Fixed Point Theory, 18(2017), No. 1, 137-146
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

APPROXIMATING FIXED POINTS OF THE COMPOSITION OF TWO RESOLVENT OPERATORS

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Abstract. Let A and B be maximal monotone operators defined on a real Hilbert space H, and let $\mathrm{Fix}(J_\mu^A J_\mu^B) \neq \emptyset$, where $J_\mu^A y := (I + \mu A)^{-1} y$ and μ is a given positive number. [H. H. Bauschke, P. L. Combettes and S. Reich, The asymptotic behavior of the composition of two resolvents, Nonlinear Anal. 60 (2005), no. 2, 283-301] proved that any sequence (x_n) generated by the iterative method $x_{n+1} = J_\mu^A y_n$, with $y_n = J_\mu^B x_n$ converges weakly to some point in $\mathrm{Fix}(J_\mu^A J_\mu^B)$. In this paper, we show that the modified method of alternating resolvents introduced in [O. A. Boikanyo, A proximal point method involving two resolvent operators, Abstr. Appl. Anal. 2012, Article ID 892980, (2012)] produces sequences that converge strongly to some points in $\mathrm{Fix}(J_\mu^A J_\mu^B)$ and $\mathrm{Fix}(J_\mu^B J_\mu^A)$.
Key Words and Phrases: Maximal monotone operator, alternating resolvents, proximal point algorithm, nonexpansive map, resolvent operator.
2010 Mathematics Subject Classification: 47J25, 47H05, 47H09, 47H10.

1. Introduction

Iterative methods have been widely used to approximate solutions of nonlinear operator inclusions of the form $0 \in Ax$, where A is a maximal monotone operator, see for example [11, 14, 18, 20, 21, 4] and the references therein. The set of solutions of this inclusion, denoted by $A^{-1}(0)$, is closed and convex. Other iterative methods have been developed to approximate solutions of the inclusion $0 \in \bigcap_{i=1}^n A_i$, where each A_i is maximal monotone (or m-accretive in the case of Banach space setting), refer to [23, 9, 22] and the references therein. Of immediate interest to us is the method of alternating projections introduced by von Neumann in the early 1930s. Given any starting point $x_0 \in H$, this method generates a sequence (x_n) iteratively by

$$x_0 \mapsto x_1 = P_{K_1} x_0 \mapsto x_2 = P_{K_2} x_1 \mapsto x_3 = P_{K_1} x_2 \mapsto x_4 = P_{K_2} x_3 \mapsto \cdots ,$$

where $P_C : H \to C$ is the projection operator onto a nonempty, closed and convex subset C. In his paper, von Neumann showed that if K_1 and K_2 are subspaces of H, then (x_n) will converge strongly to the point in $K_1 \cap K_2$ that is closest to the starting point x_0. For recent proofs of this classical result, we refer the reader to [2, 12]. If K_1 and K_2 are two arbitrary nonempty, closed and convex subsets in H with nonempty intersection, then the sequence (x_n) generated from the method of alternating projections converges weakly to a point in $K_1 \cap K_2$ [8], but strong convergence cannot be obtained in general [10, 13]. Since the projection operator coincides with the resolvent of a normal cone, one can extend this iterative method as follows: Given any starting point $x_0 \in H$, generate a sequence (x_n) iteratively as

$$x_{2n+1} = J_\mu^A x_{2n} \quad \text{for } n = 0, 1, \dots, \tag{1.1}$$

$$x_{2n} = J_\mu^B x_{2n-1} \quad \text{for } n = 1, 2, \dots, \tag{1.2}$$

where A and B are two maximal monotone operators and μ is a positive real number. In this case, it can be shown that the above sequence converges weakly to some point in $A^{-1}(0) \cap B^{-1}(0)$, provided that this set is not empty, see for example [6]. Note that strong convergence of this method fails in general, (the same counter example given in [10] applies). Bauschke et al. [1] proved a weak convergence result of the method of alternating resolvents (1.1), (1.2) to some point in $\text{Fix}(J_\mu^A J_\mu^B)$, provided that the fixed point set of the composition mapping $J_\mu^A J_\mu^B$ is nonempty. We emphasize that if K_1 and K_2 are two nonempty, closed and convex subsets in H, then the set $K_1 \cap K_2$ coincides with the set $\text{Fix}(P_{K_1} P_{K_2})$. However, the fixed point set $\text{Fix}(J_\mu^A J_\mu^B)$ is larger than the set $A^{-1}(0) \cap B^{-1}(0)$, see for example [6, Remark 5].

