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LOOP SPACE HOMOLOGY OF ELLIPTIC SPACES

by

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Dedication

To the magnificent four,

Pelo, Botho, Oteng Jr and my late grandfather John.

Contents

Abstract

In this thesis, we use the theory of minimal Sullivan models in rational homotopy theory to study the partial computation of the Lie bracket structure of the string homology on a formal elliptic space. In the process, we show the total space of the unit sphere tangent bundle $S^{2m-1} \to E \xrightarrow{p} G_{k,n}(\mathbb{C})$ over complex Grassmannian manifolds $G_{k,n}(\mathbb{C})$ for $2 \leq k \leq n/2$, where $m = k(n-k)$ is not formal. This is done by exhibiting a non trivial Massey triple product. On the other hand, let $\phi : (\wedge V, d) \to (B, d)$ be a surjective morphism between commutative differential graded algebras, where V is finite dimensional, and consider (B, d) a module over $\wedge V$ via the mapping ϕ. We show that the Hochschild cohomology $HH^*(\wedge V; B)$ can be computed in terms of the graded vector space of positive ϕ-derivations.

Given a Koszul Sullivan extension $(\wedge V, d) \overset{f}{\rightarrowtail} (\wedge V \otimes \wedge W, d) = (C, d)$, we show that if $(\wedge V, d)$ is an elliptic 2-stage Postnikov tower Sullivan algebra, and if the natural homomorphism of the differential graded algebras $(C, d) \to (\wedge W, \bar{d})$ is surjective in homology, then the natural graded linear map $HH^*(f) : HH^*(\wedge V; \wedge V) \to HH^*(\wedge V; C)$, induced in Hochschild cohomology by the inclusion $(\wedge V, d) \overset{f}{\rightarrowtail} (C, d)$, is injective. In particular, if X is an elliptic 2-stage Postnikov tower, and $(\wedge V, d)$ is the minimal Sullivan model of X, then $HH^*(f) : \mathbb{H}_*(X^{S^1}; \mathbb{Q}) \to HH^*(\wedge V; C)$ is injective, where X^{S^1} is the space of free loops on X, and $\mathbb{H}_*(X^{S^1}; \mathbb{Q})$ is the loop space homology.

1 Introduction

1.1 Rational homotopy theory

We begin by reviewing some basic facts of rational homotopy and notations that will be used throughout. After that, we continue reviewing some topics that are more specific to this work. We also state problems and significance of the study.

In this thesis, all topological spaces should be of the rational homotopy type of simply connected CW-complexes of finite type. One of the main problems in topology is to understand when two topological spaces X and Y are similar or dissimilar. Intuitively, two topological spaces X and Y are similar if there are continuous maps $f : X \leftrightarrows Y : g$ such that both compositions are equal to the identities. In other words, f is an homeomorphism. The problem then is to describe the equivalence classes of spaces under homeomorphism, which is referred to as a *classification problem* (see (Gallier & Quaintance, 2016)). Classification of all topological spaces or continuous maps between them is a very difficult task (Dieudonné, 1989; Gallier & Quaintance, 2016). However, the reaction to this fundamental difficulty was the creation of algebraic topology, whose main role is to associate "algebraic invariants"to different types of spaces, so that homeomorphic spaces have "isomorphic"algebraic invariants. If two spaces X and Y happen to have some different algebraic invariant objects, then they are not homeomorphic. Typical examples of such algebraic invariants are the singular homology groups $H_i(X)$ and cohomology groups which are abelian groups arising from a possibly infinite sequence called a *chain complex* that is built from singular chains and cochains, and the homotopy groups $\pi_n(X)$. Of interest are the homotopy groups of spheres $\pi_n(S^k)$. Despite their well-known definition, the groups $\pi_n(S^k)$ are not easy to compute and most of them are still unknown. Although they are hard to compute, these algebraic invariants provide an accurate and deep understanding of the geometric and analytic behavior of topological spaces and the continuous maps between them. For example, the notion of a fibration where one of the main properties is the long exact sequence of homotopy groups which relates homotopy groups of different topological spaces.

On the other hand, in the rational homotopy setting, the computations of these algebraic invariants are simplified. The coarser rational homotopical classification is somewhat easier. According to (Félix, Halperin, & Thomas, 2001), rational homotopy theory is the study of algebraic invariants and properties of topological spaces X and continuous maps f that depend only on the rational homotopy type of the space and the homotopy class of the map. That is, one studies topological spaces with rational homotopy equivalences. Hence, the equivalent spaces will have equal algebraic invariants.

In this paragraph we recall some useful definitions and notations on continuous mappings. Let I denote the unit interval $[0,1]$. Given two maps $f,g : X \to Y$, they are said to be *homotopic,* denoted by $f \simeq g$ if there is a continuous map that is referred to as a homotopy $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Basically, a homotopy is a continuous one-parameter family of maps from X to Y. That is, we imagine a parameter t

as representing time, then f deforms continuously into g, as t goes from zero to one. A map $f : X \to Y$ is a *homotopy equivalence* if it has a homotopy inverse h, that is, if $h \circ f \simeq id_X$ and $f \circ h \simeq id_Y$. A map f is said to be *null-homotopic* if it is homotopic to a constant map, $f \simeq c$. A space X is *contractible* if it is homotopy equivalent to a point, that is, if the retraction $X \to *$ is homotopic to the identity. The *fundamental group* $\pi_1(X,x_0)$ is the group of homotopy classes of paths γ from $I \to X$ such that $\gamma(0) = \gamma(1) = x_0$. Given points x and y of the space X, a *path* in X from x to y is a continuous map $f : I \to X$ such that $f(0) = x$ and $f(1) = y$. A space X is said to be *path-connected* if every pair of points of X can be joined by a path in X. A space X is said to be *simply connected* if it is a path-connected space and if $\pi_1(X,x_0) = 0$, for every $x_0 \in X$. The free path space of a topological space X is the mapping space $PX = \{\gamma : I \to X\}$. The based path space of a based space (X,x_0) is the mapping space $P_*X = \{\gamma : I \to X : \gamma(0) = x_0\}$ of paths in X that start at the base point. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and let X be a topological space with basepoint x_0. The based loop space is $\Omega X = \{\gamma : S^1 \to X : \gamma \text{ continuous}, \gamma(1) = x_0\}$, and the free loop space of X is the mapping space $X^{S^1} = \{\gamma : S^1 \to X\}$. If $f : X \to Y$ is a map between topological spaces, then the space of mappings from X to Y is denoted by $\operatorname{map}(X,Y)$, and the component of f in the space of mappings from X to Y is denoted by $\operatorname{map}(X,Y;f)$. In the case where X and Y have basepoints, then the associated mapping space is $\operatorname{map}_*(X,Y)$.

Now that the notation has been established, we shall continue with the classical algebraic invariants, but instead of considering $H_i(X)$ and $\pi_n(X)$, $n \geq 2$, we consider the rational homology groups $H_i(X;\mathbb{Q})$ (and cohomology groups) and the rational homotopy groups $\pi_n(X) \otimes \mathbb{Q}$. These groups are $\mathbb{Q}$-vector spaces and they contain no torsion information. This disadvantage of losing some information is compensated by the fact that these algebraic invariants are easier to compute. The first steps towards this theory date back to the work of Serre in the 1950s when he successfully computed the torsion-free part of $\pi_n(S^k)$ for all n and k (Serre, 1953). The results were remarkably easy and structured. The fact that the rational homotopy groups of spheres are so easy to compute led other mathematicians to believe that there could be a simpler, more explicit, and more complete description for all of the rational homotopy theory. The first success stories are the discoveries by Quillen (1969) and D. Sullivan (1977) that associate to a simply connected CW-complex X of finite type an explicit algebraic model. This gave computational power to rational homotopy theory. Sullivan algebras and models make the computational approach to rational homotopy theory effective, wherein the rational homotopy type of a simply connected space is identified with minimal Sullivan models. More precisely, if $(\wedge V, d)$ is a Sullivan model for a space X then we have an isomorphism of graded algebras $H^*(\wedge V, d) \cong H^*(X;\mathbb{Q})$ (Félix et al., 2001, §Introduction). The theory of Sullivan algebras and models is the main subject of this thesis.

1.1.1 Formality and fibrations

Here we state the notion of formality and study the lifting problem of a map to lead into the concept of a fibration.

Definition 1.1.2. (Félix, Oprea, & Tanré, 2008a) A simply connected space X of finite type over $\mathbb{Q}$ is called formal if there is a quasi-isomorphism $(\wedge V, d) \to H^*(\wedge V, d)$, where $(\wedge V, d)$ is the minimal Sullivan model of X.

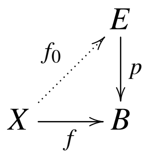
Examples of formal spaces include spheres, projective complex spaces, homogeneous spaces G/H where G and H have the same rank and compact Kähler manifolds (see (D. Sullivan, 1977)).

Definition 1.1.3. (Félix et al., 2008a) Let (A, d) be a commutative differential graded algebra (cdga for short) with cohomology $H^*(A, d)$. Let a, b, and c be cohomology classes in $H^*(A, d)$ whose products $a \cdot b = b \cdot c = 0$. Choose cocycles x, y and z representing a, b and c respectively. Then there are elements v and w such that $dv = xy$ and $dw = yz$. The element $vz - (-1)^{|x|} xw$ is a cocycle whose cohomology class depends on the choice of v and w. The set $\langle a, b, c \rangle$ of all cohomology classes $vz - (-1)^{|x|} xw$ is called the *triple Massey product* of a, b and c. The triple Massey product is trivial if $0 \in \langle a, b, c \rangle$. Let I be the ideal generated by a and c in $H^*(A, d)$. The set $\langle a, b, c \rangle$ projects to a single element in $H^*(A, d)/I$. Moreover, this element is zero if and only if the triple Massey product is trivial.

Theorem 1.1.4. *(Félix et al., 2008a) If X has a non-trivial triple Massey product then X is* *not formal.*

The definition of a fibration begins from an important problem of algebraic topology called the lifting problem.

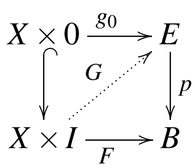
Definition 1.1.5. Suppose that $p : E \to B$ and $f : X \to B$ are maps and there is a continuous map $f_0 : X \to E$ such that $pf_0 = f$



then we say that f can be *lifted* to E and we call f_0 a lifting of f.

Definition 1.1.6. Suppose that $p : E \to B$ is a map. Then p is said to have the *homotopy lifting property* with respect to a space X if given maps $g_0 : X \times \{0\} \to E$ and $F : X \times I \to B$ such that $F(x,0) = p(g_0(x))$ for $x \in X$, then there is a map $G : X \times I \to E$ with $G(x,0) = g_0(x)$ for $x \in X$ and $pG = F$ where I is the unit interval $[0,1]$.

The commutative diagram below visualizes this situation.



We are now in a position to define a fibration.

Definition 1.1.7. A map $p : E \to B$ is called a fibration if p has the homotopy lifting property with respect to every space X. The fibration is sometimes called *Hurewicz fibration.* Furthermore, if the mapping p has the homotopy lifting property with respect to CW-complexes then it is called *Serre fibration.* Here E is the *total space* and B is the *base space* of the fibration p.

In the sequel, a fibration will mean a Serre fibration.

Definition 1.1.8. If $p : E \to B$ is a fibration then for $b \in B$, $p^{-1}(b)$ is the fibre of p over b.

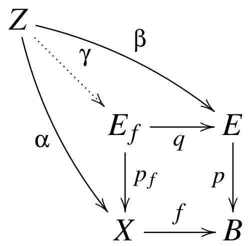
Example 1.1.9. Let $p : B \times F \to B$ be the projection. Then p is the canonical example of a fibration, called the trivial fibration.

Example 1.1.10. (Félix et al., 2008a, Page 79) Define $p : P_*X \to X$ by $p(\gamma) = \gamma(1)$. It is a fibration of fibre ΩX that is called the path space fibration.

Definition 1.1.11. Suppose $f : X \to B$ is a continuous map and $p : E \to B$ a fibration. The pullback of p over f is defined by letting E_f be the set $\{(x,y) \in X \times E : f(x) = p(y)\}$. Projection maps of E_f give the following diagram

$$\begin{array}{ccc} E_f & \longrightarrow & E \\ \Big\downarrow{\scriptstyle p_f} & & \Big\downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

If $\alpha : Z \to X$, and $\beta : Z \to E$ are such that $f \circ \alpha = p \circ \beta$, there is a unique map $\gamma : Z \to E_f$ such that all triangles and squares in the diagram below commute.



Proposition 1.1.12. (Félix et al., 2001, Page 24) The induced map $p_f : E_f \to X$ is a fibration.

Definition 1.1.13. (McCleary, 2001, Page 148) A fibration $X \xrightarrow{i} E \xrightarrow{p} B$ is said to be totally non cohomologous to zero (TNCZ for short) if the induced map in rational cohomology $H^*(i) : H^*(E;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is surjective. It is equivalent to the fact that the Serre spectral sequence (E_r, d_r) collapses at the E_2-level, i.e., $H^*(E;\mathbb{Q}) \cong H^*(B;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ as graded vector spaces. Moreover, $H^*(p) : H^*(B;\mathbb{Q}) \to H^*(E;\mathbb{Q})$ is injective. If p is trivial then $H^*(E;\mathbb{Q}) \cong H^*(B;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ as an algebra and $H^*(i)$ is the projection on the second factor, hence $H^*(i)$ is surjective. The TNCZ condition is a way of expressing that a fibration is close to being trivial.

1.1.14 Free loop space homology

The main point of this section - and of this thesis - is the free loop space $X^{S^1} = \mathrm{map}(S^1, X)$. Free loop spaces are certainly not as popular, but it has become increasingly clear over the recent years, following the seminal work of Chas and Sullivan (1999), that they play a vital role in rational homotopy theory with a viewpoint to *string topology*, a field of research that uses most of the modern techniques of algebraic topology and relates them to several other areas of mathematics influenced by theoretical physics (see (Cohen & Voronov, 2005; Roger, 2009)).

Let X be a closed, simply connected manifold of dimension m and $\mathrm{map}(S^1, X)$ the space of free loops on X. The loop space homology of X denoted by $\mathbb{H}_*(X^{S^1})$ is the ordinary homology of X^{S^1} with a shift of degrees by m, i.e., $\mathbb{H}_*(X^{S^1}) = H_{*+m}(X^{S^1})$ equipped with an associative and graded commutative product $\mu : \mathbb{H}_p(X^{S^1}) \otimes \mathbb{H}_q(X^{S^1}) \to \mathbb{H}_{p+q}(X^{S^1})$, called loop product (Chas & Sullivan, 1999). In addition, the action of S^1 on X^{S^1} by rotation, $\varphi : S^1 \times X^{S^1} \to X^{S^1}$, $(\theta, \gamma(\cdot)) \mapsto \gamma(\cdot + \theta)$ also induces an operator of degree +1, $\Delta : H_p(X^{S^1}) \to H_{p+1}(X^{S^1}), \quad \Delta([u]) = \varphi_*([S^1] \otimes [u])$.

In their seminal paper, Chas and Sullivan (1999) discovered a wealth of structure on the shifted homology $\mathbb{H}_*(X^{S^1}) = H_{*+m}(X^{S^1})$. They showed that the loop product together with Δ makes $\mathbb{H}_*(X^{S^1}) = H_{*+m}(X^{S^1})$ a Batalin-Vilkovsky algebra (BV algebra for short). In particular $\mathbb{H}_*(X^{S^1})$ is a Gerstenhaber algebra. On the other hand, if A is a differential graded algebra, then the Hochschild cohomology of A, denoted by $HH^*(A;A)$, is a Gerstenhaber algebra (Gerstenhaber, 1963). Cohen and Jones (2002) showed that, when coefficients are taken in a field $\mathbb{k}$, there is an isomorphism of graded vector spaces $\Phi : \mathbb{H}_*(X^{S^1}) \to HH^*(C^*X; C^*X)$, where C^*X is the algebra of singular cochains of X. In Félix, Thomas, and Vigué (2004); Félix and Thomas (2008); Félix, Thomas, and Vigué (2008), the authors showed that Φ is an isomorphism of Gerstenhaber algebras when X is simply connected

CW-complex of finite type and $\Bbbk = \mathbb{Q}$. Furthermore, Félix and Thomas (2008) showed that Φ is an isomorphism of BV-algebras. If $(\wedge V, d)$ is the minimal Sullivan model of X, then there is an isomorphism of Gerstenhaber algebras (Félix, Menichi, & Thomas, 2005) $HH^*(C^*X; C^*X) \cong HH^*(\wedge V; \wedge V)$.

1.2 Statement of Problem

In this section, we give brief introductions of the minor problems and the main problem of study in this thesis.

1.2.1 On unit sphere tangent bundle over complex Grassmannians

Here we state our first minor problem on the formality of the total space of the unit sphere tangent bundle over complex Grassmannian manifolds.

Let $G_{k,n}(\mathbb{C})$ denote the complex Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. In (Banyaga, Gatsinzi, & Massamba, 2018), it has been shown that for $k = 1$, which corresponds to the complex projective space $\mathbb{C}P(n)$, the total space of its unit sphere tangent bundle is formal. We show that this result is no longer true when $2 \leq k \leq n/2$.

1.2.2 Hochschild cohomology of a Sullivan model of mapping spaces

Here we state our second minor problem on the Hochschild cohomology of a Sullivan model of mapping spaces.

Let $f : X \to Y$ be a continuous map between simply connected CW-complexes of finite type, and $\phi : (\wedge V, d) \to (B, d)$ a surjective Sullivan model of f. In (Gatsinzi, 2010), it has been shown that there is an isomorphism $HH^*(\wedge V; \wedge V) \cong H_*(\wedge_{\wedge V}(s^{-1}\operatorname{Der}\wedge V), d')$, where $s^{-1}\operatorname{Der}\wedge V$ is the desuspended differential graded Lie algebra of derivations of $(\wedge V, d)$. If V is finite dimensional, we show that $HH^*(\wedge V; B)$ can be computed in terms of the graded vector space $\operatorname{Der}(\wedge V, B; \phi)$ of positive ϕ-derivations.

1.2.3 Loop space homology of elliptic spaces

Here we state our main problem on the loop space homology of elliptic spaces.

Let $(\wedge V, d)$ be a Sullivan algebra. The Hochschild cohomology space $HH^*(\wedge V; \wedge V)$ has the well-known cup product operations under which it is a graded commutative algebra, and a graded Lie bracket making it a graded Lie algebra of degree -1. These turn $HH^*(\wedge V; \wedge V)$ into a Gerstenhaber algebra (Gerstenhaber, 1963). On the other hand, let X be a closed, simply connected manifold of dimension m and X^{S^1} the space of free loops on X. If $(\wedge V, d)$ is the minimal Sullivan model of X where V is finite dimensional, then there is an isomorphism of Gerstenhaber algebras $\mathbb{H}_*(X^{S^1}; \mathbb{Q}) \to HH^*(\wedge V; \wedge V)$ (Gatsinzi, 2016). In addition, let $(\wedge V, d) \overset{f}{\rightarrowtail} (C, d)$ be a morphism of cdga's. Then C is considered as a $\wedge V$-module by the action induced by f. One would like to understand the structure of $HH^*(\wedge V; \wedge V)$ as a graded vector space, as a ring, and as a graded Lie algebra, and also the structure of $HH^*(\wedge V; C)$. In turn, such information about cohomology sheds light on the structure of the algebra $\wedge V$ itself and on its bimodules C. Therefore, under some assumptions, we show that there is an injective natural graded linear map $HH^*(f)$ between the Hochschild cohomologies $HH^*(\wedge V;\wedge V)$ and $HH^*(\wedge V;C)$.

Moreover, if $X_Y^{S^1}$ is the pullback of the fibration $p : X^{S^1} \to X$, where $p(\gamma) = \gamma(0)$ along the inclusion $Y \rightarrowtail X$, as shown in the diagram below;

$$\begin{array}{ccc} X_Y^{S^1} & \longrightarrow & X^{S^1} \\ \downarrow & & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & X, \end{array}$$

then we get a morphism of algebras (Félix et al., 2004)

$$\mathbb{H}_*(X^{S^1}) \overset{\cong}{\to} HH^*(C^*X;C^*X) \to HH^*(C^*X;C^*Y) \overset{\cong}{\to} \mathbb{H}_*(X_Y^{S^1}).$$

Let $h : Y \to X$ be a map between simply connected spaces and f a cdga model of h. The computation of $HH^*(\wedge V;C)$ and the study of $HH^*(\wedge V;\wedge V) \to HH^*(\wedge V;C)$ is a tool to understand the morphism $\mathbb{H}_*(X^{S^1}) \to \mathbb{H}_*(X_Y^{S^1})$.

1.3 Significance of the Study

The result on unit tangent sphere bundles over complex Grassmannians together with the result shown by (Banyaga et al., 2018) completes the study of the formality of the total spaces of the unit sphere tangent bundles over complex Grassmann manifolds. The result on the Hochschild cohomology of a Sullivan model of mapping spaces will be useful for computations for homotopy theorists in general and rational homotopy theorists in particular. In addition, the result on the natural graded linear map induced in the Hochschild cohomology by a certain Koszul-Sullivan extension is interesting and it should have applications in rational homotopy theory from a string topology perspective. The study mainly acknowledges the developments and applications of algebraic models to geometry in general, and this has been an important goal in algebraic topology in recent years.

2 Literature review

We begin this chapter with a review of basic definitions and notations. We refer to (Félix et al., 2001, 2008a) for details.

