \xi)}{[1 + \overline{G}(x; \xi)]^{2\alpha(k+1)+1}}. \end{aligned}$$

Furthermore, considering the expansion

$$[1 + \overline{G}(x; \xi)]^{-(2\alpha(k+1)+1)} = \sum_{p=0}^{\infty} (-1)^p \binom{2\alpha(k+1)+1}{p} \overline{G}^p(x; \xi),$$

we obtain

$$\begin{aligned} f_{X_{(1)}}(x) \left[F_{X_{(1)}}(x)\right]^{i+l-1} &= \sum_{n=1}^{\infty} \sum_{s,k,p,w=0}^{\infty} \binom{b(n+w)-1}{k} \binom{i+l-1}{s} \binom{2\alpha(k+1)+1}{p} \\ &\times 4\alpha b \theta (-1)^{s+k+p} \frac{na_n d_{w,s} \theta^{n+w}}{(C(\theta))^{w+1}} (G(x; \xi))^{2\alpha(k+1)-1} \overline{G}^p(x; \xi) g(x; \xi) \\ &= \sum_{n=1}^{\infty} \sum_{s,k,p,m,w=0}^{\infty} \binom{b(n+w)-1}{k} \binom{i+l-1}{s} \binom{2\alpha(k+1)+1}{p} \\ &\times 4\alpha b \theta (-1)^{s+k+p+m} \frac{na_n d_{w,s} \theta^{n+w}}{(C(\theta))^{w+1}} \binom{2\alpha(k+1)-1}{m} \overline{G}^{p+m}(x; \xi) g(x; \xi) \\ &= \sum_{n=1}^{\infty} \sum_{s,k,p,m,q,w=0}^{\infty} \binom{b(n+w)-1}{k} \binom{i+l-1}{s} \binom{2\alpha(k+1)+1}{p} \\ &\times 4\alpha b \theta (-1)^{s+k+p+m+q} \frac{na_n d_{w,s} \theta^{n+w}}{(C(\theta))^{w+1}} \binom{2\alpha(k+1)-1}{m} \binom{p+m}{q} \\ &\times G^q(x; \xi) g(x; \xi) \\ &= \sum_{n=1}^{\infty} \sum_{s,k,p,m,q,w=0}^{\infty} \binom{b(n+w)-1}{k} \binom{i+l-1}{s} \binom{2\alpha(k+1)+1}{p} \\ &\times 4\alpha b \theta (-1)^{s+k+p+m+q} \frac{na_n d_{w,s} \theta^{n+w}}{(C(\theta))^{w+1}} \binom{2\alpha(k+1)-1}{m} \binom{p+m}{q} \\ &\times \left(\frac{q+1}{q+1}\right) G^q(x; \xi) g(x; \xi) \\ &= \sum_{q=0}^{\infty} C_{q+1} g_{q+1}(x; \xi). \end{aligned}$$

Therefore, the distribution of the i^{th} order statistics from the TII-TL-EHL-GPS CoD is given

by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \sum_{q=0}^{\infty} \binom{n-i}{l} (-1)^l C_{q+1} g_{q+1}(x; \xi), \quad (3.15)$$

where $g_{q+1}(x; \xi) = (q+1)g(x; \xi)G^q(x; \xi)$ is an Exp-G distribution with power parameter $(q+1)$ and the linear component

$$\begin{aligned} C_{q+1} &= \sum_{n=1}^{\infty} \sum_{s,k,p,m,w=0}^{\infty} \binom{b(n+w)-1}{k} \binom{i+l-1}{s} \binom{2\alpha(k+1)+p}{p} \\ &\times (-1)^{s+k+p+m+q} \frac{n a_n d_{w,s} \theta^{n+w}}{(C(\theta))^{w+1}} \binom{2\alpha(k+1)-1}{m} \binom{p+m}{q} \left(\frac{4\alpha b \theta}{q+1} \right). \end{aligned}$$

3.4.5 Rényi Entropy

RE (Rényi [116]) quantifies the uncertainty or randomness of a system. It is well known that RE generalizes Shannon entropy (Shannon [122]). RE for the TII-TL-EHL-GPS CoD is given by

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^{\infty} [f_{X_{(1)}}(x; b, \alpha, \theta, \xi)]^v dx \right), v \neq 1, v > 0. \quad (3.16)$$

Note that

$$\begin{aligned} [f_{X_{(1)}}(x; b, \alpha, \theta, \xi)]^v &= (4\alpha b \theta)^v \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^{v(b-1)} (G(x; \xi))^{v(2\alpha-1)} \\ &\times \left(\frac{C' \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)}{C(\theta)} \right)^v \frac{g^v(x; \xi)}{[1 + \overline{G}(x; \xi)]^{v(2\alpha+1)}}, \end{aligned}$$

and by applying the power series distribution raised to a positive integer (see Gradshteyn and Ryzhik [45] for details)

$$\left(\frac{C' \left(\theta \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^b \right)}{C(\theta)} \right)^v = \sum_{z=0}^{\infty} d_{z,v} \theta^z \frac{1}{C(\theta)^v} \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^{bz},$$

where $d_{z,v} = (zb_0)^{-1} \sum_{h=1}^n [v(h+1) - z] b_h d_{z-h,v}$ and $d_{0,v} = b_0^v$, we have

$$\begin{aligned} [f_{X_{(1)}}(x; b, \theta, \xi)]^v &= \sum_{z=0}^{\infty} \frac{n d_{z,v} \theta^{v+z}}{(C(\theta))^v} (4\alpha b)^v \left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)} \right)^{2\alpha} \right]^{b(v+z)-v} \\ &\times (G(x; \xi))^{v(2\alpha-1)} \frac{g^v(x; \xi)}{[1 + \overline{G}(x; \xi)]^{v(2\alpha+1)}}. \end{aligned}$$

Considering the generalized binomial series expansion

$$\left[1 - \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)}\right)^{2\alpha}\right]^{b(v+z)-v} = \sum_{j=0}^{\infty} \binom{b(v+z)-v}{j} (-1)^j \left(\frac{G(x; \xi)}{1 + \overline{G}(x; \xi)}\right)^{2\alpha j}.$$

we have

$$\begin{aligned} [f_{X_{(1)}}(x; b, \alpha, \theta, \xi)]^v &= \sum_{z=0}^{\infty} \sum_{j=0}^{\infty} \binom{b(v+z)-v}{j} (-1)^j \frac{nd_{z,v} \theta^{v+z}}{(C(\theta))^v} (4\alpha b)^v \\ &\times (G(x; \xi))^{2\alpha(v+j)-v} \frac{g^v(x; \xi)}{[1 + \overline{G}(x; \xi)]^{2\alpha(v+j)+v}}. \end{aligned}$$

Furthermore, considering the expansion

$$[1 + \overline{G}(x; \xi)]^{-(2\alpha(v+j)+v)} = \sum_{p=0}^{\infty} (-1)^p \binom{2\alpha(v+j)+v-1+p}{p} \overline{G}^p(x; \xi),$$

we write

$$\begin{aligned} [f_{X_{(1)}}(x; b, \alpha, \theta, \xi)]^v &= \sum_{z=0}^{\infty} \sum_{j,p=0}^{\infty} \binom{b(v+z)-v}{j} (-1)^{j+p} \frac{nd_{z,v} \theta^{v+z}}{(C(\theta))^v} (4\alpha b)^v \\ &\times \binom{2\alpha(v+j)+v-1+p}{p} (G(x; \xi))^{2\alpha(v+j)-v} \overline{G}^p(x; \xi) g^v(x; \xi) \\ &= \sum_{z,j,p,k=0}^{\infty} \binom{b(v+z)-v}{j} (-1)^{j+p+k} \frac{nd_{z,v} \theta^{v+z}}{(C(\theta))^v} (4\alpha b)^v \\ &\times \binom{2\alpha(v+j)+v-1+p}{p} \binom{2\alpha(v+j)-v}{k} \overline{G}^{p+k}(x; \xi) g^v(x; \xi) \\ &= \sum_{z=0}^{\infty} \sum_{j,p,k,q=0}^{\infty} \binom{b(v+z)-v}{j} (-1)^{j+p+k+q} \frac{nd_{z,n} \theta^{v+z}}{(C(\theta))^v} (4\alpha b)^v \\ &\times \binom{2\alpha(v+j)+v-1+p}{p} \binom{2\alpha(v+j)-v}{k} \binom{p+k}{q} \overline{G}^q(x; \xi) g^v(x; \xi). \end{aligned}$$

Therefore, RE for TII-TL-EHL-GPS CoD is given by

$$I_R(v) = \frac{1}{1-v} \log \left(\sum_{q=0}^{\infty} w_q^* e^{(1-v)I_{REG}} \right), \quad (3.17)$$

where $I_{REG} = \frac{1}{1-v} \log \left(\int_0^{\infty} [(1+q/v)g(x; \psi)G^{q/v}]^v dx \right)$ is RE for an Exp-G distribution with power parameter $(q/\nu + 1)$ and

$$\begin{aligned} w_q^* &= \sum_{z,j,p,k=0}^{\infty} \binom{b(v+z)-v}{j} (-1)^{j+p+k+q} \frac{zd_{z,v} \theta^{v+z}}{(C(\theta))^v} (4\alpha b)^v \\ &\times \binom{2\alpha(v+j)+v-1+p}{p} \binom{2\alpha(v+j)-v}{k} \binom{p+k}{q} \frac{1}{(1+q/\nu)^v}. \end{aligned} \quad (3.18)$$

Consequently, RE for TII-TL-EHL-GPS CoD can be obtained from RE of the Exp-G distribution.

3.5 Some Special Cases of the TII-TL-EHL-GPS CoD

In this section, we study some special cases of the TII-TL-EHL-GPS CoD. The plots of the pdfs and hrfs as well as plots of SK and KS for some selected values of parameters are exhibited in Figures: 3.1 to 3.12 to illustrate the versatility and flexibility of the special cases of the TII-TL-EHL-GPS CoD. Though we can generate several new distributions, here we will consider only two parent distributions, namely, LLoG and Weibull distributions. Among the power series CoD, we will consider the Poisson and logarithmic distributions. The LLoG distribution has cdf and pdf given by $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$, for $c > 0$ and $x > 0$. The Weibull distribution have cdf and pdf given by $G(x; \lambda) = 1 - e^{-x^\lambda}$ and $g(x; \lambda) = \lambda x^{\lambda-1}e^{-x^\lambda}$ for $\lambda > 0$ and $x > 0$.

3.5.1 Type II Topp-Leone Exponentiated Half Logistic-Log-Logistic Logarithmic (TII-TL-EHL-LLoGL) Distribution

The cdf and pdf of TII-TL-EHL-LLoGL distribution are respectively, given by,

$$F_{X(1)}(x) = 1 - \frac{\log \left(1 - \theta \left[1 - \left(\frac{1 - (1 + x^c)^{-1}}{1 + (1 + x^c)^{-1}} \right)^{2\alpha} \right]^b \right)}{\log(1 - \theta)}, \quad (3.19)$$

and

$$\begin{aligned} f_{X(1)}(x) &= 4\alpha b \theta \left[1 - \left(\frac{1 - (1 + x^c)^{-1}}{1 + (1 + x^c)^{-1}} \right)^{2\alpha} \right]^{b-1} \frac{\left(1 - \theta \left[1 - \left(\frac{1 - (1 + x^c)^{-1}}{1 + (1 + x^c)^{-1}} \right)^{2\alpha} \right]^b \right)}{-\log(1 - \theta)} \\ &\times \frac{cx^{c-1}(1 + x^c)^{-2}}{[1 + (1 + x^c)^{-1}]^{2\alpha+1}} (1 - (1 + x^c)^{-1})^{2\alpha-1}, \end{aligned} \quad (3.20)$$

for $x > 0$, $b, \alpha, c, \theta > 0$.

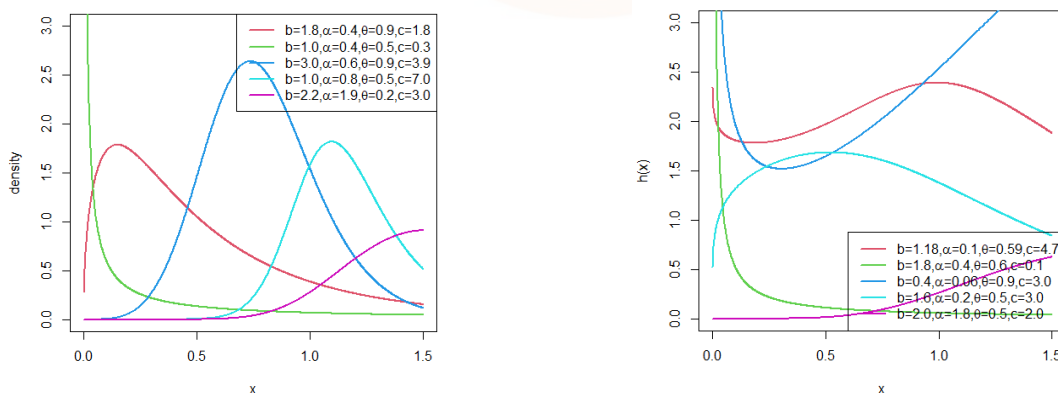


Figure 3.1: Pdf and Hrf Plots of the TII-TL-EHL-LLoGL Distribution

Figure 3.1 shows the plots for the pdf and hrf for the TII-TL-EHL-LLoGL distribution for selected values of parameters. The density function has reverse-J, uni-modal, negative-skewed, positive-skewed and J shapes, while the hrf exhibits BT, UBT, BT-UBT, decreasing and increasing shapes.

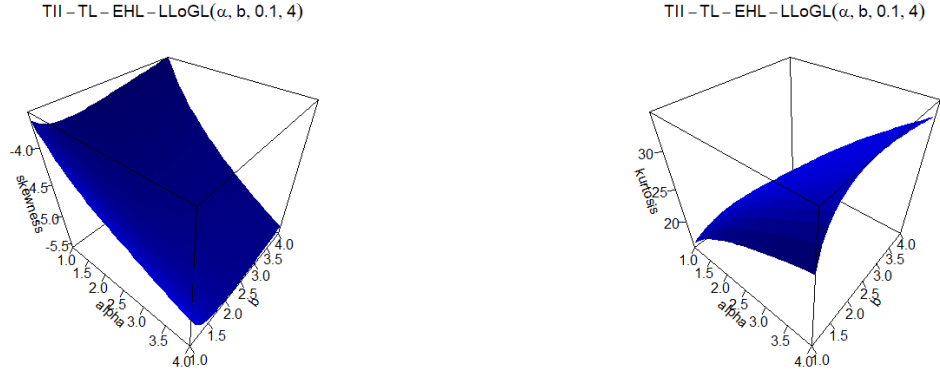


Figure 3.2: SK and KS Plots of the TII-TL-EHL-LLoGL Distribution



Figure 3.3: SK and KS Plots of the TII-TL-EHL-LLoGL Distribution

3D plots of SK and KS for the TII-TL-EHL-LLoGL distribution are given in Figures 3.2 and 3.3. We observe that

- When the parameters θ and c are fixed, the SK decrease and the KS increase as α and b increase.
- When the parameters θ and b are fixed, the SK decrease and the KS increase as α and c increase.

3.5.2 Type II Topp-Leone Exponentiated Half Logistic-Log-Logistic Poisson (TII-TL-EHL-LLoGP) Distribution

The cdf and pdf of TII-TL-EHL-LLoGP distribution are respectively given by,

$$F_{X(1)}(x) = 1 - \frac{\exp\left(\theta \left[1 - \left(\frac{(1+x^c)^{-1}}{1+(1+(1+x^c)^{-1})}\right)^{2\alpha}\right]^b\right) - 1}{\exp(\theta) - 1}, \quad (3.21)$$

and

$$\begin{aligned}
 f_{X(1)}(x) &= 4\alpha b\theta \left[1 - \left(\frac{1 - (1+x^c)^{-1}}{1 + (1+x^c)^{-1}} \right)^{2\alpha} \right]^{b-1} (1 - (1+x^c)^{-1})^{2\alpha-1} \\
 &\times \frac{\exp\left(\theta \left[1 - \left(\frac{1 - (1+x^c)^{-1}}{1 + (1+x^c)^{-1}} \right)^{2\alpha} \right]^b\right)}{\exp(\theta) - 1} \frac{cx^{c-1}(1+x^c)^{-2}}{[1 + (1+x^c)^{-1}]^{2\alpha+1}},
 \end{aligned} \tag{3.22}$$

for $x > 0$ and $b, \alpha, c, \theta > 0$.

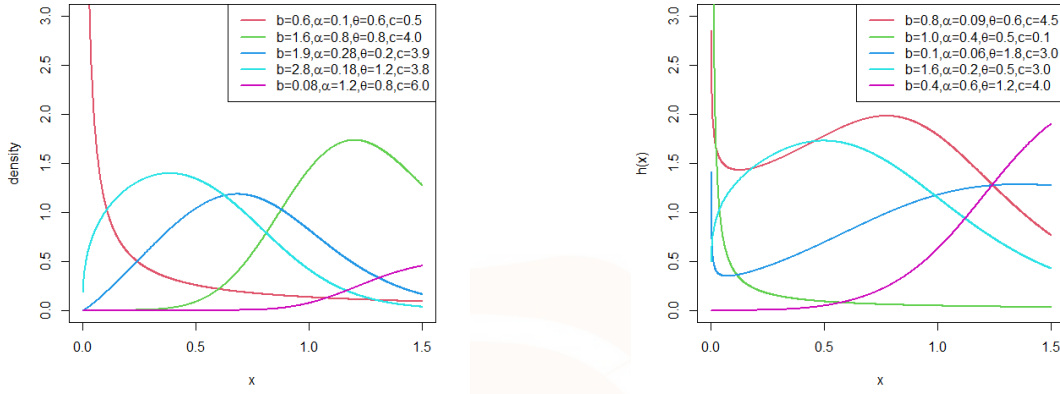


Figure 3.4: Pdf and Hrf Plots of TII-TL-EHL-LLoGP Distribution

In Figure 3.4, we presents the plots of the pdf and hrf of the TII-TL-EHL-LLoGP distribution. The pdf has different shapes comprising of almost symmetric, reverse J-shaped, right and left-skewed. The hrf of the TII-TL-EHL-LLoGP distribution shows BT, BT-UBT, UBT, decreasing and increasing shapes.

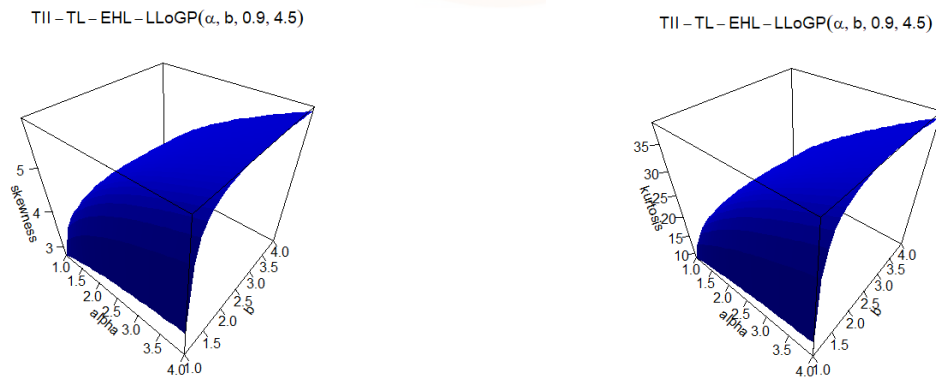


Figure 3.5: SK and KS Plots of the TII-TL-EHL-LLoGP Distribution



Figure 3.6: SK and KS Plots of the TII-TL-EHL-LLoGP Distribution

3D plots of SK and KS for the TII-TL-EHL-LLoGP distribution are given in Figures 3.5 and 3.6. We observe that

- When the parameters c and θ are fixed, the SK and KS increase as α and b increase.
- When the parameters θ and b are fixed, the SK and KS decrease then increase as α and c increase.

3.5.3 Type II-Exponentiated Half Logistic- Weibull Logarithmic (TII-TL-EHL-WL) Distribution

In this subsection, we present the TII-TL-EHL-WL distribution. The cdf of TII-TL-EHL-WL distribution is defined by,

$$F_{X_{(1)}}(x) = 1 - \frac{\log \left(1 - \theta \left[1 - \left(\frac{1-e^{-x^\lambda}}{1+e^{-x^\lambda}} \right)^{2\alpha} \right]^b \right)}{\log(1-\theta)},$$

and the corresponding pdf is given by

$$\begin{aligned} f_{X_{(1)}}(x) &= 4ab\theta \frac{\lambda x^{\lambda-1} e^{-x^\lambda}}{[1+e^{-x^\lambda}]^{2\alpha+1}} \left[1 - \left(\frac{1-e^{-x^\lambda}}{1+e^{-x^\lambda}} \right)^{2\alpha} \right]^{b-1} (1-e^{-x^\lambda})^{2\alpha-1} \\ &\times \frac{\left(1 - \theta \left(\left[1 - \left(\frac{1-e^{-x^\lambda}}{1+e^{-x^\lambda}} \right)^{2\alpha} \right]^b \right) \right)^{-1}}{-\log(1-\theta)}, \end{aligned} \quad (3.23)$$

for $x > 0$ and $b, \alpha, \theta, \lambda > 0$.

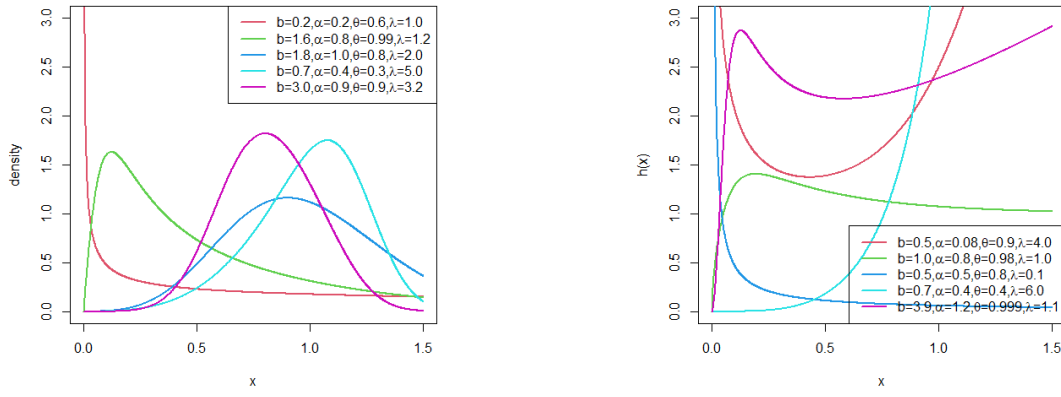


Figure 3.7: Pdf and Hrf Plots of the TII-TL-EHL-WL Distribution

Figure 3.7 shows the graphs of the pdf and hrf of the TII-TL-EHL-WL distribution. The pdf shapes of the TII-TL-EHL-WL distribution are negative-skewed, positive-skewed, reverse J-shape and almost symmetric. The hrf has different shapes including the BT, UBT, UBT-BT, increasing and decreasing.



Figure 3.8: SK and KS Plots of the TII-TL-EHL-WL Distribution



Figure 3.9: SK and KS Plots of the TII-TL-EHL-WL Distribution

3D plots of SK and KS for the TII-TL-EHL-WL distribution are given in Figures 3.8 and 3.9. We observe that the TII-TL-EHL-WL distribution can accommodate various levels of SK and KS.

3.5.4 Type II Topp-Leone-Exponentiated Half Logistic-Weibull Poisson (TII-TL-EHL-WP) Distribution

The cdf of TII-TL-EHL-WP distribution is given by

$$F_{X_{(1)}}(x) = 1 - \frac{\exp\left(\theta \left[1 - \left(\frac{1-e^{-x^\lambda}}{1+e^{-x^\lambda}}\right)^{2\alpha}\right]^b\right) - 1}{\exp(\theta) - 1}, \quad (3.24)$$

and the corresponding pdf is given by

$$f_{X_{(1)}}(x) = 4\alpha b\theta \left[1 - \left(\frac{1-e^{-x^\lambda}}{1+e^{-x^\lambda}}\right)^{2\alpha}\right]^{b-1} (1-e^{-x^\lambda})^{2\alpha-1} \frac{\exp\left(\theta \left[1 - \left(\frac{1-e^{-x^\lambda}}{1+e^{-x^\lambda}}\right)^{2\alpha}\right]^b\right)}{\exp(\theta) - 1} \frac{\lambda x^{\lambda-1} e^{-x^\lambda}}{[1+e^{-x^\lambda}]^{2\alpha+1}}, \quad (3.25)$$

for $x > 0$ and $b, \alpha, \lambda, \theta > 0$.

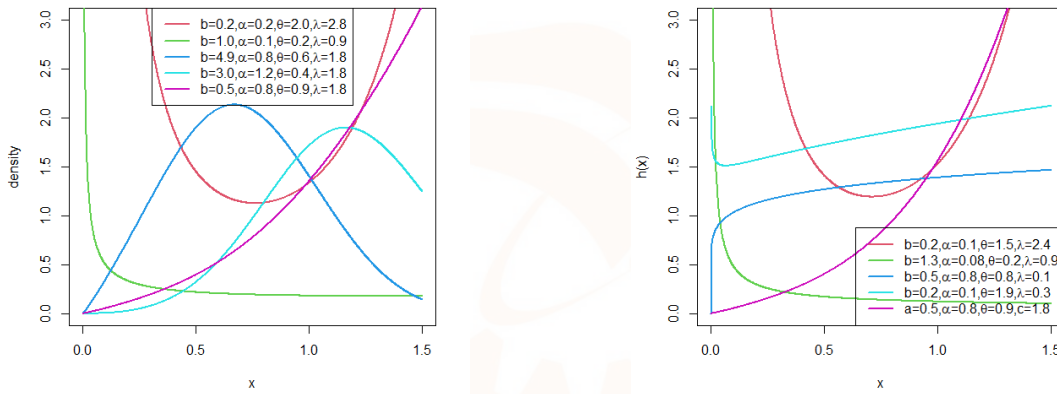


Figure 3.10: Pdf and Hrf Plots of TII-TL-EHL-WP Distribution

Figure 3.10 shows the pdf and hrf of the TII-TL-EHL-WP distribution. We have various shapes for the pdf including the symmetric, U-shape, right-skewed, increasing and reverse-J shapes. The hrf in Figure 3.10 shows bathtub, decreasing and increasing shapes.



Figure 3.11: SK and KS Plots of TII-TL-EHL-WP Distribution



Figure 3.12: SK and KS Plots of TII-TL-EHL-WP Distribution

3D plots of SK and KS for the TII-TL-EHL-WP distribution are given in Figures 3.11 and 3.12. We observe that

- When we fix the parameters θ and λ , the SK and KS increase as α and b increase.
- When we fix the parameters λ and b , the SK and KS decrease and then increase as α and θ increase.

3.6 Maximum Likelihood Estimation

Here we use the ML estimation technique to find the estimates of the parameters of the TII-TL-EHL-GPS CoD. Let $X_i \sim TII - TL - EHL - GPS(b, \alpha, \theta, \xi)$ and $\Delta = (b, \alpha, \theta, \xi)^T$ be the vector of unknown parameters. The total log-likelihood $\ell = \ell(\Delta)$ function is given by

$$\begin{aligned} \ell(\Delta) = & n \ln(4ab\theta) + (b-1) \sum_{i=1}^n \ln \left[1 - \left(\frac{G(x_i; \xi)}{1 + \overline{G}(x_i; \xi)} \right)^{2\alpha} \right] + (2\alpha - 1) \sum_{i=1}^n \ln(G(x_i; \xi)) \\ & + \sum_{i=1}^n \ln \left(C' \left(\theta \left[1 - \left(\frac{G(x_i; \xi)}{1 + \overline{G}(x_i; \xi)} \right)^{2\alpha} \right]^b \right) \right) - \sum_{i=1}^n \ln(C(\theta)) \\ & - (2\alpha + 1) \sum_{i=1}^n \ln[1 + \overline{G}(x_i; \xi)] + \sum_{i=1}^n \ln(g(x_i; \xi)). \end{aligned}$$

Elements of score vectors are given in the Appendix. The FIM of the TII-TL-EHL-GPS distribution is a $(q+3) \times (q+3)$ matrix given by $\mathbf{J}(\Delta) = [\mathbf{J}_{\theta_i, \theta_j}]_{(q+3) \times (q+3)} = E(-\frac{\partial^2 \log L(\Delta)}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, \dots, (q+3)$. The total FIM $n\mathbf{J}(\Delta)$ can be approximated by

$$\mathbf{I}_n(\Delta) \approx \left[-\frac{\partial^2 \ell(\Delta)}{\partial \theta_i \partial \theta_j} \Big|_{\Delta = \hat{\Delta}} \right]_{(q+3) \times (q+3)}, \quad i, j = 1, 2, \dots, (q+3). \quad (3.26)$$

For a given set of observations, we obtain the total FIM $n\mathbf{J}(\Delta)$ after the convergence of the Newton-Raphson technique. When the conditions for parameters in the interior of the parameter space, but not on the boundary, are met, the asymptotic distribution becomes $\sqrt{n}(\hat{\Delta} - \Delta)$ is $N_{q+3}(0, J(\Delta)^{-1})$, where $J(\Delta) = \lim_{n \rightarrow \infty} n^{-1} I_n(\Delta)$ is the unit information matrix. This asymptotic behavior remains valid if the unit information matrix is replaced by the average sample information matrix evaluated at $\hat{\Delta}$, say $n^{-1} I_n(\hat{\Delta})$. The estimated asymptotic multivariate normal $N_{q+3}(\Delta, I_n(\hat{\Delta})^{-1})$ distribution of $\hat{\Delta}$ can be used to construct CI for the parameters. An

100(1 - ϕ)% asymptotic CI for each parameter Δ_r is given by

$$ACI_r = \left(\hat{\Delta}_r - z_{\frac{\phi}{2}} \sqrt{\widehat{I}_{rr}}, \hat{\Delta}_r + z_{\frac{\phi}{2}} \sqrt{\widehat{I}_{rr}} \right), \quad (3.27)$$

where $\widehat{I}_{rr}$ is the (r, r) diagonal element of $I_n(\hat{\Delta})^{-1}$ for $r = 1, 2, \dots, (q + 3)$ and $z_{\frac{\phi}{2}}$ is the $1 - \frac{\phi}{2}$ quantile of the standard normal distribution.

3.7 Simulations

The efficiency and consistency of the ML estimators are assessed by conducting a simulation study. We use R software to obtain the numerical values of the ML estimators. To investigate the behavior of the ML estimators for the TII-TL-EHL-WL distribution, we simulated N=1000 times with sample sizes, $n = 50, 100, 200, 400, 800$ and 1600. We computed the mean ML estimates, ABias and RMSE, respectively.

In Tables 3.2 and 3.3, we give the MCS results for the TII-TL-EHL-WL distribution. We note that when sample size, n increases, the mean estimates converges to the true parameter values, and the ABias and RMSE of the estimated parameters decrease. The simulation results demonstrate the accuracy of the ML estimation method for estimating the parameters of the TII-TL-EHL-WL distribution, and support its reliability for modeling real-world data. The results also suggest that the TII-TL-EHL-WL distribution can be used in a variety of applications.

Table 3.2: MCS Results for TII-TL-EHL-WL Distribution

parameter	n	(0.4,0.7,0.4,0.4)			(2.0,2.0,2.4,0.9)			(0.2,0.7,0.7,1.0)		
		Mean	RMSE	ABias	Mean	RMSE	ABias	Mean	RMSE	ABias
b	50	0.5121	0.3093	0.1121	2.1570	0.5432	0.1570	1.1866	4.7556	0.9866
	100	0.4779	0.2952	0.0779	2.1020	0.5336	0.1020	0.7972	2.3363	0.5972
	200	0.4414	0.1666	0.0414	2.0911	0.4263	0.0911	0.4532	1.2037	0.2532
	400	0.4121	0.0916	0.0122	2.0862	0.3439	0.0862	0.3108	0.4093	0.1108
	800	0.4077	0.0631	0.0077	2.0431	0.2742	0.0431	0.2277	0.0833	0.0277
	1600	0.4057	0.0529	0.0057	2.0388	0.2551	0.0388	0.2101	0.0469	0.0101
α	50	2.1563	0.9174	1.4563	2.1835	0.5775	0.1835	1.2074	0.9824	0.5074
	100	1.0888	0.7575	0.3883	2.1638	0.5133	0.1638	1.0241	0.7307	0.3241
	200	0.9459	0.6973	0.2459	2.1563	0.4218	0.1563	0.8570	0.5316	0.1570
	400	0.8706	0.5865	0.1706	2.1485	0.2957	0.1485	0.7870	0.2730	0.0870
	800	0.7440	0.3003	0.0440	2.1178	0.2455	0.1178	0.7281	0.0991	0.0281
	1600	0.7139	0.1387	0.0139	2.0956	0.2141	0.0956	0.7031	0.0448	0.0031
θ	50	0.6428	0.3573	0.2428	2.9083	1.7623	0.5083	0.6049	0.2506	-0.0950
	100	0.5988	0.3130	0.1988	2.8930	1.5268	0.4930	0.6050	0.2276	-0.0949
	200	0.5568	0.2799	0.1568	2.7875	1.4745	0.3875	0.6116	0.1962	-0.0883
	400	0.5024	0.2539	0.1024	2.7604	1.2393	0.3604	0.6210	0.1958	-0.0789
	800	0.4606	0.2309	0.0606	2.6699	1.0650	0.2699	0.6247	0.1753	-0.0752
	1600	0.4208	0.1902	0.0208	2.4358	0.7464	0.0358	0.6582	0.1274	-0.0417
λ	50	0.7386	0.3974	0.3386	0.8291	0.1958	-0.0708	0.8613	0.2843	-0.1387
	100	0.6531	0.3697	0.2531	0.8395	0.1718	-0.0604	0.8784	0.2463	-0.1215
	200	0.6316	0.3301	0.2316	0.8561	0.1557	-0.0438	0.9374	0.1706	-0.0626
	400	0.6004	0.3196	0.2004	0.8842	0.1061	-0.0157	0.9310	0.1489	-0.0690
	800	0.5374	0.2634	0.1374	0.8915	0.0762	-0.0084	0.9702	0.0792	-0.0298
	1600	0.4806	0.2286	0.0806	0.8941	0.0529	-0.0058	0.9864	0.0441	-0.0136

Table 3.3: MCS Results for TII-TL-EHL-WL Distribution

parameter	n	(0.4,1.9,0.7,2.0)			(0.9,0.6,0.6,0.9)			(1.8,0.9,0.6,2.0)		
		Mean	RMSE	ABias	Mean	RMSE	ABias	Mean	RMSE	ABias
b	50	2.8927	10.5524	2.4927	0.7616	0.2828	-0.1383	2.2479	0.9523	0.4479
	100	2.0506	8.6979	1.6506	0.7813	0.2607	-0.1186	2.0549	1.2847	0.2549
	200	1.4856	6.6262	1.0856	0.7978	0.2350	-0.1021	2.0225	1.2091	0.2225
	400	1.1231	3.4609	0.7231	0.8310	0.1846	-0.0689	2.0076	0.9201	0.2076
	800	0.5357	0.9713	0.1357	0.8410	0.1573	-0.0589	1.8810	0.9192	0.0810
	1600	0.4241	0.3080	0.0241	0.8710	0.1012	-0.0289	1.8596	0.6092	0.0596
α	50	2.3186	0.7308	0.4186	0.5215	0.3005	-0.0784	0.8359	0.2313	-0.0640
	100	2.2354	0.6726	0.3354	0.5735	0.2970	-0.0264	0.8448	0.1844	-0.0551
	200	2.1791	0.6592	0.2791	0.5850	0.2774	-0.0149	0.8493	0.1922	-0.0507
	400	2.1246	0.6106	0.2246	0.5848	0.2403	-0.0151	0.8539	0.1679	-0.0460
	800	2.0892	0.5545	0.1892	0.5797	0.2165	-0.0202	0.8671	0.1252	-0.0328
	1600	2.0714	0.4708	0.1714	0.5928	0.1752	-0.0071	0.8717	0.1003	-0.0282
θ	50	0.6735	0.2053	-0.0264	0.7927	0.3496	0.1927	0.6824	0.2731	0.0824
	100	0.6745	0.1912	-0.0254	0.7620	0.3477	0.1620	0.6528	0.2466	0.0528
	200	0.6776	0.1778	-0.0223	0.6761	0.2865	0.0761	0.6348	0.2308	0.0348
	400	0.6780	0.1615	-0.0219	0.6756	0.2425	0.0756	0.6194	0.2174	0.0194
	800	0.6843	0.1316	-0.0156	0.6529	0.2209	0.0529	0.6074	0.2018	0.0074
	1600	0.7050	0.0893	0.0050	0.6447	0.1748	0.0447	0.5952	0.1718	-0.0047
λ	50	2.2384	1.0161	0.2384	0.7864	0.2366	-0.1135	2.2893	1.5705	0.2893
	100	2.1726	0.7442	0.1726	0.8102	0.2154	-0.0897	2.2740	1.5533	0.2740
	200	2.1640	0.5908	0.1640	0.8645	0.1325	-0.0354	2.2601	1.3300	0.2601
	400	2.1235	0.4642	0.1235	0.8801	0.0938	-0.0198	2.2001	1.3300	0.2001
	800	2.0661	0.3653	0.0661	0.8954	0.0651	-0.0045	2.1343	1.1212	0.1343
	1600	2.0282	0.2780	0.0282	0.8987	0.0504	-0.0012	1.9845	0.7634	-0.0155

3.8 Applications

In this section, we use two real datasets to illustrate the applicability and flexibility of the TII-TL-EHL-WL distribution. The ML estimation technique is used to estimate the model parameters via the **mle** function and the **goodness.fit** function in R software which provides the GoF statistics.

To verify which distribution fits better to these datasets, we use the following GoF statistics, namely, the AIC, BIC and CAIC. The CVM and AD GoF statistics, reported by (Chen and Balakrishnan [26]) are also presented. The model with smaller values of these GoF statistics is regarded as the best fitting model. The K-S GoF statistic, its associated PV and SS (Chambers et al. [24]) from the PPs are also used to assess the model performance. The model with the smaller K-S value and the highest PV for the K-S statistic, is regarded as the best fitting model.

We also utilize the LRT statistic to assess the TII-TL-EHL-WL distribution and compare it with its nested models, as presented in Tables 3.6 and 3.9. For example, when testing for $b = 1$, the LRT statistic is $\Lambda = 2[\ln(L(\hat{b}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})) - \ln(L(\tilde{1}, \tilde{\alpha}, \tilde{\theta}, \tilde{\lambda}))]$, where $\hat{b}$, $\hat{\alpha}$, $\hat{\theta}$, and $\hat{\lambda}$ are the unrestricted estimates, and $\tilde{1}$, $\tilde{\alpha}$, $\tilde{\theta}$, $\tilde{\lambda}$, are the restricted estimates. The LRT test nullifies the hypothesis if $\Lambda > \chi^2$, where χ^2 signifies the upper 100 α % point of the χ^2 distribution with 1 degree of freedom.

We compare the TII-TL-EHL-WL distribution to several non-nested models. The non-nested distributions are: TIIEHLW distribution presented by Al-Mofleh et al. [7], TIIGIE-BIII distribution introduced by Jamal et al. [56], EMW distribution proposed by Carrasco et al. [23], and

OEHLBXII distribution given by Aldahlan and Afify [9]. The density functions of TIIHLW, TIIGIE-BIII, EMW and OEHLBXII distribution are given by

$$f_{TIIHLW}(x; \alpha, \lambda, \gamma, \delta) = 2\alpha\lambda\delta\gamma x^{\gamma-1}e^{-\delta x^\gamma} \left[1 - e^{-\delta x^\gamma}\right]^{\lambda-1} \frac{[1 - [1 - e^{-\delta x^\gamma}]^\lambda]^{\alpha-1}}{[1 + [1 - e^{-\delta x^\gamma}]^\lambda]^\alpha},$$

for $\alpha, \lambda, \gamma, \delta > 0, x > 0$,

$$\begin{aligned} f_{TIIGIE-BIII}(x; \lambda, \theta, c, k) &= \frac{\lambda\theta c k x^{-c-1}(1+x^{-c})[1 - (1+x^{-c})^{-k}]^{\theta-1}}{([1 - (1+x^{-c})^{-k}]^\theta)^2} \\ &\times e^{-\lambda \frac{[1 - (1+x^{-c})^{-k}]^\theta}{1 - [1 - (1+x^{-c})^{-k}]^\theta}}, \end{aligned} \quad (3.28)$$

for $\lambda, \theta, c, k > 0, x > 0$,

$$f_{EMW}(x; \gamma, \delta, \lambda, \theta) = \gamma (\delta + \lambda\theta^\lambda) x^{\lambda-1} e^{-(\delta x + \theta x^\lambda)} \left(1 - e^{-(\delta x + \theta x^\lambda)}\right)^{\gamma-1},$$

for $\gamma, \delta, \lambda, \theta > 0, x > 0$,

and

$$f_{OEHLBXII}(x; \alpha, \lambda, a, b) = \frac{2\alpha\lambda ab x^{a-1} e^{\lambda(1-(1+x^a)^b)} \left(1 - e^{\lambda(1-(1+x^a)^b)}\right)^{\alpha-1}}{(1+x^a)^{-b+1} \left(1 + e^{\lambda(1-(1+x^a)^b)}\right)^{\alpha+1}}, \quad (3.29)$$

for $\alpha, \lambda, a, b > 0$ and $x > 0$.

3.8.1 Aircraft Windshield Data

In this section, the aircraft windshield dataset is used to assess the applicability of the TII-TL-EHL-WL distribution compared to the nested and non-nested models. This dataset represent the failure rates of 84 aircraft windscreen models and it was used in the literature by Anabike et al. [16]. The non-structural outer layer is damaged and the heating system fails to function when the aircraft windscreen fails. The data observations are: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.1902, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663. The data have whiskers that are long at the top and bottom, suggesting potential of outliers.

The estimated Var-Cov matrix is

$$\begin{pmatrix} 0.0039 & 0.0107 & -0.0030 & -0.0233 \\ 0.0107 & 0.0411 & -0.0046 & -0.0652 \\ -0.0030 & -0.0040 & 0.0294 & 0.0141 \\ -0.0233 & -0.0652 & 0.0141 & 0.1466 \end{pmatrix},$$

and the approximate 95% CI for b, α, θ and λ are given by 0.1408 ± 0.1224 , 0.3565 ± 0.3974 , $1.5188 \times 10^{-06} \pm 0.3361$ and 2.1361 ± 0.75006 , respectively. In Table 3.4, we present the parameter estimates of the models and their SE in parenthesis.

Table 3.4: Estimates for Aircraft Windshield Data

Model	Estimates			
	b	α	θ	λ
$TII - TL - EHL - WL(b, \alpha, \theta, \lambda)$	0.1408 (0.0543)	0.3565 (0.7691)	1.5188×10^{-06} (0.2193)	2.1361 (0.3212)
$TII - TL - EHL - WL(1, \alpha, \theta, \lambda)$	-	0.9564 (0.1061)	9.562×10^{-06} (0.0113)	0.9877 (0.4965)
$TII - TL - EHL - WL(b, 1, \theta, \lambda)$	0.4531 (4.0244)	-	8.7407×10^{-09} (0.0108)	1.2539 (0.1420)
$TII - TL - EHL - WL(b, \alpha, \theta, 1)$	0.8916 (0.1324)	1.4777 (0.2280)	3.718×10^{-07} (0.1538)	-
$TII - TL - EHL - WL(1, 1, \theta, \lambda)$	-	-	1.3236×10^{-08} (0.0132)	0.9158 (0.0488)
$TII - TL - EHL - WL(1, \alpha, \theta, 1)$	-	0.1209 (0.0131)	4.619×10^{-11} (0.0008)	-
TIEHLW	a	λ	δ	γ
	1.2762	1.5300	7.3129	0.0346
	2.0347×10^{-06}	1.6970×10^{-06}	(0.0253)	(0.0026)
TIIGIE-BIII	λ	θ	c	k
	2.8818	0.2172	8.5710	2.5263×10^{-05}
	(8.0559×10^{-06})	(0.0162)	(3.6472×10^{-04})	(2.6534×10^{-05})
EMW	γ	δ	λ	θ
	3.5605	0.7579	17.0010	0.0001
	(0.6110)	(0.0769)	(4.6677×10^{-17})	(7.9358×10^{-12})
OEHLBXII	α	λ	a	b
	0.1903	2.5052×10^{-05}	4.6048	1.6549
	(0.0218)	(5.2166×10^{-06})	(0.0010)	(0.0024)

Table 3.5 demonstrates that the TII-TL-EHL-WL model exhibits superior fit to the aircraft windshield data, as indicated by its smaller values of the AIC, BIC, and CAIC, AD and CVM in comparison to the other models. These results suggest the effectiveness of the TII-TL-EHL-WL distribution.

