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Analysis of the Axisymmetric far wake

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Abstract. **The axisymmetric turbulent far wake of a long slender cylinder is being studied. Classical analytical solutions describing the wake width and the velocity defect are presented. The governing equations have been solved with appropriate boundary conditions and a scaling analysis to obtain a similarity equation for the mean velocity distributions. Solutions of the velocity distributions are presented in terms of the hypergeometric function known as the Whittaker function. Validation of the analysis is done with available experimental data.**

Keywords. *Axisymmetric flow, far wake, mean velocity, half wake thickness.*

I. INTRODUCTION

The wake is the region that is produced behind an object placed in a freestream and manifest itself in the form of a velocity deficit profile. It is in this region that the flow is usually dominated by separation and reattachment or trailing edge singularities [1]. Wakes are known to have a nonlinear spread rate but are however similar to other free shear flows like jets and plumes which exhibit a self-similarity beyond a certain downstream distance such that, a characteristic length and velocity can be used to scale all distance and velocity of the flow [2].

The wake region can be divided into three regions; the near wake, outer near wake and the far wake. The near wake and the outer near wake exist in the locus immediately after the trailing edge where the flow regime is highly influenced by the initial conditions of the boundary layer at the trailing edge of the body [1, 3], whilst the far wake is the region further downstream of the trailing edge where the flow properties are generally expected to be less influenced by the trailing edge conditions. A schematic depiction of the flow regions is shown in figure 1 below.

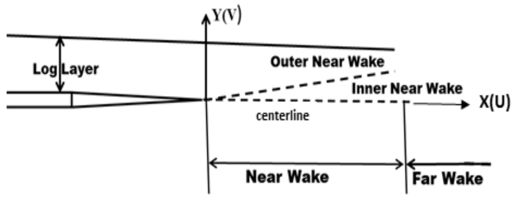

Fig. 1. Different flow regions of the wake

The far wake is an intrinsic flow region which has found itself as a subject of interest. The literature involves both the study of the two dimensional wake and the axisymmetric case [4-11]. In the far wake, it is observed; from the mentioned studies, that there is a centerline velocity decay which is coupled by a wake extent growth. Basically, the region is characterized by slower rates of growth of wake width and a decay of the centerline velocity than in the near wake. Both these characteristics exist in the 2 dimensional and the axisymmetric case. However, in the two dimensional case, the wake width and the centerline velocity defect are known to grow linearly as $x^{\frac{1}{2}}$ and $x^{\frac{-1}{2}}$, [4-6], where as investigations of the axisymmetric case have shown that the centerline velocity defect decays as $x^{\frac{-2}{3}}$ and the wake width grows as $x^{\frac{1}{3}}$ [2,7]. Understanding the axisymmetric far wake has indeed attracted a lot of keenness; experimental studies have been conducted in order to explore behavioral and flow properties in the far wake, among them, Carmody [8], Gibson et al [9], Chevray [10] and Bevilaqua and Lykoudis [11]. In these works, similarity profiles of the flow formed within a particular downstream distance. Various self similarity distances were obtained corresponding to different geometries, which one could assume, the shape of the geometry to be the attributing factor. For instance, Chevray [10] obtained mean flow of a spheroid in the range $3 < {}^{x}/_{D} < 6$ while Carmody's results indicate similarity at ${}^{x}/_{D} = 15$ when studying the axisymmetric wake of a disc, and Gibson et al [9] reported decay of mean and variance of velocity and temperature down to ${}^{x}/_{D} = 60$, when they investigated a sphere using hot wires and pitot tubes. Bevilaqua and Lykoudi's [11] also made very significant observations when they studied the self preservation of the axisymmetric wake on a sphere and

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a porous disc with the same drag. They observed that, self preservation is a process which develops gradually with downstream distances, and first to preserve is the mean velocity profile, then the Reynolds stresses and later the turbulent moments. Bevilaqua and Lykoudi [11] further challenges the common believe of Townsend [6] that, turbulence forgets how it was created. Results from [11] indicate that the self similarity of mean velocity and Reynolds stress profiles are different and are not in the same manner. Rind [12] also studies the effects of free stream turbulence on wakes using direct numerical simulation and wind tunnel experiments.