2.1 Basic Definitions and Notations

In this section, all vector spaces and algebras are taken over the field $\mathbb{Q}$ of rational numbers.

Definition 2.1.1. A lower or homologically graded vector space V is a direct sum of vector spaces, that is, $V = \oplus_i V_i$, where $i \in \mathbb{Z}$. If $V = \oplus_{i\geq 0} V_i$, then we say V is non negatively graded. Likewise, $V^\bullet = \oplus_i V^i$ is called cohomologically graded. The elements of V^i are homogeneous elements of degree i and we write $|x| = i$ if $x \in V^i$. We say V is of finite type if each V^i is finite dimensional. We use the standard convention $V^i := V_{-i}$.

Definition 2.1.2. The suspension sV of the graded vector space V is the graded vector space defined by $(sV)^n = V^{n+1}$ or $(sV)_n = V_{n-1}$ for all n.

Definition 2.1.3. A graded algebra A is a sum $A = \underset{i\geq 0}{\oplus} A^i$ together with a graded multiplication $A^i \otimes A^j \to A^{i+j}$ such that $x \otimes y \mapsto xy$ and has $1 \in A^0$. It is graded commutative if for any homogeneous elements x and y,

$$xy = (-1)^{|x||y|} yx.$$

If A is a graded algebra equipped with a linear differential map $d : A^n \to A^{n+1}$ such that $d \circ d = 0$ and

$$d(xy) = (dx)y + (-1)^{|x|}x(dy),$$

then (A,d) is called a differential graded algebra and d is called a differential. Moreover, if A is also a graded commutative algebra, then (A,d) is a commutative differential graded algebra. It is said to be connected if $A^0 \cong \mathbb{Q}$.

Definition 2.1.4. Let A be a graded algebra. A (left) A-module M is a sum $M = \bigoplus_{i\geq 0} M^i$, where M^i is a vector space, together with an action $A^i \otimes M^j \to M^{i+j}$, $x \otimes m \mapsto xm$, such that $x(ym) = (xy)m$ and $1m = m$ for all $x, y \in A$ and $m \in M$. Analogously, M is a (right) A-module if the associative multiplication $M^i \otimes A^j \to M^{i+j}$, $m \otimes x \mapsto mx$, satisfies $(mx)y = m(xy)$. Moreover, if $x(my) = (xm)y$, then M is an A-bimodule.

Example 2.1.5. If $V = \bigoplus_i V^i$ is a graded vector space, then the tensor algebra $T(V) = \oplus_{n\geq 0} T^n(V)$ defined by setting $T^n(V) = V \otimes \cdots \otimes V$ for $n \geq 1$ factors and $T^0(V) = \mathbb{Q}$ is a graded algebra.

Definition 2.1.6. A free commutative graded algebra A is the quotient of the tensor algebra $T(V)$ on the graded vector space V by the ideal generated by the elements $x \otimes y - (-1)^{|x||y|} y \otimes x$ where x and y are the homogeneous elements of $T(V)$.

Definition 2.1.7. A *morphism* of differential graded algebras $f : (A,d) \to (B,d)$ is a family of linear maps $f : A^n \to B^n$ such that $fd = df$ and $f(ab) = f(a)f(b)$. It induces a morphism $H^*(f): H^*(A) \to H^*(B)$ of graded algebras, where $H(A)$ is the *homology algebra* $H(A,d) = \ker d / \operatorname{Im} d$ of the differential graded algebra (A,d).

Definition 2.1.8. A (left) module M over a differential graded algebra (A,d) is an (A,d)-module M equipped with a linear differential map $d: M^n \to M^{n+1}$ with $d \circ d = 0$ and

$$d(xm) = (dx)m + (-1)^{|x|}x(dm),$$

for $x \in A$ and $m \in M$. If $f: (A,d) \to (B,d)$ is a morphism of differential graded algebras, and (M,d) is a differential (B,d)-module, then (M,d) is an (A,d)-module via the action induced by f. That is, if we define $a \cdot m = f(a) \cdot m$ for all $a \in A$ and $m \in M$, we obtain an (A,d)-module structure on (M,d).

Let A be a commutative graded algebra and M a $\mathbb{Z}$-graded A-module. Denote by $T_A(M)$ the A-tensor algebra. The symmetric algebra $\wedge_A M$ is the commutative graded algebra obtained as the quotient of $T_A(M)$ by the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$, where $x, y \in T_A(M)$ are homogeneous elements. The symmetric product induces a graded commutative algebra structure on $\wedge_A(M)$. Moreover, if Z is a $\mathbb{Q}$-vector space, then there is a canonical isomorphism of commutative graded algebras

$$\Phi: \wedge_A(A \otimes Z) \to A \otimes \wedge_{\mathbb{Q}} Z.$$

Definition 2.1.9. Let (A,d) be a differential graded algebra. A differential graded module

(M,d) over (A,d) is said to be free if it is free as an A-module, and the basis is made up of cycles.

Definition 2.1.10. A differential (A,d)-module (M,d) is said to be semi-free if there is a filtration $F_0M \subset F_1M \subset \cdots \subset M$ such that each $F_iM/(F_{i-1}M)$ is free on a basis of cycles.

Definition 2.1.11. A graded Lie algebra is a graded vector space $L = \underset{i}{\oplus} L_i, i \in \mathbb{Z}$ equipped with a bilinear map

$$[-,-] : L_i \otimes L_j \to L_{i+j}$$

such that

(1) $[x,y] = -(-1)^{|x||y|}[y,x]$, (antisymmetry).

(2) $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|z||y|}[z,[x,y]] + (-1)^{|x||y|}[y,[z,x]] = 0$, (Jacobi identity).

Example 2.1.12. (D. Sullivan, 1977) $\pi_*(\Omega X) \otimes \mathbb{Q}$ endowed with the Samelson product is a graded Lie algebra.

Definition 2.1.13. Let L be a graded Lie algebra. If L is equipped with a linear map $\delta : L_i \to L_{i-1}$ such that $\delta \circ \delta = 0$, then (L,δ) is called a differential graded Lie algebra.

Definition 2.1.14. The cohomology ring of a topological space is usually defined by means of the chain complex generated by singular k-simplices $\Delta^k \to X$. Recall that a singular k-

simplex in a space X is a continuous map $\Delta^k \to X$, where $k \geq 0$ and

$$\begin{aligned}\Delta^k &= \left\{x = (x_0, x_1, x_2, \ldots, x_k) \in \mathbb{R}^{k+1} : x_i \geq 0 \text{ and } \sum x_i = 1\right\} \\ &= \left\{\sum x_i v_i, x_i \geq 0 \text{ and } \sum x_i = 1\right\},\end{aligned}$$

where $v_0 = (1,0,\ldots,0), v_1 = (0,1,0,\ldots,0), \ldots, v_k = (0,0,\ldots,1)$ are the *vertices* of Δ^k. Moreover, Δ^k is simply written as $[v_0,\ldots,v_k]$.

Let $S_k(X)$ for $k \geq 0$ be the set of singular k-simplices on a space X. Denote by $CS_k(X;\mathbb{Q})$, the free $\mathbb{Q}$-module with basis $S_k(X)$, and the *singular chain complex* of X is the chain complex $CS_*(X;\mathbb{Q}) = \oplus_{k\geq 0} CS_k(X;\mathbb{Q})$, with differential $d = \sum_i (-1)^i \partial_i$. The set maps $\partial_i : CS_k(X;\mathbb{Q}) \to CS_{k-1}(X;\mathbb{Q})$ and $\delta_i : CS_k(X;\mathbb{Q}) \to CS_{k+1}(X;\mathbb{Q})$ are called the *face and degeneracy maps* (see (Félix et al., 2001, §4)). Let $DS_{k+1}(X;\mathbb{Q})$ be the submodule of $CS_{k+1}(X;\mathbb{Q})$ spanned by degenerate singular k-simplices. Since $DS_*(X;\mathbb{Q})$ is a subcochain complex of $CS_*(X;\mathbb{Q})$ and $H(DS_*(X;\mathbb{Q})) = 0$ (see (Félix et al., 2001, §4)). Then, the quotient chain complex

$$C_*(X;\mathbb{Q}) = CS_*(X;\mathbb{Q})/DS_*(X;\mathbb{Q})$$

is called the *normalised singular chain complex of* X. Its homology, denoted $H_*(X;\mathbb{Q})$, is the *singular homology of* X with $\mathbb{Q}$-coefficients. In particular, the *normalised singular cochain complex* of a topological space X is the cochain complex

$$C^*(X;\mathbb{Q}) = \mathrm{Hom}(C_*(X;\mathbb{Q}),\mathbb{Q}),$$

where for $f \in C^k(X;\mathbb{Q})$, then $C^k(X;\mathbb{Q}) = \mathrm{Hom}(C_k(X;\mathbb{Q}),\mathbb{Q})$ and $df = -(-1)^{|f|} fd$. Its cohomology, denoted as $H^*(X;\mathbb{Q})$ is known as the *singular cohomology of* X with coefficients in $\mathbb{Q}$. It turns out that it is a graded ring with respect to the well-known cup product operations, and it is called the cohomology ring of the topological space X.

The algebra $C^*(X;\mathbb{Q})$ is almost never commutative, since the singular cochains over $\mathbb{Q}$ are not commutative. However, since $\mathbb{Q}$ is a field of characteristic zero, it turns out that we may replace $C^*(X;\mathbb{Q})$ by a genuinely commutative cochain algebra in which all torsion and divisibility phenomena have been removed, allowing one to focus on the rational information in the original space (Griffiths & Morgan, 1981; Félix et al., 2001, §1 and §10). More precisely, there is a naturally defined commutative cochain algebra $A_{PL}(X)$ of rational polynomial differential forms on X and natural cochain algebra quasi-isomorphisms

$$C^*(X;\mathbb{Q}) \xrightarrow{\simeq} D(X) \xleftarrow{\simeq} A_{PL}(X;\mathbb{Q}),$$

where $D(X)$ is a third natural commutative cochain algebra. Thus, $C^*(X;\mathbb{Q})$ is homotopy commutative, and there is a natural isomorphism of graded algebras $H^*(X;\mathbb{Q}) = H^*(A_{PL}(X;\mathbb{Q}))$.

2.2 Review of current status of research problems

In this section, we give reviews of the current status of our research problems. We begin with two minor problems and the main problem.

2.2.1 Sphere bundles over Kähler manifolds

Here we review our first minor problem based on unit sphere bundles over Kähler manifolds studied in (Banyaga et al., 2018).

The authors consider the sphere bundle $S^{2m-1} \to E \to X^{2m}$, where the base X is a Kähler manifold and determine a suitable condition on X to make the total space E formal. Thus, it is shown that the sphere bundle over the complex projective space $\mathbb{C}P(n) = G_{1,n+1}(\mathbb{C})$ is formal. We also mention here that there is much of work in the literature on formality or non-formality relations amongst spaces involved in a fibration sequence (see for example, (Blair, 2010; Biswas, Fernández, Muñoz, & Tralle, 2016; Boyer & Galicki, 2008; Hajduk & Tralle, 2014; Hatakeyama, 1963; Muñoz & Tralle, 2015; Tievsky, 2008)). However, according to the result of (Banyaga et al., 2018), the case on the unit sphere tangent bundle $S^{2m-1} \to E \to G_{k,n}(\mathbb{C})$, for $2 \leq k \leq n/2$, is not known. Here $m = k(n-k)$. Therefore, we study the case for $2 \leq k \leq n/2$. On one hand, since complex Grassmann manifolds are symplectic manifolds, and Kähler manifolds are the well-known example of symplectic manifolds. For a more general result, one can use the same approach to study the formality of the unit sphere bundle over Kähler manifolds with non-monogenic cohomology in place of the complex Grassmann manifolds. Thus, our study provides another addition to the already existing results on formality relations amongst spaces involved in a fibration sequence.

Remark 2.2.2. Following the result of (Banyaga et al., 2018), we might expect to do more by considering the following cases:

(i) Is it possible to indicate more general results or more general classes of space like complex flag manifolds, or maybe products of complex projective spaces, in place of complex Grassmann manifolds for which the same approach might work?

(ii) Are there any kinds of spaces that the current approach does not work for?

2.2.3 Hochschild cohomology of Sullivan algebras and mapping spaces

Here we review our second minor problem based on Hochschild cohomology of Sullivan algebras and mapping spaces studied in (Gatsinzi, 2019).

Let $f : X \to Y$ be a map between simply connected CW-complexes of finite type, where $H^*(Y;\mathbb{Q})$ is finite dimensional and $\phi : (\wedge V, d) \to (B, d)$ a Sullivan model of f, with (B, d) as a module over $\wedge V$ via the mapping ϕ. In (Gatsinzi, 2019), it was shown that there is a canonical injection $\pi_*(\Omega\,\mathrm{map}(X,Y;f)) \otimes \mathbb{Q} \to HH^*(\wedge V; B)$. In addition, $\pi_*(\Omega\,\mathrm{map}(X,Y;f)) \otimes \mathbb{Q} \to H_*(s^{-1}\,\mathrm{Der}(\wedge V, B;\phi))$ is an isomorphism. The result is a generalization of the inclusion $\pi_*(\Omega\,\mathrm{map}(X,X;1_X)) \otimes \mathbb{Q} \to HH^*(\wedge V; \wedge V)$ (see (Félix & Thomas, 2004, Theorem 2) and (Gatsinzi, 2013, Theorem 1.1)). However, to our knowledge, it is not known in the literature whether the injection $H_*(s^{-1}\operatorname{Der}(\wedge V,B;\phi)) \to HH^*(\wedge V;B)$ extends to an isomorphism of graded commutative algebras $H_*(\Psi) : H_*(\wedge_{\wedge V} s^{-1}\operatorname{Der}(\wedge V,B;\phi),d_0) \to HH^*(\wedge V;B)$. It turns out that this happens when ϕ is surjective. More precisely, we show that if ϕ is surjective, then the Hochschild cohomology $HH^*(\wedge V;B)$ can be computed in terms of the graded vector space of positive ϕ-derivations.

Remark 2.2.4. (i) Here we illustrate the reason why ϕ surjective works. The necessity of ϕ being surjective is shown by the following counterexample. Let $\phi : \wedge V = (\wedge x_3, 0) \to (\wedge(y_3,y_7),0) = B$, where $\phi(x_3) = y_3$. This is a Sullivan model of the projection $S^3 \times S^7 \xrightarrow{p} S^3$. As all differentials are zero $HH^*(\wedge V;B) \cong \operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, B)$. Moreover, $s^{-1}\operatorname{Der}(\wedge V,B;\phi) = < s^{-1}(x_3,1) > = < z_2 >$, and $\wedge_{\wedge V} s^{-1}\operatorname{Der}(\wedge V, B;\phi) \cong \wedge x_3 \otimes \wedge z_2$. This implies that $\operatorname{Im}\Psi \cong \operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge y_3)$. Hence, Ψ is not surjective.

(ii) On the other hand, let $\phi : (\wedge V,d) \to (B,d)$ is a surjective morphism between cdga's where V finite dimensional and $I = \ker\phi$. Then, we do not know if the isomorphism $s^{-1}\operatorname{Der}(\wedge V,B;\phi) \cong (\wedge V/I) \otimes s^{-1}V^*$ extends into an isomorphism of cdga's

$$(\wedge_{\wedge V} s^{-1}\operatorname{Der}(\wedge V,B;\phi),d_0) \to ((\wedge V/I) \otimes \wedge s^{-1}V^*, d_1),$$

where $V^* = \operatorname{Hom}(V,\mathbb{Q})$ denotes the dual space of V.

2.2.5 Rational string topology

Here we review our main problem based on Hochschild cohomology of a Sullivan algebra studied in (Félix et al., 2005; Gatsinzi, 2010).

The cohomology theory of algebras associated to a space X was introduced by Hochschild (1945) as a tool for studying the homological properties of algebras. The right setup for this is the language of algebraic deformation theory, which is governed by the Hochschild cohomology as a graded Lie algebra under the Gerstenhaber bracket. Therefore, the first step towards understanding an algebra's deformation theory begins with a depiction of the Gerstenhaber bracket (Shepler & Witherspoon, 2012).

Recall that if $(\wedge V, d)$ is the minimal Sullivan model of X, then there is an isomorphism of Gerstenhaber algebras (Félix et al., 2005) $HH^*(C^*X; C^*X) \cong HH^*(\wedge V; \wedge V)$. Hence, $\mathbb{H}_*(X^{S^1})$ can be computed in terms of $(\wedge V, d)$. In Gatsinzi (2010), if $(\wedge V, d)$ is the minimal Sullivan model of X where V is finite dimensional, then the Gerstenhaber structure of $HH^*(\wedge V; \wedge V)$ can be computed in terms of derivations of $(\wedge V, d)$. Moreover, if V is finite dimensional, then $HH^*(\wedge V; \wedge V)$ is the homology of a Gerstenhaber algebra $(\wedge V \otimes \wedge s^{-1}V^*, d)$, where V^* is a positively lower graded dual of V, and it is isomorphic to the loop space homology $\mathbb{H}_*(X^{S^1})$ (Gatsinzi, 2016). Hence, there are isomorphisms of Gerstenhaber algebras

$$\mathbb{H}_*(X^{S^1}; \mathbb{Q}) \xrightarrow{\cong} HH^*(\wedge V; \wedge V) \xleftarrow{\cong} H_*(\wedge_{\wedge V} L, d_0) \xrightarrow{\cong} H_*(\wedge V \otimes \wedge Z, d),$$

where $L = s^{-1}\,\mathrm{Der}\wedge V$ and $Z = s^{-1}V^{*}$. On the other hand, a given Koszul Sullivan extension $(\wedge V,d) \overset{f}{\rightarrowtail} (C,d)$ induces a natural graded linear map $HH^{*}(f) : HH^{*}(\wedge V;\wedge V) \to HH^{*}(\wedge V;C)$. The algebraic structure of $HH^{*}(\wedge V;\wedge V)$ as a graded vector space, as a ring, and as a graded Lie algebra is well understood. In the same way, one would be interested in understanding the structure of $HH^{*}(\wedge V;C)$. Following this line of thinking, we are interested in examining $HH^{*}(f)$ because to our knowledge not much is known about the Hochschild cohomology of Koszul-Sullivan extensions.

Therefore, we study the properties of the natural graded linear map $HH^{*}(f)$. More precisely, we show that, if $F \to E \overset{p}{\to} X$ is a TNCZ fibration, where X is an elliptic 2-stage Postnikov tower and $f : (\wedge V,d) \rightarrowtail (C,d)$ a KS-model of p, then the induced map in Hochschild cohomology $HH^{*}(f) : HH^{*}(\wedge V;\wedge V) \to HH^{*}(\wedge V;C)$ is injective. In particular, since $HH^{*}(\wedge V;\wedge V)$ and $\mathbb{H}_{*}(X^{S^1};\mathbb{Q})$ are isomorphic, where $(\wedge V,d)$ is the minimal Sullivan model of X, we deduce that $HH^{*}(f) : \mathbb{H}_{*}(X^{S^1};\mathbb{Q}) \to HH^{*}(\wedge V;C)$ is injective.

Remark 2.2.6. It is not known whether $HH^{*}(f)$ is injective when X is an n-stage Postnikov tower, where $n \geq 3$.

3 Methodology

In this thesis, we rely on the theory of minimal Sullivan models in rational homotopy theory, spectral sequences, L_∞-models of function spaces and Hochschild cohomology of an algebra for which (Félix et al., 2001, 2008a; McCleary, 2001; Lada & Markl, 1995; Buijs, Félix, & Murillo, 2011, 2013; Burghelea & Vigué, 1985; Félix et al., 2004; Gatsinzi, 2010, 2016) are our main references.

3.1 Sullivan algebras and models

Here we introduce the theory of Sullivan algebras and models.

Sullivan algebras and models resemble some ideas from de Rham cohomology (D. P. Sullivan, 2005), which has a more geometric approach. That is when de Rham proved that $H^*(A_{DR}(M)) \cong H^*(M;\mathbb{R})$ for the differential algebra of C^∞-differential forms $A_{DR}(M)$ on a smooth manifold M, it immediately provided a link between the C^∞-differential forms on a smooth manifold to algebraic invariants of the smooth manifold. It is noted in Félix et al. (2008a, §Preface) that in his seminal paper D. Sullivan (1977) suggests in his remark that even within the world of topology, there is more topological information in $A_{DR}(M)$. It is difficult to detect this information, since in the de Rham algebra, we might suspect that some information is contained in two different entities; that is, the product of forms and the exterior derivative of a form. However, for easier detection of the corresponding topological information in the de Rham algebra, one looks for a simplification $\mathcal{M}_M$ of the de Rham algebra with an explicit differential graded algebra morphism $\mathcal{M}_M \to A_{DR}(M)$ inducing an isomorphism in cohomology. This was motivated by the fact that for a compact connected Lie group G there is a subdifferential algebra of bi-invariant forms $\Omega(G)$, inside the de Rham algebra $A_{DR}(M)$ such that the canonical inclusion $\Omega(G) \rightarrowtail A_{DR}(M)$ induces an isomorphism in cohomology.