Table 3.5: GoF Statistics for Aircraft Windshield Data

Model	GoF Statistics								
	$-2\log(L)$	AIC	CAIC	BIC	CVM	AD	SS	K-S	PV
$TII-TL-EHL-WL(b, \alpha, \theta, \lambda)$	256.9335	264.9335	265.4398	274.6567	0.0452	0.4648	0.0754	0.07251	0.6869
$TII-TL-EHL-WL(1, \alpha, \theta, \lambda)$	291.5790	297.5790	297.8793	304.8717	0.1158	1.0518	2.8602	0.2957	8.279×10^{-07}
$TII-TL-EHL-WL(b, 1, \theta, \lambda)$	271.3606	277.3606	277.6602	284.6527	0.1287	1.1414	0.2474	0.1118	0.1921
$TII-TL-EHL-WL(b, \alpha, \theta, 1)$	272.9926	278.9926	279.2926	286.2850	0.1448	1.2570	0.2335	0.1138	0.2267
$TII-TL-EHL-WL(1, 1, \theta, \lambda)$	286.3563	290.3563	290.5045	295.2180	0.1297	1.1543	1.8282	0.2515	4.857×10^{-05}
$TII-TL-EHL-WL(1, \alpha, \theta, 1)$	552.0824	556.0824	556.2318	560.9453	0.0563	0.5885	20.3361	0.7835	2.2×10^{-16}
TIIHLW	273.4006	281.4006	281.9070	291.1239	0.1452	1.2577	0.1421	0.0963	0.4169
TIIGIE-BIII	291.2455	299.2455	299.7518	308.9688	25.6811	163.8699	23.8512	0.9994	2.2×10^{-16}
EMW	279.6810	287.6810	288.1874	297.4043	0.2106	1.6946	5.6282	0.1209	0.1712
OEHLBXII	267.3613	275.3613	275.8676	285.0846	0.2278	1.3241	4.2449	0.1066	0.2952

The fitted pdf plots and PPs for the aircraft windshield dataset are shown in Figure 3.13. These fitted plots shows that the TII-TL-EHL-WL distribution fits the data better than the non-nested models.

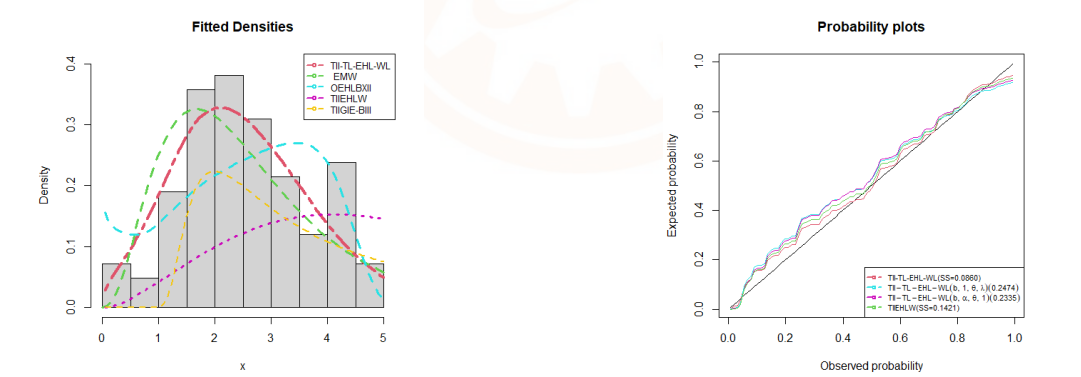


Figure 3.13: Fitted Pdf Plots and PPs for Aircraft Windshield Data

Table 3.6: LRT Results for Aircraft Windshield Data

Aircraft Windshield Data		
Model	df	$\chi^2(p-value)$
$TII-TL-EHL-WL(1, \alpha, \theta, \lambda)$	1	34.2485(< 0.00001)
$TII-TL-EHL-WL(b, 1, \theta, \lambda)$	1	14.0301 (0.00018)
$TII-TL-EHL-WL(b, \alpha, \theta, 1)$	1	15.6621 (0.00076)
$TII-TL-EHL-WL(1, 1, \theta, \lambda)$	2	29.0256 (< 0.00001)
$TII-TL-EHL-WL(1, \alpha, \theta, 1)$	2	294.7519 (< 0.00001)

Table 3.6 shows the LRT results for testing selected nested models against TII-TL-EHL-WL distribution. Based on the LRT results, we conclude that there are significant differences between each of the sub-models and the main distribution. Figure 3.14 suggest the TTT scaled plot with an increasing hrf.

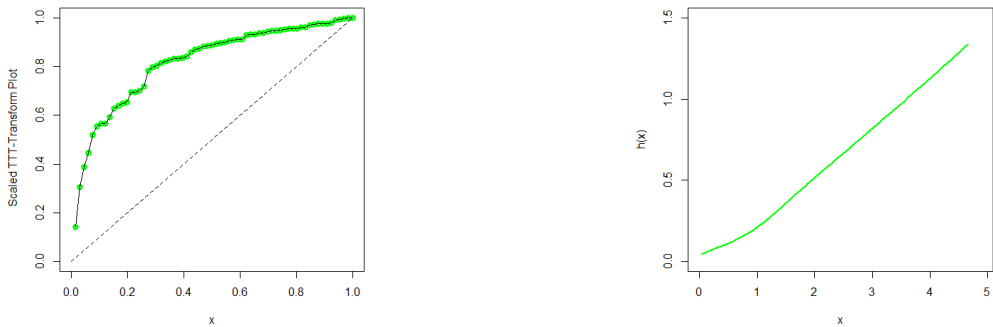


Figure 3.14: Fitted TTT and Hrf Plots for Aircraft Windshield Data

The closeness observed between the fitted K-M survival and ECDF curves, as depicted in Figure 3.15, indicates that the model fits well with the aircraft windshield data.

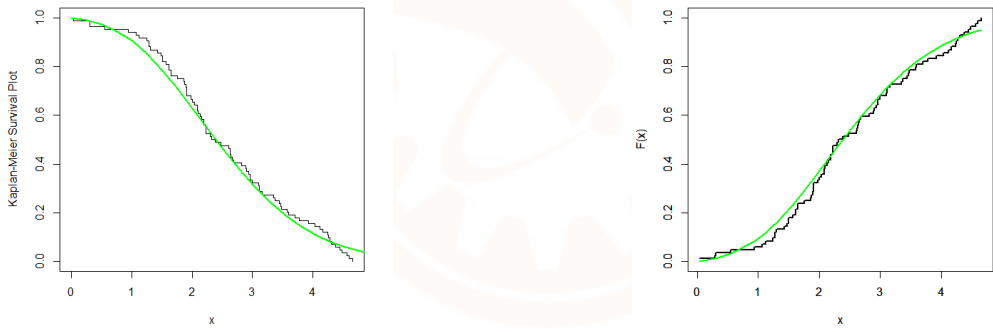


Figure 3.15: Fitted K-M Survival and ECDF Plots for Aircraft Windshield Data

We present the profile log-likelihood function of the ML estimates of TII-TL-EHL-WL distribution on Figure 3.16 for aircraft windshield data.

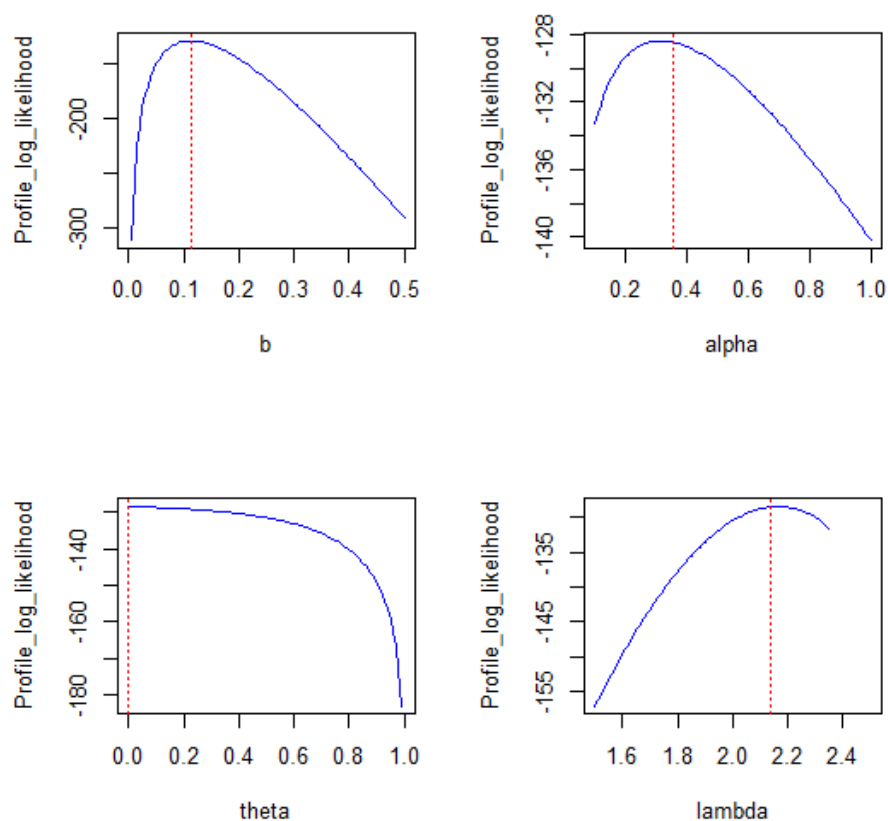


Figure 3.16: Profile Log-Likelihood Function Plots of the Parameters of the TII-TL-EHL-WL Distribution on Aircraft Windshield Data

Figure 3.16 shows the ML estimates for TII-TL-EHL-WL distribution exists and can be obtained uniquely.

3.8.2 Growth Hormone Data

The second dataset represent the time from when the growth hormone medication is administered to children until they reach the target age. The dataset was applied by Oluyede et al. [95]. The data are: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70. The data is right-skewed.

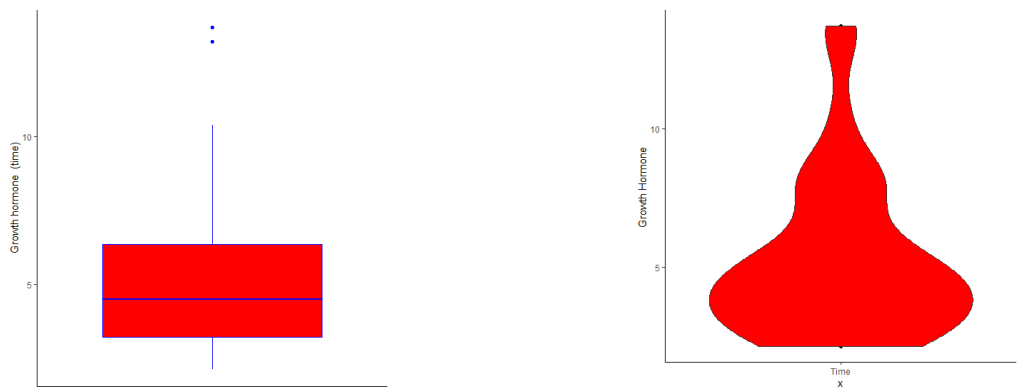


Figure 3.17: Boxplot and Violin Plots of Growth Hormone Data

The estimated Var-Cov matrix is given by

$$\begin{pmatrix} 0.0516 & -0.2712 & -0.0024 & -0.0831 \\ -0.2712 & 5.8117 & 0.0411 & 0.4998 \\ -0.0024 & 0.0411 & 0.0297 & 0.0020 \\ -0.4383 & 0.4998 & 0.0020 & 0.1381 \end{pmatrix},$$

and the approximate 95% CI for b, α, θ and λ are given by $0.2157 \pm 0.4453, 5.2318 \pm 4.725, 8.7308 \times 10^{-07} \pm 0.3382$ and 1.1933 ± 0.7284 , respectively.

Table 3.7 shows the estimates of parameters and their SE in their parenthesis.

Table 3.7: Estimates for Growth Hormone Data

Model	Estimates			
	b	α	θ	λ
$TII - TL - EHL - WL(b, \alpha, \theta, \lambda)$	0.2157 (0.2272)	5.2318 (2.4108)	8.7308×10^{-07} (0.1725)	1.1933 (0.3716)
$TII - TL - EHL - WL(1, \alpha, \theta, \lambda)$	-	0.5017 (0.0855)	6.5456×10^{-12} (0.0011)	0.1074 (0.0179)
$TII - TL - HL - WL(b, 1, \theta, \lambda)$	0.2973 (4.0244)	-	1.0197×10^{-08} (0.0108)	0.9676 (0.1420)
$TII - TL - EHL - WL(b, \alpha, \theta, 1)$	0.9853 (0.9992)	0.1272 (0.1299)	1.1975×10^{-09} (1.0060)	-
$TII - TL - EHL - WL(1, 1, \theta, \lambda)$	-	-	1.8615×10^{-09} (0.0075)	0.8954 (0.040)
$TII - TL - EHL - WL(1, \alpha, \theta, 1)$	-	1.002 (0.1694)	3.5213×10^{-09} (0.0115)	-
TII-EHLW	a	λ	δ	γ
	40302.0000 (6.3713×10^{-08})	2341.0000 (1.7885×10^{-05})	5.0264 (0.0421)	0.0362 (0.0040)
TIIGIE-BIII	λ	θ	c	k
	2.8653 (0.9874)	0.1125 (0.0400)	8.2200 (2.7417)	0.1214 (0.0966)
EMW	γ	δ	λ	θ
	6.5130 (2.2629)	0.4820 (0.0798)	17.0010 (1.7949×10^{-16})	0.0001 (3.0495×10^{-11})
OEHLBXII	α	λ	a	b
	0.4336 (0.1786)	0.0042 (0.0091)	14.1603 (0.0007)	0.1912 (0.0594)

Table 3.8 shows that the TII-TL-EHL-WL distribution has the smallest values for the AIC, CAIC, BIC, CVM, and AD, compared to the values for the other models considered. The TII-TL-EHL-WL distribution also has the highest PV for the K-S statistic. Hence, the TII-TL-EHL-WL distribution is the best-fitting model for the growth hormone dataset, based on these GoF statistics.

Table 3.8: GoF Statistics for Growth Hormone Data

Model	Statistics								
	$-2 \log(L)$	AIC	$CAIC$	BIC	CVM	AD	SS	$K - S$	PV
$TII - TL - EHL - WL(b, \alpha, \theta, \lambda)$	155.1309	163.1309	164.4642	169.3522	0.0401	0.2823	0.0453	0.0830	0.9693
$TII - TL - EHL - WL(1, \alpha, \theta, \lambda)$	324.2835	330.2835	331.0662	334.9580	0.0602	0.4129	2.4827	0.4939	7.6710×10^{-08}
$TII - TL - HL - WL(b, 1, \theta, \lambda)$	171.3356	177.3356	178.1099	182.0017	0.0751	0.5036	0.5009	0.2531	0.0225
$TII - TL - EHL - WL(b, \alpha, \theta, 1)$	414.4252	420.4252	421.1992	425.0910	0.13653	0.8652	10.8531	0.93975	2.2×10^{-16}
$TII - TL - EHL - WL(1, 1, \theta, \lambda)$	240.3300	244.3300	244.7062	247.4419	0.0861	0.5691	5.9052	0.6148	6.444×10^{-12}
$TII - TL - EHL - WL(1, \alpha, \theta, 1)$	281.4692	285.4692	285.8413	288.5770	0.0913	0.6002	7.1196	0.6736	3.231×10^{-14}
TII-EHLW	162.6945	170.6945	172.0528	176.9409	0.1346	0.8534	0.1226	0.1369	0.5278
TIIGIE-BIII	178.7655	186.7655	187.6124	192.5005	12.1159	70.4451	7.5697	0.8367	$< 2.2 \times 10^{-16}$
EMW	158.1951	166.1951	167.5284	172.4165	0.0703	0.4747	0.1358	0.1013	0.8656
OEHLBXII	183.2824	191.2824	192.6157	197.5038	0.2655	1.6321	0.3454	0.1966	0.1337

The fitted pdf plots and PPs support the GoF statistics presented in Table 3.8 and confirm that the TII-TL-EHL-WL distribution fits the growth hormone data better than the nested and non-nested models.

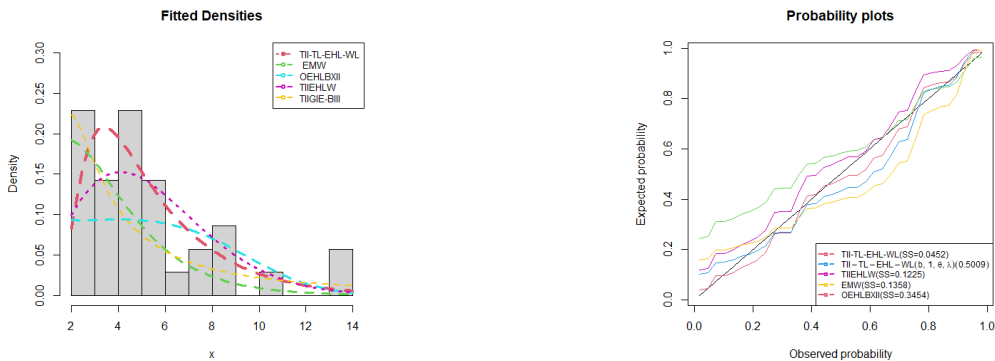


Figure 3.18: Fitted Pdf Plots and PPs for Growth Hormone Data

Table 3.9: LRT Results for Growth Hormone Data

Growth Hormone Data Set		
Model	df	$\chi^2(p-value)$
$TII-TL-EHL-WL(1, \alpha, \theta, \lambda)$	1	169.1526 (< 0.00001)
$TII-TL-EHL-WL(b, 1, \theta, \lambda)$	1	16.2047 (0.00057)
$TII-TL-EHL-WL(b, 1, \alpha, \theta,)$	1	259.2943 (< 0.00001)
$TII-TL-EHL-WL(1, 1, \theta, \lambda)$	2	85.1991 (< 0.00001)
$TII-TL-EHL-WL(1, \alpha, \theta, 1)$	2	126.3383 (< 0.00001)

Table 3.9 gives the LRT results for the growth hormone data. We conclude that there are significant differences between each of the sub-models and TII-TL-EHL-WL distribution based on the LRT tests at 5% level of significance. Figure 3.18 gives the TTT scaled plot with an increasing hrf.

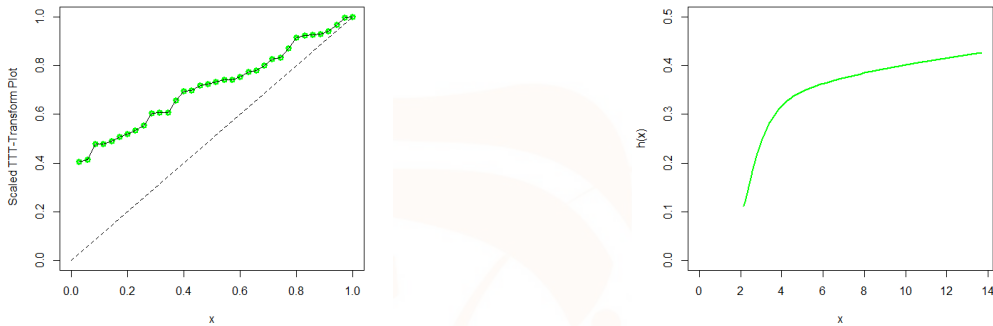


Figure 3.19: Fitted TTT and Hrf Plots for Growth Hormone Data

The fitted K-M survival and ECDF are shown in Figure 3.19.

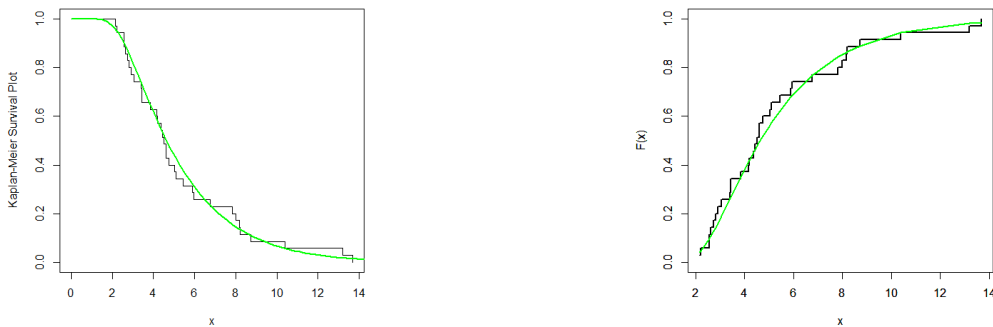


Figure 3.20: Fitted K-M Survival and ECDF Plots for Growth Hormone Data

The ML estimates of the TII-TL-EHL-WL distribution exists and can be uniquely obtained, as shown in Figure 3.20.

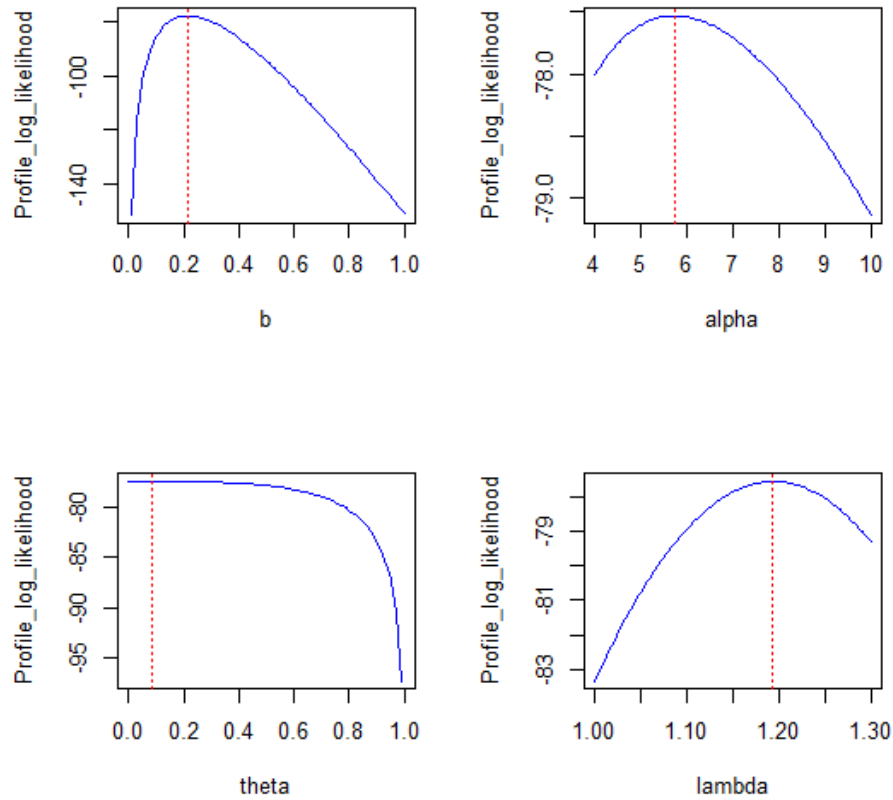


Figure 3.21: Profile Log-Likelihood Function Plots of the Parameters of the TII-TL-EHL-WL Distribution on Growth Hormone Data

3.9 Conclusions

We proposed and presented the TII-TL-EHL-GPS CoD. The statistical properties of the new CoD are derived and these include the QF, moments, distribution of order statistics, and RE. The model parameters are estimated using ML estimation technique. The MCS was conducted to assess the performance of the ML estimates. Two real datasets are applied to a special case, the TII-TL-EHL-WL distribution to demonstrate the usefulness and flexibility of the proposed CoD. The TII-TL-EHL-GPS CoD has significant potential for fulfilling current societal needs, particularly in the fields of aviation safety and growth hormone research. Aircraft windshields are essential for safe air travel, and the TII-TL-EHL-GPS CoD can accurately capture almost symmetric data, which can improve the analysis and modeling of these aircraft windshields. This leads to more accurate assessments of the lifespan and durability of aircraft windshields, informing more effective maintenance schedules and reducing the risk of failure during flight. Growth hormone data is right skewed distributed, and the flexibility of the TII-TL-EHL-GPS CoD enables better analysis and interpretation of such data. The TII-TL-EHL-GPS CoD enhance the ability to model growth hormone data and can advance personalized approaches to hormone-related therapies and treatments.

4. The Harris Odd Burr Type X-G Family of Distributions with Applications

The paper is accepted by the *Scopus Indexed International Journal of Operational Research*.

4.1 Introduction

In recent years, researchers have been developing generalized families of distributions by combining existing models. In this chapter, we combine the Harris-G FoD (Aly and Benkherouf [14]) with the Burr type X-G FoD (Yousof et al. [140]). This combination creates a model with greater adaptability and flexibility, allowing it to capture the complexities of real-life data more effectively. This expanded capability is valuable given the widespread use of the Burr type X distribution in diverse fields such as agricultural production, health, income, and reliability analysis.

The cdf and pdf of the Harris-G FoD (Pinho et al. [110]) are given in Equation (1.21) and Equation (1.22), respectively.

The cdf and pdf of the odd Burr type X-G FoD are given

$$F_{OBX-G}(x; \beta, \psi) = \left[1 - \exp \left(- \left(\frac{G(x; \psi)}{\bar{G}(x; \psi)} \right)^2 \right) \right]^\beta, \quad (4.1)$$

and

$$f_{OBX-G}(x; \beta, \psi) = 2\beta \left[1 - \exp \left(- \left(\frac{G(x; \psi)}{\bar{G}(x; \psi)} \right)^2 \right) \right]^{\beta-1} \exp \left(- \left(\frac{G(x; \psi)}{\bar{G}(x; \psi)} \right)^2 \right) \frac{G(x; \psi)}{\bar{G}(x; \psi)^3} g(x; \psi),$$

for $x \geq 0$, $\beta > 0$, and parameter vector ψ .

The introduction of the new FoD, Harris-OBX-G, presents several advantages over existing distributions, particularly in terms of flexibility and applicability. The Harris-OBX-G FoD inherits the advantageous properties of both families, while offering unique features tailored to specific data characteristics. For instance, it provides a wide range of hrf shapes, including monotonic, BT, UBT, and UBT-BT, making it suitable for modeling almost symmetric, right-skewed, and left-skewed datasets. This versatility allows for more accurate representation of real-world data scenarios compared to distributions limited in their hrf shapes.

The special cases under this new FoD is obtained when the baseline distribution is changed to existing distribution, say LLoG, Weibull, and uniform distribution. In this chapter, we discuss three special cases of Harris-OBX-G FoD namely, the Harris-OBX-LLoG distribution, Harris-OBX-W distribution, and Harris-OBX-U distribution. We also provide the plots of the density functions, hrf, as well as SK and KS to illustrate the nature of shapes that the Harris-OBX-G

FoD can accommodate when modeling different datasets. We use the ML estimation technique to estimate the model parameters, and the consistency of the ML estimators is evaluated using simulations. The applicability and versatility of the Harris-OBX-G FoD are demonstrated by applying one special case (Harris-OBX-W distribution) to two real datasets from different areas. The Harris-OBX-W distribution outperforms the models used for comparisons in all the selected datasets.

4.2 The New FoD

In this section, we develop the new FoD. Inserting Equation (4.2) into Equation (1.21), we obtain the new Harris-OBX-G FoD. The cdf of the new Harris-OBX-G FoD is given as

$$F_{Harris-OBX-G}(x; \delta, v, \beta, \psi) = 1 - \left[\frac{\delta \left(1 - [1 - \exp(-K(x; \psi))]^\beta\right)^v}{1 - \bar{\delta} \left(1 - [1 - \exp(-K(x; \psi))]^\beta\right)^v} \right]^{\frac{1}{v}}, \quad (4.2)$$

where $K(x; \psi) = \left(\frac{G(x; \psi)}{\bar{G}(x; \psi)}\right)^2$, for $\delta, v, \beta > 0$, $x \geq 0$, $\bar{\delta} = 1 - \delta$ and parameter vector ψ . The pdf corresponding to Equation (4.2) is

$$\begin{aligned} f_{Harris-OBX-G}(x; \delta, v, \beta, \psi) &= 2\beta\delta^{\frac{1}{v}} \left(1 - \bar{\delta} \left(1 - [1 - \exp(-K(x; \psi))]^\beta\right)^v\right)^{-\left(\frac{1}{v}+1\right)} \\ &\times [1 - \exp(-K(x; \psi))]^{\beta-1} \exp(-K(x; \psi)) \\ &\times \frac{G(x; \psi)}{(\bar{G}(x; \psi))^3} g(x; \psi), \end{aligned} \quad (4.3)$$

where $K(x; \psi) = \left(\frac{G(x; \psi)}{\bar{G}(x; \psi)}\right)^2$, for $\delta, v, \beta > 0$, $x \geq 0$, $\bar{\delta} = 1 - \delta$ and parameter vector ψ . The hrf of the Harris-OBX-G FoD is given by

$$\begin{aligned} h(x; \delta, v, \beta, \psi) &= 2\beta \left(1 - \bar{\delta} \left(1 - [1 - \exp(-K(x; \psi))]^\beta\right)^v\right)^{-1} \\ &\times [1 - \exp(-K(x; \psi))]^{\beta-1} \exp(-K(x; \psi)) \\ &\times \left(1 - [1 - \exp(-K(x; \psi))]^\beta\right)^{-1} \frac{G(x; \psi)}{(\bar{G}(x; \psi))^3} g(x; \psi), \end{aligned}$$

where $K(x; \psi) = \left(\frac{G(x; \psi)}{\bar{G}(x; \psi)}\right)^2$, for $\delta, v, \beta > 0$, $x \geq 0$, $\bar{\delta} = 1 - \delta$ and parameter vector ψ .

4.3 Identifiability

To establish identifiability of the Harris-OBX-G FoD, let $\Delta_1 = (\delta_1, v_1, \beta_1, \psi_1)$ and $\Delta_2 = (\delta_2, v_2, \beta_2, \psi_2)$. The likelihood ratio are given

$$\begin{aligned}
 \frac{f_1(\Delta_1)}{f_2(\Delta_2)} &= \left[\frac{\beta_1 \delta_1^{\frac{1}{v_1}}}{\beta_2 \delta_2^{\frac{1}{v_2}}} \right] \left(1 - \bar{\delta}_1 \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) \right]^{\beta_1} \right)^{v_1} \right)^{-\left(\frac{1}{v_1} + 1\right)} \\
 &\times \left(1 - \bar{\delta}_2 \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) \right]^{\beta_2} \right)^{v_2} \right)^{\frac{1}{v_2} + 1} \\
 &\times \left[1 - \exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) \right]^{\beta_1 - 1} \left[1 - \exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) \right]^{-(\beta_2 - 1)} \\
 &\times \frac{\exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) G(x; \psi_1) g(x; \psi_1)}{(\bar{G}(x; \psi_1))^3} \\
 &\times \frac{(\bar{G}(x; \psi_2))^3}{\exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) G(x; \psi_2) g(x; \psi_2)}.
 \end{aligned}$$

The log-likelihood ratio is

$$\begin{aligned}
 \ln \left(\frac{f_1(\Delta_1)}{f_2(\Delta_2)} \right) &= \ln \left[\frac{\beta_1 \delta_1^{\frac{1}{v_1}}}{\beta_2 \delta_2^{\frac{1}{v_2}}} \right] + (\beta_1 - 1) \ln \left[1 - \exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) \right] \\
 &- \left(\frac{1}{v_1 + 1} \right) \ln \left(1 - \bar{\delta}_1 \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) \right]^{\beta_1} \right)^{v_1} \right) \\
 &+ \left(\frac{1}{v_2 + 1} \right) \ln \left(1 - \bar{\delta}_2 \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) \right]^{\beta_2} \right)^{v_2} \right) \\
 &- (\beta_2 - 1) \ln \left[1 - \exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) \right] - \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 + \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \\
 &+ \ln[G(x; \psi_1)] + \ln[g(x; \psi_1)] - 3 \ln[\bar{G}(x; \psi_1)] - \ln[G(x; \psi_2)] \\
 &- \ln[g(x; \psi_2)] + 3 \ln[\bar{G}(x; \psi_2)].
 \end{aligned}$$

The derivative of the log-likelihood ratio is

$$\begin{aligned}
 \frac{d}{dx} \ln \left(\frac{f_1(\Delta_1)}{f_2(\Delta_2)} \right) &= (\beta_1 - 1) \frac{D_1}{\left[1 - \exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) \right]} - (\beta_2 - 1) \frac{D_2}{\left[1 - \exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) \right]} \\
 &- \left(\frac{1}{v_1 + 1} \right) \frac{D_1 \bar{\delta}_1 v_1 \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) \right]^{\beta_1} \right)^{v_1 - 1}}{\left(1 - \bar{\delta}_1 \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) \right]^{\beta_1} \right)^{v_1} \right)} \\
 &+ \left(\frac{1}{v_2 + 1} \right) \frac{D_2 \bar{\delta}_2 v_2 \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) \right]^{\beta_2} \right)^{v_2 - 1}}{\left(1 - \bar{\delta}_2 \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) \right]^{\beta_2} \right)^{v_2} \right)} \\
 &- 2 \frac{G(x; \psi_1)}{(\bar{G}(x; \psi_1))^3} g(x; \psi_1) + 2 \frac{G(x; \psi_2)}{(\bar{G}(x; \psi_2))^3} g(x; \psi_2) + \frac{g(x; \psi_1)}{G(x; \psi_1)} \\
 &+ \frac{g'(x; \psi_1)}{g(x; \psi_1)} + 3 \frac{g(x; \psi_1)}{\bar{G}(x; \psi_1)} - \frac{g(x; \psi_2)}{G(x; \psi_2)} - \frac{g'(x; \psi_2)}{g(x; \psi_2)} - 3 \frac{g(x; \psi_2)}{\bar{G}(x; \psi_2)},
 \end{aligned}$$

where

$$D_1 = 2\beta_1 \left[1 - \exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) \right]^{\beta_1} \exp \left(- \left(\frac{G(x; \psi_1)}{\bar{G}(x; \psi_1)} \right)^2 \right) \frac{G(x; \psi_1)}{(\bar{G}(x; \psi_1))^2} g(x; \psi_1)$$

and

$$D_2 = 2\beta_2 \left[1 - \exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) \right]^{\beta_2} \exp \left(- \left(\frac{G(x; \psi_2)}{\bar{G}(x; \psi_2)} \right)^2 \right) \frac{G(x; \psi_2)}{(\bar{G}(x; \psi_2))^2} g(x; \psi_2).$$

Now, when

$$\frac{d}{dx} \ln \left(\frac{f_1(\Delta_1)}{f_2(\Delta_2)} \right) = 0,$$

it implies that $\Delta_1 = \Delta_2$, that is, the parameters of the Harris-OBX-G FoD exist and can be uniquely determined based on a given dataset.

4.4 Quantile Function

The QF of the Harris-OBX-G FoD can be obtained by solving the non-linear equation:

$$F_{Harris-OBX-G}(x; \delta, v, \beta, \psi) = 1 - \left[\frac{\delta \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi)}{\bar{G}(x; \psi)} \right)^2 \right) \right]^{\beta} \right)^v}{1 - \bar{\delta} \left(1 - \left[1 - \exp \left(- \left(\frac{G(x; \psi)}{\bar{G}(x; \psi)} \right)^2 \right) \right]^{\beta} \right)^v} \right]^{\frac{1}{v}} = u,$$

$0 \leq u \leq 1$, that is, (after some simplifications)

$$G(x; \psi) = \left(\left(-\ln \left(1 - \left[1 - \left(\frac{(1-u)^v}{\delta - (1-u)^v [\delta - 1]} \right)^{\frac{1}{v}} \right]^{\frac{1}{\beta}} \right) \right)^{\frac{-1}{2}} + 1 \right)^{-1}.$$

Therefore, the QF of the Harris-OBX-G FoD is given by

$$Q_X(u) = G^{-1} \left[\left(\left(-\ln \left(1 - \left[1 - \left(\frac{(1-u)^v}{\delta - (1-u)^v [\delta - 1]} \right)^{\frac{1}{v}} \right]^{\frac{1}{\beta}} \right) \right)^{\frac{-1}{2}} + 1 \right)^{-1} \right]. \quad (4.4)$$

It follows therefore that random numbers can be generated via Equation (4.4) for a given parent cdf G.

4.5 Linear Representation

This subsection is devoted to expressing the pdf of Harris-OBX-G FoD as an infinite linear combination of Exp-G distribution, using the generalized binomial series expansion

$$(1-a)^{-q} = \sum_{i=0}^{\infty} \binom{q+i-1}{i} a^i,$$

which is valid for $|a| < 1$ and $q > 0$. If $\delta \in (0, 1)$, we can obtain

$$f_{Harris-OBX-G}(x; \delta, v, \beta, \psi) = f_{OBX-G}(x; \beta, \psi) \sum_{m=0}^{\infty} b_m (F_{OBX-G}(x; \beta, \psi))^m, \quad (4.5)$$

where

$$b_m = \delta^{\frac{1}{v}} (-1)^m \sum_{i=0}^{\infty} \binom{i + \frac{1}{v}}{i} \bar{\delta}^i \binom{vi}{m}. \quad (4.6)$$

For $\delta > 1$, we write Equation (4.3) as

$$f_{Harris-OBX-G}(x; \delta, v, \beta, \psi) = f_{OBX-G}(x; \beta, \psi) \sum_{m=0}^{\infty} C_m (F_{OBX-G}(x; \beta, \psi))^m, \quad (4.7)$$

where

$$C_m = \frac{1}{\theta^{i+1}} \left[\sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{l+m} (\theta - 1)^i \binom{\frac{1}{v}}{i} \binom{i}{l} \binom{lv}{m} \right], \quad (4.8)$$

where $\theta = \frac{1}{\delta}$ and $\bar{\theta} = 1 - \frac{1}{\delta}$.

Considering

$$\begin{aligned}
 f_{OBX-G}(x; \beta, \psi) (F_{OBX-G}(x; \beta, \psi))^m &= 2\beta \left[1 - \exp \left(- \left(\frac{G(x; \psi)}{\overline{G}(x; \psi)} \right)^2 \right) \right]^{\beta(m+1)-1} \\
 &\times \exp \left(- \left(\frac{G(x; \psi)}{\overline{G}(x; \psi)} \right)^2 \right) \frac{G(x; \psi)}{(\overline{G}(x; \psi))^3} g(x; \psi) \\
 &= 2\beta \sum_{j=0}^{\infty} \binom{\beta(m+1)-1}{j} (-1)^j \frac{G(x; \psi)}{(\overline{G}(x; \psi))^3} g(x; \psi) \\
 &\times \exp \left(- (j+1) \left(\frac{G(x; \psi)}{\overline{G}(x; \psi)} \right)^2 \right) \\
 &= 2\beta \sum_{j,k=0}^{\infty} \binom{\beta(m+1)-1}{j} (-1)^j \frac{(-(j+1))^k}{k!} \\
 &\times \frac{(G(x; \psi))^{2k+1}}{(\overline{G}(x; \psi))^{2k+3}} g(x; \psi) \\
 &= 2\beta \sum_{j,k,s=0}^{\infty} \binom{\beta(m+1)-1}{j} (-1)^{j+s} \frac{(-(j+1))^k}{k!} \\
 &\times \binom{2k+1}{s} (\overline{G}(x; \psi))^{s-(2k+3)} g(x; \psi) \\
 &= 2\beta \sum_{j,k,s,p=0}^{\infty} \binom{\beta(m+1)-1}{j} (-1)^{j+s+p} \frac{(-(j+1))^k}{k!} \\
 &\times \binom{2k+1}{s} \binom{s-(2k+3)}{p} G^p(x; \psi) g(x; \psi), \quad (4.9)
 \end{aligned}$$

for $\delta \in (0, 1)$, Equation (4.3) can be written as

$$f_{Harris-OBX-G}(x; \delta, v, \beta, \psi) = \sum_{p=0}^{\infty} d_{p+1} g_{p+1}(x; \psi), \quad (4.10)$$

where

$$\begin{aligned}
 d_{p+1} &= 2\beta \sum_{m,j,k,s=0}^{\infty} b_m \binom{\beta(m+1)-1}{j} (-1)^{j+s+p} \frac{(-(j+1))^k}{(p+1)k!} \\
 &\times \binom{2k+1}{s} \binom{s-(3+2k)}{p}, \quad (4.11)
 \end{aligned}$$

b_m is given by Equation (4.6) and $g_{p+1}(x; \psi) = (p+1)G^p(x; \psi)g(x; \psi)$ is the Exp-G distribution with the power parameter $(p+1)$.

Furthermore, for $\delta > 1$, Equation (4.3) can be written as

$$f_{Harris-OBX-G}(x; \delta, v, \beta, \psi) = \sum_{k,l=0}^{\infty} d_{p+1}^* g_{p+1}(x; \psi), \quad (4.12)$$

where

$$\begin{aligned}
 d_{p+1}^* &= 2\beta \sum_{m,j,k,s=0}^{\infty} C_m \binom{\beta(m+1)-1}{j} (-1)^{j+s+p} \frac{(-(j+1))^k}{(p+1)k!} \\
 &\times \binom{2k+1}{s} \binom{s-(3+2k)}{p},
 \end{aligned} \tag{4.13}$$

where C_m is given by Equation (4.8) and $g_{p+1}(x; \psi) = (p+1)G^p(x; \psi)g(x; \psi)$ is the Exp-G distribution with the power parameter $(p+1)$.

4.6 Moments and Generating Functions

Let $X \sim \text{Harris-OBX-G}(\delta, v, \beta, \psi)$ and $Y_{p+1} \sim \text{Exp-G}(p+1, \psi)$, then the r^{th} raw moment, μ_r' can be obtained from Equations (4.10) and (4.12). For $\delta \in (0, 1)$,

$$\mu_r' = E(X^r) = \sum_{p=0}^{\infty} d_{p+1} E(Y_{p+1}^r),$$

where d_{p+1} is as defined in Equation (4.11) and $E(Y_{p+1}^r)$ is the r^{th} moment of an Exp-G distribution with power parameter $(p+1)$. For $\delta > 1$,

$$\mu_r' = E(X^r) = \sum_{p=0}^{\infty} d_{p+1}^* E(Y_{p+1}^r),$$

where d_{p+1}^* is as defined in Equation (4.13) and $E(Y_{p+1}^r)$ is the r^{th} moment of an Exp-G distribution with power parameter $(p+1)$. For $\delta \in (0, 1)$, the MGF of X is given by:

$$M_X(t) = \sum_{p=0}^{\infty} d_{p+1} E(e^{tY_{p+1}}),$$

where $E(e^{tY_{p+1}})$ is the MGF of the Exp-G distribution with power parameter $(p+1)$ and d_{p+1} is as defined in Equation (4.11). For $\delta > 1$, the MGF is

$$M_X(t) = \sum_{p=0}^{\infty} d_{p+1}^* E(e^{tY_{p+1}}),$$

where $E(e^{tY_{p+1}})$ is the MGF of the Exp-G distribution with power parameter $(p+1)$ and d_{p+1}^* is as defined in Equation (4.13).

