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The Split Equality Fixed Point Problem for Quasi-Pseudo-Contractive Mappings Without Prior Knowledge of Norms

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ABSTRACT

Recently, Chang et al. (2015) constructed an algorithm that converges weakly to the solution of the split equality fixed point problem for quasi-pseudo-contractive mappings under some suitable conditions. They also showed that strong convergence is obtained in the case when the quasi-pseudo-contractive mappings are semi-compact. In this article, we construct an algorithm for quasi-pseudo-contractive mappings that always converge strongly to some solution of the split equality fixed point problem under mild conditions. We mention that we do not require the quasi-pseudo-contractive mappings to be semi-compact to obtain strong convergence. The algorithm does not require any prior knowledge of operator norms. The result of this article provides a unified framework for this type of problems.

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1. Introduction

Let C and D be closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Consider two bounded linear operators $A : H_1 \to H_3$ and $B : H_2 \to H_3$, where H_3 is another real Hilbert space. The split equality feasibility problem consists of finding two points $x \in C$ and $y \in D$ such that $Ax = By$. (For more information on this problem, we refer the reader to [1–3]). Such a problem allows asymmetric and partial relations between the variables x and y, and covers many problems such as decomposition methods for PDEs, applications in game theory, and intensity-modulated radiation therapy. These broad applications caught the attention of many researchers, and eventually leading to various research output for the split equality feasibility problem, see for example [4–12].

Note that if one of the operators, say $B = I$ (the identity mapping on H_2), and $H_2 = H_3$, then the split equality feasibility problem reduces to the split feasibility problem that was first introduced by Censor and Elfving [13] in

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1994. The latter algorithm was then used for inverse problems that arise from phase retrievals, and in medical image reconstruction [14–16]. Several convergent algorithms have been constructed that converge weakly to the solution of the split feasibility problem, see for example [17–19]. In [20], the first author constructed a strongly convergent algorithm for the split feasibility problem.

Another interesting case to consider, with regard to the split equality feasibility problem, was introduced recently by Moudafi [21], and it basically makes use of the assumptions $C = \mathrm{Fix}(T)$ and $D = \mathrm{Fix}(S)$ for some nonlinear operators $T : H_1 \to H_1$ and $S : H_2 \to H_2$, with nonempty fixed point sets $\mathrm{Fix}(T)$ and $\mathrm{Fix}(S)$. The resulting problem is sometimes referred to as the split equality fixed point problem. Some literature is already available for this problem, see for instance [1, 3, 22]. We note that the algorithms constructed in the cited papers only converge weakly. On the other hand, weak convergence is not effective for practical implementation of the algorithm.

In 2015, Zhao [24] introduced the following iterative process for the class of quasi-nonexpansive mappings:

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ v_n = y_n - \gamma_n B^*(By_n - Ax_n), \\ y_{n+1} = \alpha_n v_n + (1-\alpha_n)Sv_n, \end{cases} \tag{1.1}$$

and proved that under certain assumptions, the algorithm aforementioned converges weakly to a solution of the split equality fixed point problem without prior knowledge of norms of A and B.

Recently, Chang et al. [24] introduced the following iterative process for the class of quasi-pseudo contractive mappings:

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)[(1-\xi_n)I + \xi_n T[(1-\eta_n)I + \eta_n T]]u_n, \\ v_n = y_n - \gamma_n B^*(By_n - Ax_n), \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)[(1-\xi_n)I + \xi_n S[(1-\eta_n)I + \eta_n S]]v_n, \end{cases} \tag{1.2}$$

and proved that under certain assumptions, the algorithm aforementioned converges weakly to a solution of the split equality fixed point problem. In addition, Chang et al. [24] showed that strong convergence is guaranteed if the maps S and T are semi-compact.

Motivated by the above works, we propose an iterative method for the class of quasi-pseudo-contractive mappings that always converge strongly to some solution of the split equality fixed point problem. More precisely, if C and D are nonempty, closed and convex subsets of real Hilbert spaces

H_1 and H_2, respectively, and $u, x_0 \in H_1, v, y_0 \in H_2$ are chosen arbitrarily, we define a sequence (x_n, y_n) in $H_1 \times H_2$ iteratively by

$$\begin{cases} u_n = P_C\big[x_n - \gamma_n A^*(Ax_n - By_n)\big], \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[(1-\xi_n)I + \xi_n T[(1-\eta_n)I + \eta_n T]]u_n, \\ v_n = P_D\big[y_n - \gamma_n B^*(By_n - Ax_n)\big], \\ y_{n+1} = \alpha_n v + (1-\alpha_n)[(1-\xi_n)I + \xi_n S[(1-\eta_n)I + \eta_n S]]v_n, \end{cases} \tag{1.3}$$

where $T : C \to H_1$ and $S : D \to H_2$ are quasi-pseudo-contractive mappings, $\{\alpha_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are sequences of real numbers in (0, 1) and $\gamma_n > 0$ for all $n \geq 0$. It is known that the class of quasi-pseudo-contractive mappings is more general than the class of quasi-contractive mappings, directed mappings, and demicontractive mappings. Also from a practical point of view, weak convergence is not effective compared to strong convergence. As a special case, we will show that the projections in Equation (1.3) can be dropped and still obtain strong convergence results for this type of operators without using additional conditions on the parameters defining the constructed sequence. We observe that the algorithm does not require any prior knowledge of operator norms of A and B. Hence, our results provide a unified framework for the study of the split equality fixed point problem and quasi-pseudo-contractive mappings. As an application, we use our algorithm to approximate minimum norm elements that solve the split equality fixed point problem.

2. Preliminaries

In the sequel, H represents a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. It is known that for any $x, y \in H$, the inequalities

$$a) \quad 2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2; \tag{2.1}$$

$$b) \quad ||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle, \tag{2.2}$$

hold true. In addition, it can be proved easily that if α, β, γ are any real numbers in (0, 1) with $\alpha + \beta + \gamma = 1$, then for any $x, y, z \in H$, we have

$$\begin{aligned} ||\alpha x + \beta y + \gamma z||^2 = \alpha||x||^2 + \beta||y||^2 + \gamma||z||^2 - \alpha\beta||x-y||^2 \\ - \alpha\gamma||x-z||^2 - \beta\gamma||y-z||^2, \end{aligned} \tag{2.3}$$

(see e.g., [25, 26]). If C is a nonempty, closed, and convex subset of H, then one can define the metric projection mapping $P_C : H \to C$ by

$$||P_C x - x|| = \inf_{y \in C} ||x - y||.$$

For any $x \in H$, it can be shown that

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{2.4}$$

for all $y \in C$, (see e.g., [27, 28]).