The current study is concerned with a description of the far wake of a long slender cylinder with a sharp trailing edge, with initial conditions in terms of self-similarity solutions of the governing equations. This will thus provide a rationale and addressing questions regarding performances of similar axisymmetric bodies such as autonomous mobile axisymmetric devices in both aero and marine design and research. Engineers and designers could find optimal criteria in designing both aero and hydro driven devices from a better understanding and point of view and applying relevant knowledge such as, idyllic ways of minimization of drag, optimal body design enabling reduction of propulsive power for an efficient operation of an autonomous water vehicle etc. We present the classical results of the wake width and the centerline velocity by applying a reasonable asymptoticity on the governing equations of motion. The classical results are obtained which are also seen in the works of [2, 7, 12] when they studied an axisymmetric shape in the form of a sphere. Agrawal and Prasad [2] used the Gaussian profile as the best fit to describe velocity defect and gave an equation that describes the streamwise velocity. Rind [12] provides an analysis of equations describing the development of shear flow for incompressible axisymmetric far wake where [12] applies the momentum flux equation after a scaling analysis to obtain similarity solutions and obtains expressions for the mean centerline velocity and the wake width growth. The analysis of [12] follows the general description given by Tennekes and Lumley [13]. This study shall follow the analysis presented in [13] of which we shall further obtain a similarity solution for the mean velocity defect of the far wake.

II. ANALYSIS

In this section, we present an analysis on the equations of motion to provide solutions describing the mean velocity defect and the wake width of the far wake. The analysis is also given by [13]. The equations of motion are given by

$$\frac{1}{y}\frac{\partial yV}{\partial y} + \frac{\partial U}{\partial x} = 0. \tag{1}$$

$$yU\frac{\partial U}{\partial x} + yV\frac{\partial U}{\partial y} = \frac{1}{\rho}\frac{\partial\{y\tau\}}{\partial y}. \tag{2}$$

where the isotropic form of the shearing stress is given by $\tau = \rho\epsilon\frac{\partial U}{\partial y}$ and ϵ represents the eddy viscosity.

In the far wake stream, it can be assumed that the streamwise and cross stream mean velocities are small compared with the freestream velocity. Hence the minimum velocity defect which occurs at the outer layer of the wake should be zero, while the maximum velocity is measured from the centerline. It is also assumed that the pressure gradient is negligible. Hence, with these assumptions, the equation of motion to describe the flow in the far wake of a body of revolution reduces to,

$$U_\infty\frac{\partial \bar{U}}{\partial x} = \frac{1}{y\rho}\frac{\partial (y\tau)}{\partial y}. \tag{3}$$

Where $\bar{U} = (U_\infty - U)$.

Following [USA], the assumptions made are that the velocity distribution in different crossections of the wake are similar and that the wake radial width b And the velocity defect on the center line vary with x according to some power law. The assumptions can be expressed as

$$y = \eta x^n \tag{4}$$

$$b = kx^n \tag{5}$$

$$\bar{U} = U_\infty\frac{F(\eta)}{x^m}. \tag{6}$$

Where m and n are unknown constants. Applying Prandtl's mixing length hypothesis, $\epsilon = L^2\left|\frac{\partial U}{\partial y}\right|$, and the turbulent stress will be given as

$$\tau = \rho L^2\left|\frac{dU}{dy}\right|\frac{dU}{dy}. \tag{7}$$

L defines the mixing length. Using equation (6), we obtain

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$$\frac{dU}{dy} = \frac{U_\infty}{x^{m+n}} F'(\eta) \qquad (8)$$

(') represents differentiation with respect to η. Applying Prandtl's assumption that the mixing length is proportional to the wake width

$$L \propto b \Longrightarrow L \propto x^n. \qquad (9)$$

Then from equation (),

$$\tau \propto x^{2n} \left|\frac{U_\infty F'(\eta)}{x^{m+n}}\right| \frac{U_\infty}{x^{m+n}} F'(\eta). \qquad (10)$$

Or

$$\tau = C_1 x^{-2m} [F'(\eta)]^2 \,. \qquad (11)$$

Which after substitution and simplification, we get the expression for $\frac{1}{y\rho}\frac{\partial(y\tau)}{\partial y}$ as

$$\frac{1}{y\rho}\frac{\partial(y\tau)}{\partial y} \approx x^{-(2m+n)} \left[\frac{(F'(\eta))^2}{\eta} + 2F'(\eta)F''(\eta)\right]. \qquad (12)$$

Also we have

$$U_\infty \frac{\partial U}{\partial x} \approx x^{-(m+1)} [mF(\eta) + n\eta F'(\eta)]. \qquad (13)$$