4.7 Distribution of Order Statistics

The pdf of the h^{th} order statistic $X_{h:n}$ from Harris-OBX-G FoD with pdf $f(x; \delta, v, \beta, \psi) = f(x)$ is given by

$$f_{h:n}(x; \delta, v, \beta, \psi) = n!f(x) \sum_{z=0}^{n-h} \frac{(-1)^z}{(h-1)!(n-h)!} [F(x)]^{z+h-1}. \quad (4.14)$$

If $\delta \in (0, 1)$, we have

$$\begin{aligned} f(x) [F(x)]^{z+h-1} &= \frac{\delta^{\frac{1}{v}} f_{OBX-G}(x; \beta, \psi)}{[1 - \delta(\bar{F}_{OBX-G}(x; \beta, \psi))^v]^{1+\frac{1}{v}}} \left(1 - \frac{\delta^{\frac{1}{v}} \bar{F}_{OBX-G}(x; \beta, \psi)}{[1 - \delta(\bar{F}_{OBX-G}(x; \beta, \psi))^v]^{\frac{1}{v}}} \right)^{z+h-1} \\ &= \sum_{q=0}^{\infty} \binom{z+h-1}{q} (-1)^q \frac{\delta^{\frac{q+1}{v}} f_{OBX-G}(x; \beta, \psi) (\bar{F}_{OBX-G}(x; \beta, \psi))^q}{[1 - \delta(\bar{F}_{OBX-G}(x; \beta, \psi))^v]^{1+\frac{q+1}{v}}} \\ &= \sum_{m=0}^{\infty} R_m f_{OBX-G}(x; \beta, \psi) (F_{OBX-G}(x; \beta, \psi))^m, \end{aligned} \quad (4.15)$$

where

$$R_m = \sum_{q,i=0}^{\infty} (-1)^{q+m} \delta^{\frac{q+1}{v}} \binom{z+h-1}{q} \binom{\frac{q+1}{v} + i}{i} \bar{\delta}^i \binom{vi+q}{m}. \quad (4.16)$$

For $\delta > 1$, we replace $\theta = \frac{1}{\delta}$. Then, we can write

$$f(x) [F(x)]^{z+s-1} = \sum_{m=0}^{\infty} R_m^* f_{OBX-G}(x; \beta, \psi) (F_{OBX-G}(x; \beta, \psi))^m, \quad (4.17)$$

$$R_m^* = (-1)^m \frac{1}{\theta^{i+1}} \left[\sum_{q,i=0}^{\infty} \sum_{l=0}^i (-1)^{l+q} \binom{z+h-1}{q} (\theta-1)^i \binom{i+\frac{q+1}{v}}{i} \binom{i}{l} \binom{lv+q}{m} \right]. \quad (4.18)$$

Now, we follow the steps leading to Equation (4.9), for $\delta \in (0, 1)$, by substituting Equation (4.15) into Equation (4.14), we get

$$f_{h:n}(x; \delta, v, b, \xi) = n! \sum_{z=0}^{n-h} \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^z}{(h-1)!(n-h)!} R_m d_{p+1} g_{p+1}(x; \psi),$$

where R_m is given by Equation (4.16), d_{p+1} is given in equation (4.11) and $g_{p+1}(x; \psi) = (p+1)G^p(x; \psi)g(x; \psi)$ is the Exp-G distribution with the power parameter $(p+1)$. Furthermore, for $\delta > 1$, we get

$$f_{h:n}(x; \delta, v, b, \xi) = n! \sum_{z=0}^{n-h} \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^z}{(h-1)!(n-h)!} R_m^* d_{p+1}^* g_{p+1}(x; \psi),$$

where R_m^* is given by equation (4.18), d_{p+1}^* is given in Equation (4.13) and $g_{p+1}(x; \psi) = (p+1)G^p(x; \psi)g(x; \psi)$ is the Exp-G distribution with the power parameter $(p+1)$. (See appendix for details).

4.8 Rényi Entropy

In this subsection, RE for Harris-OBX-G FoD is presented. An entropy is a measure of uncertainty or variation of a random variable. RE for $\delta \in (0, 1)$ is given by

$$\begin{aligned} I_R(w) &= (1-w)^{-1} \log \left[2\beta \sum_{m,j,k,s,p=0}^{\infty} b_m^* \binom{\beta(m+1)-1}{j} (-1)^{j+s+p} \frac{(-(j+1))^k}{k!} \right. \\ &\quad \times \left. \binom{2k+1}{s} \binom{s-(3+2k)}{p} \frac{1}{\left[1 + \frac{p+1}{w}\right]^w} \int_0^{\infty} \left(\left[1 + \frac{p+1}{w}\right] G^{\frac{p+1}{w}}(x; \psi) g(x; \psi) \right)^w dx \right] \\ &= (1-w)^{-1} \log \left[\sum_{p=0}^{\infty} \tau_{p+1} \exp((1-w)I_{REG}) \right], \end{aligned}$$

for $v > 0, v \neq 1$, where

$$\begin{aligned} \tau_{p+1} &= 2\beta \sum_{m,j,k,s=0}^{\infty} b_m^* \binom{\beta(m+1)-1}{j} (-1)^{j+s+p} \frac{(-(j+1))^k}{k!} \\ &\quad \times \binom{2k+1}{s} \binom{s-(3+2k)}{p} \frac{1}{\left[1 + \frac{p+1}{w}\right]^w}, \end{aligned}$$

$b_m^* = \delta^{\frac{w}{v}} (-1)^m \sum_{i=0}^{\infty} \left(w(1+\frac{1}{v}) + i - 1 \right) \bar{\delta}^i \binom{vi}{m}$, and $I_{REG} = \frac{1}{1-w} \log \left(\int_0^{\infty} \left(\left[1 + \frac{p+1}{w}\right] G^{\frac{p+1}{w}}(x; \psi) g(x; \psi) \right)^w dx \right)$ is the RE of the Exp-G distribution with power parameter $(\frac{p+1}{w} + 1)$. See appendix for details.

Furthermore, for $\delta > 1$, RE is given by

$$\begin{aligned} I_R(w) &= (1-w)^{-1} \log \left[2\beta \sum_{m,j,k,s,p=0}^{\infty} C_m^* \binom{\beta(m+1)-1}{j} (-1)^{j+s+p} \frac{(-(j+1))^k}{k!} \right. \\ &\quad \times \left. \binom{2k+1}{s} \binom{s-(3+2k)}{p} \frac{1}{\left[1 + \frac{p+1}{w}\right]^w} \right. \\ &\quad \times \left. \int_0^{\infty} \left(\left[1 + \frac{p+1}{w}\right] G^{\frac{p+1}{w}}(x; \psi) g(x; \psi) \right)^w dx \right] \\ &= (1-w)^{-1} \log \left[\sum_{p=0}^{\infty} \tau_{p+1}^* \exp((1-w)I_{REG}) \right], \end{aligned}$$

where

$$\begin{aligned} \tau_{p+1}^* &= 2\beta \sum_{m,j,k,s=0}^{\infty} C_m^* \binom{\beta(m+1)-1}{j} (-1)^{j+s+p} \frac{(-(j+1))^k}{k!} \\ &\times \binom{2k+1}{s} \binom{s-(3+2k)}{p} \frac{1}{\left[1 + \frac{p+1}{w}\right]^w}, \end{aligned}$$

$$C_m^* = (-1)^m \frac{1}{\theta^{i+1}} \left[\sum_{i=0}^{\infty} \sum_{l=0}^i (-1)^l (\theta - 1)^i \binom{w(1+\frac{1}{v})+i-1}{i} \binom{i}{l} \binom{lv}{m} \right], \theta = \frac{1}{\delta},$$

and $I_{REG} = \frac{1}{1-w} \log \left(\int_0^{\infty} \left[1 + \frac{p+1}{w} \right] G^{\frac{p+1}{w}}(x; \psi) g(x; \psi) dx \right)^w$ is the RE of the Exp-G distribution with power parameter $(\frac{p+1}{w} + 1)$.

As a result, RE for Harris-OBX-G FoD can be obtained from RE of the Exp-G distribution. (See appendix for more information).

4.9 Some Special Cases

In this section, we look at the LLoG, Weibull, and uniform distributions as parent distributions for the Harris-OBX-G FoD.

4.9.1 The Harris Odd Burr Type X-Log-Logistic (Harris-OBX-LLoG) Distribution

Suppose that the LLoG distribution is used as the parent or baseline distribution with the pdf and cdf given by $g(x; c) = cx^{c-1}(1+x^c)^{-2}$, and $G(x; c) = 1 - (1+x^c)^{-1}$, respectively, for $c > 0$ and $x > 0$, then the cdf and pdf of the Harris-OBX-LLoG distribution are given by

$$F_{Harris-OBX-LLoG}(x; \delta, v, \beta, c) = 1 - \left[\frac{\delta \left(1 - [1 - \exp(-Z(x; c))]^\beta \right)^v}{1 - \bar{\delta} \left(1 - [1 - \exp(-Z(x; c))]^\beta \right)^v} \right]^{\frac{1}{v}},$$

and

$$\begin{aligned} f_{Harris-OBX-LLoG}(x; \delta, v, \beta, c) &= 2\beta\delta^{\frac{1}{v}} \frac{1 - (1+x^c)^{-1}}{((1+x^c)^{-1})^3} cx^{c-1}(1+x^c)^{-2} \\ &\times \left(1 - \bar{\delta} \left(1 - [1 - \exp(-Z(x; c))]^\beta \right)^v \right)^{-\left(\frac{1}{v}+1\right)} \\ &\times [1 - \exp(-Z(x; c))]^{\beta-1} \\ &\times \exp(-Z(x; c)), \end{aligned}$$

where $Z(x; c) = \left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}} \right)^2$, for $\delta, v, \beta, c > 0$, $x > 0$, and $\bar{\delta} = 1 - \delta$. The hrf is given by

$$\begin{aligned}
 h(x; \delta, v, \beta, c) &= 2\beta \left(1 - \bar{\delta} \left(1 - [1 - \exp(-Z(x; c))]^\beta \right)^v \right)^{-1} \\
 &\times [1 - \exp(-Z(x; c))]^{\beta-1} \exp(-Z(x; c)) \\
 &\times \left(1 - [1 - \exp(-Z(x; c))]^\beta \right)^{-1} \\
 &\times \frac{1 - (1 + x^c)^{-1}}{((1 + x^c)^{-1})^3} c x^{c-1} (1 + x^c)^{-2},
 \end{aligned}$$

$$Z(x; c) = \left(\frac{1 - (1 + x^c)^{-1}}{(1 + x^c)^{-1}} \right)^2, \text{ for } \delta, v, \beta, c > 0, x > 0, \text{ and } \bar{\delta} = 1 - \delta.$$

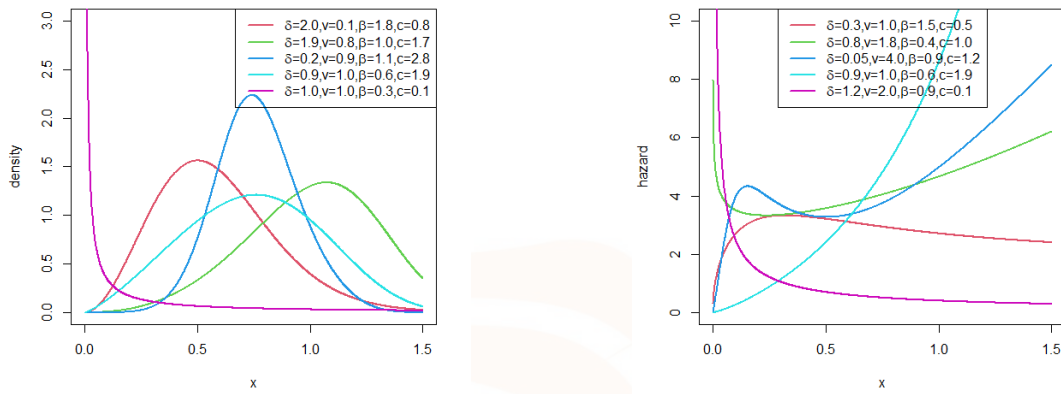


Figure 4.1: Pdf and Hrf Plots of Harris-OBX-LLoG Distribution

Figure 4.1 shows the plots of the pdf and hrf of the Harris-OBX-LLoG distribution. The pdf shows almost symmetric, negative-skewed, positive-skewed, and reverse-J shapes. The hrf has various shapes such as decreasing, increasing, UBT, BT, and UBT-BT.



Figure 4.2: SK and KS Plots of Harris-OBX-LLoG Distribution



Figure 4.3: SK and KS of Harris-OBX-LLoG Distribution

Figures 4.2 and 4.3 show 3D graphs of SK and KS for the Harris-OBX-LLoG distribution. We notice that

- When the parameters δ and v are fixed, the SK and KS of Harris-OBX-LLoG distribution increase as β and c increase.
- When the parameters δ and β are fixed, the SK and KS of Harris-OBX-LLoG distribution increase as v and c increase.

4.9.2 The Harris Odd Burr Type X-Weibull (Harris-OBX-W) Distribution

If the Weibull distribution is considered as the parent distribution with the pdf and cdf given by $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$ and $G(x; \lambda) = 1 - e^{-x^\lambda}$, respectively, for $\lambda > 0$, then the cdf and pdf of the Harris-OBX-W distribution are given by

$$F_{Harris-OBX-W}(x; \delta, v, \beta, \lambda) = 1 - \left[\frac{\delta \left(1 - [1 - \exp(-W(x; \lambda))]^\beta \right)^v}{1 - \bar{\delta} \left(1 - [1 - \exp(-W(x; \lambda))]^\beta \right)^v} \right]^{\frac{1}{v}},$$

and

$$\begin{aligned} f_{Harris-OBX-W}(x; \delta, v, \beta, \lambda) &= 2\beta\delta^{\frac{1}{v}} \left(1 - \bar{\delta} \left(1 - [1 - \exp(-W(x; \lambda))]^\beta \right)^v \right)^{-\left(\frac{1}{v}+1\right)} \\ &\times [1 - \exp(-W(x; \lambda))]^{\beta-1} \exp(-W(x; \lambda)) \\ &\times \frac{1 - e^{-x^\lambda}}{(e^{-x^\lambda})^3} \lambda x^{\lambda-1} e^{-x^\lambda}, \end{aligned}$$

where $W(x; \lambda) = \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}} \right)^2$, for $\delta, v, \beta, \lambda > 0$, $x > 0$, and $\bar{\delta} = 1 - \delta$. The hrf of the Harris-OBX-W distribution is given by

$$\begin{aligned}
 h(x; \delta, v, \beta, \lambda) &= 2\beta \left(1 - \bar{\delta} \left(1 - [1 - \exp(-W(x; \lambda))]^\beta\right)^v\right)^{-1} \\
 &\times [1 - \exp(-W(x; \lambda))]^{\beta-1} \exp\left(-\left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}}\right)^2\right) \\
 &\times \left(1 - [1 - \exp(-W(x; \lambda))]^\beta\right)^{-1} \frac{1 - e^{-x^\lambda}}{(e^{-x^\lambda})^3} \lambda x^{\lambda-1} e^{-x^\lambda},
 \end{aligned}$$

where $W(x; \lambda) = \left(\frac{1 - e^{-x^\lambda}}{e^{-x^\lambda}}\right)^2$, for $\delta, v, \beta, \lambda > 0$, $x > 0$, and $\bar{\delta} = 1 - \delta$.

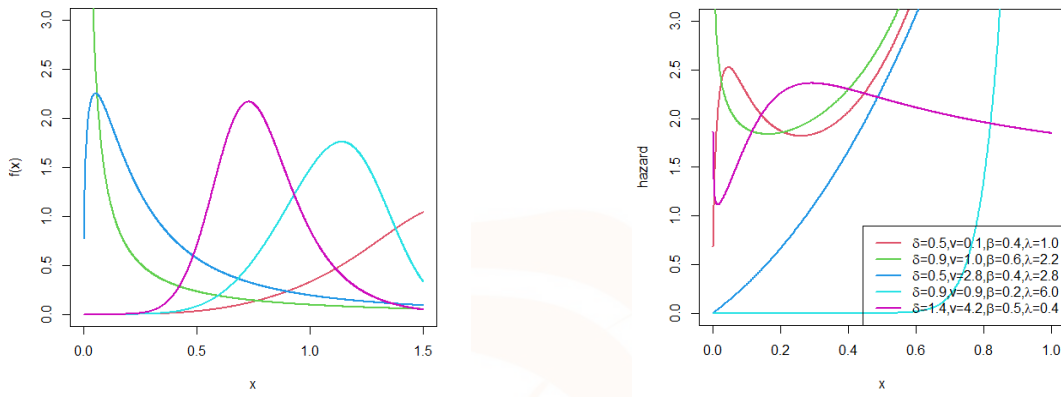


Figure 4.4: Pdf and Hrf Plots of Harris-OBX-W Distribution

Figure 4.4 shows the plots of the pdf and hrf of the Harris-OBX-W distribution. The pdf shows almost symmetric, negative-skewed or positive-skewed, and reverse-J shapes. The hrf comes in a variety of shapes, including increasing, BT, UBT-BT and BT-UBT.



Figure 4.5: SK and KS Plots of Harris-OBX-W Distribution



Figure 4.6: SK and KS Plots of Harris-OBX-W Distribution

Figures 4.5 and 4.6 show 3D graphs of SK and KS for the Harris-OBX-W distribution. We notice that

- When the parameters δ and v are fixed, the SK and KS of Harris-OBX-W distribution increase as β and λ increase.
- When the parameters β and λ are fixed, the SK and KS of Harris-OBX-W distribution increase as δ and v increase.

4.9.3 The Harris Odd Burr Type X-Uniform (Harris-OBX-U) Distribution

Taking uniform distribution as the parent distribution with the pdf and cdf given by $g(x; \theta) = \frac{1}{\theta}$ and $G(x; \theta) = \frac{x}{\theta}$, respectively, for $\theta > 0$ and $x > 0$, then the cdf and pdf of the Harris-OBX-U distribution are given by

$$F_{Harris-OBX-U}(x; \delta, v, \beta, \theta) = 1 - \left[\frac{\delta \left(1 - \left[1 - \exp \left(- \left(\frac{x}{1 - \frac{x}{\theta}} \right)^2 \right) \right]^\beta \right)^v}{1 - \bar{\delta} \left(1 - \left[1 - \exp \left(- \left(\frac{x}{1 - \frac{x}{\theta}} \right)^2 \right) \right]^\beta \right)^v} \right]^{\frac{1}{v}},$$

and

$$\begin{aligned} f_{Harris-OBX-U}(x; \delta, v, \beta, \theta) &= 2\beta\delta^{\frac{1}{v}} \left(1 - \bar{\delta} \left(1 - [1 - \exp(-Q(x; \theta))]^\beta \right)^v \right)^{-(\frac{1}{v}+1)} \\ &\times [1 - \exp(-Q(x; \theta))]^{\beta-1} \exp(-Q(x; \theta)) \\ &\times \frac{\frac{x}{\theta}}{\left(1 - \frac{x}{\theta}\right)^3} \frac{1}{\theta}, \end{aligned}$$

where $Q(x; \theta) = \left(\frac{x}{1 - \frac{x}{\theta}} \right)^2$, for $\delta, v, \beta, \theta > 0$, $x > 0$, and $\bar{\delta} = 1 - \delta$. The hrf of the Harris-OBX-U distribution is given by

$$\begin{aligned} h(x; \delta, v, \beta, \theta) &= 2\beta \left(1 - \bar{\delta} \left(1 - [1 - \exp(-Q(x; \theta))]^\beta \right)^v \right)^{-1} \\ &\times [1 - \exp(-Q(x; \theta))]^{\beta-1} \exp(-Q(x; \theta)) \\ &\times \left(1 - [1 - \exp(-Q(x; \theta))]^\beta \right)^{-1} \frac{\frac{x}{\theta}}{\left(1 - \frac{x}{\theta} \right)^3} \frac{1}{\theta}, \end{aligned}$$

where $Q(x; \theta) = \left(\frac{x}{1 - \frac{x}{\theta}} \right)^2$, for $\delta, v, \beta, \theta > 0$, $x > 0$, and $\bar{\delta} = 1 - \delta$.

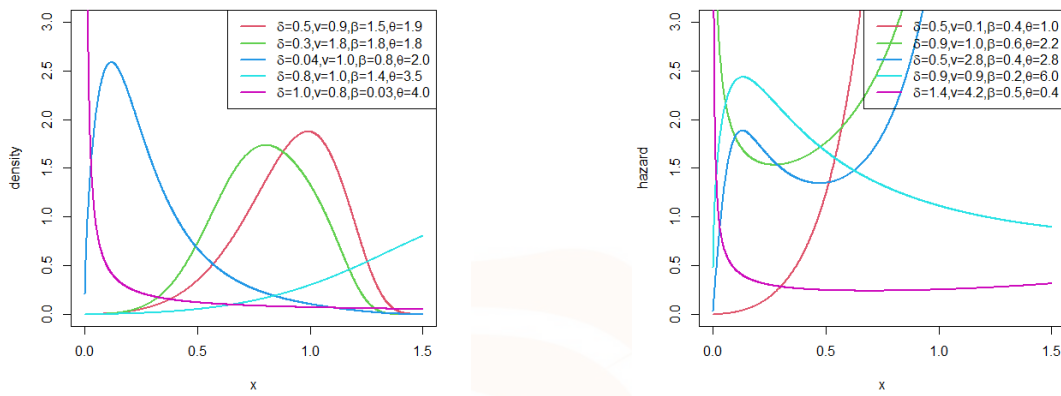


Figure 4.7: Pdf and Hrf Plots of Harris-OBX-U Distribution

Figure 4.7 shows the plots of the pdf and hrf of the Harris-OBX-U distribution for selected values of the model parameters. The pdf shows almost symmetric, negative-skewed or positive-skewed, and reverse-J shapes. The hrf comes in a variety of shapes, including increasing, decreasing, UBT, BT, and UBT-BT.



Figure 4.8: SK and KS of Harris-OBX-U Distribution



Figure 4.9: SK and KS Plots of Harris-OBX-U Distribution

Figures 4.8 and 4.9 show 3D graphs of SK and KS for the Harris-OBX-U distribution. We notice that

- When the parameters β and v are fixed, the SK and KS of Harris-OBX-U distribution decrease as δ and θ decrease.
- When the parameters δ and v are fixed, the SK and KS of Harris-OBX-U distribution decrease as β and θ increase.

4.10 Maximum Likelihood Estimation

Let $\Delta = (\delta, v, \beta, \psi)^T$ be the parameter vector from Harris-OBX-G FoD. Then the log-likelihood $\ell = \ell(\Delta)$ of the Harris-OBX-G FoD is given by

$$\begin{aligned} \ell &= n \ln(2b\delta^{\frac{1}{v}}) - \left(\frac{1}{v} + 1\right) \sum_{i=1}^{\infty} \ln \left(1 - \bar{\delta} \left(1 - [1 - \exp(-K(x; \psi))]^{\beta}\right)^v\right) \\ &+ (\beta - 1) \sum_{i=1}^{\infty} \ln [1 - \exp(-K(x; \psi))] - 2 \sum_{i=1}^{\infty} \left(\frac{G(x_i; \psi)}{\bar{G}(x_i; \psi)}\right) \\ &+ \sum_{i=1}^{\infty} \ln(g(x_i; \psi)) - 3 \sum_{i=1}^{\infty} \ln(\bar{G}(x_i; \psi)), \end{aligned}$$

where $K(x; \psi) = \left(\frac{G(x; \psi)}{\bar{G}(x; \psi)}\right)^2$. The ML estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the non-linear equation $\left(\frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial v}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \psi_k}\right)^T = \mathbf{0}$. These partial derivatives and the equations derived from them are not in closed form, but they can be solved using R, MATLAB, and SAS software using an iterative method such as the Newton-Raphson procedure for a given baseline cdf $G(x; \psi)$. The multivariate normal distribution $N_{q+3}(\mathbf{0}, (J(\hat{\Delta}))^{-1})$, where $\mathbf{0}$ is the zero mean vector and $(J(\hat{\Delta}))^{-1}$ is the observed FIM evaluated at $\hat{\Delta}$, can be used to construct CI and confidence regions for the individual model parameters as well as survival function and hrf. The elements of score vector are given in the appendix.

4.11 Simulation Study

In this section, we examine the performance of the Harris-OBX-W distribution. The MCS results for various sample sizes ($n = 25, 50, 100, 200, 400$, and 800) are presented, in order to evaluate the accuracy of the ML estimates. We run the simulations $N=3000$ times and calculate the mean estimates, ABias, and RMSE. The ABias and RMSE, for an estimated parameter $\hat{\eta}$, are given by:

$$ABias(\hat{\eta}) = \frac{\sum_{i=1}^N \hat{\eta}_i}{N} - \eta, \quad \text{and} \quad RMSE(\hat{\eta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\eta}_i - \eta)^2}{N}},$$

respectively. The Equation (4.4) can be used to generate random samples for the simulation study. The QF, which provides the inverse mapping of the cdf, can be utilized to convert uniform random variables into random variables following the Harris-OBX-W distribution. This allows for the generation of random samples necessary for conducting the simulations.

Tables 4.1 and 4.2 shows the results of the MCS study for selected parameter values obtained with the statistical computing software R. The MCS results shows that as sample size, n increases, the mean parameter estimates tend to approach the true parameter values. The RMSE and ABias are observed to decrease as sample size, n increases.

Table 4.1: MCS for Harris-OBX-W Distribution

parameter	n	(0.5,0.5,0.5,1.0)			(2.0,0.7,2.0,0.2)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
δ	25	0.4016	0.2877	-0.0983	3.7262	13.8376	1.7262
	50	0.4184	0.2235	-0.0815	3.4295	6.2526	1.4295
	100	0.4407	0.1497	-0.0592	2.7715	2.9104	0.7715
	200	0.4615	0.1204	-0.0384	2.5315	1.5950	0.5315
	400	0.4735	0.1138	-0.0264	2.2495	0.8030	0.2495
	800	0.4815	0.0939	-0.0184	2.0431	0.2973	0.0431
v	25	0.9819	0.9877	0.4819	0.9389	1.1283	0.2389
	50	0.8791	0.8701	0.3791	0.9017	1.0785	0.2014
	100	0.7893	0.6804	0.2893	0.8857	1.0453	0.1857
	200	0.6500	0.4341	0.1500	0.8769	0.7562	0.1769
	400	0.5869	0.3138	0.0869	0.8035	0.5310	0.1035
	800	0.5383	0.2404	0.0383	0.7164	0.2927	0.0164
β	25	0.7123	0.4938	0.2123	1.8049	1.3061	-0.1950
	50	0.6760	0.3365	0.1760	1.8347	0.9450	-0.1652
	100	0.6280	0.2245	0.1280	1.8513	0.6951	-0.1486
	200	0.6022	0.1676	0.1022	1.8771	0.5145	-0.1228
	400	0.5886	0.1224	0.0886	1.9384	0.3476	-0.0615
	800	0.4987	0.0898	-0.0012	1.9853	0.1904	-0.0146
λ	25	1.5229	1.3085	0.5229	0.2591	0.1202	0.0591
	50	1.3122	0.9347	0.3122	0.2279	0.0678	0.0279
	100	1.2906	0.7413	0.2906	0.2104	0.0443	0.0104
	200	1.2026	0.5175	0.2026	0.2038	0.0335	0.0038
	400	1.1434	0.3781	0.1434	0.2001	0.0232	0.0007
	800	1.0616	0.2337	0.0616	0.1995	0.0183	-0.0005

Table 4.2: MCS Results for Harris-OBX-W Distribution

parameter	n	(1.3,0.5,1.2,1.0)			(0.5,0.5,1.0,1.0)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
δ	25	2.9703	8.1558	1.6703	0.3889	0.2214	-0.1110
	50	2.5723	5.2248	1.2723	0.4178	0.1765	-0.0821
	100	2.1274	2.8450	0.8274	0.4411	0.1438	-0.0588
	200	1.8042	1.5116	0.5042	0.4692	0.1098	-0.0307
	400	1.5771	0.7132	0.2771	0.4821	0.0884	-0.0178
	800	1.3835	0.2580	0.0835	0.4888	0.0774	-0.0111
v	25	1.1318	1.0870	0.6318	1.0706	1.0304	0.5706
	50	0.9883	0.9109	0.4883	0.9408	0.8648	0.4408
	100	0.9200	0.7581	0.4200	0.6671	0.4228	0.1671
	200	0.7813	0.5778	0.2813	0.5844	0.3227	0.0844
	400	0.7063	0.4593	0.2063	0.5480	0.2637	0.0480
	800	0.5428	0.2456	0.0428	0.5296	0.2321	0.0296
β	25	1.5733	2.1885	0.3733	0.7365	1.5659	-0.2635
	50	1.3480	1.6083	0.1480	0.8409	0.5363	-0.1591
	100	1.2704	0.5080	0.0704	0.9578	0.2991	-0.0421
	200	1.2609	0.3572	0.0609	0.9685	0.2281	-0.0314
	400	1.2472	0.2777	0.0472	0.9942	0.1718	-0.0057
	800	1.2018	0.1647	0.0018	0.9971	0.1214	-0.0029
λ	25	1.7213	1.9267	0.7213	1.8199	2.0033	0.8199
	50	1.5717	1.5072	0.5717	1.3945	0.9332	0.3945
	100	1.3757	0.9104	0.3757	1.2611	0.6446	0.2611
	200	1.2424	0.5677	0.2424	1.1491	0.3636	0.1491
	400	1.1455	0.3390	0.1455	1.0651	0.2203	0.0651
	800	1.0663	0.2094	0.0663	1.0195	0.1348	0.0195

4.12 Applications

In this section, we apply the Harris-OBX-W distribution as outlined in Subsection 4.9.2 to two real-world datasets to illustrate its applicability and utility. The `mle()` function in the `stats4` package is used to fit the model parameters via the method of ML estimation by maximizing the log-likelihood. The `AdequacyModel` package is used to assess performance of the model.

The GoF statistics of the Harris-OBX-W distribution are compared to the nested and several non-nested distributions. The parameter estimates (SE in parentheses) are provided. We use the following GoF statistics: $(-2\ln(L))$, AIC, BIC and CAIC. Some additional useful GoF statistics are: CVM, AD, K-S, its associated PV, and SS ([24]) from the PPs. Chen and Balakrishnan [26] discussed CVM and AD GoF statistics. The model with the smallest GoF statistics value and the highest PV is considered the best fitting model.

The LRT statistic is used for assessing the Harris-OBX-W distribution against its sub-models. The null hypothesis is rejected if the value of the LRT statistic is greater than the critical value of the χ^2 distribution, with the appropriate degrees of freedom.

The TTT scaled plots was presented. We also present plots of the fitted pdfs, ECDF plots, PPs, K-M survival plots, and hrf plots to compare the flexibility of the Harris-OBX-W distribution to competing models.

The Harris-OBX-W distribution is compared to the following non-nested models: the Harris-EW

distribution by Pinho et al. [111] with pdf given as,

$$f(x; \theta, k, \beta, \lambda) = \frac{\theta^{\frac{1}{k}} \beta \lambda (\lambda x)^{\beta-1} \exp(-(\lambda x)^\beta)}{[1 - \bar{\theta} \exp(-k(\lambda x)^\beta)]^{\frac{1}{k+1}}},$$

for $\theta, k, \beta, \lambda > 0$ and $x > 0$, the Harris-EL distribution by Pinho et al. [111] with pdf given as,

$$f(x; \theta, k, \beta, \gamma) = \frac{\theta^{\frac{1}{k}} \beta \gamma (1 + \beta x)^{k\gamma-1}}{[(1 + \beta x)^{k\gamma} - \bar{\theta}]^{\frac{1}{k+1}}},$$

for $\theta, k, \beta, \gamma > 0$ and $x > 0$, the TL-MO-W distribution by Ahmed et al. [4] with pdf given as,

$$f(x; b, \alpha, \lambda, \beta) = \frac{2b\alpha^2 \beta \lambda^\beta x^{\beta-1} \exp(-2(\lambda x)^\beta)}{(1 - \bar{\alpha} \exp(-(\lambda x)^\beta))^3} \left(1 - \left(1 - \bar{\alpha} \exp(-(\lambda x)^\beta)\right)^2\right)^{b-1}, \quad (4.19)$$

for $b, \alpha, \lambda, \beta > 0$ and $x > 0$, the TL-GW distribution by Aryal et al. [17] with pdf given as,

$$\begin{aligned} f(x; \alpha, \theta, \lambda, \beta) &= 2\alpha\theta\beta\lambda^\beta x^{\beta-1} \exp(-(\lambda x)^\beta) (1 - \exp(-(\lambda x)^\beta))^{\theta\alpha-1} \left[1 - (1 - \exp(-(\lambda x)^\beta))^\theta\right]^\theta \\ &\times \left[2 - (1 - \exp(-(\lambda x)^\beta))^\theta\right]^{\alpha-1}, \end{aligned} \quad (4.20)$$

for $\alpha, \theta, \lambda, \beta > 0$ and $x > 0$, and the MO-OW-LLoG distribution by Makubate et al. [68] with pdf given as,

$$\begin{aligned} f(x; \beta, \delta, \lambda, c) &= \frac{\beta \lambda \delta c x^{c-1} (1 + x^c)^{-2} (1 - (1 + x^c)^{-1})^{\beta-1}}{(1 + x^c)^{-(\beta+1)}} \\ &\times \frac{\exp\left(-\lambda \left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]^\beta\right)}{1 - \left[(1 - \delta) \exp\left(-\lambda \left[\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right]^2\right)\right]}, \end{aligned}$$

for $\beta, \delta, \lambda, c > 0$ and $x > 0$.

4.12.1 Bladder Cancer Remission Data

The Harris-OBX-W distribution was fitted the remission times (months) of 128 individuals diagnosed with bladder cancer. Chamunorwa et al. [25] used this data. The observations are as follows: 0.08, 4.98, 25.74, 3.7, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.5, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.4, 11.98, 6.54, 6.93, 6.94, 0.4, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26, 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.5, 2.02, 0.2, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26, 4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.9, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07. The data is skewed to the right, with a few outliers at the high end.

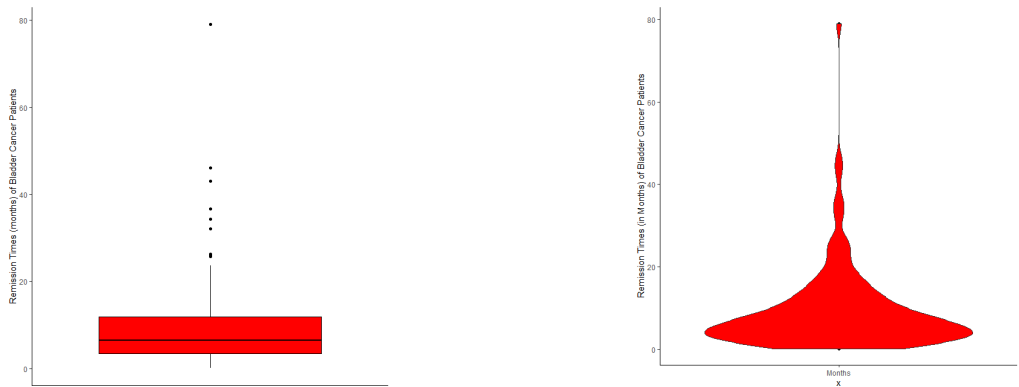


Figure 4.10: Boxplot and Violin Plots of Bladder Cancer Remission Data

The estimated Var-Cov matrix is

$$\begin{pmatrix} 0.6478 & 0.3744 & 2.5827 & -0.0030 \\ 0.3744 & 0.2888 & 1.4937 & -0.0031 \\ 2.5827 & 1.4937 & 10.2970 & -0.0212 \\ -0.0030 & -0.0031 & -0.0122 & 0.00004 \end{pmatrix},$$

and the approximate 95% CI for δ, v, β and λ are given by 25.5265 ± 1.5775 , 1.7000 ± 1.0527 , 11.6932 ± 6.2894 and 0.0813 ± 0.0137 , respectively. Table 4.3 gives the parameter estimates of the models and their SE in parenthesis.

Table 4.3: Estimates for Bladder Cancer Remission Data

Model	Estimates			
	δ	v	β	λ
$Harris - OBX - W(\delta, v, \beta, \lambda)$	25.5265 (0.8078)	1.7000 (0.5371)	11.6932 (3.2088)	0.0813 (0.0069)
$Harris - OBX - W(1, v, \beta, \lambda)$	- (5.7259 $\times 10^{-13}$)	0.0198 (5.1680)	50.4305 (0.0033)	0.0705 (0.0033)
$Harris - OBX - W(\delta, 1, \beta, \lambda)$	0.0055 (4.5032 $\times 10^{-08}$)	- (0.0071)	0.0351 (0.0040)	0.0938 (0.0040)
$Harris - OBX - W(\delta, v, 1, \lambda)$	11.0089 (11.0089)	0.0848 (0.0848)	- (0.1433)	0.1433 (0.1433)
$TL - GW(\alpha, \theta, \lambda, \beta)$	900.4500 (9.6316 $\times 10^{-06}$)	0.0154 (0.0062)	0.0325 (0.0096)	1.0810 (0.3416)
$MO - OW - LLoG(\beta, \lambda, \delta, c)$	0.0362 (0.0034)	1.4222 (0.1327)	28.9037 (0.0031)	12.7954 (0.7570)
$TL - MO - W(b, \delta, \gamma, \lambda)$	3.2159 (1.1027)	3.1049 $\times 10^{03}$ (7.6784 $\times 10^{-05}$)	0.1137 (0.0168)	6.431 (0.4241)
$Harris - EL(\theta, \beta, k, \gamma)$	2.6414 $\times 10^{07}$ (1.4754 $\times 10^{-09}$)	1.7064 $\times 10^{05}$ (2.6316 $\times 10^{-07}$)	0.2862 (0.0350)	3.9266 (0.4455)
$Harris - EW(\theta, k, \beta, \lambda)$	0.0085 (0.1059)	0.0351 (0.3989)	1.0004 (0.2036)	0.0009 (0.0276)

The Harris-OBX-W distribution has the smallest values of all the GoF statistics, indicating that it fits the bladder cancer remission data better than the nested and non-nested models used for comparison, as shown in Table 4.4. The PV of the K-S statistic is also the highest, and the SS value from the PPs is the lowest for the Harris-OBX-W distribution. This suggest that it outperforms the nested and non-nested models.

Table 4.4: GoF Statistics for Bladder Cancer Remission Data

Model	Statistics								
	$-2\log L$	AIC	$CAIC$	BIC	CVM	AD	$K-S$	PV	SS
$Harris-OBX-W(\delta, v, \beta, \lambda)$	818.4059	826.4059	826.7311	837.8140	0.0128	0.0795	0.0297	0.9999	0.0123
$Harris-OBX-W(1, v, \beta, \lambda)$	835.2582	841.2582	814.8143	844.7346	0.1782	1.1789	0.0798	0.3876	0.2196
$Harris-OBX-W(\delta, 1, \beta, \lambda)$	824.7552	830.7552	830.9487	839.3113	0.0913	0.5386	0.0524	0.8731	0.0551
$Harris-OBX-W(\delta, v, 1, \lambda)$	875.1950	881.1950	881.3898	889.7524	0.2827	1.6761	0.2440	4.8060×10^{-07}	1.9496
$TL-GW(\alpha, \theta, \lambda, \beta)$	824.1344	832.1344	832.4596	843.5425	0.0695	0.4664	0.0535	0.8568	0.0655
$MO-OW-LLoG(\beta, \lambda, \delta, c)$	826.5262	834.5662	834.8914	845.9743	0.1182	0.6938	0.0666	0.6199	0.0777
$TL-MO-W(b, \delta, \gamma, \lambda)$	822.8106	830.8401	831.1653	842.2482	0.0446	0.3196	0.0471	0.9389	2.0032
$Harris-EL(\theta, \beta, k, \gamma)$	824.1733	832.1733	832.4985	843.5814	0.0348	0.2196	0.0836	0.3326	0.1500
$Harris-EW(\theta, k, \beta, \lambda)$	827.9063	835.9063	836.2315	847.3145	0.0983	0.5919	0.0909	0.2400	0.2062

Figure 4.11 shows the fitted pdf and PPs for the bladder cancer remission dataset. These fitted plots support the GoF statistics in Table 4.4, indicating that the Harris-OBX-W distribution outperforms other models fitted to the bladder cancer remission dataset.

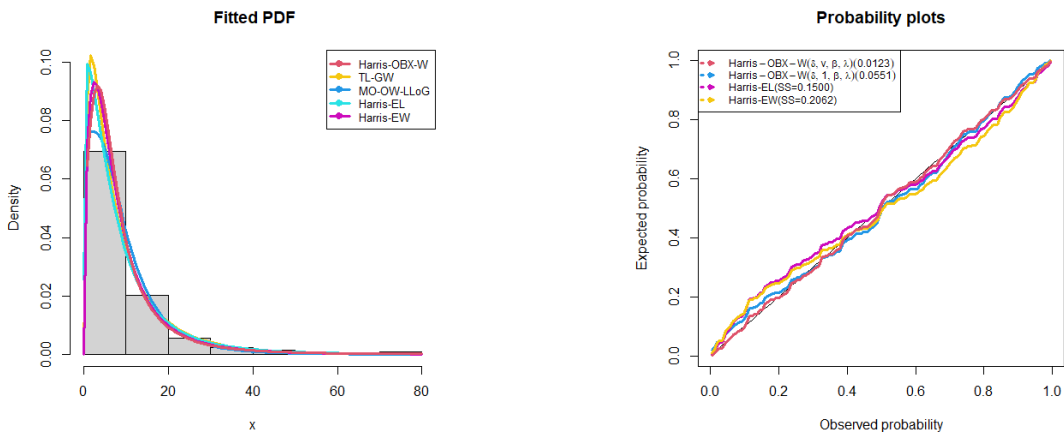


Figure 4.11: Fitted Densities Plots and PPs for Remission Times of Cancer Patients Data

Figure 4.12 illustrates the TTT scaled plot with an UBT hrf, indicating the suitability of the Harris-OBX-W distribution for fitting the bladder cancer remission dataset.

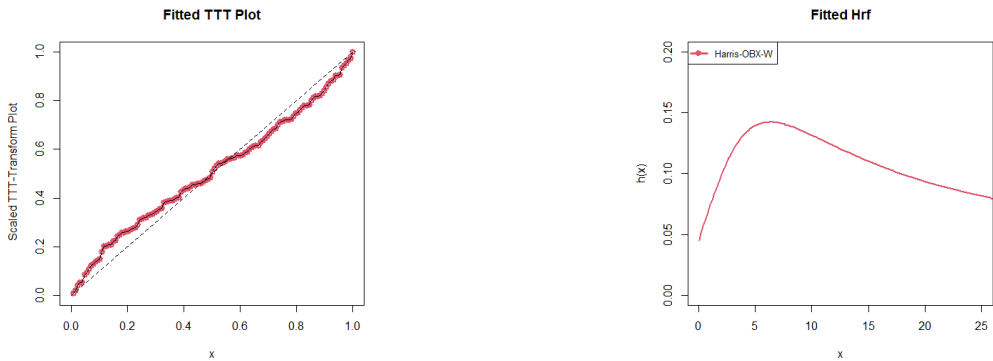


Figure 4.12: Fitted TTT and Hrf Plots for Bladder Cancer Remission Data

The K-M survival and ECDF curves for the bladder cancer remission dataset are shown in Figure 4.13. We conclude that our model outperforms the other models since the empirical and fitted K-M survival curves are close to each other.

