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A new Lindley-Burr XII power series distribution: model, properties and applications

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ABSTRACT

A new generalized class of distributions called the Lindley-Burr XII Power Series (LBXIIPS) distribution is proposed and explored. This new class of distributions contain some special cases such as Lindley-Burr XII Poisson (LBXIIP), Lindley-Burr XII Logarithmic (LBXIIL), Lindley-Burr XII Binomial (LBXIIB) and their sub-models among others. Some structural properties of the new distribution including moments, probability weighted moments, distribution of the order statistics and entropy are derived. Maximum likelihood estimation technique is used to estimate the model parameters. A simulation study to examine the bias and mean square error of the maximum likelihood estimators is presented and finally, an application to a real data set in order to illustrate the usefulness of the new distribution is given.

1. Introduction

Recently, many distributions have been developed in the literature by compounding some known continuous distributions such as Weibull, Burr XII and exponentiated exponential with power series distributions such as Poisson, logarithmic, geometric and binomial distributions as special cases [1]. Oluyede et al. [2] studied the new Burr XII-Weibull-logarithmic distribution for survival and lifetime data analysis where they compounded the Burr XII-Weibull distribution with the power series distribution. They demonstrated the usefulness and applicability of the new distribution and conclude that the distribution is more flexible than other nested and non-nested models. Silva and Cordeiro [3] presented results on the Burr Power series distribution. Chahkandi and Ganjali [4] studied the exponentiated power series and Exponential-Logarithmic was introduced by Kuş [5] while Lu and Shi [6] proposed the Weibull-geometric and Weibull-Poisson. Recently Morais and Bareto-Souza [7] studied A compound class of Weibull and power series distribution.

The Lindley distribution was first introduced by Lindley [8] as a one parameter distribution mainly used for modeling waiting and survival times data with the probability density function (pdf) given by

$$f_L(x;\theta) = f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\,; x>0, \theta>0. \quad (1)$$

An additional parameter was later introduced by Shanker et al. [9] with the pdf given by

$$f_L(x;\theta,\alpha) = f(x;\theta,\alpha) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x}\,; x>0, \theta>0, \alpha>\theta. \quad (2)$$

It can be easily seen that when $\alpha = 1$, the distribution reduces to a one parameter Lindley distribution. On the other hand, the Burr XII (Burr) was first introduced by Burr [10] as a two-parameter distribution. The Burr distribution can be used to model the household income data, insurance risk data, flood levels and failure data.

This paper is organized as follows. The LBXIIPS distribution is presented in section 2. In section 3, moments and generating functions are presented. Probability Weighted Moments are given in section 4 while the distribution of order statistics and entropy are presented in section 5. Maximum likelihood estimates of the model parameters are given in section 6. Some special cases of the proposed distribution are given in section 7 and simulation studies are provided in section 8. In section 9 application of the special case of the proposed distribution is presented and lastly the concluding remarks in section 10.

2. Lindley-Burr XII power series class of distributions

In this section, we present the Lindley-Burr XII power series (LBXIIPS) class of distributions. Recently, Mdlongwa et al. [11] presented results on a new distribution called the Burr-Modified Weibull (BMW) distribution which brought into attention the Burr-Weibull distribution as a special case of BMW when $\lambda = 0$. The motivation for this family of distributions is their use in reliability analysis as well as flexi-

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bility in shapes of the pdf and hazard rate function. Oluyede et al. [12] studied and presented the Burr-Weibull power series class of distributions. Now, suppose we have a random variable say Y that is Y_i, for $i = 1, \dots, N$ which denote the time to failure of the device due to the i^{th} defect and if Y_i's are independent and identically distributed (iid), Lindley-Burr XII (LBXII) random variables are independent of N which is a truncated power series random variable, then the time to the first failure can be modeled by a distribution in the class of LBXIIPS distributions. If Y is a random variable following LBXII with parameters $\lambda, c, k > 0$, its cumulative distribution function (cdf) is given by

$$G_{LBXII}(y;\lambda,c,k) = G(y;\lambda,c,k) = 1 - \frac{1+\lambda+\lambda y}{1+\lambda}\frac{e^{-\lambda y}}{(1+y^c)^k},$$

for $\lambda, c, k > 0$ and $y \geq 0$. The corresponding pdf and the survival function are given by

$$g_{LBXII}(y;\lambda,c,k) = \frac{(1+y^c)^{-k}}{1+\lambda}e^{-\lambda y}\left[\lambda^2(1+y^c) + \frac{(1+\lambda+\lambda y)kcy^{c-1}}{1+y^c}\right]$$

and

$$S_{LBXII}(y;\lambda,c,k) = \frac{1+\lambda+\lambda y}{1+\lambda}\frac{e^{-\lambda y}}{(1+y^c)^k},$$

respectively.

Now, let N be a discrete random variable following a power series distribution assumed to be truncated at zero, whose probability mass function (pmf) is given by

$$P(N=n) = \frac{a_n\theta^n}{C(\theta)}, \; n = 1, 2, \dots,$$

with the coefficients a_n depending only on n, $C(\theta) = \sum_{i=1}^{\infty} na_n = \theta^n$ for $\theta > 0$, such that $C(\theta)$ is finite and $a_{n_{n\geq 1}}$ a sequence of positive real numbers. The power series distributions include binomial, Poisson, geometric and logarithmic distributions, Johnson et al. [13].

Let $X = Y_{(1)} = \min(Y_1, \dots, Y_N)$, the conditional distribution of X given $N = n$ is given by

$$G_{X|N=n}(x) = 1 - \prod_{i=1}^{n}(1-G(x)) = 1 - S^n(x) = 1 - \left[\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right]^n,$$

then the marginal cdf of X, say F_θ is given by

$$F_\theta(x) = 1 - \frac{C(\theta S(x))}{C(\theta)} = 1 - \frac{C\left(\theta\left(\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right)\right)}{C(\theta)}.$$

Remark. Let $C'(\theta)$ be the derivative of $C(\theta)$, that is, $C'(\theta) = \sum_{i=1}^{\infty} na_n = \theta^{n-1}$, then the marginal pdf of X, say f_θ is given by

$$f_\theta(x) = \frac{dF_\theta(x)}{dx} = \frac{\theta g(x) C'(\theta S(x))}{C(\theta)},$$

with the hazard and reverse hazard functions as

$$h_\theta(x) = \frac{f_\theta(x)}{S_\theta(x)} = \theta g(x)\frac{C'(\theta S(x))}{C(\theta S(x))}$$

and

$$\tau_\theta(x) = \frac{f_\theta(x)}{F_\theta(x)} = \theta g(x)\frac{C'(\theta)S(x)}{C(\theta) - C(\theta S(x))},$$

respectively, where $S_\theta(x) = 1 - F_\theta(x)$.