Note that Equation (2.4) is equivalent to

$$||P_C x - y||^2 \leq ||x - y||^2 - ||x - P_C x||^2, \tag{2.5}$$

for all $y \in C$. In this case, we say that the projection operator is firmly quasi-nonexpansive. In fact one can show that the projection mapping is firmly nonexpansive, meaning that it satisfies

$$||P_C x - P_C y||^2 \leq ||x - y||^2 - ||(I - P_C)x - (I - P_C)y||^2,$$

for all $x, y \in H$. For further reading on the properties of firmly nonexpansive operators, in general, and of metric projections, in particular, we refer the reader to the excellent book by Goebel and Reich [27].

We remark that if H_1, H_2 are real Hilbert spaces, then $H := H_1 \times H_2$ is also a real Hilbert space with inner product

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \forall (x_1, y_1), (x_2, y_2) \in H_1 \times H_2,$$

and that

$$(x_n, y_n) \rightharpoonup (x^*, y^*) \text{ implies } x_n \rightharpoonup x^* \text{ and } y_n \rightharpoonup y^*. \tag{2.6}$$

Moreover, if C is a nonempty, closed, and convex subset of H, $(u, v) \in H$ and $(u^*, v^*) = P_C(u, v)$, then from inequality (2.4), we obtain that

$$\langle (u, v) - (u^*, v^*), (x, y) - (u^*, v^*) \rangle \leq 0, \forall (x, y) \in C. \tag{2.7}$$

Recall that for a nonempty subset C of H, an operator $T : C \to C$ is said to be an L-Lipschitzian if

$$||Tx - Ty|| \leq L||x - y||,$$

for all $x, y \in C$. In particular, if $L = 1$, then T is said to be nonexpansive. An operator $T : C \to C$ is said to be pseudo-contractive if

$$\langle Tx - Ty, x - y \rangle \leq ||x - y||^2,$$

for all $x, y \in C$. It is known that an operator T is pseudo-contractive if and only if $I - T$ is monotone. In addition, T is pseudo-contractive if and only if

$$||Tx - Ty||^2 \leq ||x - y||^2 + ||(I - T)x - (I - T)y||^2,$$

for all $x, y \in C$. If the fixed point set of an operator $T : C \to C$, denoted by Fix(T), is nonempty and

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + ||Tx - x||^2,$$

is satisfied for all $x \in C$ and $x^* \in \text{Fix(T)}$, then in that case T is said to be quasi-pseudo-contractive. An operator T is said to be demiclosed at zero if, for any sequence $\{x_n\}$ in C, the conditions $\{x_n\}$ converges weakly to $\hat{x}$ and $\{Tx_n\}$ converges strongly to zero imply that $T\hat{x} = 0$.

We next give some lemmas that will be useful in the sequel.

Lemma 2.1. (Chang et al. [24]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, and $T : C \to C$ be an L-Lipschitzian mapping with $L \geq 1$. Denote*

$$G := (1-\xi)I + \xi T[(1-\eta)I + \eta T], \quad \textit{where} \quad 0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}.$$

Then $Fix(T) = Fix(T[(1-\eta)I + \eta T]) = Fix(G)$.

In [29], it was shown that the map $G : C \to C$ defined in Lemma 2.1 above is quasi-nonexpansive. Our interest is to verify that G satisfies inequality (2.8) mentioned in the following lemma. This fact will be used in proving our main result.

Lemma 2.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, and $T : C \to C$ be a quasi-pseudo-contractive and L-Lipschitzian mapping with $L \geq 1$, and* $\mathrm{Fix(T)} \neq \emptyset$. *Denote*

$$G := (1-\xi)I + \xi T[(1-\eta)I + \eta T], \quad \textit{where} \quad 0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}.$$

Then for any $u \in H$ and $p \in$ $\mathrm{Fix(T)}$,

$$||Gu-p||^2 \leq ||u-p||^2 - \xi\eta(1-2\eta-L^2\eta^2)||u-Tu||^2. \tag{2.8}$$

In particular, the mapping G is quasi-nonexpansive.

Proof. Let $p \in \mathrm{Fix(T)}$. Then from Equation (2.3),

$$\begin{aligned}
||Gu-p||^2 &= ||(1-\xi)(u-p) + \xi\{T[(1-\eta)I + \eta T]u-p\}||^2 \\
&= (1-\xi)||u-p||^2 + \xi||T[(1-\eta)I + \eta T]u-p||^2 \\
&\quad -\xi(1-\xi)||u-T[(1-\eta)I + \eta T]u||^2.
\end{aligned}$$

Since T is quasi-pseudo-contractive,

$$\begin{aligned}
||T[(1-\eta)I + \eta T]u-p||^2 \leq{}& ||[(1-\eta)I + \eta T]u-p||^2 \\
&+ ||[(1-\eta)I + \eta T]u - T[(1-\eta)I + \eta T]u||^2.
\end{aligned} \tag{2.9}$$

Again from Equation (2.3) and the quasi-pseudo-contractive property of T, we have

$$\begin{aligned}||[(1-\eta)I+\eta T]u-p||^2 &= ||(1-\eta)(u-p)+\eta(Tu-p)||^2\\ &= (1-\eta)||u-p||^2+\eta||Tu-p||^2-\eta(1-\eta)||u-Tu||^2\\ &\le (1-\eta)||u-p||^2+\eta(||u-p||^2+||u-Tu||^2)\\ &\quad -\eta(1-\eta)||u-Tu||^2\\ &= ||u-p||^2+\eta^2||u-Tu||^2. \end{aligned} \tag{2.10}$$

