Faculty of Sciences
Department of Mathematics & Statistical Sciences

Challenges related to beef-cattle pricing between a western block and a third world country

by

Paul Kundai Ziwakaya
Reg. No: 17100072
MSc (Pure and Applied Mathematics), BIUST, Botswana

A Thesis Submitted to the Faculty of Sciences in Partial Fulfilment of the Requirements for the Award of the Degree of Master of Science in Pure and Applied Mathematics of BIUST

Supervisor:

Prof. Edward M. Lungu
Department of Mathematics & Statistical Sciences,
Faculty of Sciences
June 06, 2021

DECLARATION REGARDING THE WORK AND COPYRIGHT

I, Paul Kundai Ziwakaya declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Botswana International University of Science and Technology will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Signature:.. Date: 15/06/2021

CERTIFICATION

This undersigned certifies that he has read and hereby recommends for acceptance by the College of Science a dissertation/thesis titled: CHALLENGES RELATED TO BEEF-CATTLE PRICING BETWEEN A WESTERN BLOCK AND A THIRD WORLD COUNTRY, in fulfilment of the requirements for the Degree of Master of Science in Pure and Applied Mathematics of BIUST.

Supervisor:

Prof. Edward M. Lungu
Department of Mathematical and Statistical Sciences
College of Science, BIUST
Email: lungue@biust.ac.bw

Signature:...   Date: 15/6/2021...

List of submitted papers: Modeling Botswana Beef-cattle Price Dynamics (*Journal of Mathematical Finance (JMF)*).

ABSTRACT

This study considers a business scenario involving three participants. The first participant, in this case the farmer sells the cattle to the second participant, in this case the Botswana Meat Commission (BMC) which, in turn sells the processed beef to the European Union (EU). The BMC offers a price, $S_1(t)$, to the farmer for the cattle. The EU offers a price, $S_2(t)$, to the BMC. To estimate the parameters spot prices of beef-cattle from the BMC and the EU markets for years 1992 to 2018 were used. This study considers the relationship between $S_1(t)$ and $S_2(t)$, how the two prices are correlated and whether there is any seasonal dependence on the prices.

First, we constructed a bivariate model which showed strong correlation between the prices $S_1(t)$ and $S_2(t)$ and that the residue between $S_1(t)$ and $S_2(t)$ is explained by a white noise process. Based on this result, we constructed a stochastic model for the farmer price, $S_1(t)$, of the Ornstein-Uhlenbeck type where the EU price is assumed as the mean. Several scenarios arose from the stochastic model covering a number of possibilities ranging from collapse of the business for the farmer to thriving business with mean reverting returns.

The study gives insight why the BMC which is caught in between the farmer and the EU is experiencing operational problems financially.

Keywords: *Stochastic mean, Mean-reversion, Geometric Mean Reversion, Beef-cattle price, Quasi-stability, Cointegration, Transfer function.*

DEDICATIONS

This work is dedicated to my wife, kids, mom and dad, as well as the Simons Foundation. I thank you all for leading me into the life that I desire.

ACKNOWLEDGEMENTS

My deepest gratitude goes to my supervisor Professor Edward M. Lungu for guiding me in exploring this research area and for advising my thesis work along the way. His broad knowledge and attitude towards research will serve as an academic role model to me. I am, in particular, grateful for his endless support in my studies. I have been privileged to be supported by a fellowship during my graduate study and I would like to thank the Simons Foundation, through the Administrative Board, the Research and Graduate Studies in Mathematics and its Applications (RGSMA): A Network Approach for funding my studies towards my MSc Mathematics programme. I would also like to thank all the members in the Department of Applied Mathematics and Statistical Sciences at the Botswana International University of Science and Technology (BIUST) for enriching me with the necessary tools some of which I used in this thesis.

Finally, I would like to thank my family for their unconditional love and support. My wonderful wife, Rachel and children have been a constant source of encouragement and have stood by me through all the challenges of completing this thesis.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 Background of Study

The Botswana-European Union (EU) beef-cattle trade is an outstanding example of co-operation between a Western block and a third world country (Botswana). Although this trade showed promise, it has of late exhibited cracks as the EU has forged more trade agreements with other beef exporters such as South America, Australia, New Zealand. An example of this occurred when the EU stopped all meat imports from Botswana and demanded a thorough clean up of the abattoirs [1]. The beef-cattle industry is of strategic importance to Botswana as a major source of foreign currency. At the time the EU took this decision, the beef export industry was Botswana's second biggest earner of foreign currency behind diamonds. In recent years tourism has overtaken the beef export industry as the quantity of beef exported has declined. To understand the performance of the beef industry, numerous models incorporating spot prices, demand, quality, supply and seasonality as well as convenience yield behavior in the industry have been developed [2, 3, 4]. Piot-Lepetit [2] examined the price evolution of porcine and bovine for the EU and its Member States (MS) and concluded that a higher dispersion of prices at the MS level did not yield an acceptable price process between the two parties. Several other studies have made contributions, and have arrived at similar conclusions citing the unpredictability of the dynamics of the industry as one of the reasons. Although the intuitive approaches have met with numerous shortcomings, such as, inability to account for stochasticity in variables, these approaches have nevertheless permitted one to reach useful preliminary conclusions.

The export market in Botswana is dominated by a government parastatal organisation called the Botswana Meat Commission (BMC). BMC has experienced substantial changes in prices during the period 1966 to 2018 with major surges occuring during the past three decades [5, 6, 4, 7]. The beef-cattle prices dramatically changed from $BWP4.56$ in 1992 to nearly $BWP32.20$ in recent years (although the weak exchange rate inflates the current prices), see Table C.1 in Appendix A1 for an overview. Between the period of 1992 to 2018, the price upswing decelerated resulting in decreased beef-cattle prices. Jeffiris [5] analysed the changes in prices for the Botswana beef-cattle industry and found out that changes in the EU have significant impacts on the cattle prices in Botswana. Botswana meat exports to the EU increased from 13245 tonnes in 1968 to 29368 tonnes in 2017 [8]. During the period 1968 to 1990's, Botswana enjoyed unlimited preferential market access to the EU. This has changed as the country now competes with countries such as Brazil, Australia, Argentina, China and the United States of America (USA) [4].

The competition for the EU market has introduced uncertainty in the beef prices. Tothova [9] investigated price volatility in order to determine whether volatility had increased after some time at the EU and global levels. She compared price volatility in relation to other economic variables such as stocks, spot prices, volume of trade and so on. The results showed that events from the past have an impact on the present price variability. The question in the case of Botswana is what historical events have had significant effects on the Botswana -EU beef-cattle trade.

Important contributions have been made by Schwartz [10] who analyzed three stochastic models of commodity prices that considered mean reversion. The first model was a one-factor model in which the logarithm of the spot price of the commodity followed a mean reverting process. The second model considered convenience yield as a second stochastic factor of the commodity, which followed a mean reverting process. The third model incorporated stochastic interest rates. The investigation uncovered solid mean reversion in the commodity prices. Schwartz and Smith [11] formulated a two-factor model of commodity prices that permitted variability in the mean level and mean-reversion in short-term prices. They modeled spot-prices as a Brownian motion process that can be decomposed into short-term and long-term components based on their individual dynamics. By separating short-term and long-term price features and utilizing futures prices to differentiate between them, Schwartz and Smith [11] provided a conceptual model for developing richer models of commodity price movements.

In this thesis we incorporated the aspect of mean reversion and a time dependent stochastic mean reverting process for the prices. Unlike stock prices which generally exhibit upward trends in the long run as investors benefit from the long term dividend yields and earnings, beef prices, show a level dependent behaviour over a long period since supply and demand dictate the prices. However, one of the fundamental distinguishing characteristics exhibited by beef prices is mean-reverting behavior (see for example, Schwartz [10], Casassus and Collin-Dufresne [12], Bessembinder et al. [13], Pyndick [14], and Routledge et al. [15], for empirical proof supporting the utilization of mean-reversion for commodity prices). We have used a Geometric Mean Reversion process (GMR) to model the pricing process of Botswana beef-cattle. This approach was pioneered by Dixit and Pindyck [16] in an economic context. We study the evolution of prices in the Botswana beef-cattle industry using a classical Geometric Mean Reversion process, which has been used to model commodity prices by various authors in finance and economics. We obtain certain regularity results which guarantee positivity and perform some numerical analysis for the four distinct cases.

1.2 Statement of the problem

At the beginning of post-independence years (1966), the Botswana beef-cattle industry captured a respectable position of the European market. However, charges centred on administrative extravagance, and debasement as well as inflation, diseases and drought related costs have resulted in low production and increasing disputable issues around pricing for the monoponsy, BMC. Amid the early 1980s, concern was raised in the Botswana parliament regarding the performance of the BMC. It was obvious that the Botswana beef-cattle sector had endured volatile beef-cattle prices. Beef-cattle prices like several agricultural items, are unstable in nature [9]. They depend on a number of components (state factors) which incorporate quality of meat, climate, number of cattle in the catchment region, diseases, competition, social and political factors. BMC's producer prices are based on the EU market whose procedures and protocols are difficult to adhere to. An unpredictable situation prevails for the future of the beef-cattle industry in Botswana, as BMC is compounded by two conflicting goals namely, increasing profit and capacity development. In the mean time, the government of Botswana faces contradicting objectives of keeping up high and acceptable prices for farmers and at the same time making the industry viable. The nature of the European Union (EU) markets makes the Botswana beef-cattle industry vulnerable to exogenous shocks. Beef price change in the EU market has a huge economic effects on the Botswana beef-cattle producers and the overall economy through the income multiplier effects. Since Botswana' beef-cattle industry is largely affected by exogenous factors in addition to issues confronting the government in making its citizens live better lives, it is necessary to develop models which will address the eventual fate of Botswana's beef-cattle sector. The models will likewise give important information to comprehend the dynamic systems of the beef-cattle industry. Also, this will furnish strategic managers and planners with a framework for predicting the impacts on the beef-cattle production and exports to the EU because of the changes in significant factors that forces price changes. These models will enable BMC to make enlightened choices with regard to beef-cattle pricing.

Meanwhile, as the factors are known, very few literature is found at the moment in context of Botswana beef cattle pricing that incorporate these factors. The works of Ndzinge etal [17], Hubbard, Michael and Morrison, J Stephen [18], Mulale, Kutlwano [19], van Engelen, Anton etal [20] and many more captured the pricing problems around the Botswana beef-cattle. However, they did not look at the disturbances that are found in the Botswana beef-cattle industry and volatile nature of beef prices. To this end, we are motivated to formulate and incorporate aspects of mean reversion and a time dependent stochastic mean process for the beef-cattle price model.

1.3 Aim and objectives of the study

1.3.1 Aim of the Study

The aim is to develop a model that demonstrates a coherent framework on how the European Union (EU) beef prices translate into the Botswana cattle prices vice versa.

1.3.2 Objectives

The objectives of this investigation are as follows:

1. To examine evidence of causality and cointegration between the market price of live cattle in Botswana and the prices of beef in the European Union market.
2. To construct a stochastic model that will translate into a fair pricing policy among the actors in the Botswana beef-cattle industry.
3. To illustrate the impact of mean reversion and how it affects the evolution of beef-cattle prices.

1.4 Scope of the study

The study covers the issues around the pricing process of beef-cattle of Botswana for the three actors namely producer (farmer), Botswana Meat Commission (BMC) and export market (European Union (EU)). We investigate the dynamics of the beef-cattle pricing which is affected by several factors such as beef supply, demand for foreign currency. Factors such as diseases, natural disasters and socio-political influence are discussed in literature and are not explicitly included in our model.

1.5 Justification

In models of price evolution of commodities (in particular beef-cattle), information on the disturbances in the system form a crucial ingredient in the fixing of testable expectations about the time path of prices. As the demand and supply for beef as well as the arrival of other forms of information in the EU market, the price of the beef-cattle in Botswana (for BMC) will also change. The fundamental rule is that beef-cattle prices, like any other commodities, respond to the noise that is in the market. Consequently, the prices tend to follow a random walk and mean reverting to the long-run equilibrium price. This study provides an opportunity to explore the inadequately understood world regarding the use of stochastic processes on commodity valuation which are the objects of active research in financial mathematics.

1.6 Limitations

A significant part of the information that was required for precise estimates have not been made accessible to this investigation; some information which enrich this study does not exist. In particular BMC failed to provide the monthly beef-cattle prices which are most commonly used for serving as a base indicator for pricing in the future. This essentially influences the authenticity of the estimates as well as the technique for making them. The methodology received in this thesis is to make various exceptionally basic projections. This presents a picture which is dependent on the information accessible to the investigation. Despite the limitation mentioned above, the estimates and hence the reliability of the results can be improved as data becomes available.

1.7 Keywords

Definition 1. ***Geometric Mean Reversion (GMR)***

Let $S(t)$ be a stochastic process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ and $B(t)$ be a standard Weiner process. Then the classical GMR is given by,

$$dS(t) = \kappa(m - S(t))S(t)dt + \sigma S(t)dB(t) \tag{1.1}$$

Where κ is the level dependent mean reversion speed, m is the mean reversion level and σ is the level dependent volatility.

Remark: An important feature of (1.1) is that its dynamics is tied to the mean reversion m and that the process is constantly given inertia to tend to its mean reversion level.

Definition 2. ***Quasi-stability***

Quasi-stability is defined as a situation where although, the relative equilibrium point is determined, the system is not in permanent equilibrium.

Example 1. *The relative prices of most agricultural commodities are determined, but the system is not in equilibrium in the traditional sense. The excess demand and supply will not be zero, and the absolute prices are continuously increasing or falling.*

Definition 3. ***White noise***

A white noise is a type of time series with, mean zero, constant standard deviation and the correlation between lags equal to zero. One of the useful property of a white noise is that it is not predictable.

Definition 4. ***Beef-cattle***

Beef cattle are cattle raised for meat production. The meat of mature or almost mature cattle is mostly known as beef.

Definition 5. ***Transfer function***

Transfer function model is a model describing the estimation of a time series S_t called the output based on at least one associated input time series X_t. In general, equation (1.2) is a transfer model for a single input.

$$S_t = \beta_0 + \nu(\beta)X_t + \varepsilon \tag{1.2}$$

Definition 6. ***Cointegration***

*Cointegration is the long-run relationship between at least two variables. Let $S_t = (S_{1,t}, S_{2,t}, ..., S_{n,t}$ be a vector of time series processes, then S_t is integrated of vector **1**, $\mathbb{I}(1)$, if $\triangle S_t$, is a linear process, with $C(1) = \sum_{i=0}^{\infty} C_i \neq 0$. Let $\beta \neq 0$ be a vector, such that $\beta' S_t$ is stationary, then S_t is cointegrated with cointegration vector β.*

Example 2. *Consider a bivariate cointegration process given by,*

$$S_{1,t} = \alpha \sum_{i=1}^{t} \varepsilon_{1,i} + \varepsilon_{2,t} \tag{1.3}$$

$$S_{2,t} = \rho \sum_{i=1}^{t} \varepsilon_{1,i} + \varepsilon_{3,t}. \tag{1.4}$$

The system (1.3)-(1.4) is a cointegration $\mathbb{I}(1)$ process with $\beta = (\rho, -\alpha)'$, because $\triangle S_{1,t} = \alpha\varepsilon_{1,t} + \triangle\varepsilon_{2,t}$, $\triangle S_{2,t} = \rho\varepsilon_{1,t} + \triangle\varepsilon_{3,t}$ and $\rho S_{1,t} - \alpha S_{2,t} = \rho\varepsilon_{2,t} - \alpha\varepsilon_{3,t}$ are stationary.

Outline of the thesis

The rest of this thesis is organized as follows; chapter 2 present some mathematical concepts that were used in this thesis. A study of past researches pertinent to the current investigation is briefly reviewed in chapter 3. In chapter 4, the dynamic price evolution model on which this investigation is based is formulated. The model incorporates time series data on the cattle prices (Farmer-BMC) and beef prices (BMC-EU). The first sections of chapter 4, discusses the specifications of the model and empirical study prior to the stochastic model development. The fifth, chapter 5, presents a summary of results. In chapter 6, we provide concluding remarks and discuss policy implications of the study.

Subsequently, in Appendices A, B, D and C we provide supplementary materials that are not essential to the investigation itself, however which might be useful in giving a more thorough comprehension understanding of some concepts as well as cumbersome information that we felt should not be included in the body of the thesis.

Summary

In this chapter we gave a brief background information on the study as an approach to figure out the problem. The chapter subsequently described the aims and objectives as well as justification, limitations of the current investigation and gives the structure that this thesis adopts.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

In this Chapter we will define some basic concepts of stochastic calculus and probability. Stochastic calculus deals with functions of time $t \in [0, T]$, henceforth, in this chapter we present some of the concepts of the infinitesimal stochastic calculus utilized in this thesis. The following is based on [21, 22, 23, 24, 25].

2.1 Probability Space

A probability space is a triple, $(\Omega, \mathcal{F}, \mathbb{P})$, where;

1. Ω is a non-empty set which represents all possible outcomes called sample space.
2. $\mathcal{F}$ is a collection of subsets of Ω, $(\mathcal{F} \subset \Omega)$ called $\sigma - algebra$ or event space on Ω with the following properties,
   (a) $\emptyset \in \mathcal{F}$,
   (b) For any, $F^c \in \mathcal{F}$, where $F^c = \Omega \backslash F$ is the complement of F relative to Ω,
   (c) For any,
   $$F_1, F_2, \ldots \in \mathcal{F} \implies F = \bigcup_{i=1}^{\infty} F_i \in \mathcal{F}.$$
   The pair $(\Omega, \mathcal{F})$ is called a measurable space.
3. $\mathbb{P}$ is the probability function $\mathbb{P} : \mathcal{F} \to [0, 1]$ that assigns probabilities to the events $\mathcal{F}$ over Ω satisfying the following axioms:
   (a) $\mathbb{P}(\emptyset) = 0$ (unlikely to happen), $\mathbb{P}(\Omega) = 1$ (almost surely, $a.s.$),
   (b) If $F_1, F_2, \ldots \in \mathcal{F}$ and $\{F_i\}_{i=1}^{\infty}$ are disjoint i.e. $F_i \cap F_j = \emptyset$ if $i \neq j$, then
   $$\mathbb{P}(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mathbb{P}(F_i)$$
   .

Definition 7. ***Filtered Probability Space***

Let $\mathbf{T}$ *be a totally ordered index set (often* $[0, T]$, $\mathbb{R}^+$ *or* $\mathbb{N}$*). For every* $t \in \mathbf{T}$, *let* $\mathcal{F}_t$ *be a sub* $\sigma - algebra$ *of* $\mathcal{F}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$, *then* $\{\mathcal{F}_t\}_{t \in \mathbf{T}}$ *is called a filtration, if* $\mathcal{F}_n \leq \mathcal{F}_m \leq \mathcal{F}$ *for all* $n \leq m$ *and* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ *is the filtered probability space.*

So filtrations are indexed families of $\sigma - algebras$ that are ordered increasingly [24]. $\mathcal{F}_t$ represents the set of all events which can happen or the information which is accessible in the market up to time t.

Definition 8. ***Stochastic process***

A stochastic process is a collection of random variables $\{X_t\}_{t\in T}$ defined on the probability space $(\Omega\ ,\ \mathcal{F}\ ,\ \mathbb{P})$ and assuming values in $\mathbb{R}^n$. T is usually the halfline $[0,\infty)$ $\omega \to X_t(\omega)$ for each $t \in T$. $t \to X(\omega)$ and $(t,\ \omega) \to X(\omega)$, $t \in T$. The distribution of the process $X = \{X_t\}_{t\in T}$ are the measures $\mu_{t_1,t_2,\ldots,t_k}$ defined on $\mathbb{R}^n$, $k = 1, 2, \ldots$ by $\mu_{t_1,t_2,\ldots,t_k}(F_1 \times F_2 \times \ldots \times F_k) = \mathbb{P}[X_{t_1} \in F_1, \ldots X_{t_k} \in F_k]$; $t_i \in T$.