The quantile function can be obtained by inverting $F_\theta(x) = u$, $0 \leq u \leq 1$ and is equivalent to solving the non-linear equation

$$-lnC(\theta S(x)) + lnC(\theta) + ln(1-u) = 0. \tag{3}$$

Random numbers can be generated from equation (3).

Some special cases of the LBXIIPS class of distributions are given in Table 1.

Table 1. Some special cases of the LBXIIPS class of distributions.

Distribution	a_n	$C(\theta)$	cdf
LBXII Poisson	$(n!)^{-1}$	$e^\theta - 1$	$1 - \frac{e^{\left(\theta(1+\lambda+\lambda x)(1+\lambda)^{-1}e^{-\lambda x}(1+x^c)^{-k}\right)}-1}{e^\theta - 1}$
LBXII Geometric	1	$\theta(1-\theta)^{-1}$	$1 - \frac{\theta(1-\theta)^{-1}\left((1+\lambda+\lambda x)(1+\lambda)^{-1}e^{-\lambda x}(1+x^c)^{-k}\right)}{\theta(1-\theta)^{-1}}$
LBXII Logarithmic	n^{-1}	$-\log(1-\theta)$	$1 - \frac{\log\left(1-\theta\left((1+\lambda+\lambda x)(1+\lambda)^{-1}e^{-\lambda x}(1+x^c)^{-k}\right)\right)}{\log(1-\theta)}$
LBXII Binomial	$\binom{m}{n}$	$(1+\theta)^m - 1$	$1 - \frac{\left(1+\theta\left((1+\lambda+\lambda x)(1+\lambda)^{-1}e^{-\lambda x}(1+x^c)^{-k}\right)\right)^m}{(1+\theta)^m - 1}$

3. Moments and generating functions

3.1. Moments

The r^{th} moment of the LBXIIPS class of distributions is given by

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r \frac{\theta g(x) C'(\theta S(x))}{C(\theta)}dx \\
&= \int_0^\infty x^r \frac{\theta}{C(\theta)}\sum_{n=1}^{\infty} na_n\theta^{n-1}S^{n-1}(x)g(x)dx \\
&= \sum_{n=0}^{\infty}\frac{(n+1)a_{n+1}\theta^{n+1}}{C(\theta)}\int_0^\infty x^r g(x)S^n(x)dx \\
&= \sum_{n=0}^{\infty}\frac{(n+1)a_{n+1}\theta^{n+1}}{C(\theta)}\int_0^\infty x^r\left(\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right)^n g(x)dx \\
&= \sum_{n=0}^{\infty}\frac{(n+1)a_{n+1}\theta^{n+1}}{C(\theta)}\int_0^\infty x^r\frac{(1+\lambda+\lambda x)^n e^{-\lambda nx}}{(1+\lambda)^n(1+x^c)^{kn}}g(x)dx.
\end{aligned}$$

Considering the expansion

$$(1+\lambda+\lambda x)^n = \sum_{p=0}^{\infty}(1+\lambda)^{n-p}(\lambda x)^p\binom{n}{p},$$

we get

$$\begin{aligned}
E(X^r) = &\sum_{n,p=0}^{\infty}\frac{(n+1)a_{n+1}\theta^{n+1}}{C(\theta)}(1+\lambda)^{n-p}\binom{n}{p}\lambda^P \\
&\times\int_0^\infty \frac{x^{r+P}e^{-\lambda nx}}{(1+\lambda)^n(1+x^c)^{kn}}g(x)dx.
\end{aligned}$$

Substituting $g(x)$, we have

$$\begin{aligned}
E(X^r) = &\sum_{n,p=0}^{\infty}\frac{(n+1)a_{n+1}\theta^{n+1}\lambda^p}{C(\theta)(1+\lambda)^p}(1+\lambda)^{n-p}\binom{n}{p} \\
&\times\int_0^\infty \frac{x^{r+P}e^{-\lambda(n+1)x}}{(1+x^c)^{kn}}\frac{(1+x^c)^{-k}}{1+\lambda} \\
&\times\left(\lambda^2(1+x^c) + \frac{(1+\lambda+\lambda x)}{1+x^c}kcx^{c-1}\right)dx.
\end{aligned}$$

Expanding

$$e^{-\lambda(n+1)x} = \sum_{q=0}^{\infty}\frac{(-1)^q(n+1)^q\lambda^q x^q}{q!},$$

we have

$$\begin{aligned}
E(X^r) = &\sum_{n,p,q=0}^{\infty}\frac{a_{n+1}\theta^{n+1}\lambda^{p+q}(n+1)^{q+1}(-1)^q}{C(\theta)(1+\lambda)^{p+1}q!}\binom{n}{p} \\
&\times\int_0^\infty \frac{x^{r+p+q}}{(1+x^c)^{k(n+1)}}\left(\lambda^2(1+x^c) + \frac{(1+\lambda+\lambda x)}{1+\lambda}kcx^{c-1}\right)dx.
\end{aligned}$$

Splitting the function

$$E(X^r) = \sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q}(n+1)^{q+1}(-1)^q}{C(\theta)(1+\lambda)^{p+1}q!}\binom{n}{p}\int_0^{\infty}\frac{x^{r+p+q}}{(1+x^c)^{k+n-1}}dx$$
$$+\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q}(n+1)^{q+1}(-1)^q k}{C(\theta)(1+\lambda)^{p+1}q!}\binom{n}{p}$$
$$\times\int_0^{\infty}\frac{x^{r+p+q}(1+\lambda+\lambda x)cx^{c-1}}{(1+x^c)^{k+n+1}}dx.$$

Let $t=(1+x^c)^{-1} \Rightarrow \frac{t}{dx} = -cx^{c-1}(1+x^c)^{-2}$ and $x = -\left(\frac{1-t}{t}\right)^{\frac{1}{c}}$.