On the other hand, from

$$\begin{aligned}(1-\eta)u+\eta Tu-T[(1-\eta)I+\eta T]u &= (1-\eta)(u-T[(1-\eta)I+\eta T]u)\\ &\quad +\eta(Tu-T[(1-\eta)I+\eta T]u),\end{aligned}$$

and Equation (2.3), we have

$$\begin{aligned}||(1-\eta)u+\eta Tu-T[(1-\eta)I+\eta T]u||^2 &= (1-\eta)||u-T[(1-\eta)I+\eta T]u||^2\\ &\quad +\eta||Tu-T[(1-\eta)I+\eta T]u||^2\\ &\quad -\eta(1-\eta)||u-Tu||^2.\end{aligned}$$

Since T is an L-Lipschitzian mapping,

$$\begin{aligned}||Tu-T[(1-\eta)I+\eta T]u||^2 &\le L^2||u-[(1-\eta)u+\eta Tu]||^2\\ &= L^2\eta^2||u-Tu||^2.\end{aligned}$$

Using the condition $\xi<\eta,$ we have

$$\begin{aligned}||(1-\eta)u+\eta Tu-T[(1-\eta)I+\eta T]u||^2 &\le (1-\xi)||u-T[(1-\eta)I+\eta T]u||^2\\ &\quad -\eta(1-\eta-L^2\eta^2)||u-Tu||^2.\end{aligned}$$

From this last inequality and inequalities (2.9) and (2.10), we have

$$\begin{aligned}||T[(1-\eta)I+\eta T]u-p||^2 &\le \{||u-p||^2+\eta^2||u-Tu||^2\}\\ &\quad +(1-\xi)||u-T[(1-\eta)I+\eta T]u||^2\\ &\quad -\eta(1-\eta-L^2\eta^2)||u-Tu||^2\\ &= ||u-p||^2+(1-\xi)||u-T[(1-\eta)I+\eta T]u||^2\\ &\quad -\eta(1-2\eta-L^2\eta^2)||u-Tu||^2.\end{aligned}$$

Therefore,

$$\begin{aligned}||Gu-p||^2 &\le (1-\xi)||u-p||^2+\xi\{||u-p||^2+(1-\xi)||u-T[(1-\eta)I+\eta T]u||^2\}\\ &\quad -\xi\eta(1-2\eta-L^2\eta^2)||u-Tu||^2-\xi(1-\xi)||u-T[(1-\eta)I+\eta T]u||^2\\ &= ||u-p||^2-\xi\eta(1-2\eta-L^2\eta^2)||u-Tu||^2.\end{aligned}$$

The last inequality aforementioned is exactly Equation (2.8).

Note that if

$$0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}},$$

then $\xi\eta(1-2\eta-L^2\eta^2) > 0$. Therefore, from Equation (2.8), we derive

$$||Gu-p|| \le ||u-p||.$$

This shows that G is quasi-nonexpansive. The proof of the Lemma is complete. □

We conclude this section by giving two lemmas that will be used in proving our main result.

Lemma 2.3. (Xu [30]). *Let (s_n) be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1-a_n)s_n + a_n b_n + c_n, \quad n \ge 0,$$

where $(a_n), (b_n), (c_n)$ satisfy the conditions: (i) $(a_n) \subset [0,1]$, with $\sum_{n=0}^{\infty} a_n = \infty$, (ii) $c_n \ge 0$ for all $n \ge 0$ with $\sum_{n=0}^{\infty} c_n < \infty$, and (iii) $\limsup_{n\to\infty} b_n \le 0$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. (Maingé [31]). *Let (s_k) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence (s_{k_j}) of (s_k) such that $s_{k_j} < s_{k_j+1}$ for all $j \ge 0$. Define an integer sequence $(m_k)_{k\ge k_0}$ as*

$$m_k = \max\{k_0 \le l \le k : s_l < s_{l+1}\}.$$

Then $m_k \to \infty$ as $k \to \infty$, and for all $k \ge k_0$

$$\max\{s_{m_k}, s_k\} \le s_{m_k+1}. \tag{2.11}$$

3. Main results

Throughout this article, we use the notation $K_1 \subseteq K_2$ to indicate that K_1 is a subset of K_2. We assume that the set

$$\Gamma = \{(x,y) \in \mathrm{Fix(T)} \times \mathrm{Fix(S)} | Ax = By\}$$

is nonempty, where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators and $T : D(T) \subseteq H_1 \to H_1$ and $S : D(S) \subseteq H_2 \to H_2$ are nonlinear maps.

We now prove our main theorem.

Theorem 3.1. *Let C and D be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Assume that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators with adjoints A^* and B^*, respectively, where H_3 is another real Hilbert space. Suppose that the mappings $T : C \to C$ and $S : D \to D$ are quasi-pseudo-contractive and L-Lipschitzian with Lipschitz constants $L > 1$, and $\Gamma \neq \emptyset$. If $u, x_0 \in H_1$ and $v, y_0 \in H_2$ are chosen arbitrarily, then the sequence $\{(x_n, y_n)\}$ defined by Equation (1.3), with*

(a) $0 < a < \xi_n < \eta_n < b < \frac{1}{1+\sqrt{1+L^2}}$, *for all* $n \geq 0$,
(b) $\alpha_n \in (0, 1)$,
(c) $0 < \delta < \gamma_n < \mu < \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n||^2 + ||B^*(Ax_n - By_n)||^2}$, *for* $n \in \Omega$,

otherwise, $\gamma_n = \gamma$ *(*γ *being any nonnegative value), where the set of the indexes* $\Omega = \{n \in \mathbb{N} : Ax_n - By_n \neq 0\}$*, is a bounded sequence in* $H_1 \times H_2$*.*

Proof. Let $(p, q) \in \Gamma$. That is, $p \in \mathrm{Fix}(T), q \in \mathrm{Fix}(S)$ and $Ap = Bq$. Denote

$$G_n := (1-\xi_n)I + \xi_n T[(1-\eta_n)I + \eta_n T].$$

Then from Lemma 2.2,

$$||G_n u_n - p||^2 \leq ||u_n - p||^2 - \xi_n \eta_n (1 - 2\eta_n - L^2 \eta_n^2)||u_n - Tu_n||^2. \tag{3.1}$$