Example 3. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. let $\mathcal{F} = \sigma(X_k|k \leq n)$ be a $\sigma - algebra$ where, $\sigma(X_k|k \leq n)$ denotes the $\sigma - algebra$ generated by the random variables $X_1, X_2, \cdots, X_n$. Then, $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a filtration , since by definition all $\mathcal{F}_n$ are $\sigma - algebras$ and $\sigma(X_k|k \leq n) \subseteq \sigma(X_k|k \leq n+1)$.*

Next section (section 2.2), presents the $\mathcal{L}^p-$spaces which are spaces of random variables whose p^{th} power is integrable.

2.2 The $\mathcal{L}^p$-spaces

If $X : \Omega \to \mathbb{R}^n$ is a random variable we define the $\mathcal{L}^p - norm$ of X as follows,

$$\begin{aligned} ||X||_p = ||X||_{L_{(P)}} &= \left(\int_\Omega |X(\omega)|^p d\mathbb{P}(\omega)\right)^{\frac{1}{p}} \\ &= \mathbb{E}[|X|^p]^{\frac{1}{p}}, if\ p \in [1,\infty), \end{aligned} \tag{2.1}$$

where $\mathbb{E}[.]$ is the expectation operator. If $P = \infty$, then we set the size of X as,

$$||X||_\infty = ||X||_{\infty L_{(p)}} = \inf\{N \in \mathbb{R},\ \ |X(\omega)| \leq N,\ a.s.\}$$

The corresponding L^p-spaces are defined by

$$\mathcal{L}^P(P) = \mathcal{L}^P(\Omega) = \{X : \Omega \to \mathbb{R}^n, ||X||_p \leq \infty\}.$$

With this norm the $\mathcal{L}^P$-spaces are Banach spaces (a complete normed linear spaces). If $p = 2$ then $\mathcal{L}^2$-space is a Hilbert space (complete linear product space with inner product) and $(X, Y)_{L^2(P)} = \mathbb{E}[X, Y]$, for all $(X, Y) \in \mathcal{L}^2(P)$.

Definition 9. ***Independent sets***

Two subsets A,$B \in \mathcal{F}$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B).$$

A collection $\mathcal{F} = \{\mathcal{H}_i \; ; i \in I\}$ of families $\mathcal{H}_i$ of measurable sets is independent if,

$$\mathbb{P}(H_{i_1} \cap H_{i_2} \cap ... \cap H_{i_k}) = \mathbb{P}(H_{i_1}) \times \mathbb{P}(H_{i_2}) \times ... \times \mathbb{P}(H_{i_k}),$$

for all choices of $H_{i_1} \in \mathbb{H}_{i_1}$, $...H_{i_k} \in \mathbb{H}_{i_k}$ with different indecies $i_1, i_2, ...i_k$.

2.3 Stochastic process

Recently, seemingly random changes in financial related activities have motivated the use of stochastic processes in modeling the associated dynamics [25].

Theorem 1. ***Kolmogrov extension theorem***

For all $t_1, t_2, ..., t_k$, $k \in \mathbb{N}$, let $\mathcal{V}_{t_1,...t_k}$ be the probability measures on $\mathbb{R}^{nk}$ such that

$$\mathcal{V}_{t\sigma(1),...,t\sigma(k)}(F_1 \times ... \times F_k) = \mathcal{V}_{t1,...tk}(F_{\sigma^{-1}(1)} \times ... \times F_{\sigma^{-1}(k)}),$$

for all permutations of σ on $\{1, 2, ..., k\}$ and for all $m \in \mathbb{N}$

$$\mathcal{V}_{t1,...tk}(F_1 \times .... \times Fk) = \mathcal{V}_{t_1,...,t_k,t_{k+1},...t_{k+m}}(F_1 \times ... \times F_k \times \mathbb{R}^n \times ... \times \mathbf{R}^n),$$

where the set on the right hand side has a total number of $k+m$ factors. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and stochastic process $\{X_t\}_{t\in[0,T]}$ on Ω, $\{X_t\} : \Omega \longmapsto \mathbb{R}^n$, such that for all $t_i \in T, k \in \mathbb{N}$ and all Borel sets F,

$$\mathcal{V}_{t1,...,tk}(F_1 \times ... \times F_k) = \mathbb{P}[X_t 1 \in F_1, ..., Xtk \in Fk].$$

Theorem 1 is used to prove the existence of Brownian motion, as a finite dimensional random variables that follows a Gaussian distribution, satisfying the consistency conditions, see [24, 21, 22] for more details.

Definition 10. ***Brownian Motion***

Brownian motion B_t, refers to various physical phenomena in which some quantity is constantly undergoing small, random fluctuations. It was named after Robert Brown a Scottish botanist who first studied such fluctuations in 1827, (see [21] for mathematical description of the motion).

A stochastic process $\{B_t\}_{t\in[0,T]}$ is called a standard Brownian motion if, it is Gaussian Markov process, it is a martingale,it has continuous paths and it is a process with stationary independent increments (see Appendix B.3). For several characterization of a Brownian motion see [26, 21] and references in there. Figure 2.1 shows a sample path of a Brownian motion.

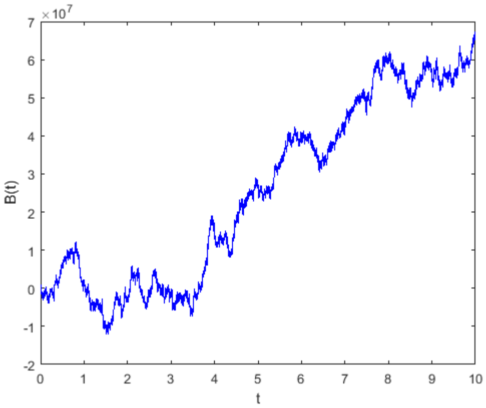


Figure 2.1: A sample path of Brownian motion

Theorem 2. ***Kolmogrov's Continuity Theorem***

Suppose the process $X = \{X_t\}$ satisfies the following conditions: for all $T \geq 0$ there exists constants α, β, D such that, $E[|X_t - X_s|^\alpha] \leq D.|t-s|^{1+\beta}, 0 \leq s, t \in T$, then there exists a continuous a version of $\{X_t\}$ that guarantees that a stochastic process satisfies certain constraints on the moments of its increments will be continuous.

2.3.1 Stochastic Integral

For adapted process $\sigma(t)$ with $\int_0^\infty \sigma(s)^2 ds < \infty$ we can define a stochastic integral with respect to Brownian motion as

$$\int_0^\infty \sigma(s)dW(s) = \underbrace{\lim_{n\to\infty}\sum_{i=1}^{n}\sigma(t_i-1)(W(t_1)-W(t_{i-1})}_{limit\,of\,Riemann\,Sums} \tag{2.2}$$

for partitions $t_i = \frac{i}{n}t$.
Note that $\int_0^t \sigma(s)dW(s)$ is a continuous martingale if $E[\int_0^\infty \sigma(s)^2 dW(s)] < \infty$. A stochastic integral is a model of the risk part of the return of an asset.

2.4 Itô Calculus

This section is on the work which extends the methods of classical calculus to stochastic processes such as Brownian Motion. It was named after the work of Kiyoshi Itô 1985 [26] who developed ways to solve stochastic differential processes.

Definition 11. *Let, $\mathcal{V} = \mathcal{V}(s,t)$ be a class of functions $f(t,\omega) : [0,\infty) \to \mathbb{R}$ such that:*

1. $(t,\omega) \to f(t,\omega)$ *in* $B \times \mathcal{A} - measurable$ *where* B *denotes the Borel* $\sigma - algebra$
2. $f(t,\omega)$ *is* $\mathcal{A} - adpted$
3. $E[\int_s^T f^2(t,\omega)ds] < \infty$

2.4.1 Itô Process

An Itô process $X(t)$ is adapted and of the form:

$$X(t) = \underbrace{X(0)}_{Initial\,value} + \underbrace{\int_0^t \mu(s)ds}_{drift} + \underbrace{\int_0^t \sigma(s)dW(s)}_{martingale\,noise\,(volatility)} \tag{2.3}$$

Equation 2.3 can be written in differential form as:

$$dX(t) = \mu dt + \sigma(t)dB(t; \quad X(0) = x \tag{2.4}$$

2.4.2 Itô's Formula

Let $f(x) \in C^2)$ (i.e. f(x) is twice continuously differentiable) function and an Itô process,

$$dX(t) = \mu(t)dt + \sigma(t)dB(t); \quad X(0) = x$$

$F(x)$ is again an Itô process with the decomposition

$$dX(t) = \frac{\partial F(X(t))}{\partial X} + \underbrace{\frac{1}{2}\frac{\partial^2 F(X(t)}{\partial X^2}\sigma(t)^2 dt}_{second\,order\,correction\,term} \tag{2.5}$$

Theorem 3. ***The 1-dimensional Itô formula***

Let $X(t)$ be an Itô process given by

$$dX_t = \mu dt + B_t dB_t; \quad X(0) = x$$

let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$ (i.e. g is twice continuously differentiable function on $[0,\infty) \times \mathbb{R}$, then we can define a new random variable called $Y_t = g(t, X_t)$ is again an Itô process and satisfies:

$$dY_t = \frac{\partial g(t,X_t)}{\partial t}dt + \frac{\partial g(t,X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 g(t,X_t)}{\partial x^2} \tag{2.6}$$

where $(dX_t)^2 = dX_t \cdot dX_t$ which is computed according to the rules $dt \cdot dt = dt \cdot dB_t = dB_t \cdot d_t = 0$ and $dB_t \cdot dB_t = dt$. Equation 2.6 is a 1-dimensional Itô formula.

Theorem 4. ***Multi-dimensional Itô formula***

Let $B(t,\omega) = (B(_1(t,\omega), B_2(t,\omega)...B_m(t,\omega)$ denote m-dimensional Brownian motion. If each of the process $U_i(t,\omega)$ and $V_{i,j}$ satisfies the following conditions:

1. $\mathbb{P}[\int_0^t V^2(s,\omega)ds, \infty, \forall t \geq 0] = 1$ (a.s).
2. If X_t is an Itô process, we assume that u is U_t-adapted and $\mathbb{P}[\int_0^t |U(s,\omega)|ds] < \infty$ and $\mathbb{P}[\int_0^t |U(s,\omega)|ds < \infty, \forall t \geq 0] = 1$ (a.s.). Then the equation in (4.1) can be written as,

$$dX_t = Udt + V_1 dB_t + V_2 dB_t + \cdots$$

Then we can form the following Itô process,

$$\begin{cases} dX_t = U_1 dt + V_{11}dB_1 + V_{12}dB_2 + \cdots + V_{1m}dB_m \\ \vdots \\ dX_n = U_n dt + V_{n1}dB_1 + V_{n2}dB_2 + \cdots + V_{nm}dB_m \end{cases}$$

In Matrix form

$$dX_t = U_t dt + V_{nm} dB_t \tag{2.7}$$

Where,

$$X_t = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}; \qquad U_t = \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}; \qquad V_{nm} = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1m} \\ V_{21} & V_{22} & \cdots & V_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ V_{n1} & V_{n2} & \cdots & V_{nm} \end{pmatrix}$$

and, $dB_t = \begin{pmatrix} dB_1(t) \\ \vdots \\ dB_m(t) \end{pmatrix}$

Then the following theorem is the general Itô formula.

Theorem 5. ***The General Itô Formula***

Let $dX_t = U_t dt + V_{nm} dB_t$ be an m-dimensional Itô process and let $g(t,x) = (g_1(t,x), \cdots g_p(t,x)$ be a C^2 map from $[0,\infty) \times \mathbf{R}$ into $\mathbb{R}^P$, Then the process $Y(t,\omega) = g(t, X(t))$ is again an

Itô process whose component number k is given by:

$$dY_k = \frac{\partial g(t,x)}{\partial t}dt + \sum_i \frac{\partial g_k(t,x)}{\partial x_i}dx_i + \frac{1}{2}\sum_{ij}\frac{\partial^2 g_k(t,x)}{\partial x_i \partial x_j}dx_i dx_j \tag{2.8}$$

2.4.3 Martingale

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $B(t)$ be a standard n-dimensional Brownian motion under the probability measure $\mathbb{P}$ and define $\mathcal{F}_t = \sigma(\{B(u) : 0 \leq u \leq t\})$ to be the σ-algebra generated by $B(u)$ up to time t.

Theorem 6. *Martingale Representation*

- *Suppose $M(t)$ is an $\mathcal{F}_t$-martingale under the probability measure $\mathbb{P}$ and that*

  $$\mathbb{E}_{\mathbb{P}}[\int_0^T M(t)^2 dt] < \infty$$

  . Then there exists a unique n-dimensional, previsible process $m(t)$ (i.e. $m(t)$ is $\mathcal{F}_t$-measurable) such that

  $$M(t) = M(0) + \int_0^t m(u)'dB(t) \text{ or } dM(t) = m(t)'dB(t)$$

- *Suppose that the n-dimensional diffussions $M^{(1)}(t)$ and $M^{(2)}(t)$ are $\mathcal{F}_t$-measurable under the probability measure $\mathbb{P}$ with a stochastic differential equation*

  $$dM^{(i)}(t) = S_i(t)dB(t).$$

  $S_1(t)$ and $S_2(t)$ are $n \times n$ volatility matrices that can be dependent on $M^{(1)}(t)$ and $M^{(2)}(t)$ respectively. Suppose also that $S_1(t)$ is a non-singular for all $0 \leq t \leq T$ almost surely (a.s.). Then there exists a unique, previsible $n \times n$ matrix process $\phi(t) = (\phi_{ij})_{i,j=1}^n$ such that

  $$dM^{(2)}(t) = \phi(t)dM^{(1)}(t) \text{ or } dM_i^{(2)}(t) = \sum_{j=1}^n \phi_{ij}dM_j^{(1)}(t).$$

A stochastic process $\{X_t\}_{t\geq 0}$ is called a continuous-time martingale with respect to the filtration if:

1. $E[X_t] < \infty, \forall t \geq 0,$
2. X_t is adapted to $\mathcal{A}_t$,

3. $E[X_t \setminus \mathcal{A}_t) = X_s \, \forall \, 0 \leq s \leq t$.

CHAPTER 3

LITERATURE REVIEW

This chapter is twofold, and covers the following; (i) detail on commodity price models and some techniques utilized in their development and (ii) work related to beef-cattle industry of Botswana. We will start by reviewing some of the most classical price models, to build a historical literature perspective. Then, we shall present commodity price models that explore the concepts used in this area. Finally, we shall review specific work related to our study of beef-cattle production in Botswana.

3.1 Commodity Price Models

Pricing models have been the focal area for quite a long time and have prompted a wide number of modeling techniques to support the inquiry. This field bridges the gap between pricing and agriculture commodities production [27]. In this section, we present historical price models and related approaches utilized in their development. Let Ω be a set of all possible price outcomes. Denote by $S(t)$ a continuous random price process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{\{t\in[0,T]\}}, \mathbb{P})$, where $\mathcal{F}$ is the sigma algebra over Ω, $\{\mathcal{F}\}_{\{t\in[0,T]\}}$ is the information generated by the process $S(t)$ (adapted to the filtration $\{\mathcal{F}\}_{\{t\in[0.T]\}}$) over the time horizon $[0, T]$ and $\mathbb{P}$ is the probability measure.

Understanding the stochastic behaviour of agriculture commodity prices plays a vital role in deciding the type of model to be used to assess asset investment. Stochasticity of commodities plays a significant role in formulating models for pricing financial contingent claims on the commodity, as well as in evaluating investments [10]. Usually the simplest stochastic differential equation (SDE) that is used to present the underlying price is the Arithmetic Brownian motion of the form,

$$dS(t) = \mu dt + \sigma dB(t), \quad \alpha, \sigma \; are \; constants, \tag{3.1}$$

In integral form; (3.1) can be represented as

$$S(t) = S(0) + \int_0^t \mu dt + \int_0^t \sigma dB(t), \tag{3.2}$$

which for μ and σ constants simplifies to

$$S(t) = S(0) + \mu t + \sigma B(t). \tag{3.3}$$

In (3.1), μ is the drift, σ is the diffusion and $B(t)$ is the standard Brownian motion. Since $S(t) \sim N(S_0 + \mu, \sigma^2 t)$, $(S(t) - S(0)) \sim N(\mu t, \sigma^2 t)$, $S(t)$, (3.1) is also called drifted Brownian Motion. The application of Arithmetic Brownian motion dates back to 1900 when Bachelier defended his thesis, "Théorie de la Spéculation" [28]. Bachelier's model, also called the Arithmetic Brownian motion, received recognition during the 1960s, when few financial specialists speculated that stock prices evolve randomly because of the efficient market hypothesis, which can be stated as "at a given moment, the price relies just upon the price at that time and not on its set of experiences of varieties". Samuelson [29], the recipient of the Nobel Prize in Economics, introduced Bachelier's theories to modern financial economics. Over a century after Bachelier's thesis, his ideas still remain relevant to mathematical finance. Bachelier, laid the foundation for several fields of stochastic calculus such as Markov processes, weak convergence in functional spaces and Brownian motion. Most of the contributions made by Bachelier to mathematical finance are documented in the book by de Montessus [30] , where there is a whole chapter devoted to probabilistic methods in finance based upon Bachelier's work.

Although Arithmetic Brownian motion is one of the most basic stochastic process, the price may become negative which violates the condition of limited liability. Samuelson, [29], noticed this possibility arising from Bachelier's model, that it could predict negative stock prices, which gives a possibility that an investor can gain or lose, thereby suggesting a non-arbitrage market. To overcome these drawbacks several authors for example Samuelson [29], Black and Scholes [31] have suggested the use of geometric Brownian motion in modeling asset prices. In mathematical finance, the famous models utilized to explain the stochastic process of prices are the mean reverting model and the geometric Brownian motion model.

A Geometric Brownian Motion (gBM) which represent a simple recipe for pricing is given by:

$$dS(t) = \mu S(t) d_t + \sigma S(t) dB(t). \tag{3.4}$$

Using Itô' formula, the gBM in (3.4) has the explicit solution,

$$S(t) = S_0 \exp[(\mu - \frac{\sigma^2}{2})t + \sigma B(t)]. \tag{3.5}$$

Hence the price $S(t)$ is lognormally distributed.

Remark: For all values of μ, $exp[(-\frac{\mu^2}{2})t + \mu B(t)]$ is an exponential martingale on $t \geq o$ and $exp[-\frac{\mu^2}{2}t + \mu B(t)] \rightarrow 1 \;\; as \;\; t \rightarrow \infty$. Then the n^{th} moment,

$$\mathbb{E}^n[S(t)] = S_0^n \exp[n\mu t + \frac{\sigma^2}{2} n(n-1)t] \tag{3.6}$$

The price $S(t)$ has the mean $\mathbb{E}[S(t)] = S_0 e^{\mu t}$ and the variance,

$$Var[S(t)] = S_0^2 e^{2\mu t}[e^{\sigma^2 t} - 1].$$

Therefore, the average of the prices increases exponentially and is independent of the parameter σ. If $n = 1$ in equation (3.6), the expected price of $S(t)$ evolve like fixed-income asset with a compound interest rate μ. However, commodity prices has variability as a result of randomness of the presents of disturbances in the market and could result in drop of price causing a big loss [3].