Recently, an attempt was made in [3, 6, 5] to modify algorithm (1.1), (1.2) in order to enforce strong convergence to some point in $A^{-1}(0) \cap B^{-1}(0)$. One such modification introduced in [3] defines a sequence (x_n) iteratively by

$$x_{2n+1} = \alpha_n u + (1 - \alpha_n) J_\mu^A x_{2n} + e_n \quad \text{for } n = 0, 1, \dots, \tag{1.3}$$

$$x_{2n} = J_\mu^B (\lambda_n u + (1 - \lambda_n) x_{2n-1} + e_n') \quad \text{for } n = 1, 2, \dots, \tag{1.4}$$

where $\alpha_n, \lambda_n \in [0, 1]$, (e_n) and (e_n') are sequences of computational errors and μ is a positive real number. Our purpose in this paper is to investigate strong convergence of the iterative method (1.3), (1.4) to some point in $\text{Fix}(J_\mu^A J_\mu^B)$. Note that the set $\text{Fix}(J_\mu^A J_\mu^B)$ is in general larger than the set $A^{-1}(0) \cap B^{-1}(0)$, see for example, Remark 5 [6].

2. Preliminary Results

Let H be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Consider a nonlinear (and possibly set-valued) operator $A : D(A) \subset H \to H$ whose graph is $G(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$. The operator A is called (i) monotone if $\langle x - \overline{x}, y - \overline{y} \rangle \geq 0$ for all $(x, y), (\overline{x}, \overline{y}) \in G(A)$ and (ii) maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone operator. Let K be a nonempty, closed and convex subset of H. The normal cone to K at the point z, denoted by $N_K(z)$, is the set $\{w \in H | \langle w, z - v \rangle \geq 0 \forall v \in K\}$. It is known that N_K is maximal monotone. Given any maximal monotone operator A and a positive real number c, one can always define the map $J_c^A : H \to H$ by $x \mapsto (I + cA)^{-1} x$, where I is the identity operator. This map is called the resolvent operator of A. It is well known that the Yosida approximation of A, an operator defined by $A_c := c^{-1}(I - J_c^A)$ is maximal monotone for every $c > 0$. The weak ω-limit set of a sequence (x_n), denoted by $\omega_{\mathtt{w}}((x_n))$, is the set

$$\omega_{\mathtt{w}}((x_n)) = \{x \in H : x_{n_k} \rightharpoonup x \text{ for some subsequence } (x_{n_k}) \text{ of } (x_n)\}.$$

The notation $x_n \to x$ will be used to indicate that the sequence (x_n) converges strongly to x whereas $x_n \rightharpoonup x$ will be used to indicate that (x_n) converges weakly to x.

The following two lemmas will be useful in proving our main results.

Lemma 2.1 (Boikanyo and Moroşanu [7]). *Let (s_n) be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)(1-\lambda_n)s_n + \alpha_n b_n + \lambda_n c_n + d_n, \quad n \geq 0, \tag{2.1}$$

where (α_n), (λ_n), (b_n), (c_n) and (d_n) satisfy the conditions: (i) $\alpha_n, \lambda_n \in [0,1]$, *with* $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$, (ii) $\limsup_{n\to\infty} b_n \leq 0$, (iii) $\limsup_{n\to\infty} c_n \leq 0$, *and* (iv) $d_n \geq 0$ *for all $n \geq 0$ with $\sum_{n=0}^{\infty} d_n < \infty$. Then $\lim_{n\to\infty} s_n = 0$.*

Remark 2.2. It can be easily verified that if $\lim_{n\to\infty} \alpha_n = 0$, then $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$ if and only if $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Lemma 2.3 (Maingé [16]). *Let (s_n) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence (s_{n_j}) of (s_n) such that $s_{n_j} < s_{n_j+1}$ for all $j \geq 0$. Define an integer sequence $(\tau(n))_{n\geq n_0}$ as*

$$\tau(n) = \max\{n_0 \leq k \leq n : \ s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$

$$\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}. \tag{2.2}$$

3. Main Results

Let (α_n) and (λ_n) be non-zero sequences of real numbers in $(0,1)$, and suppose that $v_0, u \in H$ are given. Consider the sequence (v_n) generated iteratively by