In his seminal paper, D. Sullivan (1977) successfully constructed a cdga of the form $(\wedge V, d)$. The cdga $(\wedge V, d)$ is called a *Sullivan algebra*. It is a commutative cochain algebra with $V = \bigcup\limits_{k\geq 0} V(k)$, where $V(0) \subset V(1) \subset \cdots$ such that $d(V(0)) = 0$ and $d(V(k)) \subset \wedge V(k-1)$. It is minimal if $dV \subset \wedge^{\geq 2} V := \wedge^+ V. \wedge^+ V$. This means that the differential d of any element of V is a "polynomial"in $\wedge V$ with no linear term (Félix et al., 2001, §12). There is also the smallest possible sub-differential algebra of forms with the same cohomology as the model $(\wedge V, d)$ (see (Félix et al., 2008a, §Preface)). It is a minimal model, in which we may ask what geometrical invariants can be detected in it. This is a functor from algebra to geometry that together with forms, serves as a dictionary between the algebraic and the geometrical worlds. *However, it is studied over the rationals and not over the reals*. As a result, the de Rham algebra is being replaced by other types of forms. This new construction is of great advantage since it allows for an extension of the usual theory from manifolds to topological spaces (Félix et al., 2008a, §Preface). The right setup for this is found in the work of (D. Sullivan, 1977) who constructs a contravariant functor $X \rightsquigarrow A_{PL}(X)$ from the homotopy category of simply connected rational spaces of finite type to the homotopy category of simply connected rational cgdas of finite type. It is a commutative differential graded cochain algebra of rational polynomial differential forms on X that uniquely determines the rational homotopy type of X.

Definition 3.1.1. Two commutative cochain algebras (A,d) and (B,d) are *weakly equivalent* if they are connected by a chain

$$(A,d) \xrightarrow{\simeq} (C(0),d) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} (C(k),d) \xleftarrow{\simeq} (B,d)$$

of quasi-isomorphisms of commutative cochain algebras. Such a chain is called a *weak equivalence* between (A,d) and (B,d).

Definition 3.1.2. A *commutative model* for X is a commutative cochain algebra (A,d) which is weakly equivalent to $A_{PL}(X;\mathbb{Q})$.

If (A,d) is a commutative differential graded algebra of which the cohomology is connected and finite in each degree, then there is a homomorphism $m : (\wedge V,d) \to (A,d)$ which induces an isomorphism in cohomology. A minimal Sullivan model of a simply connected space X is a minimal Sullivan model of $A_{PL}(X)$ (D. Sullivan, 1977). More precisely $H^*(X;\mathbb{Q}) \cong H^*(\wedge V,d)$ as graded algebras, and $V^n = \mathrm{Hom}(\pi_n(X),\mathbb{Q})$ as graded vector spaces.

Example 3.1.3. (Félix et al., 2008a)

(i) The minimal Sullivan model of an odd dimensional sphere S^{2n+1} is given by $(\wedge x, 0)$ for $|x| = 2n+1$.

(ii) The minimal Sullivan model for an even dimensional sphere S^{2n} is given by $(\wedge(x,y), d)$ for $|x| = 2n$, $|y| = 4n-1$, $dx = 0$, and $dy = x^2$.

(iii) The Sullivan minimal model of $\mathbb{C}P(n)$ is given by $(\wedge(x,y), d)$ with $|x| = 2$, $|y| = 2n+1$, and $dx = 0, dy = x^{n+1}$.

Definition 3.1.4. (Félix et al., 2008a) Let (A,d) be a commutative model for a closed, simply connected n-dimensional manifold X. A *Poincaré duality model* for X is a cdga (A,d) that satisfies the following properties.

1. $A^p = 0$ for all $p > n$,
2. $A^0 = \mathbb{Q}$ and $A^1 = 0$,
3. the bilinear form $A^i \otimes_{\mathbb{Q}} A^{n-i} \to A^n$, $a \otimes b \to ab$ is non degenerate, that is, an element $a \in A^i$ is zero if and only if $ab = 0$ for all $b \in A^{n-i}$.

Moreover, if (A,d) is a *Poincaré duality model* for a closed, simply connected n-dimensional manifold X, then there is an element $\omega \in A^n$ such that $A^n = \mathbb{Q}\omega$. Such an element is called the fundamental class of A.

Definition 3.1.5. A commutative cochain algebra of the form $(B \otimes \wedge V, d)$ is a *relative Sullivan algebra* whenever

(a) $(B,d) = (B \otimes 1, d)$ is a sub-cochain algebra, and $H^0(B) = \mathbb{Q}$.

(b) $1 \otimes V = V = \oplus_{p \in \mathbb{N}} V^p$.

(c) $V = \bigcup_{k=0}^{\infty} V(k)$ where $V(k)$ is an increasing sequence of graded subspaces such that $d : V(0) \to B$ and $d : V(k) \to B \otimes \wedge V(k-1)$, for $k \geq 0$.

Definition 3.1.6. Let $(B \otimes \wedge V, d)$ be a relative Sullivan algebra, then we say $(B \otimes \wedge V, d)$ is minimal if Im $d \subset B^+ \otimes \wedge V + B \otimes \wedge^{\geq 2} V$.

Definition 3.1.7. A morphism of commutative differential graded algebras $(B,d) \rightarrowtail (B \otimes \wedge V, d) \to (\wedge V, \bar{d})$ is called a Koszul Sullivan extension (KS-extension for short) if $(B \otimes \wedge V, d)$ is a relative Sullivan algebra.

In general, for any morphism of commutative cochain algebras

$$\varphi : (B,d) \to (C,d)$$

with $H^0(B) = H^0(C) = \mathbb{Q}$, and $H^1(\varphi)$ is injective, there is a KS-extension $(B,d) \to (B \otimes \wedge V, d)$ and a quasi-isomorphism $\psi : (B \otimes \wedge V, d) \to (C,d)$ such that the following diagram commutes (see (Félix et al., 2001, Prop 14.3))

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & C \\ & \searrow & \uparrow {\scriptstyle \psi} \, {\scriptstyle \cong} \\ & & (B \otimes \wedge V, d), \end{array}$$

we call ψ a relative Sullivan model of φ. Moreover, $(B \otimes \wedge V, d)$ can be chosen to be minimal and a minimal relative Sullivan model unique upto isomorphism (see (Félix et al., 2001, Prop 14.12)).

Note that, if $F \xrightarrow{i} E \xrightarrow{p} X$ is a fibration, where X is simply connected, then the KS-extension $(B,d) \overset{f}{\rightarrowtail} (B \otimes \wedge V, d)$ is called a Koszul Sullivan model of p (Félix et al., 2001, §15). If the fibration p is trivial, then $(B \otimes \wedge V, d)$ is quasi-isomorphic to $(B,d) \otimes (\wedge V, \bar{d})$.

By analogy, we will call E a rational 2-stage Postnikov tower if there is a non trivial fibration $F \xrightarrow{i} E \xrightarrow{p} X$ where F and X are products of Eilenberg-Mac Lane spaces. In this case, a Koszul Sullivan model of $A_{PL}(p)$ is of the form $(\wedge V, 0) \rightarrowtail (\wedge V \otimes \wedge W, d)$.

Definition 3.1.8. Let (A,d) be a differential graded algebra. A derivation θ of A of degree k is a linear mapping $\theta : A^n \to A^{n-k}$ such that $\theta(ab) = \theta(a)b + (-1)^{k|a|} a\theta(b)$.

Denote by $\mathrm{Der}_k A$ the vector space of all derivation of degree k and $\mathrm{Der} A = \oplus_k \mathrm{Der}_k A$. The Lie bracket induces a graded Lie algebra structure on $\mathrm{Der} A$. On the other hand, $(\mathrm{Der} A, \delta)$ is a differential graded Lie algebra (D. Sullivan, 1977) with the differential δ defined in the usual way by

$$\delta\theta = [d, \theta] = d \circ \theta - (-1)^k \theta \circ d.$$

Furthermore, $\mathrm{Der} A$ is a differential graded A-module via the action $(a\theta)(x) = a\theta(x)$. If $\theta_1 \in \mathrm{Der}_k A$ and $a \in A^i$, then $a\theta_1 \in \mathrm{Der}_{k-i} A$. That is, $\mathrm{Der} A$ is a graded A-module which satisfies the relation

$$[d, a\theta] = (da)\theta + (-1)^{|a|}a[d,\theta].$$

Let $\theta_1, \theta_2 \in \mathrm{Der}A$ and $a \in A$. Then

$$[\theta_1, a\theta_2] = \theta_1(a)\theta_2 + (-1)^{|a||\theta_1|}a[\theta_1,\theta_2] \tag{1}$$

(see (Gatsinzi, 2016, 2017)). Let $(\wedge V, d)$ be a Sullivan algebra where V is spanned by $\{v_1 \ldots, v_k\}$. Consider $\theta_i = (v_i, 1)$ the unique derivation of $\wedge V$ defined by $\theta_i(v_j) = \delta_{ij}$. Then the graded $\wedge V$-module $\mathrm{Der}\wedge V$ is spanned by $\{\theta_1, \ldots, \theta_k\}$.

3.2 Spectral sequences

Here we introduce the study of spectral sequences.

Definition 3.2.1. A bigraded module is a family $M = \oplus_{(p,q)\in\mathbb{Z}\times\mathbb{Z}} M_{p,q}$ of modules over a commutative ring $\Bbbk$ indexed by pairs of integers (p,q).

Definition 3.2.2. A differential bigraded module (M,d) is a bigraded module $M = \oplus_{(p,q)\in\mathbb{Z}\times\mathbb{Z}} M_{p,q}$ together with a bigraded map $d : M \to M$ called the differential, such that $d \circ d = 0$ and d has bidegree $(-r, r-1)$ for some r.

Definition 3.2.3. The homology $H(M,d)$ of a differential bigraded module (M,d) is defined by:

$$H_{p,q}(M,d) = \frac{\ker d_{p,q}}{\mathrm{Im}\, d_{p-a,q-b}}$$

where (a,b) is the bidegree of the differential d.

Combining these notions, we can now define spectral sequences.

Definition 3.2.4. A *homology spectral sequence* starting at E^n is a sequence

$$(E^r, d^r)_{r \geq n}$$

in which $E^r = \{E^r_{p,q}\}$ is a bigraded module, d^r is a differential in E^r of bidegree $(-r, r-1)$ and $H(E^r) \overset{\cong}{\to} E^{r+1}$ is an isomorphism of bigraded modules.

A *cohomology spectral sequence* is a sequence $(E_r, d_r)_{r \geq n}$ in which $E_r = \{E_r^{p,q}\}$, d^r a differential of bidegree $(r, -r+1)$ and $H(E_r) \overset{\cong}{\to} E_{r+1}$.

Definition 3.2.5. A *first quadrant* cohomology spectral sequence is a sequence of differential bigraded modules (E_r, d_r) in which each $E_r = \{E_r^{p,q}\}_{p \geq 0, q \geq 0}$ for $r = 1, 2, 3, \ldots$, is equipped with a differential, $d_r \circ d_r = 0$, of bidegree $(r, -r+1)$,

$$d_r : E_r^{p,q} \to E_r^{p+r, q+1-r}.$$

Thus, for all $r \geq 1$, $E_{r+1}^{*,*} \cong H(E_r^{*,*}, d_r)$. That is,

$$E_{r+1}^{p,q} = (\operatorname{Ker} d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}) / (\operatorname{Im} d_r : E_r^{p-r,q-1+r} \to E_r^{p,q}).$$

We call the r^{th} stage of such an object its E_r-level. Moreover, consider $E_r^{p,q}$ for $r >$

$\max(p, q+1)$, then for $k \geq 0$ we have $E_{r+k}^{p,q} = E_r^{p,q}$ see (McCleary, 2001). We denote this common graded module by $E_\infty^{p,q}$.

Definition 3.2.6. (McCleary, 2001; Félix et al., 2001). A *filtration* $\Im$ on an A-module M is a family of submodules $\{F^pM\}_{p\in\mathbb{Z}}$ of M, such that

$$\cdots \subset F^{p-1}M \subset F^pM \subset F^{p+1}M \subset \cdots \subset M$$

is an increasing sequence of submodules or

$$\cdots \subset F^{p+1}M \subset F^pM \subset F^{p-1}M \subset \cdots \subset M$$

is a decreasing sequence of submodules. Then, M is called a filtered graded module, and denoted by $(M, \Im)$. We can collapse the filtered module $(M, \Im)$ to its *associated bigraded module* $E_0^{*,*}(M)$ given by

$$E_0^{p,q}(M) = F^pM^{p+q}/F^{p+1}M^{p+q},$$

where p is the *filtration degree*, q is the *complementary degree* and $\Im$ is decreasing or increasing respectively

Example 3.2.7. (Félix et al., 2001) *Relative Sullivan algebras.*

Let $(B \otimes \wedge V, d)$ be a relative Sullivan algebra, filtered by the degree of B, that is

$$F^p(B \otimes \wedge V) = B^{\geq p} \otimes \wedge V, \ p \geq 0.$$

The associated bigraded module is given by

$$E_0^{*,*}(B \otimes \wedge V, d) = (B, 0) \otimes (\wedge V, \bar{d}).$$

We now define the notion of convergence.

Definition 3.2.8. (McCleary, 2001; Félix et al., 2001) Consider a cohomology spectral sequence $(E_r^{p,q}, d_r)$. If for each (p,q) there is an integer s such that

$$E_s^{p,q} \cong E_{s+1}^{p,q} \cong \cdots \cong E_r^{p,q} \cong \cdots \ r \geq s,$$

then the spectral sequence $(E_r^{p,q}, d_r)$ is *convergent.* In this case the bigraded module $E_\infty^{p,q}$ is defined by $E_\infty^{p,q} = E_r^{p,q}$, $r \geq s$. The spectral sequence *collapses* at E_r-level if $d_i = 0$, $i \geq r$. In this case E_∞ is defined and $E_r = E_\infty$.

The spectral sequence converges to H^* if there is a filtration on H^* such that $E_\infty^{p,q} = E_0^{p,q}(H^*)$.

Theorem 3.2.9. *(McCleary, 2001; Félix et al., 2008a) (**The cohomology Serre spectral sequence**) Let $\mathbb{k}$ be a field. Suppose $F \to E \xrightarrow{p} X$ is a fibration, where X is simply connected and F is path connected. There is a first quadrant spectral sequence of algebras $(E_r^{*,*}, dr)$*

such that

$$E_2^{p,q} \cong H^p(X;\Bbbk) \otimes H^q(F;\Bbbk),$$

converging to $H^*(E;\Bbbk)$ *as an algebra.*

3.3 L_∞-models of function spaces

L_∞ algebras were introduced by Lada and Markl (1995) and L_∞ models of function spaces were studied by Félix et al. in Buijs et al. (2011, 2013). Here we provide details on their notion.

Definition 3.3.1. Let S_k be the symmetric group. A permutation $\sigma \in S_k$ is called an $(i, k-i)$ shuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(k)$ where $i = 1, \ldots, k$. The Koszul sign $\varepsilon(\sigma)$ of a permutation $\sigma \in S_k$ is determined by

$$x_1 \wedge \cdots \wedge x_k = \varepsilon(\sigma) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)},$$

where the subscripts indicate the degrees of the graded objects $x_1, \ldots, x_k$.

Definition 3.3.2. (Buijs et al., 2011) An L_∞ algebra or a strongly homotopy Lie algebra is a graded vector space $L = \oplus_i L_i$ endowed with a collection of linear maps

$$\ell_k : L^{\otimes k} \to L$$

of degree $k-2$ for $k \geq 1$, called brackets, such that

(1) ℓ_k are skew-symmetric, that is, for any k-permutation σ

$$\ell_k(x_{\sigma(1)},\ldots,x_{\sigma(k)}) = \mathrm{sgn}(\sigma)\varepsilon(\sigma)\ell_k(x_{\sigma(1)},\ldots,x_{\sigma(k)}),$$

where $\mathrm{sgn}(\sigma)$ is the sign of σ.

(2) The Jacobi identities are generalised as follows;

$$\sum_{i+j=k+1}\sum_{\sigma}\mathrm{sgn}(\sigma)\varepsilon(\sigma)(-1)^{i(j-1)}\ell_j(\ell_i(x_{\sigma(1)},\ldots,x_{\sigma(i)}),x_{\sigma(i+1)},\ldots,x_{\sigma(k)}) = 0,$$

where $\sigma \in S(i,k-i)$.

In particular, if $\ell_k = 0$ for $k \geq 3$, we recover the notion of a differential graded Lie algebra where $[x,y] := \ell_2(x,y)$ and $\delta x = \ell_1(x)$.

There is a bijection between L_∞ structures on L and codifferentials $d_k : \wedge^p(sL) \to \wedge^{p-k+1}(sL)$ of degree -1 on the coalgebra $\wedge sL$ such that $d^2 = 0$ where $d = d_1 + \cdots + d_k + \cdots$ (Lada & Markl, 1995).

Definition 3.3.3. Let $\phi : (A,d_A) \to (B,d_B)$ be a morphism of cdga's. A ϕ-derivation of degree k is a linear mapping $\theta : A^n \to B^{n-k}$ for which $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b)$.

Denote by $\mathrm{Der}_n(A,B;\phi)$ the vector space of ϕ-derivations of degree n and by $\mathrm{Der}(A,B;\phi) = \oplus_n \mathrm{Der}_n(A,B;\phi)$ the graded vector space of all ϕ-derivations. We consider only derivations of positive degree. The differential graded vector space of ϕ-derivations is denoted by $(\mathrm{Der}(A,B;\phi),\partial)$, where the differential ∂ is defined by $\partial\theta = d_B\theta - (-1)^{|\theta|}\theta d_A$. If $\wedge V = B$ and $\phi = 1_{\wedge V}$, then we get the Lie algebra of derivations $\mathrm{Der}\wedge V$. Moreover, whenever $A = (\wedge V, d)$ is a Sullivan algebra, we note that, there is an isomorphism of graded vector spaces

$$\mathrm{Der}(\wedge V, B;\phi) \cong \mathrm{Hom}(V,B).$$

If $\{v_i\}$ is a basis of V, and $\{b_j\}$ is a basis of B, then the graded vector space $\mathrm{Der}(\wedge V, B;\phi)$ is spanned by the unique ϕ-derivation θ denoted by (v_i, b_j) such that

$$\begin{cases} \theta_i(v_i) = b_j, & b_j \in B, \\ \theta_i(v_j) = 0, & i \neq j. \end{cases}$$

Moreover, let $A = (\wedge V, d)$ be a minimal Sullivan algebra where V is finite dimensional. A morphism of cdga's $\phi : (\wedge V, d) \to (B,d)$ induces a linear mapping

$$\Phi : \mathrm{Der}\wedge V \to \mathrm{Der}(\wedge V, B;\phi),$$

by post composing by ϕ.

Define $\widetilde{\mathrm{Der}}(A,B;\phi)$ as follows (Buijs et al., 2013),

$$\widetilde{\mathrm{Der}}_i(A,B;\phi) = \begin{cases} \mathrm{Der}_i(A,B;\phi), & i>1, \\ \{\theta \in \mathrm{Der}_1(A,B;\phi) : \partial\theta = 0\}, & i=1. \end{cases}$$

Let $(\wedge V,d)$ be a Sullivan algebra and $\theta_1,\dots,\theta_k \in \widetilde{\mathrm{Der}}(\wedge V,B;\phi)$ be ϕ-derivations of respective degrees $n_1,\dots,n_k$, we define their bracket $[\theta_1,\dots,\theta_k] \in \widetilde{\mathrm{Der}}(\wedge V,B;\phi)$ of length k by

$$[\theta_1,\dots,\theta_k](v) = (-1)^{\eta}\sum\sum_{i_1,\dots,i_k} \varepsilon\phi(v_1\dots\hat{v_{i_1}}\dots\hat{v_{i_k}}\dots v_n)\theta_1(v_{i_1})\dots\theta_k(v_{i_k}),$$

where $dv = \sum v_1\dots v_n$, $\eta = n_1+\dots+n_{k-1}$ and ε is the suitable sign given by the Koszul convention. These operations may be desuspended to define linear maps ℓ_k for $k \geq 1$ each of degree $k-2$ on $s^{-1}\widetilde{\mathrm{Der}}(\wedge V,B;\phi)$ by

$$\ell_1(s^{-1}\theta) = -s^{-1}\partial\theta,\ \ell_k(s^{-1}\theta_1,\dots,s^{-1}\theta_k) = (-1)^{\beta}s^{-1}[\theta_1,\dots,\theta_k],$$

where $\beta = \frac{k^2-k}{2} + \sum_{i=1}^{k-1}(k-i)|\theta_i|$ (Buijs et al., 2013). It is shown that $(s^{-1}\,\mathrm{Der}(\wedge V,B;\phi),$ $\ell_k)$ is an L_∞ algebra (Buijs et al., 2013). Also, an L_∞ algebra is an L_∞ model of a simply connected space X, if it is a model of the differential graded Lie algebra $\mathcal{C}_*(X)$. Here $\mathcal{C}_*$ denotes the Quillen functor that associates to any simply connected space X, a differential graded Lie algebra $\mathcal{C}_*(X)$, which yields an equivalence between the homotopy category of simply connected rational spaces and that of reduced differential graded Lie algebras over the rationals (Félix et al., 2001, Chapter 22).

Proposition 3.3.4. (Buijs et al., 2013) Let $f : X \to Y$ be a map between simply connected spaces having the rational homotopy type of a CW complex of finite type and $\phi : (\wedge V, d) \to (B, d)$ a Sullivan model of f. Then $(s^{-1}\widetilde{\mathrm{Der}}(\wedge V, B; \phi), \ell_k)$ is an L_∞ model of $\mathrm{map}(X, Y; f)$.

3.4 Hochschild cohomology

Here we give details on the Hochschild cohomology of an algebra.