Let us now examine the individual price, for the sample properties of $S(t)$. By the law of the iterated logarithm, [32] we can easily show that,

$$\lim_{t \to \infty} \frac{1}{t} \log S(t) = \mu - \frac{\sigma^2}{2} \tag{3.7}$$

It is interesting to observe that an individual who holds the asset long enough would almost certainly be ruined if $\mu < \frac{\sigma^2}{2}$ even though in this case the average of the price (see (3.6) is increasing. Geometric Brownian motion models are commonly used in valuing underlying assets, agricultural commodities are way better modeled by mean reverting processes [3].

The most recognized feature of commodity prices is their mean-reversion characteristic. Recently, commodity modeling of price evolution has taken advantage of mean-reversion behaviour. The Ornstein–Uhlenbeck type process, is the basic stochastic process that explains the characteristic of commodity prices to revert toward a long-term equilibrium value [33]. The Ornstein Uhlenbeck type process is continuous in probability and stationary [34]. Gibson and Schwartz [35] were among the first to apply the idea of mean reversion in commodity modeling to study convenience yield and prices of crude oil contingent claims. They developed a two factor model for pricing commodities on the spot price of crude oil and concluded that spot price is a significant determinant of price evolution, but not the only a single factor as convenience yield is considered another factor (see [35] for details).

In general, let $S(t)$ be the price at time t and let μ be the level dependent mean reversion speed, then a price dynamics represented by the Stochastic Differential Equation (SDE) in (3.8) is a mean reverting process (also called an arithmetic Ornstein-Uhlenbeck type process [33]):

$$dS(t) = \mu(\lambda - S(t))dt + \sigma S(t)dB(t). \tag{3.8}$$

Here, if $S(t)$ increases above a mean reversion value $\lambda > 0$, then $\mu(\lambda - S(t) < 0$. This makes $dS(t) < 0$ and $S(t)$ decrease and if $S(t) < 0 < \lambda)$, then $\mu(\lambda - S(t) > 0$, $dS(t) > 0$ and $S(t)$ increase. Therefore, $S(t)$ will eventually move towards λ and $\mathbb{E}[S(t)] \to \mu$ as $t \to \infty$. The solution to equation (3.8) is a stochastic process $S(t)$ given by:

$$S(t) = S_0 \exp[-(\lambda + \frac{\sigma^2}{2} + \sigma B_t] + \mu\lambda \int_0^t \exp[-(\lambda + \frac{\sigma^2}{2}(t-s) + \sigma(B(t) - B(s)]ds. \tag{3.9}$$

Clearly from equation (3.9) $S(t) > 0$ $a.s.$ as long as $S_0 > 0$ and the expectation and variance (see [23] for the characterization of the distribution) are given by,

$$\mathbb{E}[S(t)] = \lambda + e^{-\mu t}(S_0 - \lambda), \tag{3.10}$$

$$\lim_{t\to\infty} \mathbb{E}[S(t)] = \lambda,$$

and

$$Var[S(t)] = \frac{\sigma^2}{2}(1 - e^{-2t}).$$

Mean reverting processes have assumed a huge function over time in price dynamic models. Alos and Ewald [36] noted three important economic objectives that equation (3.8) have, which are;

- The mean reversion level λ can be seen conceivably as price, which is controlled by economic forces such as supply and demand.
- The mean reversion speed μ measure of how fast the economy respond to noises from the mean reversion level λ.
- The uncertainty in the economy is represented in the volatility term σ. In the the absence of volatility, mean reversion level λ and equilibrium of the dynamic are equal.

Important applications of mean reversion in life include stochastic volatility models (see [37, 36, 38] and the reference in there for more details) and interest rate models such as Vasizcek [39] and Cox-Ingersoll-Ross (CIR)([40]) while financial applications include commodity pricing, focusing on geometric mean reversion and Ornstein-Uhlenbeck (OU) processes.

The underlying process of the geometric mean reversion (GMR) is given by;

$$dS(t) = \mu(\lambda - lnS(t))S(t)dt + \sigma S(t)dW(t) \tag{3.11}$$

with μ,λ and σ strictly positive constants and $dB(t)$ is the standard Brownian motion increment.

The differences of GMR to the classical mean reversion models Ornstein-Uhlenbeck and Cox-Ingersoll-Ross, is that the Ornstein-Uhlenbeck has the weakness of taking negative values and therefore it is not reasonable for many economic applications, where prices are typically thought to be positive. A significant improvement is the CIR process. The CIR process remains strictly positive for all time t, for certain constraints, see Alos and Ewald [36] for more details and other coefficient constraints ensuring certain smoothness properties of the CIR process. Clearly from equation (3.11), once $S(t)$ is at zero, it will stay there forever, which economically means an absolute loss of value in that project. Hence, positivity of the system over the time horizon is often a necessary condition for technical reasons. Our main interest in the GMR process lies in the feature that beef-cattle prices neither exponentially grow nor decline, but revert to the long term equilibrium mean. In our case we used a GMR, discussed the stability for several scenarios and checked for positivity of the system using the lyapunov function.

3.1.1 Stochastic Stability Concepts

Once a dynamic system is presented using stochastic differential equation, the traditional definition of stability (for example Lyapunov stability) is no longer applicable [41]. In this subsection, the definitions of stochastic stability are given (see Khasminskii [42] for details). There are at least three types of stochastic stability namely moment stability, almost sure stability and stability in probability.

Definition 12. *The trivial solution to a system of stochastic differential equations is said to be stochastically stable or stable in probability if, for every pair of $\epsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon, r, t_0)$, such that*

$$\mathbb{P}|x(t; t_0.x_0)| < r, \ \forall t \geq t_0 \geq 1 - \epsilon,$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable.

Definition 13. *The trivial solution to a system of stochastic differential equations is said to be asymptotically stable if it is stochastically stable and moreover, for every $\epsilon \in (0,1)$,* *there exists a* $\delta = \delta(\epsilon, t_0) > 0$ *such that*

$$\mathbb{P}(\lim_{t\to\infty} x(t, t_0, x_0) = 0) \geq 1 - \epsilon,$$

whenever $|x_0| > \delta$.

Definition 14. *The trivial solution to a system of stochastic differential equations is said to be asymptotically stable if it is stochastically stable in the large if it is stochastically stable and moreover, for* $x_0 \in \mathbb{R}^m$

$$\mathbb{P}(\lim_{t\to\infty} x(t, t_0, x_0) = 0) = 1,$$

whenever $|x_0| > \delta$.

Definition 15. *The trivial solution to a system of stochastic differential equations is said to be* p^{th} *moment exponentially stable if there is a pair of positive constants* α *and* C *such that*

$$\lim_{t\to\infty} \mathbb{E}[|x(t, t_0, x_0)|^p] \leq C|x_0|^P e^{-\alpha(t-t_0)}\ t \geq t_0,$$

for all $x_0 \in \mathbb{R}^m$.

The dynamic systems is called stochastically mean stable if $p = 1$, which means the mean value of the system dynamics is bounded and stochastically mean square stable if $p = 2$ which means the mean square deviation of the system dynamics is bounded [42]. Asfaw et al [43] investigated the effect of adding noise on the stability of deterministic system. They discovered that the solution of the stochastic model does not tend to any nearby steady state (i.e. it becomes quasi stable around the deterministic steady state).

3.2 Beef-cattle Industry of Botswana

Botswana is located in the Southern part of Africa enclosed by South Africa, Namibia, Zimbabwe and Angola, and has an area of $566,703km^2$, of which 70% is the Kalahari. The country is characterized by prolonged drought spells with erratic rains. The soils are very poor making the carrying capacity very low and more suitable for beef-cattle farming than any other farming activities [44]. The dynamics in climate have significant impacts on beef-cattle sector. Thornton et al. [45] provided a table that depicted productivity changes due to direct and indirect effects of climate. These authors noted that an increase in adverse climate variability is more likely to cause an increase in production risks and reducing the farmer's ability to mitigate the risks.

Beef-cattle production in Botswana consists of two systems of farming namely traditional and commercial farming. Traditional cattle farming relies on uncontrolled grazing due to open access to vast natural rangeland resources, whilst commercial farming involves cattle ranching in either leasehold or freehold land [1, 4]. The traditional system comprise of a large proportion of the cattle population of about 80% of the country's cattle from 1979 to 2017 [46]. Cattle farming is one of the important sources of income in Botswana, especially for the rural people where opportunities for generating money is limited. The industry has a significant impact on the economy as a non-mineral source of foreign currency for Botswana. The impact of prevailing droughts on the cattle production is that farmers through the support of the government rely on drilling expensive boreholes and importing folder from neighbouring countries [4].

The Botswana beef-cattle industry has experienced substantial changes in prices over the past years with major surges in 2010/2011 and 2015/2017 [6, 5, 4]. The prices dramatically increased from BWP4.56 through 1992, reaching their peak level of about BWP34.67 in recent years. In the third quarter of 2010, the price upswing decelerated resulting in decreased beef-cattle prices in the midst of a foot and mouth disease and economic crisis. A similar price pattern emerged in 2016/2017 when the beef price slowly began to climb but dropped suddenly at the end of that period [47]. These price movements coincided with sharp rises in inflation, prolonged drought spell, BMC mismanagement and political influences. Sharp changes in beef-cattle prices were not uncommon, but it is the short period between the recent two price surges that has drawn concerns and raised questions. One may ask what were the causes of the fluctuations in the EU beef prices and what are the prospects for future price movements to the Botswana beef-cattle farmers would be? Were the trends driven by fundamental changes in global supply and demand relationships that may bring about different outcome? What are its implication on the Botswana beef-cattle industry?

Several authors [1, 5, 48] have discussed the factors behind the sharp Botswana beef-cattle price increases in the past, though no consensus has been reached on the cause of these phenomena. Rapid economic growth in Botswana, years of underinvestment in beef-cattle, low beef-cattle production, poor rains, speculative and financial influences are among factors cited leading to high fluctuations of Botswana beef-cattle prices [1]. The price spikes were also associated with increased price volatility in beef-cattle prices globally. Increasing volatility has been a concern for most cattle producers in Botswana and for other agents along the value chain as it renders planning very difficult for all market participants. Price volatility can have long run impact on the incomes of many Botswana beef-cattle producers and can make trading and planning difficult for the BMC. Aizenman and Pinto [49], have argued that higher volatility results in overall welfare loss, though some may benefit from higher volatility. Sudden changes and long run trend movements in beef-cattle prices present serious challenges to market participants especially to exporting developing countries.

In Botswana beef-cattle farming, fluctuations in beef prices in the EU market is of particular importance as can be noted from different perspective [5]. Firstly, most of the poor households in Botswana spend portions of their incomes on beef. Secondly, most farm households are small scale farm households who fully rely on the sale of cattle in order to cover their basic needs and expenditures like health and education. Beef-cattle price volatility thus feeds directly into the dynamics of poverty in Botswana. This is so since high beef prices can play a major role in moving many non vulnerable non-poor households into poverty. Since these households devotes a large proportion of their budgets on beef, beef price shocks can easily pre-empt their income moving them from sustainability into poverty [48].

The major issues concerning cattle prices have been the subject of concern for consumers and farmers. An unpredictable situation might prevail in the future for the industry as BMC is confronted with two conflicting goals of increasing profit and capacity development. In the mean time, the government of Botswana faces contradicting objectives of keeping up regional prices for farmers whereas making the industry profitable. The encounters of 1983/84, 2010-2012, and 2015-2017 were dumbfounding for the national utility, BMC. This had been checked by a horde of restrictions on the trade such as the EU stringent procedures and protocols, foot and mouth disease, subsidation among BMC abattoirs and dry spell related costs which had brought down carcass weight, drilling of expensive boreholes [48]. Since its establishment in 1965, BMC had an obligation of buying and slaughtering all cattle made accessible at the most elevated prices and lowest possible costs [47]. Moreover, it is statutorily mandated to disseminate to cattle farmers all excesses above the legitimately stipulated savings. The suggestion is that decreases in beef-cattle prices and increment in BMC expenses must be passed onto cattle farmers in lower costs unless they are transitory and can be smoothed out utilizing the government support. BMC has no control over prices within the foreign markets nor over drought-related costs, diseases and inflation [4]. It has control over managerial and operational costs and being a monoponsy locally it is fundamental that the government screen these carefully. Finally, as BMC suppliers have been accustomed to price increments, a number of current changes have been making cattle prices to go down.

The sudden and unexpected fluctuations in the EU beef prices in recent years has drawn the attention of policy makers for the Botswana beef-cattle industry and this has led to the debate about the future reliability of the EU markets as a lucrative market. Botswana meat exports to the EU increased from 13245 tonnes in 1968 to 29368 tonnes in 2017 [8]. During the period 1968 to 1990's, Botswana enjoyed unlimited preferential market access to the EU. This has changed as the country now competes with countries such as Brazil, Australia, Argentina, China and the United States of America (USA) [4]. This competition for the EU market has introduced uncertainty in the beef prices. Tothova [9] investigated price volatility in order to determine whether volatility had increased after some time at the EU and global levels. She compared price volatility in relation to other economic variables such as stocks, spot prices, volume of trade and so on. The results showed that events from the past have an impact on the present price variability. The question in the case of Botswana is what historical events have significant effects on the Botswana -EU beef-cattle trade. The fear of further spells of volatility in beef-cattle prices has prompted efforts in designing and proposing price stabilization mechanism both by the EU and the BMC. This fear has been driven by the recognition that a new set of forces may be driving beef-cattle prices and their volatility. These forces emerge from linkages between the beef-cattle industry and the government of Botswana, currency exchange, inflation and collectively with the wider macro-economy, which together render beef-cattle industry much more exposed to shocks.

In this way it is important to understand the impact of the cattle prices on Batswana's wealth and how well the BMC can optimally price live cattle for the harmony of all stakeholders in the Botswana beef-cattle industry, which is relevant to this investigation. The discovery of diamonds at the end of 1980s had been a turn around for Botswana. However, as experts [50, 51, 52] had predicted the replenishment of the precious stones in the near future, henceforth, the need for a thorough research for other possible sectors that can sustainably support the Batswana in the near future. The government of Botswana is now increasing its endeavors to diversify the economy to reduce its over reliance on mining by supporting development of non-mining sectors. It is therefore relevant that this study is going to add value to the understanding the market for beef-cattle which is one of the area that the government of Botswana can give first priority.

Understanding the extent to which domestic agricultural commodity markets in respond to changes in foreign prices is a fundamental issue when analysing performance in agricultural markets. Price transmission from the EU to the BMC is crucial in determining the integration of economic agents into the market process. Transmission of price between markets are founded on ideas related to competitive pricing behaviour. In spatial terms, the concept of the Law of One Price and market integration provided by price determination models discussed earlier. Samuelson [53] postulates that price transmission between markets is complete when commodity sold on foreign country and domestic markets differ only by transfer costs, when converted to a common currency. He developed a model that predicts that changes in demand and supply conditions in one market will affect prices in other markets as price equilibrium is restored through spatial arbitrage.

The absence complete pass-through of price changes from one market to another (market integration) has significant impacts on the economy [54]. Incomplete price transmission arising either as a result of trade policies, or as a result of transaction costs results in less price information available to market agents and may lead to decisions that result in inefficient outcomes. Price transmission studies are apparently an exercise assessing the predictions of economic theory on how changes in one market are transmitted to another as well as the extent to which markets function efficiently. Price transmission mechanisms feature to a large extent in agricultural equilibrium models, such as the World Food Model of the United Nations (UN) Food and Agriculture Organization (FAO) and other models such as the that developed by Tyers and Anderson [55]. In these models the price transmission consist of key issues and play an fundamental role in determining the magnitude and direction of economic welfare effects of trade policy situations (for details of price transmission mechanisms see Piero 2004 [56]).

Price transmission models suggest that, if two markets are linked, excess supply or demand shocks in one market will have an equal impact on price in both markets [53]. The import tariffs, will allow foreign price changes transmitted to domestic markets in comparative terms. Thus an increase in domestic prices is proportional to an increase in the international price provided that tariff levels remain unchanged. However, if the tariff level is exorbitant, changes in the foreign price would not be or partly transmitted to the domestic market, thus obliterating opportunities for spatial arbitrage and resulting in the two prices moving independently of each other, as if an import ban was implemented [57].

Markets serves many functions which include borrowing, lending, information dissemination, risk sharing, efficiency and liquidity [58]. For the purpose of this thesis, the market provide information for current spot price and expected beef-cattle prices in the future. The existence of spot prices is crucial for the derivation of derivative market. Derivative are financial contracts contingent on the realization of a spot market price. In the course of the thesis, the spot prices for two markets which are assumed to be cointegrated are used as instruments in pricing beef-cattle among the three actors, farmer, BMC and the EU. Financial theory assume markets to be free of arbitrage, i.e. no financial product providing a free lunch or a free lottery exist [59]. There is an intuitive explanation which is presented next.

Arbitrage is defined as a trading strategy that takes advantage of two or more securities being mispriced relatively to each other [59]. Financial theory distinguishes two kinds of arbitrage, the so called "free lunch" and the "free lottery" (see Øksendal 6^{th} edition [60]). A free lunch provides its owner a deterministic positive positive profit without usage of capital whereas a free lottery provides the participation at the lottery with positive expected outcome [61]. If two products providing the same future outcome are connected with different current prices, every agent is interested in buying one of the contracts would rather buy the cheaper one. On the other hand, every agent interested in selling the contracts would rather sell the more valuable one. An increasing demand (supply) for cheap (the more valuable) contract will increase (decrease) the price of the respective contract until both are traded at the same price. The effect is called the law of one price

[57].

A situation relevant to this thesis considers no-arbitrage on the Botswana beef-cattle market. Assume BMC buys cattle which is required at time $t = 1$. The current time is $t = 0$. In general, there are two simple strategies in order to buy one cattle at $t = 1$. The cost of these strategies are composed of purchasing costs p_0 (the deterministic spot price at $t = 0$ and the holding cost h per cattle and period which are assumed to be payable at $t = 0$. Second, wait until $t = 1$ and buy at the future spot price $p_1$which is stochastic from the view of $t = 0$ but will realize just before the purchase is made at $t = 1$. The expected value of $p - 1$ is $\mathbb{E}(p_1)$. Payments are assumed to be discounted with a rate of p per period. The two strategies are illustrated as; absence of arbitrage claims equality of the two prices, (Farmer-BMC ($S_1(t)$) and the BMC-EU price ($S_2(t)$))

$$S_1(t)= S_2(t) \Leftrightarrow S_0 + h= \rho\mathbb{E}(S_1). \tag{3.12}$$

Equation (3.12) represents a tight connection between parameters of the stochastic process spatial market integration i.e. domestic and international prices are related via a no-arbitrage condition (see Góes [62] for more explanation). Spatial market integration refers to a situation in which prices of a commodity in spatially separated markets move together due to arbitrage and the price signals and information are transmitted smoothly across the markets. With free flow of information in a competitive market, difference in prices of a product in the two markets would be equal to or less than transportation cost between them. Hence, spatial market performance may be evaluated in terms of the relationship between the prices in spatially separated markets. Estimation of bivariate correlation coefficients between price changes in different markets has been employed as the most common methodology [57, 63] for testing market integration.