From the equations above

$$x^{-(m+1)} [mF(\eta) + n\eta F'(\eta)] \approx x^{-(2m+n)} \left[\frac{(F'(\eta))^2}{\eta} + 2F'(\eta)F''(\eta)\right]. \qquad (14)$$

For both sides of this equation to be of the same order of x it requires that

$$x^{-(m+1)} = x^{-(2m+n)} \,. \qquad (15)$$

Which yields

$$m + n = 1. \qquad (16)$$

We obtain a second equation between and m and n by making use of the equation describing the drag coefficient. The drag coefficient is constant and it is assumed that at the far wake the velocity defect is much smaller than the free stream velocity. The drag coefficient is calculated as

$$C_D = \frac{D_r}{{}^1\!/_2\, \rho U_\infty A} \,. \qquad (17)$$

D_r defines the drag and the drag in this case is calculated by the relationship between aerodynamic drag and the momentum change given by,

$$D_r = \rho \int_0^{2\pi} \int_0^{\infty} y U_\infty \bar{U} dy d\theta. \qquad (18)$$

Based on the stated assumptions equation (17) simplifies to

$$D_r = 2\pi\rho U_\infty \int_0^{\infty} \bar{U} dy \,. \qquad (19)$$

From equation (6) and (19) we get after simplification

$$\int_0^{\infty} F(\eta)\, \eta x^{2n-m} d\eta \;= C_4. \qquad (20)$$

Hence

$$x^{2n-m} = 1. \qquad (21)$$

Where C_4 defines a constant. Therefore

$$2n - m = 0. \qquad (22)$$

Solving equations (18) and (24) gives

$$m = \frac{2}{3} \text{ and } n = \frac{1}{3}. \qquad (23)$$

These values are the same as those found by [5]. Hence we can write the equations (4-6) as

$$y = \eta x^{\frac{1}{3}}. \qquad (24)$$

$$b = k x^{\frac{1}{3}}. \qquad (25)$$

$$\bar{U} = U_\infty \frac{F(\eta)}{x^{\frac{2}{3}}}. \qquad (26)$$

III. SIMILARITY SOLUTION FOR MEAN VELOCITY PROFILE

In this section, we seek to find a solution that will describe the velocity profile in the far wake region after making a constant assumption on the eddy viscosity. A second order differential equation is

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obtained after applying appropriate transformations and introducing an eddy viscosity model which appropriately describes the flow at the far wake. In order to get a solution of equations (1) and (2), we seek a similarity of solution of the form

$$\bar{U} = W_0 F(\eta), \tau = \rho W_0^2 g(\eta). \qquad (27)$$

Where $\bar{U} = U_\infty - U$ and $\eta = y/b$, and W_0 and b are the local velocity and length scales, respectively. From equation (1) we get an equation of V as

$$V = \frac{-U'_\infty \eta b}{2} + \frac{bW_0'}{\eta}\int F\eta d\eta - \frac{W_0 b'}{\eta}\int F'\eta d\eta. \qquad (28)$$

Substituting the expression for V into equation (2) yields,

$$-U_\infty U_\infty' + \eta F'\left\{\frac{U_\infty b'}{W_0} + \frac{bU_\infty'}{2W_0} - Fb'\right\} - F\left\{\frac{bU_\infty W_0'}{W_0^2} + \frac{bU_\infty}{W_0} - \frac{bFW_0'}{W_0}\right\} - \frac{F'}{\eta}\left\{\int \frac{W_0'}{W_0}Fd\eta - \int b'F'\eta d\eta\right\} - \frac{(\eta g)'}{\eta} = 0. \qquad (29)$$

At large distances from the body, the velocity defect is small, that is, $W_0 \ll U_\infty$, and the terms of second order are also small and hence the equation (5) reduces to;

$$\eta F'\left(\frac{U_\infty b'}{W_0}\right) - F\left(\frac{W_0' U_\infty b}{W_0^2}\right) - \frac{(\eta g)'}{\eta} = 0. \qquad (30)$$