Let (A, d) be an augmented differential graded cochain algebra over a field $\Bbbk$ of characteristic zero and $\bar{A} = \ker(\varepsilon : A \to \Bbbk)$. The Hochschild cohomology of A with coefficients in A is defined as $\mathrm{Ext}_{A^e}(A, A)$ where A is an $A^e = A \otimes A^{op}$-module under the action $(a \otimes b)c = (-1)^{|c||b|}acb$ for $a, b, c \in A$ (see (Gatsinzi, 2019)). If (P, d) is a right differential graded A-module and $(N.d)$ a left graded differential A-module, the definition of the *two sided (normalised) bar construction* on (A, d) is as follows (see for instance (Félix et al., 2005, 2004)). It is the complex

$$(\mathbb{B}(P; A; N), D) = (\oplus_k \mathbb{B}_k(P; A; N), D)$$

with

$$\mathbb{B}_k(P; A; N) = P \otimes T^k(s\bar{A}) \otimes N, \ k \geq 1.$$

A generic element $p[a_1|a_2|\cdots|a_k]n$ in $\mathbb{B}_k(P;A;N)$ has (upper) degree $|p|+|n|+\sum_{i=1}^{k}(|sa_i|)$. If $k=0$, then $p[\,]n = p\otimes 1\otimes n \in P\otimes T^0(s\bar{A})\otimes N$. The differential D decomposes into two terms $D = d_0 + d_1$ as follows, $d_0 : \mathbb{B}_k(P;A;N) \to \mathbb{B}_k(P;A;N)$, with

$$d_0(p[a_1|a_2|\cdots|a_k]n) = d(p)[a_1|a_2|\cdots|a_k]n - \sum_{i=1}^{k}(-1)^{\varepsilon_i} p[a_1|a_2|\cdots|d(a_i)|\cdots|a_k]n$$
$$+(-1)^{\varepsilon_{k+1}} p[a_1|a_2|\cdots|a_k]d(n),$$

and $d_1 : \mathbb{B}_k(P;A;N) \to \mathbb{B}_{k-1}(P;A;N)$, is given by

$$d_1(p[a_1|a_2|\cdots|a_k]n) = (-1)^{|p|} pa_1[a_2|\cdots|a_k]n + \sum_{i=2}^{k}(-1)^{\varepsilon_i} p[a_1|a_2|\cdots|a_{i-1}a_i|\cdots|a_k]n$$
$$-(-1)^{\varepsilon_k} p[a_1|a_2|\cdots|a_{k-1}]a_k n,$$

where $\varepsilon_i = |p| + \sum_{j<i}(sa_j)$. There is a canonical projection $\varphi : \mathbb{B}(A;A;A) \to A$ defined by $\varphi([\,]) = 1$ and $\varphi([a_1|\cdots|a_k]) = 0$ if $k>0$ which provides a semi-free resolution of A as an A^e-module (Félix, Halperin, & Thomas, 1995). Thus, $HH^*(A;A)$ is the homology of the normalized Hochschild cochain complex

$$(C^*(A;A), D) = \mathrm{Hom}_{A^e}(\mathbb{B}(A;A;A), A) \cong (\mathrm{Hom}(T^k(s\bar{A}), A), D_0 + D_1).$$

The differential $D_0 + D_1$ is defined as follows (Félix et al., 2005).

$$(D_0 f)([a_1|a_2|\cdots|a_k]) = d(f([a_1|a_2|\cdots|a_k])) + \sum_{i=1}^{k} (-1)^{\bar{\varepsilon}(i)} (f([a_1|a_2|\cdots|a_k])$$

and

$$(D_1 f)([a_1|a_2|\cdots|a_k]) = -(-1)^{|sa_1||f|} a_1 f([a_2|\cdots|a_k]) + (-1)^{\bar{\varepsilon}(k)} f([a_1|\cdots|a_{k-1}]) a_k + \sum_{i=2}^{k} (-1)^{\bar{\varepsilon}(i)} f([a_1|\cdots|a_{i-1}a_i|\cdots|a_k]),$$

where $\bar{\varepsilon}(i) = |f| + |sa_1| + \cdots |sa_{i-1}|$.

As $T^k(s\bar{A})$ is a graded coalgebra, the complex $C^*(A;A)$ is endowed with a product, making it a differential graded algebra. For $f \in C^p(A;A)$ and $g \in C^q(A;A)$,

$$(f \circ g)(a_1|\cdots|a_{p+q}) = (-1)^{\varepsilon(p)} f(a_1|\cdots|a_p) \cdot g(a_{p+1}|\cdots|a_{p+q}),$$

where $\varepsilon(p) = |g|(|sa_1| + \cdots + |sa_p|)$. This product induces a well-defined product in Hochschild cohomology $\smile : HH^p(A;A) \otimes HH^q(A;A) \to HH^{p+q}(A;A)$ which turns the graded $\Bbbk$-vector space $HH^*(A;A) = \oplus_{n\geq 0} HH^p(A;A)$ into a graded commutative algebra $HH^*(A;A)$ (see (Gerstenhaber, 1963, Corollary 1)).

Definition 3.4.1. (Félix et al., 2005) A (graded) Gerstenhaber algebra is a commutative graded algebra A equipped with a bracket

$$A_i \otimes A_j \to A_{i+j+1},\ x \otimes y \mapsto \{x, y\}$$

such that

(i) the suspension of A is a graded Lie algebra with bracket

$$(sA)_i \otimes (sA)_j \to (sA)_{i+j},\ sx \otimes sy \mapsto -(-1)^{|x|} s\{x, y\},$$

(ii) the product is compatible with the bracket, i.e. for $a, b, c \in A$,

$$\{a, bc\} = \{a, b\}c + (-1)^{|b|(|a|+1)} b\{a, c\}.$$

Moreover, Gerstenhaber (1963) defined a bracket called the Gerstenhaber bracket on $C^*(A; A)$, inducing a graded Gerstenhaber algebra structure on $HH^*(A; A)$ (Gerstenhaber, 1963). The bracket is defined by the formula

$$\{f, g\} = f \bar{\circ} g - (-1)^{|f||g|} g \bar{\circ} f,$$

where

$$(f\bar{\circ}g)([a_1|a_2\cdots|a_k]) = \sum_{0\leq i\leq j\leq k}(-1)^{\varepsilon(i)}f([a_1|\cdots|a_i|g([a_{i+1}|\cdots|a_j])|a_{j+1}|\cdots|a_k]),$$

and $\varepsilon(i) = |g|(|sa_1|+\cdots+|sa_i|)$ on $C^*(A;A)$.

Example 3.4.2. Exterior algebra of a graded Lie algebra. If L is a graded Lie algebra, then $\wedge_*(L)$ is naturally a Gerstenhaber algebra, for exterior product and natural prolongation of the bracket of L (see, (Roger, 2009)).

Definition 3.4.3. (Félix & Thomas, 2008) A Batalin-Vilkovisky algebra is a commutative graded algebra, A together with a linear map (called a BV-operator)

$$\Delta : A_k \to A_{k+1}$$

such that

(1) $\Delta^2 = 0$

(2) A is a Gerstenhaber algebra with the bracket defined by

$$\{a,a'\} = (-1)^{|a|}\left(\Delta(aa') - \Delta(a)a' - (-1)^{|a|}a\Delta(a')\right).$$

3.5 Hochschild cohomology of a Sullivan algebra

Here we provide details on Hochschild cohomology of a Sullivan algebra.

As $HH^*(A;A) := \mathrm{Ext}_{A^e}(A,A)$, therefore in order to compute the Hochschild cohomology of a differential graded algebra A, it suffices to find a free resolution of A as an $A \otimes A^{op}$-module. In particular, for any commutative differential graded algebra $(\wedge V, d)$ and any differential graded $\wedge V$-module B, the Hochshild cohomology $HH^*(\wedge V;B)$ is computed with ease using technical tools introduced in rational homotopy theory. This was first observed by Burghelea and Vigué (1985) and enhanced by Burghelea and Vigué-Poirrier (1988). Many other authors have made use of this material. In Félix et al. (2001), for the minimal Sullivan algebra $(\wedge V, d)$, one considers a relative Sullivan model of the multiplication $m : (\wedge V \otimes \wedge V, d^{'}) \to (\wedge V, d)$, where $d^{'} = d \otimes 1 + 1 \otimes d$. Such a model is given by the following commutative diagram

$$\begin{array}{ccc} (\wedge V \otimes \wedge V, d^{'}) & \xrightarrow{\quad m \quad} & (\wedge V, d) \\ & \searrow & \cong \uparrow \varphi \\ & & (\wedge V \otimes \wedge V \otimes \wedge sV, D), \end{array} \quad ,$$

where $sV^n = V.^{n+1}$ In (Félix et al., 2001, §15), the differential in $\wedge V \otimes \wedge V \otimes \wedge sV$ is defined by

$$D(v \otimes 1 \otimes 1) = dv \otimes 1 \otimes 1,\ D(1 \otimes v \otimes 1) = 1 \otimes dv \otimes 1,$$

and $D(1 \otimes 1 \otimes sv)$ is defined by induction on the degree of v by the formula

$$D(1 \otimes 1 \otimes sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 + \sum_{i=1}^{\infty} \frac{(sD)^i}{i!}(v \otimes 1 \otimes 1). \tag{2}$$

Here, s is the derivation of degree -1 on $\wedge V \otimes \wedge V \otimes \wedge sV$ defined as

$$s(v \otimes 1 \otimes 1) = s(1 \otimes v \otimes 1) = 1 \otimes 1 \otimes sv,\ s(1 \otimes 1 \otimes sv) = 0.$$

Example 3.5.1. (Félix et al., 2001) Consider $X = \mathbb{C}P(2)$ of which the minimal Sullivan model is $(\wedge(x_2, x_5), d),\ dx_2 = 0,\ dx_5 = x_2^3$. Then using (2) we have;

$$D(1 \otimes 1 \otimes sx_2) = x_2 \otimes 1 \otimes 1 - 1 \otimes x_2 \otimes 1,$$

$$D(1 \otimes 1 \otimes sx_5) = x_5 \otimes 1 \otimes 1 - 1 \otimes x_5 \otimes 1 + 1 \otimes x_2^2 \otimes sx_2 + x_2 \otimes x_2 \otimes sx_2 + x_2^2 \otimes 1 \otimes sx_2.$$

Proposition 3.5.2. (Félix et al., 2001) $\varphi : (\wedge V \otimes \wedge V \otimes \wedge sV, D) \xrightarrow{\cong} (\wedge V, d)$ is a semi-free resolution of $(\wedge V, d)$ as a $\wedge V \otimes \wedge V$ differential module.

Therefore, the Hochschild cohomology $HH^*(\wedge V; \wedge V)$ is given by the homology of the following complex

$$(C^*(\wedge V; \wedge V), D) \cong (\mathrm{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge sV, \wedge V), D).$$

Thus, for any for any $\wedge V$-module B, $HH^*(\wedge V;B)$ is the homology of a complex of the form $(\mathrm{Hom}_{\wedge V\otimes\wedge V}(\wedge V\otimes\wedge V\otimes\wedge sV,B),D)$. Moreover, $(\mathrm{Hom}_{\wedge V\otimes\wedge V}(\wedge V\otimes\wedge V\otimes\wedge sV,B),D)$ is isomorphic to $\mathrm{Hom}_{\wedge V}(\wedge V\otimes\wedge sV,B)$, where $(\wedge V\otimes\wedge sV,D)$ is a Sullivan algebra such that $Dv=dv$, $D(sv)=-\mathcal{S}(dv)$ and $\mathcal{S}$ is the unique derivation on $\wedge V\otimes\wedge sV$ defined by $\mathcal{S}v=sv$ and $\mathcal{S}sv=0$. If $(\wedge V,d)$ is the minimal Sullivan model of X, then $(\wedge V\otimes\wedge sV,D)$ is a Sullivan model of the free loop space X^{S^1} (Félix et al., 2001, §15). It is obtained as a push out in the following diagram.

$$\begin{array}{ccc}(\wedge V\otimes\wedge V,d') & \xrightarrow{\quad m\quad} & (\wedge V,d)\\ \Big\downarrow i & & \Big\downarrow\\ (\wedge V\otimes\wedge V\otimes\wedge sV,D) & \xrightarrow{m'} & (\wedge V\otimes\wedge sV,D).\end{array}$$

The composition with m' gives an isomorphism of complexes $\mathrm{Hom}_{\wedge V}(\wedge V\otimes\wedge sV,\wedge V)\to\mathrm{Hom}_{\wedge V\otimes\wedge V}(\wedge V\otimes\wedge V\otimes\wedge sV,\wedge V)$. Moreover, the differential D satisfies the condition $D(\wedge V\otimes\wedge^n sV)\subset\wedge V\otimes\wedge^n sV$. Hence each subspace $(\mathrm{Hom}_{\wedge V}(\wedge V\otimes\wedge^n sV,\wedge V),D)$ is a sub cochain complex of $(\mathrm{Hom}_{\wedge V}(\wedge V\otimes\wedge sV,\wedge V),D)$. This gives a Hodge type decomposition of the Hochschild cohomology $HH^*(\wedge V;\wedge V)=\oplus_{n\geq 0}HH^*_{(n)}(\wedge V;\wedge V)$. In particular, for any $\wedge V$-module B, $HH^*(\wedge V;B)=\oplus_{n\geq 0}HH^*_{(n)}(\wedge V;B)$ (see (Loday, 1998, Page 184) and (Gatsinzi, 2010, 2019)). There is also an isomorphism of graded vector spaces $\mathrm{Hom}_{\wedge V}(\wedge V\otimes sV,\wedge V)\cong\mathrm{Der}\wedge V$, and $\mathrm{Hom}_{\wedge V}(\wedge V\otimes sV,B)\cong\mathrm{Der}(\wedge V,B;\phi)$ (see (Gatsinzi, 2010, 2019)).

3.6 Gerstenhaber structure on the Hochschild cohomology

Here we review the Gerstenhaber structure on $HH^*(\wedge V;\wedge V)$.

Let $A = (\wedge V, d)$ be a Sullivan algebra. The Gerstenhaber bracket on $HH^*(\wedge V;\wedge V)$ is defined by identifying the homology of $(C^*(A;A), D)$ with the exterior algebra of the desuspended differential graded Lie algebra of derivations of A (Gatsinzi, 2010, 2016). The following are more general results.

Theorem 3.6.1. *(Gatsinzi, 2010) If $A = (\wedge V, d)$ is a Sullivan algebra, and $L = s^{-1}\,\mathrm{Der}A$, then there is a mapping $\phi : (\wedge_A L, d_0) \to (C^*(A;A), D)$ which induces an isomorphism of graded Gerstenhaber algebras in homology.*

Theorem 3.6.2. *(Gatsinzi, 2016) Let $A = (\wedge V, d)$ be a minimal Sullivan algebra where V is finite dimensional and $Z = s^{-1}V^*$. Then, $\varphi : (\wedge_A L, d_0) \to (A \otimes \wedge Z, D)$ extends to an isomorphism of differential graded Gerstenhaber algebras.*

3.7 The free loop space homology spectral sequence

We review here a spectral sequence that is useful to compute the loop space homology of certain spaces as given in (Gatsinzi, 2016).

Let X be a simply connected closed manifold of dimension m of which V is finite dimensional and $(\wedge V, d) = (\wedge(V_0 \oplus \cdots \oplus V_{n-1}), d)$ its minimal Sullivan model, where $dV_i \subset \wedge(V_0 \oplus \cdots \oplus V_{i-1})$. Let $Z = Z_0 \oplus \cdots \oplus Z_p$, where $Z_k = s^{-1}V^*_{n-k}$. Filter $\wedge V \otimes \wedge Z$ by $F_p =$

$\wedge V \otimes \wedge(Z_0 \oplus \cdots \oplus Z_p)$. . It verifies

$$\wedge V = F_0 \subset F_1 \subset \cdots \subset F_n = \wedge V \otimes \wedge Z.$$

This filtration yields a spectral sequence of Gerstenhaber algebras for which $E^1 = H^*(\wedge V) \otimes \wedge Z$ and which converges to $H_*(\wedge V \otimes \wedge Z, d) \cong \mathbb{H}_*(X^{S^1}; \mathbb{Q})$.

4 Preliminary results

In this chapter, we study the following problems.

(i) The partial computation of the Lie bracket structure of the string homology on a formal elliptic space.

(ii) The formality of the total space of the unit sphere tangent bundle $S^{2m-1} \to E \xrightarrow{p} G_{k,n}(\mathbb{C})$ over complex Grassmannian manifolds $G_{k,n}(\mathbb{C})$, for $2 \leq k \leq n/2$, where $m = k(n-k)$, by exhibiting a non trivial Massey triple product in $H^*(E;\mathbb{Q})$.

(iii) The Hochschild cohomology of a Sullivan model of mapping spaces.

4.1 On the Lie bracket structure of the string homology on a formal elliptic space

In this section, we consider the Chas-Sullivan loop space homology $\mathbb{H}_*(X^{S^1})$ of a formal elliptic space X and show that the centre of the graded Lie algebra $s\mathbb{H}_*(X^{S^1};\mathbb{Q})$ is non trivial.

Definition 4.1.1. A Lie group G is a group that is also a smooth manifold such that the multiplication $\mu : G \times G \to G$ and the inverse map $g \mapsto g^{-1}$ are both smooth.

Example 4.1.2. (i) The general linear group $GL(n;\mathbb{C})$, which is the group of $n \times n$ invertible matrices with entries in $\mathbb{C}$ under matrix multiplication, is a Lie group of dimension $(2n)^2$.

(ii) A matrix $A \in M_n(\mathbb{C})$ is unitary, if $AA^* = I$, where $A^* = \bar{A}^T$. The subgroup of $GL(n;\mathbb{C})$ consisting of all unitary matrices is a compact Lie group called the unitary group, and denoted by $U(n)$.

The subgroup of $U(n)$, of matrices with determinant 1 is called the special unitary group, and it is denoted $SU(n)$.

(iii) The $n \times n$ quarternionic symplectic group, $Sp(n)$ is defined by

$$Sp(n) = \{A \in GL(n;\mathbb{H}) : AA^* = I\}.$$

Observe that both $U(n)$ and $SU(n)$ are subgroups of $Sp(n)$.

Definition 4.1.3. Let G be a compact connected Lie group with a closed subgroup H. The coset space G/H admits a differentiable structure and it is called a homogeneous space.

Example 4.1.4. The quaternionic Grassmannian $G_{k,n}(\mathbb{H})$ of k-dimensional vector subspaces of $\mathbb{H}^n$ is a homogeneous space as $G_{k,n}(\mathbb{H}) \cong Sp(n)/(Sp(k) \times Sp(n-k))$ for $1 \leq k < n$.

Below we compute the free loop space homology for some homogeneous spaces. Recall that, if X is homogeneous space, then its minimal Sullivan model is given by

$$(A,d) = (\wedge(b_1,\ldots,b_n,a_1,\ldots,a_m),d),$$

where $db_i = 0$ and $da_i \in \wedge(b_1, \ldots, b_n)$ (see (Félix et al., 2008a)). Thus, to compute the loop space homology of X we consider a complex of the form $(A \otimes \wedge(z_1, \ldots, z_m, u_1, \ldots, u_n), d)$ where $z_j = s^{-1}a_j^*$, $u_i = s^{-1}b_i^*$, $dz_j = 0$ and $du_i = \sum \frac{\partial f_j}{\partial b_i} z_j$ with $f_j = db_j$. Further, if A is a minimal Sullivan model of a simply connected compact oriented m-manifold X of which $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, then there is a filtration on $A \otimes \wedge(z_1, \ldots, z_m, u_1, \ldots, u_n)$ which yields a spectral sequence of Gerstenhaber algebras for which $E^1 = H^*(A) \otimes \wedge(z_1, \ldots, z_m, u_1, \ldots, u_n)$ and which converges to $H_*(A \otimes \wedge(z_1, \ldots, z_m, u_1, \ldots, u_n), d) \cong \mathbb{H}_*(X^{S^1}; \mathbb{Q})$ (see (Gatsinzi, 2016)). In particular, if $X = G/K$ is a homogeneous space of which G and K have an equal rank, then

$$\mathbb{H}_*(X^{S^1}; \mathbb{Q}) \cong H_*(H^*(A) \otimes \wedge(Z_0 \oplus Z_1), d),$$

where $dZ_0 = 0$ and $dZ_1 \subset H^+ \otimes Z_0$ (see (Gatsinzi, 2016)).