Recent advances in mathematical finance using time series analysis, especially those related to studies in market cointegration have led to an explosion in the literature of commodity pricing [57, 63, 62]. Goodwin and Schroeder [64] studied spatial price behaviour in regional United States of America (USA) cattle markets in relation to overall market performance using cointegration tests of regional price series. They found out that several markets over the 1980 through 1987 were not integrated and that distances between markets, industry cointegration ratios, market volumes and market types have significant influences on cointegration relationship between markets. They suggested that markets that are not integrated may convey inaccurate price information that may distort producer marketing decisions and contribute to to inefficient product movement. This condition is relevant for this study considering transactions between market participants in the Botswana beef-cattle industry who rely on each other for their price strategies. Hence, the present study was conducted to examine the performance of Botswana beef-cattle in terms of the EU beef market.

Summary

The chapter reviewed various aspects related to the current study. Initially, stochastic processes that are used in modeling commodity prices are examined. Among all the processes that were discussed, mean-reverting and geometric mean-reverting processes were found to be more suitable in modeling commodity prices. One of the main interest in mean-reversion lies in the feature that commodity prices neither exponentially grow nor decline, but fluctuate about the long term equilibrium mean. A brief discussion on the concept of stochastic stability was done. There are several ways to define stochastic stability for the solutions of different special cases for the dynamics system. For example assume that, $x_i(0) \neq 0$. If

$$\lim_{t \to \infty} |x_i(t)| = 0$$

with probability 1 (almost surely ($a.s.$), the $x_i(t) \equiv 0$ is said to be asymptotically stochastically stable whereas if

$$\lim_{t \to \infty} \mathbb{E}[|x_i(t)|^2] = 0,$$

then $x_i = 0$ is said to be mean-square stable. Some stochastic differential equations are both asymptotically stochastically stable and mean-square stable while others maybe asymptotically stable but not mean-square stable. Lastly, we gave a brief discussion on the Botswana beef-cattle industry. The beef-cattle industry consist of two markets namely the farmer-BMC and BMC-EU market. Market characteristics for the Botswana beef-cattle industry were defined and their related financial theory inherent were discussed.

CHAPTER 4

MODELING BOTSWANA BEEF-CATTLE PRICE DYNAMICS

The dynamics in the beef-cattle industry of Botswana alarm a good pricing strategy for better economic and socioeconomic growth. In this chapter we incorporated the aspect of stochastic mean equilibrium level to make the mean-reversion more realistic in modelling the beef-cattle price for Botswana. Initially we present some preliminary results based on the prices presented in Tables C.1 (see Appendix C). Since the beef-cattle industry for Botswana is segmented into two markets namely farmer-Botswana Meat Commission (BMC) and BMC-European Union (EU) market, we discuss the performance of the industry based on the price acceptable to the European market. The stochastic and the deterministic parts for our model were calibrated using historic prices and least squares respectively (see Appendix C).

4.1 Model Formulation

In our model we assume that the information available to BMC is the average price of beef $\overline{S}(t)$, whose future trend is not known. In general equation (4.1), below represents the evolution of price $S(t)$ at time t based on the EU price.

$$S(t) = \overline{S}_i(t) + a_i,\ i = 1, 2, \tag{4.1}$$

where, $\overline{S}_i(t)$ is the mean price and a_i is a random variable.

4.1.1 Preliminary Analysis

Prior to our stochastic model we present some preliminary results associated with the price data sets. We noted that the Botswana beef-cattle industry is divided into several markets segments. The results presented in this study are based on the most lucrative market (the EU).

Let $S_1(t)$ be the price BMC pays the farmer (Farmer-BMC price) and $S_2(t)$ be the price EU pays BMC (BMC-EU price). Figure 4.1 shows time plot for annual averaged beef-cattle prices for Farmer-BMC and BMC-EU from the year 1992 to 2018.

Note that while the ratio of the price of Farmer-BMC to BMC-EU was 0.65 in 1992 this ratio declined to 0.30 in 2015. This could be a result of competition in the beef market

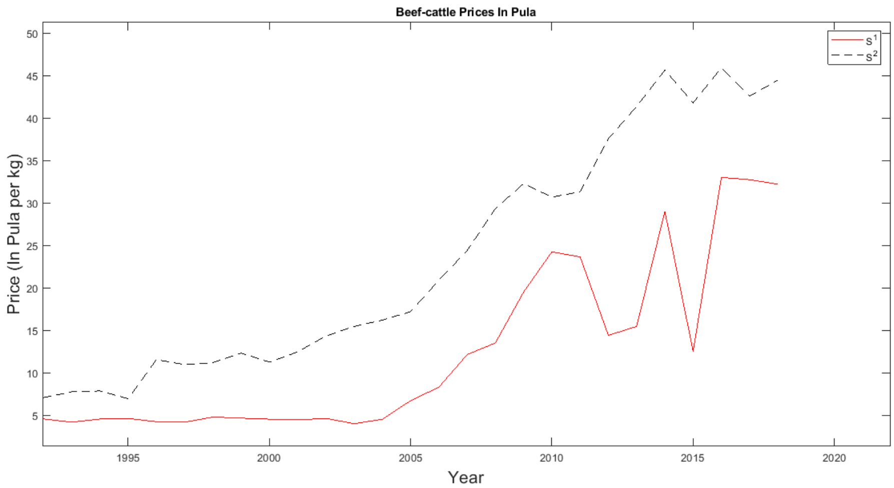


Figure 4.1: Time plot for Farmer-BMC prices and BMC-EU prices for the years 1992 to 2018

where the seller had to accept a lower price to secure the export quarter. The mean and standard deviation for each of the two price processes are given in Table 4.1 (see Table C.1 in Appendix A1 for an overview of data that was used).

	$S_1(t)$	a_1	$S_2(t)$	a_2
Mean	12.4133	0.0033	23.3407	0
Standard deviation	10.2059	10.2059	13.9738	13.9738

Table 4.1: Mean and standard deviation for $S_1(t)$, $S_2(t)$ and their respective residuals

We conclude that the random variable a_i, $i = 1, 2$ is purely white noise.

The Pearson's correlation coefficient for the two price processes are given by,

$$\begin{aligned} r &= \frac{\frac{\sum_{i=1}^{n}(S_1(t)-\overline{S}_1)(S_{2,i}-\overline{S}_2)}{n-1}}{\sqrt{\sum_{i=1}^{n}(S_1(t)-\overline{S}_1)^2\sum_{i=1}^{n}(S_{2,i}-\overline{S}_2)^2}} \\ &= 0.8999. \end{aligned} \tag{4.2}$$

This confirms a strong positive correlation between the Farmer-BMC price $S_1(t)$ and BMC-EU price $S_2(t)$ suggested by Figure 4.1. The results can be interpreted to mean that BMC passes on to farmers the gain or loss it incurs from trading with the EU.

The relationship between $S_1(t)$ and $S_2(t)$ is of interest. We assume a relationship between $S_1(t)$ and $S_2(t)$ of the type:

$$S_2(t) = \omega(B)S_{1,t-b}. \tag{4.3}$$

Where, B is the backward shift operator and b is the delay parameter. Our task is to find the transfer function $\omega(B)$ and b which for this relationship holds.

In the next subsection we discuss the concept of linear prices translation in the Botswana beef-cattle industry using the Box and Jenkins procedure [65].

4.1.2 Price translation in the Botswana beef-cattle industry

Suppose there exists N price observations $S_i(t)$, $i = 1, 2$, where $S_1(t)$ and $S_2(t)$ then, these prices may be regarded at equispaced intervals on the time horizon $[0, T]$ (yearly averages in this case). These observations may be denoted pairwise as $(S_{1,1}, S_{2,1}), (S_{1,2}, S_{2,2}), ..., (S_{1,N}, S_{2,N})$ and finite realization of a discrete bivariate process with the $S_1(t)$ as the independent variable and $S_2(t)$ as the dependent variable (see Table C.2 in Appendix A4 for the notations used throughout this paper). We need to find the weights $\{w_k\}$ (response functions), where $k = 0, 1, ...$ of the pricing process

$$S_{2,t} = w(B)S_{1,t-b}, \tag{4.4}$$

where $w(B) = v_0 - v_1B - v_2B^2-, ...$ is called the transfer function. Let, $s_{1,t} = \bigtriangledown^d s_{1,t}$ and $s_{2,t} = \bigtriangledown^d s_{2,t}$ be incremental changes for the Farmer-BMC prices and BMC-EU prices, respectively. The constant d denotes the degree of differencing, $\bigtriangledown = (1 - B)$, then for any series $\{M_t\}$, $M_{t-b} = B^b M_t$, we can show on differencing equation (4.4) that $s_{2,t}$ and $s_{1,t}$ satisfy the same transfer function model as do $S_{2,t}$ and $S_{1,t}$ i.e.

$$s_{2,t} = w(B)s_{1,t-b}. \tag{4.5}$$

Writing equation (4.5) (the linear filter) in a parsimonious way as in Box and Jenkins [66] we have,

$$\delta(B)s_{2,t} = \rho(B)s_{1,t-b}. \tag{4.6}$$

Where

$$\delta(B) = 1 - \delta_1 B - \delta_2 B^2 - ...\delta_r B^r \tag{4.7}$$
$$\rho(B) = \rho_0 - \rho_1 B - \rho_2 B^2 - ...\rho_h B^h. \tag{4.8}$$

We compared equations (4.5) and (4.6) to obtain:

$$
\begin{aligned}
w(B) &= \delta^{-1}(B)\rho(B) && (4.9)\\
w_j &= 0,\ j < b && (4.10)\\
w_j &= \delta_1 w_{j-1} + \delta_2 w_{j-2} + ... + \delta_r wj - r - w_0,\ j = 0, 1, ..., b && (4.11)\\
w_j &= \delta_1 w_{j-1} + \delta_2 w_{j-2} + ... + \delta_r wj - r - \rho_{j-b},\ j = (b+1), ... + (b+h) && (4.12)\\
w_j &= \delta_1 w_{j-1} + \delta_2 w_{j-2} + ... + \delta_r wj - r,\ j > (b+h). && (4.13)
\end{aligned}
$$

Theoretically, a plot of the weights w_k, $k = 0, 1, ...$ against lag k provides a pictorial representation of the impulse response function. In reality, however (considering the beef-cattle industry of Botswana in particular) the system involves noise or disturbances whose net effect influences the predicted model by an amount η_t, so that the combined translation function-noise model may be written as;

$$s_{2,t} - \delta^{-1}(B)\rho(B)s_{1,t-b} = \eta_t. \tag{4.14}$$

Where $s_{2,t}$ and $s_{1,t}$ are stationary time series for d differencing. Using the Box and Jenkins [65] pre-whitening procedure we can fit an ARIMA model to the differenced input series $S_{1,t}$ as our initial procedure as;

$$\phi(B)s_{1,t} = \theta(B)\alpha_t, \tag{4.15}$$

where the variable α_t represents a pure white noise process, $\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$ and $\theta(B) = 1 - \sum_{j=1}^{p} \theta_j B^j$ are moving average and autoregressive polynomials respectively. Note that from (4.15) the transformations $\theta(B)\theta^{-1}(B)$ transforms the correlated series of the dependent variable $s_{1,t}$ to the uncorrelated pure random process α_t such that,

$$\alpha_t = \theta^{-1}(B)\phi(B)s_{1,t}. \tag{4.16}$$

Transforming, to the BMC-EU prices (output series), we obtain

$$\beta_t = \phi(B)\theta^{-1}(B)s_{2,t}. \tag{4.17}$$

Calculating, the cross-covariance function of the filtered input and output (α_t and β_t and multiplying both sides of equation (4.15) by $\phi(B)\theta^{-1}(B)$ gives:

$$\beta_t = w(B)\alpha_t + \varepsilon_t. \tag{4.18}$$

Where, $\varepsilon_t = \phi(B)\theta^{-1}n_t$ is the transformed noise series. Multiplying both sides of equation (4.18) by α_{t-k} and taking expectation, noting that α_t and n_t are uncorrelated yields:

$$w_t = \frac{\mathbb{E}[\alpha_{1-b}, \beta_t]\sigma_\beta}{\sigma_\alpha}. \tag{4.19}$$

Where σ_β^2 and σ_α^2 are the variances of α_t and β_t respectively, and $\mathbb{E}[\alpha_{1-b}, \beta_t]$ is the cross-covariance function at lag k. The estimate of the impulse response function (w_k) determined as outlined above are found to be reliable [65] and we used them as a basis for estimating constants r, h and b. Furthermore, considering the orders of parameters r and h of $\delta(B)$ and $\rho(B)$ of equation (4.12), we seek to identify Bivariate Autoregressive Integrated Moving Average (BARIMA) models that describes the noise at the BMC-EU prices $S_2(t)$ (the output).

If we consider $b = 0$, $\delta^{-1}(B) = 1$, $\rho(B) = \rho = 1$, equation (4.14) is transformed into

$$S_1(t) - S_2(t) = \eta_t. \tag{4.20}$$

In Figure 4.2 is a scatter plot of BMC-EU prices against Farmer-BMC prices.

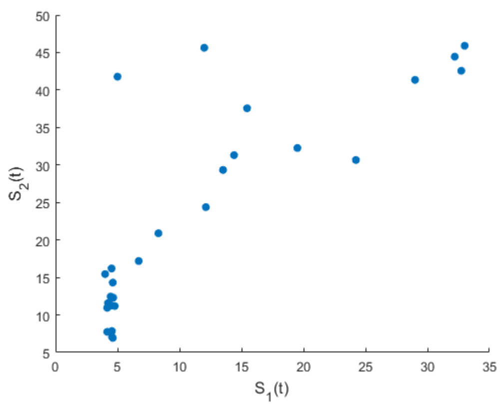


Figure 4.2: Scatter plot of BMC-EU prices against Farmer-BMC prices

A scatter plot of $S_1(t)$ against $S_2(t)$ (Figure 4.2) justify the subsequent section for which a dynamic model for the beef-cattle industry for Botswana is presented.

4.2 Dynamics of the price: Stochastic Model

The bivariate relationship between the Farmer-BMC and BMC-EU prices in section 4.1.2 have shown that the difference between the two prices is a white noise process (fig. 4.2). In this section, we propose to formulate the Farmer-BMC price as an Ornsten-Uhlenbeck process with the BMC-EU price as the stochastic mean to which the Farmer-BMC price reverts. Let $S(t) = (S_1(t), S_2(t))$ be a price process, where $S_1(t)$ represent the Farmer-BMC price and $S_2(t)$ represents the BMC-EU price. Denote by, $B_1(t)$ and $B_2(t)$ the noise for the Farmer-BMC and the BMC-EU prices, respectively. The price process $S(t)$ is assumed to mimic a GMR as in Ewald and Yang [36].

For simplicity of notation, we define the state variables $(x_1, x_2) = (S_1(t), S_2(t))$ where x_2 is assumed to be the level mean of the price process $x_1(t)$. The GMR model can be written as:

$$dx_1 = \kappa(x_2 - x_1)x_1 dt + \sigma_1 x_1 dB_1 \tag{4.21}$$

$$dx_2 = \sigma_2 x_2 dB_2. \tag{4.22}$$

Where, $\sigma_1 > 0, \quad \sigma_2 > 0$ and $dB_1 dB_2 = \rho dt$ and κ is the level dependent mean reverting speed. Note that we assume the mean to be purely stochastic, since the BMC has no knowledge about the price it will be offered by the EU. We can write the system (4.21)-(4.22) in matrix form as

$$dx_t = f(x_t)dt + \sigma(x_t)dB, \tag{4.23}$$

where $x_t = (x_1(t), x_2(t))^T$

$$f(x_t) = \begin{pmatrix} \kappa(x_2 - x_1)x_1 \\ 0 \end{pmatrix}; \qquad \sigma(x_t) = \begin{pmatrix} \sigma_1 x_1 & 0 \\ 0 & \sigma_2 x_2 \end{pmatrix}; \qquad dB = \begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}.$$

The solution of (4.23) is not unique due to nonlinear terms $x_1 x_2$ and x_1^2 in the drift of the equation (4.21). We want to define conditions on $f(x_t)$ for the system (4.23) to have a unique solution.

Remark: The solution of the stochastic model (4.23) does not tend to any nearby steady state (see for example Asfaw et al [43]). The stability analysis in this study refers to quasi stability about a deterministic steady state $\overline{x}$ so that instead of considering individual paths of $x_t - \overline{x}$, we shall evaluate the expectation of $|x_t - \overline{x}|^2$.

Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2_+, x_i \geq 0, i = 1, 2\}$ be a positive region under study and consider the system (4.23) with initial values $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2_+$. It is easy to show that (4.23) can be modified (see, Gard [67] for details) such that the nonlinear terms $x_1 x_2$ and x_1^2 are replaced by linearly bounded terms $g_t(x_1) g_t(x_2)$, where

$$g_\epsilon(s) = \begin{cases} 0 & \text{if } s < 0 \\ s & \text{if } 0 \leq s \leq \frac{1}{\epsilon} \\ \frac{1}{\epsilon} & \text{if } s > \frac{1}{\epsilon} \end{cases}$$

and ϵ is an arbitrary small number. We denote the modified $f(x_t)$ in (4.23) by $f^\epsilon(x_t^\epsilon)$ and consider the following stochastic system

$$dx_t^\epsilon = f^\epsilon(x_t^\epsilon)dt + \sigma(x_t^\epsilon)dB. \tag{4.24}$$

Clearly (see Øksendal 6^{th} edition [60]),

$$|f^\epsilon(x) - f^\epsilon(\overline{x})| \leq K_\epsilon |x - \overline{x}|$$

$$|\sigma(x) - \sigma(\overline{x})| \leq K|x - \overline{x}|$$

For all x, $\overline{x} \in \mathbb{R}^2$ where K_ϵ and K are constants.

Denote by τ_ϵ the next exit time of x_t^ϵ from the domain

$$\mathbb{R}^2 \cap \left\{ \max_{1 \leq i \leq 2} x_i < \frac{1}{\epsilon} \right\},$$

If this domain contains the initial point x^0 then the system (4.21)-(4.22) possesses a nonnegative solution for $t > 0$ in this domain.

4.3 Lemma 1 (Positivity Region)

For any finite $T > 0$ the solution x_t^ϵ of the system (4.24) with initial condition $x^0 \in \mathbb{R}_t^2$ remains in $\Omega \in \mathbb{R}_t^2$ for all $t < T \wedge \tau_\epsilon$ (where $T \wedge \tau_\epsilon$ denotes the smaller between T and τ_ϵ) so that the components x_t^ϵ satisfy $x_{i,t} > 0$, if $t \wedge T, \tau_\epsilon$ for $i = 1, 2$. Furthermore,

$\mathbb{P}(\tau_\epsilon < T) < C_o\tau_\epsilon$, where C_0 is a constant independent of x^0.