So that

$$\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q+2}(n+1)^{q+1}(-1)^q}{C(\theta)(1+\lambda)^{p+1}q!}\binom{n}{p}\int_0^{1}\left(\frac{1-t}{t}\right)^{\frac{r+p+q-c+1}{c}}t^{3-k-n}dt.$$

The first part of the function is given by

$$\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q+2}(n+1)^{q+1}(-1)^q}{C(\theta)(1+\lambda)^{p+1}q!c}\binom{n}{p}$$
$$\times B\left(\frac{r+p+q-c+1}{c}, 3-k-n-\frac{r+q+p-c+1}{c}\right),$$

and the second part as

$$\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q}(n+1)^{q+1}(-1)^q k}{C(\theta)(1+\lambda)^{p}q!}\binom{n}{p}\int_0^{1}\left(\frac{1-t}{t}\right)^{\frac{r+p+q}{c}}t^{1-k-n}dt.$$

$$+\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q+1}(n+1)^{q+1}(-1)^q}{C(\theta)(1+\lambda)^{p+1}q!k}\binom{n}{p}\int_0^{1}\left(\frac{1-t}{t}\right)^{\frac{r+p+q+1}{c}}t^{1-k-n}dt,$$

so that the r^{th} moment of the LBXIIPS class of distributions is given as

$$E(X^r) = \sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q+2}(n+1)^{q+1}(-1)^q}{C(\theta)(1+\lambda)^{p+1}q!c}\binom{n}{p}$$
$$\times B\left(\frac{r+p+q-c+1}{c}, 3-k-n-\frac{r+q+p-c+1}{c}\right)$$
$$+\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q}(n+1)^{q+1}(-1)^q k}{C(\theta)(1+\lambda)^{p}q!}\binom{n}{p}$$
$$\times B\left(\frac{r+p+q}{c}, 1-k-n-\frac{r+q+p}{c}\right)$$
$$+\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q+1}(n+1)^{q+1}(-1)^q}{C(\theta)(1+\lambda)^{p+1}q!k}\binom{n}{p}$$
$$\times B\left(\frac{r+p+q+1}{c}, 1-k-n-\frac{r+q+p+1}{c}\right),$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function.

3.2. Conditional moments

It is essential to know about the conditional moments when dealing with lifetime models. The r^{th} conditional moment of a random variable X following a LBXIIPS class of distributions is defined by

$$E(X^r|X>t) = \frac{1}{\bar{F}(t)}\int_t^{\infty}x^r f_{LBXIIPS}(x)dx$$
$$= \frac{1}{\bar{F}(t)}\frac{\theta}{C(\theta)}\sum_{n=1}^{\infty}na_n\theta^{n-1}\int_t^{\infty}x^r\left(\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right)^n$$
$$\times\theta\left[\frac{(1+x^c)^{-k}}{1+\lambda}e^{-\lambda x}\left(\left(\lambda^2(1+x^c)\right)+\frac{kcx^{c-1}(1+\lambda+\lambda x)}{1+x^c}\right)\right]dx$$
$$= \frac{1}{\bar{F}(t)}\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q+2}(n+1)^{q+1}(-1)^q}{C(\theta)(1+\lambda)^{p+1}q!c}\binom{n}{p}$$
$$\times B\left(\frac{r+p+q-c+1}{c}, 3-k-n-\frac{r+q+p-c+1}{c}\right)$$
$$+\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q}(n+1)^{q+1}(-1)^q k}{C(\theta)(1+\lambda)^{p}q!}\binom{n}{p}$$
$$\times B\left(\frac{r+p+q}{c}, 1-k-n-\frac{r+q+p}{c}\right)$$
$$+\sum_{n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1}\lambda^{p+q+1}(n+1)^{q+1}(-1)^q}{C(\theta)(1+\lambda)^{p+1}q!k}\binom{n}{p}$$
$$\times B\left(\frac{r+p+q+1}{c}, 1-k-n-\frac{r+q+p+1}{c}\right),$$

where $B(a,b)$ is the incomplete beta function.

4. Probability weighted moments

Given a random variable X with a cdf $F(x)$, the probability weighted moments (PWMs) are defined as

$$\eta_{s,i} = E\left(X^s[F(X)]^i\right) = \int_0^{\infty}x^s(F(x))^i f(x)dx.$$

They are mainly used in the estimation of parameters for a probability function hence the PWMs of a LBXIIPS is derived as follows:

$$\eta_{s,i} = E\left(X^s[F(X)]^i\right) = \int_0^{\infty}x^s(F(x))^i f(x)dx$$
$$= \int_o^{\infty}\left(1-\frac{C\left(\theta\left(\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{(1+\lambda)(1+x^c)^k}\right)\right)}{C(\theta)}\right)^i\frac{\theta g(x)C'(\theta S(x))}{C(\theta)}dx.$$

Expanding

$$\left[1-\frac{C\left(\theta\left(\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right)\right)}{C(\theta)}\right]^i$$
$$= \sum_{m=0}^{n}(-1)^i\binom{i}{m}\left(\frac{C\left(\theta\left(\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right)\right)}{C(\theta)}\right)^m,$$

such that

$$\left(\frac{C\left(\theta\left(\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right)\right)}{C(\theta)}\right)^m = \sum_{z,m=0}^{\infty}\frac{d_{z,m}}{C(\theta)}\theta^z\left(\frac{(1+\lambda+\lambda x)e^{-\lambda x}}{1+\lambda(1+x^c)^k}\right)^z,$$

where $d_{z,m} = \left(zb_o\right)^{-1}\sum_{n=1}^{z}[m(h+1)-z]b_h\, d_{z-h,m}$ and $d_{o,z} = b_o^z$ by applying the power series raised to a positive integer (Gradshetyn and Ryzhik). Substituting $f(x)$, we get

$$\sum_{z,m,n=0}^{\infty}\frac{a_{n+1}\theta^{n+1+z}\lambda^{p+q+2}(n+1)(-1)^m d_{z,m}}{C(\theta)^{z+1}}\binom{i}{m}\left(\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right)^{n+z}$$
$$\times\frac{(1+x^c)^{-k}}{1+\lambda}\left(\lambda^2(1+x^c)+\frac{(1+\lambda+\lambda x)}{1+x^c}kcx^{c-1}\right).$$

Expanding

$$(1+\lambda+\lambda x)^{n+z} = \sum_{p=0}^{\infty}\binom{n+z}{p}(1+\lambda)^{n-p+2}(\lambda x)^p,$$

$$e^{-\lambda(n+z+1)x} = \sum_{q=0}^{\infty}(-1)^q\frac{(n+z+1)^q}{q!}(\lambda)^q x^q,$$

so that

$$\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}}{C(\theta)^{z+1}(1+\lambda)^{n+z+1}q!} \times \binom{i}{m}\binom{n+z}{p}\frac{x^{p+q}}{(1+x^c)^{n+z+k}} \times \left[\frac{(1+x^c)^{-k}}{1+\lambda}\left(\lambda^2(1+x^c)+\frac{(1+\lambda+\lambda x)kcx^{c-1}}{1+x^c}\right)\right].$$