Example 4.1.5. (Félix et al., 2008a; Gatsinzi, 2016) Let $X = \mathbb{C}P(n)$ of which the minimal model is $A = (\wedge(b_2, a_{2n+1}), d)$, $db_2 = 0$, $da_{2n+1} = b_2^{n+1}$. Thus,

$$\mathbb{H}_*(\mathbb{C}P(n), \mathbb{Q}) \cong H_*((\wedge b_2)/(b_2^{n+1}) \otimes \wedge(z_1, z_{2n}), d),\ dz_{2n} = 0,\ dz_1 = (n+1)b_2^n z_{2n}.$$

Homology classes are

$$\{b^j z_{2n}^k, b^i z_1, b^i z_1 z_{2n}^k,\ k \geq 0,\ 0 \leq j \leq n-1,\ 1 \leq i \leq n\}.$$

Example 4.1.6. We consider the Sullivan minimal model of

$$X = Sp(5)/(Sp(2) \times Sp(3))$$

which is given by

$$A = (\wedge(b_4, b_8, a_{15}, a_{19}), d)$$

where $db_i = 0$, and $da_{15} = 2b_8b_4^2 + b_4^4 - b_8^2$, $da_{19} = 2b_8b_4^3 + b_8^2b_4$. Consider the ideal $I = (2b_8b_4^2 + b_4^4 - b_8^2, 2b_8b_4^3 + b_8^2b_4)$. It follows thus, $H^*(\wedge V, d) = \wedge(b_4, b_8)/I$. Hence, there is a quasi-isomorphism

$$f : (\wedge(b_4, b_8, a_{15}, a_{19}), d) \xrightarrow{\cong} H^*(\wedge V, d).$$

Thus, $Sp(5)/(Sp(2) \times Sp(3))$ is formal. The rational cohomology is given by classes of

$$\{1, b_4, b_4^2, b_8, b_4^3, b_4b_8, b_8b_4^2, b_4^4, [2b_8b_4^2 + b_4^4] = [b_8^2],$$

$$[b_4^5] = [-2b_4^3b_8] = [b_8^2b_4], [b_4^6] = [b_4^2b_8^2] = [b_8^3]\}.$$

Thus, to compute the loop space homology of $X = Sp(5)/(Sp(2) \times Sp(3))$ we consider a complex of the form

$$(A \otimes \wedge(z_{14}, z_{18}, u_3, u_7), d),$$

where $dz_i = 0$, and

$$du_3 = (4b_4b_8 + 4b_4^3)z_{14} + (6b_8b_4^2 + b_8^2)z_{18}, du_7 = (2b_4^2 - 2b_8)z_{14} + (2b_4^3 + 2b_4b_8)z_{18}.$$

In some lower degrees, the loop space homology is given by classes of

$$\{1, z_{14}, z_{18}, b_4, b_4^2, b_8, b_4^3, b_4b_8, b_8b_4^2, b_4^4, [2b_8b_4^2 + b_4^4] = [b_8^2], b_4z_{14}, b_4^2z_{14}, b_8z_{14}, b_4^3z_{14},$$

$$b_4b_8z_{14}, b_8b_4^2z_{14}, b_4^4z_{14}, [b_4^5] = [-2b_4^3b_8] = [b_8^2b_4], [b_4^6] = [b_4^2b_8^2] = [b_8^3], b_8^2z_{14}, b_4^5z_{14}, b_4^6z_{14},$$

$$b_4z_{18}, b_4^2z_{18}, b_8z_{18}, b_4^3z_{18}, b_4b_8z_{18}, b_8b_4^2z_{18}, b_4^4z_{18}, b_8^2z_{18}, b_4^5z_{18}, b_4^6z_{18},$$

$$(4b_4b_8 + 4b_4^3)z_{14}, (6b_8b_4^2 + b_8^2)z_{18}, (2b_4^2 - 2b_8)z_{14}, (2b_4^3 + 2b_4b_8)z_{18}\}.$$

Definition 4.1.7. Let L be a Lie algebra. The *centre* $Z(L)$ is defined by

$$Z(L) = \{x \in L : [x, y] = 0, \forall y \in L\}.$$

In (Gatsinzi, 2016, Theorem 13), it is shown that, if X is a simply connected homogeneous space of which $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, then the graded Lie algebra $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$ is not nilpotent. In addition, we establish the following result.

Theorem 4.1.8. *If X is a simply connected formal homogeneous space of which $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, then the centre of $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$ is non trivial.*

Proof. Let X be a homogeneous space of which $(\wedge V, d) = (\wedge(V_0 \oplus V_1), d)$ is its minimal Sullivan model, where V is finite dimensional and $dV_0 = 0$, $dV_1 \subseteq \wedge V_0$. Denote by $< v_1, v_2, \ldots, v_n >$ the vector space generated by a finite basis $\{v_i\}$ of V. Write $V_0^{\text{even}} = \mathbb{Q} < p_1, \ldots, p_q >= P$, $V_0^{\text{odd}} = \mathbb{Q} < w_1, \ldots, w_r >= W$, and $V_1^{\text{odd}} = \mathbb{Q} < y_1, \ldots, y_p >= Y$, so that $(\wedge(V_0 \oplus V_1), d) \overset{\cong}{\to} (\wedge(P \oplus Y), d) \otimes (\wedge W, 0)$, and $dP = 0$, $dY \subseteq \wedge P$. The associated minimal Sullivan model $(\wedge V, d)$ is called a pure Sullivan algebra. Homogeneous spaces are pure. Moreover, since X is a formal homogeneous space, then $p = q$, and we have

$$H^*(\wedge V, d) = \frac{\wedge(p_1, \ldots, p_p)}{(\alpha_1, \ldots, \alpha_p)} \otimes \wedge(w_i, \ldots, w_r),$$

where $(\alpha_1, \ldots, \alpha_p)$ is a regular sequence in $\wedge P$. Hence, X as a formal homogeneous space admits a minimal Sullivan model of the form $A = (\wedge V, d) = (\wedge(P \oplus Y), d) \otimes (\wedge W, 0)$, where $dP = 0$, $dy_k = \alpha_k$. Further, there is filtration (see (Gatsinzi, 2016)) on $(A \otimes \wedge s^{-1}V^*, d)$ which yields a spectral sequence of Gerstenhaber algebras for which $E^1 = H^*(A) \otimes \wedge Z$, where $Z = Z_0 \oplus Z_1$, and $Z_0 = s^{-1}V_1^*$, $Z_1 = s^{-1}V_0^*$. Let $L = s^{-1}\operatorname{Der} A$. Given $a \in \wedge_A^0 L = A$, $\theta_1, \theta_2 \in \operatorname{Der} A$, and for $x, \beta \in L$, and using

$$[\theta_1, a\theta_2] = \theta_1(a)\theta_2 + (-1)^{|a||\theta_1|} a[\theta_1, \theta_2] \tag{3}$$

we have

$$\{x, a\} = -(-1)^{|a|}(sx)(a). \tag{4}$$

Hence

$$\{x, a\beta\} = \{x, a\}\beta + (-1)^{|a|(|x|+1)} a\{x, \beta\}, \quad (5)$$

(see (Gatsinzi, 2017)). Moreover $\wedge_A L$ and $(A \otimes \wedge(Z_0 \oplus Z_1), d)$ are isomorphic as differential Gerstenhaber algebras. It is sufficient to check that $\wedge Z_0$ is abelian in $(H^*(A) \otimes \wedge Z, d)$. It follows from equation (4), that for all $z_i, z_j \in Z_0$ we have

$$\{z_i, z_j\} = -(-1)^{|z_i|} s(z_i)(z_j) = 0.$$

Furthermore, using (5), for all $a_i \neq 0 \in H^*(A)$, $v_i \neq 0 \in H^*(A \otimes \wedge(Z_0 \oplus Z_1))$ and $z_i \in Z_0$, one gets

$$\begin{aligned} \{a_i z_i, v_i\} &= a_i\{z_i, v_i\} + (-1)^{|a_i|(|v_i|+1)} z_i\{a_i, v_i\}, \\ &= 0, \text{ as } \{v_i, z_i\} = \{a_i, v_i\} = 0, \text{ using (4)}. \end{aligned}$$

Also, using (4), if $x_i \neq 0$ is a cocycle in $H^*(A) \otimes \wedge^+(Z_0 \oplus Z_1)$, then $\{Z_0, x_i\} = 0$. Hence $\wedge Z_0$ is abelian. Thus, the algebra $\wedge Z_0$ is in the centre of $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$. □

Remark 4.1.9. The formality condition on X is necessary for $\wedge Z_0$ to be in the centre of $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$. We consider the minimal Sullivan model of the non formal homogeneous space $X = Sp(6)/SU(6)$ which is given by

$$A = (\wedge(x_6, x_{10}, b_{15}, b_{19}, b_{23}), d), dx_i = 0, db_{15} = x_6 x_{10}, db_{19} = x_{10}^2, db_{23} = x_6^4.$$

The rational cohomology is given by classes of

$$\{1, x_6, x_{10}, x_6^2, x_6^3, x_6 b_{19} - x_{10} b_{15}, x_6^2 b_{19} - x_6 x_{10} b_{15},$$

$$x_6^3 b_{15} - x_{10} b_{23}, x_6^3 b_{19} - x_6^2 x_{10} b_{15}, x_6^4 b_{19} - x_6^3 x_{10} b_{15}\}.$$

The loop space homology of $X = Sp(6)/SU(6)$ is computed from the complex

$$(A \otimes \wedge(z_{14}, z_{18}, z_{22}, b_5, b_9), d), dz_i = 0, db_5 = x_{10} z_{14} + 4x_6^3 z_{22}, db_9 = x_6 z_{14} + 2x_{10} z_{18},$$

which is isomorphic to $(A \otimes (Z_0 \oplus Z_1), d)$, where $dZ_0 = 0, dZ_1 \subseteq A \otimes Z_0$. It contains $H^*(X) \otimes \wedge(z_{14}, z_{18}, z_{22})/I$ where is I is the ideal generated by $\{db_5, db_9\}$. In some lower degrees, the loop space homology is given by classes of

$$\{1, z_{14}, z_{18}, z_{22}, x_6, x_{10}, x_6^2, x_6^3,$$

$$x_6 b_{19} - x_{10} b_{15}, x_6^2 b_{19} - x_6 x_{10} b_{15},$$

$$x_6^3 b_{15} - x_{10} b_{23}, x_6^3 b_{19} - x_6^2 x_{10} b_{15}, x_6^4 b_{19} - x_6^3 x_{10} b_{15},$$

$$x_6z_{14}, x_6z_{18}, x_6z_{22}, x_{10}z_{14}, x_{10}z_{18}, x_{10}z_{22}, x_6^2z_{14},$$

$$x_6^2z_{18}, x_6^2z_{22}, x_6^3z_{14}, x_6^3z_{18}, x_6^3z_{22}, x_{10}z_{14}, 4x_6^3z_{22}, x_6z_{14}, 2x_{10}z_{18},$$

$$x_6b_5 - (z_{14}b_{15} + 4z_{22}b_{23}), x_{10}b_5 - (z_{14}b_{19} + 4x_6^2z_{22}b_{15}),$$

$$x_{10}b_9 - (z_{14}b_{15} + 2z_{18}b_{19})\}.$$

Then for $z_i \in Z_0$, $x_i \neq 0 \in H^*(A \otimes \wedge(Z_0 \oplus Z_1), d)$. The non zero brackets for $k \geq 1$ include

$$\{z_{14}, (x_6b_{19} - x_{10}b_{15})z_i^k\} = x_{10}z_i^k, \{z_{14}, (x_6^3b_{15} - x_{10}b_{23})z_i^k\} = -x_6^3z_i^k,$$

$$\{z_{18}, (x_6b_{19} - x_{10}b_{15})z_i^k\} = -x_6z_i^k, \{z_{18}, (x_6^2b_{19} - x_6x_{10}b_{15})z_i^k\} = -x_6^2z_i^k,$$

$$\{z_{18}, (x_6^3b_{19} - x_6^2x_{10}b_{15})z_i^k\} = -x_6^3z_i^k, \{z_{22}, (x_6^3b_{15} - x_{10}b_{23})z_i^k\} = x_{10}z_i^k.$$

Hence $\wedge Z_0$ is not in the centre of $(H^*(A \otimes \wedge(Z_0 \oplus Z_1)), d)$.

Definition 4.1.10. (Kirillov, 2008, §5.4) Let L be a Lie algebra. Set $L^{(0)} := L$ and, for $k \geq 1$, define the k-th derived algebra of L as $L^{(k)} := [L^{(k-1)}, L^{(k-1)}]$. Then L is called solvable if $L^{(k)} = 0$ for some k.

Proposition 4.1.11. Let $L = (\wedge(w_1, \ldots, w_r) \otimes \wedge(z_1, \ldots, z_r), d = 0) \subseteq (H^*(A) \otimes \wedge Z, d)$, where $|w_i|$ is odd, and $z_i = s^{-1}w_i^*$. Then L is solvable.

Proof. Let $a,b \in L$. Then we have

$$\{a,b\} = \sum_i (-1)^{|a|} \frac{\partial^2}{\partial w_i \partial z_i}(ab).$$

Hence, if $a \in \wedge^k(w_1,\dots,w_r)\otimes\wedge(z_1,\dots,z_r)$, and $b \in \wedge^l(w_1,\dots,w_r)\otimes\wedge(z_1,\dots,z_r)$, with $k+l \leq r$. Then, $\{a,b\} \subseteq \wedge^{k+l-1}(w_1,\dots,w_r)\otimes\wedge(z_1,\dots,z_r)$. Thus,

$$L^{(1)} = \{L,L\} \subseteq \wedge^{\leq r-1}(w_1,\dots,w_r)\otimes\wedge(z_1,\dots,z_r),$$

$$L^{(2)} = \{L^{(1)},L^{(1)}\} \subseteq \wedge^{\leq r-2}(w_1,\dots,w_r)\otimes\wedge(z_1,\dots,z_r),$$

$$L^{(3)} = \{L^{(2)},L^{(2)}\} \subseteq \wedge^{\leq r-3}(w_1,\dots,w_r)\otimes\wedge(z_1,\dots,z_r).$$

Continuing iterating this process yields $L^{(r)} = \{L^{(r-1)},L^{(r-1)}\} \subseteq \wedge(z_1,\dots,z_r)$, which implies that $L^{(r+1)} = \{L^{(r)},L^{(r)}\} = 0$, and L is solvable. □

In conclusion, we have the following result.

Corollary 4.1.12. If X is a simply connected formal homogeneous space of which $\pi_*(X)\otimes\mathbb{Q}$ is finite dimensional, then

(1) $s\mathbb{H}_*(X^{S^1};\mathbb{Q})$ is not nilpotent,

(2) the centre of $s\mathbb{H}_*(X^{S^1};\mathbb{Q})$ is non trivial.

4.2 On the unit sphere tangent bundles over complex Grassmannians

Let $G_{k,n}(\mathbb{C})$ denote the Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. In (Banyaga et al., 2018), it was shown that the total space of the unit sphere tangent bundle over the complex projective space $\mathbb{C}P(n) = G_{1,n}(\mathbb{C})$ is formal. As $G_{k,n}(\mathbb{C}) \cong G_{n-k,n}(\mathbb{C})$, we will assume $k \leq n/2$. In this section, we show that the total space of the unit sphere tangent bundle over $G_{k,n}(\mathbb{C})$ is not formal, for $2 \leq k \leq n/2$.

The complex Grassmannian $G_{k,n}(\mathbb{C})$ is a homogeneous space as $G_{k,n}(\mathbb{C}) \cong U(n)/(U(k) \times U(n-k))$ for $1 \leq k < n$, where $U(n)$ is the unitary group. It is a symplectic manifold of dimension $2m$, where $m = k(n-k)$. The method to compute a Sullivan model of the homogeneous space $G_{k,n}(\mathbb{C})$ is given in details in (Greub, Halperin, & Vanstone, 1976; Murillo, 1999). Let $S^{2m-1} \to E \xrightarrow{p} G_{k,n}(\mathbb{C})$ be the unit sphere tangent bundle and $(\wedge V, d)$ a Sullivan model of $G_{k,n}(\mathbb{C})$. A relative minimal model of p is given by

$$(\wedge V, d) \overset{\iota}{\rightarrowtail} (\wedge V \otimes \wedge x_{2m-1}, d') \to (\wedge x_{2m-1}, 0),$$

with $d'v = dv$ for $v \in V$ and $d'x_{2m-1} = z$, as $[z]$ is the Euler class of the tangent bundle (Félix et al., 2008a, Page 82). Moreover, if $[\omega] \in H^{2m}(\wedge V, d)$ is the fundamental class of $G_{k,n}(\mathbb{C})$, then $[z] = \chi(G_{k,n}(\mathbb{C})) \cdot [\omega]$, where $\chi(G_{k,n}(\mathbb{C}))$ is the Euler characteristic of $G_{k,n}(\mathbb{C})$ (see

(Bott & Tu, 1982, Proposition 11.24)). As $\chi(G_{k,n}(\mathbb{C})) \neq 0$, there is a quasi-isomorphism

$$(\wedge V \otimes \wedge x_{2m-1}, d') \to (\wedge V \otimes \wedge x_{2m-1}, D),$$

where $Dv = dv$ for $v \in V$ and $Dx_{2m-1} = \omega$.

Remark 4.2.1. For the general case, a Sullivan model of $G_{k,n}(\mathbb{C})$ for $1 \leq k < n$ is given by (see (Murillo, 1999))

$$(\wedge(b_2, b_4, \dots, b_{2k}, x_2, x_4, \dots, x_{2(n-k)}, y_1, y_3, \dots, y_{2n-1}), d) \tag{6}$$

with

$$db_i = 0 = dx_j,\ dy_{2p-1} = \sum_{p_1+p_2=p} b_{2p_1} \cdot x_{2p_2},\ 1 \leq p \leq n.$$

Lemma 4.2.2. Where $2 \leq k \leq n/2$, the minimal Sullivan model of $G_{k,n}(\mathbb{C})$ is given by

$$(\wedge(b_2, \dots, b_{2k}, y_{2(n-k)+1}, \dots, y_{2n-1}), d),\ dy_{2n-1} = b_{2k} r$$

where $r \notin (b_{2k})$.

Proof. Consider a Sullivan model of $G_{n,k}(\mathbb{C})$ from (6)

$$(\wedge(b_2, b_4, \dots, b_{2k}, x_2, x_4, \dots, x_{2(n-k)}, y_1, y_3, \dots, y_{2n-1}), d).$$

Then

$$dy_1 = b_2 + x_2,\ dy_3 = b_4 + x_4 + b_2x_2, \ldots, dy_{2n-1} = b_{2k}x_{2(n-k)}.$$

The model is not minimal as the linear part is not zero. To find its minimal Sullivan model, we make a change of variable $t_2 = b_2 + x_2$ and replace x_2 by $t_2 - b_2$ wherever it appears in the differential. This gives an isomorphic Sullivan algebra

$$(\wedge(b_2, t_2, b_4, \ldots, b_{2k}, x_4, \ldots, x_{2(n-k)}, y_1, y_3, \ldots, y_{2n-1}), d)$$

where

$$dy_1 = t_2,\ dy_3 = b_4 + x_4 + b_2(t_2 - b_2), \ldots, dy_{2n-1} = b_{2k}x_{2(n-k)}.$$

As the ideal generated by y_1 and t_2 is acyclic, the above Sullivan algebra is quasi-isomorphic to

$$(\wedge(b_2, b_4, \ldots, b_{2k}, x_4, \ldots, x_{2(n-k)}, y_3, \ldots, y_{2n-1}), d)$$

where

$$dy_3 = b_4 + x_4 - b_2^2, \ldots, dy_{2n-1} = b_{2k}x_{2(n-k)}.$$

One continues in this fashion to get the minimal Sullivan model

$$(\wedge(b_2,\ldots,b_{2k},y_{2(n-k)+1},\ldots,y_{2n-1}),d)$$

with

$$dy_{2n-1}=b_{2k}r$$

where $r\in\wedge(b_2,\ldots,b_{2k})$ and $[r]\neq 0$ in $H^*(G_{k,n}(\mathbb{C}),\mathbb{Q})$ as $|r|=2(n-k)$ and there is no coboundary of degree less than $2(n-k)$. In particular, $[r]\neq[b_{2k}]$. □

Theorem 4.2.3. *If* $2\leq k\leq n/2$ *and* $m=k(n-k)$, *then the total space of the unit sphere tangent bundle*

$$S^{2m-1}\to E\to G_{k,n}(\mathbb{C})$$

is not formal.