Proof

We want to prove first that a non-negative solution does not exit the positive domain Ω. We introduce a Lyapunov type function

$$V = x_1 - \kappa_1 lnx_1 + x_2 - \kappa_2 lnx_2, \tag{4.25}$$

where, $\kappa_1 > 0$ and $\kappa_2 > 0$ are constants.

$$\begin{aligned}
dV &= (1-\frac{\kappa_1}{x_1})dx_1 + (1-\frac{\kappa_2}{x_2})dx_2 + \frac{1}{2}\frac{\kappa_1}{x_1^2}(dx_1)^2 + \frac{1}{2}\frac{\kappa_2}{x_2^2}(dx_2)^2 \\
&= (1-\frac{\kappa_1}{x_1})[\kappa(x_1-x_2)x_1dt + \sigma_1x_1dB_1] + (1-\frac{\kappa_2}{x_2})\sigma_2x_2dB_2 + \frac{1}{2}\kappa_1\sigma_1^2dt + \frac{1}{2}\kappa_2\sigma_2^2dt \\
&= [\kappa x_1x_2 - \kappa_1x_1^2 - \kappa_1\kappa x_2 + \kappa_1\kappa x_1 + \frac{1}{2}\kappa_1\sigma_1^2 + \frac{1}{2}\kappa_2\sigma_2^2]dt - \kappa_1\sigma_1dB_1 - \kappa_2\sigma_2dB_2 \\
&+ \sigma_1x_1dB_1 + \sigma_2x_2dB_2 \\
&\leq [\kappa x_1x_2 + \kappa_1\kappa x_1 + \frac{1}{2}\kappa_1\sigma_1^2 + \frac{1}{2}\kappa_2\sigma_2^2]dt + \sigma_1x_1dB_1 + \sigma_2x_2dB_2 \\
&= \mathcal{M}(t)dt + \sum_{i=1}^{2}\sigma_ix_idB_i.
\end{aligned} \tag{4.26}$$

where

$$\mathcal{M}(t) = \kappa x_1x_2 + \kappa_1\kappa x_1 + \frac{1}{2}\kappa_1\sigma_1^2 + \frac{1}{2}\kappa_2\sigma_2^2. \tag{4.27}$$

Integrating (4.26), we obtain

$$\int_0^{T\wedge\tau_\epsilon} dV \leq \int_0^{T\wedge\tau_\epsilon} \mathcal{M}ds + \sum_{i=1}^{2}\int_0^{T\wedge\tau_\epsilon} \sigma_ix_idB_i. \tag{4.28}$$

Taking expectation we obtain

$$\mathbb{E}[V(T\wedge\tau_\epsilon)] \leq \mathbb{E}[V(x^0)] + \mathcal{M}(T\wedge\tau_\epsilon). \tag{4.29}$$

Note that if the path x_t^ϵ is such that it exits $\mathbb{R}_t^2$ at $T\wedge\tau_\epsilon$ then by definition of (4.25) the function V becomes ∞ at the exit point. In view of (4.26) the probability is zero. This completes the proof.

4.3.1 Special cases

We consider the following four cases of the system (4.21)-(4.22):

$$\begin{cases} Case\,1: & \sigma_1 = 0,\, \sigma_2 = 0, \\ Case\,2: & \sigma_1 = 0,\, \sigma_2 \neq 0, \\ Case\,3: & \sigma_1 \neq 0,\, \sigma_2 = 0, \\ Case\,4: & \sigma_1 \neq 0,\, \sigma_2 \neq 0. \end{cases}$$

Lemma 2

For the case $\sigma_1 = \sigma_2 = 0$, the system (4.21)-(4.22) possesses two equilibrium points one of which is stable and the other unstable.

Proof

Take $\sigma_1 = \sigma_2 = 0$, the system (4.21)-(4.22) becomes

$$\begin{aligned} \frac{dx_1}{dt} &= \kappa(x_2 - x_1)x_1 \\ dx_2 &= 0 \implies x_2 = m,\, m\,is\,a\,constant. \end{aligned} \tag{4.30}$$

Equation (4.30) has two equilibrium points $(0, m)$ and (m, m) The transient solution of equation (4.30) for $x_1(t)$, for $x_2(t) = m$, is given by

$$x_1 = \frac{m}{1 + e^{-\kappa m t}}. \tag{4.31}$$

Clearly, as $t \to \infty$, $x_1 \to m$ implying that the movement is away from the unstable equilibrium point $(0, m)$ towards the stable equilibrium point (m, m).

Remark: When there is no volatility the price $x_1(t)$ would remain constant about the mean price m which economically is not a good situation for either the Farmer-BMC price or the BMC-EU price as neither the farmer nor BMC would generate sufficient capital to expand.

Lemma 3

For the case $\sigma_1 = 0$, $\sigma_2 \neq 0$, the system (4.21)-(4.22) possesses one stable equilibrium point $(0, 0)$.

Proof

For $\sigma_1 = 0$, $\sigma_2 \neq 0$, the system (4.21)-(4.22) becomes,

$$\begin{aligned} dx_1 &= \kappa(x_2 - x_1)x_1 dt \quad (4.32) \\ dx_2 &= \sigma_2 x_2 dB_2. \quad (4.33) \end{aligned}$$

Equation (4.33) yields the solution

$$x_2 = \exp[-\frac{1}{2}\sigma_2^2 t + \sigma_2 B_2] \quad (4.34)$$

$$\mathbb{E}[x_2] = \exp[-\frac{1}{2}\sigma_2^2 t] \to 0 \, as \, t \to \infty. \quad (4.35)$$

Solving for x_1 in terms of x_2, we obtain

$$\begin{aligned} x_1 &= \frac{x_2 \exp[\kappa x_2 t]}{1 + \exp[\kappa x_2 t]} \\ &= \frac{\exp[-\frac{1}{2}\sigma_2^2 t + \sigma_2 B_2] \exp[\kappa \exp[-\frac{1}{2}\sigma_2^2 t + \sigma_2 B_2] t]}{1 + \exp[\kappa \exp[-\frac{1}{2}\sigma_2^2 t + \sigma_2 B_2] t]}. \quad (4.36) \end{aligned}$$

As $t \to \infty$, $x_2 \to 0$, $x_1 \to 0$.

Remark: For $\sigma_1 = 0$, $\sigma_2 \neq 0$ the system (4.21)-(4.22) possesses one stable equilibrium point $(0, 0)$ which implies the collapse of business. This solution is unrealistic but it cautions against decisions such as price control without taking into account supply and demand. This is the case in the African beef-cattle market where fixing prices is usually a political decision.

Lemma 4

For $\sigma_1 \neq 0, \quad \sigma_2 = 0$ the system (4.21)-(4.22) is reducible to an Ornstein-Uhlenbeck type process, which yields Ornstein-Uhlenbeck gains or losses.

Proof

Note that for $\sigma_1 \neq 0, \quad \sigma_2 = 0$ the system (4.21)-(4.22) becomes,

$$dx_1 = \kappa(m - x_1)x_1 dt + \sigma_1 x_1 dB_1, \tag{4.37}$$

which can be written in the form,

$$dx_1 = f(x_1)dt + g(x_1)dB_1, \tag{4.38}$$

where,

$$f(x_1) = \kappa(m - x_1)x_1 \, and \, g(x_1) = \sigma_1. \tag{4.39}$$

Defining,

$$A(x) = \frac{f(x)}{g(x)} - \frac{1}{2}g'(x), \tag{4.40}$$

then based on Gard (1988, chapter 4) [67], equation (4.37) is reducible to a standard Ornstein-Uhlenbeck process if,

$$\left(\frac{(gA')'}{A'}\right)' = 0. \tag{4.41}$$

From (4.39), it is easy to show that

$$\begin{aligned}\left(\frac{(gA')'}{A'}\right)' &= \left(\frac{g'A' + gA"}{A'}\right)' \\ &= \left(\frac{g'A'}{A'}\right)' \\ &= \frac{A'(g'A')' - (g'A')A"}{(A')^2} \\ &= \frac{A'(g'A" + A'g") - (g'A')A"}{(A')^2} \\ &= 0. \end{aligned} \quad (4.42)$$

Following Gard (1988, chapter 4) [67], the solution of (4.37) is given by,

$$x_1 = \frac{exp\{(m - \frac{1}{2}\sigma_1^2)t + \sigma_1^2 B_{1,t}\}}{x_{1,0}^{-1} + m\int_0^t exp\{(\kappa m - \frac{1}{2}\sigma_1^2)s + \sigma_1 B_{1,s}\}ds}. \quad (4.43)$$

If we rewrite (4.37) as,

$$\frac{dx_1}{x_1} = \kappa(m - x_1)dt + \sigma_1 dB_1, \quad (4.44)$$

then, we can see that the left hand side of (4.44) represents the gains or losses, while the right hand side represents an Ornstein-Uhkenbeck process.

Proposition 1

For $\sigma_1 \neq 0, \quad \sigma_2 \neq 0$, the system (4.21)-(4.22) yields Ornstein-Uhlenbeck gains or losses that revert to a fluctuating mean, m.

Remark: The system (4.21)-(4.22) can be written as,

$$\frac{dx_1}{x_1} = \kappa(x_2 - x_1)dt + \sigma_1 dB_1 \quad (4.45)$$
$$dx_2 = \sigma_2 x_2 dB_2. \quad (4.46)$$

Proof

The approach used in lemma (4) can be used to prove proposition (1) and the system (4.45)-(4.46) has the following solution,

$$
\begin{aligned}
x_1 &= \frac{exp\{(x_2 - \frac{1}{2}\sigma_1^2)t + \sigma_1^2 B_{1,t}\}}{x_{1,0}^{-1} + x_2 \int_0^t exp\{(\kappa x_2 - \frac{1}{2}\sigma_1^2)s + \sigma_1 B_{1,s}\}ds} && (4.47)\\
x_2 &= \exp[-\frac{1}{2}\sigma_2^2 t + \sigma_2 B_2]. && (4.48)
\end{aligned}
$$

As in lemma 4, the relative gains or losses in price (right hand side of (4.45) is of Ornstein-Uhlenbeck type and are level dependent (mean reverting to x_2 given in (4.34). We present the solution for $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$ in chapter 4 numerically.

4.4 Numerical Approximation and Model Calibration

In line with least squares approximation, the system (4.21)-(4.22) can be be written as follows,

$$
\begin{aligned}
x_{1,t+1} &= x_{1,t} + \kappa(x_2 - x1, t)x_{1,t} + \sigma_1 x_{1,t}\epsilon_{1,t} && (4.49)\\
x_{2,t+1} &= \sigma_2 x_{2,t}\epsilon_{2,t} && (4.50)
\end{aligned}
$$

Where $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are standard normal variables. The technique for least squares is tied in estimating parameters by minimizing the squared errors of historical data, from one perspective, and their expected values on the other. The approach we used takes the form of a regression problem, where the variation in one variable, called the dependent variable Y, can be partly explained by the variation in the other variables, called independent variables say X. To estimate the parameters κ, x_2, σ_1 and σ_2 we used weekly spot prices of beef-cattle from the EU markets for years 1992 to 2018. The choice of 1992 as the initial observation was mainly conditioned by the availability of data. We got weekly beef prices from an official website of the EU [68]. The prices were converted into Botswana Pula (BWP) using the exchange rates we obtained from [69]. Assuming constant parameters during the time period of estimation and rewrite the system (4.21)-(4.22) as,

$$
\begin{aligned}
\frac{x_{1,t+1}x_{1,t}}{x_{1,t}} &= \kappa x_2 - \kappa x_{1,t} + \sigma_1\epsilon_{1,t} && (4.51)\\
x_{2,t_1} &= x_{2,t}\sigma_2\epsilon_{2,t} && (4.52)
\end{aligned}
$$

Both equations (4.51) and (4.52) bears the characteristics of a linear regression model, with the gain and loss given by $\frac{x_{1,t+1}x_{1,t}}{x_{1,t}}$ and $x_{2,t+1}$ as dependent variables, $x_{1,t}$ and $x_{2,t}$ as explanatory variables. Following the work by [70], the estimates of κ and volatility (σs) are obtained as the negative of the coefficient in front of $x_{1,t}$ and the standard error of the regression respectively (see Table 4.2 for results).

Parameter	κ	$m(x_2)$	$x_1(0)$	σ_1	σ_2
EU Beef-cattle Price	0.036	32.35	4.56	0.124	0.0210

Source: Authors' estimates based on the EU market [68]

Table 4.2: Parameter estimates for beef-cattle prices based on the EU

As a good start to the numerical methods for the system (4.21)-(4.22) we consider the Euler-Mayurama method to simulate the stochastic differential equations. Given the system (4.21)-(4.22), the Euler-Maruyama method generates a discrete sequence $x_t = \{x_t^j\}_{j\in\{1,\dots,d\}}$, which approximates the system on the interval $[0,T]$. From the system (4.21)-(4.22) we have,

$$\begin{aligned} x^j_{1,t_{i+1}} &= x^j_{1,t_i} + \kappa(x^j_{2,t_i} - x_{1,t_i})x^j_{1,t_i} \triangle_{t,i} + x^j_{1,t_i}\sigma_1(W^1_{t_i+1} - W^2_{t_i}) \\ &= x^j_{1,t_i} + \kappa(x^j_{2,t_i} - x_{1,t_i})x^j_{1,t_i} \triangle_{t,i} + \sigma_1 x^j_{1,t_i}\sqrt{t_{i+1} - t_1\epsilon^j} \qquad (4.53) \\ x^j_{2,t_{i+1}} &= x^j_{2,t_i} + x^j_{2,t_i}\sigma_2(W^2_{t_i+1} - W^2_{t_i}) \\ &= \sigma_2 x^j_{2,t_i}\sqrt{t_{i+1} - t_1\epsilon^j} \qquad (4.54) \end{aligned}$$

Where $X_{t,0} = X_0$ ($X_{t,0} = (x_{1,t_0}, x_{2,t_0})$, $\triangle_{t,i} = t_{i+1} - t_i$, for $i = 0, \dots, N$ and ϵ is a Gaussian process with mean 0 and co-variance $\mathbb{I}d$. It is an iterative technique as the solution of the SDE is changed at every step.

Summary

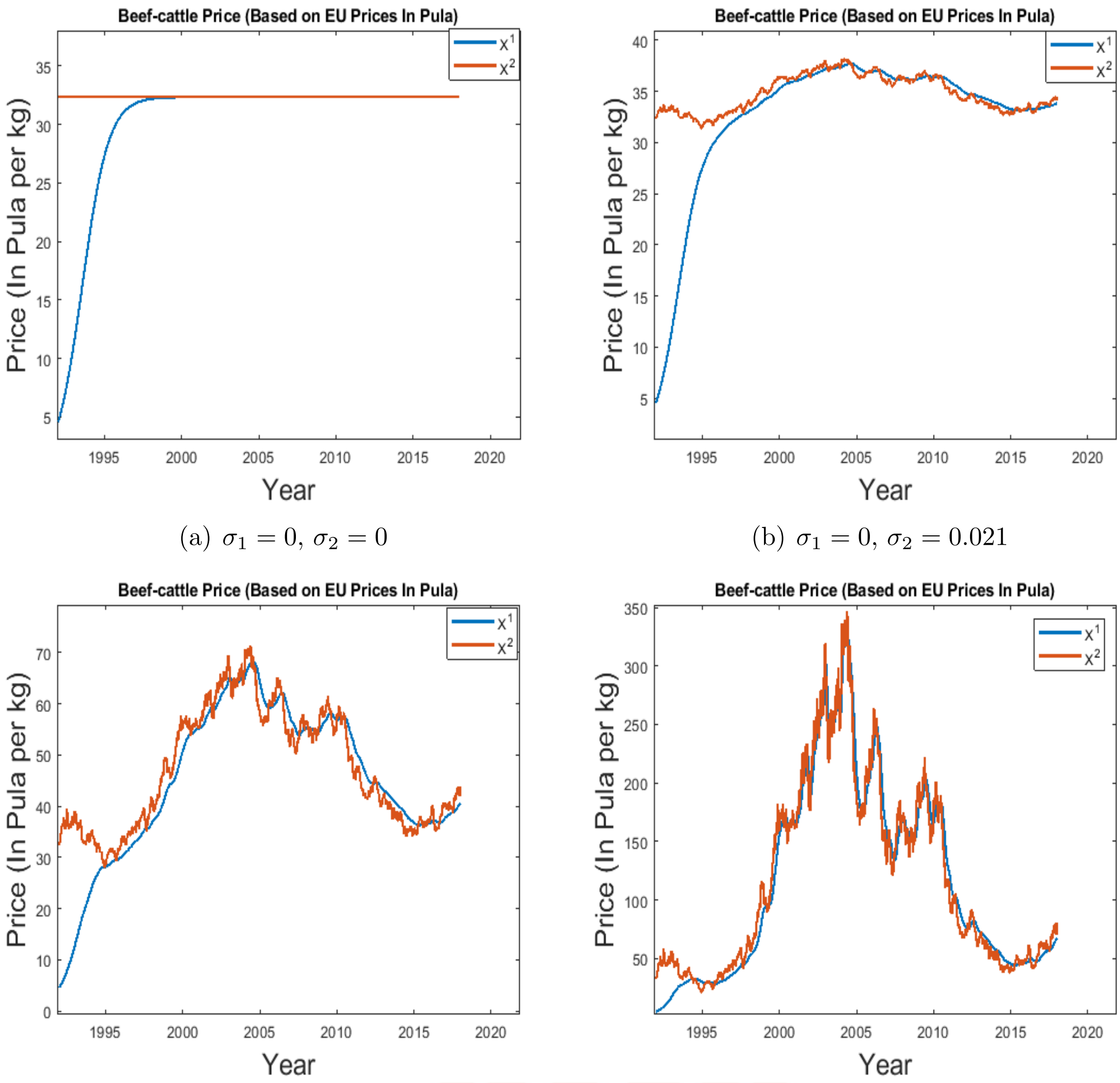
The chapter examined the behaviour of beef-cattle prices in Botswana using the Geometric mean reversion process. The beef-cattle industry in Botswana consists of two markets namely the farmer-Botswana Meat Commission (BMC) and the BMC-European market. A preliminary bivariate analysis was carried out to analyze the farmer-BMC and BMC-EU prices for the year 1992 to 2018. The analysis revealed the presence of a white noise process in the difference between the two prices. As a result, the farmer- BMC price was modeled using the Geometric mean reversion process and the BMC-EU price was modeled using a pure stochastic process. The solution of the model formulated is not unique, so the conditions on the drift coefficient that ensure the existence of a unique solution were discussed. A positivity result for the model was stated and proved using a Lyapunov function. The stability of stochastic differential equations was discussed. In particular, different cases had been considered which include no volatility in the model, no volatility in the farmer-BMC model, no volatility in the BMC-EU model and volatility in the model.

CHAPTER 5

RESULTS

In this chapter, we present the results of all the special cases presented in section 4.3.1. We explore the impact of changes in level of noises (changes in σ_1 and σ_2) on, among other things , the stability of the steady states and its implications.

Figure 5.1(a), shows the Botswana beef-cattle prices for the case $\sigma_1 = 0, \quad \sigma_2 = 0$. The price x_1 increases gradually to the mean level x_2, and stays at that level in agreement with lemma 2. Figure 5.1(b), shows the result for $\sigma_1 = 0$, $\sigma_2 \neq 0$. The price, x_1, increases to a fluctuating mean level x_2. The equilibrium point is not a constant unlike the situation in figure 5.1(a), because the noise degrades the stability (this is quasi-stable). Note that this case represents a scenario when the volatility is very low. Figures 5.1(c) and 5.1(d) show that for as long as σ_1 remains zero, increasing the volatility σ_2 increases the magnitude of the fluctuations in both x_1 and x_2, but the two prices remain close to one another. Quasi stability of x_1 and x_2 is still maintained.