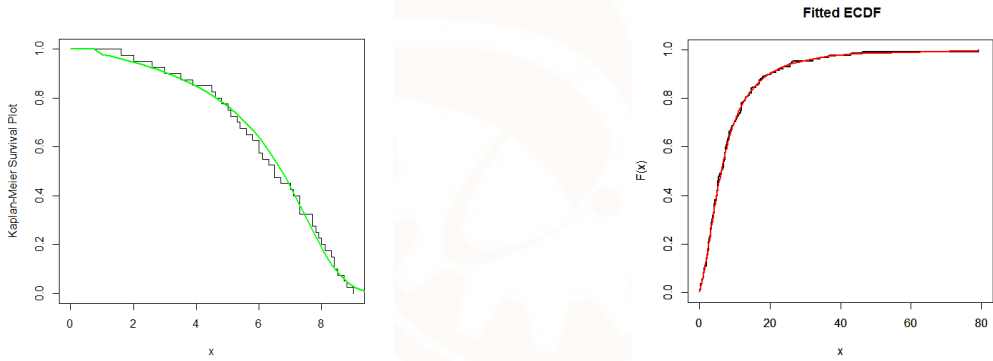


Figure 4.13: Fitted K-M Survival and ECDF Plots for Bladder Cancer Remission Data

We also conducted the LRT in Table 4.5 to compare the performance of Harris-OBX-W distribution with the nested models.

Table 4.5: LRT For Bladder Cancer Remission Data

Model	df	$\chi^2(p - value)$
$Harris - OBX - W(1, v, \beta, \lambda)$	1	16.8523 (0.00004)
$Harris - OBX - W(\delta, 1, \beta, \lambda)$	1	6.3493 (0.011741)
$Harris - OBX - W(\delta, v, 1, \lambda)$	1	56.7891 (< 0.00001)

According to the LRT results in Table 4.5, we conclude that there are significant differences between the nested models and the Harris-OBX-W distribution. Consequently, the Harris-OBX-W distribution outperforms its nested models at 5% level of significance. The profile

log-likelihood function of the ML estimates of δ, v, β , and λ on bladder cancer remission dataset are presented in Figure 4.14.

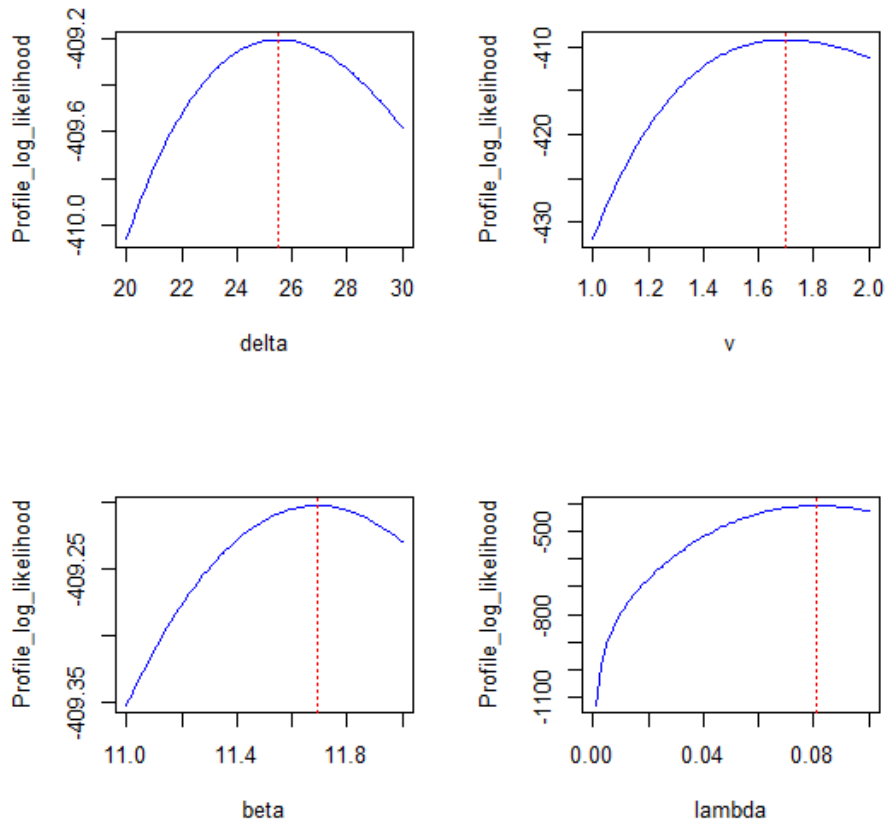


Figure 4.14: Plots of the Profile Log-Likelihood Function of the Parameters of the Harris-OBX-W Distribution on Bladder Cancer Remission Data

In Figure 4.14, we observe that the ML estimates for the Harris-OBX-W distribution exists and can be obtained uniquely on bladder cancer remission data.

4.12.2 Guinea Pigs Data

In the final dataset below, 72 guinea pigs exposed to virulent tubercle bacilli are given the survival times (measured in days). These dataset was obtained from Anabike et al. [16]. The data are as follows: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376. The data is skewed to the right with outliers.

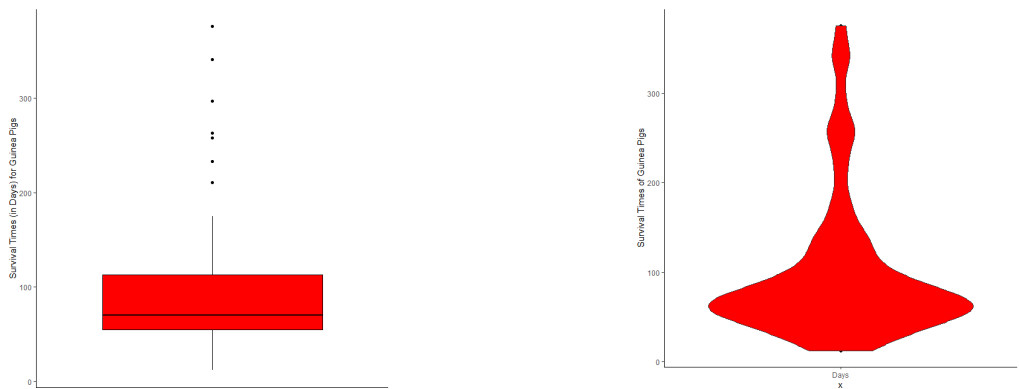


Figure 4.15: Boxplot and Violin Plots of Guinea Pigs Data

The estimated Var-Cov matrix is given by

$$\begin{pmatrix} 2.5251 \times 10^{-04} & -0.0190 & 5.3609 \times 10^{-05} & 5.2375 \times 10^{-05} \\ -0.0190 & 1.4414 & -0.0040 & -0.0039 \\ 5.3609 \times 10^{-05} & -0.0040 & 1.1381 \times 10^{-05} & 1.1119 \times 10^{-05} \\ 5.2375 \times 10^{-05} & -0.0039 & 1.1119 \times 10^{-05} & 1.2072 \times 10^{-05} \end{pmatrix},$$

and the approximate 95% CI for δ, v, β , and λ are given by 20.3150 ± 0.0310 , 3.0110 ± 2.3532 , 325.0200 ± 0.0066 and 0.0627 ± 0.0068 , respectively. In Table 4.6, the estimated parameters are accompanied by their SE in parenthesis.

Table 4.6: Estimates for Guinea Pigs Data

Model	Estimates			
	δ	v	β	λ
$Harris - OBX - W(\delta, v, \beta, \lambda)$	20.3150 (0.01589)	3.0110 (1.2006)	325.0200 (0.0033)	0.0627 (0.0034)
$Harris - OBX - W(1, v, \beta, \lambda)$	- (4.0644 $\times 10^{-19}$)	0.0198 (1.3033 $\times 10^{-08}$)	885.9500 (0.0012)	0.0614 (0.0012)
$Harris - OBX - W(\delta, 1, \beta, \lambda)$	8.4139 $\times 10^4$ (3.5580 $\times 10^{-08}$)	- (0.0374)	0.0799 (0.0030)	0.0737 (0.0030)
$Harris - OBX - W(\delta, v, 1, \lambda)$	47.2859 (9.3815)(0.0682)	0.1234 -	- (0.0136)	0.1056 (0.0136)
$TL - GW(\alpha, \theta, \lambda, \beta)$	900.5600 (8.9988 $\times 10^{-06}$)	0.0746 (0.0765)	0.01693 (0.0240)	0.5825 (0.3061)
$MO - OW - LLoG(\beta, \lambda, \delta, c)$	0.0578 (0.0055)	0.6155 (0.1309)	99.2290 (0.0004)	8.0155 (1.7627 $\times 10^{-05}$)
$TL - MO - W(b, \delta, \gamma, \lambda)$	4.5534 (3.0599)	3.0435 $\times 10^5$ (1.6793 $\times 10^{-06}$)	0.0966 (0.0201)	8.6491 (1.0744)
$Harris - EL(\theta, \beta, k, \gamma)$	1.1739 $\times 10^7$ (6.5472 $\times 10^{-06}$)	29.1430 (15.4530)	0.7776 (0.1131)	2.6554 (0.3256)
$Harris - EW(\theta, k, \beta, \lambda)$	371.6700 (2.5740 $\times 10^{-06}$)	94.4230 (7.9918 $\times 10^{-05}$)	1.1138 (0.0918)	0.0098 (0.0010)

According to Table 4.7, the Harris-OBX-W distribution has the largest PV for the K-S statistic and the smallest values of the GoF statistics. Thus, in the case of the guinea pigs dataset, the Harris-OBX-W distribution stands out as a better fit compared to other distributions examined.

Table 4.7: GoF Statistics for Guinea Pigs Data

Model	Statistics								
	$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	$K-S$	$P-VALUE$	SS
$Harris-OBX-W(\delta, v, \beta, \lambda)$	777.5322	785.5322	786.1293	794.6389	0.0777	0.4572	0.0768	0.7894	0.0742
$Harris-OBX-W(1, v, \beta, \lambda)$	782.1969	788.1969	788.5498	795.0269	0.1345	0.7716	0.1207	0.2446	0.1369
$Harris-OBX-W(\delta, 1, \beta, \lambda)$	786.0623	792.0623	792.4153	798.8923	0.2511	1.3959	0.0968	0.5086	0.1765
$Harris-OBX-W(\delta, v, 1, \lambda)$	825.5378	831.5378	831.8913	838.3683	0.6025	3.3371	0.2851	0.0001	23.9441
$TL-GW(\alpha, \theta, \lambda, \beta)$	780.3251	788.3251	788.9222	797.4318	0.1446	0.7883	0.1001	0.4655	0.1371
$MO-OW-LLoG(\beta, \lambda, \delta, c)$	791.4526	799.4526	800.0497	808.5593	0.3331	1.8615	0.1201	0.2494	0.2477
$TL-MO-W(b, \delta, \gamma, \lambda)$	779.8760	787.8760	788.4731	796.9827	0.1200	0.6665	0.0934	0.5554	0.1127
$Harris-EL(\theta, \beta, k, \gamma)$	782.4166	790.4166	791.1093	799.6189	0.1569	0.8629	0.1025	0.4347	0.1715
$Harris-EW(\theta, k, \beta, \lambda)$	791.5002	799.5002	800.0972	808.6069	0.3086	1.6661	0.1415	0.1118	0.4122

The smaller SS value observed in Figure 4.16 suggests that the Harris-OBX-W distribution provides a better fit for the guinea pigs data compared to both the nested and non-nested models.

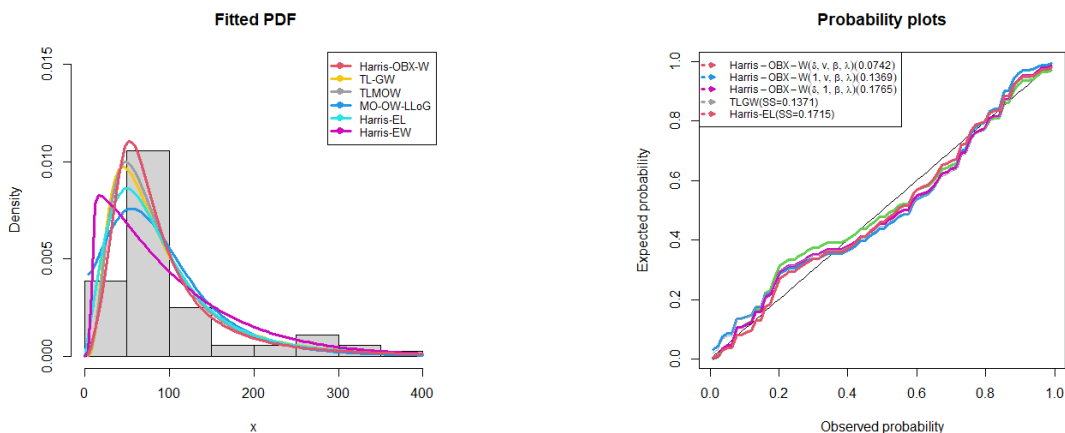


Figure 4.16: Fitted Densities Plots and PPs for Guinea Pigs Data

The TTT scaled plot with an UBT hrf, as depicted in Figure 4.17, illustrates the suitability of the Harris-OBX-W distribution for fitting guinea pigs data.

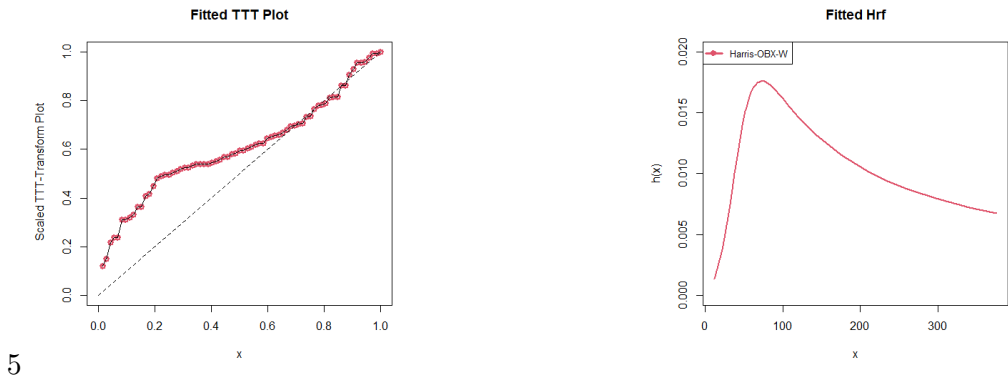


Figure 4.17: Fitted TTT and Hrf Plots for Guinea Pigs Data

Figure 4.18 displays the K-M survival and ECDF curves for the guinea pigs data. We draw the conclusion that the Harris-OBX-W distribution outperforms the other models, as evidenced by the close alignment between the observed and fitted K-M survival curves.

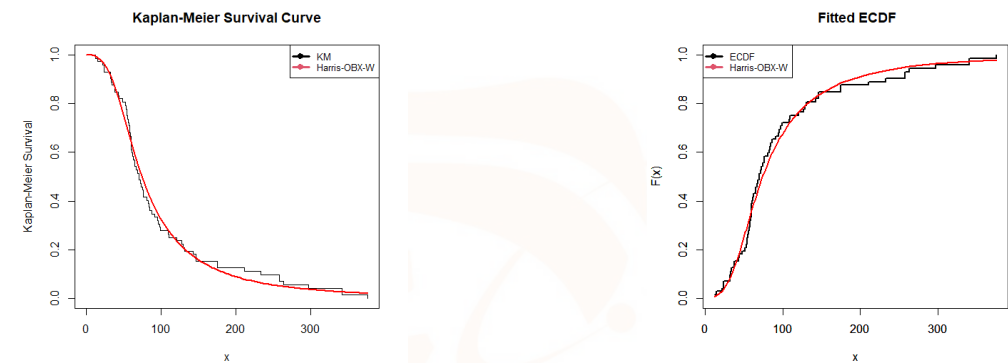


Figure 4.18: Fitted K-M Survival and ECDF Plots for Guinea Pigs Data

For the guinea pigs dataset, we run the LRT to assess how well the Harris-OBX-W distribution and its nested models fit the data.

Table 4.8: LRT Results for Guinea Pigs Data

	Model	df	$\chi^2(p - value)$
	$Harris - OBX - W(1, v, \beta, \lambda)$	1	4.6647 (0.030789)
	$Harris - OBX - W(\delta, 1, \beta, \lambda)$	1	8.5301 (0.003493)
	$Harris - OBX - W(\delta, v, 1, \lambda)$	1	48.0056 (< 0.00001)

From Table 4.8, we conclude that there are significant differences between Harris-OBX-W $(1, v, \beta, \lambda)$ and Harris-OBX-W distributions, Harris-OBX-W $(\delta, 1, \beta, \lambda)$ and Harris-OBX-W distributions as well as between Harris-OBX-W $(\delta, v, 1, \lambda)$ and Harris-OBX-W distributions at the 5% level of significance. The profile log-likelihood function for the ML estimates of δ, v, β , and λ for the guinea pigs data is shown in Figure 4.19.

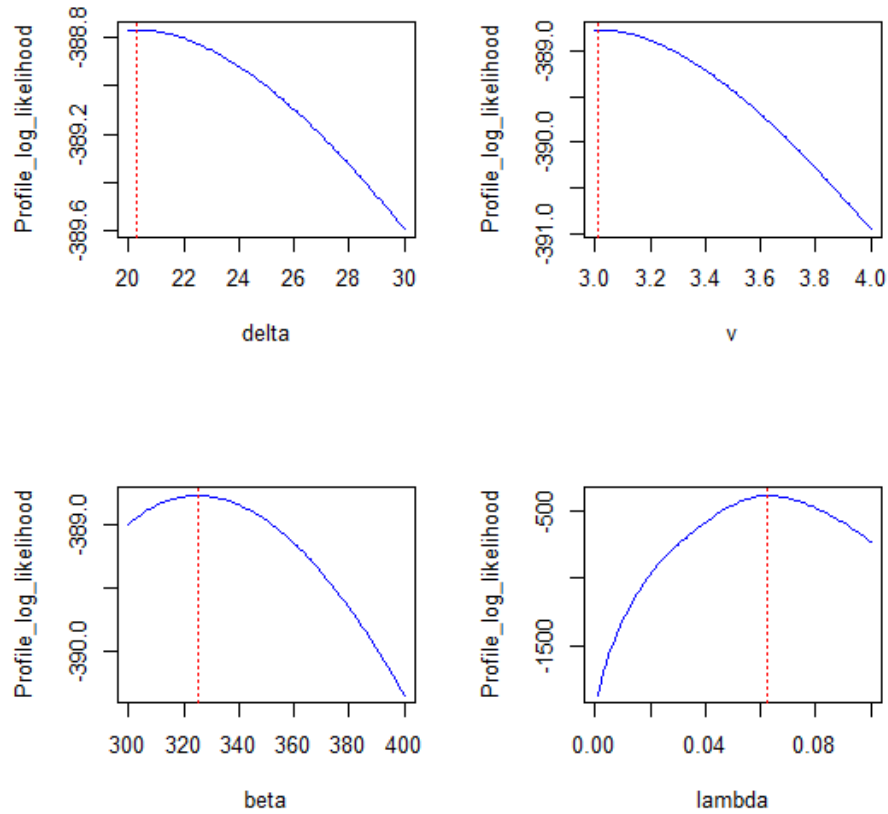


Figure 4.19: Plots of the Profile Log-Likelihood Function of the Parameters of the Harris-OBX-W Distribution on Guinea Pigs Data

Figure 4.19 shows that the ML estimates for the Harris-OBX-W distribution from guinea pigs data exist and can be achieved uniquely.

4.13 Conclusions

This chapter examines the Harris-OBX-G FoD in detail. Some statistical and mathematical properties of the new FoD are obtained. The ML estimation method was used to estimate and evaluate the unknown parameters of the new FoD. Among the three special cases of the Harris-OBX-G FoD, we applied the Harris-OBX-W distribution to MCS studies and applications. Two real-world datasets are used to assess the utility and adaptability of the Harris-OBX-W distribution. According to the simulation results, the ML estimation method performs admirably in estimating the parameters of the Harris-OBX-W distribution. Our applications also demonstrate that the Harris-OBX-W distribution is more efficient than other models in capturing the non-monotonic hrf of the datasets. These results demonstrate the potential of the Harris-OBX-W distribution for modeling real-world survival data and its applicability in medical research.

5. The Marshall-Olkin-Exponentiated Half Logistic-Type II Topp-Leone-G Family of Distributions with Applications

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5.1 Introduction

The Marshall-Olkin method have been widely used in modeling various real-life situations due to its ability to add flexibility by incorporating additional parameters. We introduce a new family of flexible distributions to model various lifetime datasets, based on the generalization proposed by Marshall and Olkin [74] and adopting the EHL-TII-TL-G FoD as the baseline distribution. This generalization makes the distribution more flexible, especially when analysing the tail properties, and it is also useful in improving the GoF. The primary motivation lies in the new FoD's capability to accommodate both monotonic and non-monotonic hrf in practical applications.

The mathematical and statistical properties of MO-EHL-TII-TL-G FoD are explored. Statistical properties of the new FoD considered are the QF, moments and MGF, distribution of the order statistics and RE. The ML estimation technique is used for estimating the model parameters and MCS experiments are carried out to evaluate the performance of the ML estimators. Finally, we give examples of real-life data to demonstrate the usefulness and suitability of the MO-EHL-TII-TL-G FoD.

Existing distributions often exhibit limitations in their ability to adequately capture the complexity of real-world data due to their constrained parametrization. The proposed MO-EHL-TII-TL-G FoD offers enhanced flexibility for accurate representation of diverse datasets, particularly those with long- tailed, heavy-tailed, short-tailed, as well as non-monotonic hrf.

5.2 The Model, Sub-Families and Some Statistical Properties

This section is concerned with some statistical properties of the MO-EHL-TII-TL-G FoD including sub-families, hrf, QF, density expansion, moments, MGF, order statistics and RE.

5.2.1 The Model

The cdf and pdf of the EHL-TII-TL-G FoD are given by

$$F(x; \alpha, b, \xi) = \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b} \right)^\alpha \quad (5.1)$$

and

$$f(x; \alpha, b, \xi) = \frac{4\alpha b g(x; \xi) G(x; \xi) [1 - G^2(x; \xi)]^{b-1} \left(1 - [1 - G^2(x; \xi)]^b\right)^{\alpha-1}}{\left(1 + [1 - G^2(x; \xi)]^b\right)^{\alpha+1}}, \quad (5.2)$$

respectively, for $\alpha, b > 0$, $x \geq 0$, and parameter vector ξ .

Considering the cdf given in Equation (5.1) and the pdf given in Equation (5.2), and inserting them into Equations (1.23) and (1.24), we obtain the MO-EHL-TII-TL-G FoD with the cdf given by

$$F(x; \delta, \alpha, b, \xi) = 1 - \frac{\delta \left[1 - \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b}\right)^\alpha\right]}{1 - \bar{\delta} \left[1 - \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b}\right)^\alpha\right]}, \quad (5.3)$$

for $\delta, \alpha, b > 0$, $x \geq 0$, and parameter vector ξ . The corresponding pdf is given by

$$f(x; \delta, \alpha, b, \xi) = \frac{4\delta\alpha b g(x; \xi) G(x; \xi) [1 - G^2(x; \xi)]^{b-1} \left(1 - [1 - G^2(x; \xi)]^b\right)^{\alpha-1}}{\left(1 + [1 - G^2(x; \xi)]^b\right)^{\alpha+1} \left(1 - \bar{\delta} \left[1 - \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b}\right)^\alpha\right]\right)^2}. \quad (5.4)$$

The hrf is given by

$$\begin{aligned} h_F(x) &= \frac{4\alpha b g(x; \xi) G(x; \xi) [1 - G^2(x; \xi)]^{b-1} \left(1 - [1 - G^2(x; \xi)]^b\right)^{\alpha-1}}{\left(1 + [1 - G^2(x; \xi)]^b\right)^{\alpha+1} \left(1 - \bar{\delta} \left[1 - \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b}\right)^\alpha\right]\right)} \\ &\times \left[1 - \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b}\right)^\alpha\right]^{-1}, \end{aligned} \quad (5.5)$$

for $\delta, \alpha, b > 0$, $x \geq 0$, $\bar{\delta} = 1 - \delta$ and parameter vector ξ .

5.2.2 Identifiability

We demonstrate the identifiability property for the MO-EHL-TII-TL-G FoD. We consider $\Delta_1 = (\delta_1, \alpha_1, b_1, \xi_1)$ and $\Delta_2 = (\delta_2, \alpha_2, b_2, \xi_2)$. The likelihood ratio is

$$\begin{aligned}
\frac{f_1(\Delta_1)}{f_2(\Delta_2)} &= \left(\frac{\delta_1 \alpha_1 b_1}{\delta_2 \alpha_2 b_2} \right) [1 - G^2(x; \xi_1)]^{b_1-1} [1 - G^2(x; \xi_2)]^{-(b_2-1)} \left(1 - [1 - G^2(x; \xi_1)]^{b_1} \right)^{\alpha_1-1} \\
&\times \left(1 - [1 - G^2(x; \xi_2)]^{b_2} \right)^{-(\alpha_2-1)} \left(1 + [1 - G^2(x; \xi_1)]^{b_1} \right)^{-(\alpha_1+1)} \\
&\times \left(1 + [1 - G^2(x; \xi_2)]^{b_2} \right)^{\alpha_2+1} \left(1 - \bar{\delta}_1 \left[1 - \left(\frac{1 - [1 - G^2(x; \xi_1)]^{b_1}}{1 + [1 - G^2(x; \xi_1)]^{b_1}} \right)^{\alpha_1} \right] \right)^{-2} \\
&\times \left(1 - \bar{\delta}_2 \left[1 - \left(\frac{1 - [1 - G^2(x; \xi_2)]^{b_2}}{1 + [1 - G^2(x; \xi_2)]^{b_2}} \right)^{\alpha_2} \right] \right)^2 G(x; \xi_1) g(x; \xi_1) \\
&\times (G(x; \xi_2))^{-1} (g(x; \xi_2))^{-1}.
\end{aligned}$$

The log-likelihood ratio is

$$\begin{aligned}
\ln \left(\frac{f_1(\Delta_1)}{f_2(\Delta_2)} \right) &= \ln \left(\frac{\delta_1 \alpha_1 b_1}{\delta_2 \alpha_2 b_2} \right) - 2 \ln \left(1 - \bar{\delta}_1 \left[1 - \left(\frac{1 - [1 - G^2(x; \xi_1)]^{b_1}}{1 + [1 - G^2(x; \xi_1)]^{b_1}} \right)^{\alpha_1} \right] \right) \\
&+ 2 \ln \left(1 - \bar{\delta}_2 \left[1 - \left(\frac{1 - [1 - G^2(x; \xi_2)]^{b_2}}{1 + [1 - G^2(x; \xi_2)]^{b_2}} \right)^{\alpha_2} \right] \right) + (b_1 - 1) \ln [1 - G^2(x; \xi_1)] \\
&- (b_2 - 1) \ln [1 - G^2(x; \xi_2)] - (\alpha_1 + 1) \ln [1 + (1 - G^2(x; \xi_1))^{b_1}] \\
&+ (\alpha_2 + 1) \ln [1 + (1 - G^2(x; \xi_2))^{b_2}] + (\alpha_1 - 1) \ln [1 - (1 - G^2(x; \xi_1))^{b_1}] \\
&- (\alpha_2 - 1) \ln [1 - (1 - G^2(x; \xi_2))^{b_2}] + \ln G(x; \xi_1) + \ln g(x; \xi_1) \\
&- \ln G(x; \xi_2) + \ln g(x; \xi_2).
\end{aligned}$$

Take the derivative of the log-likelihood ratio, we get

$$\begin{aligned}
 \frac{d}{dx} \left[\ln \left(\frac{f_1(\Delta_1)}{f_2(\Delta_2)} \right) \right] &= -(b_1 - 1) \frac{2G(x; \xi_1)g(x; \xi_1)}{[1 - G^2(x; \xi_1)]} + (b_2 - 1) \frac{2G(x; \xi_2)g(x; \xi_2)}{[1 - G^2(x; \xi_2)]} \\
 &\quad - \frac{2\bar{\delta}_1 K_1}{\left(1 - \bar{\delta}_1 \left[1 - \left(\frac{1 - [1 - G^2(x; \xi_1)]^{b_1}}{1 + [1 - G^2(x; \xi_1)]^{b_1}}\right)^{\alpha_1}\right]\right)} + \frac{2\bar{\delta}_2 K_2}{\left(1 - \bar{\delta}_2 \left[1 - \left(\frac{1 - [1 - G^2(x; \xi_2)]^{b_2}}{1 + [1 - G^2(x; \xi_2)]^{b_2}}\right)^{\alpha_2}\right]\right)} \\
 &\quad + (\alpha_1 - 1) \frac{2b_1 [1 - G^2(x; \xi_1)]^{b_1 - 1} G(x; \xi_1)g(x; \xi_1)}{\left(1 - [1 - G^2(x; \xi_1)]^{b_1}\right)} \\
 &\quad - (\alpha_2 - 1) \frac{2b_2 [1 - G^2(x; \xi_2)]^{b_2 - 1} G(x; \xi_2)g(x; \xi_2)}{\left(1 - [1 - G^2(x; \xi_2)]^{b_2}\right)} \\
 &\quad - (\alpha_2 + 1) \frac{2b_2 [1 - G^2(x; \xi_2)]^{b_2 - 1} G(x; \xi_2)g(x; \xi_2)}{\left(1 + [1 - G^2(x; \xi_2)]^{b_2}\right)} \\
 &\quad + (\alpha_1 + 1) \frac{2b_1 [1 - G^2(x; \xi_1)]^{b_1 - 1} G(x; \xi_1)g(x; \xi_1)}{\left(1 + [1 - G^2(x; \xi_1)]^{b_1}\right)} \\
 &\quad + \frac{g(x; \xi_1)}{G(x; \xi_1)} + \frac{g'(x; \xi_1)}{g(x; \xi_1)} - \frac{g(x; \xi_2)}{G(x; \xi_2)} - \frac{g'(x; \xi_2)}{g(x; \xi_2)},
 \end{aligned} \tag{5.6}$$

where

$$K_1 = \frac{4\alpha_1 b_1 g(x; \xi_1) G(x; \xi_1) [1 - G^2(x; \xi_1)]^{b_1 - 1} \left(1 - [1 - G^2(x; \xi_1)]^{b_1}\right)^{\alpha_1 - 1}}{\left(1 + [1 - G^2(x; \xi_1)]^{b_1}\right)^{\alpha_1 + 1}},$$

and

$$K_2 = \frac{4\alpha_2 b_2 g(x; \xi_2) G(x; \xi_2) [1 - G^2(x; \xi_2)]^{b_2 - 1} \left(1 - [1 - G^2(x; \xi_2)]^{b_2}\right)^{\alpha_2 - 1}}{\left(1 + [1 - G^2(x; \xi_2)]^{b_2}\right)^{\alpha_2 + 1}}.$$

When $\frac{d}{dx} \left[\ln \left(\frac{f_1(\Delta_1)}{f_2(\Delta_2)} \right) \right] = 0$, the parameters of the MO-EHL-TII-TL-G FoD exist and can be uniquely determined, if $\Delta_1 = \Delta_2$.

5.2.3 Sub-Families of MO-EHL-TII-TL-G Family of Distributions

This subsection introduces some new and well-known sub-families of the MO-EHL-TII-TL-G FoD.

- When $\delta = 1$, we obtain the baseline EHL-TII-TL-G FoD with the cdf given in Equation (5.1).
- When $\alpha = 1$, we obtain MO-HL-TII-TL-G FoD with the cdf

$$F(x; \delta, b, \xi) = 1 - \frac{\delta \left[1 - \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b} \right) \right]}{1 - \bar{\delta} \left[1 - \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b} \right) \right]},$$

for $\delta, b > 0$, $x \geq 0$, and parameter vector ξ .

- When $b = 1$, we obtain the new FoD with the cdf

$$F(x; \delta, \alpha, \xi) = 1 - \frac{\delta \left[1 - \left(\frac{G^2(x; \xi)}{1 + [1 - G^2(x; \xi)]} \right)^\alpha \right]}{1 - \bar{\delta} \left[1 - \left(\frac{G^2(x; \xi)}{1 + [1 - G^2(x; \xi)]} \right)^\alpha \right]},$$

for $\delta, \alpha > 0$, $x \geq 0$, and parameter vector ξ .

- When $\alpha = b = 1$, $x \geq 0$, we obtain the new FoD with the cdf

$$F(x; \delta, \xi) = 1 - \frac{\delta \left[1 - \left(\frac{G^2(x; \xi)}{1 + [1 - G^2(x; \xi)]} \right) \right]}{1 - \bar{\delta} \left[1 - \left(\frac{G^2(x; \xi)}{1 + [1 - G^2(x; \xi)]} \right) \right]},$$

for $\delta > 0$, $x \geq 0$, and parameter vector ξ .

- When $\delta = \alpha = 1$, we obtain the new HL-TII-TL-G FoD with the cdf

$$F(x; b, \xi) = \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b} \right),$$

for $b > 0$, $x \geq 0$, and parameter vector ξ .

- When $\delta = b = 1$, we obtain the new FoD with the cdf

$$F(x; \alpha, \xi) = \left(\frac{G^2(x; \xi)}{1 + [1 - G^2(x; \xi)]} \right)^\alpha,$$

for $\alpha > 0$, $x \geq 0$, and parameter vector ξ .

- When $\delta = \alpha = b = 1$, we obtain the new FoD with the cdf

$$F(x; \xi) = \left(\frac{G^2(x; \xi)}{1 + [1 - G^2(x; \xi)]} \right),$$

for $x \geq 0$, and parameter vector ξ .

5.2.4 Quantile Function

Let X be a random variable with cdf defined by Equation (5.3). The QF ($Q_X(u)$) is defined by $F_X(Q_X(u)) = u, 0 \leq u \leq 1$. Assuming,

$$A = \frac{1 - u}{(\delta + (1 - u)\bar{\delta})},$$

then the QF of the MO-EHL-TII-TL-G FoD is given by,

$$Q_X(u) = G^{-1} \left[\left(1 - \left(\frac{1 - (1 - A)^{\frac{1}{\alpha}}}{1 + (1 - A)^{\frac{1}{\alpha}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{2}} \right]. \quad (5.7)$$

It follows therefore that random numbers can be generated from the MO-EHL-TII-TL-G FoD using Equation (5.7), for specified baseline cdf G .

5.2.5 Expansion of Density Function

In this subsection, we derive the statistical properties of the MO-EHL-TII-TL-G FoD using general results for the Marshall and Olkin's FoD by Barreto-Souza et al. [20]. Considering

$$f_{MO-EHL-TII-TL-G}(x; \delta, \alpha, b, \xi) = \frac{\delta f_{EHL-TII-TL-G}(x; \alpha, b, \xi)}{(1 - \delta \bar{F}_{EHL-TII-TL-G}(x; \alpha, b, \xi))^2}, \quad (5.8)$$

we can write Equation (5.4) as

$$f_{MO-EHL-TII-TL-G}(x; \delta, \alpha, b, \xi) = \frac{f_{EHL-TII-TL-G}(x; \alpha, b, \xi)}{\delta [1 - \frac{\delta-1}{\delta} F_{EHL-TII-TL-G}(x; \alpha, b, \xi)]^2}, \quad (5.9)$$

where $F_{EHL-TII-TL-G}(x; \alpha, b, \xi)$ and $f_{EHL-TII-TL-G}(x; \alpha, b, \xi)$ are as given in Equations (5.1) and (5.2), respectively. We apply the series expansion

$$(1 - z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^j, \quad (5.10)$$

which is valid for $|z| < 1$ and $k > 0$. If $\delta \in (0, 1)$, we can obtain

$$\begin{aligned} f_{MO-EHL-TII-TL-G}(x; \delta, \alpha, b, \xi) &= f_{EHL-TII-TL-G}(x; \alpha, b, \xi) \\ &\times \sum_{j=0}^{\infty} \sum_{k=0}^j w_{j,k} (F_{EHL-TII-TL-G}(x; \alpha, b, \xi))^{j-k}, \end{aligned} \quad (5.11)$$

where $w_{j,k} = w_{j,k}(\delta) = \delta(j+1)(1-\delta)^j(-1)^{j-k} \binom{j}{k}$.

For $\delta > 1$, we have

$$\begin{aligned} f_{MO-EHL-TII-TL-G}(x; \delta, \alpha, b, \xi) &= f_{EHL-TII-TL-G}(x; \alpha, b, \xi) \\ &\times \sum_{j=0}^{\infty} v_j F_{EHL-TII-TL-G}^j(x; \alpha, b, \xi), \end{aligned} \quad (5.12)$$

where $v_j = v_j(\delta) = \frac{(j+1)(1-1/\delta)}{\delta}$.

For $\delta \in (0, 1)$, Equation (5.4) becomes

$$\begin{aligned}
 f_{MO-EHL-TII-TL-G}(x; \delta, a, b, \xi) &= \sum_{j,i,q,p=0}^{\infty} \sum_{k=0}^j w_{j,k} \binom{\alpha(j-k+1)+i}{i} \binom{\alpha(j-k+1)-1}{q} \\
 &\times \binom{b(i+q+1)-1}{p} (-1)^{i+q+p} 4\alpha b \left(\frac{2(p+1)}{2(p+1)} \right) \\
 &\times g(x; \xi) G^{2p+1}(x; \xi) \\
 &= \sum_{p=0}^{\infty} C_{2(p+1)} g_{2(p+1)}(x; \xi),
 \end{aligned} \tag{5.13}$$

where $g_{2(p+1)}(x; \xi) = (2(p+1)) (G(x; \xi))^{2p+1} g(x; \xi)$ is the Exp-G distribution with power parameter $(2(p+1))$ and

$$\begin{aligned}
 C_{2(p+1)} &= \sum_{j,i,q=0}^{\infty} \sum_{k=0}^j w_{j,k} \binom{\alpha(j-k+1)+i}{i} \binom{\alpha(j-k+1)-1}{q} \\
 &\times \binom{b(i+q+1)-1}{p} (-1)^{i+q+p} \left(\frac{4\alpha b}{2(p+1)} \right).
 \end{aligned} \tag{5.14}$$

Thus, the pdf of MO-EHL-TII-TL-G FoD can be written as an infinite linear combination of Exp-G distribution. Furthermore, for $\delta > 1$, Equation (5.4) can be written as

$$\begin{aligned}
 f_{MO-EHL-TII-TL-G}(x; \delta, a, b, \xi) &= \sum_{j,i,q,p=0}^{\infty} v_j \binom{\alpha(j+1)+i}{i} \binom{\alpha(j+1)-1}{q} \\
 &\times \binom{b(i+q+1)-1}{p} (-1)^{i+q+p} 4\alpha b \left(\frac{2(p+1)}{2(p+1)} \right) \\
 &\times G^{2p+1}(x; \xi) g(x; \xi) \\
 &= \sum_{p=0}^{\infty} b_{2(p+1)} g_{2(p+1)}(x; \xi),
 \end{aligned} \tag{5.15}$$

where $g_{2(p+1)}(x; \xi) = (2(p+1)) (G(x; \xi))^{2p+1} g(x; \xi)$ is the Exp-G distribution with power parameter $2(p+1)$ and

$$\begin{aligned}
 b_{2(p+1)} &= \sum_{j,i,q=0}^{\infty} v_j \binom{\alpha(j+1)+i}{i} \binom{\alpha(j+1)-1}{q} \\
 &\times \binom{b(i+q+1)-1}{p} (-1)^{i+q+p} \left(\frac{4\alpha b}{2(p+1)} \right).
 \end{aligned} \tag{5.16}$$

Thus, the pdf of MO-EHL-TII-TL-G FoD can be written as an infinite linear combination of Exp-G distribution with a power parameter $(2(p+1))$. See Appendix for derivations.

5.2.6 Moments and Generating Functions

Let $X \sim MO - EHL - TII - TL - G(x; \delta, \alpha, b, \xi)$, $Y_{2(p+1)} \sim E - G(2(p+1), \xi)$ then the η^{th} raw moment, μ_{η}' moment can be obtained from Equations (5.13) and (5.15). For $\delta \in (0, 1)$,

$$E(X^{\eta}) = \sum_{p=0}^{\infty} C_{2(p+1)} E(Y_{2(p+1)}^{\eta}),$$

where $C_{2(p+1)}$ is as defined in Equation (5.14) and $E(Y_{2(p+1)}^{\eta})$ denotes the η^{th} moment of an Exp-G distribution with power parameter $(2(p+1)) > 0$. For $\delta > 1$,

$$E(X^{\eta}) = \sum_{p=0}^{\infty} b_{2(p+1)} E(Y_{2(p+1)}^{\eta}), \quad (5.17)$$

where $b_{2(p+1)}$ is as defined in Equation (5.16) and $E(Y_{2(p+1)}^{\eta})$ denotes the η^{th} moment of an Exp-G distribution with power parameter $(2(p+1)) > 0$. The MGF of X is given as follows:

For $\delta \in (0, 1)$,

$$M_X(t) = \sum_{p=0}^{\infty} C_{2(p+1)} E(e^{tY_{2(p+1)}}),$$

where $E(e^{tY_{2(p+1)}})$ is the MGF of the Exp-G distribution with power parameter $(2(p+1)) > 0$ and $C_{2(p+1)}$ is as defined in Equation (5.14). For $\delta > 1$,

$$M_X(t) = \sum_{p=0}^{\infty} b_{2(p+1)} E(e^{tY_{2(p+1)}}),$$

where $E(e^{tY_{2(p+1)}})$ is the MGF of the Exp-G distribution with power parameter $(2(p+1)) > 0$ and $b_{2(p+1)}$ is as defined in Equation (5.16).

5.2.7 Distribution of Order Statistics

In this subsection, we present the distribution of the i^{th} order statistic. Suppose that $X_1, X_2, \dots, X_n$ are iid random variable distributed according to Equation (5.4). The pdf of the i^{th} order statistic $X_{i:n}$ is given by

$$f_{i:n}(x; \delta, \alpha, b, \xi) = \sum_{p=0}^{\infty} e_{2(p+1)} g_{2(p+1)}(x; \xi),$$

for $\delta \in (0, 1)$, where, $g_{2(p+1)}(x; \xi) = (2(p+1)) (G(x; \xi))^{2p+1} g(x; \xi)$ is the Exp-G density function with power parameter $(2(p+1)) > 0$ and linear component

$$e_{2(p+1)} = \sum_{j,q,m=0}^{\infty} \sum_{l=0}^{n-i} \sum_{k=0}^j U_{j,l,k} \binom{\alpha(j+l-k+i)+q}{q} \binom{\alpha(j+l-k+i)-1}{m} \\ \times \binom{b(q+m+1)-1}{p} (-1)^{q+m+p} \left(\frac{4\alpha b}{2(p+1)} \right).$$

Furthermore, for $\delta > 1$, we get

$$f_{i:n}(x; \delta, \alpha, b, \xi) = \sum_{m=0}^{\infty} d_{2(m+1)} g_{2(m+1)}(x; \xi),$$

where $g_{2(m+1)}(x; \xi) = 2(m+1) [G(x; \xi)]^{2m+1} g(x; \xi)$ is the Exp-G density function with power parameter $(2(m+1))$ and linear component

$$d_{2(m+1)} = \sum_{j,q,k=0}^{\infty} \sum_{l=0}^{n-i} c_{j,l} \binom{\alpha(j+l+i)+q}{q} \binom{\alpha(j+l+i)-1}{k} (-1)^{q+k+m} \\ \times \binom{b(q+k+1)-1}{m} \left(\frac{4\alpha b}{2(m+1)} \right).$$

Please see the appendix for further details on how the distribution of the order statistics is derived.