On the other hand, from Equation (1.3), the firmly nonexpansive property of projections and Equation (2.1), we have

$$\begin{aligned} ||u_n - p||^2 &\leq ||x_n - \gamma_n A^*(Ax_n - By_n) - p||^2 - ||x_n - \gamma_n A^*(Ax_n - By_n) - u_n||^2 \\ &= ||x_n - p||^2 + \gamma_n^2 ||A^*(Ax_n - By_n)||^2 - 2\gamma_n \langle x_n - p, A^*(Ax_n - By_n)\rangle \\ &\quad - ||x_n - u_n - \gamma_n A^*(Ax_n - By_n)||^2 \\ &= ||x_n - p||^2 + \gamma_n^2 ||A^*(Ax_n - By_n)||^2 - 2\gamma_n \langle Ax_n - Ap, Ax_n - By_n\rangle \\ &\quad - ||x_n - u_n - \gamma_n A^*(Ax_n - By_n)||^2 \\ &= ||x_n - p||^2 + \gamma_n^2 ||A^*(Ax_n - By_n)||^2 - \gamma_n ||Ax_n - Ap||^2 \\ &\quad - \gamma_n ||Ax_n - By_n||^2 + \gamma_n ||By_n - Ap||^2 \\ &\quad - ||x_n - u_n - \gamma_n A^*(Ax_n - By_n)||^2. \end{aligned} \tag{3.2}$$

Therefore,

$$\begin{aligned} ||G_n u_n - p||^2 \leq\ & ||x_n - p||^2 + \gamma_n^2 ||A^*(Ax_n - By_n)||^2 - \gamma_n ||Ax_n - Ap||^2 \\ & - \gamma_n ||Ax_n - By_n||^2 + \gamma_n ||By_n - Ap||^2 \\ & - \xi_n \eta_n (1 - 2\eta_n - L^2 \eta_n^2)||u_n - Tu_n||^2 \\ & - ||x_n - u_n - \gamma_n A^*(Ax_n - By_n)||^2. \end{aligned} \tag{3.3}$$

Moreover, from $x_{n+1} = \alpha_n u + (1-\alpha_n) G_n u_n$, equation (2.3) and inequality (3.1), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\alpha_n (u - p) + (1-\alpha_n)(G_n u_n - p)||^2 \\ &= \alpha_n ||u - p||^2 + (1-\alpha_n)||G_n u_n - p||^2 \\ &\quad - \alpha_n (1-\alpha_n)||G_n u_n - u||^2, \end{aligned}$$

and

$$||x_{n+1}-p||^2 \leq \alpha_n||u-p||^2 + (1-\alpha_n)||u_n-p||^2. \tag{3.4}$$

Similarly, setting

$$H_n := (1-\xi_n)I + \xi_n S[(1-\eta_n)I + \eta_n S],$$

we have $y_{n+1} = \alpha_n v + (1-\alpha_n)H_n v_n$, and again from Lemma 2.2,

$$||H_n v_n - q||^2 \leq ||v_n-q||^2 - \xi_n\eta_n(1-2\eta_n-L^2\eta_n^2)||v_n-Sv_n||^2.$$

From Equation (1.3), the firmly nonexpansive property of projections, and Equation (2.1), we have

$$\begin{aligned}
||v_n-q||^2 &\leq ||y_n-\gamma_n B^*(By_n-Ax_n)-q||^2 - ||y_n-\gamma_n B^*(By_n-Ax_n)-v_n||^2 \\
&= ||y_n-q||^2 + \gamma_n^2||B^*(By_n-Ax_n)||^2 - 2\gamma_n\langle y_n-q, B^*(By_n-Ax_n)\rangle \\
&\quad - ||y_n-v_n-\gamma_n B^*(By_n-Ax_n)||^2 \\
&= ||y_n-q||^2 + \gamma_n^2||B^*(By_n-Ax_n)||^2 - 2\gamma_n\langle By_n-Bq, By_n-Ax_n\rangle \\
&\quad - ||y_n-v_n-\gamma_n B^*(By_n-Ax_n)||^2 \\
&= ||y_n-q||^2 + \gamma_n^2||B^*(By_n-Ax_n)||^2 - \gamma_n||By_n-Bq||^2 \\
&\quad - \gamma_n||Ax_n-By_n||^2 + \gamma_n||Ax_n-Bq||^2 \\
&\quad - ||y_n-v_n-\gamma_n B^*(By_n-Ax_n)||^2.
\end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned}
||H_n v_n-q||^2 \leq\ & ||y_n-q||^2 + \gamma_n^2||B^*(By_n-Ax_n)||^2 - \gamma_n||By_n-Bq||^2 \\
& - \gamma_n||Ax_n-By_n||^2 + \gamma_n||Ax_n-Bq||^2 \\
& - \xi_n\eta_n(1-2\eta_n-L^2\eta_n^2)||v_n-Sv_n||^2 \\
& - ||y_n-v_n-\gamma_n B^*(By_n-Ax_n)||^2,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
||y_{n+1}-q||^2 &= \alpha_n||v-q||^2 + (1-\alpha_n)||H_n v_n-q||^2 - \alpha_n(1-\alpha_n)||v-H_n v_n||^2 \\
&\leq \alpha_n||v-q||^2 + (1-\alpha_n)||v_n-q||^2.
\end{aligned} \tag{3.7}$$

Now, adding inequalities (3.2) and (3.5), using the fact that $Ap = Bq$ and noting assumptions on γ_n, we obtain

$$\begin{aligned}
||u_n-p||^2 + ||v_n-q||^2 &\leq ||x_n-p||^2 + ||y_n-q||^2 - \gamma_n[2||Ax_n-By_n||^2 \\
&\quad - \gamma_n(||A^*(Ax_n-By_n)||^2 + ||B^*(By_n-Ax_n)||^2)] \\
&\leq ||x_n-p||^2 + ||y_n-q||^2.
\end{aligned}$$

Now, adding (3.4) and (3.7) yields

$$\begin{aligned}||x_{n+1}-p||^2+||y_{n+1}-q||^2 &\le \alpha_n\left[||u-p||^2+||v-q||^2\right]\\ &\quad+(1-\alpha_n)\left[||x_n-p||^2+||y_n-q||^2\right]\\ &\le \max\{||x_n-p||^2+||y_n-q||^2, ||u-p||^2+||v-q||^2\}\\ &\quad\vdots\\ &\le \max\{||x_1-p||^2+||y_1-q||^2, ||u-p||^2+||v-q||^2\}.\end{aligned}$$