$$v_{2n+1} = \alpha_n u + (1-\alpha_n)J_\mu^A v_{2n} \quad \text{for } n = 0,1,\dots, \tag{3.1}$$

$$v_{2n} = J_\mu^B(\lambda_n u + (1-\lambda_n)v_{2n-1}) \quad \text{for } n = 1,2,\dots, \tag{3.2}$$

for any $\mu > 0$, where A and B are maximal monotone operators. We investigate (in Theorem 3.2 below) the convergence properties of the sequence (v_n) to some fixed point of the composition mappings $J_\mu^A J_\mu^B$ and $J_\mu^B J_\mu^A$.

Let us note that if $\text{Fix}(J_\mu^A J_\mu^B)$ is non-empty, then so is $\text{Fix}(J_\mu^B J_\mu^A)$. Indeed, if $\text{Fix}(J_\mu^A J_\mu^B) \neq \emptyset$, then we can find $p \in H$ such that $p = J_\mu^A J_\mu^B p$. Since J_μ^B is single valued and defined on the whole space H, it then follows that $J_\mu^B p = J_\mu^B(J_\mu^A J_\mu^B p)$. Setting $z = J_\mu^B p$, we see that $z = J_\mu^B J_\mu^A z$. That is, $z \in \text{Fix}(J_\mu^B J_\mu^A)$, and so $\text{Fix}(J_\mu^B J_\mu^A) \neq \emptyset$. Similarly, it can be shown that if $\text{Fix}(J_\mu^B J_\mu^A)$ is non-empty, then so is $\text{Fix}(J_\mu^A J_\mu^B)$. This note can be summarized in the following remark.

Remark 3.1. Let A and B be maximal monotone operators, and μ be any positive real number. Then $\text{Fix}(J_\mu^A J_\mu^B) \neq \emptyset \Leftrightarrow \text{Fix}(J_\mu^B J_\mu^A) \neq \emptyset$.

Theorem 3.2. *Let* $A : D(A) \subset H \to 2^H$ *and* $B : D(B) \subset H \to 2^H$ *be maximal monotone operators with* $Fix(J_\mu^A J_\mu^B) =: S \neq \emptyset$*. For arbitrary but fixed vectors* $v_0, u \in H$*, let* (v_n) *be the sequence generated by (3.1), (3.2), where* $\alpha_n, \lambda_n \in (0,1)$ *and* $\mu > 0$*. Assume that* $\lim_{n\to\infty} \alpha_n = 0$*,* $\lim_{n\to\infty} \lambda_n = 0$*, and either* $\sum_{n=0}^\infty \alpha_n = \infty$ *or* $\sum_{n=0}^\infty \lambda_n = \infty$*. Then the subsequence (i)* (v_{2n+1}) *of* (v_n) *converges strongly to the point* $q \in S$ *that is nearest to* u*, and (ii)* (v_{2n}) *of* (v_n) *converges strongly to the point* $z = J_\mu^B q$ *in* $Fix(J_\mu^B J_\mu^A)$*.*

Proof. (The proof of the following theorem makes use of some ideas of the papers [16, 19, 7, 3]). Let p be any point in $\text{Fix}(J_\mu^A J_\mu^B)$. Then from (3.1), we have

$$\begin{aligned} \|v_{2n+1} - p\| &\leq \alpha_n \|u - p\| + (1 - \alpha_n) \left\|J_\mu^A v_{2n} - p\right\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n) \left\|v_{2n} - J_\mu^B p\right\|, \end{aligned} \tag{3.3}$$

where the last inequality follows from the fact that the resolvent operator $J_\mu^A : H \to H$ is nonexpansive. But from the nonexpansive property of J_μ^B, we have from (3.2)

$$\begin{aligned} \left\|v_{2n} - J_\mu^B p\right\| &\leq \|\lambda_n(u - p) + (1 - \lambda_n)(v_{2n-1} - p)\| \\ &\leq \lambda_n \|u - p\| + (1 - \lambda_n) \|v_{2n-1} - p\|. \end{aligned}$$

Therefore, from this inequality and (3.3), we get

$$\begin{aligned} \|v_{2n+1} - p\| &\leq [\alpha_n + (1 - \alpha_n)\lambda_n] \|u - p\| + (1 - \alpha_n)(1 - \lambda_n) \|v_{2n-1} - p\| \\ &= [1 - (1 - \alpha_n)(1 - \lambda_n)] \|u - p\| + (1 - \alpha_n)(1 - \lambda_n) \|v_{2n-1} - p\|. \end{aligned}$$