Therefore, the PWMs of the LBXIIPS is given by

$$\begin{aligned}
&E\left(X^s[F(X)]^i\right)\\
&=\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q+2}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}c}{C(\theta)^{z+1}(1+\lambda)^{n+z+1}q!}\\
&\times\binom{i}{m}\binom{n+z}{p}\int\frac{x^{s+p+q}}{(1+x^c)^{n+z+k-1}}dx\\
&+\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}k}{C(\theta)^{z+1}(1+\lambda)^{n+z}q!}\\
&\times\binom{i}{m}\binom{n+z}{p}\int_0^{\infty}\frac{x^{s+p+q}cx^{c-1}}{(1+x^c)^{n+z+k+1}}dx\\
&+\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q+1}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}k}{C(\theta)^{z+1}(1+\lambda)^{n+z+1}q!}\\
&\times\binom{i}{m}\binom{n+z}{p}\int_0^{\infty}\frac{x^{s+p+q+1}cx^{c-1}}{(1+x^c)^{n+z+k+1}}dx\\
&=\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q+2}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}c}{C(\theta)^{z+1}(1+\lambda)^{n+z+1}q!}\\
&\times\binom{i}{m}\binom{n+z}{p}\frac{1}{c}B\left(\frac{s+p+q}{c},3-n-z-k-\frac{s+p+q}{c}\right)\\
&+\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}k}{C(\theta)^{z+1}(1+\lambda)^{n+z}q!}\\
&\times\binom{i}{m}\binom{n+z}{p}B\left(\frac{s+p+q}{c},1-n-z-k-\frac{s+p+q}{c}\right)\\
&+\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}k}{C(\theta)^{z+1}(1+\lambda)^{n+z}q!}\\
&\times\binom{i}{m}\binom{n+z}{p}B\left(\frac{s+p+q+1}{c},1-n-z-k-\frac{s+p+q+1}{c}\right).
\end{aligned}$$

5. Distribution of order statistics and entropy

In this section, we present the distribution of the i^{th} order statistics and the Rényi entropy.

5.1. Order statistics

Let $X_1,\ldots,X_n$ be a random sample from the LBXIIPS distribution, using some results from Section 4 the pdf of the i^{th} order statistic from the LBXIIPS distribution is given by

$$f_{i,n}(x)=\frac{f_{LBXIIPS}(x)}{B(i,n-i+1)}\sum_{j=0}^{n-j}\binom{n-i}{j}\left[F_{LBXIIPS}(x)\right]^{j+i-1}.$$

Using the results from PWMs for $f(x)F^i(x)$, the i^{th} order statistic of the LBXIIPS class of distributions is given as

$$\begin{aligned}
&=\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q+2}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}c}{C(\theta)^{z+1}(1+\lambda)^{n+z+1}q!}\\
&\times\binom{j+i-1}{m}\binom{n+z}{p}\\
&\times\frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-j}\binom{n-i}{j}\frac{1}{c}B\left(\frac{p+q}{c},3-n-z-k-\frac{p+q}{c}\right)\\
&+\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}k}{C(\theta)^{z+1}(1+\lambda)^{n+z}q!}\\
&\times\binom{j+i-1}{m}\binom{n+z}{p}\\
&\times\frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-j}\binom{n-i}{j}B\left(\frac{p+q}{c},1-n-z-k-\frac{p+q}{c}\right)\\
&+\sum_{z,m,n,p,q=0}^{\infty} \frac{a_{n+1}\theta^{n+1+z}(1+\lambda)^{n-p}\lambda^{p+q}(n+1)(n+q+1)^q(-1)^{m+q}d_{z,m}k}{C(\theta)^{z+1}(1+\lambda)^{n+z}q!}\\
&\times\binom{j+i-1}{m}\binom{n+z}{p}\\
&\times\frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-j}\binom{n-i}{j}B\left(\frac{p+q+1}{c},1-n-z-k-\frac{p+q+1}{c}\right),
\end{aligned}$$

where $\left[F_{LBXIIPS}\right](x;\lambda,c,k,\theta)=1-\frac{C\left(\theta\left(\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right)\right)}{C(\theta)}$, $\lambda,c,k,\theta>0$.

5.2. Entropy

Rényi entropy [14] of a random variable X following the LBXIIPS distribution is given by

$$I_R(\nu)=\frac{1}{1-\nu}\log\left(\int_0^{\infty}\left[f_{LBXIIPS}(x;\lambda,c,k,\theta)\right]^{\nu}dx\right), \nu\neq 1, \nu>0.$$

Note that, Rényi entropy tends to Shannon entropy as $\nu\to 1$.

$$\begin{aligned}
f^{\nu}_{LBXIIPS}(x)&=\left(\theta g(x)\frac{C'(\theta S(x))}{C(\theta)}\right)^{\nu}\\
&=\theta^{\nu}g^{\nu}(x)\left(\sum_{n=1}^{\infty}\frac{a_n\theta^{n-1}S^{n-1}(x)}{C(\theta)}\right)^{\nu}\\
&=\theta^{\nu}g^{\nu}(x)\sum_{z=0}^{\infty}\frac{d_{s,\nu}\theta^{(n-1)z}S^{(n-1)z}(x)g^{\nu}(x)}{C^{\nu}(\theta)}\\
&=\sum_{z=0}^{\infty}\frac{\theta^{(n-1)z+\nu}}{C^{\nu}(\theta)}d_{s,\nu}g^{\nu}(x)S^{(n-1)z}(x)\\
&=\sum_{z=0}^{\infty}\frac{\theta^{(n-1)z+\nu}}{C^{\nu}(\theta)}d_{s,\nu}g^{\nu}(x)\left(\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)^k}\right)^{(n-1)z}.
\end{aligned}$$