Therefore, $\{||x_n-p||^2+||y_n-q||^2\}$ is bounded. Hence, $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently, $\{u_n\}, \{v_n\}, \{Tu_n\}$ and $\{Sv_n\}$ are bounded. □

Theorem 3.2. *Let C and D be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Assume that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators with adjoints A^* and B^*, respectively, where H_3 is another real Hilbert space. Suppose that the maps $T : C \to C$ and $S : D \to D$ are quasi-pseudo-contractive and L-Lipschitzian with Lipschitz constants $L > 1$, and $\Gamma \neq \emptyset$. Assume that both I – T and I – S are demiclosed at zero. If $u, x_0 \in H_1$ and $v, y_0 \in H_2$ are chosen arbitrarily, then the sequence $\{(x_n, y_n)\}$ defined by Equation (1.3), with*

(a) $0 < a < \xi_n < \eta_n < b < \frac{1}{1+\sqrt{1+L^2}}$, *for all* $n \ge 0$,
(b) $\alpha_n \in (0, 1)$, *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(c) $0 < \delta < \gamma_n < \mu < \frac{2||Ax_n-By_n||^2}{||A^*(Ax_n-By_n||^2+||B^*(Ax_n-By_n)||^2}$, *for* $n \in \Omega$,

otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of the indexes $\Omega = \{n \in \mathbb{N} : Ax_n - By_n \neq 0\}$, converges strongly to an element $(\hat{p}, \hat{q}) = P_\Gamma(u, v)$.

Proof. The sequence $\{(x_n, y_n)\}$ is a bounded sequence in $H_1 \times H_2$, see Theorem 3.1. Take $(\hat{p}, \hat{q}) = P_\Gamma(u, v)$. If we denote

$S_n(\hat{p}, \hat{q}) := ||x_n-\hat{p}||^2 + ||y_n-\hat{q}||^2$ and $G_n := (1-\xi_n)I + \xi_n T[(1-\eta_n)I + \eta_n T]$,

then from property (2.2) and the definition of x_n, we have

$$\begin{aligned}||x_{n+1}-\hat{p}||^2 &= ||\alpha_n(u-\hat{p}) + (1-\alpha_n)(G_n u_n-\hat{p})||^2\\ &\le (1-\alpha_n)||G_n u_n-\hat{p}||^2 + 2\alpha_n\langle u-\hat{p}, x_{n+1}-\hat{p}\rangle.\end{aligned}$$

Using inequality (3.3), we get

$$\begin{aligned} ||x_{n+1}-\hat{p}||^2 \le{}& (1-\alpha_n)\Big[||x_n-\hat{p}||^2+\gamma_n^2||A^*(Ax_n-By_n)||^2\Big]\\ &-(1-\alpha_n)\gamma_n\big[||Ax_n-Ap||^2+||Ax_n-By_n||^2-||By_n-Ap||^2\big]\\ &+2\alpha_n\langle u-\hat{p},x_{n+1}-\hat{p}\rangle\\ &-(1-\alpha_n)\xi_n\eta_n(1-2\eta_n-L^2\eta_n^2)||u_n-Tu_n||^2\\ &-(1-\alpha_n)||x_n-u_n-\gamma_nA^*(Ax_n-By_n)||^2. \end{aligned} \tag{3.8}$$

Similarly, from property (2.2) and the definition of y_n, we have

$$\begin{aligned} ||y_{n+1}-\hat{q}||^2 &= ||\alpha_n(v-\hat{q})+(1-\alpha_n)(H_nv_n-\hat{q})||^2\\ &\le (1-\alpha_n)||H_nv_n-\hat{q}||^2+2\alpha_n\langle v-\hat{q},y_{n+1}-\hat{q}\rangle, \end{aligned}$$

where $H_n := (1-\xi_n)I+\xi_nS[(1-\eta_n)I+\eta_nS]$. Using inequality (3.6), we get

$$\begin{aligned} ||y_{n+1}-\hat{q}||^2 \le{}& (1-\alpha_n)\Big[||y_n-\hat{q}||^2+\gamma_n^2||B^*(By_n-Ax_n)||^2\Big]\\ &-(1-\alpha_n)\gamma_n\big[||By_n-Bq||^2+||Ax_n-By_n||^2-||Ax_n-Bq||^2\big]\\ &+2\alpha_n\langle v-\hat{q},y_{n+1}-\hat{q}\rangle\\ &-(1-\alpha_n)\xi_n\eta_n(1-2\eta_n-L^2\eta_n^2)||v_n-Sv_n||^2\\ &-(1-\alpha_n)||y_n-v_n-\gamma_nB^*(By_n-Ax_n)||^2. \end{aligned} \tag{3.9}$$

From condition (a) of the theorem, we can find a positive constant C such that

$$(1-\alpha_n)\xi_n\eta_n(1-2\eta_n-L^2\eta_n^2) > C.$$

Adding inequalities (3.8) and (3.9), and making use of the above condition, we get

$$\begin{aligned} S_{n+1}(\hat{p},\hat{q}) \le{}& (1-\alpha_n)S_n(\hat{p},\hat{q})+2\alpha_n\big[\langle u-\hat{p},x_{n+1}-\hat{p}\rangle+\langle v-\hat{q},y_{n+1}-\hat{q}\rangle\big]\\ &-(1-\alpha_n)\gamma_n\big[2||Ax_n-By_n||^2-\gamma_n(||A^*(Ax_n-By_n)||^2+||B^*(Ax_n-By_n)||^2)\big]\\ &-C\big[||Tu_n-u_n||+||Sv_n-v_n||^2\big]\\ &-(1-\alpha_n)||x_n-u_n-\gamma_nA^*(Ax_n-By_n)||^2\\ &-(1-\alpha_n)||y_n-v_n-\gamma_nB^*(By_n-Ax_n)||^2. \end{aligned} \tag{3.10}$$