By a simple induction argument, we arrive at

$$\|v_{2n+1} - p\| \leq \left[1 - \prod_{k=1}^n (1 - \alpha_k)(1 - \lambda_k)\right] \|u - p\| + \|v_1 - p\| \prod_{k=1}^n (1 - \alpha_k)(1 - \lambda_k).$$

Therefore, if either $\sum_{k=0}^\infty \alpha_k = \infty$ or $\sum_{k=0}^\infty \lambda_k = \infty$, then we derive the boundedness of the subsequence (v_{2n+1}) of (v_n). Note that if (v_{2n+1}) is bounded, then so is (v_{2n}). Hence the sequence (v_n) is bounded.

Now let $q := P_S u$ and $z := J_\mu^B q$. Then $q = J_\mu^A J_\mu^B q$ and $q = J_\mu^A z$. Since the inequality

$$\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y\rangle$$

holds true for all $x, y \in H$, we have from (3.1)

$$\begin{aligned} \|v_{2n+1} - q\|^2 &\leq (1 - \alpha_n) \left\|J_\mu^A v_{2n} - q\right\|^2 + 2\alpha_n \langle u - q, v_{2n+1} - q\rangle \\ &\leq (1 - \alpha_n) \left[\|v_{2n} - z\|^2 - \left\|\left(I - J_\mu^A\right) v_{2n} - \left(I - J_\mu^A\right) z\right\|^2\right] \\ &+ 2\alpha_n \langle u - q, v_{2n+1} - q\rangle \\ &= (1 - \alpha_n) \left[\|v_{2n} - z\|^2 - \left\|v_{2n} - J_\mu^A v_{2n} - z + q\right\|^2\right] \\ &+ 2\alpha_n \langle u - q, v_{2n+1} - q\rangle, \end{aligned} \tag{3.4}$$

where the second inequality follows from the fact that the resolvent of a maximal monotone operator A is firmly nonexpansive. If we denote $w_n := \lambda_n u + (1 - \lambda_n) v_{2n-1}$, then using the firmly nonexpansive property of J_μ^B, we have from (3.2)

$$\begin{aligned}\|v_{2n}-z\|^2 &= \left\|J_\mu^B w_n - J_\mu^B q\right\|^2\\ &\leq \|w_n-q\|^2 - \left\|\left(I-J_\mu^B\right)w_n - \left(I-J_\mu^B\right)q\right\|^2\\ &= \lambda_n^2\|u-q\|^2 + 2\lambda_n(1-\lambda_n)\langle u-q, v_{2n-1}-q\rangle\\ &+ (1-\lambda_n)\|v_{2n-1}-q\|^2 - \|w_n-v_{2n}-q+z\|^2.\end{aligned}$$

This inequality together with (3.4) implies that

$$\begin{aligned}\|v_{2n+1}-q\|^2 &\leq (1-\alpha_n)(1-\lambda_n)\|v_{2n-1}-q\|^2 + \lambda_n b_n + \alpha_n c_n\\ &- (1-\alpha_n)\left(\left\|v_{2n}-J_\mu^A v_{2n}-z+q\right\|^2 + \|w_n-v_{2n}-q+z\|^2\right)\end{aligned} \tag{3.5}$$

where $b_n := (1-\alpha_n)[\lambda_n\|u-q\|^2 + 2(1-\lambda_n)\langle u-q, v_{2n-1}-q\rangle]$ and $c_n := 2\langle u-q, v_{2n+1}-q\rangle$. Note that if we denote $s_n := \|v_{2n-1}-P_S u\|^2$, then we can find a positive constant M such that

$$s_{n+1}-s_n + \left\|v_{2n}-J_\mu^A v_{2n}-z+q\right\|^2 + \|w_n-v_{2n}-q+z\|^2 \leq (\alpha_n+\lambda_n)M. \tag{3.6}$$

Our aim is to show that (s_n) converges to zero strongly. In order to prove this, we shall consider two possible cases on the sequence (s_n) of real numbers.