Proof. The minimal Sullivan model of $G_{k,n}(\mathbb{C})$ is given by $(\wedge V,d)=(\wedge(b_2,\ldots,b_{2k},y_{2(n-k)+1},\ldots,y_{2n-1}),d)$ and $(\wedge x_{2m-1},0)$ is the model of S^{2m-1}. Let $[b^*_{2k}]$ in $H^{2(n-k)}(\wedge V,d)$ be the Poincaré dual of $[b_{2k}]$ in $H^*(G_{k,n}(\mathbb{C}),\mathbb{Q})$ and $\omega=[b_{2k}b^*_{2k}]$ the fundamental class of $G_{k,n}(\mathbb{C})$. Since the Euler characteristic of $G_{k,n}(\mathbb{C})$, $\chi(G_{k,n}(\mathbb{C}))\neq 0$, a relative minimal model for the unit sphere tangent bundle $S^{2m-1}\to E\to G_{k,n}(\mathbb{C})$ is given by

$$(\wedge V,d)\overset{\iota}{\rightarrowtail}(\wedge V\otimes\wedge x_{2m-1},D)\to(\wedge x_{2m-1},0),$$

with $Dv = dv$ for $v \in V$ and $Dx_{2m-1} = \omega$. By Lemma 4.2.2, there is $[r] \in H^{2n-2k}(G_{k,n}(\mathbb{C}), \mathbb{Q})$ the class of smallest degree such that $H^*(\iota)([b_{2k}]) \cdot H^*(\iota)([r]) = 0$ in $H^*(E; \mathbb{Q})$, where $r \notin (b_{2k})$. We show that the triple Massey product $\langle H^*(\iota)([b^*_{2k}]), H^*(\iota)([b_{2k}]), H^*(\iota)([r])\rangle$ in $H^*(E; \mathbb{Q})$ is not trivial. It is represented by the cocycle

$$rx_{2m-1} - b^*_{2k} y_{2n-1}.$$

To show that it is not a coboundary, we use an argument in the Leray-Serre spectral sequence for the unit sphere tangent bundle $S^{2m-1} \to E \to G_{k,n}(\mathbb{C})$. In (Félix et al., 2001, Chapter 18), the Leray-Serre spectral sequence is obtained by filtering $(\wedge V \otimes \wedge x_{2m-1}, D)$ by the degree of $\wedge V$; that is,

$$F^p(\wedge V \otimes \wedge x_{2m-1}) = (\wedge V)^{\geq p} \otimes \wedge x_{2m-1},\ p = 0, 1, 2, \ldots$$

and the associated bigraded module is given by

$$\begin{aligned} E_0^{p,q} &= (\wedge V)^{\geq p} \otimes \wedge x_{2m-1} / (\wedge V)^{\geq (p+1)} \otimes \wedge x_{2m-1} \\ &\cong (\wedge V)^p \otimes \wedge x_{2m-1}. \end{aligned}$$

Moreover, $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ is zero and $d_1 : (\wedge V)^p \otimes \wedge x_{2m-1} \to (\wedge V)^{p+1} \otimes \wedge x_{2m-1}$ is $d \otimes 1$. Therefore, $E_2^{p,*} = H^p(\wedge V, d) \otimes \wedge x_{2m-1}$. Thus, $[rx_{2m-1} - b^*_{2k} y_{2n-1}] \cong [rx_{2m-1}]$ at

$E_2^{2(n-k),q}$ and we have $E_2 = E_3 = \cdots = E_{2m}$. In particular, $E_{2m}^{2(n-k),2m-1} \cong H^{2(n-k)}(\wedge V, d) \otimes \mathbb{Q} < x_{2m-1} >$. Moreover, $d_{2m} : E_{2m}^{2(n-k),2m-1} \to E_{2m}^{2(n-k)+2m,0}$ is zero, for degree reasons. Hence, the element $rx_{2m-1} \in E_{2m}^{2(n-k),2m-1}$ is a d_{2m}-cocycle. Moreover, it cannot be a d_{2m}-coboundary because $E_{2m}^{2(n-k)-2m,4m-2} = 0$. Hence the class $[rx_{2m-1}]$ is not zero at $E_{2m+1} = E_\infty$. This is a non zero triple Massey product in $H^*(E;\mathbb{Q})$. Therefore, E is not formal. □

Remark 4.2.4. We recall here that Kähler manifolds are the best known example of symplectic manifolds $X = M^{2m}$. Therefore, one might conclude that the cocycle $rx_{2m-1} - b_{2k}^* y_{2n-1}$ represents a non-zero cohomology class by considering the bigraded model $(\wedge(b_2,\ldots,b_{2k},y_{2(n-k)+1},\cdots,y_{2n-1},x_{2m-1}),d)$, where x_{2m-1} is the only generator of lower degree 2 (see (Halperin & Stasheff, 1979, §3)). As $rx_{2m-1} - b_{2k}^* y_{2n-1}$ is a sum of elements of respective degrees lower than 2 and 1, it cannot be a coboundary. Hence, the same approach can be applied to show that the total space of the unit sphere tangent bundle over a formal symplectic manifold X is not formal if $H^*(X;\mathbb{Q})$ is non-monogenic.

Example 4.2.5. The minimal Sullivan model of $G_{2,4}(\mathbb{C})$ is given by $(\wedge(b_2,b_4,y_5,y_7),d)$, where $db_2 = db_4 = 0$, $dy_5 = -b_2^3 + 2b_2b_4$, $dy_7 = b_2^4 - 3b_2^2b_4 + b_4^2$ as h_j is the $2j$-th degree term in the Taylor expansion of $(1+b_2+b_4)^{-1}$ (Hoffman, 1982; Charkaborty & Sankaran, 2014). With $\chi(G_{2,4}(\mathbb{C})) = 5$, the total space of the unit sphere bundle $S^7 \to E \to G_{2,4}(\mathbb{C})$ will have a relative minimal model of the form $(\wedge(b_2,b_4,y_5,y_7,a_7),D)$ with $Db_i = 0$, $Dy_5 = b_2(b_2^2 - 2b_4)$, $Dy_7 = b_2^4 - 3b_2^2b_4 + b_4^2$ and $Da_7 = b_2^4$. Take $a = H^*(\iota)([b_2^3])$, $b = H^*(\iota)([b_2])$ and $c = H^*(\iota)([b_2^2 - 2b_4])$ cohomology classes in $H^*(E,\mathbb{Q})$. The products $a \cdot b = b \cdot c = 0$.

The triple Massey product set $\langle a,b,c\rangle$ is represented by the cocycle $(b_2^2-2b_4)a_7-b_2^3y_5$ of degree 11 which cannot be a coboundary for degree reasons. Thus, the triple Massey product set $\langle a,b,c\rangle$ is non-trivial.

4.3 Hochschild cohomology of a Sullivan model of mapping spaces

Let $\phi:(\wedge V,d)\to(B,d)$ be a surjective morphism between commutative differential graded algebras, where V is finite dimensional. We consider (B,d) as module over $\wedge V$ via the action induced by the mapping ϕ. In this section, we show that the Hochschild cohomology $HH^*(\wedge V;B)$ can be computed in terms of the graded vector space of positive ϕ-derivations. We begin with the following results for ϕ-derivations.

Proposition 4.3.1. If $\phi:(\wedge V,d)\to(B,d)$ is a morphism of cdga's, then $\mathrm{Der}(\wedge V,B;\phi)$ is a graded differential module over $(\wedge V,d)$.

Proof. Let $a\in\wedge V$ and $\theta\in\mathrm{Der}(\wedge V,B;\phi)$. Define $a\theta$ by $(a\theta)(x)=\phi(a)\theta(x)$. This action makes $\mathrm{Der}(\wedge V,B;\phi)$ a graded module of $\wedge V$. Then

$$\begin{aligned}(\partial(a\theta))(x)&=d((a\theta)(x))-(-1)^{|a|+|\theta|}a\theta(dx)\\&=(da)\theta(x)+(-1)^{|a|}a(d\theta(x))-(-1)^{|a|+|\theta|}a\theta(dx)\\&=(da)\theta(x)+(-1)^{|a|}(ad\theta(x)-(-1)^{|\theta|}a\theta(dx))\\&=(da)\theta(x)+(-1)^{|a|}a(\partial\theta)(x).\end{aligned}$$

Hence $\partial(a\theta) = (da)\theta + (-1)^{|a|}a(\partial\theta)$, that is, $\mathrm{Der}(\wedge V, B; \phi)$ is graded differential module over $(\wedge V, d)$. □

Proposition 4.3.2. (Gatsinzi, 2019) Let $\phi : (\wedge V, d) \to (B, d)$ be a surjective morphism between cdga's where V is finite dimensional and $I = \mathrm{Ker}\,\phi$. Then $\mathrm{Der}(\wedge V, B; \phi) \cong (\wedge V/I) \otimes V^*$.

On the other hand, consider the desuspension $s^{-1}\theta \in s^{-1}\,\mathrm{Der}(\wedge V, B; \phi)$. We note there is an induced $\wedge V$-module structure on $s^{-1}\,\mathrm{Der}(\wedge V, B; \phi)$ defined by $a(s^{-1}\theta) = (-1)^{|a|}s^{-1}(a\theta)$.

Proposition 4.3.3. Let $\phi : (\wedge V, d) \to (B, d)$ be a surjective morphism between cdga's where V is finite dimensional and $I = \mathrm{Ker}\,\phi$. Then $s^{-1}\,\mathrm{Der}(\wedge V, B; \phi) \cong (\wedge V/I) \otimes s^{-1}V^*$ as $\wedge V$-modules.

Proof. If $\{v_1, \ldots, v_n\}$ is a basis of V, then in $\mathrm{Der}(\wedge V, B; \phi)$ we denote the derivation $(v_i, 1)$ by v_i^*. As ϕ is surjective, there is $a_i \in \wedge V$ such that $\phi(a_i) = b_i$, where $b_i \in B$. Let $\theta \in \mathrm{Der}(\wedge V, B; \phi)$. Then, $\theta = \sum a_i v_i^*$. Therefore, $\mathrm{Der}(\wedge V, B; \phi)$ is generated by V^*. This gives a surjective linear map $\wedge V \otimes V^* \to \mathrm{Der}(\wedge V, B; \phi)$ of which the kernel is $I \otimes V^*$. Hence by the first isomorphism theorem $\mathrm{Der}(\wedge V, B; \phi) \cong (\wedge V/I) \otimes V^*$. We have that, $s^{-1}\theta = \sum(-1)^{|a_i|}a_i s^{-1}v_i^*$. Therefore, $s^{-1}\,\mathrm{Der}(\wedge V, B; \phi) \cong (\wedge V/I) \otimes s^{-1}V^*$. □

In (Gatsinzi, 2019, Lemma 15), it is shown that there is an isomorphism of differential graded vector spaces $\mathrm{Hom}_{\wedge V}(\wedge V \otimes sV, B) \cong \mathrm{Der}(\wedge V, B; \phi)$. We recall the following result.

Theorem 4.3.4. *(Gatsinzi, 2019) Let $f : X \to Y$ be a map between simply connected spaces having the rational homotopy type of a CW complex of finite type and $\phi : (\wedge V, d) \to (B, d)$ a Sullivan model of f. Then there is a canonical injection*

$$\iota : \pi_*(\Omega\,\mathrm{map}(X,Y;f)) \otimes \mathbb{Q} \to HH^*(\wedge V; B).$$

Moreover, there is a natural isomorphism $\pi_*(\Omega\,\mathrm{map}(X,Y;f)) \otimes \mathbb{Q} \to HH^*_{(1)}(\wedge V; B)$.

Moreover, let $\phi : (\wedge V, d) \to (B, d)$ be a surjective morphism between cdga's where V is finite dimensional. Consider the commutative differential graded algebra

$$\wedge_{\wedge V} s^{-1}\,\mathrm{Der}(\wedge V, B; \phi) = T_{\wedge V}(s^{-1}\,\mathrm{Der}(\wedge V, B; \phi))/I$$

where I is the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$ for $x, y \in T_{\wedge V}(s^{-1}\,\mathrm{Der}(\wedge V, B; \phi))$. The differential ∂' on $s^{-1}\,\mathrm{Der}(\wedge V, B; \phi)$ extends to the differential d_0 on

$$\wedge_{\wedge V} s^{-1}\,\mathrm{Der}(\wedge V, B; \phi) = \wedge V \oplus s^{-1}\,\mathrm{Der}(\wedge V, B; \phi) \oplus \wedge^2_{\wedge V} s^{-1}\,\mathrm{Der}(\wedge V, B; \phi) \oplus \cdots$$

by the Leibniz rule. Our aim is to relate the commutative graded algebra $H_*(\wedge_{\wedge V} s^{-1}\,\mathrm{Der}(\wedge V, B; \phi), d_0)$ to $HH^*(\wedge V; B)$. In particular, the following result extends Theorem 4.3.4.

Theorem 4.3.5. *If* $\phi : (\wedge V, d) \to (B, d)$ *is a surjective morphism between cdga's, where V is finite dimensional, then there is an algebra isomorphism* $\Psi : (\wedge_{\wedge V} s^{-1}\operatorname{Der}(\wedge V, B; \phi), d_0) \to (\operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, B), D)$.

The proof is given by the following Lemmas. Let $\phi : (\wedge V, d) \to (B, d)$ be a morphism of cdga's. Assume V is finite dimensional and let $\{v_i, \ldots, v_n\}$, be a homogeneous linear basis of V. Define the map

$$\psi : (s^{-1}\operatorname{Der}(\wedge V, B; \phi), \partial') \to (\operatorname{Hom}_{\wedge V}(\wedge V \otimes sV, B), D)$$

by $\psi(s^{-1}\theta)(sv) = (-1)^{|\theta|}\theta(v)$ for $v \in V$.

Lemma 4.3.6. The map ψ commutes with differentials.

Proof. Let $s^{-1}\theta \in s^{-1}\operatorname{Der}(\wedge V, B; \phi)$. Then $\partial'(s^{-1}\theta) = -s^{-1}\partial\theta$ and $|\partial\theta| = |\theta| - 1$. Hence

$$\begin{aligned}
\psi(\partial' s^{-1}\theta)(sv) &= \psi(-s^{-1}\partial\theta)(sv) \\
&= -(-1)^{|\theta|-1}(\partial\theta)(v) \\
&= (-1)^{|\theta|}(d\theta(v) - (-1)^{|\theta|}\theta(dv)).
\end{aligned}$$

Moreover,

$$\begin{aligned} D(\psi(s^{-1}\theta))(sv) &= d\psi(s^{-1}\theta)(sv) + (-1)^{|\theta|}\psi(s^{-1}\theta)(dsv) \\ &= (-1)^{|\theta|}d\theta(v) - (-1)^{|\theta|}\psi(s^{-1}\theta)(Sdv) \\ &= (-1)^{|\theta|}(d\theta(v) - (-1)^{|\theta|}\theta(dv)). \end{aligned}$$

Therefore

$$D\psi(s^{-1}\theta) = \psi(\partial' s^{-1}\theta).$$

Thus, ψ commutes with differentials. □

Further, we follow (Cattaneo & Felder, 2007, §4) and (Gatsinzi, 2010) for this construction. The map

$$\psi : s^{-1}\operatorname{Der}(\wedge V, B; \phi) \to \operatorname{Hom}_{\wedge V}(\wedge V \otimes sV, B)$$

can be canonically extended to

$$\psi_r : (\wedge^r_{\wedge V} s^{-1}\operatorname{Der}(\wedge V, B; \phi), d_0) \to (\operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge^r sV, B), D)$$

for $r \geq 1$ by the formula

$$\psi_r(\alpha_1 \wedge \cdots \wedge \alpha_r)(sx_1 \wedge \cdots \wedge sx_r) = \sum_{\sigma \in S_r} \varepsilon(\sigma)\psi(\alpha_1)(sx_{\sigma(1)}) \cdots \psi(\alpha_r)(sx_{\sigma(r)}), \qquad (7)$$

where $\varepsilon(\sigma)$ is the Koszul sign of the permutation $\sigma \in S_r$. Put $\psi_0 = 1_{\wedge V}$, $\psi_1 = \psi$ and define

$$\Psi : (\wedge_{\wedge V} s^{-1} \operatorname{Der}(\wedge V, B; \phi), d_0) \to (\operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, B), D)$$

by $\Psi = \oplus_{r \geq 0} \psi_r$.

Lemma 4.3.7. The map Ψ is a morphism of graded algebras.

Proof. Recall that if V is a graded vector space, then $\wedge sV$ is a cocommutative coalgebra under the comultiplication Δ defined by

$$\begin{aligned} &\Delta(sx_1 \wedge \cdots \wedge sx_r) \\ &\quad = \sum_{p=0}^{r} \frac{1}{p!(r-p)!} \sum_{\sigma \in S_r} \varepsilon(\sigma) sx_{\sigma(1)} \wedge \cdots \wedge sx_{\sigma(p)} \otimes sx_{\sigma(p+1)} \wedge \cdots \wedge sx_{\sigma(r)} \end{aligned}$$

(Félix et al., 2001, §22). As B is a graded algebra with multiplication μ then for $x, y \in \operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, B)$ the product, $x \cdot y$ given by the composition $\wedge sV \xrightarrow{\Delta} \wedge sV \otimes \wedge sV \xrightarrow{x \otimes y} B \otimes B \xrightarrow{\mu} B$, endows the graded module $\operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, B)$ a structure of a graded algebra (see, (Félix et al., 2005)). For simplicity, let $f = \alpha_1 \wedge \cdots \wedge \alpha_p \in \wedge^p_{\wedge V} s^{-1} \operatorname{Der}(\wedge V, B; \phi)$ and $g = \alpha_{p+1} \wedge \cdots \wedge \alpha_r \in \wedge^{r-p}_{\wedge V} s^{-1} \operatorname{Der}(\wedge V, B; \phi)$ for $\alpha_i \in s^{-1} \operatorname{Der}(\wedge V, B; \phi)$. To show that ψ_r is a morphism of graded algebras we only need to show that $\psi_p(f)\psi_{r-p}(g) = \psi_r(fg)$. Applying the multiplication μ via the composition $\mu \circ (\psi_p(f) \otimes \psi_{r-p}(g)) \circ \Delta$ and using (7) one gets

$$\begin{aligned}
&\psi_p(f)\psi_{r-p}(g)(sx_1 \wedge \cdots \wedge sx_r) \\
&= \frac{1}{p!(r-p)!} \sum_{\sigma \in S_r} \varepsilon(\sigma)\psi_p(f)(sx_{\sigma(1)} \wedge \cdots \wedge sx_{\sigma(p)})\psi_{r-p}(g)(sx_{\sigma(p+1)} \\
&\quad \wedge \cdots \wedge sx_{\sigma(r)}) \\
&= \frac{1}{p!(r-p)!} \Bigg[\sum_{\tau \in S_p} \varepsilon(\sigma)\varepsilon(\tau)\psi(\alpha_1)(sx_{\tau(\sigma(1))}) \cdots \psi(\alpha_p)(sx_{\tau(\sigma(p))}) \cdot \\
&\quad \sum_{\kappa \in S_{r-p}} \varepsilon(\sigma)\varepsilon(\kappa)\psi(\alpha_{p+1})(sx_{\kappa(\sigma(p+1))}) \cdots \psi(\alpha_r)(sx_{\kappa(\sigma(r))}) \Bigg] \\
&= \frac{1}{p!(r-p)!} \Bigg[\sum_{\substack{\tau \in S_p \\ \kappa \in S_{r-p}}} \varepsilon(\sigma)\varepsilon(\tau)\varepsilon(\kappa)\psi(\alpha_1)(sx_{\tau(\sigma(1))}) \cdots \psi(\alpha_p)(sx_{\tau(\sigma(p))}) \\
&\quad \psi(\alpha_{p+1})(sx_{\kappa(\sigma(p+1))}) \cdots \psi(\alpha_r)(sx_{\kappa(\sigma(r))}) \Bigg] \\
&= \sum_{\sigma' \in S_r} \varepsilon(\sigma')\psi(\alpha_1)(sx_{\sigma'(1)}) \cdots \psi(\alpha_p)(sx_{\sigma'(p)})\psi(\alpha_{p+1})(sx_{\sigma'(p+1)}) \cdots \\
&\quad \psi(\alpha_r)(sx_{\sigma'(r)}), \text{ where } \sigma' = (\tau \times \kappa) \circ \sigma \text{ and } \tau \times \kappa \in S_p \times S_{r-p} \subseteq S_r. \\
&= \psi_r(fg)(sx_1 \wedge \cdots \wedge sx_r).
\end{aligned}$$

Hence $\psi_p(f)\psi_{r-p}(g) = \psi_r(fg)$. Thus, ψ_r is a morphism of graded algebras. □

Lemma 4.3.8. The map ψ_r commutes with differentials.

Proof. Denote by $f = \alpha_1 \wedge \cdots \wedge \alpha_r \in \wedge^r_{\wedge V} s^{-1}\operatorname{Der}(\wedge V, B; \phi)$ for $\alpha_i \in s^{-1}\operatorname{Der}(\wedge V, B; \phi)$.