Considering

$$(1+\lambda+\lambda x)^{(n-1)z}=\sum_{p=0}^{\infty}\binom{(n-z)z}{p}(1+\lambda)^{(n-1)z-p}(\lambda x)^p,$$

$$e^{-\lambda((n-1)z+1)x}=\sum_{q=0}^{\infty}\frac{(-1)^q\lambda^q x^q((n-1)z+1)^q}{q!}$$

such that

$$\sum_{p,q,z=0}^{\infty}\frac{\theta^{(n-1)z+\nu}(-1)^q d_{s,\nu}}{C^{\nu}(\theta)q!}((n-1)z+1)^q(1+\lambda)^{(n-1)z-p}\lambda^{p+q}\binom{(n-1)z}{p}x^{p+q} \times\left(\lambda^2(1+x^c)+\frac{(1+\lambda+\lambda x)kcx^{c-1}}{1+x^c}\right),$$

$$\begin{aligned}
&\int_0^{\infty}f^{\nu}(x)dx\\
&=\sum_{p,q,z=0}^{\infty}\frac{\theta^{(n-1)z+\nu}(-1)^q d_{s,\nu}}{C^{\nu}(\theta)q!}((n-1)z+1)^q(1+\lambda)^{(n-1)z-p}\lambda^{p+q+2}\\
&\times\binom{(n-1)z}{p}\int_0^{\infty}\frac{x^{p+q}}{(1+x^c)^{k(z+1)-1}}dx
\end{aligned}$$

$$+\sum_{p,q,z=0}^{\infty}\frac{\theta^{(n-1)z+v}(-1)^q d_{s,v}}{C^v(\theta)q!}((n-1)z+1)^q(1+\lambda)^{(n-1)z-p+1}\lambda^{p+q}\binom{(n-1)z}{p}$$
$$\times\int_0^\infty\frac{x^{q+p}kcx^{c-1}}{(1+x^c)^{k(z+1)+1}}dx$$
$$+\sum_{p,q,z=0}^{\infty}\frac{\theta^{(n-1)z+v}(-1)^q d_{s,v}}{C^v(\theta)q!}((n-1)z+1)^q(1+\lambda)^{(n-1)z-p}\lambda^{p+q+1}\binom{(n-1)z}{p}$$
$$\times\int_0^\infty\frac{x^{q+p+1}kcx^{c-1}}{(1+x^c)^{k(z+1)+1}}dx.$$

Hence, Rényi entropy for the LBXIIPS distribution is given by

$$I_R(v)=\frac{1}{1-v}\log\left[\sum_{p,q,z=0}^{\infty}\frac{\theta^{(n-1)z+v}(-1)^q d_{s,v}}{C^v(\theta)q!}((n-1)z+1)^q(1+\lambda)^{(n-1)z-p}\right.$$
$$\times\lambda^{p+q}\binom{(n-1)z}{p}\frac{\lambda^2}{c}B\left(\frac{q+p+c-1}{c},3-k(z+1)-\frac{q+p+c-1}{c}\right)$$
$$+\sum_{p,q,z=0}^{\infty}\frac{\theta^{(n-1)z+v}(-1)^q d_{s,v}}{C^v(\theta)q!}((n-1)z+1)^q(1+\lambda)^{(n-1)z-p}\lambda^{p+q}\binom{(n-1)z}{p}$$
$$\times(1+\lambda)B\left(\frac{q+p}{c},1-k(z+1)-\frac{q+p}{c}\right)$$
$$+\sum_{p,q,z=0}^{\infty}\frac{\theta^{(n-1)z+v}(-1)^q d_{s,v}}{C^v(\theta)q!}((n-1)z+1)^q(1+\lambda)^{(n-1)z-p}\lambda^{p+q}\binom{(n-1)z}{p}$$
$$\left.\times\lambda kB\left(\frac{q+p+1}{c},1-k(z+1)-\frac{q+p+1}{c}\right)\right].$$

Here $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}$ is the incomplete beta function.

6. Estimation and inference

Let $X\sim\text{LBXIIPS}(\lambda,c,k,\theta)$ and $\Delta=(\lambda,c,k,\theta)^T$ be the parameter vector. Based on a random sample of size n, the log likelihood function $\ell=\ell(\Delta)$ is

$$l_n=n\ln(\theta)-n\ln[C(\theta)]+\sum_{i=1}^{\infty}ln\left[g(x_i)\right]+\sum_{i=1}^{\infty}ln\left[C'(\theta S(x_i))\right]$$
$$=n\ln(\theta)-n\ln(C(\theta))+\sum_{i=1}^{\infty}ln\left[\frac{(1+x_i^c)^{-k}e^{-\lambda x_i}}{1+\lambda}\right]$$
$$+\sum_{i=1}^{\infty}ln\left(\lambda^2(1+x_i^c)+\frac{kcx_i^{c-1}(1+\lambda+\lambda x_i)}{1+x_i^c}\right)$$
$$+\sum_{i=1}^{\infty}ln\left[C'\left(\theta\left(\frac{(1+\lambda+\lambda x_i)}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)^k}\right)\right)\right].$$

The elements of the score vector are given by

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}-\frac{C'(\theta)}{C(\theta)}+\sum_{i=1}^{\infty}\frac{C''\left(\theta\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)^k}\right)\right)\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)^k}}{C'\left(\theta\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)^k}\right)\right)},$$

$$\frac{\partial\ell}{\partial\lambda}=\sum_{i=1}^{\infty}\frac{(1+x_i^c)^{-k}\left(x_i(1+\lambda)^{-1}e^{-\lambda x_i}-e^{-\lambda x_i}(1+\lambda)^{-2}\right)}{(1+x_i^c)^{-k}e^{-\lambda x_i}(1+\lambda)^{-1}}$$
$$+\frac{2\lambda(1+x_i^c)+ckx_i^{c-1}(1+x_i)(1+x_i^c)^{-1}}{\lambda^2(1+x_i^c)+ckx_i^{c-1}(1+\lambda+\lambda x_i)(1+x_i^c)^{-1}}$$
$$+\frac{C''(\theta(-e^{-\lambda x_i}\lambda^2x_i^2-e^{-\lambda x_i}\lambda x_i^2-e^{-\lambda x_i}\lambda^2x_i^2-2e^{-\lambda x_i}\lambda x_i))}{C'\left(\theta\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)^k}\right)\right)}$$