In order to determine the velocity profile F, it is necessary to invoke a turbulence closure model. Thus, for example, we introduce a constant eddy viscosity;

$$g = \frac{-\nu_\tau}{W_0 b}F'. \qquad (31)$$

Under this assumption we get a solution for (30) as

$$F = \frac{K_1 \exp(\frac{-B_2\eta^2}{4})}{\eta} Whittaker\ M\left(\frac{B_2+B_3}{2B_2}, 0, \frac{-B_2\eta^2}{2}\right) + \frac{K_2 \exp(\frac{-B_2\eta^2}{4})}{\eta} Whittaker\ W\left(\frac{B_2+B_3}{2B_2}, 0, \frac{-B_2\eta^2}{2}\right). \qquad (32)$$

Where K_1 and K_2 are constants.

$$B_2 = \frac{U_\infty b' b}{\nu_\tau} \qquad (33)$$

$$B_3 = \frac{W_0' U_\infty b^2}{W_0 \nu_\tau} \qquad (34)$$

Following [14], The Whittaker functions are solution to the differential equation (34), defined by

$$WhittakerM = e^{\frac{-B_2\eta^2}{4}}\eta^{\frac{1}{2}}M\left(-\frac{B_3}{2B_2}, 1, \frac{-B_2\eta^2}{2}\right). \qquad (35)$$

And

$$WhittakerW = e^{\frac{-B_2\eta^2}{4}}\eta^{\frac{1}{2}}U\left(-\frac{B_3}{2B_2}, 1, \frac{-B_2\eta^2}{2}\right) \qquad (36)$$

M and U are related to the Kummer function and are given by

$$M\left(\frac{-B_3}{2B_2}, 1, \frac{-B_2\eta^2}{2}\right) = \sum_{n=1}^{\infty}\left(\frac{-B_3}{2B_2}\right)_n \frac{(^{-B_2\eta^2}/_2)^n}{(n!)^2} \qquad (37)$$

$$U\left(\frac{-B_3}{2B_2}, 1, \frac{-B_2\eta^2}{2}\right) = \frac{1}{\Gamma(B_2)}\left\{M\left(\frac{-B_3}{2B_2}, 1, \frac{-B_2\eta^2}{2}\right)\ln(\frac{B_2\eta^2}{2}) + \sum_{n=1}^{\infty}\left(\frac{-B_3}{2B_2}\right)_n \frac{(^{-B_2\eta^2}/_2)^n}{(n!)^2}[\psi\left(\frac{-B_3}{2B_2}+n\right) + \psi(1+n)]\right\} \qquad (38)$$

Here $\psi\left(\frac{-B_3}{2B_2}+n\right)$ is the logarithmic derivative of the gamma function Γ and is given by

$$\psi\left(\frac{-B_3}{2B_2}+n\right) = \frac{1}{(n-1)+\left(\frac{-B_3}{2B_2}\right)} + \frac{1}{(n-2)+\left(\frac{-B_3}{2B_2}\right)} + \cdots + \frac{1}{\left(\frac{-B_3}{2B_2}\right)} + \psi\left(\frac{-B_3}{2B_2}\right) \qquad (39)$$

And

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$$\left(\frac{-B_3}{2B_2}\right)_n = \left(\frac{-B_3}{2B_2}\right)\left(\frac{-B_3}{2B_2}-1\right)\dots\left(\frac{-B_3}{2B_2}+n-1\right). \quad (40)$$

In the neighborhood of $\eta = \infty$, the functions $M\left(\frac{-B_3}{2B_2},1,\frac{-B_2\eta^2}{2}\right)$ and $U\left(\frac{-B_3}{2B_2},1,\frac{-B_2\eta^2}{2}\right)$ behave asymptomatically and for $\eta \to \infty$, their behavior are given by

$$M\left(\frac{-B_3}{2B_2},1,\frac{-B_2\eta^2}{2}\right) = \frac{1}{\Gamma\left(\frac{-B_3}{2B_2}\right)} e^{\frac{-B_2\eta^2}{2}} \left(\frac{-B_2\eta^2}{2}\right)^{\frac{-B_3}{2B_2}-1} [1 + O(\eta^{-1})] \quad (41)$$

And

$$U\left(\frac{-B_3}{2B_2},1,\frac{-B_2\eta^2}{2}\right) = \left(\frac{-B_2\eta^2}{2}\right)^{\frac{B_3}{2B_2}} [1 + O(\eta^{-1})]. \quad (42)$$

Hence our solution for large η becomes

$$F = \left(\frac{-B_2\eta^2}{2}\right)^{\frac{-B_3}{2B_2}} e^{\frac{-B_2\eta^2}{2}} \left\{\frac{K_1}{\sqrt{\eta}\Gamma}\left(\frac{-B_2\eta^2}{2}\right)^{-1} e^{\frac{-B_2\eta^2}{2}} + \left(\frac{-B_2\eta^2}{2}\right)^{\frac{B_3}{B_2}} \frac{K_2}{\sqrt{\eta}}\right\} + O(\eta^{-1}) \quad (43)$$

From equation (43), it is clear that the exponential term will dominate more than the other terms in the equation. Hence we can conclude that the similarity mean velocity defect decays exponentially for large values of the similarity variable η for very large streamwise distances.