Then

$$d_0 f = \sum_i (-1)^{|\eta_i|} \alpha_1 \wedge \cdots \wedge \partial' \alpha_i \wedge \cdots \wedge \alpha_r$$

where $|\eta_i| = \sum_{k<i} |\alpha_k|$. Let $\xi_i = \sum_{k=1}^{i-1} |sx_{\sigma(k)}|$. Then,

$$\begin{aligned}
&\psi_r(d_0 f)(sx_1 \wedge \cdots \wedge sx_r) \\
&\quad = \psi_r\left(\sum_i (-1)^{|\eta_i|} \alpha_1 \wedge \cdots \wedge \partial' \alpha_i \wedge \cdots \wedge \alpha_r\right)(sx_1 \wedge \cdots \wedge sx_r) \\
&\quad = \sum_i (-1)^{|\eta_i|} \psi_r(\alpha_1 \wedge \cdots \wedge \partial' \alpha_i \wedge \cdots \wedge \alpha_r)(sx_1 \wedge \cdots \wedge sx_r) \\
&\quad = \sum_i (-1)^{|\eta_i|} \sum_{\sigma \in S_r} \varepsilon(\sigma)(-1)^{\xi_i} \psi(\alpha_1)(sx_{\sigma(1)}) \cdots \psi(\partial' \alpha_i)(sx_{\sigma(i)}) \cdots \psi(\alpha_r)(sx_{\sigma(r)}) \\
&\quad = \sum_i (-1)^{|\eta_i|} \sum_{\sigma \in S_r} \varepsilon(\sigma)(-1)^{\xi_i} \psi(\alpha_1)(sx_{\sigma(1)}) \cdots \left[D\psi(\alpha_i)(sx_{\sigma(i)})\right] \cdots \psi(\alpha_r)(sx_{\sigma(r)}) \\
&\quad = \sum_i (-1)^{|\eta_i|} \sum_{\sigma \in S_r} \varepsilon(\sigma)(-1)^{\xi_i} \psi(\alpha_1)(sx_{\sigma(1)}) \cdots \Big[d(\psi(\alpha_i)(sx_{\sigma(i)})) + (-1)^{|f|} \\
&\qquad \psi(\alpha_i)(Dsx_{\sigma(i)})\Big] \cdots \psi(\alpha_r)(sx_{\sigma(r)}) \\
&\quad = \sum_i (-1)^{|\eta_i|} \sum_{\sigma \in S_r} \varepsilon(\sigma)(-1)^{\xi_i} \psi(\alpha_1)(sx_{\sigma(1)}) \cdots \Big[d(\psi(\alpha_i)(sx_{\sigma(i)})) - (-1)^{|f|} \\
&\qquad \psi(\alpha_i)(\mathcal{S}dx_{\sigma(i)})\Big] \cdots \psi(\alpha_r)(sx_{\sigma(r)}) \\
&\quad = \sum_i (-1)^{|\eta_i|} \sum_{\sigma \in S_r} \varepsilon(\sigma)(-1)^{\xi_i} \psi(\alpha_1)(sx_{\sigma(1)}) \cdots d(\psi(\alpha_i)(sx_{\sigma(i)})) \cdots \psi(\alpha_r) \\
&\qquad (sx_{\sigma(r)}) - (-1)^{|f|} \sum_i (-1)^{|\eta_i|} \sum_{\sigma \in S_r} \varepsilon(\sigma)(-1)^{\xi_i} \psi(\alpha_1)(sx_{\sigma(1)}) \cdots \psi(\alpha_i)(\mathcal{S}dx_{\sigma(i)}) \\
&\qquad \cdots \psi(\alpha_r)(sx_{\sigma(r)}).
\end{aligned}$$

Moreover,

$$
\begin{aligned}
&D(\psi_r(f)(sx_1 \wedge \cdots \wedge sx_r)) \\
&\quad = d\left(\sum_{\sigma \in S_r} \varepsilon(\sigma)\psi(\alpha_1)(sx_{\sigma(1)}) \cdots \psi(\alpha_r)(sx_{\sigma(r)})\right) + (-1)^{|f|}\psi_r(f)(D(sx_1 \wedge \cdots \wedge sx_r)) \\
&\quad = \sum_i (-1)^{|\eta_i|} \sum_{\sigma \in S_r} \varepsilon(\sigma)(-1)^{\xi_i}\psi(\alpha_1)(sx_{\sigma(1)}) \cdots d(\psi(\alpha_i)(sx_{\sigma(i)})) \cdots \psi(\alpha_r) \\
&\qquad (sx_{\sigma(r)}) - (-1)^{|f|}\sum_i (-1)^{|\eta_i|} \sum_{\sigma \in S_r} \varepsilon(\sigma)(-1)^{\xi_i}\psi(\alpha_1)(sx_{\sigma(1)}) \cdots \psi(\alpha_i)(\mathcal{S}dx_{\sigma(i)}) \\
&\qquad \cdots \psi(\alpha_r)(sx_{\sigma(r)}).
\end{aligned}
$$

Therefore

$$\psi_r(d_0 f) = D(\psi_r(f)).$$

We deduce that ψ_r commutes with differentials. □

Lemma 4.3.9. If $\phi : (\wedge V, d) \to (B, d)$ is a surjective morphism between cdga's where V is finite dimensional, then

$$\psi_r : (\wedge^r_{\wedge V} s^{-1}\operatorname{Der}(\wedge V, B; \phi), d_0) \to (\operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge^r sV, B), D)$$

is bijective.

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of V. Consider the ϕ-derivation $\theta_i = (v_i, 1) \in \operatorname{Der}(\wedge V, B; \phi)$ and the corresponding $\bar{\theta}_i = s^{-1}\theta_i \in s^{-1}\operatorname{Der}(\wedge V, B; \phi)$. Assume $\{sv_{i_1} \wedge \ldots \wedge sv_{i_r}\}$ is a basis of $\wedge^r sV$ for $i_1 \leq i_2 \leq \cdots \leq i_r$ and $r \geq 1$. We denote by $\sum(sv_{i_1} \wedge \cdots \wedge sv_{i_r}, b_{i_1} \ldots_{i_r})$ the element $f \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge^r sV, B)$ such that $f(sv_{i_1} \wedge \cdots \wedge sv_{i_r}) = b_{i_1} \ldots_{i_r}$ and zero on other elements of the basis of $\wedge^r sV$. As ϕ is surjective, there exists $a_{i_1} \ldots_{i_r} \in \wedge V$ such that $\phi(a_{i_1} \ldots_{i_r}) = b_{i_1} \ldots_{i_r}$. Let $\bar{\theta} = \sum_{i_1 \ldots i_r} a_{i_1} \ldots_{i_r} \cdot \bar{\theta}_{i_1} \wedge \cdots \wedge \bar{\theta}_{i_r} \in \wedge^r_{\wedge V} s^{-1} \mathrm{Der}(\wedge V, B; \phi)$. Define

$$\gamma_r : (\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge^r sV, B), D) \to (\wedge^r_{\wedge V} s^{-1} \mathrm{Der}(\wedge V, B; \phi), d_0)$$

by $\gamma_r(f) = \sum a_{i_1} \ldots_{i_r} \cdot \bar{\theta}_{i_1} \wedge \cdots \wedge \bar{\theta}_{i_r}$. It is easily verified that both compositions of ψ_r and γ_r equal to the identities. Hence, γ_r is the inverse of ψ_r, and we deduce that each ψ_r is bijective, for each $r \geq 1$. □

As an application of Theorem 4.3.5, we show that Hochschild cohomology algebra of a surjective Sullivan model $\phi : (\wedge V, d) \to (B, d)$ of a based map $f : X \to Y$ between simply connected finite CW-complexes contains a polynomial algebra.

Definition 4.3.10. (Lupton & Smith, 2007) Given a commutative differential graded algebra map $\phi : (\wedge V, d) \to (B, d)$, post-composition by the augmentation $\varepsilon : B \to \mathbb{Q}$ gives a map of chain complexes $\varepsilon_* : \mathrm{Der}(\wedge V, B; \phi) \to \mathrm{Der}(\wedge V, \mathbb{Q}; \varepsilon)$ given by $\varepsilon_*(\varphi)(v) = \varepsilon(\varphi(v))$ for $\varphi \in \mathrm{Der}(\wedge V, B; \phi)$. We define the evaluation subgroups of ϕ by $G_n(\wedge V, B; \phi) = \mathrm{Im}\{H(\varepsilon_*) : H_n(\mathrm{Der}(\wedge V, B; \phi)) \to H_n(\mathrm{Der}(\wedge V, \mathbb{Q}; \varepsilon))\}$.

Moreover, if ϕ is a Sullivan model of a based map $f : X \to Y$ between simply connected finite CW-complexes, then $G_n(\wedge V, B; \phi) \cong \pi_*(ev) \otimes \mathbb{Q}$, where $ev : \mathrm{map}(X, Y; f) \to Y$ is the evaluation at the base point (Félix et al., 2001; Lupton & Smith, 2007).

The following result extends (Gatsinzi, 2010, Theorem 11).

Theorem 4.3.11. *If $\phi : (\wedge V, d) \to (B, d)$ is a surjective Sullivan model of a based map $f : X \to Y$ between simply connected finite CW-complexes, then the commutative graded algebra $HH^*(\wedge V; B)$ contains a polynomial algebra as a graded sub algebra.*

Proof. Let $(\wedge V, d) = (\wedge(v_1, \ldots, v_n), d)$ be a minimal Sullivan model with $|v_1| \leq |v_2| \leq \cdots \leq |v_n|$. In $\mathrm{Der}(\wedge V, B; \phi)$, let $v_i^* = (v_i, 1)$. Since Y is a finite CW-complex, then $|v_n|$ is odd (Félix et al., 2001, §29). Thus, in this case v_n^* is odd and a simple calculation shows that $\partial v_n^* = 0$. Further, v_n^* cannot be a boundary for degree reasons. Hence $[v_n^*]$ represents a non zero homology class in $H_*(\mathrm{Der}(\wedge V, B; \phi))$ and $H(\varepsilon_*)([v_n^*]) \neq 0$. Moreover, let $s^{-1}v_n^* \in s^{-1}\mathrm{Der}(\wedge V, B; \phi)$. Then $[s^{-1}v_n^*]$ represents a non zero homology class in $H_*(s^{-1}\mathrm{Der}(\wedge V, B; \phi))$. Further, let $\theta_1 = \psi(s^{-1}v_n^*) \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes sV, B)$. Define $\theta_k \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge^k sV, B)$ by $\theta_k(sv_n \wedge \cdots \wedge sv_n) = 1$ and zero on other elements of a basis of $\wedge^k(sV)$. Then, we claim by contradiction that $[\theta_k] \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge^k sV, B)$ is a non zero cohomology class. Assume there exists $\gamma_k \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge^k sV, B)$ such that $D\gamma_k = \theta_k$. Hence,

$$
\begin{aligned}
\theta_k(sv_n)^k &= D\gamma_k(sv_n)^k \\
&= d\gamma_k(sv_n)^k - (-1)^{|\gamma_k|}\gamma_k(D(sv_n)^k) = 1.
\end{aligned}
$$

Moreover, $d\gamma_k(sv_n)^k \in B^{\geq 2}$. If $dv \neq 0$, then $dv \in \wedge^{\geq 2}V$. Hence, $\mathcal{S}(dv) \in \wedge^+ V \otimes sV$. Therefore, $\gamma_k D(sv_n)^k = \gamma_k(-\mathcal{S}dv_n \wedge sv_n \wedge \cdots \wedge sv_n) \subseteq \wedge^+ V \otimes \wedge^k(sV)$. Thus, $\gamma_k(D(sv_n)^k) \in B^{\geq 2}$. Hence a contradiction. Therefore, θ_k is a non zero cohomology class. Moreover, $\theta_1^k = k!\theta_k \neq 0$. We deduce that $HH^*(\wedge V;B)$ contains a sub algebra $\mathbb{Q}[\theta_1]$ isomorphic to $\wedge(s^{-1}v_n^*)$.

□

Remark 4.3.12. The proof of the above result could be adapted to show that $HH^*(\wedge V;B)$ contains a polynomial algebra over $s^{-1}G_*(\wedge V,B;\phi)$.

The following Example illustrates the result above.

Example 4.3.13. Consider the inclusion $G_{2,4}(\mathbb{C}) \rightarrowtail G_{2,5}(\mathbb{C})$ between complex Grassmannians. A Sullivan model of the inclusion is given by

$$\phi : \wedge V = (\wedge(a_2,a_4,a_7,a_9),d) \to (\wedge(b_2,b_4,b_5,b_7),d) = B,$$

where $da_2 = da_4 = 0$, $da_7 = a_4^2 - 3a_2^2a_4 + a_2^4, da_9 = 4a_2^3a_4 - 3a_2a_4^2 - a_2^5$, and $db_2 = db_4 = 0$, $db_5 = 2b_2b_4 - b_2^3, db_7 = b_4^2 - 3b_2^2b_4 + b_2^4$. Further, consider the ideal

$$I = (2b_2b_4 - b_2^3, b_4^2 - 3b_2^2b_4 + b_2^4).$$

Then, $H^*(B,d) = \wedge(b_2,b_4)/I$. Hence, there is a quasi-isomorphism

$$\bar{\phi} : \wedge V = (\wedge(a_2,a_4,a_7,a_9),d) \to H^*(B,d),$$

where $\phi(a_2) = b_2, \phi(a_4) = b_4, \phi(a_7) = 0$ and $\phi(a_9) = 0$. Thus $\bar{\phi}$ is surjective. Moreover, define $\theta_7 = (a_7, 1), \theta_9 = (a_9, 1) \in \mathrm{Der}(\wedge V, H^*(B); \bar{\phi})$. One verifies that $\partial\theta_7 = \partial\theta_9 = 0$. Thus, $[\theta_7]$ and $[\theta_9]$ are non zero cohomology classes in $H_*(\mathrm{Der}(\wedge V, H^*(B); \bar{\phi}))$. A simple calculation shows that $\theta_2 = (a_2, 1), \theta_4 = (a_4, 1)$ are not cocycles in $\mathrm{Der}(\wedge V, H^*(B); \bar{\phi})$. Let $\bar{\theta} = s^{-1}\theta_7$, then $\psi(\bar{\theta}^k)$ is a non zero cohomology class in $\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge^k sV, B)$, for all $k \geq 1$. It cannot be coboundary because for $\beta_k \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge^k sV, B)$, then $D\beta_k(sa_7 \wedge \cdots \wedge sa_7) = \psi(\bar{\theta}^k)(sa_7 \wedge \cdots \wedge sa_7) \in (sa_7)$, where (sa_7) is the ideal generated by sa_7. But $\psi(\bar{\theta}^k)(sa_7 \wedge \cdots \wedge sa_7) = k!$. In a similar way, $(s^{-1}\theta_9)^k \neq 0$. Therefore, $HH^*(\wedge V; B)$ contains a polynomial algebra isomorphic to $\wedge(s^{-1}\theta_7, s^{-1}\theta_9) \cong \wedge(s^{-1}G_*(\wedge V, B; \phi))$.

5 Main result

We begin this chapter by providing details on main result which extends (Gatsinzi, 2016). A space X and its model $(\wedge V, d)$ are called elliptic, if and only if V and $H^*(\wedge V, d)$ are both finite dimensional. Topologically, this means that both $\pi_*(X) \otimes \mathbb{Q}$ and $H^*(X;\mathbb{Q})$ are finite-dimensional $\mathbb{Q}$-vector spaces (Félix et al., 2001, §32). For instance homogeneous spaces are elliptic. Let X be an elliptic space. A space X is an n-stage Postnikov tower and its minimal Sullivan model is given by $(\wedge V, d) = (\wedge(V_0 \oplus \cdots \oplus V_{n-1}), d)$, where $dV_i \subset \wedge(V_0 \oplus \cdots \oplus V_{i-1})$. Homogeneous spaces are elliptic 2-stage Postnikov towers. Our aim is to study properties of the map induced in Hochschild cohomology by a Koszul Sullivan model (KS-model for short) of a TNCZ fibration with base an elliptic 2-stage Postnikov tower. Hence, our main result reads as follows.

Theorem 5.0.1. *If $F \to E \xrightarrow{p} X$ is a TNCZ fibration, where X is an elliptic 2-stage Postnikov tower and $f : (\wedge V, d) \rightarrowtail (C, d)$ a KS-model of p, then the induced map in Hochschild cohomology*

$$HH^*(f) : HH^*(\wedge V; \wedge V) \to HH^*(\wedge V; C)$$

is injective.

5.1 Hochschild cohomology of certain Koszul Sullivan extensions

We present some results about Hochschild cohomology for the kind of commutative differential graded algebras that arise as minimal models in rational homotopy theory.

Let $(\wedge V, 0)$ be a Sullivan algebra where V is finite dimensional. We consider a KS-model $f : (\wedge V, 0) \rightarrowtail (\wedge V \otimes \wedge x_{2k+1}, d) = C$ of commutative differential graded cochain algebras, where $dx_{2k+1} \in \wedge V$ is non zero. It induces a homomorphism of Hochschild cochain complexes

$$\Phi : \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V) \to \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C).$$

If $g \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V)$, then $\Phi(g)$ is the composition of $\wedge V$-modules

$$(\wedge V \otimes \wedge sV, 0) \xrightarrow{g} (\wedge V, 0) \overset{f}{\rightarrowtail} (\wedge V \otimes \wedge x_{2k+1}, d).$$

Moreover, for $g \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C)$

$$\begin{aligned}(Dg)(sv) &= d_C \cdot g(sv) - (-1)^{|g|} g(D(sv)),\\ &= d_C g(sv) - (-1)^{|g|} g(-\mathcal{S}dv),\\ &= d_C g(sv) - (-1)^{|g|} g(0),\\ &= d_C g(sv).\end{aligned}$$

Hence, g is a cocycle if and only if $d_C(g(sv)) = 0$.

Theorem 5.1.1. *Let $f : (\wedge V, 0) \rightarrowtail (\wedge V \otimes \wedge x_{2k+1}, d) = C$ be a KS-model, where $dx_{2k+1} \in \wedge V$ is non zero. Then,*

$$HH^*(\wedge V; C) \cong \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, H^*(C)).$$

Proof. We consider the complex

$$\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C).$$

Let g be a cocycle in $\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C)$. Given a homogeneous basis $\{w_i\}$ of $\wedge sV$ and $g(w_i) = b_i$, we have that,

$$(Dg)(w_i) = d_C g(w_i) = d_C(b_i) = 0.$$

Therefore, g is a cocycle if and only if each b_i is a cocycle in C. We show that, g is not a coboundary. We claim this by contradiction. Assume g is a coboundary, then there is a g_1 such that $Dg_1 = g$. Hence

$$(Dg_1)(w_i) = d_C g_1(w_i) = g(w_i) = b_i.$$

Thus, b_i is a coboundary, this is a contradiction. Hence, g is a coboundary if and only if each b_i is a coboundary. Conversely, if $b_i \in C$ is a cocycle if and only if g is a cocycle. Hence,

$$HH^*(\wedge V; C) \cong \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, H^*(C)).$$

In particular,

$$HH^*_{(1)}(\wedge V; C) \cong \mathrm{Hom}(sV, H^*(C)).$$

□

Theorem 5.1.2. *Let* $f : (\wedge V, 0) \rightarrowtail (\wedge V \otimes \wedge x_{2k+1}, d)$ *be a KS-model, where* $dx_{2k+1} \in \wedge V$ *is non zero. Then, the induced homomorphism of Hochschild cochain complexes* $\Phi :$ $\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V) \to \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C)$ *satisfies*

$$\ker H(\Phi) \cong \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, P \cap \wedge V)$$

where P *is the subspace of coboundaries of* C*.*

Proof. We consider

$$\Phi : \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V) \to \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C).$$

Assume, $\Phi(g) = f \circ g$ is a coboundary in $\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C)$. That is, $f \circ g = dg'$ where

$g' \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C)$. Then for a homogeneous basis $\{w_i\}$ of $\wedge sV$,

$$(Dg')(w_i) = d_C(g'(w_i)) = d_C(\alpha_i),$$

where $\alpha_i = g'(w_i)$. Hence, $(f \circ g)(w_i) = d_C(\alpha_i)$. Thus, $f \circ g \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, P \cap \wedge V)$. Conversely, if $f \circ g \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, P \cap \wedge V)$, then $H(\Phi)([g]) = 0$. Thus, $[g] \in \ker H(\Phi)$ and

$$\ker H(\Phi) \cong \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, P \cap \wedge V).$$

$\square$

Theorem 5.1.3. *If $f : (\wedge V, d) \rightarrowtail (\wedge V, d) \otimes (\wedge W, \bar{d}) = (C, d)$ is a trivial KS-model, where W is finite dimensional, then*

$$HH^*(f) : HH^*(\wedge V; \wedge V) \to HH^*(\wedge V; C)$$

is injective.

Proof. Consider the induced map

$$\Psi : \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V) \to \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C).$$

If $g \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V)$, then $\Psi(g)$ is the composition

$$(\wedge V \otimes \wedge sV, D) \xrightarrow{g} (\wedge V, d) \overset{f}{\rightarrowtail} (\wedge V, d) \otimes (\wedge W, \bar{d}).$$

Let $[g] \in \ker HH^*(f)$, we have that $f \circ g \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C)$ is a coboundary. That is, there is $g' \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, C)$ such that $Dg' = f \circ g$. For a homogeneous basis $\{w_i\}$ of $\wedge sV$ we have $(Dg')(w_i) = (f \circ g)(w_i)$.

Hence

$$\begin{aligned}(Dg')(w_i) &= d(g'(w_i)) - (-1)^{|g'|} g'(D(w_i)) \\ &= (f \circ g)(w_i).\end{aligned}$$

We can decompose g' as $g'(w_i) = a_i \otimes 1 + \alpha_n$, where $\alpha_n \in \wedge V \otimes \wedge^+ W$ and $a_i \in \wedge V$. Hence $d(g'(w_i)) = da_i \otimes 1 + d\alpha_n$. Therefore,

$$da_i \otimes 1 + d\alpha_n - (-1)^{|g'|} g'(D(w_i)) = (f \circ g)(w_i). \tag{8}$$

Moreover, consider the projection

$$(\wedge V, d) \otimes (\wedge W, \bar{d}) \xrightarrow{p} (\wedge V, d).$$

Define

$$g'' : (\wedge V \otimes \wedge sV, D) \to (\wedge V, d)$$

by $g''(w_i) = (p \circ g')(w_i) = a_i$. We show that $(Dg'')(w_i) = g(w_i)$.

$$\begin{aligned}
(Dg'')(w_i) &= d(g''(w_i)) - (-1)^{|g''|} g''(D(w_i)), \\
&= d(p \circ g')(w_i) - (-1)^{|g''|}(p \circ g')(D(w_i)), \\
&= p(f \circ g)(w_i), \text{ using } (8), \\
&= g(w_i).
\end{aligned}$$

We have that, $(Dg'')(w_i) = g(w_i)$, which implies that g is a coboundary in $\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V)$. Therefore, $[g] = 0$. Hence, $HH^*(f)$ is injective. □

5.2 Mapping spaces and fibrations over spheres

In this section, for simplicity, we consider a based map $p : E \to S^{2n}$, together with certain fibrations over spheres, and show that they yield injective maps in Hochschild cohomology. Let $p : E \to S^{2n}$ be a based map and $f : (\wedge V, d) = (\wedge(x_{2n}, x_{4n-1}), d) \to (B, \bar{d})$ a Sullivan model of p. We study the following induced homomorphism of Hochschild cochain complexes

$$\Phi : \mathrm{Hom}_{\wedge V}(\wedge V \otimes sV, \wedge V) \to \mathrm{Hom}_{\wedge V}(\wedge V \otimes sV, B).$$

We prove the following.

Theorem 5.2.1. *Let* $p : E \to S^{2n}$ *be a based map and* $f : (\wedge(x_{2n}, x_{4n-1}), d) \to (\wedge V, \bar{d}) = (B, d)$ *a Sullivan model of* p. *Then*

$$f_* : \mathrm{Der} \wedge (x_{2n}, x_{4n-1}) \to \mathrm{Der}(\wedge(x_{2n}, x_{4n-1}), B; f)$$

is injective in homology.