$$\frac{\partial l}{\partial c}=-\sum_{i=1}^{\infty}\frac{ke^{-\lambda x_i}x_i^{c-1}(1+x_i^c)^{-k}ln(1+x_i^c)(1+\lambda)^{-1}}{(1+x_i^c)^{-k}e^{-\lambda x_i}(1+\lambda)^{-1}}$$
$$+\frac{\lambda^2x_i^{c-1}+k(1+\lambda+\lambda x_i)\left(\frac{x_i^{c-1}}{1+x_i^c}\frac{x_i^c ln(x_i^c)c}{(1+x_i^c)^2}\right)}{\lambda^2(1+x_i^c)+kcx_i^{c-1}(1+\lambda+\lambda x_i)(1+x_i^c)^{-1}}$$
$$+\frac{C''\left(\theta\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)^k}\right)\right)\theta kx_i^{c-1}ln(1+x_i^c)}{C'\left(\theta\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)^k}\right)\right)},$$

and

$$\frac{\partial l}{\partial k}=-\sum_{i=1}^{\infty}\frac{e^{-\lambda x_i}(1+x_i^c)^{-k}ln(1+x_i^c)(1+\lambda)^{-1}}{(1+x_i^c)^{-k}e^{-\lambda x_i}(1+\lambda)^{-1}}$$
$$+\frac{cx_i^c(1+\lambda+\lambda x_i)(1+x_i^c)^{-1}}{\lambda^2(1+x_i^c)+kcx_i^c(1+\lambda+\lambda x_i)(1+x_i^c)^{-1}}$$
$$+\frac{C''\left(\theta\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)^k}\right)\right)\theta ln(1+x_i^c)}{C'\left(\theta\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)^k}\right)\right)}.$$

The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ are obtained by solving the non-linear equation $\left(\frac{\partial l}{\partial\theta},\frac{\partial l}{\partial\lambda},\frac{\partial l}{\partial c},\frac{\partial l}{\partial k}\right)=0$, using the numerical method such as Newton-Raphson procedure.

6.1. Other methods of parameter estimation

Other methods of parameter estimation include the following.

6.1.1. Least squares method

Ordinary least squares parameter estimates are obtained by minimizing the function

$$Q(\Delta|y)=\sum_{i=1}^{n}\left(G(y_{i:n}|\Delta)-\frac{i}{n+1}\right)^2.$$

The solutions to the nonlinear equations

$$\left(\frac{\partial Q(\Delta|y)}{\partial\alpha},\frac{\partial Q(\Delta|y)}{\partial\beta},\frac{\partial Q(\Delta|y)}{\partial b},\frac{\partial Q(\Delta|y)}{\partial\xi_k}\right)^T=0,$$

give the least squares parameter estimates for the LBXIIPS distribution.

6.1.2. Cramér-von Mises method

Cramér-von Mises method involves minimizing the function

$$CVM(\Delta|y)=\frac{1}{12n}+\sum_{i=1}^{n}\left(G(y_{i:n}|\Delta)-\frac{2i-1}{2n}\right)^2,$$

with respect to the parameters $(\alpha,\beta,b,\xi_k)^T$.

6.1.3. Weighted least squares method

We obtain weighted least squares parameter estimates for the LBXIIPS distribution by minimizing the function

$$W(\Delta|y)=\sum_{i=1}^{n}\frac{1}{Var[G(y_{i:n})]}\left(G(y_{i:n}|\Delta)-\frac{i}{n+1}\right)^2$$
$$=\sum_{i=1}^{n}w_i\left(G(y_{i:n}|\Delta)-\frac{i}{n+1}\right)^2,$$

where $w_i=\frac{(n+1)^2(n+2)}{i(n-i+1)}$. The solution to the non-linear system of equations

$$\left(\frac{\partial W(\Delta|y)}{\partial\alpha},\frac{\partial W(\Delta|y)}{\partial\beta},\frac{\partial W(\Delta|y)}{\partial b},\frac{\partial W(\Delta|y)}{\partial\xi_k}\right)^T=0,$$

gives weighted least squares parameter estimates for the LBXIIPS distribution.

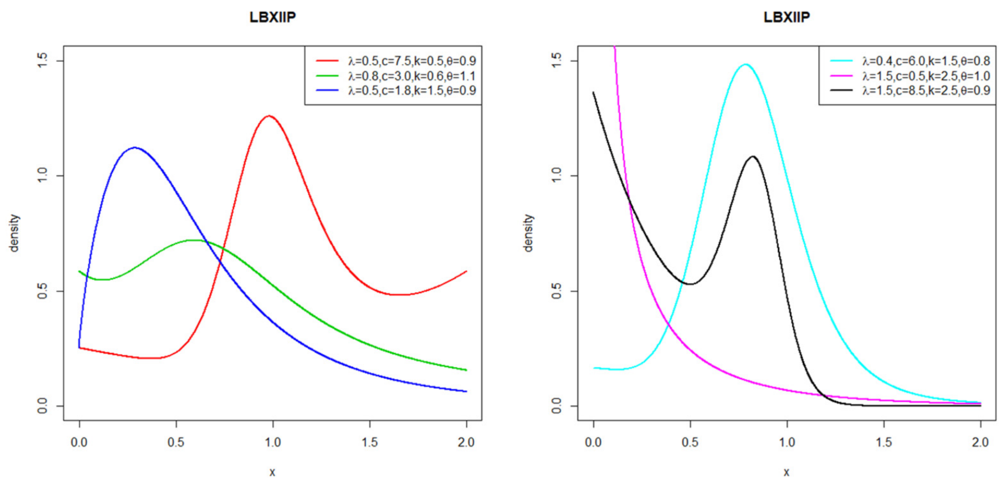

Fig. 1. Plots of the LBXIIP density function.

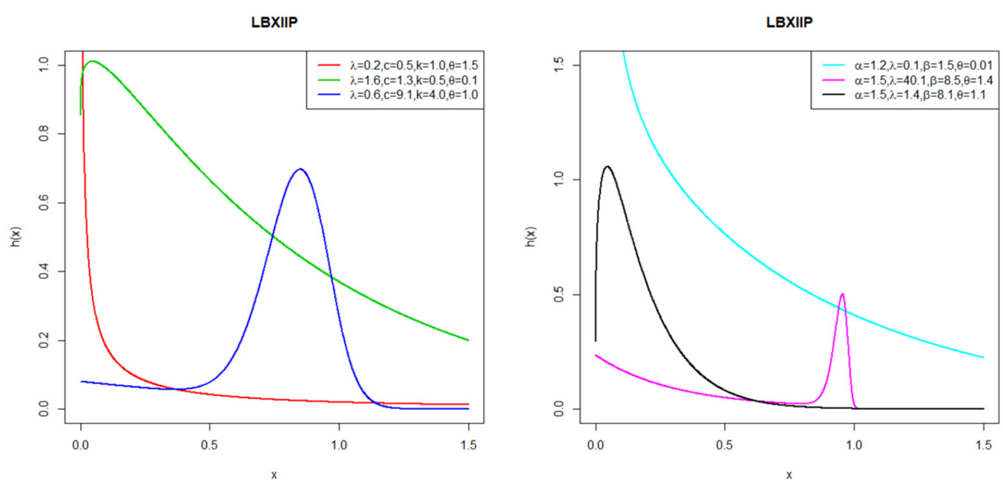

Fig. 2. Plots of the LBXIIP hazard function.