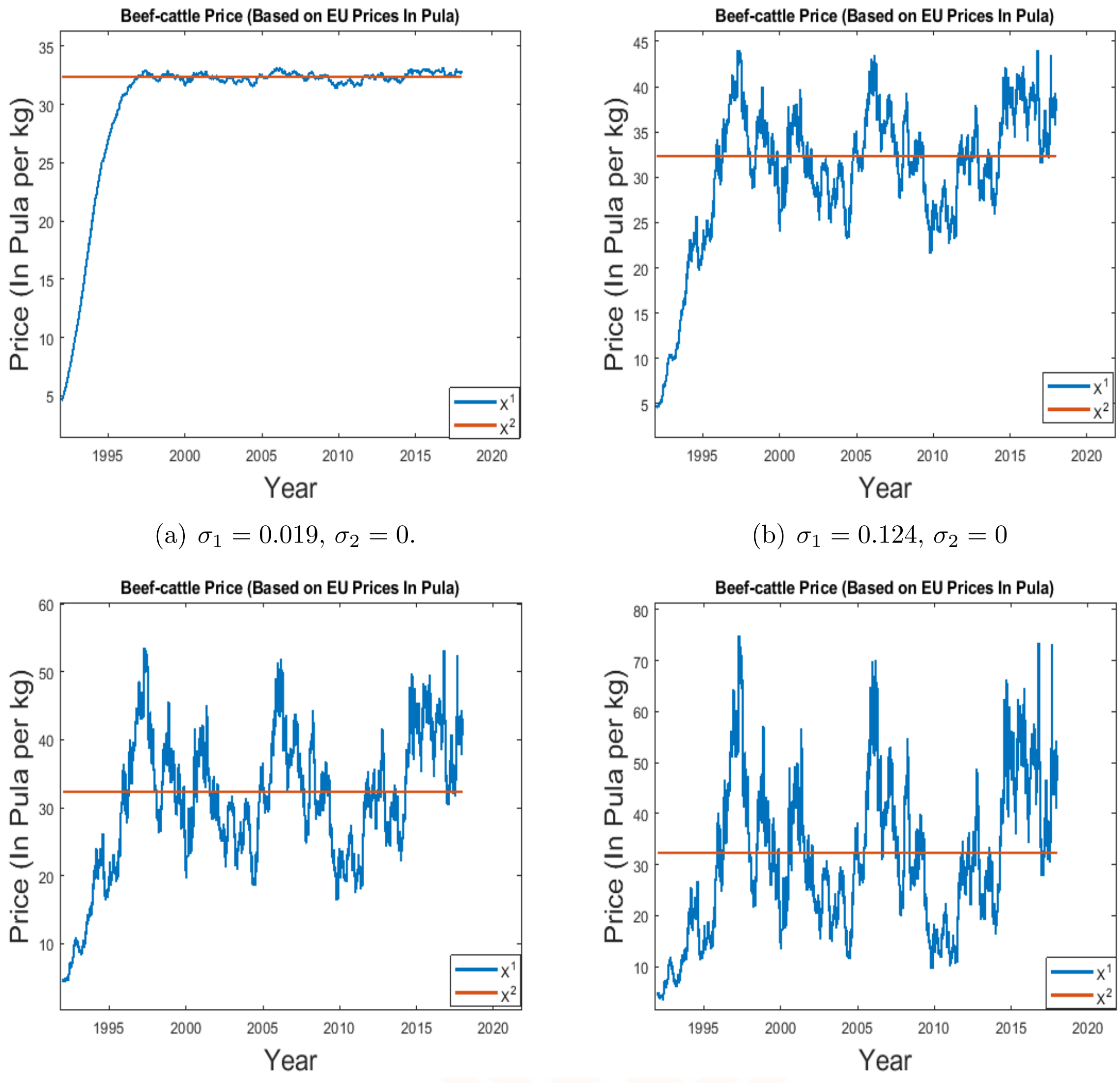


(a) $\sigma_1 = 0$, $\sigma_2 = 0$ (b) $\sigma_1 = 0$, $\sigma_2 = 0.021$

(c) $\sigma_1 = 0$, $\sigma_2 = 0.1$ (d) $\sigma_1 = 0$, $\sigma_2 = 0.3$

Figure 5.1: Botswana beef-cattle prices following a deterministic GMR

Figure 5.2 shows a scenario when the volatility of the price x_1 is varied and the volatility of x_2 remains zero. Figure 5.2(a) shows that the price x_1 increases and fluctuates about the constant price x_2. The fluctuations are very small for small values of σ_1. This represents a stable equilibrium point which is degraded by the noise.



(a) $\sigma_1 = 0.019$, $\sigma_2 = 0$.

(b) $\sigma_1 = 0.124$, $\sigma_2 = 0$

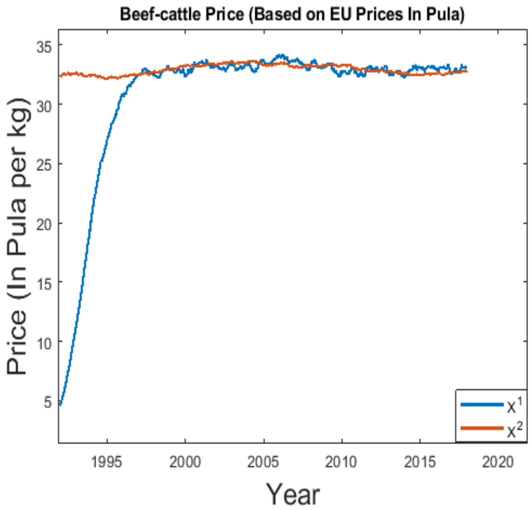
(c) $\sigma_1 = 0.4$, $\sigma_2 = 0$

(d) $\sigma_1 = 0.7$, $\sigma_2 = 0$

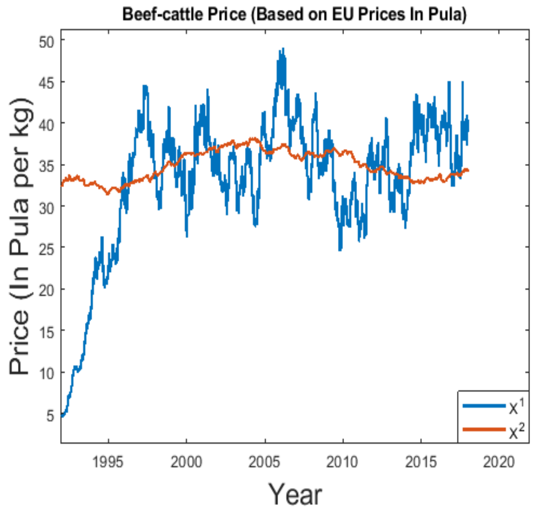
Figure 5.2: Botswana beef-cattle prices following Stochastic GMR and mean-reversion level

As we increase the volatility σ_1 the fluctuations about a constant mean increase significantly as depicted in figures 5.2(b), 5.2(c) and 5.2(d). Economically, an investor receiving a price x_1 subjected to a constant mean x_2 is subjected to a fair market with non-arbitrage conditions as one is equally subjected to price increase as well as price decrease about the mean.

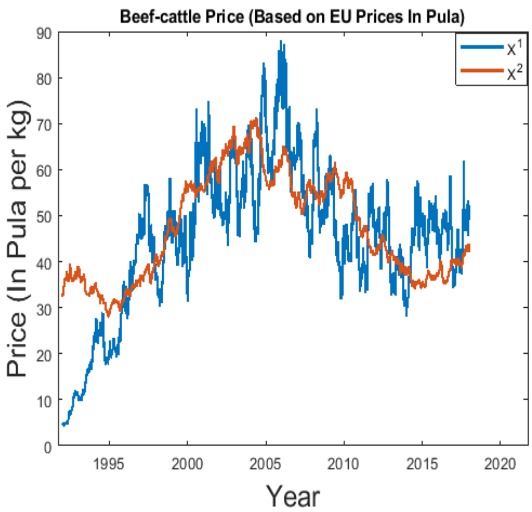
Figure 5.3 presents a scenario when both volatilities σ_1 and σ_2 are non-zero. It can be seen that if both volatilities are non-zero then both price processes fluctuate. Like the scenario in figure 5.2 there are no arbitrage opportunities as the price x_1 is as much above x_2 as it is below. However, if both noises (σ_1 and σ_2) are large (figure 5.3(d)) even if the difference between the volatilities is large the two prices do not differ significantly. The advantage between the prices, depicted by figures 5.3(c) and 5.3(d) vanishes as the two price processes seem to overlap.



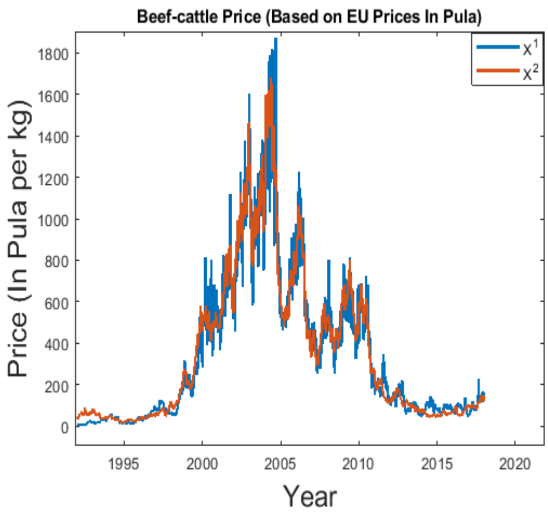
(a) $\sigma_1 = 0.019$, $\sigma_2 = 0.005$



(b) $\sigma_1 = 0.124$, $\sigma_2 = 0.021$



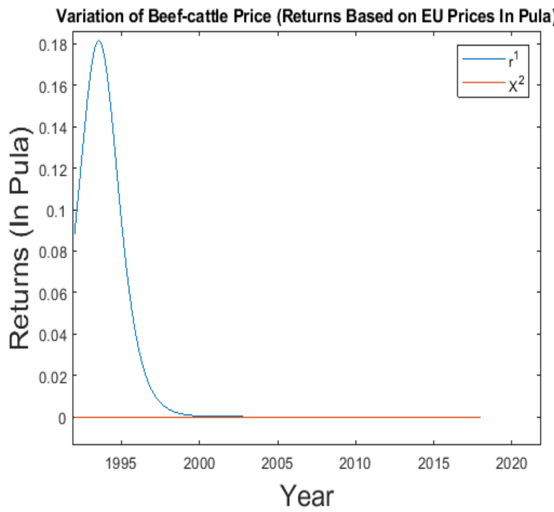
(c) $\sigma_1 = 0.1$, $\sigma_2 = 0.4$



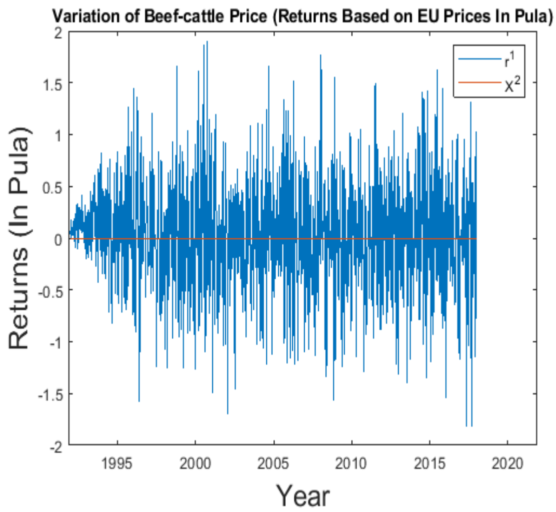
(d) $\sigma_1 = 0.5$, $\sigma_2 = 0.9$

Figure 5.3: Botswana beef-cattle prices following Stochastic GMR and mean-reversion level for varying σ_1 and σ_2

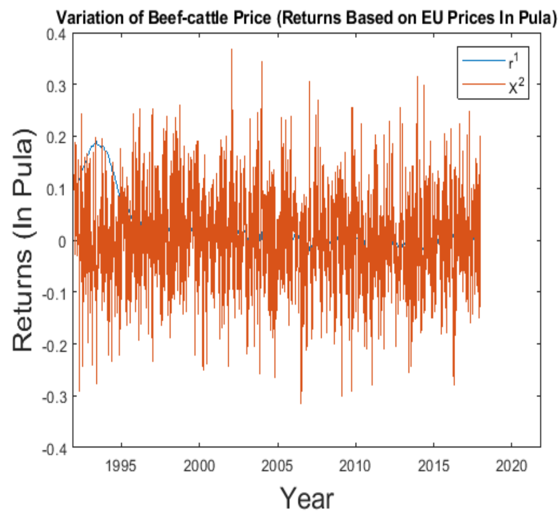
Figure 5.4 shows the returns (r_1) on the price x_1 for varying values of σ_1 and σ_2. Figure 5.4(a) represents a scenario when both σ_1 and σ_2 are zero.



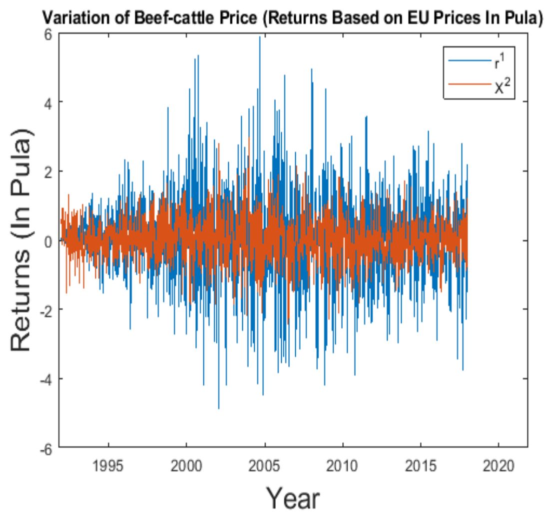
(a) $\sigma_1 = 0$, $\sigma_2 = 0$



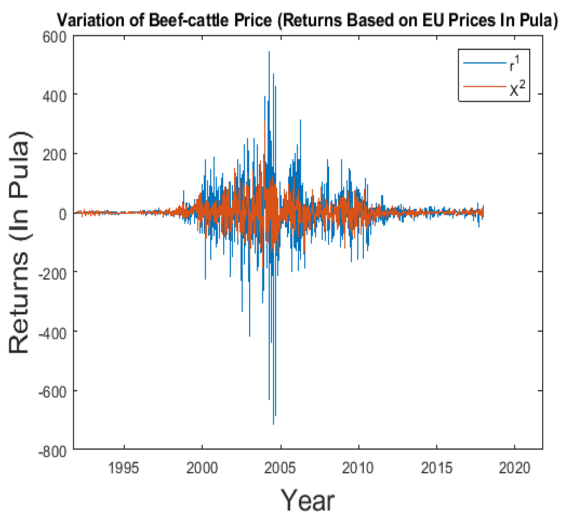
(b) $\sigma_1 = 0.124$, $\sigma_2 = 0$



(c) $\sigma_1 = 0$, $\sigma_2 = 0.021$



(d) $\sigma_1 = 0.2$, $\sigma_2 = 0.1$



(e) $\sigma_1 = 0.7$, $\sigma_2 = 0.5$

Figure 5.4: The variation of beef-cattle price (returns)

The mean price x_2 remains constant (assumed to be zero in this case) but r_1 increases initially to a peak but declines to zero and remains zero. In the long run if there is no noise, there is no benefit from the sell of cattle but more importantly the BMC is subjected to no growth in returns. Figure 5.4(b) shows that when there is noise in the BMC price then the cattle price will fluctuate about the BMC beef price which remains constant. Farmers can now gain or loose as the cattle price is above the beef price as it is below. The most ideal scenarios are given by figures 5.4(d) and 5.4(e) where both the cattle price and the beef price fluctuate and consequently removing any arbitrage opportunities.

Summary

In this chapter, numerical results were presented which shows the impact of varying the volatility in the model. The preferred scenario for the farmer is when the EU mean price is kept constant. However, economically, the most ideal scenarios are when both prices fluctuate which eliminate any arbitrage opportunities.

CHAPTER 6

CONCLUSION AND DISCUSSION

We have shown through a bivariate analysis in section 4.1.2 that the price the EU offers to the BMC and the price the BMC offers to the cattle farmers are highly correlated with the difference between them explained by a white noise process. This suggests that the middle agent, in this case the BMC, is left with insignificant amounts to support their operations such as maintenance and salaries. Based on this conclusion, we formulated a stochastic model of the Ornstein-Uhlenbeck type with the BMC-EU price as the stochastic mean of the Farmer-BMC price. We have shown that when both volatilities are zero ($\sigma_1 = \sigma_2 = 0$) the Farmer-BMC price increases to the mean price and remains at that level. This situation is of course uneconomical for BMC as they operate as an intermediary agent which is passing on to the farmer everything received from selling beef to the EU. BMC would obviously not manage to maintain their operations as they would have no capital to service their processing plant and pay salaries. BMC has in recent years run into operational problems, with operations subsidized from government handouts. The situation described for $\sigma_1 = \sigma_2 = 0$ could partly explain some of the difficulties the company is subjected to.

When the volatility for the farmers' price is kept at $\sigma_1 = 0$, but the volatility σ_2 is nonzero, the farmers' price fluctuates about the EU mean price. There is then a quasi but variable stability in the sense that the prices $x_1(t)$ and $x_2(t)$ remain close. However, when the volatility σ_2 grows large the difference in the two prices vanishes. This situation is again not conducive to the BMC. When the EU mean price is kept constant (by taking the volatility equal to zero) but the farmer price varies by increasing the volatility of the farmer price σ_1, there is a big difference between the farmer price and the EU price. We believe that this is the preferred scenario for the farmer. It can be achieved by among other things, increasing and diversifying the number of beef consumers and cattle buyers locally and regionally. The farmers should look for local and regional buyers of their cattle instead of relying on the BMC. The BMC, in turn, should source regional markets for their beef instead of relying on the EU. Sub-Saharan Africa is a large market comprising of over 1 billion consumers. This is the market that BMC should target. We believe that government intervention is one of the reasons the price that the BMC offers to the farmers is kept artificially high because the government is trying to eradicate poverty by artificially keeping the price of cattle high without due consideration of what price the EU is offering the BMC.

REFERENCES

[1] Christopher Stevens and Jane Kennan. Botswana beef exports and trade policy. 2005.

[2] Isabelle Piot-Lepetit and Robert M'Barek. Methods to analyse agricultural commodity price volatility. In *Methods to analyse agricultural commodity price volatility*, pages 1–11. Springer, 2011.

[3] Helyette Geman. *Commodities and commodity derivatives: modeling and pricing for agriculturals, metals and energy*. John Wiley & Sons, 2009.

[4] Kanar Dizyee, Derek Baker, and Karl M Rich. A quantitative value chain analysis of policy options for the beef sector in botswana. *Agricultural Systems*, 156:13–24, 2017.

[5] Keith Jefferis. Price responsiveness of cattle supply in botswana. *Policy briefing paper. Southern Africa Global Competitiveness Hub, Gaborone. Available at: http://www. econsult. co. bw/tempex/Price Responsiveness of Cattle Supply in Botswana. pdf*, 2007.

[6] Dikgang Stephen Gosalamang. *Econometric analysis of supply response among beef farmers in Botswana*. PhD thesis, University of Limpopo (Turfloop Campus), 2010.

[7] Statistics Botswana. Annual agricultural survey report 2017. january 2017. *Statistics Botswana. Gaborone, Botswana*, 2017.

[8] Botswana - meat exports quantity. https://knoema.com/atlas/Botswana/topics/Agriculture/Trade-Export-Quantity/Meat-exports-quantity. Accessed: 2018-02-06.

[9] Monika Tothova. Main challenges of price volatility in agricultural commodity markets. In *Methods to analyse agricultural commodity price volatility*, pages 13–29. Springer, 2011.

[10] Eduardo S Schwartz. The stochastic behavior of commodity prices: Implications for valuation and hedging. *The journal of finance*, 52(3):923–973, 1997.

[11] Eduardo Schwartz and James E Smith. Short-term variations and long-term dynamics in commodity prices. *Management Science*, 46(7):893–911, 2000.

[12] Jaime Casassus and Pierre Collin-Dufresne. Stochastic convenience yield implied from commodity futures and interest rates. *The Journal of Finance*, 60(5):2283–2331, 2005.