<u>CASE I</u>: (s_n) is eventually decreasing (i.e., there exists $N \geq 0$ such that (s_n) is decreasing for all $n \geq N$). In this case, (s_n) is convergent. Letting $n \to \infty$ in (3.6), we get

$$\lim_{n\to\infty}\|w_n-v_{2n}+z-q\| = 0 = \lim_{n\to\infty}\left\|v_{2n}-J_\mu^A v_{2n}-z+q\right\|. \tag{3.7}$$

On the other hand,

$$\begin{aligned}\left\|(I-J_\mu^A J_\mu^B)w_n\right\| &= \left\|w_n - J_\mu^A v_{2n}\right\|\\ &\leq \|w_n-v_{2n}+z-q\| + \left\|v_{2n}-J_\mu^A v_{2n}-z+q\right\|,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\left\|(I-J_\mu^A J_\mu^B)w_n\right\| = 0. \tag{3.8}$$

If (w_{n_k}) is a subsequence of (w_n) converging weakly to some $w \in H$, then it follows from the demiclosed property of $(I-J_\mu^A J_\mu^B)$ that the weak limit $w \in Fix J_\mu^A J_\mu^B$, (see for example [17, p. 20]). Thus $\omega_{\mathtt{w}}((v_{2n+1})) = \omega_{\mathtt{w}}((w_n)) \subset S$. Now take a subsequence (v_{2n_l+1}) of (v_{2n+1}) converging weakly to some $\overline{w} \in S$ such that

$$\limsup_{n\to\infty}\langle u-q, v_{2n+1}-q\rangle = \lim_{l\to\infty}\langle u-q, x_{2n_l+1}-q\rangle.$$

Then, we have from one of the properties of projections

$$\limsup_{n\to\infty}\langle u-q, v_{2n+1}-q\rangle = \langle u-q, \overline{w}-q\rangle \leq 0.$$

Since $\lambda_n \to 0$ as $n \to \infty$, the above inequality implies that

$$\limsup_{n\to\infty} b_n \leq 0.$$

From (3.5), we have

$$\|v_{2n+1}-q\|^2 \leq (1-\alpha_n)(1-\lambda_n)\|v_{2n-1}-q\|^2+\lambda_n b_n+\alpha_n c_n.$$

Using Lemma 2.1 we get $\|v_{2n+1}-q\| \to 0$ as $n\to\infty$. That is, (s_n) converges to zero strongly.

<u>CASE II</u>: (s_n) is not eventually decreasing, that is, there is a subsequence (s_{n_j}) of (s_n) such that $s_{n_j} < s_{n_j+1}$ for all $j \geq 0$. We then define an integer sequence $(\tau(n))_{n\geq n_0}$ as in Lemma 2.3 so that $s_{\tau(n)} \leq s_{\tau(n)+1}$ for all $n \geq n_0$. It then follows from (3.6) that

$$\lim_{n\to\infty}\left\|v_{2\tau(n)}-J_\mu^A v_{2\tau(n)}-z+q\right\|=0=\lim_{n\to\infty}\left\|w_{\tau(n)}-v_{2\tau(n)}-q+z\right\|.$$

From these two limits, we derive

$$\left\|w_{\tau(n)}-J_\mu^A v_{2\tau(n)}\right\| \leq \left\|w_{\tau(n)}-v_{2\tau(n)}-q+z\right\|+\left\|v_{2\tau(n)}-J_\mu^A v_{2\tau(n)}-z+q\right\| \to 0,$$

as $n\to\infty$. In addition, we have

$$\begin{aligned}\left\|v_{2\tau(n)-1}-J_\mu^A v_{2\tau(n)}\right\| &\leq \left\|v_{2\tau(n)-1}-w_{\tau(n)}\right\|+\left\|w_{\tau(n)}-J_\mu^A v_{2\tau(n)}\right\| \\ &= \lambda_{\tau(n)}\left\|v_{2\tau(n)-1}-u\right\|+\left\|w_{\tau(n)}-J_\mu^A v_{2\tau(n)}\right\| \to 0,\end{aligned}$$

as $n\to\infty$. Therefore, from (3.1) we get

$$\begin{aligned}\left\|v_{2\tau(n)+1}-v_{2\tau(n)-1}\right\| &\leq \alpha_{\tau(n)}\left\|u-v_{2\tau(n)-1}\right\|+(1-\alpha_{\tau(n)})\left\|J_\mu^A v_{2\tau(n)}-v_{2\tau(n)-1}\right\| \\ &\to 0,\end{aligned}$$