Next, we show that the sequence $\{S_n(\hat{p},\hat{q})\}$ converges strongly to zero. Note that the convergence of $\{S_n(\hat{p},\hat{q})\}$ to zero implies that

$$\lim_{n\to\infty}||x_n-\hat{p}||=0 \quad\text{and}\quad \lim_{n\to\infty}||y_n-\hat{q}||=0,$$

which in turn imply that $x_n\to\hat{p}$ and $y_n\to\hat{q}$ as $n\to\infty$. In order to derive strong convergence of $\{S_n(\hat{p},\hat{q})\}$ to zero, we consider two possible cases on $\{S_n(\hat{p},\hat{q})\}$.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that the sequence of real numbers $\{S_n(\hat{p}, \hat{q})\}$ is decreasing for all $n \geq n_0$. It then follows that $\{S_n(\hat{p}, \hat{q})\}$ is convergent. Since the sequences $\{x_n\}$ and $\{y_n\}$ are bounded, it follows from inequality (3.10), the fact that $\{\alpha_n\}$ converges to zero as $n \to \infty$ and condition (c) of the assumption that

$$\lim_{n\to\infty}(||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2) = 0, \text{ for } n \in \Omega, \tag{3.11}$$

and

$$\lim_{n\to\infty}||x_n - u_n - \gamma_n A^*(Ax_n - By_n)|| = 0 = \lim_{n\to\infty}||y_n - v_n - \gamma_n B^*(By_n - Ax_n)||.$$

Note that $Ax_n - By_n = 0$, if $n \notin \Omega$. Thus,

$$\lim_{n\to\infty}||A^*(Ax_n - By_n)|| = \lim_{n\to\infty}||B^*(Ax_n - By_n)||) = 0. \tag{3.12}$$

In addition, from equation (3.11) and inequality (3.10) we obtain that

$$\lim_{n\to\infty}||Ax_n - By_n|| = 0. \tag{3.13}$$

Moreover,

$$||x_n - u_n|| \leq ||x_n - u_n - \gamma_n A^*(Ax_n - By_n)|| + ||\gamma_n A^*(Ax_n - By_n)||.$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty}||x_n - u_n|| = 0. \tag{3.14}$$

Similarly,

$$||y_n - v_n|| \leq ||y_n - v_n - \gamma_n B^*(By_n - Ax_n)|| + ||\gamma_n B^*(By_n - Ax_n)||.$$

Passing to the limit as $n \to \infty$, we get

$$\lim_{n\to\infty}||y_n - v_n|| = 0. \tag{3.15}$$

Furthermore, from inequality (3.10), we have

$$\lim_{n\to\infty}||u_n - Tu_n|| = 0 = \lim_{n\to\infty}||v_n - Sv_n||. \tag{3.16}$$

Now, since $\{(x_n, y_n)\}$ is bounded in $H_1 \times H_2$, there exists $(x^*, y^*) \in H_1 \times H_2$ and a subsequence $\{(x_{n_j}, y_{n_j})\}$ of $\{(x_n, y_n)\}$ such that $(x_{n_j}, y_{n_j}) \rightharpoonup (x^*, y^*)$, and

$$\begin{aligned}
&\limsup_{n\to\infty}[\langle u - \hat{p}, x_n - \hat{p}\rangle + \langle v - \hat{q}, y_n - \hat{q}\rangle] \\
&= \limsup_{n\to\infty}\langle (u, v) - (\hat{p}, \hat{q}), (x_n, y_n) - (\hat{p}, \hat{q})\rangle \\
&= \lim_{j\to\infty}\langle (u, v) - (\hat{p}, \hat{q}), (x_{n_j}, y_{n_j}) - (\hat{p}, \hat{q})\rangle.
\end{aligned} \tag{3.17}$$

But $(x_{n_j}, y_{n_j}) \rightharpoonup (x^*, y^*)$ implies that $x_{n_j} \rightharpoonup x^*$ and $y_{n_j} \rightharpoonup y^*$. Hence from Equations (3.14) and (3.15), we have $u_{n_j} \rightharpoonup x^*$ and $v_{n_j} \rightharpoonup y^*$, respectively. Now, since $(I-T)$ and $(I-S)$ are demiclosed at zero, from Equation (3.16) we get $x^* \in F(T)$ and $y^* \in F(S)$.

Next, we show that $Ax^* = By^*$. Observe that

$$\begin{aligned}||Ax^*-By^*||^2 &= ||(Ax^*-Ax_{n_j} + By_{n_j}-By^*) + (Ax_{n_j}-By_{n_j})||^2\\ &\leq ||Ax_{n_j}-By_{n_j}||^2 + 2\langle Ax^*-By^*, Ax^*-Ax_{n_j} + By_{n_j}-By^*\rangle,\end{aligned}$$

where the inequality follows from inequality (2.2). Since $x_{n_j} \rightharpoonup x^*$ and $y_{n_j} \rightharpoonup y^*$ as $j \to \infty$, it follows that $Ax_{n_j} \rightharpoonup Ax^*$ and $By_{n_j} \rightharpoonup By^*$ as $j \to \infty$. Taking limits on both sides, and making use of Equation (3.13), we get

$$\begin{aligned}||Ax^*-By^*||^2 &\leq \limsup_{j\to\infty} ||Ax_{n_j}-By_{n_j}||^2\\ &\quad + 2\limsup_{j\to\infty}\langle Ax^*-By^*, Ax^*-Ax_{n_j} + By_{n_j}-By^*\rangle\\ &= 0.\end{aligned}$$

This inequality implies that $Ax^* = By^*$. That is $(x^*, y^*) \in \Gamma$. Consequently, from equation (3.17) and inequality (2.7) we derive that

$$\begin{aligned}&\limsup_{n\to\infty} \left[\langle u-\hat{p}, x_n-\hat{p}\rangle + \langle v-\hat{q}, y_n-\hat{q}\rangle\right]\\ &= \langle (u,v) - (\hat{p},\hat{q}), (x^*,y^*) - (\hat{p},\hat{q})\rangle \leq 0.\end{aligned} \tag{3.18}$$

Now from Equation (1.3), we have

$$\begin{aligned}||x_{n+1}-x_n|| &\leq \alpha_n||u-x_n|| + (1-\alpha_n)||G_n u_n-x_n||\\ &\leq \alpha_n M' + ||G_n u_n-x_n||,\end{aligned}$$

for some positive constant M'. Note that

$$\begin{aligned}||G_n u_n-x_n|| &= ||u_n-x_n + \xi_n(T[(1-\eta_n)I + \eta_n T]u_n-u_n)||\\ &\leq ||u_n-x_n|| + \xi_n||u_n-Tu_n||\\ &\quad + \xi_n||Tu_n-T[(1-\eta_n)I + \eta_n T]u_n||.\end{aligned}$$