[13] Hendrik Bessembinder, Jay F Coughenour, Paul J Seguin, and Margaret Monroe Smoller. Mean reversion in equilibrium asset prices: Evidence from the futures term structure. *The Journal of Finance*, 50(1):361–375, 1995.

[14] Robert S Pindyck. The dynamics of commodity spot and futures markets: a primer. *The energy journal*, 22(3), 2001.

[15] Bryan R Routledge, Duane J Seppi, and Chester S Spatt. Equilibrium forward curves for commodities. *The Journal of Finance*, 55(3):1297–1338, 2000.

[16] Avinash K Dixit and Robert S Pindyck. Vinvestment under uncertaintyv. *V xiv*, 468, 1994.

[17] Loftus O Ndzinge, John M Marsh, and R Clyde Greer. Herd inventory and slaughter supply response of botswana beef cattle producers. *Journal of Agricultural Economics*, 35(1):97–107, 1984.

[18] Michael Hubbard and J Stephen Morrison. Current issues in cattle pricing and marketing in botswana. 1985.

[19] Kutlwano Mulale. The challenges to sustainable beef production in botswana: implications on rangeland management. In *17th Symposium of the Farming Systems Association*, 2000.

[20] Anton van Engelen, Patrick Malope, John Keyser, and David Neven. Botswana agrifood value chain project: beef value chain study. *FAO: Rome, Italy*, 2013.

[21] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[22] Hans Föllmer, Alexander Schied, et al. Probabilistic aspects of finance. *Bernoulli*, 19(4):1306–1326, 2013.

[23] Steven E Shreve. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer Science & Business Media, 2004.

[24] Achim Klenke. *Probability theory: a comprehensive course*. Springer Science & Business Media, 2013.

[25] Michael J Steele. *Stochastic calculus and financial applications*, volume 45. Springer Science & Business Media, 2012.

[26] Kiyosi Itô. *On stochastic differential equations*, volume 4. American Mathematical Soc., 1951.

[27] Pirrong Craig. Cambridge Unversity Press, 2011.

[28] Louis Jean-Baptiste Alphonse Bachelier. Theorie de la speculation, gauthier-villars, paris, w: P. cootner, the random character of stock market prices, 1900.

[29] Paul A Samuelson. Economic theory and mathematics–an appraisal. *The American Economic Review*, 42(2):56–66, 1952.

[30] Robert de Montessus. Sur les fractions continues algébriques. *Bulletin de la Société Mathématique de France*, 30:28–36, 1902.

[31] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.

[32] Michael J Wichura. Some strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments. *The Annals of Probability*, pages 272–296, 1973.

[33] Nattiya Chaiyapo and Nattakorn Phewchean. An application of ornstein-uhlenbeck process to commodity pricing in thailand. *Advances in Difference Equations*, 2017(1):179, 2017.

[34] George E Uhlenbeck and Leonard S Ornstein. On the theory of the brownian motion. *Physical review*, 36(5):823, 1930.

[35] Rajna Gibson and Eduardo S Schwartz. Stochastic convenience yield and the pricing of oil contingent claims. *The Journal of Finance*, 45(3):959–976, 1990.

[36] Christian-Oliver Ewald and Zhaojun Yang. Geometric mean reversion: formulas for the equilibrium density and analytic moment matching. *Available at SSRN 999561*, 2007.

[37] Christian-Oliver Ewald and Zhaojun Yang. Utility based pricing and exercising of real options under geometric mean reversion and risk aversion toward idiosyncratic risk. *Mathematical Methods of Operations Research*, 68(1):97–123, 2008.

[38] Shijie Deng. *Stochastic models of energy commodity prices and their applications: Mean-reversion with jumps and spikes.* University of California Energy Institute Berkeley, 2000.

[39] Oldrich A Vasicek. Credit valuation, 1984.

[40] John C Cox, Jonathan E Ingersoll Jr, and Stephen A Ross. An intertemporal general equilibrium model of asset prices. *Econometrica: Journal of the Econometric Society*, pages 363–384, 1985.

[41] Hao Jiang, Hui Liu, Linlin Wu, and Haixiang Xu. Literature review of power system stochastic stability. In *2016 International Conference on Probabilistic Methods Applied to Power Systems (PMAPS)*, pages 1–4. IEEE, 2016.

[42] Rafail Khasminskii. *Stochastic stability of differential equations*, volume 66. Springer Science & Business Media, 2011.

[43] Manalebish Debalike Asfaw, Semu Mitiku Kassa, and Edward M Lungu. Stochastic plant–herbivore interaction model with allee effect. *Journal of Mathematical Biology*, 79(6-7):2183–2209, 2019.

[44] Xavier Irz and Colin Thirtle. Dual technological development in botswana agriculture: A stochastic input distance function approach. *Journal of Agricultural Economics*, 55(3):455–478, 2004.

[45] Phillip K Thornton, Jeannette van de Steeg, An Notenbaert, and Mario Herrero. The impacts of climate change on livestock and livestock systems in developing countries: A review of what we know and what we need to know. *Agricultural systems*, 101(3):113–127, 2009.

[46] Statistics Botswana. Botswana environment statistics. *Water and Climate Digest*, 2017.

[47] Bmc-lobatse. https://bmc.bw/. Accessed: 2019-06-27.

[48] Benjamin K Acquah, Karabo Dudu Modise, et al. The impact of foot and mouth disease on the beef industry in botswana: The case of tonota village. *American Scientific Research Journal for Engineering, Technology, and Sciences (ASRJETS)*, 39(1):105–125, 2018.

[49] Joshua Aizenman and Brian Pinto. *Managing economic volatility and crises: A practitioner's guide*. Cambridge University Press, 2005.

[50] James AH Campbell. Financing diamond projects. *Journal of the Southern African Institute of Mining and Metallurgy*, 119(2):139–147, 2019.

[51] Roman Grynberg. Coal exports and the diversification of botswana's economy. 2012.

[52] Peter Joyce. *This is Botswana*. Penguin Random House South Africa, 2016.

[53] Paul A Samuelson. Spatial price equilibrium and linear programming. *The American economic review*, 42(3):283–303, 1952.

[54] Christopher B Barrett and Jau Rong Li. Distinguishing between equilibrium and integration in spatial price analysis. *American Journal of Agricultural Economics*, 84(2):292–307, 2002.

[55] Rodney Tyers Tyers and Kym Anderson. *Disarray in world food markets: A quantitative assessment*. cambridge University press, 1992.

[56] Piero Conforti. Price transmission in selected agricultural markets. *FAO Commodity and trade policy research working paper*, 7, 2004.

[57] Paul L Fackler and Barry K Goodwin. Spatial price analysis. *Handbook of agricultural economics*, 1:971–1024, 2001.

[58] Stanley Mishkin, Frederic. The economics of money, banking, and financial markets. *Mishkin Frederic–Addison Wesley Longman*, 2004.

[59] John Hull et al. *Options, futures and other derivatives/John C. Hull*. Upper Saddle River, NJ: Prentice Hall,, 2009.

[60] Bernt Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Universitext. Springer Berlin Heidelberg, 2010.

[61] Hans Föllmer, Alexander Schied, and Terry J Lyons. Stochastic finance. an introduction in discrete time. *The Mathematical Intelligencer*, 26(4):67–68, 2004.

[62] Carlos Góes and Troy Matheson. Domestic market integration and the law of one price in brazil. *Applied Economics Letters*, 24(5):284–288, 2017.

[63] Jyotish Prakash Basu and Soumyananda Dinda. Market integration: An application of error correction model to potato market in hooghly district, west bengal. *Indian Journal of Agricultural Economics*, 58(4):742–751, 2003.

[64] Barry K Goodwin and Ted C Schroeder. Cointegration tests and spatial price linkages in regional cattle markets. *American Journal of Agricultural Economics*, 73(2):452–464, 1991.

[65] George Box. Box and jenkins: time series analysis, forecasting and control. In *A Very British Affair*, pages 161–215. Springer, 2013.

[66] George Box and Gwilym Jenkins. Time series analysis-forecasting and control. san francisco: Holden day. 553 p. 1970.

[67] Thomas C Gard. *Introduction to Stochastic Differential Equations. Monographs and Text-books in pure and applied mathematics*. Dekker, Inc, 1988.

[68] European commission, beef prices. https://ec.europa.eu/info/food-farming-fisheries/farming/facts-and-figures/markets/overviews/market-observatories/meat/beef-statistics_en. Accessed: 2018-05-06.

[69] Euro Botswana pula exchange rate history (EUR BWP). https://currencies.zone/historic/euro/botswana-pula. Accessed: 2018-10-12.

[70] Dessislava A Pachamanova and Frank J Fabozzi. Recent trends in equity portfolio-construction analytics. *The Journal of Portfolio Management*, 40(3):137–151, 2014.

[71] Gabriel A Costa Lima and Saul B Suslick. Estimation of volatility of selected oil production projects. *Journal of petroleum science and engineering*, 54(3-4):129–139, 2006.

[72] Emmanuel Gobet. Introduction to stochastic calculus and to the resolution of pdes using monte carlo simulations-lectures notes of xv spanish-french school on numerical simulation in physics and engineering. 2012.

[73] Eric Chin, Dian Nel, et al. *Problems and Solutions in Mathematical Finance: Stochastic Calculus*. John Wiley & Sons, 2014.

APPENDIX A

BASIC CALCULUS

A.1 Taylor Series

Consider an analytical function $f(x)$, then for small $\triangle$,

$$\begin{aligned} f(x_0+\triangle) &= f(x_0)+\frac{d}{dx}f(x_0)\triangle+\frac{1}{2!}\frac{d^2}{dx^2}f(x_0)\triangle^2+\frac{1}{3!}\frac{d^3}{dx^3}f(x_0)\triangle^3+\cdots \\ &= \sum_{n-0}^{\infty}\frac{d^n}{dx^n}f(x_0)(x-x_0). \end{aligned} \tag{A.1}$$

if $f(x,y)$ is an analytical function of x and y, then for small $\triangle x$ and $\triangle y$,

$$\begin{aligned} f(x_0+\triangle x, y_0+\triangle y) &= f(x-0,y_0)+f_x(x_0,y_0)\triangle x+f_y(x_0,y_0)\triangle y \\ &+ \frac{1}{2!}[f_{xx}(x_0,y_0)(\triangle x)^2+2f_{xy}(x_0,y_0)\triangle x\triangle y+f_{yy}(\triangle y)^2] \\ &+ \frac{1}{3!}[f_{xxx}(x_0,y_0)(\triangle x)^3+3f_{xxy}(x_0,y_0)(\triangle x)^2\triangle y \\ &+ 3f_{xyy}(x_0,y_0)\triangle x(\triangle y)^2+f_{yyy}(x_0,y_0)(\triangle y)^3]+\cdots \end{aligned} \tag{A.2}$$

A.2 Integrals

If $g(x)$ is the derivative of $G(x)$, then (A.3) is an indefinite integral,

$$\int g(x)dx = G(x)+c,\ c\,is\,a\,constant, \tag{A.3}$$

where $G'(x)=\frac{d}{dx}G(x)=g(x)$. If $g(x)$ is continuous function then, $\frac{d}{dx}\int g(x)dx = g(x)$. If $g(x)$ is the derivative of $G(x)$ on a closed interval $[a,b]$ then,

$$\int_a^b g(x) = -\int_b^a g(x) = G(b)-G(a) \tag{A.4}$$

where $G'(x)=\frac{d}{dx}G(x)=g(x)$. Equation (A.4) is called a definite integral.

Suppose $f(x)$ and $g(x)$ are integrable functions on the interval $[a, b]$ then,

1. $\int_a^a f(x) = 0$,
2. $\int_a^b [\lambda f(x) + \rho g(x)]dx = \lambda \int_a^b f(x)dx + \rho \int_a^b g(x)dx$,
3. $\int_a^c g(x)dx = \int_a^b g(x)dx + \int_b^c g(x)dx$, $b \in [a, c]$.

Now lets consider two basic integration techniques, namely integration by parts (equation (A.5)) and integration by substitution (equation (A.6)).

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx \tag{A.5}$$

where, $f'(x) = \frac{d}{dx}f(x)$ and $g'(x) = \frac{d}{dx}g(x)$. Now if $u(x)$ is a continuous function and $v'(x)$ is continuous on $[a, b]$, then (A.6) is the integration by substitution.

$$\int_{v(b)}^{v(a)} u(x)dx = \int_a^b u(v(z))v'(z)dz. \tag{A.6}$$

Suppose $a(x)$ and $b(x)$ are continuous functions of x, and $g(t)$ is a continuous function then the derivatives of definite integrals are given by,

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(t)dt = g(b(x))\frac{d}{dx}b(x) - g(a(x))\frac{d}{dx}a(x). \tag{A.7}$$

if $f(x)$ is a differentiable function, then

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x, b(x))\frac{d}{dx}b(x) - f(x, a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}dt \tag{A.8}$$

APPENDIX B

PROBABILITY CONCEPTS

B.1 Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A : \Omega \to \mathbb{R}^n$ be a random variable such that $\mathbb{E}[|A|] < \infty$. Assume that the random variable is integrable i.e.

$$\int_{\Omega} |A(\omega)| \cdot d\mathbb{P}(\omega) < \infty.$$

A very important quantity is the expectation of A.

Definition 16. *The expectation of A, ($\mathbb{E}[A]$) with respect to the probability measure $\mathbb{P}$ is given by*

$$\mathbb{E}[A] = \int_{\Omega} |A(\omega)| \cdot d\mathbb{P}(\omega) = \int_{\mathbb{R}^n} a \cdot d\mathbb{P}_A(a) \tag{B.1}$$

If $\mathcal{B} \subset \mathcal{F}$ is a sub σ-algebra, then the conditional expectation of A given by $\mathcal{B}$, denoted by $\mathbb{E}[A|\mathcal{B}]$, is defined by

Definition 17. $\mathbb{E}[A|\mathcal{B}]$ *is the (a.s. unique) function from $\Omega \to \mathbb{R}^n$ satisfying :*

1. $\mathbb{E}[A|\mathcal{B}]$ *is $\mathcal{B}$-measurable*
2. $\int_B \mathbb{E}[A|\mathcal{B}]d\mathbb{P} = \int_B Ad\mathbb{P}$, *for all $B \in \mathcal{B}$*

The existence and uniqueness of conditional expectation comes from the Radon-Nikodyn theorem, (see Appendix A of Øksendal 6^{th} edition [60] for details).

B.2 The Multivariate Normal Distribution

let $\{X_n\}_{n\in\mathbb{N}}$ be a set of random variables where $\mathbb{E}[X_i] = \mu_i$, $Cov[X_i, X_j] = v_{ij}$ for $i, j = 1 \cdots n$. Let $\mu = (\mu_1 \cdots \mu_n)'$ and $V = v_{ij}$ be a $n \times n$ covariance matrix. We will assume that V is non-singular. X has a multivariate Normal distribution if it has the density function

$$f(x) = (2\pi)^{-\frac{n}{2}}|V|^{-\frac{1}{2}}exp[-\frac{1}{2}(x-\mu)'V^{-1}(x-\mu)].$$

This is sometimes written as $X \sim \mathcal{MVN}(\mu, V)$

B.3 Independence of Brownian Motion Increments

A standard Brownian motion B_t has independent increments, i.e.

$$B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}, \cdots, B_{t_j} - B_{t_{j-1}},$$

are independent for all $0 \leq t_1 < t_2 < \cdots < t_i < \cdots < t_j$.

Proof

To prove this we use the fact that normal random variables are independent if and only if they are uncorrelated.
$E^x[(B_{t_i} - B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})]$ for $t_i < t_j$
$= E^x[B_{t_i}B_{t_j} - B_{t_i}B_{t_{j-1}} - B_{t_{i-1}}B_{t_j} + B_{t_{i-1}}B_{t_{j-1}}]$
$= nt_i - nt_i - nt_{i-1} + nt_{i-1}$
$= 0$
It then follows that for any $0 \leq s < t < \infty$, the increment $B_t - B_s$ is independent of $\mathcal{F}_t$.

A Brownian motion is a Gaussian process. A stochastic process $Z(t)$ is a Gaussian process if a finite dimensional $(Z(t_1), Z(t_2), ..., Z(t_n))$ is multivariate normal (Gaussian) distributed for any $0 \leq t_1 < t_2 < ... < t_n$.

Example 4. *One of the first applications of the Brownian motion was proposed by Bachelier [28]. Bachelier used the Brownian motion to describe the evolution of prices on Paris stock exchange. Bachelier assumed that the infinitesimal price increments dX_t of a financial asset are proportional to the increments dB_t of a standard Brownian motion, that is, $dX_t = \sigma dB_t$. Starting at an initial value $X_0 = x$, the value of the process at time t is $X_t = x + \sigma Bt$.*

A major drawback of this specification is that the price has a non zero probability of getting negative. In order to solve this issue, we rather model the relative increments with respect to the prices (returns) as a standard Brownian motion, that is, $\frac{dX_t}{X_t} = \sigma dB_t$ or equivalently $dX_t = \sigma X_t dB_t$. The second expression looks like a differential equation. However, there are two difficulties:

- the variables in the equation are stochastic.
- the sample paths of B_t are not differentiable even though they are continuous.

The mathematical solution to this problem was found by Itô in the 1940's using a new kind of integral: the stochastic integral. In particular, it allows to write $X_t = x + \sigma \int_t^0 X_s dB_s$ where we integrate with respect to the random element B.

B.4 Law of the iterated logarithm

Let $\{X_n\}_{n\in\mathbb{N}} \sim \mathcal{N}(0,1)$ and let $S_n = X_1 + \cdots + X_n$. Then,

$$\lim_{n\to\infty} sup \frac{\pm S_n}{\sqrt{2nloglogn}} = 1,\ a.s. \tag{B.2}$$

Where log is the natural logarithm, *lim sup* denotes the limit superior and *a.s.* stands for "almost surely" i.e. with probability 1, ($\mathbb{P}(x = x_1) = 1$). The law of iterated logarithm operates in between the law of large numbers, the weak and the strong and they both state that the sum S_n scaled by n^{-1} converges to zero respectively in probability and almost surely:

$$\frac{S_n}{n} \xrightarrow{\mathbb{P}} 0,\ \frac{S_n}{n} \xrightarrow{a.s.} 0,\ as\ n \to \infty.$$

On the other hand, the central limit theorem states that the sum S_n scaled by the factor $n^{-\frac{1}{2}}$ converges in distribution to standard normal distribution. By Kolmogorov's zero-one law, for any fixed M, the probability that the event,

$$lim_n sup \frac{S_n}{\sqrt{n}} \geq M$$

, occurs is 0 or 1. Then,

$$\mathbb{P}\left(\lim_n sup \frac{S_n}{\sqrt{n}} \geq M\right) \geq \lim_n sup \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq M\right) \tag{B.3}$$

$$= \mathbb{P}(\mathcal{N}(0,1) \geq M) > 0. \tag{B.4}$$

So,

$$\lim_n sup \frac{S_n}{\sqrt{n}} = \infty\ (a.s.)$$

. This implies that these quantities cannot converge with probability of 1, (a.s.). In fact, they cannot even converge in probability which follows from the equality

$$\frac{S_{2n}}{\sqrt{2n}} - \frac{S_n}{\sqrt{n}} = \frac{1}{\sqrt{2}}\frac{S_{2n} - S_n}{\sqrt{n}} - \left(1 - \frac{1}{\sqrt{2}}\right)\frac{S_n}{\sqrt{n}}$$

, and the fact that the random variables $\frac{S_n}{\sqrt{n}}$ and $\frac{S_{2n}-S_n}{\sqrt{n}}$ are independent and both converge in distribution to $\mathcal{N}(0,1)$. The law of the iterated logarithm provides the scaling factor where the two limits become different:

$$\frac{S_n}{\sqrt{2nloglogn}} \xrightarrow{\mathbb{P}} 0, \;\; \frac{S_n}{\sqrt{2nloglogn}} \overset{a.s.}{\nrightarrow} 0, \;\; as\ n \to \infty.$$

Thus, although the quantity $\left|\frac{S_n}{\sqrt{2nloglogn}}\right|$ is less than any predefined $\epsilon > 0$ with probability approaching one, the quantity will nevertheless be greater than ϵ infinitely often. In fact the quantity will be visiting the neighborhoods of any point in the interval $(-1, 1)$ almost surely.