as $n\to\infty$. As in Case I, we derive $\omega_{\mathtt{w}}((v_{2\tau(n)+1})) \subset S$. As a result, we have

$$\limsup_{n\to\infty}\langle u-q, v_{2\tau(n)+1}-q\rangle \leq 0.$$

Note that we may write (3.5) as

$$\begin{aligned}\|v_{2n+1}-q\|^2 &\leq (1-\alpha_n)(1-\lambda_n)\|v_{2n-1}-q\|^2+\lambda_n\bar{b}_n \\ &+ 2[\alpha_n+\lambda_n(1-\alpha_n)]\langle u-q, v_{2n+1}-q\rangle, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned}\bar{b}_n &:= \|v_{2n-1}-v_{2n+1}\|L+\lambda_n(1-\alpha_n)[\|u-q\|^2-2\langle u-q, v_{2n+1}-q\rangle] \\ &\leq \|v_{2n-1}-v_{2n+1}\|L+\lambda_n M',\end{aligned}$$

for some positive constants L and M'. Clearly,

$$\limsup_{n\to\infty}\bar{b}_{\tau(n)} \leq 0.$$

Since $s_{\tau(n)} \leq s_{\tau(n)+1}$ for all $n \geq n_0$, we derive from (3.9)

$$\begin{aligned}s_{\tau(n)+1} &\leq 2\langle u-q, v_{2\tau(n)+1}-q\rangle+\frac{\lambda_{\tau(n)}\bar{b}_{\tau(n)}}{\lambda_{\tau(n)}(1-\alpha_{\tau(n)})+\alpha_{\tau(n)}} \\ &\leq 2\langle u-q, v_{2\tau(n)+1}-q\rangle+\bar{b}_{\tau(n)}.\end{aligned}$$

Passing to the limit as $n\to\infty$ in the above inequality, we see that $s_{\tau(n)+1} \to 0$. Hence from (2.2) it follows that $s_n \to 0$ as $n\to\infty$. That is, $v_{2n+1} \to q = P_S u$ as $n\to\infty$. This proves the result for the case when (s_n) is not eventually decreasing.

Therefore, from Case I and Case II above, we conclude that the subsequence (v_{2n+1}) of (v_n) converges strongly to some point $q \in S$ that is nearest to u.

(ii) We now show that the subsequence (v_{2n}) of (v_n) converges strongly to the point $z = J_\mu^B q$ in $\mathrm{Fix}(J_\mu^B J_\mu^A)$. Note that we have from (3.2) and the nonexpansive property of J_μ^B

$$\begin{aligned} \|v_{2n} - z\| &= \left\|J_\mu^B(\lambda_n u + (1-\lambda_n)v_{2n-1}) - J_\mu^B q\right\| \\ &\leq \lambda_n \|u - q\| + (1-\lambda_n)\|v_{2n-1} - q\|. \end{aligned}$$

Since $v_{2n+1} \to q$ as $n \to \infty$, it follows that $\|v_{2n} - z\| \to 0$ as $n \to \infty$. This shows that (v_{2n}) converges strongly to $z = J_\mu^B q$, as desired. This completes the proof of the theorem. □

We now show that strong convergence properties of the sequence (x_n) generated by the inexact iterative process (1.3), (1.4) can be derived from the convergence properties of the sequence (v_n) generated by algorithm (3.1), (3.2).

Theorem 3.3. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $Fix(J_\mu^A J_\mu^B) =: S \neq \emptyset$. For arbitrary but fixed vectors $x_0, u \in H$, let (x_n) be the sequence generated by (1.3), (1.4), where $\alpha_n, \lambda_n \in (0,1)$ and $\mu > 0$. Assume that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \lambda_n = 0$, and either $\sum_{n=0}^\infty \alpha_n = \infty$ or $\sum_{n=0}^\infty \lambda_n = \infty$. Suppose that any of the following conditions is satisfied*