Since T is an L-Lipschitzian, we have

$$\begin{aligned}||G_n u_n-x_n|| &\leq ||u_n-x_n|| + \xi_n||u_n-Tu_n||\\ &\quad + \xi_n L||u_n-[(1-\eta_n)I + \eta_n T]u_n||\\ &= ||u_n-x_n|| + \xi_n(1 + \eta_n L)||u_n-Tu_n||.\end{aligned}$$

Therefore,

$$||x_{n+1}-x_n|| \leq \alpha_n M' + ||u_n-x_n|| + \xi_n(1 + \eta_n L)||u_n-Tu_n||.$$

Passing to the limit as $n \to \infty$, and making use of Equations (3.14) and (3.16), we derive

$$\lim_{n\to\infty} ||x_{n+1}-x_n|| = 0.$$

Similarly from Equations (3.15) and (3.16), we derive

$$\lim_{n\to\infty} ||y_{n+1}-y_n|| = 0.$$

These last two limits together with inequality (3.18) imply that

$$\limsup_{n\to\infty} \left[\langle u-\hat{p}, x_{n+1}-\hat{p}\rangle + \langle v-\hat{q}, y_{n+1}-\hat{q}\rangle\right] \le 0.$$

Finally, rearranging inequality (3.10) gives

$$S_{n+1}(\hat{p},\hat{q}) \le (1-\alpha_n)S_n(\hat{p},\hat{q}) + 2\alpha_n\left[\langle u-\hat{p}, x_{n+1}-\hat{p}\rangle + \langle v-\hat{q}, y_{n+1}-\hat{q}\rangle\right].$$

Then by Lemma 2.3, it follows that $\{S_n(\hat{p},\hat{q})\}$ converges strongly to zero as $n \to \infty$. Hence $\{x_n\}$ converges strongly to $\hat{p}$ and the sequence $\{y_n\}$ converges strongly to $\hat{q}$ as $n \to \infty$. Consequently, $\{(x_n, y_n)\}$ converges strongly to $(\hat{p},\hat{q})$ as $n \to \infty$.

Case 2. Assume that there exists a subsequence $\{S_{k_i}(\hat{p},\hat{q})\}$ of $\{S_k(\hat{p},\hat{q})\}$ such that $S_{k_i}(\hat{p},\hat{q}) < S_{k_i+1}(\hat{p},\hat{q})$ for all $i \ge 0$. Then in view of Lemma 2.4, we can define a nondecreasing sequence $(m_k) \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$ and $\max\{S_{m_k}(\hat{p},\hat{q}), S_k(\hat{p},\hat{q})\} \le S_{m_k+1}(\hat{p},\hat{q})$ for all $k \in \mathbb{N}$. Since the sequences $\{x_{m_k}\}$ and $\{y_{m_k}\}$ are bounded, following the methods in Case 1, inequality (3.10) and the fact that $\{\alpha_{m_k}\}$ converges to zero as $n \to \infty$, we obtain

$$\lim_{k\to\infty} ||Ax_{m_k} - By_{m_k}|| = 0, \tag{3.19}$$

and

$$\lim_{k\to\infty} ||u_{m_k} - Tu_{m_k}|| = 0 = \lim_{k\to\infty} ||v_{m_k} - Sv_{m_k}||. \tag{3.20}$$

Moreover, as in Case 1, we can also derive the limits

$$\lim_{k\to\infty} ||x_{m_k} - u_{m_k} - \gamma_{m_k}A^*(Ax_{m_k} - By_{m_k})|| = 0,$$

and

$$\lim_{k\to\infty} ||y_{m_k} - v_{m_k} - \gamma_{m_k}B^*(By_{m_k} - Ax_{m_k})|| = 0.$$

From these limits and Equation (3.19), we derive

$$\lim_{k\to\infty} ||u_{m_k} - x_{m_k}|| = 0 = \lim_{k\to\infty} ||v_{m_k} - y_{m_k}||. \tag{3.21}$$

Now choose a subsequence $\{(x_{m_{k(l)}}, y_{m_{k(l)}})\}$ of $\{(x_{m_k}, y_{m_k})\}$ converging weakly to an element, say $(\hat{x}, \hat{y}) \in H_1 \times H_2$ with the property

$$\limsup_{k\to\infty} \left[\langle u-\hat{p}, x_{m_k}-\hat{p}\rangle + \langle v-\hat{q}, y_{m_k}-\hat{q}\rangle\right] = \lim_{l\to\infty}[\langle u-\hat{p}, x_{m_{k(l)}}-\hat{p}\rangle + \langle v-\hat{q}, y_{m_{k(l)}}-\hat{q}\rangle].$$

Then as in Case 1, we derive that $(\hat{x}, \hat{y}) \in \Gamma$. Consequently, from inequality (2.4), we have

$$\limsup_{k\to\infty} \left[\langle u-\hat{p}, x_{m_k}-\hat{p}\rangle + \langle v-\hat{q}, y_{m_k}-\hat{q}\rangle\right] = \langle u-\hat{p}, \hat{x}-\hat{p}\rangle + \langle v-\hat{q}, \hat{y}-\hat{q}\rangle \leq 0.$$

Following similar steps as in Case 1, we derive from Equations (3.19), (3.20) and (3.21)

$$\lim_{k\to\infty} ||x_{m_k+1}-x_{m_k}|| = 0 = \lim_{k\to\infty} ||y_{m_k+1}-y_{m_k}||.$$

Hence, we have

$$\limsup_{k\to\infty} \left[\langle u-\hat{p}, x_{m_k+1}-\hat{p}\rangle + \langle v-\hat{q}, y_{m_k+1}-\hat{q}\rangle\right] \leq 0.$$

Finally, making use of the inequality

$$S_{m_k}(\hat{p}, \hat{q}) \leq S_{m_k+1}(\hat{p}, \hat{q})$$

for all $k \in \mathbb{N}$, and rearranging terms in inequality (3.10), we derive

$$\alpha_{m_k} S_{m_k+1}(\hat{p}, \hat{q}) \leq 2\alpha_{m_k}\left[\langle u-\hat{p}, x_{m_k+1}-\hat{p}\rangle + \langle v-\hat{q}, y_{m_k+1}-\hat{q}\rangle\right].$$

Dividing throughout by α_{m_k} and passing to the limit as $k \to \infty$ in the resulting inequality, we obtain $S_{m_k+1}(\hat{p}, \hat{q}) \to 0$ as $k \to \infty$. Since $S_k(\hat{p}, \hat{q}) \leq S_{m_k+1}(\hat{p}, \hat{q})$, it follows that $S_k(\hat{p}, \hat{q}) \to 0$ as $k \to \infty$. That is,

$$\lim_{k\to\infty} ||x_k-\hat{p}|| = 0 = \lim_{k\to\infty} ||y_k-\hat{q}||.$$

Thus $x_k \to \hat{p}$ and $y_k \to \hat{q}$ as $k \to \infty$.