IV. COMPARISON WITH EXPERIMENTAL DATA

Analytical results given by the analysis in section 2 are compared with experimental data of Jimmenez et al [15]. They conducted experimental studies on axisymmetric body with a Reynolds number ranging between 1.1×10^6 and 67×10^6. For this study, the data pertaining to the Reynolds number of 1.1×10^6 was used. In the experimental study, the pressure gradient was negligible hence the data sets are best suit for validating this work.

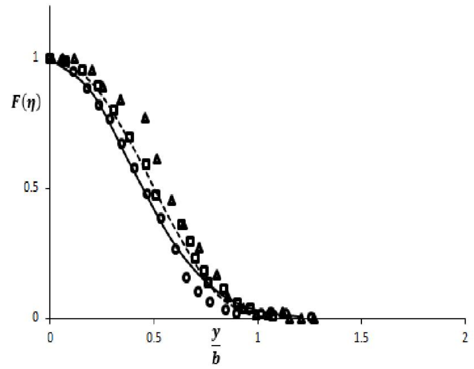

Fig. 2. Mean velocity profiles in similarity variables,

○ x/d= 15, □ x/d= 12, ▲ and x/d= 9, —— exponential decay, ---- analysis.

Figure 2 shows mean velocity profiles in the far wake flow of the data set of the three stream wise distances. Also shown in the figure is the exponential curve given by equation (46). The data fits well with the similarity equation.

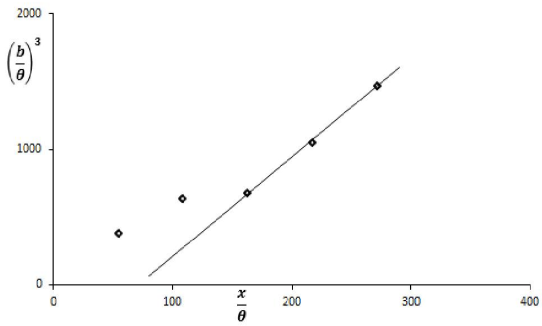

Fig. 3. Streamwise development of the wake half thickness, —— analysis.

Figure 3 shows the variation of the wake half thickness given by equation (25). In figure 3, it is seen that there is a disagreement in the data pertaining to the region of the stations in the near wake, however the data

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pertaining to the "later wake" the figure 3 shows good agreement between the data and the analytical results.

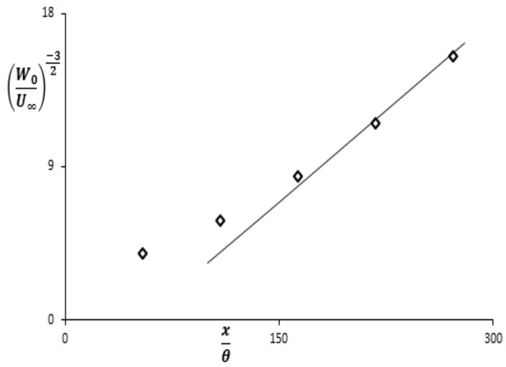

Fig. 4. Streamwise development of inverse of maximum velocity defect , —— analysis.

Figure 4 show the variation of inverse of maximum velocity defect. They show the relationship for the inverse maximum velocity deficit as given by equation (28). The experimental data shows agreement for data pertaining to the far wake region.

V. CONCLUSIONS

A theoretical analysis of the governing equations in the axisymmetric far wake of a long slender cylinder has been carried out and compared with experimental data. The results obtained show the self-similarity of the axisymmetric far wake. The half wake thickness and inverse maximum velocity defect agree very well with the data of Jimmenez et al [15]. Also shown is good agreement of the experimental data with the similarity solution to the mean velocity which is shown to decay exponentially. All in all, the results of self-similarity analysis agree well with the available experimental data of the far wake.

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