5.2.8 Rényi Entropy

RE from the MO-EHL-TII-TL-G FoD is presented in this section. Entropy is a measure of uncertainty variation for a random variable X with pdf, $f(x)$. RE is defined as

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\int_0^{\infty} f^{\nu}(x) dx \right],$$

where $\nu > 0$ and $\nu \neq 1$. Using the expansion in Equation (5.10), for $\delta \in (0, 1)$,

$$f_{MO-EHL-TII-TL-G}^{\nu}(x; \delta, \alpha, b, \xi) = \frac{\delta^{\nu} f_{EHL-TII-TL-G}^{\nu}(x; \alpha, b, \xi)}{\Gamma(2\nu)} \sum_{j=0}^{\infty} (1 - \alpha)^j \Gamma(2\nu + j) \\ \times \frac{[1 - F_{EHL-TII-TL-G}(x; \alpha, b, \xi)]^j}{j!}$$

and for $\delta > 1$,

$$f_{MO-EHL-TII-TL-G}^\nu(x; \delta, \alpha, b, \xi) = \frac{f_{EHL-TII-TL-G}^\nu(x; \alpha, b, \xi)}{\delta^\nu \Gamma(2\nu)} \sum_{j=0}^{\infty} (\delta - 1)^j \Gamma(2\nu + j) \times \frac{F_{EHL-TII-TL-G}^j(x; \alpha, b, \xi)}{j!}.$$

Thus, RE for $\delta \in (0, 1)$ and $\delta > 1$ are given by

$$I_R(\nu) = (1 - \nu)^{-1} \log \left(\sum_{j=0}^{\infty} e_j \int_0^{\infty} f_{EHL-TII-TL-G}^\nu(x; \alpha, b, \xi) \times (1 - F_{EHL-TII-TL-G}(x; \alpha, b, \xi))^j dx \right) \quad (5.18)$$

and

$$I_R(\nu) = (1 - \nu)^{-1} \log \left(\sum_{j=0}^{\infty} h_j \int_0^{\infty} f_{EHL-TII-TL-G}^\nu(x; \alpha, b, \xi) \times F_{EHL-TII-TL-G}^j(x; \alpha, b, \xi) dx \right), \quad (5.19)$$

where

$$e_j = e_j(\delta) = \frac{\delta^\nu (1 - \delta)^j \Gamma(2\nu + j)}{\Gamma(2\nu) j!}$$

and

$$h_j = h_j(\delta) = \frac{(\delta - 1)^j \Gamma(2\nu + j)}{\delta^{\nu+j} \Gamma(2\nu) j!}.$$

Now, for $\delta \in (0, 1)$, and using Equation (5.18), we have

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\sum_{q=0}^{\infty} C_{2q+\nu}^* \exp(1 - \nu) I_{REG} \right],$$

where

$$C_{2q+\nu}^* = \sum_{j,k,m=0}^{\infty} e_j \binom{\alpha(\nu + j) + \nu + k - 1}{k} \binom{\alpha(\nu + j) - \nu}{m} \times \binom{b(m + k + \nu) - \nu}{q} (-1)^{k+m+q} (4\alpha b)^\nu \frac{1}{\left(\frac{2q+\nu}{\nu} + 1\right)^\nu}, \quad (5.20)$$

and $I_{REG} = \int_0^{\infty} \left(\left(\frac{2q+\nu}{\nu} + 1 \right) g(x; \xi) [G(x; \xi)]^{\frac{2q+\nu}{\nu}} \right)^\nu dx$ is the RE of the Exp-G distribution with power parameter $\left(\frac{2q+\nu}{\nu} + 1 \right)$.

Consequently, RE for MO-EHL-TII-TL-G FoD can be obtained from RE of the Exp-G distri-

bution. Furthermore, for $\delta > 1$, we note that

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\sum_{q=0}^{\infty} b_{2q+\nu}^* \exp(1 - \nu) I_{REG} \right],$$

where

$$\begin{aligned} b_{2q+\nu}^* &= \sum_{j,p,k,m=0}^{\infty} h_j \int_0^{\infty} \binom{j}{p} \binom{\alpha(\nu+p) + \nu + k - 1}{k} \binom{\alpha(\nu+p) - \nu}{m} \\ &\times \binom{b(k+m+\nu) - \nu}{q} (-1)^{p+k+m+q} (4\alpha b)^{\nu} \frac{1}{\left(\frac{2q+\nu}{\nu} + 1\right)^{\nu}}, \end{aligned} \quad (5.21)$$

and $I_{REG} = \int_0^{\infty} \left(\left(\frac{2q+\nu}{\nu} + 1 \right) g(x; \xi) [G(x; \xi)]^{\frac{2q+\nu}{\nu}} \right)^{\nu} dx$ is the RE of the Exp-G distribution with power parameter $\left(\frac{2q+\nu}{\nu} + 1 \right)$.

Consequently, RE for MO-EHL-TII-TL-G FoD can be obtained from RE of the Exp-G distribution. See appendix for detailed derivation of the RE.

5.3 Maximum Likelihood Estimation

Let $X_i \sim MO - EHL - TII - TL - G(\delta, \alpha, b, \xi)$ and $\phi = (\delta, \alpha, b, \xi)^T$ be the vector of unknown the parameters. The total log-likelihood function is given by

$$\begin{aligned} \ell(\phi) &= n \ln(4\alpha b \delta) + (b-1) \sum_{i=1}^n \ln[1 - G^2(x; \xi)] + (\alpha-1) \\ &\times \sum_{i=1}^n \ln(1 - [1 - G^2(x; \xi)]^b) - (\alpha+1) \sum_{i=1}^n \ln(1 + [1 - G^2(x; \xi)]^b) \\ &- 2 \sum_{i=1}^n \ln \left(1 - \bar{\delta} \left[1 - \left(\frac{1 - [1 - G^2(x; \xi)]^b}{1 + [1 - G^2(x; \xi)]^b} \right)^{\alpha} \right] \right) + \sum_{i=1}^n \ln(g(x; \xi)) + \sum_{i=1}^n \ln(G(x; \xi)). \end{aligned}$$

Solving the non-linear system equations, $\mathbf{U} = \left(\frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \xi_k} \right)^T = 0$ yields the ML estimates. These equations can be obtained numerically via iterative methods such as Newton-Raphson technique using statistical software. The FIM of the MO-EHL-TII-TL-G FoD is a $(3+q) \times (3+q)$ matrix given by $\mathbf{J}(\phi) = [\mathbf{J}_{\theta_i, \theta_j}]_{(3+q) \times (3+q)} = E \left(-\frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j} \right)$, $i, j = 1, 2, 3, \dots, (3+q)$. The total FIM $\mathbf{J}_n(\hat{\phi})$ can be approximated by

$$\mathbf{J}_n(\hat{\phi}) \approx \left[- \frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \Big|_{\phi=\hat{\phi}} \right]_{(3+q) \times (3+q)}, \quad i, j = 1, 2, 3, \dots, (3+q). \quad (5.22)$$

For a given set of observations, we obtain the total FIM $\mathbf{J}_n(\hat{\phi})$ after the convergence of the Newton-Raphson technique. See elements of score vectors $\mathbf{U} = \left(\frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \xi_k} \right)$ in the appendix.

5.4 Some Special Cases of MO-EHL-TII-TL-G Family of Distributions

Here, we consider three special cases of the MO-EHL-TII-TL-G FoD. The baseline distributions used are LLoG, Weibull, and Lindley distributions.

5.4.1 Marshall-Olkin Exponentiated Half Logistic-Type II Topp-Leone Log-Logistic (MO-EHL-TII-TL-LLoG) Distribution

Consider the baseline distribution as the LLoG distribution with cdf and pdf given by $G(x; c) = 1 - (1 + x^c)^{-1}$ and $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$, for $c > 0$ and $x > 0$, respectively. The MO-EHL-TII-TL-LLoG distribution has cdf, pdf and hrf respectively given by

$$F(x; \delta, \alpha, b, c) = 1 - \frac{\delta \left[1 - \left(\frac{1 - [(1+x^c)^{-2}]^b}{1 + [(1+x^c)^{-2}]^b} \right)^\alpha \right]}{1 - \bar{\delta} \left[1 - \left(\frac{1 - [(1+x^c)^{-2}]^b}{1 + [(1+x^c)^{-2}]^b} \right)^\alpha \right]}, \quad (5.23)$$

$$f(x; \delta, \alpha, b, c) = \frac{4\delta\alpha bcx^{c-1}(1+x^c)^{-2}(1 - (1+x^c)^{-1}) [(1+x^c)^{-2}]^{b-1} \left(1 - [(1+x^c)^{-2}]^b \right)^{\alpha-1}}{\left(1 + [(1+x^c)^{-2}]^b \right)^{\alpha+1} \left(1 - \bar{\delta} \left[1 - \left(\frac{1 - [(1+x^c)^{-2}]^b}{1 + [(1+x^c)^{-2}]^b} \right)^\alpha \right] \right)^2}, \quad (5.24)$$

and

$$h_F(x; \delta, \alpha, b, c) = \frac{4\alpha bcx^{c-1}(1+x^c)^{-2}(1 - (1+x^c)^{-1}) [(1+x^c)^{-2}]^{b-1} \left(1 - [(1+x^c)^{-2}]^b \right)^{\alpha-1}}{\left(1 + [(1+x^c)^{-2}]^b \right)^{\alpha+1} \left(1 - \bar{\delta} \left[1 - \left(\frac{1 - [(1+x^c)^{-2}]^b}{1 + [(1+x^c)^{-2}]^b} \right)^\alpha \right] \right)} \times \left[1 - \left(\frac{1 - [(1+x^c)^{-2}]^b}{1 + [(1+x^c)^{-2}]^b} \right)^\alpha \right]^{-1}, \quad (5.25)$$

for $\delta, \alpha, b, c > 0$, and $x > 0$.

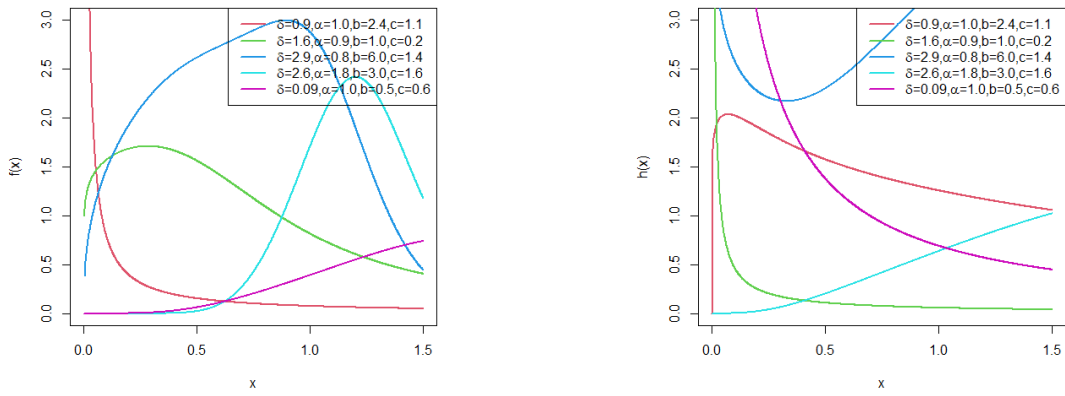


Figure 5.1: Plots of the Pdf and Hrf of the MO-EHL-TII-TL-LLoG Distribution

For given parameter values, Figure 5.1 shows the various shapes of the pdf and hrf for the MO-EHL-TII-TL-LLoG distribution. The pdf exhibits reverse-J shape, unimodal, right and left-skewed. The BT, UBT, increasing, and decreasing are some of the shapes that the hrf can take.



Figure 5.2: Plots of SK and KS of the MO-EHL-TII-TL-LLoG Distribution



Figure 5.3: Plots of SK and KS of the MO-EHL-TII-TL-LLoG Distribution

3D plots of SK and KS for the MO-EHL-TII-TL-LLoG distribution are given in Figures 5.2 and 5.3. We observe that

- If we fix the parameters c and δ , the SK and KS of MO-EHL-TII-TL-LLoG distribution increase as b and α increase.
- If we fix the parameters b and α , the SK increase and KS decrease then increase as c and δ increase.

5.4.2 Marshall-Olkin Exponentiated Half Logistic Type II Topp-Leone Weibull (MO-EHL-TII-TL-W) Distribution

Consider the baseline distribution as the Weibull distribution with the cdf and pdf given by $G(x; \lambda) = 1 - e^{-x^\lambda}$ and $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$ for $\lambda > 0$ and $x > 0$. The MO-EHL-TII-TL-W distribution has cdf, pdf and hrf respectively given by

$$F(x; \delta, \alpha, b, \lambda) = 1 - \frac{\delta \left[1 - \left(\frac{1 - [e^{-x^{2\lambda}}]^b}{1 + [e^{-x^{2\lambda}}]^b} \right)^\alpha \right]}{1 - \bar{\delta} \left[1 - \left(\frac{1 - [e^{-x^{2\lambda}}]^b}{1 + [e^{-x^{2\lambda}}]^b} \right)^\alpha \right]}, \quad (5.26)$$

$$f(x; \delta, \alpha, b, \lambda) = \frac{4\delta\alpha b\lambda x^{\lambda-1} e^{-x^\lambda} (1 - e^{-x^\lambda}) [e^{-x^{2\lambda}}]^{b-1} \left(1 - [e^{-x^{2\lambda}}]^b \right)^{\alpha-1}}{\left(1 + [e^{-x^{2\lambda}}]^b \right)^{\alpha+1} \left(1 - \bar{\delta} \left[1 - \left(\frac{1 - [e^{-x^{2\lambda}}]^b}{1 + [e^{-x^{2\lambda}}]^b} \right)^\alpha \right] \right)^2}, \quad (5.27)$$

and

$$h_F(x; \delta, \alpha, b, \lambda) = \frac{4\alpha b\lambda x^{\lambda-1} e^{-x^\lambda} (1 - e^{-x^\lambda}) [e^{-x^{2\lambda}}]^{b-1} \left(1 - [e^{-x^{2\lambda}}]^b \right)^{\alpha-1}}{\left(1 + [e^{-x^{2\lambda}}]^b \right)^{\alpha+1} \left(1 - \bar{\delta} \left[1 - \left(\frac{1 - [e^{-x^{2\lambda}}]^b}{1 + [e^{-x^{2\lambda}}]^b} \right)^\alpha \right] \right)} \times \left[1 - \left(\frac{1 - [e^{-x^{2\lambda}}]^b}{1 + [e^{-x^{2\lambda}}]^b} \right)^\alpha \right]^{-1}, \quad (5.28)$$

for $\delta, \alpha, b, \lambda > 0$, and $x > 0$.

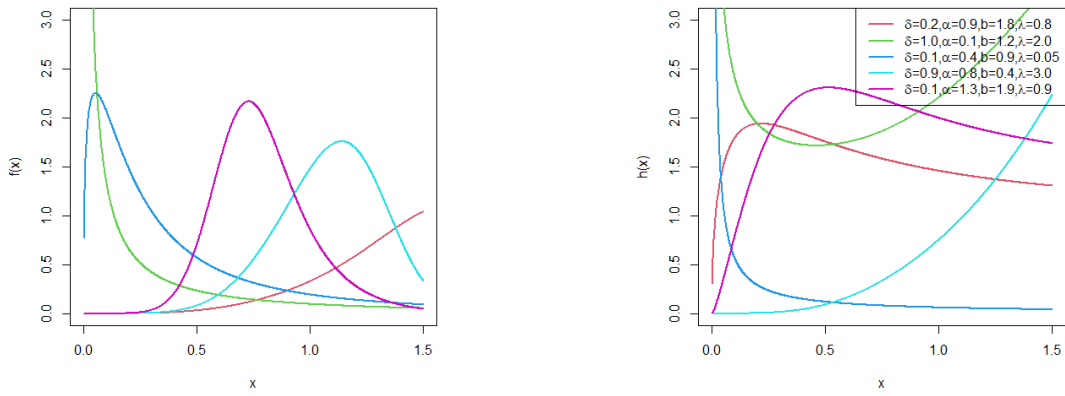


Figure 5.4: Plots of the Pdf and Hrf of the MO-EHL-TII-TL-W Distribution

The pdf and hrf for the MO-EHL-TII-TL-W distribution for various parameter values are shown in Figure 5.4. Among other shapes, the pdf can be almost symmetric, reverse J-shaped, right-skewed, or left-skewed. The hrf of the MO-EHL-TII-TL-W distribution exhibits increasing, decreasing, UBT, and BT shapes.



Figure 5.5: Plots of SK and KS of the MO-EHL-TII-TL-W Distribution



Figure 5.6: Plots of SK and KS of the MO-EHL-TII-TL-W Distribution

3D plots of SK and KS for the MO-EHL-TII-TL-W distribution are given in Figures 5.5 and 5.6. We observe that

- When we fix the parameters α and b , the SK and KS of MO-EHL-TII-TL-W distribution increase as δ and λ increase.
- If we fix the parameters λ and δ , the SK and KS of MO-EHL-TII-TL-W distribution decrease as b and α increase.

5.4.3 Marshall-Olkin Exponentiated Half Logistic-Type II Topp-Leone Lindley (MO-EHL-TII-TL-L) Distribution

Taking baseline distribution to be the Lindley distribution with cdf and pdf given by $G(x; \lambda) = 1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}$ and $g(x; \lambda) = \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}$, for $\lambda > 0$, the MO-EHL-TII-TL-L distribution has cdf, pdf and hrf respectively given by

$$F(x; \delta, \alpha, b, \lambda) = 1 - \frac{\delta \left[1 - \left(\frac{1 - [1 - (1 + \frac{\lambda x}{1+\lambda})e^{-2\lambda x}]^b}{1 + [1 - (1 + \frac{\lambda x}{1+\lambda})e^{-2\lambda x}]^b} \right)^\alpha \right]}{1 - \delta \left[1 - \left(\frac{1 - [1 - (1 + \frac{\lambda x}{1+\lambda})e^{-2\lambda x}]^b}{1 + [1 - (1 + \frac{\lambda x}{1+\lambda})e^{-2\lambda x}]^b} \right)^\alpha \right]}, \quad (5.29)$$

$$\begin{aligned} f(x; \delta, \alpha, b, \lambda) &= 4\delta\alpha b \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x} \left(1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x} \right) \\ &\times \left[1 - \left(1 - (1 + \frac{\lambda x}{1+\lambda})e^{-2\lambda x} \right) \right]^{b-1} \\ &\times \frac{\left(1 - \left[1 - \left(1 - (1 + \frac{\lambda x}{1+\lambda})e^{-2\lambda x} \right) \right]^b \right)^{\alpha-1}}{\left(1 + \left[1 - \left(1 - (1 + \frac{\lambda x}{1+\lambda})e^{-2\lambda x} \right) \right]^b \right)^{\alpha+1}} \\ &\times \frac{1}{\left(1 - \delta \left[1 - \left(\frac{1 - [1 - (1 + \frac{\lambda x}{1+\lambda})e^{-2\lambda x}]^b}{1 + [1 - (1 + \frac{\lambda x}{1+\lambda})e^{-2\lambda x}]^b} \right)^\alpha \right] \right)^2}, \end{aligned} \quad (5.30)$$

and

$$\begin{aligned}
 h_F(x; \delta, \alpha, b, \lambda) &= 4\alpha b g(x; \xi) \left(1 - \left(1 + \frac{\lambda x}{1+\lambda}\right)e^{\lambda x}\right) \left[1 - \left(1 - \left(1 + \frac{\lambda x}{1+\lambda}\right)e^{-2\lambda x}\right)\right]^{b-1} \\
 &\times \frac{\left(1 - \left[1 - \left(1 - \left(1 + \frac{\lambda x}{1+\lambda}\right)e^{-2\lambda x}\right)\right]^b\right)^{\alpha-1}}{\left(1 + \left[1 - \left(1 - \left(1 + \frac{\lambda x}{1+\lambda}\right)e^{-2\lambda x}\right)\right]^b\right)^{\alpha+1}} \\
 &\times \left(1 - \bar{\delta} \left[1 - \left(\frac{1 - \left[1 - \left(1 - \left(1 + \frac{\lambda x}{1+\lambda}\right)e^{-2\lambda x}\right)\right]^b\right)^{\alpha}}{1 + \left[1 - \left(1 - \left(1 + \frac{\lambda x}{1+\lambda}\right)e^{-2\lambda x}\right)\right]^b}\right)^{\alpha}\right]^{-1} \\
 &\times \left[1 - \left(\frac{1 - \left[1 - \left(1 - \left(1 + \frac{\lambda x}{1+\lambda}\right)e^{-2\lambda x}\right)\right]^b\right)^{\alpha}}{1 + \left[1 - \left(1 - \left(1 + \frac{\lambda x}{1+\lambda}\right)e^{-2\lambda x}\right)\right]^b}\right)^{-1}\right], \tag{5.31}
 \end{aligned}$$

for $\delta, \alpha, b, \lambda > 0$, and $x > 0$.

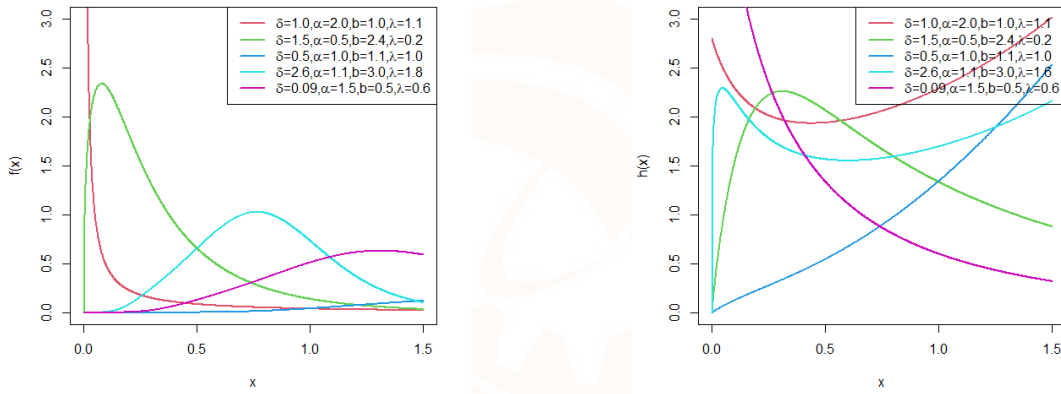


Figure 5.7: Plots of the Pdf and Hrf of the MO-EHL-TII-TL-L Distribution

The pdf and hrf for the MO-EHL-TII-TL-L distribution for various parameter values are shown in Figure 5.7. Among other shapes, the pdf can be almost symmetric, reverse J-shaped, and right-skewed. The hrf of the MO-EHL-TII-TL-L distribution exhibits increasing, decreasing, BT, UBT, and UBT-BT shapes.



Figure 5.8: Plots of SK and KS of the MO-EHL-TII-TL-L Distribution



Figure 5.9: Plots of SK and KS of the MO-EHL-TII-TL-L Distribution

3D plots of SK and KS for the MO-EHL-TII-TL-L distribution are given in Figures 5.8 and 5.9. We observe that

- When we fix the parameters δ and λ , the SK and KS increase as b and α increase.
- If we fix the parameters α and δ , the SK and KS decrease then increase as λ and b increase.

5.5 Simulation Results

The consistency of the ML estimates of the parameters δ, α, b and λ is obtained using a MCS experiment in this section. We use R software to assess the performance of ML estimators for the MO-EHL-TII-TL-W distribution. We obtain the mean ML estimates, ABias, and RMSE.

In simulation studies, the QF is used to generate random samples from a probability distribution. Specifically, the QF provides the inverse mapping of the cdf, allowing us to convert uniformly distributed random variables into random variables following the desired distribution.

1. First, generate n independent uniform random variables for $0 < u < 1$.
2. Apply the inverse QF to each of the uniform random variables to obtain random variables following the MO-EHL-TL-W distribution.
3. Repeat steps 1 and 2 to generate multiple sets of random samples.

The non-linear Equation (5.7) is solved to find the QF, and then random numbers are generated based on this function to conduct simulation studies and evaluate the performance of ML estimates.

Tables 5.1 and 5.2 shows that the ML estimation method performs well in estimating the parameters of the MO-EHL-TII-TL-W distribution. As the sample size increases, n the estimates of each parameter converge to the true value, demonstrating consistency. In addition, as the sample size, n increases, the RMSE and ABias of each parameter tend to zero, indicating that the ML estimates are accurate and precise.

Table 5.1: MCS Results for MO-EHL-TII-TL-W Distribution

(0.2, 1.0, 1.0, 0.5)					(1.0, 3.0, 0.2, 0.5)			(1.0, 3.0, 1.0, 0.5)			(0.5,0.5,7.5,2.1)		
parameter	Sample Size	Mean	RMSE	ABias	Mean	RMSE	ABias	Mean	RMSE	ABias	Mean	RMSE	ABias
δ	50	0.9255	1.5942	0.7255	2.2623	3.3920	1.2623	2.0259	2.4754	1.0259	3.1355	12.5521	2.6355
	100	0.5356	0.7252	0.3356	2.1824	3.0586	1.1824	1.6323	1.7785	0.6323	1.7574	7.2020	1.2574
	200	0.3184	0.2321	0.1184	1.8123	2.2393	0.8123	1.2865	1.1639	0.2865	1.0070	4.2526	0.5070
	400	0.2712	0.1370	0.0712	1.5170	1.5638	0.5170	1.1264	0.7542	0.1264	0.5964	0.3810	0.0964
	800	0.2246	0.0650	0.0246	1.2132	0.9617	0.2132	1.0417	0.5172	0.0417	0.5401	0.1898	0.0401
	1600	0.2197	0.0445	0.0197	1.0407	0.6114	0.0407	1.0240	0.3603	0.0240	0.5237	0.1300	0.0237
α	50	0.7722	0.3431	-0.2277	3.5319	2.1391	0.5319	3.2316	1.6782	0.2316	0.7677	2.7590	0.2677
	100	0.8078	0.2642	-0.1921	3.2101	1.7392	0.2101	3.1915	1.4523	0.1915	0.5548	0.3539	0.0548
	200	0.8940	0.1650	-0.1059	3.1356	1.5266	0.1356	3.1689	1.2071	0.1689	0.5264	0.2418	0.0264
	400	0.9256	0.1235	-0.0743	3.1234	1.2732	0.1234	3.1561	0.9598	0.1561	0.5085	0.1791	0.0085
	800	0.9679	0.0650	-0.0320	3.0798	1.0157	0.0798	3.1542	0.7405	0.1542	0.4973	0.1291	-0.0026
	1600	0.9760	0.0481	-0.0239	3.0510	0.7798	0.0510	3.0762	0.5236	0.0762	0.4963	0.1022	-0.0003
b	50	3.5222	6.6327	2.5222	2.3184	19.4767	2.1184	4.9998	24.6344	3.9998	9.4297	6.3244	1.9297
	100	1.8104	2.4473	0.8104	1.0350	4.4040	0.8350	2.7849	10.3407	1.7849	8.8589	5.3240	1.3589
	200	1.2630	0.5926	0.2630	0.6565	3.4624	0.4565	1.5317	2.6236	0.5317	8.2301	3.0656	0.7301
	400	1.1346	0.2994	0.1346	0.3557	1.2244	0.1557	1.1700	0.6805	0.1700	8.1450	2.5606	0.6450
	800	1.0481	0.1625	0.0481	0.2680	1.0926	0.0680	1.0705	0.3531	0.0705	8.0568	2.3572	0.5568
	1600	1.0186	0.1505	0.0786	0.2140	0.0481	0.0140	1.0291	0.2012	0.0291	7.5524	1.7677	0.0524
λ	50	0.7460	1.0761	0.2460	0.8238	2.2084	0.3238	0.7353	2.4931	0.2353	6.3559	44.7919	5.8559
	100	0.6021	0.1923	0.1021	0.7540	0.9591	0.2540	0.5617	0.3264	0.0617	3.2518	3.4710	1.1518
	200	0.5405	0.1103	0.0405	0.6360	0.4874	0.1360	0.5208	0.2105	0.0208	2.8223	2.5020	0.7223
	400	0.5248	0.0691	0.0248	0.5919	0.3446	0.0919	0.5081	0.1515	0.0081	2.2336	1.1998	0.1336
	800	0.5122	0.0433	0.0122	0.5309	0.2423	0.0309	0.4961	0.1096	-0.0038	2.1310	0.8847	0.0310
	1600	0.4990	0.0271	0.0009	0.5253	0.1463	0.0253	0.4998	0.0798	-0.0002	2.1193	0.6892	0.0193

Table 5.2: MCS Results for MO-EHL-TII-TL-W Distribution

parameter	Sample Size	(0.1, 0.8, 1.0, 1.2)			(1.2, 0.8, 1.0, 1.2)			(1.0, 0.8, 1.2, 1.0)		
		Mean	RMSE	ABias	Mean	RMSE	ABias	Mean	RMSE	ABias
δ	50	2.9345	14.4266	2.8345	8.4268	27.1005	7.2268	9.1651	40.0673	8.1651
	100	1.3979	4.9160	1.2979	5.6039	22.8246	4.4039	5.7877	30.1724	4.7877
	200	1.0436	4.1723	0.9436	2.7370	9.0097	1.5370	2.6377	10.9462	1.6377
	400	0.4950	1.9140	0.3950	1.6276	1.7639	0.4276	1.4400	2.0011	0.4400
	800	0.1989	0.5284	0.0989	1.3687	0.9477	0.1687	1.1777	0.9004	0.1777
	1600	0.1293	0.1187	0.0293	1.2703	0.4075	0.0703	1.0613	0.3461	0.0613
α	50	1.9454	4.6877	1.1454	2.8357	21.6069	2.0357	1.5100	5.5162	0.7100
	100	1.8016	4.0059	1.0016	1.0742	1.5974	0.2742	1.0793	1.8196	0.2793
	200	1.6933	3.1188	0.8933	0.8899	0.7780	0.0899	0.8553	0.4886	0.0553
	400	1.3292	2.0970	0.5292	0.8728	0.6358	0.0728	0.8097	0.2778	0.0097
	800	0.9577	0.8254	0.1577	0.8092	0.2577	0.0092	0.7963	0.1935	-0.0036
	1600	0.8753	0.3992	0.0753	0.8018	0.1308	0.0018	0.7995	0.1409	-0.0004
b	50	2.8547	4.6831	1.8547	2.1824	3.0116	1.18246	2.4611	3.2431	1.2611
	100	2.6421	4.2069	1.6421	1.8354	2.3396	0.8354	2.1227	2.5878	0.9227
	200	2.6267	3.9217	1.6267	1.3821	1.5179	0.3821	1.6301	1.7141	0.4301
	400	2.0982	2.9755	1.0982	1.1601	0.9232	0.1601	1.3870	1.1108	0.1870
	800	1.4867	1.7419	0.4867	1.0524	0.6202	0.0524	1.2835	0.7493	0.0835
	1600	1.0741	1.2775	0.0741	1.0400	0.4297	0.0400	1.2371	0.4995	0.0371
λ	50	1.4581	0.9802	0.2581	1.2912	0.6272	0.0912	1.2410	0.7434	0.2410
	100	1.3852	0.8714	0.1852	1.2898	0.5550	0.0898	1.1995	0.6306	0.1995
	200	1.2889	0.5237	0.0889	1.2858	0.4564	0.0858	1.1228	0.5339	0.1228
	400	1.2329	0.7096	0.0329	1.2857	0.3853	0.0857	1.1090	0.4388	0.1090
	800	1.2292	0.5947	0.0293	1.2656	0.2929	0.0656	1.0679	0.3216	0.0679
	1600	1.2194	0.4119	0.0194	1.2260	0.2090	0.0260	1.0371	0.2312	0.0371

5.6 Applications

In this section, we use two real-world datasets to assess the utility of the proposed distribution. We obtain the best-fitting model of the proposed distribution using several GoF, which include: $-2\ln(L)$, AIC, BIC and CAIC. Furthermore, we use CVM and AD statistics, given by Chen and Balakrishnan [26]. The K-S statistic, its PV, and SS (Chambers et al. [24]) from the PPs were used to assess the GoF statistics of the model. The model with the smallest values of the GoF statistics and a bigger PV for the K-S statistic is regarded as the best model.

The density functions of the following non-nested models were used for comparisons in this section.

The pdf of the MOKA distribution was introduced by Javed et al. [59] and given as,

$$f_{MO-KA}(x; a, b, \beta, \lambda) = \frac{\left(\frac{\lambda a \beta}{b} \left(\frac{x}{b}\right) a \beta\right)^{-\frac{(a+1)}{a}}}{\left(\lambda + (1 - \lambda) \left(\frac{\frac{x}{b} a b}{a + \frac{a}{b} a \beta}\right)^{\frac{1}{a}}\right)^2},$$

for $a, b, \beta, \lambda > 0$ and $x > 0$,

the pdf of the MOMW distribution was introduced by Santos-Neto et al. [119] and given as,

$$f_{MOMW}(x; \alpha, \delta, \lambda, \gamma) = \delta \alpha (\gamma + \lambda x) x^{\gamma-1} \frac{\exp[\lambda x - \alpha x^\gamma \exp(\lambda x)]}{(1 - \delta \exp[-\alpha x^\gamma \exp(\lambda x)])^2} \quad (5.32)$$

for $\alpha, \delta, \lambda, \gamma > 0$ and $x > 0$,

the pdf of the KwW distribution was introduced by Cordeiro et al. [36] and given as,

$$f(x; \alpha, \theta, \lambda, \beta) = ab\beta\alpha^\beta x^{\beta-1} e^{-\alpha x^\beta} \times (1 - e^{-\alpha x^\beta})^{a-1} (1 - (1 - e^{-\alpha x^\beta})^a)^{b-1}, \quad (5.33)$$

for $\alpha, \theta, \lambda, \beta > 0$ and $x > 0$, and

the pdf of the TII-EHL-W distribution (Al-Mofleh et al. [7]) is given in Equation (3.28), pdf of the TL-GW distribution (Aryal et al. [17]) is given in Equation (4.20), and the pdf of OEHLBXII distribution (Aldahlan and Afify [9]) is given in Equation (3.29).

The LRT statistic compares the fit of the MO-EHL-TII-TL-W distribution against its sub-models using a dataset. For example, when assessing $\delta = 1$, the LRT statistic, Λ is calculated by comparing the log-likelihoods of the unrestricted and restricted estimates. If Λ exceeds a critical value χ_{α}^2 , representing the upper 100 α % of the χ^2 distribution with 1 degree of freedom, the null hypothesis is rejected.

We use a standard graphical approach for data analysis to compare theoretical and empirical distributions, which allows us to determine whether the data can be used for a specific distribution or not. In data fitting, we also show estimated densities, ECDF, PPs, TTT scaled plot, K-M curves and fitted hrf plot to compare the flexibility of the MO-EHL-TII-TL-W distribution to competing models.

Tables 5.4 and 5.6 present the results of data analysis, parameter estimates (SE in parentheses), and GoF statistics for the various models used in this chapter.

5.6.1 Remission Times Data

This dataset represents the remission times (in months) of a random sample of 128 bladder cancer patients. The dataset was extracted from Klakattawi [63]. The data is skewed to the right, with a few outliers at the high end.

The estimated Var-Cov matrix is given by:

$$\begin{pmatrix} 7.234 \times 10^{-06} & 1.0107 \times 10^{-05} & -0.0013 & 1.7956 \times 10^{-05} \\ 1.0107 \times 10^{-05} & 1.4121 \times 10^{-05} & -0.0017 & 2.5086 \times 10^{-05} \\ -0.0013 & -0.0017 & 0.2164 & 0.0031 \\ 1.7956 \times 10^{-05} & 2.5086 \times 10^{-05} & -0.0031 & 5.8096 \times 10^{-05} \end{pmatrix},$$

and the approximate 95% CI for δ, α, b and λ are given by 322.1300 ± 0.0053 , 237.6200 ± 0.0074 , 17.8210 ± 0.9118 and 0.0994 ± 0.0149 , respectively. The parameter estimates and their SE for the models used are given in Table 5.3.

Table 5.3: Parameter Estimates for Remission Times Data

Model	Estimates			
	δ	α	b	λ
$MO - EHL - TII - TL - W(\delta, \alpha, b, \lambda)$	322.1300 (0.0026)	237.6200 (0.0037)	17.8210 (0.1465)	0.0994 (0.0076)
$MO - EHL - TII - TL - W(\delta, 1, b, \lambda)$	1.0645 (1.7411)	-	0.3390 (0.5792)	0.7478 (0.3458)
$MO - EHL - TII - TL - W(\delta, \alpha, 1, \lambda)$	0.5371 (0.0601)	0.8145 (0.0754)	-	0.3854 (0.3854)
$MO - EHL - TII - TL - W(\delta, 1, 1, \lambda)$	0.9358 (0.1119)	-	-	0.4352 (0.0201)
$MO - EHL - TII - TL - W(1, 1, 1, \lambda)$	-	-	-	0.4426 (0.0186)
KwW	a 0.4974 (0.0303)	b 330.1900 (1.3702×10^{-07})	α 0.0040 (1.3617×10^{-04})	β 2.0958 (0.0072)
TIEHL-W	a 2060.4000 (7.3009×10^{-07})	λ 54.7750 (3.1053×10^{-04})	δ 1.7135 (1.8786×10^{-02})	γ 0.0603 (3.8735×10^{03})
TL-GW	α 0.0961 (0.0657)	θ 52.0982 (0.0024)	λ 0.6751 (0.7464)	β 0.46471 (0.1395)
OEHLBXII	α 0.3925 (0.0648)	λ 0.0242 (0.0018)	a 3.6828 (1.0211)	b 0.3659 (0.0957)
MOMW	δ 0.0018 (0.0067)	λ 0.0325 (0.0014)	α 0.0001 (0.0037)	γ 1.4654 (0.0014)
MOKA	α 1.8509 (0.8780)	θ 1.1009 (0.6176)	δ 3.0349 (9.8636)	β 3.8919 (7.1463)

The MO-EHL-TII-TL-W distribution outperforms the non-nested models in terms of the GoF. This is evidenced by the smaller values of the -2log (L), AIC, CAIC, BIC, CVM, AD, and K-S statistics, and the larger value of the PV for the K-S statistic. According to the measures of GoF statistics, the MO-EHL-TII-TL-W distribution is the best-fitting model.

Table 5.4: GoF Statistics for Remission Times Data

Model	Statistics								
	$-2\log(L)$	AIC	$CAIC$	BIC	CVM	AD	$K-S$	PV	SS
$MO-EHL-TII-TL-W(\delta, \alpha, b, \lambda)$	818.9434	826.9434	827.2686	838.3516	0.0135	0.0942	0.0296	0.9999	0.0117
$MO-EHL-TII-TL-W(\delta, 1, b, \lambda)$	824.1342	830.1342	830.3277	838.6903	0.0862	0.5200	0.0650	0.6513	0.1884
$MO-EHL-TII-TL-W(\delta, \alpha, 1, \lambda)$	965.8650	971.8650	972.0585	980.4210	0.0470	0.3206	0.4836	2.2000×10^{-16}	11.4834
$MO-EHL-TII-TL-W(\delta, 1, 1, \lambda)$	872.7307	876.7308	876.8268	882.4348	0.0388	0.2548	0.2874	1.3100×10^{-09}	3.8965
$MO-EHL-TII-TL-W(1, 1, 1, \lambda)$	867.7488	869.7488	869.7806	872.6009	0.0389	0.2530	0.2742	8.7270×10^{-09}	20.6256
KwW	828.3340	836.3340	836.6592	847.7421	0.1322	0.7918	0.0700	0.5561	0.1529
TII-EHL-W	823.7092	831.7091	832.0343	843.1172	0.0794	0.4866	0.0571	0.7979	0.0812
TL-GW	823.2468	831.2468	831.5720	842.6549	0.0643	0.4199	0.0514	0.8880	0.0591
OEHLBXII	881.8121	889.8121	890.1373	901.2202	0.3274	1.9124	0.1165	0.0616	0.5482
MOMW	819.2511	827.2511	827.5763	838.6592	0.0212	0.1480	0.0329	0.9891	42.6244
MOKA	819.3114	827.3117	827.6369	838.7198	0.0159	0.0989	0.0308	0.9897	1.8130

Based on the LRT results given in Table 5.5, there are significant differences between nested models and the MO-EHL-TII-TL-W distribution. We, therefore, conclude that MO-EHL-TII-TL-W model performs better than its nested models on the remission times data.

Table 5.5: LRT Results For Remission Times Data

Remission Times Data		
Model	df	$\chi^2(p-value)$
$MO-EHL-TII-TL-W(\delta, 1, b, \lambda)$	1	5.1908 (0.0227)
$MO-EHL-TII-TL-W(\delta, \alpha, 1, \lambda)$	1	146.9216 (< 0.00001)
$MO-EHL-TII-TL-W(\delta, 1, 1, \lambda)$	2	53.7873 (< 0.00001)
$MO-EHL-TII-TL-W(1, 1, 1, \lambda)$	3	48.8054 (< 0.00001)

Figure 5.10 shows that the MO-EHL-TII-TL-W distribution is the best fit for the remission times data as compared to the selected models.

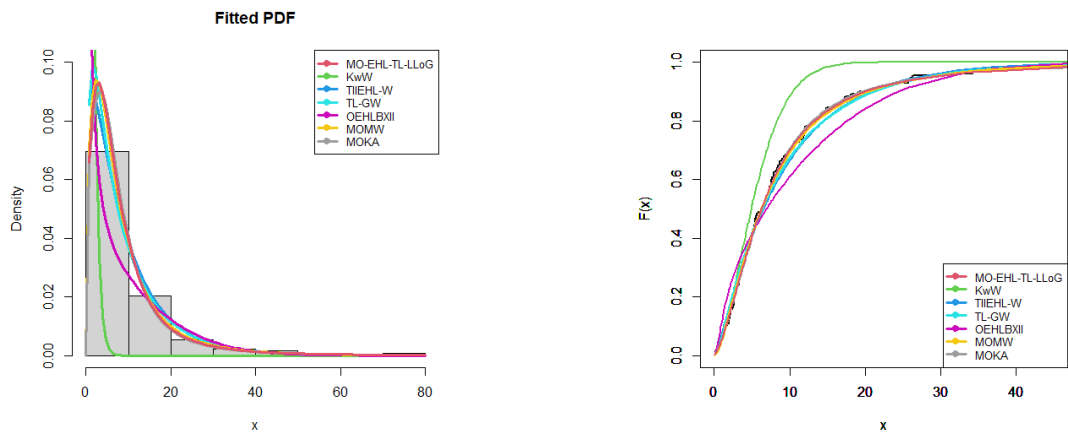


Figure 5.10: Estimated densities and ECDF for Remission Times Data

The PPs and TTT scaled plots for the proposed model are given in Figure 5.11.

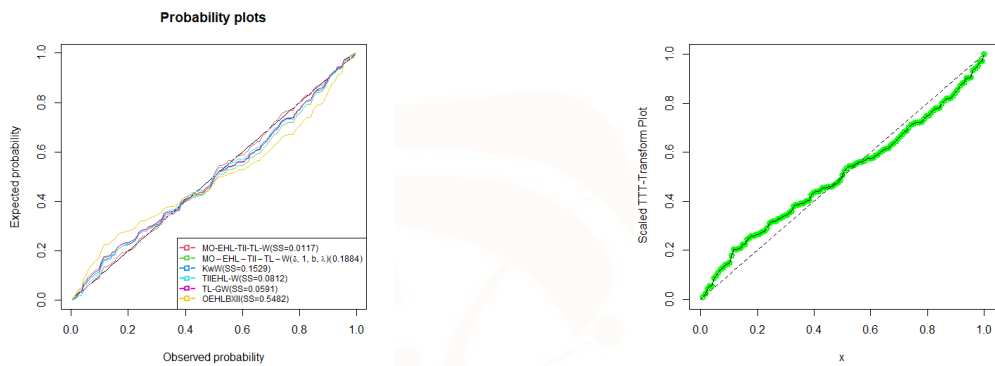


Figure 5.11: PPs and TTT Scaled Plot for Remission Times Data

Based on the TTT scaled plot there is an indication that the hrf of the proposed model exhibits an UBT shape in these data. The K-M Survival and hrf plots for the proposed model are given in Figure 5.12.