7. Some special cases

In this Section, we present two special cases of the LBXIIPS, namely, the LBXII-Poisson (LBXIIP) and LBXII-logarithmic (LBXIIL) distributions.

7.1. Lindley-Burr XII Poisson distribution

In this sub-section, we discuss some results on the Lindley-Burr XII Poisson (LBXIIP) distribution and its sub models. In Table 1 we showed that LBXIIP is a special case of the LBXIIPS class of distributions with cdf given by

$$F_{LBXIIP}(x;\lambda,c,k,\theta)=1-\frac{e^{\left(\theta(1+\lambda+\lambda x)(1+\lambda)^{-1}e^{-\lambda x}(1+x^c)^{-k}\right)}-1}{e^{\theta}-1}.$$

The corresponding pdf is given as

$$f_{LBXIIP}(x;\lambda,c,k,\theta)=\theta e^{\theta\left((1+\lambda+\lambda x)(1+\lambda)^{-1}e^{-\lambda x}(1+x^c)^{-k}\right)} \times\frac{(1+x^c)^{-k}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x^c)+\frac{kcx^{c-1}(1+\lambda+\lambda x)}{1+x^c}\right] \times\left(e^{\theta}-1\right)^{-1},$$

for $\lambda,c,k,\theta>0$ and $x\geq 0$. The hazard rate function is

$$h_{LBXIIP}(x;\lambda,c,k,\theta)=\theta e^{\theta\left((1+\lambda+\lambda x)(1+\lambda)^{-1}e^{-\lambda x}(1+x^c)^{-k}\right)} \times\frac{(1+x^c)^{-k}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x^c)+\frac{kcx^{c-1}(1+\lambda+\lambda x)}{1+x^c}\right] \times\left(e^{\theta}-1\right)^{-1} \times\left(1-\frac{e^{\left(\theta(1+\lambda+\lambda x)(1+\lambda)^{-1}e^{-\lambda x}(1+x^c)^{-k}\right)}-1}{e^{\theta}-1}\right)^{-1}.$$

The plots of the LBXIIP pdf (see Fig. 1) display right skewed, symmetric among other possible shapes, whereas for the LBXIIP hazard can be decreasing, upside down bathtub, bathtub followed by upside down bathtub shape and reverse j-shaped as indicated in Fig. 2. Table 2 gives table of quantile for selected parameter values of the LBXIIP distribution. The first 5 moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for selected parameter values of the LBXIIP distribution are given in Table 3.

3D plots of skewness and kurtosis for the LBXIIP distribution show that (see Figs. 3 and 4)

- When we fix the parameters k and θ, the skewness and kurtosis of LBXIIP increases as λ and c increases.
- When we fix the parameters λ and θ, the skewness and kurtosis of LBXIIP decreases as k and c increases.

Table 2. Table of quantile for LBXIIP distribution.

	(λ, c, k, θ)				
u	$(0.9, 3.5, 5.5, 3.1)$	$(0.7, 1, 1.5, 2)$	$(1.3, 2, 3.5, 1.6)$	$(3.1, 4.5, 3.5, 4)$	$(1.2, 2.5, 5.5, 0.9)$
0.1	0.9987	2.2843	0.7195	1.0482	0.3494
0.2	1.0100	2.4022	0.7448	1.0587	0.3761
0.3	1.0227	2.5384	0.7734	1.0707	0.4048
0.4	1.0373	2.6987	0.8066	1.0843	0.4361
0.5	1.0544	2.8923	0.8459	1.1004	0.4712
0.6	1.0753	3.1347	0.8941	1.1198	0.5118
0.7	1.1018	3.4551	0.9565	1.1447	0.5612
0.8	1.1389	3.9195	1.0452	1.1794	0.6267
0.9	1.2016	4.7416	1.1989	1.2378	0.7306

Table 3. Table of moments for LBXIIP distribution.

	(λ, c, k, θ)				
Moments	$(1, 1.2, 2.1, 2)$	$(0.4, 1, 4.5, 0.3)$	$(1, 2.2, 2.3, 3)$	$(1, 1.2, 3.4, 1.8)$	$(0.5, 1.5, 2.5, 2)$
$E(X)$	0.2909	0.2486	0.2975	0.2131	0.3519
$E(X^2)$	0.2247	0.1669	0.1587	0.1124	0.2676
$E(X^3)$	0.3434	0.2294	0.1299	0.1133	0.4009
$E(X^4)$	0.8477	0.5684	0.1614	0.1861	1.1004
$E(X^5)$	2.9461	2.2973	0.3001	0.4463	4.9419
SD	0.3742	0.3242	0.2649	0.2588	0.3791
CV	1.2863	1.3043	0.8908	1.2144	1.0770
CS	3.7502	3.9804	2.2033	3.5047	3.7758
CK	27.5727	35.3662	13.6819	25.4157	33.3597

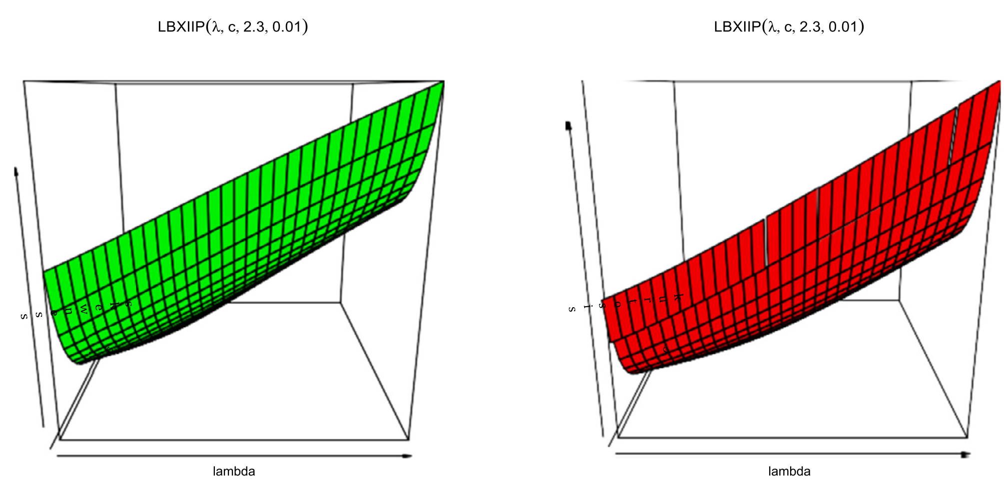

Fig. 3. 3D plots of skewness and kurtosis for LBXIIP distribution.