- *(a)* $\sum_{n=0}^\infty \|e_n\| < \infty$ *and* $\sum_{n=1}^\infty \|e'_n\| < \infty$;
- *(b)* $\sum_{n=0}^\infty \|e_n\| < \infty$ *and* $\|e'_n\|/\alpha_n \to 0$;
- *(c)* $\sum_{n=0}^\infty \|e_n\| < \infty$ *and* $\|e'_n\|/\lambda_n \to 0$;
- *(d)* $\|e_n\|/\alpha_n \to 0$ *and* $\sum_{n=1}^\infty \|e'_n\| < \infty$;
- *(e)* $\|e_n\|/\lambda_n \to 0$ *and* $\sum_{n=1}^\infty \|e'_n\| < \infty$;
- *(f)* $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\alpha_n \to 0$;
- *(g)* $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$;
- *(h)* $\|e_n\|/\lambda_n \to 0$ *and* $\|e'_n\|/\alpha_n \to 0$;
- *(i)* $\|e_n\|/\lambda_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$;
- *(j)* $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$;
- *(k)* $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$;
- *(l)* $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$;
- *(m)* $\sum_{n=0}^\infty \|e_n\| < \infty$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$;
- *(n)* $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\sum_{n=1}^\infty \|e'_n\| < \infty$.

Then the subsequence (i) (x_{2n+1}) of (x_n) converges strongly to the point $q \in S$ that is nearest to u, and (ii) (x_{2n}) of (x_n) converges strongly to the point $z = J_\mu^B q$ in $Fix(J_\mu^B J_\mu^A)$.

Proof. Clearly, the inequality

$$\|x_{2n} - v_{2n}\| \quad \leq \quad (1-\lambda_n)\,\|x_{2n-1} - v_{2n-1}\| + \|e'_n\| \tag{3.10}$$

can be derived from the equations (1.3) and (3.1), as well as the fact that the resolvent of B is nonexpansive. Similarly, from (1.4), (3.2) and the nonexpansive property of J^A_μ, we derive

$$\|x_{2n+1} - v_{2n+1}\| \quad \leq \quad (1-\alpha_n)\,\|x_{2n} - v_{2n}\| + \|e_n\|\,. \tag{3.11}$$

Now substituting (3.10) into (3.11) yields

$$\|x_{2n+1} - v_{2n+1}\| \quad \leq \quad (1-\alpha_n)(1-\lambda_n)\,\|x_{2n-1} - v_{2n-1}\| + \|e_n\| + \|e'_n\|\,.$$

Note that if the error sequence satisfy any of the conditions (a)-(i), then it readily follows from Lemma 2.1 that $\|x_{2n+1} - v_{2n+1}\| \to 0$ as $n \to \infty$. Since $v_{2n+1} \to q = P_S u$ as $n \to \infty$, it follows that $x_{2n+1} \to P_S u$ as well. Now passing to the limit in (3.10), we also derive $\|x_{2n} - v_{2n}\| \to 0$. Since $v_{2n} \to z = J^B_\mu q$ as $n \to \infty$, we conclude that $x_{2n} \to z$ as $n \to \infty$.

On the other hand, if the error sequence satisfy any of the conditions (j)-(n), then from (3.10) and (3.11), we have

$$\|x_{2n} - v_{2n}\| \quad \leq \quad (1-\alpha_{n-1})(1-\lambda_n)\,\|x_{2n-2} - v_{2n-2}\| + \|e_{n-1}\| + \|e'_n\|\,.$$

Lemma 2.1 guarantees that $\|x_{2n} - v_{2n}\| \to 0$ as $n \to \infty$. Again from the conclusion of Theorem 3.2, we conclude that $x_{2n} \to z$ as $n \to \infty$. Passing to the limit in (3.11), we derive $\|x_{2n+1} - v_{2n+1}\| \to 0$ as $n \to \infty$. Since $v_{2n+1} \to q$ as $n \to \infty$, it follows that $x_{2n+1} \to q$ as $n \to \infty$. This completes the proof of the theorem. □

Remark 3.4. We conclude by noting that any point in the fixed point set $\mathrm{Fix}(J^A_\mu J^B_\mu)$ is a solution of the inclusion relation $0 \in Ax + B_\mu x$, where B_μ is the Yosida approximation of B. Indeed,

$$p = J^A_\mu J^B_\mu p \;\Leftrightarrow\; p + \mu Ap \ni J^B_\mu p \;\Leftrightarrow\; (I - J^B_\mu)p + \mu Ap \ni 0 \;\Leftrightarrow\; B_\mu p + Ap \ni 0.$$

Note that the sum $A+B_\mu$ is maximal monotone. If the sum of two maximal monotone operators is again maximal monotone, then one can always generate a sequence that converges strongly to some zero of the sum of the two operators, refer to [23] for details. It is well known that in general, the sum of two maximal monotone operators is not maximal monotone.
The above remark leads us to the following open question.