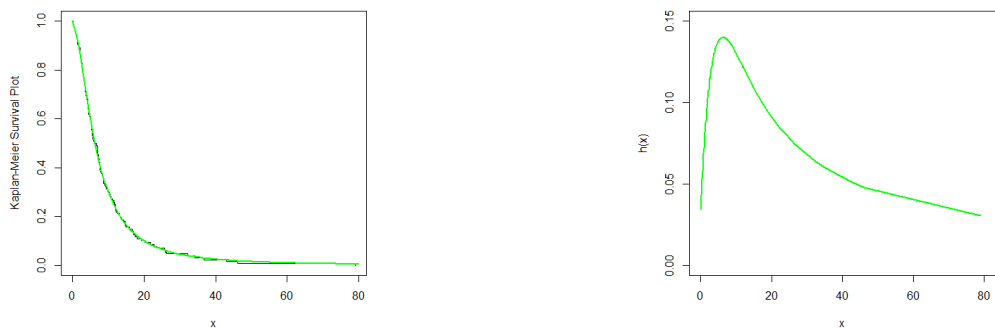


Figure 5.12: Fitted K-M Survival and Hrf for Remission Times Data

Figure 5.13 shows that the ML estimates for the MO-EHL-TII-TL-W distribution exists and

are obtained uniquely.

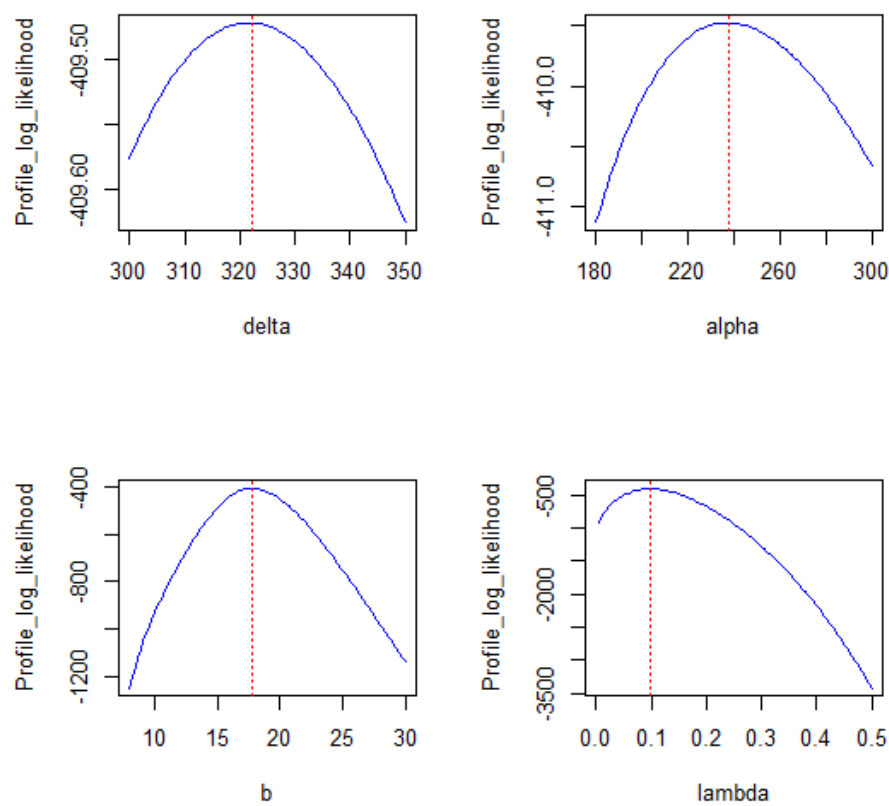


Figure 5.13: Plots of the Profile Log-Likelihood Function of the Parameters of the MO-EHL-TII-TL-W Distribution on Remission Times Data

5.6.2 Coronavirus Mortality Rate Data

This data represents the Coronavirus mortality rate data for 108 days in Mexico. The data was recorded from 4 March to 20 July 2020. Almongy et al. [12] applied this data to a new extended Rayleigh distribution. The data has few outliers and the most of the observations are clustered in the central values.

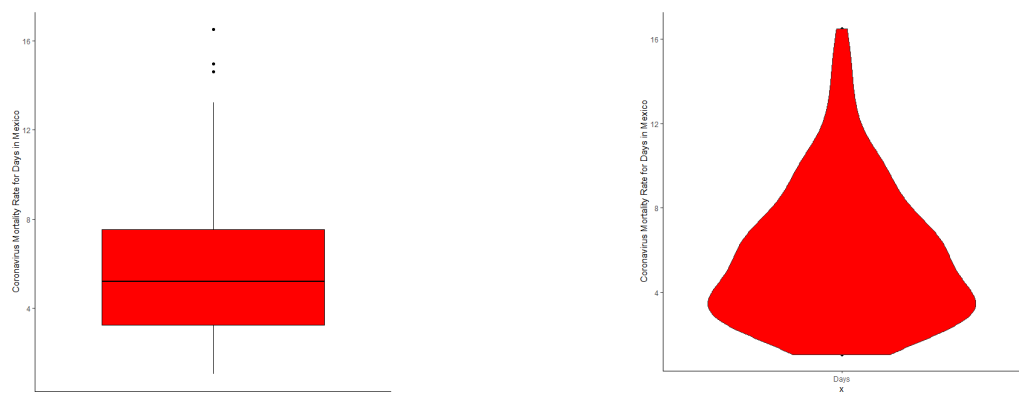


Figure 5.14: Boxplot and Violin Plots of Coronavirus Mortality Rate Data

The estimated Var-Cov matrix is given by:

$$\begin{pmatrix} 0.0770 & -0.0003 & 0.0281 & -0.0298 \\ -0.0003 & 0.2650 & 0.0639 & -0.1183 \\ 0.0281 & 0.0639 & 0.0274 & -0.0434 \\ -0.0298 & -0.1183 & -0.0434 & 0.0747 \end{pmatrix},$$

and the approximate 95% CI for δ, α, b and λ are given by 0.3510 ± 0.5441 , 1.8916 ± 1.0090 , 0.1880 ± 0.3246 and 1.1517 ± 0.5357 , respectively. The parameter estimates and their SE are given in Table 5.6.

Table 5.6: Parameter Estimates for Coronavirus Mortality Rate Data

Model	Estimates			
	δ	a	b	λ
$MO - EHL - TII - TL - W(\delta, \alpha, b, \lambda)$	0.3510 (0.2776)	1.8916 (0.5148)	0.1880 (0.1656)	1.15170 (0.2733)
$MO - EHL - TII - TL - W(\delta, 1, b, \lambda)$	2.1751 (0.7461)	-	0.4538 (0.1038)	0.8931 (0.0696)
$MO - EHL - TII - TL - W(\delta, \alpha, 1, \lambda)$	3.3183 (0.8157)	1.8257 (1.7094)	-	0.7483 (0.0606)
$MO - EHL - TII - TL - W(\delta, 1, 1, \lambda)$	2.0451 (0.4576)	-	-	0.5000 (0.0375)
$MO - EHL - TII - TL - W(1, 1, 1, \lambda)$	-	-	-	0.5606 (0.0249)
KwW	a	b	α	β
	0.3744 (0.043)	478.9600 (1.3419×10^{-04})	0.0034 (0.0011)	4.2708 (0.4896)
TII-EHL-W	a	λ	δ	γ
	392.1800 (6.3210×10^{-06})	7.0410 (4.3279×10^{-04})	1.9519 (0.0328)	0.1127 (0.0079)
TL-GW	α	θ	λ	β
	6.4034 (13.6113)	0.6732 (1.6186)	0.1649 (0.1765)	1.0315 (0.7012)
OEHLBXII	α	λ	a	b
	0.3385 (0.0614)	9.5194×10^{-04} (0.0010)	73.4560 (1.4620×10^{-5})	0.0432 (0.0057)
MOMW	α	δ	λ	γ
	12.7610 (0.4033)	181.7500 (2.1935×10^{08})	0.0039 (0.0039)	0.1538 (0.1538)
MOKA	α	θ	δ	β
	0.5040 (1.2929)	5.8166 (14.9141)	314.6575 (0.1274)	0.6993 (0.1159)

Table 5.7 shows that the MO-EHL-TII-TL-W distribution outperforms the other non-nested models considered. Specifically, it has the smallest values for -2log(L), AIC, CAIC, BIC, CVM, AD, and K-S GoF statistics, and the highest value for the PV of the K-S statistic. Therefore, the MO-EHL-TII-TL-W distribution is the most suitable.

Table 5.7: GoF Statistics for Coronavirus Mortality Rate Data

Model	$-2\log(L)$	AIC	$CAIC$	BIC	CVM	AD	$K-S$	PV	SS
$MO-EHL-TII-TL-W(\delta, \alpha, b, \lambda)$	532.6811	540.6811	541.0696	551.4098	0.0642	0.3491	0.0629	0.7854	0.0595
$MO-EHL-TII-TL-W(\delta, 1, b, \lambda)$	548.1621	554.1622	554.3930	562.2086	0.0936	0.5732	0.1009	0.2207	0.6247
$MO-EHL-TII-TL-W(\delta, \alpha, 1, \lambda)$	536.6721	542.6721	542.9028	550.7184	0.0997	0.6004	0.0806	0.4850	0.0849
$MO-EHL-TII-TL-W(\delta, 1, 1, \lambda)$	609.6720	613.6720	613.7863	619.0363	0.0656	0.3759	0.1991	0.0004	27.8210
$MO-EHL-TII-TL-W(1, 1, 1, \lambda)$	627.7503	629.7503	629.7880	632.4324	0.0620	0.3445	0.3762	0.0001	34.2998
KwW	542.9947	550.0187	550.4070	560.7472	0.0915	0.5755	0.0898	0.3478	0.1200
TIIHL-W	533.9266	541.9260	542.3143	552.6545	0.0737	0.4454	0.0758	0.5633	0.0694
TL-GW	532.2276	545.4991	545.8874	556.2276	0.1076	0.6895	0.0767	0.5528	0.1006
OEHLBXII	578.6627	587.0510	587.0510	597.3912	0.2920	1.8555	0.1234	0.0745	0.5096
MOMW	536.9594	553.9464	554.3347	564.6749	0.0922	0.5263	0.1141	0.1202	35.9577
MOKA	538.0226	546.0226	546.4109	556.7511	0.1143	0.6466	0.0739	0.5964	0.0957

Table 5.8 shows that there are significant differences between nested models and the MO-EHL-TII-TL-W distribution. We conclude that MO-EHL-TII-TL-W model performs better than its nested models on the Coronavirus mortality rate dataset.

Table 5.8: LRT Results For Coronavirus Mortality Rate Data

Coronavirus Mortality Rate Data		
Model	df	$\chi^2(p-value)$
$MO-EHL-TII-TL-W(\delta, 1, b, \lambda)$	1	15.4810 (0.000083)
$MO-EHL-TII-TL-W(\delta, \alpha, 1, \lambda)$	1	3.9910 (0.045744)
$MO-EHL-TII-TL-W(\delta, 1, 1, \lambda)$	2	77.2111 (< 0.00001)
$MO-EHL-TII-TL-W(1, 1, 1, \lambda)$	3	95.2894 (< 0.00001)

Figure 5.15 represent the plots of estimated densities, and ECDF.

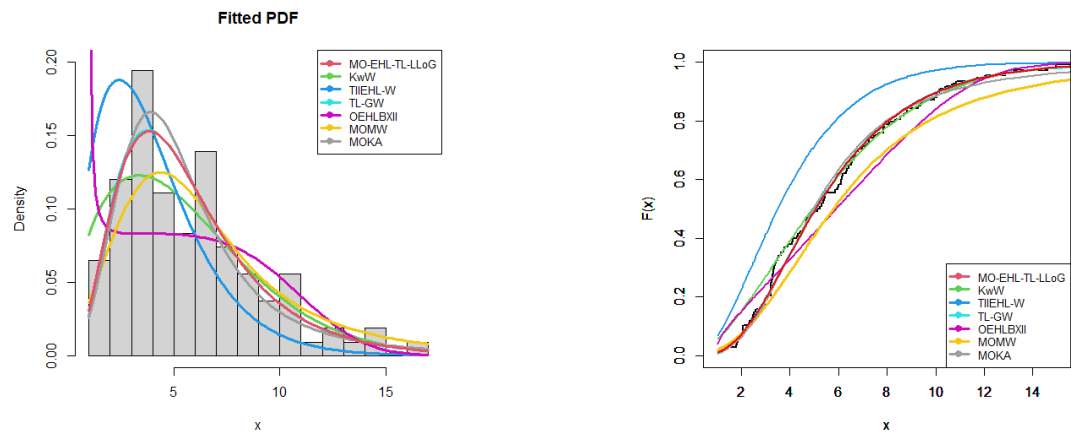


Figure 5.15: Estimated Densities and ECDF for Coronavirus Data

The PPs for the models and the TTT scaled plots are given in Figure 5.16.

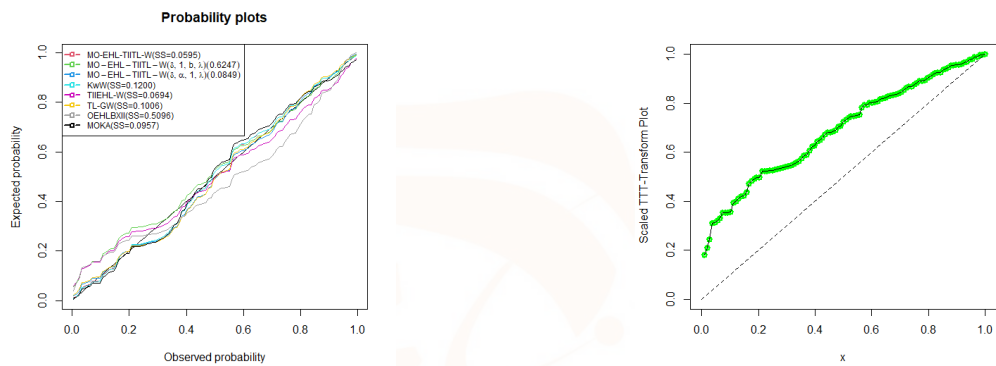


Figure 5.16: PPs and TTT Scaled Plot for Coronavirus Mortality Rate Data

The K-M survival and hrf plots for the proposed model are given in Figure 5.17. Based on the TTT scaled plot, the hrf of the MO-EHL-TII-TL-W distribution have an increasing shape.

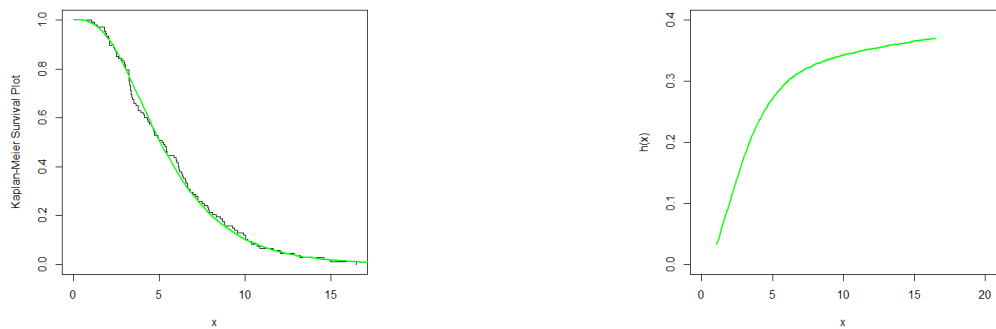


Figure 5.17: Fitted K-M Survival and Hrf for Coronavirus Mortality Rate Data

We observe that the ML estimates for the MO-EHL-TII-TL-W distribution exist and can be obtained uniquely, as shown in 5.18.

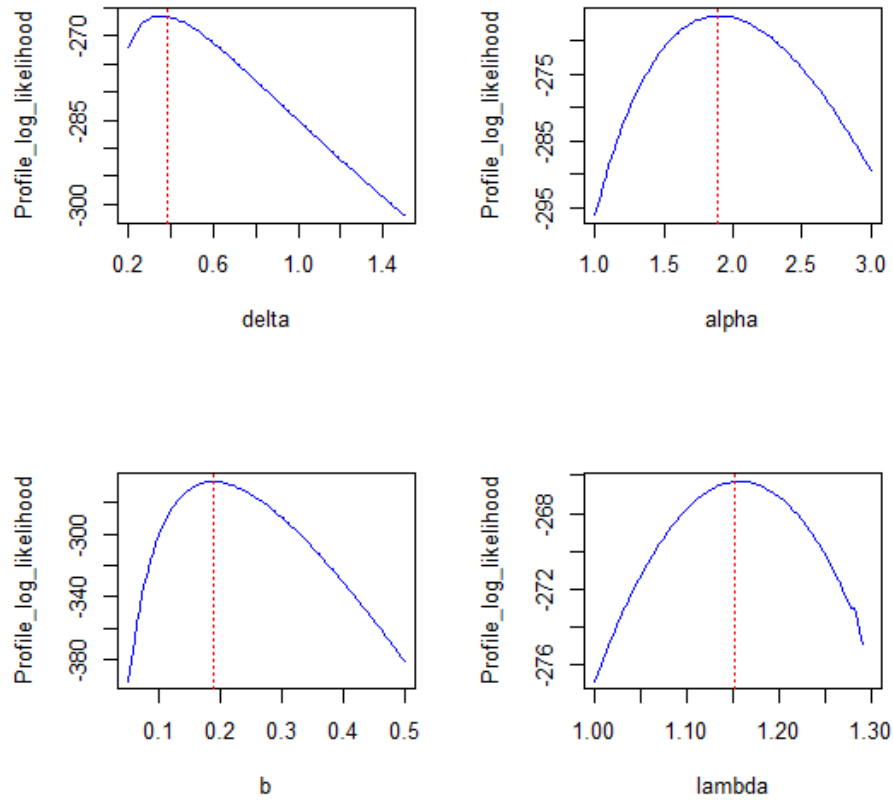


Figure 5.18: Plots of the Profile Log-Likelihood Function of the Parameters of the MO-EHL-TII-TL-W Distribution on Corona Virus Mortality Rate Data

5.7 Conclusions

We have developed and studied the statistical properties of the new FoD called the MO-EHL-TII-TL-G distribution. Statistical properties including the pdf, cdf, hrf, QF, moments, MGF, order statistics, RE, and ML estimates were presented. Some special cases of the MO-EHL-TII-TL-G FoD were also considered. We ran a MCS study to see how consistent the ML estimates were. Two real-life datasets from various areas are used to test the GoF of the MO-EHL-TII-TL-W distribution. The model can be used to explain real-life phenomena in a variety of fields. For example, the MO-EHL-TII-TL-W distribution can model rising COVID-19 mortality rates, indicating possible increases due to healthcare system overload, new variants, and delays. This hrf can help governments allocate resources, implement targeted interventions, and strengthen public health measures to combat the rising mortality rate.

6. TL-Harris-G Family of Distributions with Applications

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6.1 Introduction

The Topp-Leone distribution (Topp and Leone [134]) is characterized by a bathtub-shaped hrf. This distribution is defined on the interval $(0,1)$. The result obtained by Al-Shomrani et al. [8] expanded the Topp-Leone distribution to improve its flexibility when modeling data. In this chapter, we combine the TL distribution (Al-Shomrani et al. [8]) and Harris distribution (Harris [51]) to develop the TL-Harris-G FoD.

The motivation for developing this new FoD is based on the increased flexibility acquired by combining the Harris distribution and the TL-G distributions in modelling real-world datasets. This new FoD can be used with heavy-tailed, right or left-skewed datasets. In comparison to the Harris-G distribution, the TL-Harris-G FoD has multiple shapes of the hrf, such as monotonic, BT, UBT, and UBT-BT. The TL-Harris-G FoD can be expressed as an infinite linear combination of the Exp-G distribution and other statistical characteristics can be deduced using this property.

We examine some structural properties, such as moments, MGF, distribution of order statistics, entropy, and PWMs of the new FoD. Some special cases, such as TL-Harris-LLoG, TL-Harris-U and TL-Harris-W distributions are given. We exhibit plots of the pdfs and hrfs, as well as SK and KS to illustrate the shapes that TL-Harris-G FoD can accommodate when modeling different datasets. The ML estimation method is used to estimate model parameters. To illustrate the consistency and efficiency of parameter estimations, simulations are conducted, and real-world datasets are used to demonstrate the applicability of the proposed model. The proposed model proves to be a better alternative to some well known models. The results show that the special case of the TL-Harris-W distribution captures right-skewed datasets very well.

6.2 The Model, Sub-Models and Properties

This section introduces the TL-Harris-G FoD, as well as its sub-models and a detailed development of its statistical properties.

6.2.1 The Model

The cdf of the new TL-Harris-G FoD is given by

$$F(x; b, v, c, \xi) = \left(1 - \left(\frac{c (\overline{G}(x; \xi))^v}{1 - \bar{c} (\overline{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right)^b, \quad (6.1)$$

for $v, c > 0$, $\bar{c} = 1 - c$, $x \geq 0$, and parameter vector ξ . The corresponding pdf is given by

$$\begin{aligned} f(x; b, v, c, \xi) &= 2bc^{\frac{2}{v}} \left[1 - \left(\frac{c (\overline{G}(x; \xi))^v}{1 - \bar{c} (\overline{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]^{b-1} \\ &\times (1 - \bar{c} (\overline{G}(x; \xi))^v)^{-\left(\frac{2}{v}+1\right)} \overline{G}(x; \xi) g(x; \xi). \end{aligned} \quad (6.2)$$

The hrf is given as

$$\begin{aligned} h(x; b, v, c, \xi) &= 2bc^{\frac{2}{v}} \left[1 - \left(\frac{c (\overline{G}(x; \xi))^v}{1 - \bar{c} (\overline{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]^{b-1} (1 - \bar{c} (\overline{G}(x; \xi))^v)^{-\left(\frac{2}{v}+1\right)} \\ &\times \overline{G}(x; \xi) g(x; \xi) \left(1 - \left[1 - \left(\frac{c (\overline{G}(x; \xi))^v}{1 - \bar{c} (\overline{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]^b \right)^{-1}, \end{aligned} \quad (6.3)$$

for $v, c > 0$, $\bar{c} = 1 - c$, $x \geq 0$, and parameter vector ξ .

6.2.2 Identifiability

Identifiability ensures model parameters can be determined uniquely based on observed data. We demonstrate that the TL-Harris-G FoD is identifiable. We let $\Psi_1 = (b_1, v_1, c_1, \xi_1)$ and $\Psi_2 = (b_2, v_2, c_2, \xi_2)$. The likelihood ratio is given as:

$$\frac{f(\Psi_1)}{f(\Psi_2)} = \frac{b_1 c_1^{\frac{2}{v_1}} \left[1 - \left(\frac{c_1 (\overline{G}(x; \xi_1))^{v_1}}{1 - \bar{c}_1 (\overline{G}(x; \xi_1))^{v_1}} \right)^{\frac{2}{v_1}} \right]^{b_1-1} (1 - \bar{c}_1 (\overline{G}(x; \xi_1))^{v_1})^{-\left(\frac{2}{v_1}+1\right)} \overline{G}(x; \xi_1) g(x; \xi_1)}{b_2 c_2^{\frac{2}{v_2}} \left[1 - \left(\frac{c_2 (\overline{G}(x; \xi_2))^{v_2}}{1 - \bar{c}_2 (\overline{G}(x; \xi_2))^{v_2}} \right)^{\frac{2}{v_2}} \right]^{b_2-1} (1 - \bar{c}_2 (\overline{G}(x; \xi_2))^{v_2})^{-\left(\frac{2}{v_2}+1\right)} \overline{G}(x; \xi_2) g(x; \xi_2)}.$$

Taking the logarithm of the likelihood ratio, we have:

$$\begin{aligned}
 \ln \left(\frac{f(\Psi_1)}{f(\Psi_2)} \right) &= \ln \left(\frac{b_1 c_1^{\frac{2}{v_1}}}{b_2 c_2^{\frac{2}{v_2}}} \right) + (b_1 - 1) \ln \left(1 - \left(\frac{c_1 (\overline{G}(x; \xi_1))^{v_1}}{1 - \overline{c}_1 (\overline{G}(x; \xi_1))^{v_1}} \right)^{\frac{2}{v_1}} \right) \\
 &- (b_2 - 1) \ln \left(1 - \left(\frac{c_2 (\overline{G}(x; \xi_2))^{v_2}}{1 - \overline{c}_2 (\overline{G}(x; \xi_2))^{v_2}} \right)^{\frac{2}{v_2}} \right) + \left(\frac{2}{v_2} + 1 \right) \ln (1 - \overline{c}_2 (\overline{G}(x; \xi_2))^{v_2}) \\
 &- \left(\frac{2}{v_1} + 1 \right) \ln (1 - \overline{c}_1 (\overline{G}(x; \xi_1))^{v_1}) + \ln(\overline{G}(x; \xi_1)) + \ln(g(x; \xi_1)) \\
 &- \ln(\overline{G}(x; \xi_2)) - \ln(g(x; \xi_2)).
 \end{aligned}$$

The derivative of the log-likelihood ratio is given as

$$\begin{aligned}
 \frac{d}{dx} \left(\ln \left(\frac{f(\Psi_1)}{f(\Psi_2)} \right) \right) &= (b_1 - 1) \frac{E_1}{\left(1 - \left(\frac{c_1 (\overline{G}(x; \xi_1))^{v_1}}{1 - \overline{c}_1 (\overline{G}(x; \xi_1))^{v_1}} \right)^{\frac{2}{v_1}} \right)} - (b_2 - 1) \frac{E_2}{\left(1 - \left(\frac{c_2 (\overline{G}(x; \xi_2))^{v_2}}{1 - \overline{c}_2 (\overline{G}(x; \xi_2))^{v_2}} \right)^{\frac{2}{v_2}} \right)} \\
 &- \left(\frac{2}{v_1} + 1 \right) \frac{\overline{c}_1 v_1 [\overline{G}(x; \xi_1)]^{v_1-1} g(x; \xi_1)}{(1 - \overline{c}_1 (\overline{G}(x; \xi_1))^{v_1})} + \left(\frac{2}{v_2} + 1 \right) \frac{\overline{c}_2 v_2 [\overline{G}(x; \xi_2)]^{v_2-1} g(x; \xi_2)}{(1 - \overline{c}_2 (\overline{G}(x; \xi_2))^{v_2})} \\
 &- \frac{g(x; \xi_1)}{\overline{G}(x; \xi_1)} + \frac{g'(x; \xi_1)}{g(x; \xi_1)} + \frac{g(x; \xi_2)}{\overline{G}(x; \xi_2)} - \frac{g'(x; \xi_2)}{g(x; \xi_2)},
 \end{aligned} \tag{6.4}$$

where

$$E_1 = 2c_1^{\frac{2}{v_1}} (1 - \overline{c}_1 (\overline{G}(x; \xi_1))^{v_1})^{-\left(\frac{2}{v_1} + 1\right)} \overline{G}(x; \xi_1) g(x; \xi_1),$$

and

$$E_2 = 2c_2^{\frac{2}{v_2}} (1 - \overline{c}_2 (\overline{G}(x; \xi_2))^{v_2})^{-\left(\frac{2}{v_2} + 1\right)} \overline{G}(x; \xi_2) g(x; \xi_2).$$

Therefore, $\frac{d}{dx} \left(\ln \left(\frac{f(\Psi_1)}{f(\Psi_2)} \right) \right) = 0$, implying that the TL-Harris-G FoD is identifiable and can be uniquely determined if and only if $\Psi_1 = \Psi_2$.

6.2.3 Sub-Families of TL-Harris-G FoD

This section introduces some new and well known sub-families of TL-Harris-G FoD.

- When $b = 1$, we get a new FoD with the cdf:

$$F(x; v, c, \xi) = \left[1 - \left(\frac{c (\overline{G}(x; \xi))^v}{1 - \overline{c} (\overline{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]$$

for $v, c > 0$, $x \geq 0$, $\overline{c} = 1 - c$, and parameter vector ξ .

- When $v = 1$, we get the TL-MO-G FoD by Chipepa et al. [28] with the cdf:

$$F(x; b, c, \xi) = \left[1 - \left(\frac{c (\bar{G}(x; \xi))}{1 - \bar{c} (\bar{G}(x; \xi))} \right)^2 \right]^b$$

for $b, c > 0$, $x \geq 0$, $\bar{c} = 1 - c$, and parameter vector ξ .

- When $b = v = 1$, we get the new FoD with the cdf:

$$F(x; c, \xi) = \left[1 - \left(\frac{c (\bar{G}(x; \xi))}{1 - \bar{c} (\bar{G}(x; \xi))} \right)^2 \right]$$

for $c > 0$, $x \geq 0$, $\bar{c} = 1 - c$, and parameter vector ξ .

- When $b = c = 1$, we get the FoD with the cdf:

$$F(x; v, \xi) = \left[1 - (\bar{G}(x; \xi))^2 \right]$$

for $v > 0$, $x \geq 0$, and parameter vector ξ .

- When $v = c = 1$, we get the TL-G FoD with the cdf:

$$F(x; b, \xi) = \left[1 - (\bar{G}(x; \xi))^2 \right]^b$$

for $b > 0$, $x \geq 0$, and parameter vector ξ .

- When $b = v = c = 1$, we get the FoD with the cdf:

$$F(x; \xi) = \left[1 - (\bar{G}(x; \xi))^2 \right]$$

for $x \geq 0$ and parameter vector ξ .

6.2.4 Quantile Function

Let X be a random variable with cdf defined by Equation (6.1). The QF ($Q_X(u)$) is defined by $F_X(Q_X(u)) = u$, $0 \leq u \leq 1$. Note that

$$\left[1 - \left(\frac{c (\bar{G}(x; \xi))^v}{1 - \bar{c} (\bar{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]^b = u,$$

so that

$$\bar{G}(x; \xi) = \left[\frac{\left(\left[\left(1 - u^{\frac{1}{b}} \right)^{\frac{v}{2}} \right]^{-1} + \frac{\bar{c}}{c} \right)^{-1}}{c} \right]^{\frac{1}{v}}.$$

That is,

$$G(x; \xi) = 1 - \left[\frac{\left(\left[\left(1 - u^{\frac{1}{b}} \right)^{\frac{v}{2}} \right]^{-1} + \frac{\bar{c}}{c} \right)^{-1}}{c} \right]^{\frac{1}{v}}.$$

Therefore, the QF of the TL-Harris-G FoD is given by,

$$Q_X(u) = G^{-1} \left(1 - \left[\frac{\left(\left[\left(1 - u^{\frac{1}{b}} \right)^{\frac{v}{2}} \right]^{-1} + \frac{\bar{c}}{c} \right)^{-1}}{c} \right]^{\frac{1}{v}} \right). \quad (6.5)$$

It follows therefore that random numbers can be generated from the TL-Harris-G FoD using Equation (6.5) for specified parent cdf G.

6.2.5 Expansion of Density

Expansion of the density function of the TL-Harris-G FoD is presented in this sub-section. Using the generalized binomial series expansion.

$$\left[1 - \left(\frac{c (\bar{G}(x; \xi))^v}{1 - \bar{c} (\bar{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]^{b-1} = \sum_{k=0}^{\infty} \binom{b-1}{k} (-1)^k \left(\frac{c (\bar{G}(x; \xi))^v}{1 - \bar{c} (\bar{G}(x; \xi))^v} \right)^{\frac{2k}{v}},$$

we write the pdf of the TL-Harris-G FoD as

$$\begin{aligned} f(x; b, v, c, \xi) &= 2b \sum_{k=0}^{\infty} \binom{b-1}{k} (-1)^k c^{\frac{2(1+k)}{v}} \\ &\times (1 - \bar{c} (\bar{G}(x; \xi))^v)^{-\left(\frac{2(1+k)}{v} + 1\right)} \bar{G}^{2k+1}(x; \xi) g(x; \xi). \end{aligned}$$

Considering the expansion

$$(1 - \bar{c} (\bar{G}(x; \xi))^v)^{-\left(\frac{2(1+k)}{v} + 1\right)} = \sum_{p=0}^{\infty} (-1)^p \binom{\frac{2(1+k)}{v} + 1}{p} \bar{c}^p \bar{G}^{vp}(x; \xi),$$

we write

$$\begin{aligned}
 f(x; b, v, c, \xi) &= 2b \sum_{k,p=0}^{\infty} \binom{b-1}{k} \left(\frac{2(1+k)}{v} + p \right) (-1)^{k+p} c^{\frac{2(1+k)}{v}} \bar{c}^p \\
 &\times \bar{G}^{2k+vp+1}(x; \xi) g(x; \xi) \\
 &= 2b \sum_{k,p,u=0}^{\infty} \binom{b-1}{k} \left(\frac{2(1+k)}{v} + p \right) (-1)^{k+p+u} c^{\frac{2(1+k)}{v}} \bar{c}^p \\
 &\times \binom{2k+vp+1}{u} G^u(x; \xi) g(x; \xi) \\
 &= 2b \sum_{k,p,u=0}^{\infty} \binom{b-1}{k} \left(\frac{2(1+k)}{v} + p \right) (-1)^{k+p+u} c^{\frac{2(1+k)}{v}} \bar{c}^p \\
 &\times \binom{2k+vp+1}{u} \left(\frac{u+1}{u+1} \right) G^u(x; \xi) g(x; \xi) \\
 &= \sum_{u=0}^{\infty} b_{u+1} g_{u+1}(x; \xi), \tag{6.6}
 \end{aligned}$$

where $g_{u+1}(x; \xi) = (u+1) (G(x; \xi))^u g(x; \xi)$ is the Exp-G distribution with power parameter $(u+1)$ and

$$b_{u+1} = 2 \sum_{k,p=0}^{\infty} \binom{b-1}{k} \left(\frac{2(1+k)}{v} + p \right) (-1)^{k+p+u} \binom{2k+vp+1}{u} \left(\frac{bc^{\frac{2(1+k)}{v}} \bar{c}^p}{u+1} \right). \tag{6.7}$$

Thus, the pdf of TL-Harris-G FoD can be written as an infinite linear combination of Exp-G distribution.

6.2.6 Moments and Generating Function

If X follows the TL-Harris-G FoD and $Y \sim \text{Exp} - G(u+1)$. Then using Equation (6.6), the η^{th} raw moment, μ'_η of the TL-Harris-G FoD is obtained as

$$\mu'_\eta = E(X^\eta) = \int_{-\infty}^{\infty} x^\eta f(x; b, v, c, \xi) dx = \sum_{u=0}^{\infty} b_{u+1} E(Y^\eta),$$

where $E(Y^\eta)$ is the η^{th} moment of Exp-G distribution and b_{u+1} is given by Equation (6.7). The MGF $M(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{u=0}^{\infty} b_{u+1} M_Y(t),$$

where $M_Y(t)$ is the MGF of Y and b_{u+1} is given by Equation (6.7).

6.2.7 Order Statistics

In this section, we present the distribution of the i^{th} order statistic. Order statistics play a significant role in many areas of statistical theory and practice. Let $X_1, X_2, \dots, X_n$ be a random sample from TL-Harris-G FoD. The pdf of the i^{th} order statistic is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l f(x) [F(x)]^{i+l-1}. \quad (6.8)$$

Using Equations (6.1) and (6.3), we note that

$$\begin{aligned} f(x) [F(x)]^{i+l-1} &= 2bc^{\frac{2}{v}} \left[1 - \left(\frac{c (\overline{G}(x; \xi))^v}{1 - \bar{c} (\overline{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]^{b(i+l)-1} \\ &\times (1 - \bar{c} (\overline{G}(x; \xi))^v)^{-\left(\frac{2}{v}+1\right)} \overline{G}(x; \xi) g(x; \xi). \end{aligned}$$

Using the generalized binomial series expansion:

$$\left[1 - \left(\frac{c (\overline{G}(x; \xi))^v}{1 - \bar{c} (\overline{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]^{b(i+l)-1} = \sum_{k=0}^{\infty} \binom{b(i+l)-1}{k} (-1)^k \left(\frac{c (\overline{G}(x; \xi))^v}{1 - \bar{c} (\overline{G}(x; \xi))^v} \right)^{\frac{2k}{v}},$$

we write

$$\begin{aligned} f(x) [F(x)]^{i+l-1} &= 2b \sum_{k=0}^{\infty} \binom{b(i+l)-1}{k} (-1)^k c^{\frac{2(1+k)}{v}} \\ &\times (1 - \bar{c} (\overline{G}(x; \xi))^v)^{-\left(\frac{2(1+k)}{v}+1\right)} \overline{G}^{2k+1}(x; \xi) g(x; \xi). \end{aligned}$$

Considering the expansion

$$(1 - \bar{c} (\overline{G}(x; \xi))^v)^{-\left(\frac{2(1+k)}{v}+1\right)} = \sum_{p=0}^{\infty} (-1)^p \binom{\frac{2(1+k)}{v} + p}{p} \bar{c}^p \overline{G}^{vp}(x; \xi),$$

we have

$$\begin{aligned}
 f(x) [F(x)]^{i+l-1} &= 2b \sum_{k,p=0}^{\infty} \binom{b(i+l)-1}{k} \left(\frac{2(1+k)}{v} + p \right) (-1)^{k+p} c^{\frac{2(1+k)}{v}} \bar{c}^p \\
 &\times \bar{G}^{2k+vp+1}(x; \xi) g(x; \xi) \\
 &= 2b \sum_{k,p,u=0}^{\infty} \binom{b(i+l)-1}{k} \left(\frac{2(1+k)}{v} + p \right) (-1)^{k+p+u} \\
 &\times \binom{2k+vp+1}{u} G^u(x; \xi) g(x; \xi) c^{\frac{2(1+k)}{v}} \bar{c}^p \\
 &= 2b \sum_{k,p,u=0}^{\infty} \binom{b(i+l)-1}{k} \left(\frac{2(1+k)}{v} + p \right) (-1)^{k+p+u} c^{\frac{2(1+k)}{v}} \bar{c}^p \\
 &\times \binom{2k+vp+1}{u} \left(\frac{u+1}{u+1} \right) G^u(x; \xi) g(x; \xi) \\
 &= \sum_{u=0}^{\infty} B_{u+1} g_{u+1}(x; \xi). \tag{6.9}
 \end{aligned}$$

Therefore, the distribution of the i^{th} order statistics from TL-Harris-G FoD is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \sum_{u=0}^{\infty} \binom{n-i}{l} (-1)^l B_{u+1} g_{u+1}(x; \xi), \tag{6.10}$$

where $g_{u+1}(x; \xi) = (u+1)g(x; \xi)G^u(x; \xi)$ is an Exp-G with power parameter $(u+1)$ and linear component

$$\begin{aligned}
 B_{u+1} &= \sum_{k,p=0}^{\infty} \binom{b(i+l)-1}{k} \left(\frac{2(1+k)}{v} + p \right) (-1)^{k+p+u} \\
 &\times \binom{2k+vp+1}{u} \left(\frac{2bc^{\frac{2(1+k)}{v}} \bar{c}^p}{u+1} \right). \tag{6.11}
 \end{aligned}$$

6.2.8 Rényi Entropy

RE by Rényi [116] is an extension of Shannon entropy (Shannon [122]). RE is defined to be

$$I_R(w) = \frac{1}{1-w} \log \left(\int_0^{\infty} [f(x; b, v, c, \xi)]^w dx \right), w \neq 1, w > 0. \tag{6.12}$$

Note that

$$\begin{aligned} [f(x; b, v, c, \xi)]^w &= \left(2bc^{\frac{2}{v}}\right)^w \left[1 - \left(\frac{c(\overline{G}(x; \xi))^v}{1 - \bar{c}(\overline{G}(x; \xi))^v}\right)^{\frac{2}{v}}\right]^{w(b-1)} \\ &\times (1 - \bar{c}(\overline{G}(x; \xi))^v)^{-w(\frac{2}{v}+1)} \overline{G}^w(x; \xi) g^w(x; \xi). \end{aligned}$$

Using the generalized binomial series expansion:

$$\left[1 - \left(\frac{c(\overline{G}(x; \xi))^v}{1 - \bar{c}(\overline{G}(x; \xi))^v}\right)^{\frac{2}{v}}\right]^{w(b-1)} = \sum_{k=0}^{\infty} \binom{w(b-1)}{k} (-1)^k \left(\frac{c(\overline{G}(x; \xi))^v}{1 - \bar{c}(\overline{G}(x; \xi))^v}\right)^{\frac{2k}{v}},$$

we can write

$$\begin{aligned} [f(x; b, v, c, \xi)]^v &= (2b)^w \sum_{k=0}^{\infty} \binom{w(b-1)}{k} (-1)^k c^{\frac{2(w+k)}{v}} \\ &\times (1 - \bar{c}(\overline{G}(x; \xi))^v)^{-\left(\frac{2(k+w)}{v}+w\right)} \overline{G}^{2k+w}(x; \xi) g^w(x; \xi). \end{aligned}$$

Considering the expansion

$$(1 - \bar{c}(\overline{G}(x; \xi))^v)^{-\left(\frac{2(k+w)}{v}+w\right)} = \sum_{p=0}^{\infty} (-1)^p \binom{\frac{2(k+w)}{v} + w + p - 1}{p} \bar{c}^p \overline{G}^{vp}(x; \xi),$$

we obtain

$$\begin{aligned} [f(x; b, v, c, \xi)]^v &= (2b)^w \sum_{k,p=0}^{\infty} \binom{w(b-1)}{k} (-1)^{k+p} c^{\frac{2(w+k)}{v}} \bar{c}^p \\ &\times \overline{G}^{2k+w+vp}(x; \xi) g^w(x; \xi) \\ &= (2b)^w \sum_{k,p,u=0}^{\infty} \binom{w(b-1)}{k} (-1)^{k+p+u} c^{\frac{2(w+k)}{v}} \bar{c}^p \\ &\times \binom{2k+w+vp}{u} \binom{\frac{2(k+w)}{v} + w + p - 1}{p} \overline{G}^u(x; \xi) g^w(x; \xi). \end{aligned}$$

Therefore, RE for TL-Harris-G FoD is given by

$$I_R(v) = \frac{1}{1-w} \log \left(\sum_{u=0}^{\infty} a_u^* e^{(1-w)I_{REG}} \right), \quad (6.13)$$

where $I_{REG} = \int_0^\infty [(1 + u/w)g(x; \xi)G^{u/w}]^w dx$ is RE for an Exp-G distribution with power parameter $u/w + 1$ and

$$\begin{aligned} a_u^* &= (2b)^w c^{\frac{2(w+k)}{v}} \bar{c}^p \sum_{k,p=0}^{\infty} \binom{w(b-1)}{k} \binom{\frac{2(k+w)}{v} + w + p - 1}{p} (-1)^{k+p+u} \\ &\times \binom{2k + w + vp}{u} \frac{1}{(1 + u/w)^w}. \end{aligned} \quad (6.14)$$

Consequently, RE for TL-Harris-G FoD can be obtained from RE of the Exp-G distribution.