We have shown in both cases that the sequence $\{(x_n, y_n)\}$ generated by Equation (1.3) converges strongly to $(\hat{p}, \hat{q}) \in \Gamma$ as $n \to \infty$. This completes the proof of the theorem. □

If T is a continuous pseudo-contractive mapping on a closed and convex subset C of H, then $I - T$ is demiclosed at zero, (see for example, [32]). In this case, we have the following corollary.

Corollary 3.1. *Let C and D be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Assume that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators with adjoints A^* and B^*, respectively,* *where H_3 is another real Hilbert space. Suppose that the mappings $T : C \to C$ and $S : D \to D$ are pseudo-contractive and L-Lipschitzians with Lipschitz constants $L > 1$, and $\Gamma \neq \emptyset$. If $u, x_0 \in H_1$ and $v, y_0 \in H_2$ are chosen arbitrarily, then the sequence $\{(x_n, y_n)\}$ defined by Equation (1.3) converges strongly to an element $(\hat{p}, \hat{q}) \in \Gamma$, where $(\hat{p}, \hat{q}) = P_\Gamma(u, v)$, provided that conditions* (a)–(c) *of Theorem 3.2 are satisfied.*

We note that algorithm (1.3) and the method of proof of Theorem 3.2 provides the following theorem for approximating the minimum norm point of the set Γ. The algorithm that guarantees convergence to the point with minimum norm is

$$\begin{cases} u_n = P_C\big[x_n - \gamma_n A^*(Ax_n - By_n)\big], \\ x_{n+1} = (1-\alpha_n)[(1-\xi_n)I + \xi_n T[(1-\eta_n)I + \eta_n T]]u_n, \\ v_n = P_D\big[y_n - \gamma_n B^*(By_n - Ax_n)\big], \\ y_{n+1} = (1-\alpha_n)[(1-\xi_n)I + \xi_n S[(1-\eta_n)I + \eta_n S]]v_n. \end{cases} \tag{3.22}$$

Theorem 3.3. *Let C and D be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Assume that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators with adjoints A^* and B^*, respectively, where H_3 is another real Hilbert space. Suppose that $T : C \to C$ and $S : D \to D$ are quasi-pseudo-contractive and L-Lipschitzian mappings with Lipschitz constants $L > 1$. Assume that $\Gamma \neq \emptyset$, and both I – T and I – S are demiclosed at zero. If $x_0 \in H_1$ and $y_0 \in H_2$ are chosen arbitrarily, then the sequence $\{(x_n, y_n)\}$ defined by Equation (3.22) converges strongly to a minimum norm element $(\bar{p}, \bar{q}) \in \Gamma$, provided that conditions* (a)*–(c) of Theorem 3.2 are satisfied. That is, $(\bar{p}, \bar{q}) = P_\Gamma(0, 0)$.*

If $C = H_1$ and $D = H_2$, then algorithm (1.3) reduces to

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[(1-\xi_n)I + \xi_n T[(1-\eta_n)I + \eta_n T]]u_n, \\ v_n = y_n - \gamma_n B^*(By_n - Ax_n), \\ y_{n+1} = \alpha_n v + (1-\alpha_n)[(1-\xi_n)I + \xi_n S[(1-\eta_n)I + \eta_n S]]v_n. \end{cases} \tag{3.23}$$

In this case, we have from Theorem 3.2 the following corollary.

Corollary 3.2. *Let H_1, H_2, and H_3 be real Hilbert spaces. Assume that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators with adjoints A^* and B^*, respectively. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be quasi-pseudo-contractive and L-Lipschitzian mappings with Lipschitz constants $L > 1$. Suppose that $\Gamma \neq \emptyset$, and both I – T and I – S are demiclosed at zero. If $u, x_0 \in H_1$ and $v, y_0 \in H_2$ are chosen arbitrarily, then the sequence $\{(x_n, y_n)\}$ defined by*

Equation (3.23), with conditions (a)–(c) *of Theorem 3.2, converges strongly to an element* $(\hat{p}, \hat{q}) \in \Gamma$, *where* $(\hat{p}, \hat{q}) = P_\Gamma(u, v)$.

Corollary 3.3. *Let* H_1, H_2, *and* H_3 *be real Hilbert spaces. Assume that* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *are bounded linear operators with adjoints* A^* *and* B^*, *respectively. Let* $T : H_1 \to H_1$ *and* $S : H_2 \to H_2$ *be pseudo-contractive and L-Lipschitzian mappings with Lipschitz constants* $L > 1$. *If* $u, x_0 \in H_1$ *and* $v, y_0 \in H_2$ *are chosen arbitrarily, and* $\Gamma \neq \emptyset$, *then the sequence* $\{(x_n, y_n)\}$ *defined by Equation (3.23), with conditions* (a)–(c) *of Theorem 3.2, converges strongly to an element* $(\hat{p}, \hat{q}) = P_\Gamma(u, v)$.

Remark 3.1. The above results for quasi-pseudo contractive maps are also valid for quasi nonexpansive maps, and hence our results improve and extend many results in the literature [21, 23, 24].

Remark 3.2. Our algorithm and results provide a unified framework in the study of the split common fixed point problem. We emphasize that our algorithms always converge strongly to a solution of split equality problem for the class of Lipschitzian and quasi-pseudocontractive mappings T and S without compactness assumption. The algorithm does not require the prior information about the operator norms of A and B.

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