Open Question. Can the inexact iterative process (1.3), (1.4), (or even the exact algorithm (3.1), (3.2)), be used to approximate solutions of the inclusion relation $0 \in Ax + Bx$, for arbitrary maximal monotone operators A and B?

It is worthy of note that for the case when one of the operators is α-inverse strongly monotone, López et. al. [15] introduced an algorithm that converges strongly to some solution of the above inclusion relation.

Acknowledgment. The author would like to sincerely thank Professor Simeon Reich for his valuable comments that improved the presentation of this manuscript.

References

[1] H.H. Bauschke, P.L. Combettes, S. Reich, *The asymptotic behavior of the composition of two resolvents*, Nonlinear Anal., **60**(2005), no. 2, 283-301.

[2] H.H. Bauschke, E. Matoušková, S. Reich, *Projection and proximal point methods: convergence results and counterexamples*, Nonlinear Anal., **56**(2004), no. 5, 715-738.

[3] O.A. Boikanyo, *A proximal point method involving two resolvent operators*, Abstr. Appl. Anal. 2012, Article ID 892980, 2012, 12 p.

[4] O.A. Boikanyo, G. Moroşanu, *Inexact Halpern-type proximal point algorithm*, J. Glob. Optim., **51**(2011), no. 1, 11-26.

[5] O.A. Boikanyo, G. Moroşanu, *A contraction proximal point algorithm with two monotone operators*, Nonlinear Anal., **75**(2012), no. 14, 5686-5692.

[6] O.A. Boikanyo, G. Moroşanu, *On the method of alternating resolvents*, Nonlinear Anal., **74**(2011), 5147-5160.

[7] O.A. Boikanyo, G. Moroşanu, *Strong convergence of the method of alternating resolvents*, J. Nonlinear Convex Anal., **14**(2013), no. 2, 221-229.

[8] L.M. Bregman, *The method of successive projection for finding a common point of convex sets*, Sov. Math. Dokl., **6**(1965), 688-692.

[9] L. Hu, L. Liwei, *A new iterative algorithm for common solutions of a finite family of accretive operators*, Nonlinear Anal., **70**(2009), 2344-2351.

[10] H. Hundal, *An alternating projection that does not converge in norm*, Nonlinear Anal., **57**(2004), no. 1, 35-61.

[11] S. Kamimura, W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory, **106**(2000), 226–240.

[12] E. Kopecká, S. Reich, *A note on the von Neumann alternating projections algorithm*, J. Nonlinear Convex Anal., **5**(2004), 379-386.

[13] E. Matoušková, S. Reich, *The Hundal example revisited*, J. Nonlinear Convex Anal., **4**(2003), 411-427.

[14] N. Lehdili, A. Moudafi, *Combining the proximal algorithm and Tikhonov regularization,* Optimization, **37**(1996), 239-252.

[15] G. López, V. Martín-Márquez, F. Wang, H.K. Xu, *Forward-Backward Splitting Methods for Accretive Operators in Banach Spaces*, Abstr. Appl. Anal. 2012, Article ID 109236, 2012, 25 p.

[16] P.E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal., **16**(2008), 899-912.

[17] G. Moroşanu, *Nonlinear Evolution Equations and Applications*, Reidel, Dordrecht, 1988.

[18] O. Nevanlinna, S. Reich, *Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces*, Israel J. Math., **32**(1979), 44-58.

[19] F. Wang, H. Cui, *On the contraction-proximal point algorithms with multi-parameters*, J. Glob. Optim., **54**(2012), 485-491.

[20] H.K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc., **66**(2002), no. 2, 240-256.

[21] H.K. Xu, *A regularization method for the proximal point algorithm,* J. Glob. Optim., **36**(2006), 115-125.

[22] H. Zegeye, N. Shahzad, *Strong convergence theorems for a common zero of a finite family of m-accretive mappings*, Nonlinear Anal., **66**(2007), 1161-1169.

[23] H. Zegeye, N. Shahzad, *Proximal point algorithms for finding a zero of a finite sum of monotone mappings in Banach spaces*, Abstr. Appl. Anal. 2013, Article ID 232170, 2013, 7 p.

Received: April 25, 2014; Accepted: October 23, 2014.