6.2.9 Probability Weighted Moments

The PWMs approach can be used to estimate parameters of a distribution whose inverse form is not explicitly defined. The PWMs based estimates are frequently thought to be superior to traditional moment-based estimates. When ML estimates are unavailable or difficult to compute, they are occasionally used. The PWMs are defined as follows:

$$Z_{\eta,i} = E(X^\eta [F(X)]^i) = \int_0^\infty x^\eta [F(x)]^i f(x) dx.$$

Using Equations (6.1) and (6.3), we note that

$$\begin{aligned} f(x) [F(x)]^i &= 2bc^{\frac{2}{v}} \left[1 - \left(\frac{c (\bar{G}(x; \xi))^v}{1 - \bar{c} (\bar{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]^{b(i+1)-1} \\ &\times \left(1 - \bar{c} (\bar{G}(x; \xi))^v \right)^{-\left(\frac{2}{v}+1\right)} \bar{G}(x; \xi) g(x; \xi). \end{aligned}$$

Using the generalized binomial series expansion:

$$\left[1 - \left(\frac{c (\bar{G}(x; \xi))^v}{1 - \bar{c} (\bar{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right]^i = \sum_{k=0}^{\infty} \binom{b(i+1)-1}{k} (-1)^k \left(\frac{c (\bar{G}(x; \xi))^v}{1 - \bar{c} (\bar{G}(x; \xi))^v} \right)^{\frac{2k}{v}},$$

we can write

$$\begin{aligned} f(x) [F(x)]^{i+l-1} &= 2b \sum_{k=0}^{\infty} \binom{b(i+1)-1}{k} (-1)^k c^{\frac{2(1+k)}{v}} \\ &\times \left(1 - \bar{c} (\bar{G}(x; \xi))^v \right)^{-\left(\frac{2(1+k)}{v}+1\right)} \bar{G}^{2k+1}(x; \xi) g(x; \xi). \end{aligned}$$

Considering the expansion

$$\left(1 - \bar{c} (\bar{G}(x; \xi))^v \right)^{-\left(\frac{2(1+k)}{v}+1\right)} = \sum_{p=0}^{\infty} (-1)^p \binom{\frac{2(1+k)}{v} + p}{p} \bar{c}^p \bar{G}^{vp}(x; \xi),$$

we have

$$\begin{aligned}
 f(x) [F(x)]^i &= 2b \sum_{k,p=0}^{\infty} \binom{b(i+1)-1}{k} \binom{\frac{2(1+k)}{v} + p}{p} (-1)^{k+p} c^{\frac{2(1+k)}{v}} \bar{c}^p \\
 &\times \bar{G}^{2k+vp+1}(x; \xi) g(x; \xi) \\
 &= 2b \sum_{k,p,u=0}^{\infty} \binom{b(i+1)-1}{k} \binom{\frac{2(1+k)}{v} + p}{p} (-1)^{k+p+u} \\
 &\times \binom{2k+vp+1}{u} G^u(x; \xi) g(x; \xi) c^{\frac{2(1+k)}{v}} \bar{c}^p \\
 &= 2b \sum_{k,p,u=0}^{\infty} \binom{b(i+1)-1}{k} \binom{\frac{2(1+k)}{v} + p}{p} (-1)^{k+p+u} c^{\frac{2(1+k)}{v}} \bar{c}^p \\
 &\times \binom{2k+vp+1}{u} \binom{u+1}{u+1} G^u(x; \xi) g(x; \xi) \\
 &= \sum_{u=0}^{\infty} D_{u+1} g_{u+1}(x; \xi). \tag{6.15}
 \end{aligned}$$

Therefore, the $(\eta, i)^{th}$ PMWs of the TL-Harris-G FoD is given by

$$Z_{\eta,i} = \int_0^{\infty} x^{\eta} \sum_{u=0}^{\infty} D_{u+1} g_{u+1}(x; \xi) dx = \sum_{u=0}^{\infty} D_{u+1} \int_0^{\infty} x^{\eta} g_{u+1}(x; \xi) dx,$$

where $g_{u+1}(x; \xi) = (u+1)g(x; \xi)G^u(x; \xi)$ is an Exp-G with power parameter $(u+1)$ and D_{u+1} is given Equation (6.16) below .

$$\begin{aligned}
 D_{u+1} &= \sum_{k,p=0}^{\infty} \binom{b(i+1)-1}{k} \binom{\frac{2(1+k)}{v} + p}{p} (-1)^{k+p+u} \\
 &\times \binom{2k+vp+1}{u} \left(\frac{2bc^{\frac{2(1+k)}{v}} \bar{c}^p}{u+1} \right). \tag{6.16}
 \end{aligned}$$

6.3 Some Special Cases of TL-Harris-G FoD

In this section, we consider the LLoG, Weibull, and uniform distributions as baseline distributions of the TL-Harris-G FoD.

6.3.1 Topp-Leone-Harris-Log-Logistic (TL-Harris-LLoG) Distribution

If the LLoG distribution is used as the baseline distribution with the pdf and cdf given by $g(x; \lambda) = \lambda x^{\lambda-1} (1+x^{\lambda})^{-2}$ and $G(x; \lambda) = 1 - (1+x^{\lambda})^{-1}$, respectively, for $\lambda > 0$ and $x > 0$, then the TL-Harris-LLoG distribution cdf and pdf are given by

$$F(x; b, v, c, \lambda) = \left[1 - \left(\frac{c ((1+x^{\lambda})^{-1})^v}{1 - \bar{c} ((1+x^{\lambda})^{-1})^v} \right)^{\frac{2}{v}} \right]^b \tag{6.17}$$

and

$$\begin{aligned}
 f(x; b, v, c, \lambda) &= 2bc^{\frac{2}{v}} \left[1 - \left(\frac{c((1+x^\lambda)^{-1})^v}{1 - \bar{c}((1+x^\lambda)^{-1})^v} \right)^{\frac{2}{v}} \right]^{b-1} \\
 &\times \left(1 - \bar{c}((1+x^\lambda)^{-1})^v \right)^{-\left(\frac{2}{v}+1\right)} (1+x^\lambda)^{-1} \\
 &\times \lambda x^{\lambda-1} (1+x^\lambda)^{-2},
 \end{aligned} \tag{6.18}$$

respectively, for $x > 0$, $b, v, c, \lambda > 0$, and $\bar{c} = 1 - c$. The hrf is given by

$$\begin{aligned}
 h(x; b, v, c, \lambda) &= 2bc^{\frac{2}{v}} \left[1 - \left(\frac{c((1+x^\lambda)^{-1})^v}{1 - \bar{c}((1+x^\lambda)^{-1})^v} \right)^{\frac{2}{v}} \right]^{b-1} \left(1 - \bar{c}((1+x^\lambda)^{-1})^v \right)^{-\left(\frac{2}{v}+1\right)} \\
 &\times (1+x^\lambda)^{-1} \lambda x^{\lambda-1} (1+x^\lambda)^{-2} \left(1 - \left[1 - \left(\frac{c((1+x^\lambda)^{-1})^v}{1 - \bar{c}((1+x^\lambda)^{-1})^v} \right)^{\frac{2}{v}} \right]^b \right)^{-1},
 \end{aligned} \tag{6.19}$$

for $x > 0$, $v, c, \lambda > 0$ and $\bar{c} = 1 - c$.

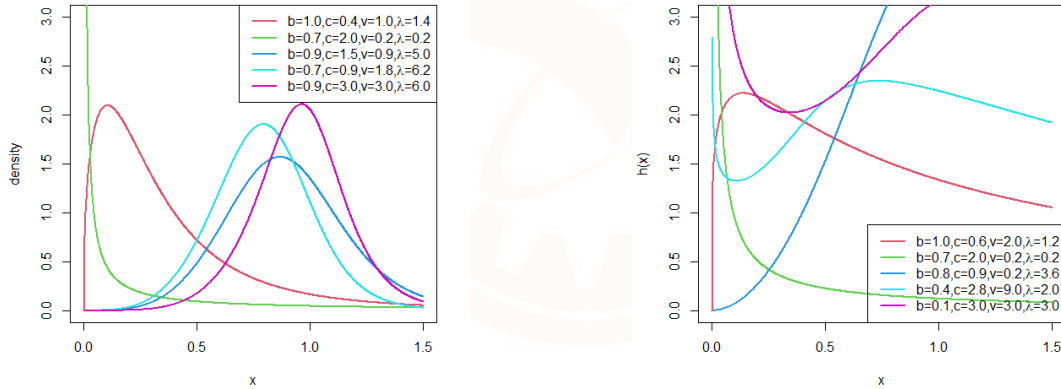


Figure 6.1: Pdf and Hrf Plots of TL-Haris-LLoG Distribution

The TL-Harris-LLoG pdf plots shows left-skewed or right-skewed, symmetric, reverse J-shape among many other shapes, whereas the TL-Harris-LLoG hrf can be increasing, decreasing, BT, and BT-UBT shapes, as shown in Figure 6.1.



Figure 6.2: SK and KS Plots of TL-Harris-LLoG Distribution for Fixed Values of Parameters c and λ



Figure 6.3: SK and KS Plots of TL-Harris-LLoG Distribution for Fixed Values of Parameters b and v

Figures 6.2 and 6.3 show 3D graphs of SK and KS for the TL-Harris-LLoG distribution. We notice that

- When the parameters c and λ are fixed, the SK and KS of TL-Harris-LLoG distribution decrease then increase as b and v increase.
- When the parameters b and v are fixed, the SK and KS of TL-Harris-LLoG distribution decrease as c and λ increase.

6.3.2 Topp-Leone-Harris-Weibull (TL-Harris-W) Distribution

Suppose that the Weibull distribution is considered as the baseline distribution with the pdf and cdf given by $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$ and $G(x; \lambda) = 1 - e^{-x^\lambda}$, respectively, for $\lambda > 0$ and $x > 0$, we obtain the TL-Harris-W distribution with cdf and pdf given by

$$F(x; b, v, c, \lambda) = \left[1 - \left(\frac{c (e^{-x^\lambda})^v}{1 - c (e^{-x^\lambda})^v} \right)^{\frac{2}{v}} \right]^b \quad (6.20)$$

and

$$\begin{aligned}
 f(x; b, v, c, \lambda) &= 2bc^{\frac{2}{v}} \left[1 - \left(\frac{c(e^{-x^\lambda})^v}{1 - \bar{c}(e^{-x^\lambda})^v} \right)^{\frac{2}{v}} \right]^{b-1} \\
 &\times \left(1 - \bar{c}(e^{-x^\lambda})^v \right)^{-\left(\frac{2}{v}+1\right)} \lambda x^{\lambda-1} e^{-2x^\lambda}
 \end{aligned} \tag{6.21}$$

for $x > 0$, $b, v, c, \lambda > 0$, and $\bar{c} = 1 - c$, respectively. The hrf is given as

$$\begin{aligned}
 h(x; b, v, c, \lambda) &= 2bc^{\frac{2}{v}} \left[1 - \left(\frac{c(e^{-x^\lambda})^v}{1 - \bar{c}(e^{-x^\lambda})^v} \right)^{\frac{2}{v}} \right]^{b-1} \left(1 - \bar{c}(e^{-x^\lambda})^v \right)^{-\left(\frac{2}{v}+1\right)} \\
 &\times \left(e^{-x^\lambda} \right) \lambda x^{\lambda-1} e^{-x^\lambda} \left(1 - \left[1 - \left(\frac{c(e^{-x^\lambda})^v}{1 - \bar{c}(e^{-x^\lambda})^v} \right)^{\frac{2}{v}} \right]^b \right)^{-1},
 \end{aligned} \tag{6.22}$$

for $x > 0$, $v, c, \lambda > 0$ and $\bar{c} = 1 - c$.

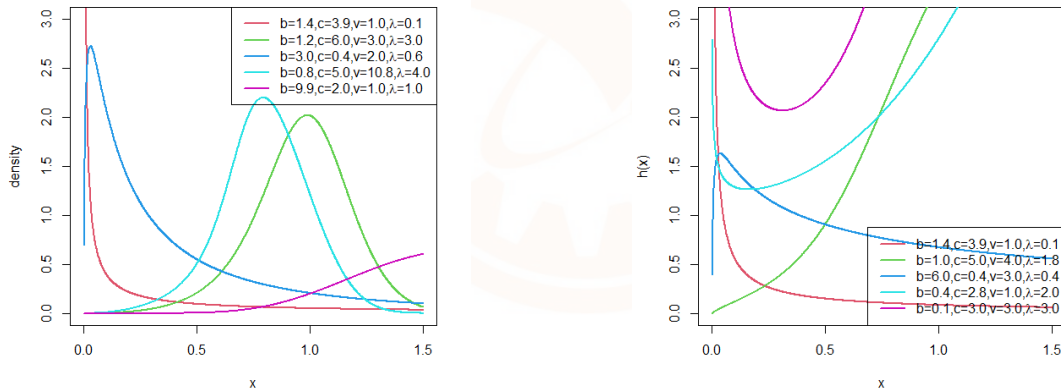


Figure 6.4: Pdf and Hrf Plots of TL-Haris-W Distribution

The plots of the pdf and hrf of the TL-Harris-W distribution are shown in Figure 6.4. The pdf takes various shapes, including almost symmetric, positive-skewed, negative-skewed, or reverse-J. The hrf graphs exhibit increasing, decreasing, BT, and UBT shapes.



Figure 6.5: SK and KS Plots of the TL-Harris-W Distribution for Fixed Values of Parameters b and v



Figure 6.6: SK and KS Plots of the TL-Harris-W distribution for Fixed Values of Parameters b and c

The 3D graphs of SK and KS for the TL-Harris-W distribution are shown in Figures 6.5 and 6.6. We observed that

- When the parameters b and v are fixed, the SK and KS of TL-Harris-W distribution increase as c and λ increase.
- When the parameters c and v are fixed, the SK and KS of TL-Harris-W distribution decrease then increase as b and λ increase.

6.3.3 Topp-Leone-Harris-Weibull (TL-Harris-U) Distribution

If we consider the uniform distribution as the parent distribution with the pdf and cdf given by $g(x; \beta) = \frac{1}{\beta}$ and $G(x; \lambda) = \frac{x}{\beta}$, respectively, for $\beta > 0$ and $x > 0$, we obtain the TL-Harris-U distribution with cdf given by

$$F(x; b, v, c, \beta) = \left[1 - \left(\frac{c(1 - x/\beta)^v}{1 - c(1 - x/\beta)^v} \right)^{\frac{2}{v}} \right]^b, \quad (6.23)$$

for $x > 0$, $v, c > 0, \beta$ and $\bar{c} = 1 - c$. The corresponding pdf is given by

$$f(x; b, v, c, \beta) = \frac{2bc^{\frac{2}{v}}}{\beta} \left[1 - \left(\frac{c(1-x/\beta)^v}{1-\bar{c}(1-x/\beta)^v} \right)^{\frac{2}{v}} \right]^{b-1} \times (1-\bar{c}(1-x/\beta)^v)^{-\left(\frac{2}{v}+1\right)} (1-x/\beta). \quad (6.24)$$

The hrf is given as

$$h(x; b, v, c, \beta) = 2bc^{\frac{2}{v}} \left[1 - \left(\frac{c(1-x/\beta)^v}{1-\bar{c}(1-x/\beta)^v} \right)^{\frac{2}{v}} \right]^{b-1} \left(1 - \bar{c} \left(1 - \frac{x}{\beta} \right)^v \right)^{-\left(\frac{2}{v}+1\right)} \times 1 - \frac{x}{\beta} g(x; \xi) \left(1 - \left[1 - \left(\frac{c(1-x/\beta)^v}{1-\bar{c}(1-x/\beta)^v} \right)^{\frac{2}{v}} \right]^b \right)^{-1}, \quad (6.25)$$

for $x > 0$, $v, c, \beta > 0$, and $\bar{c} = 1 - c$.

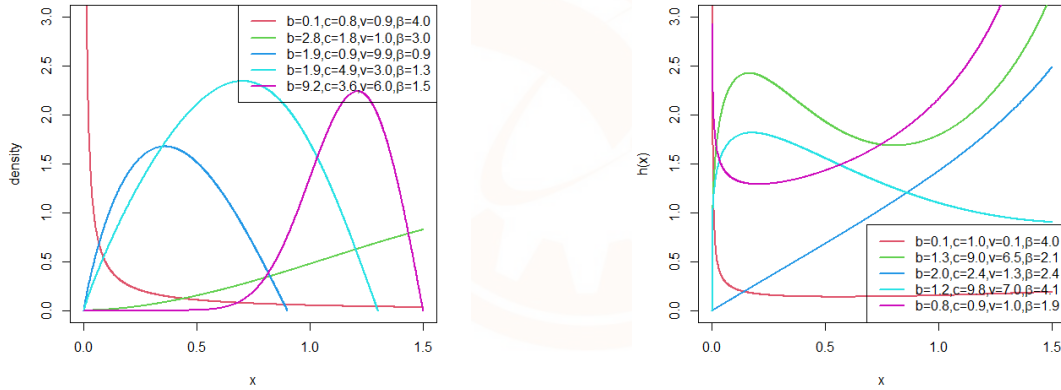


Figure 6.7: Pdf and Hrf Plots of the TL-Haris-U Distribution

The plots of the pdf for the TL-Harris-U distribution for selected parameter values are shown in Figure 6.7. The pdf can be reverse-J shaped, uni-modal, or left-skewed. The hrf of the TL-Harris-U distribution has increasing, decreasing, BT, UBT, and UBT-BT shapes.

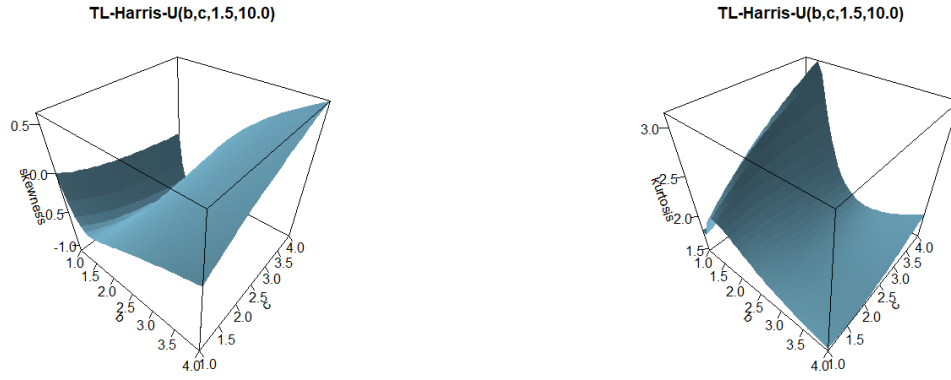


Figure 6.8: SK and KS Plots of the TL-Harris-U Distribution for Fixed Values of Parameters v and β



Figure 6.9: SK and KS of the TL-Harris-U Distribution for Fixed Values of Parameters c and β

The three dimensional graphs of SK and KS for the TL-Harris-U distribution can be found in Figures 6.8 and 6.9. We observed that

- When the parameters v and β are fixed, the SK of TL-Harris-U decrease and increase and KS of TL-Harris-U distribution decrease and then increase as b and c increase.
- When the parameters c and β are fixed, the SK of TL-Harris-U decrease and then increase and KS of TL-Harris-U distribution increase and then decrease as b and v increase.

6.4 Maximum Likelihood Estimation

Here we use the ML estimation method to find the ML estimates of the parameters of the TL-Harris-G FoD. Let $X_i \sim TL - Harris - G(b, v, c, \xi)$ and $\Delta = (b, v, c, \xi)^T$ be the vector of unknown the parameters. The total log-likelihood $\ell = \ell(\Delta)$ function based on a random sample of size n is given by

$$\begin{aligned} \ell(\Delta) = & n \ln(2bc^{\frac{2}{v}}) + (b-1) \sum_{i=1}^n \ln \left[1 - \left(\frac{c (\overline{G}(x; \xi))^v}{1 - \bar{c} (\overline{G}(x; \xi))^v} \right)^{\frac{2}{v}} \right] \\ & - \left(\frac{2}{v} + 1 \right) \sum_{i=1}^n \ln \left(1 - \bar{c} (\overline{G}(x; \xi))^v \right) + \sum_{i=1}^n \ln (\overline{G}(x; \xi)) + \sum_{i=1}^n \ln (g(x_i; \xi)). \end{aligned}$$

Elements of score vectors are presented in the Appendix. The FIM $I(\Delta)$, which is a $(q+3) \times (q+3)$ matrix:

$$I(\Delta) = \begin{pmatrix} I_{bb} & I_{bv} & I_{bc} & I_{b\xi} \\ I_{vb} & I_{vv} & I_{vc} & I_{v\xi} \\ I_{cb} & I_{cv} & I_{cc} & I_{c\xi} \\ I_{\xi b} & I_{\xi v} & I_{\xi c} & I_{\xi\xi} \end{pmatrix}.$$

The ML estimate $\hat{\Delta}$ converges in distribution to a multivariate normal distribution $N_{q+3}(\underline{0}, J(\Delta)^{-1})$, where $\underline{0}$ denotes the zero mean vector and $J(\Delta)$ is the expected FIM. Therefore, under the usual regularity conditions and assuming that the parameters are within the interior of the parameter space (but not on the boundary), we have:

$$\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} N_{q+3}(\underline{0}, J(\Delta)^{-1}),$$

where, n represents the sample size. This convergence in distribution allows us to construct asymptotic CI and confidence regions for the parameters of interest using the multivariate normal distribution $N_{q+3}(\underline{0}, J(\Delta)^{-1})$.

6.5 Simulations

To assess the consistency of ML estimators, a simulation study was carried out. We ran the simulation $N=1000$ times with sample size of $n=25, 50, 100, 150, 200, 400$ and 800 . First, we generated n uniformly distributed random variables, apply the inverse QF to obtain random variables following the TL-Harris-LLoG distribution, repeated these steps to produce multiple sets of random samples, solve the non-linear equation, and generate random numbers for the purpose of simulation studies.

The ABias and RMSE are calculated. Tables 6.1 and 6.2 shows the results of the simulations. Based on the MCS results, we conclude that the proposed model generates consistent results when estimating the parameters. This is evidenced by the mean estimates approaching the true parameter values as the sample size increases, as well as the RMSE and ABias for all parameter values decreasing as the sample size increases.

Table 6.1: MCS Results for TL-Harris-LLoG Distribution

Parameter	n	(1.0,1.5,0.8,1.0)			(1.0,0.5,1.2,0.9)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
<i>b</i>	25	2.9504	4.0683	1.9504	4.4619	6.6750	3.4619
	50	1.8383	3.3783	0.8383	1.8916	5.7008	0.8916
	100	1.2534	1.3296	0.2534	1.1773	1.1748	0.1773
	200	1.0782	0.3919	0.0782	1.0386	0.3978	0.0386
	400	1.0579	0.2587	0.0579	1.0243	0.2339	0.0243
	800	1.0304	0.1520	0.0304	1.0214	0.1823	0.0214
<i>v</i>	25	2.0248	1.3314	0.5249	1.1107	1.0040	0.6107
	50	1.7278	0.6729	0.2278	0.9607	0.8091	0.4607
	100	1.6197	0.3538	0.1197	0.8334	0.6037	0.3334
	200	1.5804	0.2531	0.0804	0.7845	0.5041	0.2845
	400	1.5475	0.1621	0.0475	0.6090	0.4625	0.1090
	800	1.5389	0.1211	0.0389	0.5617	0.3010	0.0617
<i>c</i>	25	1.5987	3.8333	0.7957	4.9684	6.0506	3.7684
	50	1.0956	1.6216	0.2956	3.1628	5.6715	1.9628
	100	0.8545	0.3977	0.0544	1.6921	2.7889	0.4821
	200	0.8433	0.2648	0.0433	1.4033	0.7547	0.2033
	400	0.8061	0.1476	0.0061	1.2741	0.3180	0.0741
	800	0.8001	0.1053	0.0001	1.2349	0.2142	0.0349
λ	25	1.3373	0.8438	0.3373	1.3769	1.0940	0.4769
	50	1.1319	0.4132	0.1319	1.1129	0.6550	0.2129
	100	1.0636	0.2110	0.0636	0.9970	0.4041	0.0970
	200	1.0447	0.1526	0.0447	0.9549	0.2586	0.0549
	400	1.0242	0.1006	0.0242	0.9099	0.1263	0.0099
	800	1.0158	0.0578	0.0158	0.9018	0.0971	0.0018

Table 6.2: MCS Results for TL-Harris-LLoG Distribution

Parameter	n	(1.0,0.5,1.2,1.0)			(0.9,1.0,1.2,1.0)		
		Mean	RMSE	ABias	Mean	RMSE	ABias
<i>b</i>	25	2.6246	6.1500	1.6246	5.3221	7.8135	4.8221
	50	1.8014	5.2523	0.8014	1.5760	3.7731	0.6761
	100	1.1741	1.3555	0.1741	1.0572	0.9610	0.1572
	200	1.0263	0.3863	0.0263	0.9474	0.3830	0.0471
	400	1.0224	0.2400	0.0224	0.9231	0.1967	0.0231
	800	1.0146	0.1869	0.0146	0.9095	0.1383	0.0095
<i>v</i>	25	1.1388	1.0520	0.6389	3.7732	4.1272	2.7732
	50	0.9711	0.7761	0.4711	1.9400	2.3545	0.9400
	100	0.8591	0.6267	0.3591	1.6436	2.1951	0.6437
	200	0.7875	0.5646	0.2875	1.3308	2.1194	0.3308
	400	0.6131	0.5104	0.1131	1.1710	1.9333	0.1710
	800	0.5427	0.4786	0.0427	1.0468	1.5977	0.0468
<i>c</i>	25	4.9616	5.1515	3.7617	4.5943	5.6578	3.3943
	50	2.2359	4.0371	1.0359	2.9915	4.5990	1.7915
	100	1.6848	2.7113	0.4848	1.6709	2.0438	0.4709
	200	1.4234	0.8171	0.2234	1.4869	1.3106	0.2839
	400	1.2874	0.4003	0.0874	1.3134	0.4687	0.1134
	800	1.2440	0.2275	0.0440	1.2744	0.3202	0.0744
λ	25	1.5294	1.2248	0.5294	1.4606	1.1456	0.4606
	50	1.1936	0.6037	0.1936	1.1299	0.4971	0.1299
	100	1.1123	0.4509	0.1123	1.0680	0.3108	0.0680
	200	1.0663	0.2928	0.0663	1.0319	0.1908	0.0319
	400	1.0131	0.1554	0.0131	1.0040	0.1070	0.0040
	800	1.0068	0.1130	0.0068	1.0007	0.0856	0.0007

6.6 Applications

The flexibility and applicability of the proposed model are demonstrated in this section. The TL-Harris-LLoG distribution is compared to the nested and non-nested distributions using two real-world data examples. The `mle()` function in the `stats4` package is used to fit model via the ML estimation method by maximizing the log-likelihood. The `AdequacyModel` package is used to assess performance of the model.

The following GoF statistics are taken into account: $-2\ln(L)$, AIC, BIC and CAIC, CVM and AD, are used to determine which distribution fits the data better. Chen and Balakrishnan [26] discussed these GoF statistics measures. The K-S GoF statistic, its associated PV, and SS (Chambers et al. [24]) are also utilized. The model with the smallest GoF statistics value and the highest PV for the K-S statistic is considered the best fitting model.

The LRT statistic is used to assess the GoF of the TL-Harris-LLoG distribution against its sub-models when analyzing empirical data. For example, when examining a parameter such as $b = 1$, the LRT statistic is $\Lambda = 2[\ln(L(\hat{b}, \hat{v}, \hat{c}, \hat{\lambda})) - \ln(L(\tilde{1}, \tilde{v}, \tilde{c}, \tilde{\lambda}))]$. If the computed Λ value exceeds the critical value derived from the χ^2 distribution, representing the upper $100\alpha\%$ percentile with 1 degree of freedom, the LRT test rejects the null hypothesis.

The TTT scaled plots has been found to be useful in studying the ageing properties of the

underlying distribution as well as in solving geometrically some stochastic maintenance problems. According to Aarset [1], the hrf increases (decreases) when the curve is concave (convex); when it is convex and then concave (concave and then convex), the hrf has a bathtub (inverse bathtub) shape.

Tables 6.3, 6.4, 6.6 and 6.7 presents the results of data analysis, parameter estimates (SE in parentheses), and GoF statistics for the various models taken into account in this paper. In data fitting, we also show fitted densities plots and PPs to compare the flexibility of the TL-Harris-LLoG distribution to competing models. The TL-Harris-LLoG distribution is compared to the following models: the TIIGIE-BIII distribution (Jamal et al. [56]) is given in Equation (3.28), the TL-MO-W distribution (Ahmed et al. [4]) is given in Equation (4.19), the MO-KA distribution (Javed et al. [59]) is given in Equation (5.32) and the TL-GW distribution (Aryal et al. [17]) is given in Equation (4.20).

6.6.1 Survival Years of Chemotherapy Patients Data

The data set contains survival times for 46 patients who received chemotherapy alone, it was applied by Moakofi et al. [77]. The data is right skewed, with outliers.

The estimated Var-Cov matrix is

$$\begin{pmatrix} 2.8408 \times 10^{-10} & 1.4919 \times 10^{-07} & 1.1278 \times 10^{-09} & -2.3365 \times 10^{-06} \\ 1.4919 \times 10^{-07} & 7.8350 \times 10^{-05} & 5.9223 \times 10^{-07} & -1.2270 \times 10^{-03} \\ 1.1278 \times 10^{-09} & 5.9223 \times 10^{-07} & 1.7041 \times 10^{-08} & -9.2749 \times 10^{-06} \\ -2.3365 \times 10^{-06} & -1.2270 \times 10^{-03} & -9.2749 \times 10^{-06} & 1.9217 \times 10^{-02} \end{pmatrix},$$

and the approximate 95% CI for b, v, c and λ are given by 66.8040 ± 0.0003 , 4.7268 ± 0.0173 , 0.0003 ± 0.0002 and 1.0941 ± 0.2717 , respectively. Table 6.3 gives estimates of parameter and their SE in parenthesis.

Table 6.3: Estimates for Survival Years of Chemotherapy Patients Data

Model	Estimates			
	b	v	c	λ
$TL - Harris - LLoG(b, v, c, \lambda)$	66.8040 (1.6855×10^{-05})	4.7268 (8.8516×10^{-03})	3.2286×10^{-04} (1.3054×10^{-05})	1.0941 (0.1386)
$TL - Harris - LLoG(1, v, c, \lambda)$	-	879.0919 (0.0035)	1.0001 (3.1256)	1.2517 (0.1449)
$TL - Harris - LLoG(b, v, 1, \lambda)$	2.6984 (0.4209)	0.0089 (7.3956×10^{-11})	-	0.5000 (0.0674)
$TL - Harris - LLoG(b, 1, 1, \lambda)$	1.7845 (0.2801)	-	-	0.6828 (0.6828)
TL-GW	α	θ	λ	β
	0.0033 (5.1366×10^{-04})	36.4550 (8.9533×10^{-08})	0.2704 (0.0212)	5.4045 (1.9046×10^{-04})
TL-MO-W	b	δ	γ	λ
	1.3838 (1.0893)	5.1480×10^{06} (1.6106×10^{-07})	0.0751 (0.0288)	15.0720 (0.8979)
MO-KA	α	θ	δ	β
	58.6260 (8.5743×10^{-06})	0.0036 (6.9932×10^{-04})	801.9500 (6.6699×10^{-06})	0.0522 (0.0382)
THIGIE-BIII	λ	θ	c	k
	7.8140 (15.3760)	1.0893 (0.6432)	1.4206 (0.8140)	0.1206 (0.1898)

Table 6.4 shows that the TL-Harris-LLoG distribution performs better than the non-nested models, since it has the smallest GoF statistics and the highest PV for the K-S statistic.

Table 6.4: Estimates and GoF Statistics for Survival Years of Chemotherapy Patients Data

Model	Statistics								
	$-2\log(L)$	AIC	$CAICC$	BIC	CVM	AD	SS	$K - S$	PV
$TL - Harris - LLoG(b, v, c, \lambda)$	117.7688	125.7688	126.7688	132.9954	0.0628	0.4474	0.0649	0.0856	0.8687
$TL - Harris - LLoG(1, v, c, \lambda)$	130.3369	136.3369	136.9223	141.7569	1.0967	6.5998	1.6989	0.4222	8.6370×10^{-08}
$TL - Harris - LLoG(b, v, 1, \lambda)$	139.8179	145.8179	146.4032	151.2379	0.0658	0.4773	0.5920	0.2709	0.0021
$TL - Harris - LLoG(b, 1, 1, \lambda)$	128.9924	132.9924	133.2781	136.6057	0.0664	0.4765	0.2387	0.1386	0.3228
TL-GW	118.0426	126.0426	127.0426	133.2693	0.1746	1.1279	0.2481	0.1535	0.2158
TL-MO-W	118.2446	126.2446	127.2446	133.4713	0.0741	0.5059	0.0699	0.0936	0.7900
MO-KA	129.0426	137.0427	138.0427	144.2693	0.1737	1.1164	0.1080	0.1126	0.5793
THIGIE-BIII	119.8395	127.8395	128.8395	135.0661	15.8083	90.2799	0.0746	0.9912	2.2000×10^{-16}

Figure 6.10 shows plots of the estimated pdf and cdf to support the results in Table 6.4, and it is clear that the TL-Harris-LLoG distribution can be chosen as the best model among the fitted models in this section.

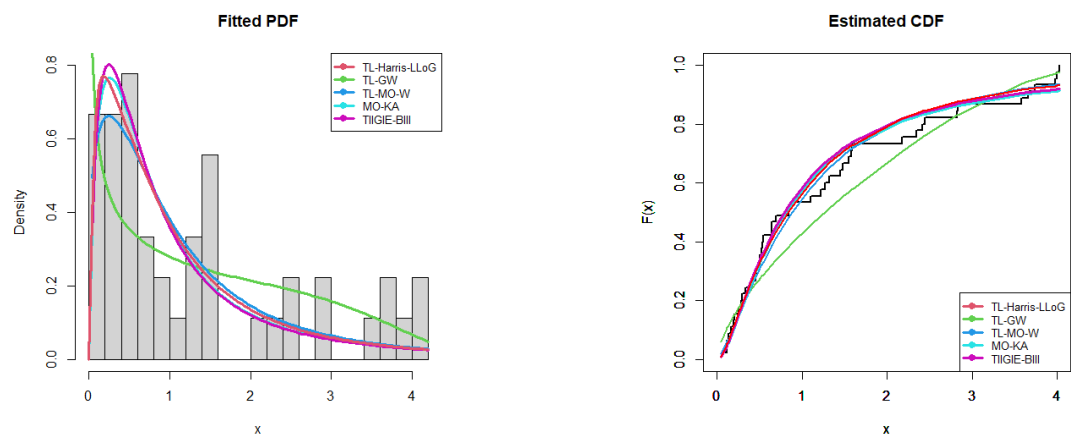


Figure 6.10: Pdf and ECDF Plots for Survival Years of Chemotherapy Patients Data

Table 6.5 gives the LRT results for survival years of chemotherapy patients data.

Table 6.5: LRT Results For Survival Years of Chemotherapy Patients Data

Survival Years of Chemotherapy Patients Data		
Model	<i>df</i>	$\chi^2(p - value)$
$TL - Harris - LLoG(1, v, c, \lambda)$	1	12.5681 (0.00392)
$TL - Harris - LLoG(b, v, 1, \lambda)$	1	22.0491 (< 0.00001)
$TL - Harris - LLoG(b, 1, 1, \lambda)$	2	11.2236 (0.003656)

There are significant differences between the nested models and the TL-Harris-LLoG distribution, according to the LRT results in Table 6.5. As a result, we conclude that the TL-Harris-LLoG model outperforms its nested models on the survival years of chemotherapy patients dataset.

Figure 6.11 depicts the PPs and TTT scaled plot. Figure 6.11 depicts the TTT scaled plot for the data, which suggests an UBT hrf shape. As a result, the TL-Harris-LLoG distribution may be an adequate model for the data.

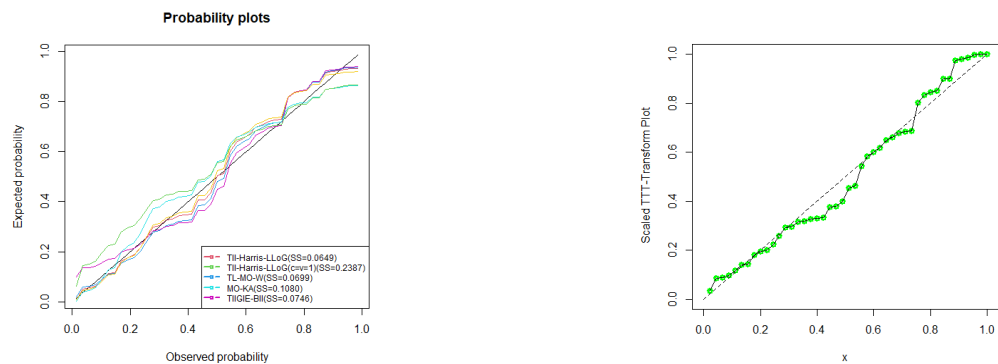


Figure 6.11: PPs and TTT Scaled Plot for Survival Years of Chemotherapy Patients Data

The K-M survival and hrf plots for the proposed model are displayed in Figure 6.12.

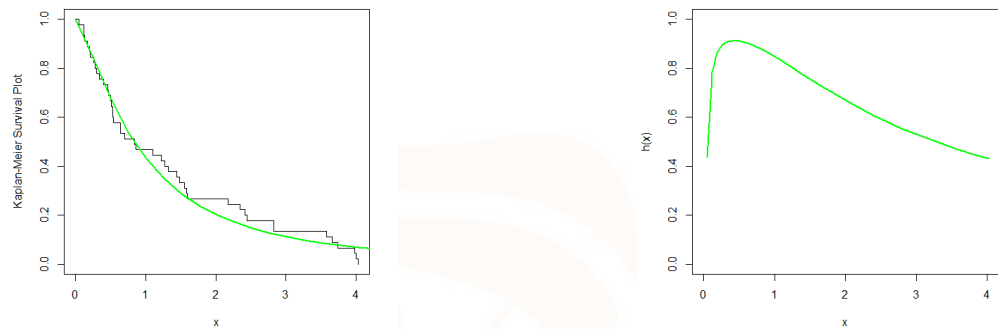


Figure 6.12: K-M Survival and Hrf Plots for Survival Years of Chemotherapy Patients Data

In Figure 6.13 we demonstrate the plots of the profile log-likelihood function of the ML estimates of b , v , c , and λ .

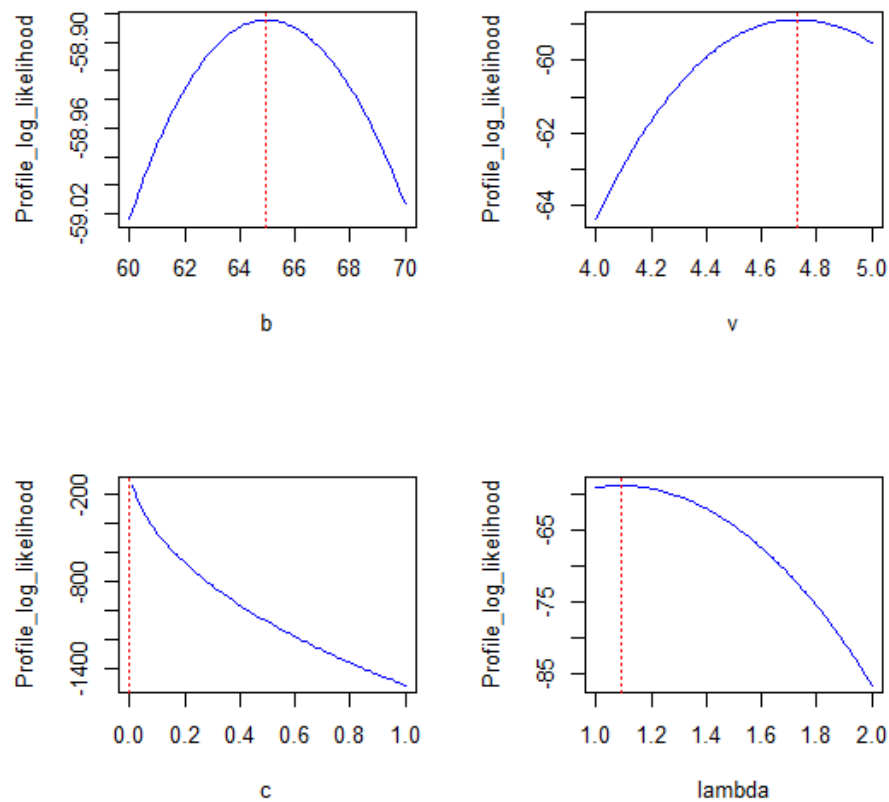


Figure 6.13: Plots of the Profile Log-Likelihood Function of the Parameters of the TL-Harris-LLoG Distribution on Survival Years of Chemotherapy Patients Data

From Figure 6.13, we observe that the ML estimates for TL-Harris-LLoG distribution are achieved uniquely.

6.6.2 Airborne Communication Transceivers Data

The second set of data was presented in by Klakattawi [63]. The data consist of 40 observations of active repair times for airborne communication transceivers. The time interval is expressed in hours. The data points are: 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50. The data is right-skewed with few outliers.

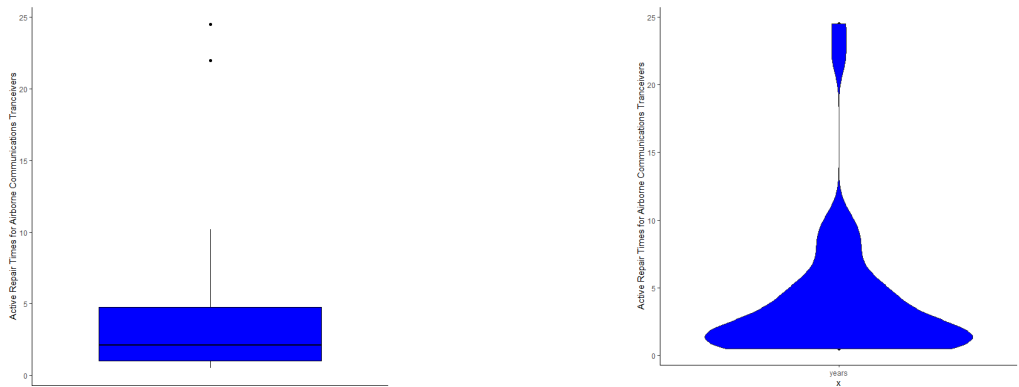


Figure 6.14: Boxplot and Violin Plots of Airborne Communication Transceivers Data

The estimated Var-Cov matrix is

$$\begin{pmatrix} 0.0024 & 0.0006 & 0.0004 & -0.0312 \\ 0.0006 & 0.0024 & 0.0019 & -0.1263 \\ 0.0004 & 0.0019 & 0.0016 & -0.1095 \\ -0.0312 & -0.1263 & -0.1095 & 7.2668 \end{pmatrix}, \tag{6.26}$$

and the approximate 95% CI for b, v, c and λ are given by $0.3029 \pm 0.0971, 0.1198 \pm 0.0968, 191.9118 \pm 0.0796$ and 9.9232 ± 5.2835 , respectively. The parameter estimates of the models for the airborne communication transceivers data are given in Table 6.6.

Table 6.6: Estimates for Airborne Communication Transceivers Data

Model	Estimates			
	b	v	c	λ
$TL - Harris - LLoG(b, v, c, \lambda)$	0.3029 (0.0495)	0.1198 (0.0494)	191.9118 (0.0406)	9.9232 (2.6957)
$TL - Harris - LLoG(1, v, c, \lambda)$	-	879.0900 (0.0037)	1.0001 (3.3153)	0.7949 (0.1046)
$TL - Harris - LLoG(b, v, 1, \lambda)$	5.8179 (0.9221)	0.0088 (290.9100)	-	1.0000 (0.1158)
$TL - Harris - LLoG(b, 1, 1, \lambda)$	2.6118 (0.4153)	-	-	0.5277 (0.0717)
TL-GW	α	θ	λ	β
	0.4743 (3.3412)	13.7643 (95.3489)	2.1959 (3.5840)	0.3900 (0.1019)
TL-MO-W	b	α	λ	β
	15.5130 (12.7570)	7.2177×10^{07} (2.2583×10^{-09})	0.0380 (0.0063)	18.8880 (0.5791)
MO-KA	α	θ	δ	β
	0.6971 (2.2332)	2.4297 (7.7829)	279.2439 (0.0364)	0.0801 (0.0363)
TIIGIE-BIII	λ	θ	c	k
	77.2007 (0.0266)	18.1216 (21.0971)	0.2011 (0.1071)	2.37451 (0.5076)

Table 6.7 demonstrates that the TL-Harris-LLoG distribution outperforms the non-nested models considered since it has the lowest GoF statistics and the highest PV for the K-S statistic.