Proof. Consider the minimal Sullivan model of S^{2n} given by $(\wedge(x_{2n}, x_{4n-1}), d)$ where $dx_{2n} = 0$ and $dx_{4n-1} = x_{2n}^2$. The Lie algebra $(\mathrm{Der} \wedge (x_{2n}, x_{4n-1}), \delta)$ is generated (as a vector space) by the derivations $\theta_{2n} = (x_{2n}, 1)$, $\theta_{2n-1} = (x_{4n-1}, x_{2n})$, $\theta_{4n-1} = (x_{4n-1}, 1)$. A straightforward calculation shows that the differential is given by $\delta\theta_{2n-1} = \delta\theta_{4n-1} = 0$, $\delta\theta_{2n} = 2\theta_{2n-1}$. Thus

$$H_*(\mathrm{Der} \wedge (x_{2n}, x_{4n-1}), \delta) = < [\theta_{4n-1}] > .$$

Therefore, $H_i(\mathrm{Der} \wedge (x_{2n}, x_{4n-1}), \delta) = \mathbb{Q}$ for $i = 4n-1$ and vanishes in all other degrees. If $\theta \in \mathrm{Der} \wedge (x_{2n}, x_{4n-1})$, then $f_*(\theta)$ is the composition $\wedge(x_{2n}, x_{4n-1}) \xrightarrow{\theta} (\wedge(x_{2n}, x_{4n-1}), d) \xrightarrow{f} (\wedge V, \bar{d})$. So, $f_*(\theta_{4n-1}) = (x_{4n-1}, 1) = \alpha_{4n-1}$. Moreover, α_{4n-1} cannot be a boundary for degree reasons. Hence $H(f_*)([\theta_{4n-1}]) \neq 0$, which implies $H(f_*)$ is injective. □

Let $(\wedge V, d)$ be a minimal model of X. If $h : X \to X_{\mathbb{Q}}$ is the rationalization of X, then $G_*(\wedge V) \cong G_*(X_{\mathbb{Q}})$ (Félix et al., 2001, Proposition 29.8). In the proof of Theorem 5.2.1, it shown that $H_*(\mathrm{Der} \wedge V) \cong G_*(S^{2n})$. Moreover, if n is odd, then $H_*(\mathrm{Der} \wedge V) \cong G_*(S^n)$.

Theorem 5.2.2. *(D. Sullivan, 1977; Lupton & Smith, 2007) Let $p : E \to B$ be a map between simply connected CW-complexes of finite type and $f : (\wedge V, d) \to (C, \bar{d})$ a Sullivan model of p. Then there are natural isomorphisms*

$$H_*(\mathrm{Der}\wedge V) \cong \pi_*(\mathrm{aut}_1(B)) \otimes \mathbb{Q}$$

$$H_*(\mathrm{Der}(\wedge V, C; f)) \cong \pi_*(\mathrm{map}(E, B; p)) \otimes \mathbb{Q},$$

where $\mathrm{aut}_1(B)$ *denotes the monoid of self homotopy equivalences of B which are homotopic to the identity.*

We deduce the following result.

Corollary 5.2.3. Let $p : E \to S^n$ be a based map. The map $p^* : \mathrm{aut}_1(S^n) \to \mathrm{map}(E, S^n; p)$ induces an injective map of rational homotopy groups

$$\pi_*(\mathrm{aut}_1(S^n)) \otimes \mathbb{Q} \to \pi_*(\mathrm{map}(E, S^n; p)) \otimes \mathbb{Q}.$$

Proof. The proof follows immediately from Theorem 5.2.2 and Theorem 5.2.1. □

Further, we remind here that in 1977, S. Halperin (Halperin, 1977) formulated a conjecture saying that every fibration with fibre an elliptic space X with evenly graded cohomology (equivalently, with positive Euler characteristic) is TNCZ. This conjecture has been verified in the following cases: if $H^*(X, \mathbb{Q})$ has at most three generators (Lupton, 1990;

Thomas, 1981), if X is a flag manifold and $X = G/H$ is a homogeneous space, where G and H have equal rank (Meier, 1981; Shiga & Tezuka, 1987). The following result uses some of these ideas supported by an example.

Theorem 5.2.4. *Given a TNCZ fibration $X \to E \xrightarrow{p} S^{2n}$ with*

$$f : (\wedge V, d) = (\wedge(x_{2n}, x_{4n-1}), d) \rightarrowtail (\wedge(x_{2n}, x_{4n-1}) \otimes \wedge W, D) = (C, d)$$

a KS-model of p, then the induced map in Hochschild cohomology

$$HH^*(f) : HH^*(\wedge V; \wedge V) \to HH^*(\wedge V; C)$$

is injective.

Proof. Let $(\wedge V, d) = (\wedge(x_{2n}, x_{4n-1}), dx_{2n} = 0, dx_{4n-1} = x_{2n}^2)$ be the minimal Sullivan model of S^{2n}. The complex $(\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V), D)$ is isomorphic to

$$(\wedge V \otimes \wedge((sx_{4n-1})^*, (sx_{2n})^*), D(sx_{4n-1})^* = 0, D(sx_{2n})^* = -2x_{2n}(sx_{4n-1})^*),$$

where $\{(sx_{4n-1})^*, (sx_{2n})^*\}$ is a basis of the dual vector space $(sV)^* = \mathrm{Hom}(sV, \mathbb{Q})$. Moreover, define $g \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V)$ by

$$g(sx_{2n}) = x_{2n}, \quad g(sx_{4n-1}) = 2x_{4n-1}.$$

A simple computation shows that g is a cocycle and $[g]$ is non zero in $HH^*(\wedge V;\wedge V)$. Then, $HH^*(\wedge V;\wedge V)$ is given by $H^*(\wedge V,d)\oplus\wedge^+(sx_{4n-1})^*\oplus <[g]>$. Moreover, for $i\geq 1$, $((sx_{4n-1})^*)^i$ is represented by $\phi_i\in \mathrm{Hom}_{\wedge V}(\wedge V\otimes\wedge sV,\wedge V)$ such that $\phi_i((sx_{4n-1})^i)=1$ and zero otherwise. We claim: $HH^*([\phi_i])\neq 0$ in $HH^*(\wedge V;C)$. By contradiction, assume there exists $\xi_i\in\mathrm{Hom}_{\wedge V}(\wedge V\otimes\wedge sV,C)$ such that $D\xi_i=\phi_i$. Hence

$$\begin{aligned}(D\xi_i)(sx_{4n-1}) &= D\xi_i(sx_{4n-1})-(-1)^{|\xi_i|}\xi_i(dsx_{4n-1})\\ &= D\xi_i(sx_{4n-1})-(-1)^{|\xi_i|}\xi_i(2x_{2n}sx_{2n})\\ &= D\xi_i(sx_{4n-1})-(-1)^{|\xi_i|}2x_{2n}\xi_i(sx_{2n})=1.\end{aligned}$$

But $D\xi_i(sx_{4n-1})\in C^{\geq 2}$. In the same way, $2x_{2n}\xi_i(sx_{2n})\in C^{\geq 2}$, which is in contradiction with $(D\xi_i)(sx_{4n-1})=1$. Hence $HH^*(f)([\phi_i])\neq 0$. Further, assume there is $g'\in\mathrm{Hom}_{\wedge V}(\wedge V\otimes\wedge sV,C)$ such that $Dg'=f\circ g$. But $(Dg')(sx_{2n})=Dg'(sx_{2n})=x_{2n}$. This implies there is $y\in\wedge W$ such that $dy=x_{2n}$ which is a contradiction as the fibration is TNCZ. Thus, $HH^*(f)([g])\neq 0$. Therefore, $HH^*_{\geq 1}(f)$ is injective. Finally, $HH^*_{(0)}(f)=H^*(f):H^*(\wedge V,d)\to H^*(C)$ is injective as $H^*(p)$ is injective. □

Example 5.2.5. Consider the KS-model $f:\wedge V=(\wedge(x_{2n},x_{4n-1}),d)\rightarrowtail(\wedge V\otimes\wedge(x_2,x_{2n-1}),$ $Dx_2=0,Dx_{2n-1}=x_2^n+x_{2n})$, where $f(x_{2n})=x_{2n}$ and $f(x_{4n-1})=x_{4n-1}$. This is a model of a fibration $\mathbb{C}P(n-1)\to E\to S^{2n}$. Moreover, $H^*(\mathbb{C}P(n-1);\mathbb{Q})\cong\wedge x_2/(x_2^{n-1})$. Thus, the cohomology $H^*(\mathbb{C}P(n-1);\mathbb{Q})$ is monogenic, i.e., it has a single even generator, so this fibration is TNCZ (see (Thomas, 1981)). Recall that,, $HH^*(\wedge V;\wedge V) \cong H^*(\wedge V,d) \oplus \wedge^+(sx_{4n-1})^* \oplus < x_{2n}(sx_{2n})^* - 2x_{4n-1}(sx_{4n-1})^* >$. Moreover, $HH^*(f)([((sx_{4n-1})^*)^i]) \neq 0$ and $HH^*(f)([x_{2n}(sx_{2n})^* - 2x_{4n-1}(sx_{4n-1})^*]) \neq 0$ for degree reasons. Further, $H^*(f)([x_{2n}]) = [x_2^n] \neq 0$. Thus, $HH^*(f)$ is injective.

5.3 Proof of the main result

We remind here the main result.

Theorem 5.3.1. *If $F \to E \xrightarrow{p} X$ is a TNCZ fibration, where X is an elliptic 2-stage Postnikov tower and $f : (\wedge V,d) \rightarrowtail (C,d)$ a KS-model of p, then the induced map in Hochschild cohomology*

$$HH^*(f) : HH^*(\wedge V;\wedge V) \to HH^*(\wedge V;C)$$

is injective.

Proof. Let $\mathcal{B}$ be an elliptic 2-stage Postnikov tower of which the minimal Sullivan model is given by $(\wedge V,d) = (\wedge(V_0 \oplus V_1),d)$ with $dV_0 = 0$ and $dV_1 \subset \wedge V_0$. Then, $HH^*(\wedge V;\wedge V)$ will be computed from a complex of the form

$$(\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V), D) \cong (\wedge V \otimes \wedge(Z_0 \oplus Z_1), d),$$

where $Z_0 = (sV_1)^* = s^{-1}V_1^*$, $Z_1 = (sV_0)^* = s^{-1}V_0^*$, and $dZ_0 = 0$, $dZ_1 \subseteq \wedge V \otimes Z_0$. Let $C = \wedge V \otimes \wedge W$. In the same way, $HH^*(\wedge V;C)$ is computed by the complex $(\wedge V \otimes \wedge W \otimes$

$\wedge(Z_0 \oplus Z_1), d)$. The Hochschild complex inclusion $C^*(\wedge V; \wedge V) \to C^*(\wedge V; C)$ is modelled by $(\wedge V \otimes \wedge(Z_0 \oplus Z_1), d) \xrightarrow{\phi} (\wedge V \otimes \wedge W \otimes \wedge(Z_0 \oplus Z_1), d)$. Our goal is to show that $HH^*(f) = H_*(\phi)$ is injective. We begin by filtering $(\wedge V \otimes \wedge(Z_0 \oplus Z_1), d)$ with the wedge degree in Z_1. That is, $F_p = (\wedge V \otimes \wedge Z_0) \otimes \wedge^{\leq p} Z_1$. We get an increasing filtration

$$\mathfrak{F} : F_0 = \wedge V \otimes \wedge Z_0 \subseteq F_1 \subseteq \cdots \subseteq F_p \subseteq F_{p+1} \subseteq \cdots \subseteq \wedge V \otimes \wedge Z_0 \otimes \wedge Z_1,$$

where $dF_p \subseteq F_{p-1}$. This yields a spectral sequence such that

$$E^0_{p,*} = (\wedge V \otimes \wedge Z_0) \otimes \wedge^p Z_1,$$

$$E^1_{p,*} = H^*(\wedge V \otimes \wedge Z_0) \otimes \wedge^p Z_1 \cong H^*(\wedge V) \otimes \wedge Z_0 \otimes \wedge^p Z_1.$$

As $(\wedge V, d) = (\wedge(V_0 \oplus V_1), d)$ with $dV_0 = 0$ and $dV_1 \subset \wedge V_0$, then the above spectral sequence collapses at E^2-level. The E^1-term, together with differentials, is pictured below

$$\begin{array}{ccc} E^1_{p,*} & \xrightarrow{\quad d_1 \quad} & E^1_{p-1,*} \\ \| & & \| \\ H^*(\wedge V) \otimes \wedge Z_0 \otimes \wedge^p Z_1 & \xrightarrow{d_1} & H^*(\wedge V) \otimes \wedge Z_0 \otimes \wedge^{p-1} Z_1. \end{array}$$

Likewise, we filter $(\wedge V \otimes \wedge W \otimes \wedge(Z_0 \oplus Z_1), d)$ by the wedge degree in Z_1, which gives rise to a spectral sequence $\{\bar{E}_r\}$. Then $\phi : (\wedge V \otimes \wedge(Z_0 \oplus Z_1), d) \to (\wedge V \otimes \wedge W \otimes \wedge(Z_0 \oplus Z_1), d)$

induces a morphism of spectral sequences as shown below.

$$\begin{array}{ccccccccc}
\longrightarrow E^1_{p,*} & \xrightarrow{d_1} & E^1_{p-1,*} & \cdots \xrightarrow{d_1} & E^1_{1,*} & \xrightarrow{d_1} & E^1_{0,*} \\
\downarrow \phi_{1,p} & & \downarrow \phi_{1,p-1} & & \downarrow \phi_{1,1} & & \downarrow \phi_{1,0} \\
\longrightarrow \bar{E}^1_{p,*} & \xrightarrow{d_1} & \bar{E}^1_{p-1,*} & \cdots \xrightarrow{d_1} & \bar{E}^1_{1,*} & \xrightarrow{d_1} & \bar{E}^1_{0,*}.
\end{array}$$

Now, to show that $HH^*(f)$ is injective, it is enough to show that $\phi_{2,*} = H_*(\phi_{1,*})$ is injective.

Consider the following commutative diagram

$$\begin{array}{ccc}
\longrightarrow H^*(\wedge V)\otimes\wedge Z_0\otimes\wedge^{p+1}Z_1 & \xrightarrow{d_1} & H^*(\wedge V)\otimes\wedge Z_0\otimes\wedge^{p}Z_1 \longrightarrow \\
\downarrow \phi_{1,p+1} & & \downarrow \phi_{1,p} \\
\longrightarrow H^*(\wedge V\otimes\wedge W)\otimes\wedge Z_0\otimes\wedge^{p+1}Z_1 & \xrightarrow{d_1} & H^*(\wedge V\otimes\wedge W)\otimes\wedge Z_0\otimes\wedge^{p}Z_1 \longrightarrow
\end{array}$$

As $H^*(p)$ is injective, then $\phi_{1,p}$ is injective for each p. Let $x \neq 0 \in H^*(\wedge V\otimes\wedge Z_0)\otimes\wedge^p Z_1$ be a d_1-cocycle. We show by contradiction that $H_*(\phi_{1,p}([x])) \neq 0$ in $H_*(H^*(\wedge V\otimes\wedge W)\otimes\wedge Z_0\otimes\wedge^p Z_1, d_1)$. Assume $\phi_{1,p}(x)$ is a coboundary. Hence, there is $y \in H^*(\wedge V\otimes\wedge Z_0)\otimes\wedge^{p+1}Z_1$ such that $d_1 y = \phi_{1,p}(x)$. Let $A = \mathrm{Im}\, H^*(\phi) \subseteq H^*(\wedge V\otimes\wedge W)$ and A' its complement. Then $y = y_1 + y_2$, where $y_1 \in A\otimes\wedge Z_0\otimes\wedge^{p+1}Z_1$ and $y_2 \in A'\otimes\wedge Z_0\otimes\wedge^{p+1}Z_1$. As $\phi_{1,p}(x) \in A\otimes\wedge Z_0\otimes\wedge^p Z_1$, and $d_1 y_2 \in A'\otimes\wedge Z_0\otimes\wedge^p Z_1$, then $d_1 y_2 = 0$. Moreover, there is $x_1 \in H^*(\wedge V\otimes\wedge Z_0)\otimes\wedge^{p+1}Z_1$ such that $\phi_{1,p+1}(x_1) = y_1$. As the diagram above commutes, one gets

$$\phi_{1,p}(d_1(x_1)) = d_1 y_1 = \phi_{1,p}(x).$$

But $\phi_{1,p}$ is injective, so we deduce that $x = d_1(x_1)$. Hence $H_*(\phi_{1,p})([x]) \neq 0$. Therefore, $H_*(\phi_{1,p})$ is injective. $\square$

As $HH^*(\wedge V;\wedge V)$ and $\mathbb{H}_*(X^{S^1};\mathbb{Q})$ are isomorphic, where $(\wedge V,d)$ is the minimal Sullivan model of X, we deduce the following result.

Corollary 5.3.2. If $F \to E \xrightarrow{p} X$ is a TNCZ fibration, where X is an elliptic 2-stage Postnikov tower and $f : (\wedge V,d) \rightarrowtail (C,d)$ a KS-model of p, then the natural graded linear map

$$HH^*(f) : \mathbb{H}_*(X^{S^1};\mathbb{Q}) \to HH^*(\wedge V;C)$$

is injective.

Example 5.3.3. Given the KS-model $f : \wedge V = (\wedge(x_2,x_5),d) \rightarrowtail (\wedge V \otimes \wedge(y_2,y_3)$, $Dy_2 = 0, Dy_3 = x_2^2 - y_2^2)$. It is a model of a fibration $\mathbb{C}P(1) \to E \to \mathbb{C}P(2)$ which is TNCZ, as the cohomology $H^*(\mathbb{C}P(1);\mathbb{Q}) \cong \wedge y_2/(y_2^2)$ is monogenic (see (Thomas, 1981)). Moreover, $\mathbb{H}_*(\mathbb{C}P(2);\mathbb{Q}) \cong H_*((\wedge x_2/(x_2^3) \otimes \wedge(z_4,z_1),d), dz_4 = 0,\ dz_1 = 3x_2^3 z_4$. Here Z_0 (resp. Z_1) is spanned by $z_4 = (sx_5)^*$ (resp. $z_1 = (sx_2)^*$). The non-zero homology classes are represented by $\{x_2^j, x_2^i z_1, x_2^i z_1 z_4^k,\ k \geq 0,\ 0 \leq j \leq 1,\ 1 \leq i \leq 2\}$ (see, (Gatsinzi, 2016)). Moreover, at the E^1-level of the spectral sequence, we have $\cdots \xrightarrow{d_1} E^1_{p,*} \xrightarrow{\phi_{1,p}} \bar{E}^1_{p,*} \xrightarrow{d_1} \cdots$ and $\phi_{1,p}$ are injective because the fibration is TNCZ. So, for every non zero homology class $y \in E^1_{p,*}$, we have that $H_*(\phi_{1,p})([y]) \neq 0$ for degree reasons. Hence, $HH^*(f)$ is injective.

5.4 Conclusion and Future Work

As we couldn't find, to the best of our knowledge, the precise references in the literature that extend some of the work of Banyaga et al. (2018) and Gatsinzi (2019, 2016), the focus of this thesis research is to serve as an extension to the work of Banyaga et al. (2018) and Gatsinzi (2019, 2016).

On the other hand, the study of Hochschild cohomology of $C = (\wedge V, d) \otimes (\wedge W, d)$ is also an interesting research subject and should have an application to rational homotopy theory with a viewpoint to string topology. Here, we consider the following problem recommended for future work. To introduce the difficulties once at a time, one should begin to establish an explicit isomorphism between $HH^*(C;C)$ and the product $HH^*(\wedge V; \wedge V) \otimes HH^*(\wedge W; \wedge W)$ as graded algebras, furthermore, as Gerstenhaber algebras (we could not obtain a specific reference of this in the literature). Secondly, one should use what is proved in (Gatsinzi, 2017) that for minimal Sullivan algebras, the cochain complex $(\wedge V \otimes \wedge s^{-1}V^*, d)$ computing Hochschild cohomology is a differential BV algebra which extends the Gerstenhaber structure. In particular, let $(\wedge V, d)$ and $(\wedge W, d)$ be two commutative differential graded algebras, where V and W are finite-dimensional. Then $(\wedge V, d) \otimes (\wedge W, d) = (\wedge(V \oplus W), d)$ should have also a relative Sullivan model. Hence, one should show that there is an isomorphism of differential BV algebras $(\wedge V \otimes \wedge s^{-1}V^*, d) \otimes (\wedge W \otimes \wedge s^{-1}W^*, d) \cong (\wedge(V \oplus W) \otimes \wedge s^{-1}(V \oplus W)^*, d)$.

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Appendix

Submitted papers

- Jean Baptiste Gatsinzi & Oteng Maphane (2022): **On unit sphere tangent bundles over complex Grassmannians,** Novi Sad Journal of Mathematics, https://doi.org/10.30755/NSJOM.10787.

- Jean Baptiste Gatsinzi & Oteng Maphane (2022): **On Hochschild cohomology of a map between commutative differential graded algebras,** *Topological Algebra and its Applications.* To appear.

- Jean Baptiste Gatsinzi & Oteng Maphane (2021): **Hochschild cohomology of certain Koszul Sullivan Extensions,** Quaestiones Mathematicae, DOI: 10.2989/16073606.2021.1977733.