APPENDIX C

BOTSWANA BEEF-CATTLE INDUSTRY

C.1 Farmer-BMC prices $S_1(t)$, BMC-EU prices $S_2(t)$ and their respective residuals

Year	$S_1(t)$	$S_1(t)-\overline{S}_1$	$S_2(t)$	$S_2(t)-\overline{S}_2$
1992	4.56	-7.85	7.06	-16.29
1993	4.17	-824	7.75	.15.61
1994	4.55	-7.86	7.85	-15.51
1995	4.62	-7.79	6.94	-16.42
1996	4.23	-8.18	11.56	-11.82
1997	4.17	-8.24	10.95	-12.41
1998	4.78	-7.63	11.19	-12.16
1999	4.65	-7.76	12.31	-11.04
2000	4.53	-7.88	11.25	-12.1
2001	4.45	-7.96	2.45	-10.91
2002	4.62	-7.97	14.32	-9.02
2003	4	-8.41	15.45	-7.9
2004	4.52	-7.89	16.2	-7.16
2005	6.71	-5.7	17.19	-6.17
2006	8.3	-4.11	20.89	-2.46
2007	12.13	-0.8	24.38	1.03
2008	13.51	1.1	29.34	5.98
2009	19.51	7.1	32.27	8.92
2010	24.23	11.82	30.67	7.32
2011	23.64	11.23	31.31	7.96
2012	14.4	1.99	37.56	14.21
2013	15.45	3.04	41.35	17.21
2014	29	16.59	45.64	22.29
2015	12.5	0.09	41.78	18.43
2016	33	20.59	45.9	22.54
2017	32.73	20.32	42.57	19.22
2018	32.2	19.79	44.45	21.1
Mean($\overline{S}=\frac{1}{n}\sum_{i=1}^{n}S_i$)	12.41			23.35

Table C.1: Farmer-BMC prices $S_1(t)$ and BMC-EU prices $S_2(t)$ for the years 1992 to 2018

Agriculture commodities are transnational products of which the Botswana Beef-cattle is not exceptional and are traded usually having historical data series. In particular, the future volatility rate σ from the system (4.21)-(4.22) is estimated from the historic volatility of beef cattle prices, based on the standard deviation of the time series [71], given by equation:

$$\sigma = \sqrt{\frac{1}{n-1}\sum (S(t) - \overline{S})^2} \tag{C.1}$$

The drift part of equation (4.21) describes the time evolution of beef-cattle prices the continually yield at a rate $\kappa(x_2 - x_1)x_1$, which is represented as a combination of mean-reversion force κ, the average of the EU historical beef prices (assumed to be stochastic) and the cattle prices (Farmer-BMC).

The sum of residuals for the two prices are;

1. Farmer-BMC price: $\sum_{i=1}^{n}(S_{1,i} - \overline{S}_1) = 0.00333$ and
2. BMC-EU price: $\sum_{i=1}^{n}(S_{2,i} - \overline{S}_2) = 0$

C.2 Plot of weekly beef prices based on the EU market

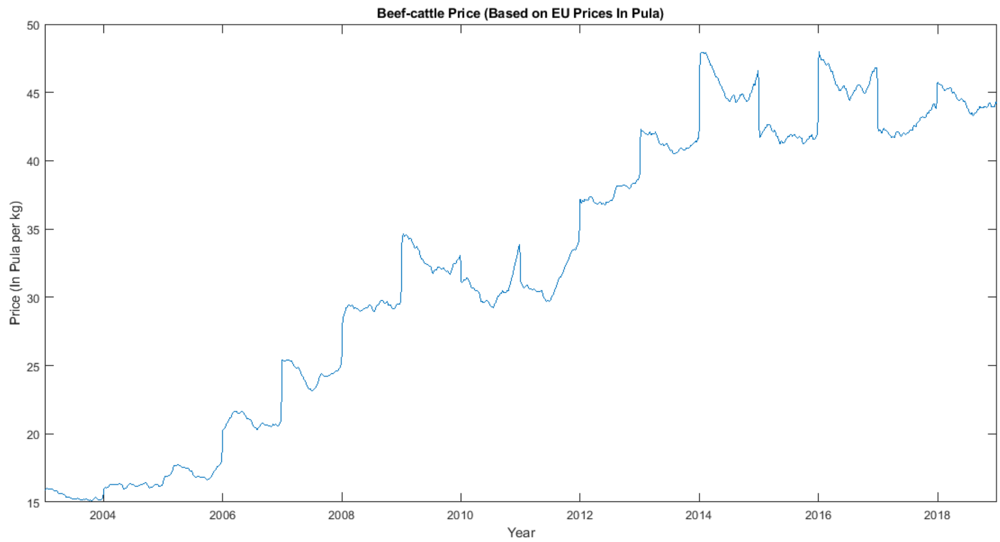


Figure C.1: European Union Beef Average Market Price

In figure C.1 are the weekly beef average market prices for the European Union as from January 2003 to December 2018. The figure depicts some changing levels and trends are neither linear nor quadratic. In order to calculate the initial value for x_2 (i.e. the mean), we sum all the observed prices starting from 2003 to 2018 and divide it by the number of observations n. The following formula was used,

$$x_2(0) \quad = \overline{x}_2 = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} x_{2,i} \tag{C.2}$$

The data used in this study consisted of $n = 835$ weekly observations and we obtained $x_2(0) = 32.35$. Agriculture commodities are transnational products of which the Botswana Beef-cattle is not exceptional and are traded usually having historical data series. In particular, the future volatility rate σ from the system (4.21)-(4.22) is estimated from the historic volatility of beef cattle prices, based on the standard deviation of the time series, given by equation:

$$\sigma = \sqrt{\frac{1}{n-1}\sum(S(t) - \overline{S})^2} \tag{C.3}$$

The drift part of equation (4.21) describes the time evolution of beef-cattle prices the continually yield at a rate $\kappa(x_2 - x_1)x_1$, which is represented as a combination of mean-reversion force κ, the average of the EU historical beef prices (assumed to be stochastic) and the cattle prices (Farmer-BMC).

In Table C.2 are the notations that were used in chapter 3.

Notation	Description
BWP	Botswana Pula
$S(t)$	Beef-cattle price at time t
$\overline{S}$	Average beef-cattle price
$S_1(t)$, $S_2(t)$	Farmer-BMC price and BMC-EU respectively price at time (for simplicity we have denoted by x_1 and x_2 respectively)
a_1, a_2	Respective residuals for S_1 and S_2
r	The Pearson's correlation coefficient
$r(t)$	Rate of return
r_1, r_2	Returns for x_1 and x_2 respectively
x'	Prime symbol (for derivative)
$\int$	Integral sign
$d(.)$	Differential sign
σ_1, σ_2	Rate of volatilities for Farmer-BMC price and BMC-EU respectively price
$\omega(B)$	Transfer function
$\bigtriangledown^d_{S_{1,t}}$, $\bigtriangledown_{S_{2,t}}$	Incremental changes for Farmer-BMC prices and BMC-EU prices respectively
$\mathbb{E}[.]$	Expectation operator
$T \wedge \tau_\epsilon$	Smaller between T and τ_ϵ, where ϵ is an arbitrary small number
$\beta_t = \phi_p(B)S_{1,t}$	Autoregressive process AR(p) of order p where, $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$
$\alpha_t = \theta_q(B)S_{2,t}$	Moving average MA(q) of order q where, $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$
$\triangle_{t,i} = t_i - t_{i-1}$	Time interval
$\mathbb{P}(.)$	Probability of (.)

Notations and symbols.

Table C.2: Table of notations and symbols

APPENDIX D

METHODS OF SOLVING STOCHASTIC DIFFERENTIAL EQUATIONS

D.1 Bachelier Model (Arithmetic Brownian Motion)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{B_t\}_{t\in[0,\infty]}$ be the standard Brownian Motion process. Then the stochastic differential equation (SDE) of the Arithmetic Brownian Motion is given by

$$dS_t = \alpha dt + \sigma dB_t. \tag{D.1}$$

Where, $B_0 = 0$ and $\alpha, \sigma > 0$ are constants satisfying certain conditions which essentially boils down to the requirement that they neither grow fast nor very too much [72]. Solving equation (D.1) is relatively straightforward as variables are nicely separated. Integrating equation (D.1) we have,

$$\begin{aligned} \int_0^t dS_s &= \int_0^t \alpha ds + \int_0^t \sigma dB_s \\ S_t - S_0 &= \alpha(t-0) + \sigma(B_t - B_0) \\ S_t &= S_0 + \alpha t + \sigma B_t. \end{aligned} \tag{D.2}$$

D.2 Geometric Brownian Motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{B_t\}_{t\in[0,\infty]}$ be the standard Brownian Motion process. Suppose S_t follows the GBM with stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma S_t dB_t, \tag{D.3}$$

where α and σ are constants. Equation (D.3) is also known as Black-Scholes model. To obtain the solution to equation (D.3) apply Itô's formula to $Y_t = lnS_t$ and taking integrals for $t \geq 0$.

Solution: Using Talyor's expansion and subsequently from Itô's formula we have

$$\begin{aligned} d(lnS_t) &= \frac{1}{2}dB_t - \frac{1}{S_t^2}(dS_t)^2 + \cdots \\ &= \alpha dt + \sigma dB_t - \frac{1}{2S_t^2}(\sigma^2 S_t^2 dt) \\ &= (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dB_t \end{aligned} \tag{D.4}$$

Taking integrals of equation (D.4) for $t \geq 0$,

$$\begin{aligned} \int_0^t d(lnS_s) &= \int_0^t (\alpha - \frac{1}{2}\sigma^2)ds + \int_0^t \sigma dB_s \\ ln(S_t) - ln(S_0) &= \int_0^t (\alpha - \frac{1}{2}\sigma^2)ds + \int_0^t \sigma dB_s \\ ln\frac{S_t}{S_0} &= \alpha_t - \frac{1}{2}\sigma^2 t + \sigma B_t, \quad B_0 = 0 \\ S_t &= S_0 exp\{\alpha t - \frac{1}{2}\sigma^2 t + \sigma B_t\}. \end{aligned} \tag{D.5}$$

D.3 Mean-Reverting

In this section we present the solutions for the Ornstein Uhlenbeck type process and Geometric mean-reverting process. For other versions of mean-reverting processes see the solution manual by Chin et .al [73].

Ornstein Uhlenbeck Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$be a probability space and let $\{B_t\}_{t\geq 0}$ be a standard Brownian Motion process. The Ornstein Uhlenbeck type process S_t is the solution to the stochastic differential equation

$$dS_t = \kappa(\theta - S_t)dt + \sigma dB_t, \tag{D.6}$$

where κ, θ and σ are constants as defined in chapter **??**. By applying $\hat{\text{I0}}$'s formula to $Y_t = e^{\kappa t}S_t$ and taking integrals we solve for S_t as follows:

Solution Using Taylor,s formula to expand $Y_t = e^{\kappa t}S_t$ and applying Itô,s formula we obtain

$$\begin{aligned} d(e^{\kappa t}S_t) &= \kappa e^{\kappa t}S_t dt + e^{\kappa t}dS_t + \frac{1}{2}\kappa^2 e^{\kappa t}(dS_t)^2 + \cdots \\ &= \kappa e^{\kappa t}S_t dt + e^{\kappa t}(\kappa(\theta - S_t))dt + \sigma dB_t) \\ &= \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}dB_t. \end{aligned} \tag{D.7}$$

Integrating equation (D.7) for $t \geq 0$ we have,

$$\begin{aligned}\int_0^t d(e^{\kappa s}S_s) &= \int_0^t \kappa\theta e^{\kappa s}ds + \int_0^t \sigma e^{\kappa s}dB_s \\ S_t &= S_t e^{-\kappa t} + \theta(1-e^{-\kappa t}) + \sigma\int_0^t e^{-\kappa(t-s)}dB_s. \end{aligned} \tag{D.8}$$

Geometric Mean Reverting Process Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion process. Suppose S_t follows the geometric mean-reverting process with stochastic differential equation

$$dS_t = \kappa(\theta - lnS_t)S_t d_t + \sigma S_t dB_t \tag{D.9}$$

where κ, θ and σ are constants. By applying Itô's formula to $Y_t = lnS_t$ equation (D.9) is reducible to an Ornstein Uhlenbeck type process.

Solution

By expanding $Y_t = lnS_t$ using Taylor's formula and subsequently applying Itô's formula, we have

$$\begin{aligned} d(lnS_t) &= \frac{1}{S_t}dS_t - \frac{1}{S_t^2}(dS_t)^2 + \cdots \\ &= \kappa(\theta - lnS_t)dt + \sigma dB_t - \frac{1}{2}dt \\ &= (\kappa(\theta - lnS_t) - \frac{1}{2}\sigma^2)dt + \sigma dB_t \\ dY_t &= (\kappa(\theta - Y_t) - \frac{1}{2}\sigma^2)dt + \sigma dBt. \end{aligned} \tag{D.10}$$

Using the same steps in solving the Ornstein Uhlenbeck type process, we apply Itô's formula on $Z_t = e^{\kappa Y_t}$ such that

$$\begin{aligned} d(e^{\kappa t}Y_t) &= \kappa e^{\kappa t}Y_t dt + e^{\kappa t}dY_t + \frac{1}{2}\kappa^2 e^{\kappa t}Y_t(dY_t)^2 + \cdots \\ &= \kappa e^{\kappa t}Y_t dt + e^{\kappa t}[(\kappa(\theta - Y_t) - \frac{1}{2}\sigma^2)dt + \sigma dB_t] \\ &= (\kappa\theta e^{\kappa t} - \frac{1}{2}\sigma^2 e^{\kappa t})dt + \sigma e^{\kappa t}dB_t \end{aligned} \tag{D.11}$$

Taking integrals of equation (D.11) from $0tot$ we have have

$$\begin{aligned} \int_0^t d(e^{\kappa s Y_s}) &= \int_0^t (\kappa\theta e^{\kappa s} - \frac{1}{2}\sigma^2 e^{\kappa s})ds + \int_0 \sigma e^{\kappa s}dB_s \\ Y_t &= Y_0 e - \kappa t + (\theta - \frac{\sigma^2}{2\kappa}(1 - e^{-\kappa t}) + \int_0^t \sigma e^{-\kappa(t-s)}dBs. \end{aligned} \tag{D.12}$$

Since $Y_t = ln(S_t)$, equation (D.12) becomes

$$\begin{aligned} ln(S_t) &= ln(S_0)e{-}\kappa t + (\theta - \frac{\sigma^2}{2\kappa}(1 - e^{-\kappa t}) + \int_0^t \sigma e^{-\kappa(t-s)} dBs \\ S_t &= exp\{ln(S_0)e{-}\kappa t + (\theta - \frac{\sigma^2}{2\kappa}(1 - e^{-\kappa t}) + \int_0^t \sigma e^{-\kappa(t-s)} dBs\} \end{aligned} \quad \text{(D.13)}$$

D.4 Change of Variables Method

The equation (4.37) in chapter 3 can be reduced to a standard Ornstein-Uhlenbeck process of the type,

$$dY_1(t) = \frac{\kappa}{\hat{\sigma}_1}(\hat{m} - Y_1)dt + \hat{\sigma}_1 d\hat{B}_1. \quad \text{(D.14)}$$

The Equation (4.37) can be solved by transforming it into linear using change of variables method (substitution) [21] and see Gard [67] for more details. We define a new stochastic process

$$Y(t) = h(t, X(t)), \quad \text{(D.15)}$$

which according to the Itô formula satisfies the stochastic differential equation,

$$dY(t) = f(t, Y(t))dt + g(t, Y(t))dB(t). \quad \text{(D.16)}$$

where, f and g are real valued functions. The equation (4.37) is reducible if there exists a function $h(t, X(t))$ satisfying,

$$\left[\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} g^2\right](t, x) = \overline{f}(t), \quad \text{(D.17)}$$

and

$$\left[\frac{\partial h}{\partial x} g\right](t, x) = \overline{g}(t), \quad \text{(D.18)}$$

$$\text{(D.19)}$$

or

$$\left[\frac{\partial h}{\partial x}\right](t,x) = \frac{\overline{g}}{g(t,x)} \tag{D.20}$$

Differentiating (D.17) with respect to x gives

$$\frac{\partial^2 h}{\partial x \partial t} + \frac{\partial}{\partial x}\left[\frac{\partial h}{\partial x} f + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} g^2\right] = 0 \tag{D.21}$$

Differentiating (D.20) with respect to t,

$$\frac{\partial^2 h}{\partial t \partial x} = \frac{g(t,x)\overline{g}^{'}(t) - \overline{g}(t)\left[\frac{\partial g}{\partial t}\right]}{g^2(t)}, \tag{D.22}$$

and we differentiate (D.20) with respect to t,

$$\frac{\partial^2 h}{\partial x^2} = -\frac{\overline{g}(t)\left(\frac{\partial g}{\partial x}\right)}{g^2(t,x)}. \tag{D.23}$$

Substituting (D.17), (D.21) and (D.22) into equation (D.20), we obtain

$$\frac{\overline{g}^{'}}{g} - \overline{g}\left[\frac{\frac{\partial g}{\partial t}}{g^2} - \frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \frac{1}{2}\frac{\partial^g}{\partial x^2}\right] = 0 \tag{D.24}$$

or

$$\overline{g}^{'} = g\overline{g}\left[\frac{\frac{\partial g}{\partial t}}{g^2} - \frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \frac{1}{2}\frac{\partial^g}{\partial x^2}\right] \tag{D.25}$$

Since the left hand side of (D.25) is independent of x it follows that,

$$\frac{\partial}{\partial x}\left[\frac{\frac{\partial g}{\partial t}}{g^2} - \frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \frac{1}{2}\frac{\partial^g}{\partial x^2}\right] = 0 \tag{D.26}$$

We argue as follows, if (D.26) holds , then $\overline{g} \neq 0$ can be computed from (D.20) since

$$g\left[\frac{\frac{\partial g}{\partial t}}{g^2} - \frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \frac{1}{2}\frac{\partial^g}{\partial x^2}\right] = constant \tag{D.27}$$

Therefore from (D.25), we have,

$$\begin{aligned} \frac{\partial \overline{g}}{\partial t} &= \eta \overline{g} \\ \Rightarrow ln\overline{g} &= \eta t \\ \overline{g} &= c_1 e^{\eta t} \end{aligned} \tag{D.28}$$

at $t = 0$, $c = c_0$, $\overline{g} = c_0 e^{\eta t}$. From equation (D.25), $\frac{\partial \overline{g}}{\partial t} = 0$, $\overline{g} = 0$.

$$\begin{aligned} A(x) &= \frac{f(x)}{g(x)} - \frac{1}{2} g"(x) \\ &= \frac{\kappa(m - x_1}{\sigma_1} - \frac{1}{2}\sigma_1 \\ &= \frac{\kappa}{\sigma_1}(m - x_1) - \frac{\sigma_1}{2} \end{aligned} \tag{D.29}$$