Table 6.7: GoF Statistics for Active Repair Times for Airborne Communication Transceivers Data

Model	Statistics								
	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	SS	$K-S$	PV
$TL-Harris-LLoG(b, v, c, \lambda)$	177.5611	185.5611	186.7039	192.3166	0.0345	0.2757	0.0328	0.0734	0.9822
$TL-Harris-LLoG(1, v, c, \lambda)$	253.4297	259.4297	260.0963	264.4963	1.3758	7.8573	7.6082	0.6500	4.2190×10^{-15}
$TL-Harris-LLoG(b, v, 1, \lambda)$	182.1691	188.1691	188.8358	193.2358	0.0682	0.4447	0.1698	0.1386	0.4259
$TL-Harris-LLoG(b, 1, 1, \lambda)$	209.8566	213.8566	214.1809	217.2344	0.0660	0.4558	1.2265	0.3381	0.0002
TL-GW	184.4911	192.4911	193.6340	199.2466	0.0902	0.6519	0.1014	0.1388	0.4241
TL-MO-W	179.7750	187.7750	188.9179	194.9179	0.0631	0.4123	0.0607	0.0946	0.8665
MO-KA	183.5981	191.5981	192.7410	198.3536	0.0861	0.5724	0.0903	0.1116	0.7015
THIGIE-BIII	179.3263	187.2638	188.4067	194.0193	14.1785	80.4512	0.0665	0.9806	2.2000×10^{-16}

The estimated pdf and cdf are shown in Figure 6.15. Among the fitted models compared, the TL-Harris-LLoG distribution is demonstrated to be the best model.

Table 6.8: LRT Results For Active Repair Times for Airborne Communication Transceivers Data

Model	df	$\chi^2(PV)$
$TL-Harris-LLoG(1, v, c, \lambda)$	1	75.8686 (< 0.00001)
$TL-Harris-LLoG(b, v, 1, \lambda)$	1	4.6080 (0.031823)
$TL-Harris-LLoG(b, 1, 1, \lambda)$	2	11.2236 (< 0.00001)

According to the LRT results in Table 6.8, there are significant differences between nested models and the TL-Harris-LLoG distribution. As a result, we conclude that the TL-Harris-LLoG model outperforms its nested models on the active repair for airborne communication transceivers dataset.

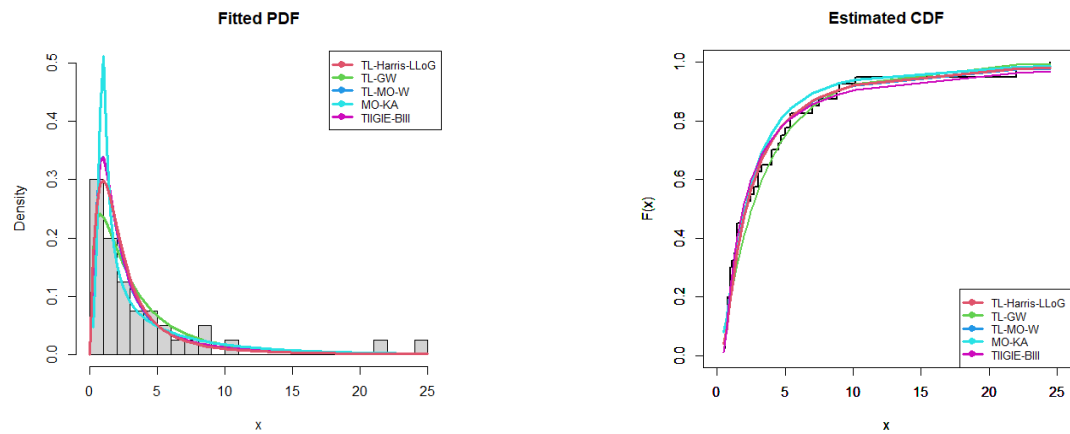


Figure 6.15: Estimated Densities and Estimated Cdf Plots for Airborne Communication Transceivers Data

The PPs and TTT scaled plots are shown in Figure 6.16. The TTT scaled plot shows the hrf with an UBT shape. As a result, the TL-Harris-LLoG distribution is a good fit for the airborne communication transceivers data.

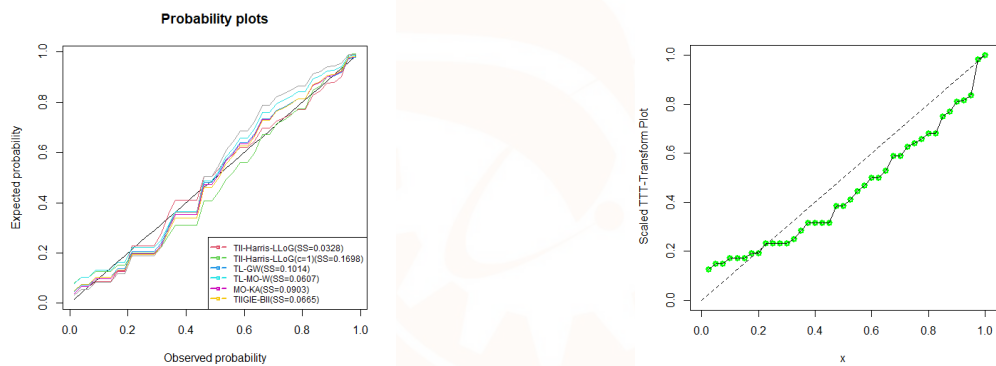


Figure 6.16: PPs and TTT Scaled Plot for Airborne Communication Transceivers Data

The K-M survival and hrf plots for the proposed model are displayed in Figure 6.17.

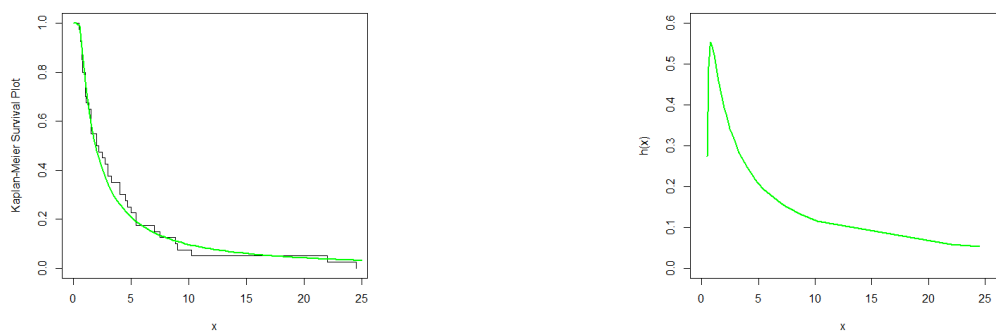


Figure 6.17: K-M Survival and Hrf Plots for Airborne Communication Transceivers Data

We show the plots of the profile log-likelihood function of the ML estimates of the TL-Harris-

LLoG distribution in Figure 6.18.

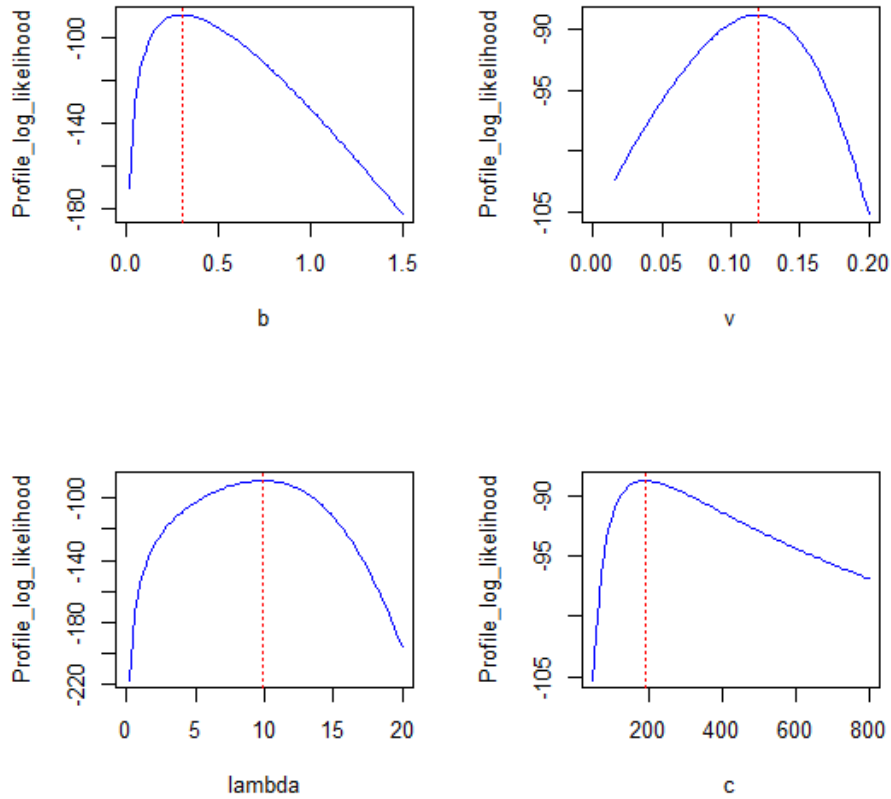


Figure 6.18: Plots of the profile Log-Likelihood Function of the Parameters of the TL-Harris-LLoG Distribution on Airborne Communication Tranceivers Data

According to Figure 6.18, the ML estimates for the TL-Harris-LLoG distribution can be obtained uniquely.

6.7 Conclusions

The TL-Harris-G FoD is proposed and studied. The probability density function of the new distribution can be expressed as an infinite linear combination of the Exp-G distributions. The statistical properties such as: moments, MGF, distribution of order statistics, RE, and PWMs were derived. The method of ML estimation was used to estimate the model parameters. The ABias and RMSE of the ML estimators of the model parameters were evaluated using MCS experiments. Furthermore, we showed that the parameters of TL-Harris-G FoD can be uniquely determined from the data. To illustrate the practical utility of the TL-Harris-G distribution, we fitted its special case, the TL-Harris-LLoG distribution, to real-world datasets. The TL-Harris-G FoD is a versatile and can be useful to model patient outcomes over time. It can accurately capture various hrf shapes, including monotonic and non-monotonic patterns, as well as different pdf shapes. This flexibility allows the TL-Harris-G distribution to represent the failure of chemotherapy treatment of patient outcomes, from initial to long-term treatment.

7. Conclusions and Areas of Further Research

7.1 Conclusions

In this thesis, we introduced and presented five (5) new generalized classes and families of distributions namely, the LLoGMWPS CoD, TII-TL-EHL-GPS CoD, MO-EHL-TII-TL-G FoD, Harris-OBX-G FoD, and TL-Harris-G FoD. These new classes and families of distributions can be expressed as linear combination of the Exp-G distribution, which allows for computations of some statistical properties. We derived the mathematical and statistical characteristics of these new classes and families of distributions, including moments and MGF, distribution of order statistics, PWMs, and RE. We also, used the ML estimation method to estimate the model parameters. Simulation experiments were used to illustrate the consistency of the ML estimates. The suitability and applicability of these new classes and families of distributions for some of the selected special cases are demonstrated by their successful fits to various real-world datasets, surpassing the performance of both nested and non-nested models applied to the same datasets.

7.2 Areas of Further Research

- Develop bivariate and multivariate generalisations of these new families and classes of distributions.
- Consider different methods of parameter estimation including weighted least squares, maximum product of spacing, percentile and Bayesian estimation.
- Extend the proposed distributions for potential applications to rare events or extreme value data sets via the use of heavy-tailed techniques.
- Apply the new distributions to censored data, including type-I and type-II progressive hybrid censored data as well as carrying out inference for the parameters of the new families of distributions based on the above mentioned types of data.
- Study characterizations of proposed distributions based on (i) truncated moments; (ii) hazard rate functions; (iii) the reverse hazard functions and (iv) conditional expectation of a function of the corresponding random variable.

A. Chapter 2 Appendix

A.1 Log-Logistic Modified Weibull Power Series (LLoGMWPS) Class of Distributions

<https://drive.google.com/file/d/1TLGwfa6Y5WdbLu7fPXkXh0dmrzCedx8s/view?usp=sharing>



B. Chapter 3 Appendix

B.1 The Type II Topp-Leone Exponentiated Half Logistic-G Power Series Class of Distributions

https://drive.google.com/file/d/18coDSic3H301SkdbqTzmX7JnVgNYri_Y/view?usp=sharing



C. Chapter 4 Appendix

C.1 Harris odd Burr-X-G Family of Distributions

<https://drive.google.com/file/d/1XpnBl56iligWKuxBFnbm83ZTVHeP4qWP/view?usp=sharing>



D. Chapter 5 Appendix

D.1 Marshall-Olkin-Exponentiated Half Logistic-Type II Topp-Leone-G Family of Distributions

<https://drive.google.com/file/d/1ZbevWBbW54anK7v1YXo0l2u1j1xbrfer/view?usp=sharing>



E. Chapter 6 Appendix

E.1 Topp-Leone Harris-G Family of Distributions

https://drive.google.com/file/d/1Fm1nXW08f6_9f0MZ953Lu-qp-KmyeWME/view?usp=sharing



Bibliography

- [1] Aarset, M. V. (1987). How to Identify a Bathtub Hazard Rate. *IEEE Transactions on Reliability*, 36(1):106–108.
- [2] Abdelkhalek, R. H. M. (2020). The Poisson Burr X Inverse Rayleigh Distribution And Its Applications. *Journal of Data Science*, 18(1):56–77.
- [3] Aboraya, M. (2021). Marshall–Olkin Lehmann Lomax Distribution: Theory, Statistical Properties, Copulas and Real Data Modeling. *Pakistan Journal of Statistics and Operation Research*, 17(2):509–530.
- [4] Ahmed, M. T., Khaleel, M. A., and Khalaf, E. K. (2020). The New Distribution (Topp Leone Marshall Olkin–Weibull) Properties with an Application. *Periodicals of Engineering and Natural Sciences (PEN)*, 8(2):684–692.
- [5] Al-Jadaani, S. H. and Al-Saiary, Z. A. (2022). Type II Topp Leone Generalized Inverted Exponential Distribution with Real Data Applications. *Asian Journal of Probability and Statistics*, 16(2):37–51.
- [6] Al-Mofleh, H., Elgarhy, M., Afify, A., and Zannon, M. (2020a). Type II Exponentiated Half Logistic Generated Family of Distributions with Applications. *Electronic Journal of Applied Statistical Analysis*, 13(2):536–561.
- [7] Al-Mofleh, H., Elgarhy, M., Afify, A., and Zannon, M. (2020b). Type II Exponentiated Half Logistic Generated Family of Distributions with Applications. *Electronic Journal of Applied Statistical Analysis*, 13(2):536–561.
- [8] Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S., and Shahbaz, M. Q. (2016). Topp-Leone Family of Distributions: Some Properties and Application. *Pakistan Journal of Statistics and Operation Research*, 12(4):443–451.
- [9] Aldahlan, M. and Afify, A. (2018). The Odd Exponentiated Half–Logistic Burr XII Distribution. *Pakistan Journal of Statistics and Operation Research*, 14(2):305–317.
- [10] Algarni, A., M. Almarashi, A., Elbatal, I., Hassan, A. S., Almetwally, E. M., M Daghistani, A., and Elgarhy, M. (2021). Type I Half Logistic Burr X-G Family: Properties, Bayesian, and Non-Bayesian Estimation under Censored Samples and Applications to COVID-19 Data. *Mathematical Problems in Engineering*, 2021(1):1–21.
- [11] Almarashi, A. M., Khan, K., Chesneau, C., and Jamal, F. (2021). Group Acceptance Sampling Plan Using Marshall-Olkin Kumaraswamy Exponential (MOKw-E) Distribution. *Processes*, 9(6):1066.

- [12] Almongy, H. M., Almetwally, E. M., Aljohani, H. M., Alghamdi, A. S., and Hafez, E. H. (2021). A New Extended Rayleigh Distribution with Applications of COVID-19 Data. *Results in Physics*, 23(60):104012.
- [13] Altun, E., Alizadeh, M., Yousof, H. M., Rasekhi, M., and Hamedani, G. G. (2021). A New Type II Half Logistic-G Family of Distributions with Properties, Regression Models, System Reliability and Applications. *Applications and Applied Mathematics*, 16(2):3.
- [14] Aly, E. A. A. and Benkherouf, L. (2011). A New Family of Distributions Based on Probability Generating Functions. *Sankhya: The Indian Journal of Statistics*, 73(1):70–80.
- [15] Alyami, S. A., Babu, M. G., Elbatal, I., Alotaibi, N., and Elgarhy, M. (2022). Type II Half-Logistic Odd Fréchet Class of Distributions: Statistical Theory and Applications. *Symmetry*, 14(6):1222.
- [16] Anabike, I. C., Igbokwe, C. P., Onyekwere, C. K., and Obulezi, O. J. (2023). Inference on the parameters of Zubair-Exponential distribution with Application to Survival Times of Guinea Pigs. *Journal of Advances in Mathematics and Computer Science*, 38(7):12–35.
- [17] Aryal, G. R., Ortega, E. M., Hamedani, G. G., and Yousof, H. M. (2017). The Topp-Leone Generated Weibull Distribution: Regression Model, Characterizations and Applications. *International Journal of Statistics and Probability*, 6(1):126–141.
- [18] Balakrishnan, N. (1985). Order statistics from the Half Logistic Distribution. *Journal of statistical computation and simulation*, 20(4):287–309.
- [19] Bantan, R. A. R., Jamal, F., Chesneau, C., and Elgarhy, M. (2020). Type II Power Topp-Leone Generated Family of Distributions with Statistical Inference and Applications. *Symmetry*, 12(1):75.
- [20] Barreto-Souza, W., Lemonte, A. J., and Cordeiro, G. M. (2013). General Results for the Marshall and Olkin’s Family of Distributions. *Anais da Academia Brasileira de Ciências*, 85(1):3–21.
- [21] Batsidis, A. and Lemonte, A. J. (2015). On the Harris Extended Family of Distributions. *Statistics*, 49(6):1400–1421.
- [22] Bello, O. A., Doguwa, S. I., Yahaya, A., and Jibril, H. M. (2021). A Type II Half Logistic Exponentiated-G Family of Distribution with Applications to Survival Analysis. *Fudma Journal OF Sciences*, 5(3):177–190.
- [23] Carrasco, J. M. F., Ortega, E. M. M., and Cordeiro, G. M. (2008). A Generalized Modified Weibull Distribution for Lifetime Modeling. *Computational Statistics and Data Analysis*, 53(2):450–462.
- [24] Chambers, J. M., Cleveland, W. S., Kleiner, B., and Tukey, P. A. (1983). *Graphical Methods for Data Analysis*. Chapman and Hall/CRC.

- [25] Chamunorwa, S., Makubate, B., Oluyede, B., and Chipepa, F. (2021). The Exponentiated Half Logistic-Log-Logistic Weibull Distribution: Model, Properties and Applications. *Journal of Statistical Modelling: Theory and Applications*, 2(1):101–120.
- [26] Chen, G. and Balakrishnan, N. (1995). A General Purpose Approximate Goodness-of-Fit Test. *Journal of Quality Technology*, 27(2):154–161.
- [27] Chipepa, F. and Oluyede, B. (2021). The Topp-Leone Odd Exponential Half Logistic-G Family of Distributions: Model, Properties and Applications. *Pakistan Journal of Statistics*, 37(3):253–277.
- [28] Chipepa, F., Oluyede, B., and Makubate, B. (2020a). The Topp-Leone-Marshall-Olkin-G Family of Distributions with Applications. *International Journal of Statistics and Probability*, 9(4):15–32.
- [29] Chipepa, F., Oluyede, B., and Makubate, B. (2021a). The New Odd Lindley-G Power Series Class of Distributions: Theory, Properties and Applications. *Afrika Statistika*, 16(3):2825–2849.
- [30] Chipepa, F., Oluyede, B., and Peter, P. O. (2021b). The Burr III-Topp-Leone-G Family of Distributions with Applications. *Heliyon*, 7(4):e06534.
- [31] Chipepa, F., Oluyede, B., and Wanduku, D. (2021c). The Exponentiated Half Logistic Odd Weibull-Topp-Leone-G: Model, Properties and Applications. *Journal of Statistical Modelling: Theory and Applications*, 2(1):15–38.
- [32] Chipepa, F., Oluyede, B., Wanduku, D., and Moakofi, T. (2022). The Exponentiated Half Logistic-Topp-Leone-G Power Series Class of Distributions: Model, Properties and Applications. In *Methods of Mathematical Modelling and Computation for Complex Systems*, volume 373, pages 341–374. Springer.
- [33] Chipepa, F., Wanduku, D., and Oluyede, B. (2020b). Half-logistic odd weibull-topp-leone-g family of distributions: Model, properties and applications. *Afrika Statistika*, 15(4):2481–2510.
- [34] Cordeiro, G. M., Alizadeh, M., and Diniz Marinho, P. R. (2016). The Type I Half-Logistic Family of Distributions. *Journal of Statistical Computation and Simulation*, 86(4):707–728.
- [35] Cordeiro, G. M., Alizadeh, M., and Ortega, E. M. M. (2014). The Exponentiated Half-Logistic Family of Distributions: Properties and Applications. *Journal of Probability and Statistics*, 2014(1):1–21.
- [36] Cordeiro, G. M., Ortega, E. M. M., and Nadarajah, S. (2010). The Kumaraswamy Weibull Distribution with Application to Failure Data. *Journal of the Franklin Institute*, 347(8):1399–1429.
- [37] Eghwerido, J. T., Ogbo, J. O., and Omotoye, A. E. (2021a). The Marshall-Olkin Gompertz Distribution: Properties and Applications. *Statistica*, 81(2):183–215.

- [38] Eghwerido, J. T., Oguntunde, P. E., and Agu, F. I. (2021b). The Alpha Power Marshall-Olkin-G Distribution: Properties, and Applications. *Sankhya: The Indian Journal of Statistics*, 85-A(1):1–26.
- [39] Eissa, F. H. and Abdulaziz, R. K. (2017). The Exponentiated Kumaraswamy-Weibull Distribution with Application to Real Data. *International Journal of Statistics and Probability*, 6(6):167–82.
- [40] Elbatal, I., Altun, E., Afify, A. Z., and Ozel, G. (2019). The Generalized Burr XII Power Series Distributions with Properties and Applications. *Annals of Data Science*, 6(3):571–597.
- [41] Elgarhy, M., Nasir, M. A., Jamal, F., and Ozel, G. (2018). The Type II Topp-Leone Generated Family of Distributions: Properties and Applications. *Journal of Statistics and Management Systems*, 21(8):1529–1551.
- [42] Esmaeili, L. and Niaparast, M. (2022). New Compounding Lifetime Distributions with Applications to Real Data. *International Journal of Modeling, Simulation, and Scientific Computing*, 13(5):2250038.
- [43] Fagbamigbe, A. F., Melamu, P., Oluyede, B. O., and Makubate, B. (2018). The Ristić and Balakrishnan Lindley-Poisson Distribution: Model, Theory and Application. *Afrika Statistika*, 13(4):1837–1864.
- [44] Freels, J. K., Timme, D. A., Pignatiello, J. J., Warr, R. L., and Hill, R. R. (2019). Maximum Likelihood Estimation for the Poly-Weibull Distribution. *Quality Engineering*, 31(4):545–552.
- [45] Gradshteyn, I. S. and Ryzhik, I. M. (2000). Table of Integrals, Series, and Products.
- [46] Gupta, R. C., Gupta, P. L., and Gupta, R. D. (1998). Modeling Failure Time Data by Lehman Alternatives. *Communications in Statistics-Theory and Methods*, 27(4):887–904.
- [47] Gupta, R. D. and Kundu, D. (2001a). Exponentiated Exponential Family: an Alternative to Gamma and Weibull Distributions. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, 43(1):117–130.
- [48] Gupta, R. D. and Kundu, D. (2001b). Exponentiated Exponential Family: An Alternative To Gamma and Weibull Distributions. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, 43(1):117–130.
- [49] Gurvich, M. R., Dibenedetto, A. T., and Ranade, S. V. (1997). A New Statistical Distribution for Characterizing the Random Strength of Brittle Materials. *Journal of Materials Science*, 32(10):2559–2564.
- [50] Handique, L. and Chakraborty, S. and Ali, M. M. (2023). *The Topp-Leone-G Power Series Distribution*. CRC Press.
- [51] Harris, T. E. (1948). Branching Processes. *The Annals of Mathematical Statistics*, pages 474–494.

- [52] Haupt, E. and Schäbe, H. (1992). A New Model for a Lifetime Distribution with Bathtub Shaped Failure Rate. *Microelectronics reliability*, 32(5):633–639.
- [53] Hjorth, U. (1980). A Reliability Distribution with Increasing, Decreasing, Constant and Bathtub-Shaped Failure Rates. *Technometrics*, 22(1):99–107.
- [54] Huang, S. and Oluyede, B. O. (2014). Exponentiated Kumaraswamy-Dagum Distribution with Applications to Income and Lifetime Data. *Journal of Statistical Distributions and Applications*, 1(1):1–20.
- [55] Ibrahim, N. A., Khaleel, M. A., Merovci, F., Kilicman, A., and Shitan, M. (2017). Weibull Burr X Distribution Properties and Application. *Pakistan Journal of Statistics*, 33(5):315–336.
- [56] Jamal, F., Chesneau, C., and Elgarhy, M. (2020). Type II General Inverse Exponential Family of Distributions. *Journal of Statistics and Management Systems*, 23(3):617–641.
- [57] Jamal, F. and Nasir, M. A. (2018). Generalized Burr X Family of Distributions. *International Journal of Mathematics and Statistics*, 19(1):55–73.
- [58] Jamal, F., Tahir, M. H., Alizadeh, M., and Nasir, M. A. (2017). On marshall-olkin burr x family of distribution. *Tbilisi Mathematical Journal*, 10(4):175–199.
- [59] Javed, M., Nawaz, T., and Irfan, M. (2019). The Marshall-Olkin Kappa Distribution: Properties and Applications. *Journal of King Saud University-Science*, 31(4):684–691.
- [60] Kang, S. and Seo, J. (2011). Estimation in an Exponentiated Half Logistic Distribution under Progressively Type-II Censoring. *Communications for Statistical Applications and Methods*, 18(5):657–666.
- [61] Karamikabir, H., Afshari, M., Alizadeh, M., and Yousof, H. M. (2021). The Odd Log-Logistic Burr-X Family of Distributions: Properties and Applications. *Journal of Statistical Theory and Applications*, 20(2):228–241.
- [62] Khalil, M. G., Hamedani, G. G., and Yousof, H. M. (2019). The Burr X Exponentiated Weibull Model: Characterizations, Mathematical Properties and Applications to Failure and Survival Times Data. *Pakistan Journal of Statistics and Operation Research*, XV(1):141–160.
- [63] Klakattawi, H. S. (2022). Survival Analysis of Cancer Patients using a New Extended Weibull Distribution. *Plos one*, 17(2):e0264229.
- [64] Kundu, D., Mitra, D., and Ganguly, A. (2017). Analysis of Left Truncated and Right Censored Competing Risks Data. *Computational Statistics & Data Analysis*, 108:12–26.
- [65] Lai, C. D., Xie, M., and Murthy, D. N. P. (2003). A Modified Weibull Distribution. *IEEE Transactions on reliability*, 52(1):33–37.
- [66] Lehmann, E. L. (2012). The Power of Rank Tests. In *Selected Works of EL Lehmann*, pages 373–390. Springer.

- [67] Madaki, U. Y., Bakar, M. R. A., and Handique, L. (2018). Beta Kumaraswamy Burr Type X Distribution and Its Properties. 2:2018080304.
- [68] Makubate, B., Chamunorwa, S., Oluyede, B., and Chipepa, F. (2021a). The Marshall-Olkin-Odd Weibull-G Family of Distributions: Model, Properties and Applications. *Eurasian Bulletin of Mathematics (ISSN: 2687-5632)*, 4(2):71–97.
- [69] Makubate, B., Chipepa, F., Oluyede, B., and Peter, P. O. (2021b). The Marshall-Olkin Half Logistic-G Family of Distributions With Applications. *International Journal of Statistics and Probability*, 10(2):120–120.
- [70] Makubate, B., Matsuokwane, M., Oluyede, B. O., Gabaitiri, L., and Chamunorwa, S. (2022). The type II Topp-Leone-G Power Series Distribution with Applications on Bladder Cancer. *Journal of the Nigerian Society of Physical Sciences*, 4(3):848–848.
- [71] Makubate, B., Oluyede, B. O., Motobetso, G., Huang, S., and Fagbamigbe, A. F. (2018). The Beta Weibull-G Family of Distributions: Model, Properties and Application. *International Journal of Statistics and Probability*, 7(2):12–32.
- [72] Makubate, B., Rannona, K., Oluyede, B., and Chipepa, F. (2021c). The Topp-Leone-G Power Series Class of Distributions with Applications. *Pakistan Journal of Statistics and Operation Research*, 17(4):827–846.
- [73] Mansoor, M., Tahir, M. H., Cordeiro, G. M., Provost, S. B., and Alzaatreh, A. (2019). The Marshall-Olkin Logistic-Exponential distribution. *Communications in Statistics-Theory and Methods*, 48(2):220–234.
- [74] Marshall, A. W. and Olkin, I. (1997). A New Methods for Adding a Parameter to a Family of Distributions with Application to the Exponential and Weibull Families. *Biometrika*, 84(3):641–652.
- [75] Moakofi, T., Oluyede, B., and Chipepa, F. (2021a). Type II Exponentiated Half-Logistic-Topp-Leone-G Power Series Class of Distributions with Applications. *Pakistan Journal of Statistics and Operation Research*, 17(4):885–909.
- [76] Moakofi, T., Oluyede, B., and Chipepa, F. (2021b). Type II Exponentiated Half-Logistic Topp-Leone Marshall-Olkin-G Family of Distributions with Applications. *Heliyon*, 7(12):e08590.
- [77] Moakofi, T., Oluyede, B., Chipepa, F., and Makubate, B. (2022). Odd Power Generalized Weibull-G Family of Distributions: Model, properties and Applications. *Journal of Statistical Modelling: Theory and Applications*, 2(1):121–142.
- [78] Moakofi, T., Oluyede, B., and Makubate, B. (2021c). Marshall-Olkin Lindley-Log-Logistic Distribution: Model, Properties and Applications. *Mathematica Slovaca*, 71(5):1269–1290.
- [79] Morais, A. L. and Barreto-Souza, W. (2011). A Compound Class of Weibull and Power Series Distributions. *Computational Statistics and Data Analysis*, 55(3):1410–1425.

- [80] Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull Family for Analyzing Bathtub Failure-Rate Data. *IEEE transactions on reliability*, 42(2):299–302.
- [81] Mukhopadhyay, A. and Basu, B. (2007). Consolidation–microstructure–property relationships in bulk nanoceramics and ceramic nanocomposites: a review. *International Materials Reviews*, 52(5):257–288.
- [82] Murthy, D. N. P., Xie, M., and Jiang, R. (2004). *Weibull Models*. John Wiley and Sons.
- [83] Nadarajah, S., Cordeiro, G. M., and Ortega, E. M. M. (2011). General Results for the Beta-Modified Weibull Distribution. *Journal of Statistical Computation and Simulation*, 81(10):1211–1232.
- [84] Nadarajah, S. and Kotz, S. (2003). Moments of Some J-shaped Distributions. *Journal of Applied Statistics*, 30(3):311–317.
- [85] Nadarajah, S. and Kotz, S. (2006). The Beta Exponential Distribution. *Reliability Engineering and System Safety*, 91(6):689–697.
- [86] Nassar, M., Kumar, D., Dey, S., Cordeiro, G. M., and Afify, A. Z. (2019). The Marshall–Olkin Alpha Power Family of Distributions with Applications. *Journal of Computational and Applied Mathematics*, 351:41–53.
- [87] Noack, A. (1950). A Class of Random Variables with Discrete Distributions. *The Annals of Mathematical Statistics*, 21(1):127–132.
- [88] Ogunde, A. A., Laoye, V. E., Ezichi, O. N., and Balogun, K. O. (2021). Harris Extended Power Lomax Distribution: Properties, Inference and Applications. *International Journal of Statistics and Probability*, 10(4):1–77.
- [89] Oluyede, B., Chamunorwa, S., Chipepa, F., and Alizadeh, M. (2022a). The Topp-Leone Gompertz-G Family of Distributions with Applications. *Journal of Statistics and Management Systems*, 25(6):1399–1423.
- [90] Oluyede, B., Chipepa, F., and Wanduku, D. (2021a). Exponentiated Half Logistic-Power Generalized Weibull-G Family of Distributions: Model, Properties and Applications. *Eurasian Bulletin of Mathematics (ISSN: 2687-5632)*, 3(3):134–161.
- [91] Oluyede, B., Chipepa, F., and Wanduku, D. (2021b). The odd Weibull-Topp-Leone-G Power Series Family of Distributions: Model, Properties and Applications. *Journal of Nonlinear Sciences and Applications*, 14:268–286.
- [92] Oluyede, B., Mdlongwa, P., Makubate, B., and Huang, S. (2019). The Burr-Weibull Power Series Class of Distributions. *Austrian Journal of Statistics*, 48(1):1–13.
- [93] Oluyede, B. and Moakofi, T. (2022). Type II Exponentiated Half-Logistic-Gompertz Topp-Leone-G Family of Distributions with Applications. *Central European Journal of Economic Modelling and Econometrics*, 14(1):415–461.

- [94] Oluyede, B., Moakofi, T., and Chipepa, F. (2022b). The Odd Power Generalized Weibull-G Power Series Class of Distributions: Properties and Applications. *Statistics in Transition new series*, 23(1):89–108.
- [95] Oluyede, B., Moakofi, T., Chipepa, F., and Makubate, B. (2020a). A New Power Generalized Weibull-G Family of Distributions: Properties and Applications. *Journal of Statistical Modelling: Theory and Applications*, 1(2):167–191.
- [96] Oluyede, B., Peter, P. O., Ndwapi, N., and Bindele, H. (2022c). The Exponentiated Half-logistic Odd Burr III-G: Model, Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, 18(1):33–57.
- [97] Oluyede, B. O., Bindele, H. F., Makubate, B., and Huang, S. (2018a). A New Generalized Log-Logistic and Modified Weibull Distribution with Applications. *International Journal of Statistics and Probability*, 7(3):72–93.
- [98] Oluyede, B. O., Huang, S., and Pararai, M. (2014). A New Class of Generalized Dagum Distribution with Applications to Income and Lifetime Data. *Journal of Statistical and Econometric Methods*, 3(2):125.
- [99] Oluyede, B. O., Makubate, B., Fagbamigbe, A. F., and Mdlongwa, P. (2018b). A New Burr XII-Weibull-Logarithmic Distribution for Survival and Lifetime Data Analysis: Model, Theory and Applications. *Stats*, 1(1):77–91.
- [100] Oluyede, B. O., Mashabe, B., Fagbamigbe, A., Makubate, B., and Wanduku, D. (2020b). The Exponentiated Generalized Power Series: Family of distributions: Theory, Properties and Applications. *Heliyon*, 6(8):e04653.
- [101] Oluyede, B. O., Motsewabagale, G., Huang, S., Warahena-Liyanage, G., and Pararai, M. (2016a). Dagum-Poisson Distribution: Model, Properties and Application. *Electronic Journal of Applied Statistical Sciences*, 9(1):169.
- [102] Oluyede, B. O., Warahena-Liyanage, G., and Pararai, M. (2016b). A New Compound Class of Log-Logistic Weibull-Poisson Distribution: Model, Properties and Applications. *Journal of Statistical Computation and Simulation*, 86(7):1363–1391.
- [103] Oluyede, B. O. and Yang, T. (2015). A New Class of Generalized Lindley Distributions with Applications. *Journal of Statistical Computation and Simulation*, 85(10):2072–2100.
- [104] Osatohanmwen, P., Oyegue, F. O., and Ogbonmwan, S. M. (2020). The T-R {Y} Power Series Family of Probability Distributions. *Journal of the Egyptian Mathematical Society*, 28(1):1–18.
- [105] Pararai, M., Warahena-Liyanage, G., and Oluyede, B. O. (2017). Exponentiated Power Lindley-Poisson Distribution: Properties and Applications. *Communications in Statistics-Theory and Methods*, 46(10):4726–4755.
- [106] Paula, A. and Joseb, K. K. (2021). Harris Generalized Linear Exponential Distribution and Its Applications. *Reliability: Theory and Applications*, 16(1 (61)):176–187.

- [107] Pelumi, E. O., Adebowale, O. A., Enahoro, A. O., Manoj, K. R., and Oluwole, A. O. (2017). The Burr X-Exponential Distribution: Theory and applications. In *Proceedings of the World Congress on Engineering*, volume 1, pages 1–7.
- [108] Pham, H. and Lai, C. (2007). On Recent Generalizations of the Weibull Distribution. *IEEE Transactions on Reliability*, 56(3):454–458.
- [109] Pinho, L. G. B., Cordeiro, G. M., and Nobre, J. S. (2012). The Gamma-Exponentiated Weibull Distribution. *Journal of Statistical Theory and Applications*, 11(4):379–395.
- [110] Pinho, L. G. B., Cordeiro, G. M., and Nobre, J. S. (2015a). On the Harris-G Class of Distributions: General Results and Application. *Brazilian Journal of Probability and Statistics*, 29(4):813–832.
- [111] Pinho, L. G. B., Cordeiro, G. M., and Nobre, J. S. (2015b). The Harris Extended Exponential Distribution. *Communications in Statistics-Theory and methods*, 44(16):3486–3502.
- [112] Rajarshi, S. and Rajarshi, M. B. (1988). Bathtub Distributions: A Review. *Communications in Statistics-Theory and Methods*, 17(8):2597–2621.
- [113] Rannona, K., Oluyede, B., Chipepa, F., and Makubate, B. (2022). The Exponentiated Odd Exponential Half Logistic-G Power Series Class of Distributions with Applications. *Journal of Statistics and Management Systems*, 25(8):1821–1848.
- [114] Rasheed, N. (2020). Topp-Leone Dagum Distribution. *Research Journal of Mathematical and Statistical Sciences*, 8(1):16–30.
- [115] Refaie, M. K. A. (2018). Burr X Exponentiated Exponential Distribution. *Journal of Statistics and Applications*, 1(2):71–88.
- [116] Rényi, A. (1961). On Measures of Entropy and Information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, volume 4, pages 547–562. Berkeley, California, USA.
- [117] Ristić, M. M. and Balakrishnan, N. (2012). The Gamma-Exponentiated Exponential Distribution. *Journal of Statistical Computation and Simulation*, 82(8):1191–1206.
- [118] Sandhya, E., Sherly, S., and Raju, N. (2005). Harris Family of Discrete Distributions. *arXiv preprint math/0506220*.
- [119] Santos-Neto, M., Bourguignon, M., Zea, L. M., Nascimento, A., and Cordeiro, G. M. (2014). The Marshall-Olkin Extended Weibull Family of Distributions. *Journal of Statistical Distributions and Applications*, 1(1):1–24.
- [120] Sengweni, W., Oluyede, B., and Makubate, B. (2021). The Exponentiated Half-Logistic Odd Lindley-G Family of Distributions with Applications. *Journal of Nonlinear Sciences and Applications*, 14(2021):287–309.

- [121] Shakhathreh, M. K., Dey, S., and Kumar, D. (2021). Inverse Lindley Power Series Distributions: A New Compounding Family and Regression Model with Censored Data. *Journal of Applied Statistics*, 49(13):3451–3476.
- [122] Shannon, C. E. (1948). A Mathematical Theory of Communication. *The Bell System Technical Journal*, 27(3):379–423.
- [123] Shrahili, M., Elbatal, M., and Muhammad, M. (2019). The Type I Half-Logistic Burr X Distribution: Theory and Practice. *Journal of Nonlinear Sciences and Applications*, 12:262–277.
- [124] Silva, G. O., Ortega, E. M. M., and Cordeiro, G. M. (2010). The Beta Modified Weibull Distribution. *Lifetime Data analysis*, 16(3):409–430.
- [125] Silva, R. B., Bourguignon, M., Dias, C. R. B., and Cordeiro, G. M. (2013). The Compound Class of Extended Weibull Power Series Distributions. *Computational Statistics and Data Analysis*, 58:352–367.
- [126] Silva, R. B. and Cordeiro, G. M. (2015). The Burr XII Power Series Distributions: A New Compounding Family. *Brazilian Journal of Probability and Statistics*, 29(3):565–589.
- [127] Sivadasa, R. and Joseb, K. K. (2022). Harris Extended Burr XII Distribution and its Applications. *Asian Journal of Statistical Sciences*, 2(1):37–46.
- [128] Soliman, A. H., Elgarhy, M. A. E., and Shakil, M. (2017). Type II Half Logistic Family of Distributions with Applications. *Pakistan Journal of Statistics and Operation Research*, 13(2):245–264.
- [129] Tahir, M. H., Cordeiro, G. M., Mansoor, M., Alzaatreh, A., and Zubair, M. (2018). A New Generalized Family of Distributions from Bounded Support. *Journal of Data Science*, 16(2):251–276.
- [130] Tahmasebi, S. and Jafari, A. A. (2015). Exponentiated Extended Weibull-Power Series Class of Distributions. *arXiv preprint arXiv:1503.08653*, 37(2):183–193.
- [131] Team, R. C. (2011). R: A Language and Environment for Statistical Computing. *R Foundation for Statistical Computing*.
- [132] Tharu, R. P., Pahari, S., Sedhai, G. P., and Dhungana, G. P. (2021). Exponentiated Marshall-Olkin Exponential Distribution: Application of COVID-19 Second Wave in Nepal. *Nepal Journal of Mathematical Sciences*, 2(2):43–56.
- [133] Thomas, S. P., Tomy, L., and Jose, K. K. (2021). Harris Extended Two Parameter Lindley Distribution and Applications in Reliability. *Reliability: Theory and Applications*, 16(3 (63)):302–321.
- [134] Topp, C. W. and Leone, F. C. (1955). A Family of J-shaped Frequency Functions. *Journal of the American Statistical Association*, 50(269):209–219.

- [135] ul Haq, M. A., Afify, A. Z., Al-Mofleh, H., Usman, R. M., Alqawba, M., and Sarg, A. M. (2021). The Extended Marshall-Olkin Burr III Distribution: Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, 17(1):1–14.
- [136] Usman, R. M. and ul Haq, M. A. (2020). The Marshall-Olkin Extended Inverted Kumaraswamy Distribution: Theory and Applications. *Journal of King Saud University-Science*, 32(1):356–365.
- [137] Usman, U., Musa, Y., Muhammad, A. B., and Zakari, Y. (2019). On the Burr X-Topp Leone Distribution. *Annals. Computer Science Series*, 17(2):206–208.
- [138] Vasileva, M., Rahneva, O., Malinova, A., and Arnaudova, V. (2021). The Odd Weibull-Topp-Leone-G Power Series Family of Distributions. *International Journal of Differential Equations and Applications*, 20(1):43–58.
- [139] Volodin, N. K. W. B. A. (2018). The Topp-Leone Generalized Exponential Power Series Distribution with Applications. *JPSS*, 16(2):197–208.
- [140] Yousof, H. M., Afify, A. Z., Hamedani, G. G., and Aryal, G. R. (2017). The Burr X Generator of Distributions for Lifetime Data. *Journal of Statistical Theory and Applications*, 16(3):288–305.
- [141] ZeinEldin, R. A., Hashmi, S., Elsehety, M., and Elgarhy, M. (2020). Type II Half Logistic Kumaraswamy Distribution with Applications. *Journal of Function Spaces*, 2020.
- [142] ZeinEldin, R. A., Jamal, F., Chesneau, C., and Elgarhy, M. (2019). Type II Topp-Leone Inverted Kumaraswamy Distribution with Statistical Inference and Applications. *Symmetry*, 11(12):1459.
- [143] Zografos, K. and Balakrishnan, N. (2009). On Families of Beta-and Generalized Gamma-Generated Distributions and Associated Inference. *Statistical Methodology*, 6(4):344–362.