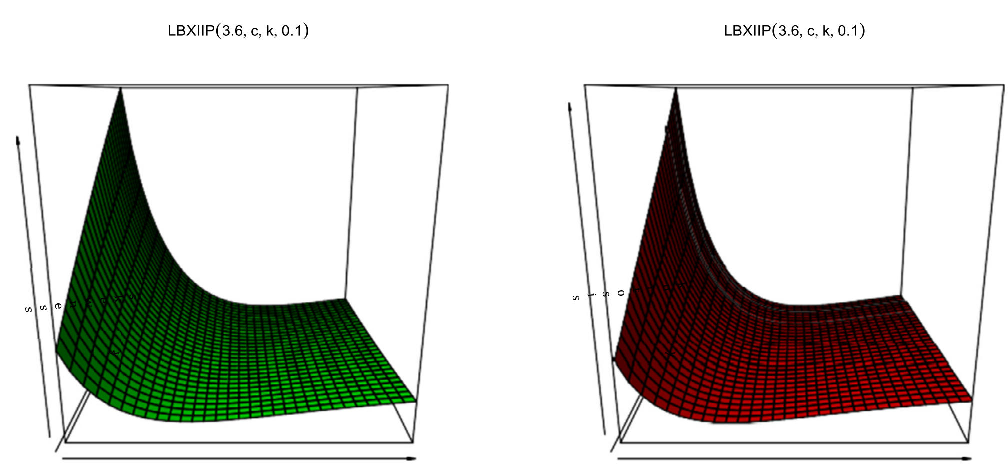

Fig. 4. 3D plots of skewness and kurtosis for LBXIIP distribution.

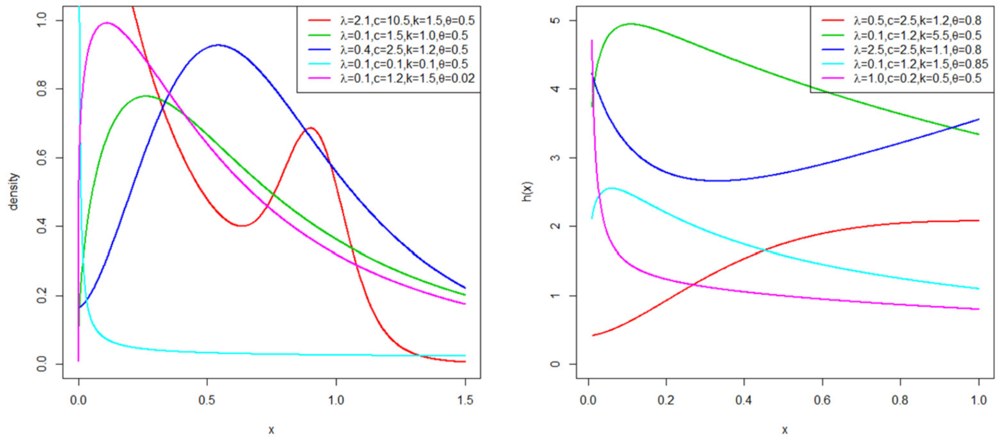

Fig. 5. Plots of the LBXIIL pdf and hrf.

Table 4. Table of quantile for LBXIIL distribution.

u	(λ, c, k, θ) (1.9, 1.5, 5.5, 0.1)	(0.7, 1, 1.5, 0.5)	(1.3, 2, 3.5, 0.6)	(3.1, 4.5, 3.5, 0.4)	(1.2, 2.5, 5.5, 0.9)
0.1	0.0421	0.0423	0.0674	0.0348	0.0413
0.2	0.0768	0.0931	0.1222	0.0744	0.0839
0.3	0.1112	0.1550	0.1741	0.1201	0.1272
0.4	0.1473	0.2325	0.2268	0.1734	0.1721
0.5	0.1869	0.3323	0.2833	0.2364	0.2203
0.6	0.2326	0.4664	0.3472	0.3113	0.2743
0.7	0.2885	0.6582	0.4239	0.4009	0.3380
0.8	0.3637	0.9626	0.5254	0.5089	0.4196
0.9	0.4867	1.5644	0.6886	0.6496	0.5435

7.1.1. Sub-models of the LBXIIP distribution

In this subsection, we discuss some of the sub-models of the LBXIIP distribution.

- We obtain Lindley Poisson (LP) distribution with $k = 0$.
- When $\lambda = 0$, we obtain the Burr-XII Poisson (BXIIP) distribution.
- If $k = 1$, we obtain Lindley-Log-logistic Poisson (LLLoGP) distribution.
- If $k = 1$, and $\lambda = 0$, we obtain Log-logistic Poisson (LLoGP) distribution.
- If $c = 1$, we obtain Lindley-Lomax Poisson (LLP) distribution.
- If $c = 1$, and $\lambda = 0$, we obtain Lomax-Poisson (LP) distribution.
- When $\theta \to 0^+$, we obtain the Lindley-Burr XII (LBXII) distribution.
- When $\theta \to 0^+$, $\lambda \to 0^+$, we obtain the Burr XII (BXII) distribution.

7.2. Lindley-Burr XII logarithmic distribution

In this sub-section, we discuss some results on the Lindley-Burr XII Logarithmic (LBXIIL) distribution and its sub models. In Table 1 we showed that LBXIIL is a special case of the LBXIIPS class of distributions with cdf given by

$$F_{LBXIIL}(x; \lambda, c, k, \theta) = 1 - \frac{\log\left(1 - \theta\left((1+\lambda+\lambda x)(1+\lambda)^{-1} e^{-\lambda x}(1+x^c)^{-k}\right)\right)}{\log(1-\theta)},$$

and the corresponding pdf defined by

$$f_{LBXIIL}(x; \lambda, c, k, \theta) = \frac{\theta e^{-\lambda x} \frac{(1+x^c)^{-k}}{1+\lambda}\left[\lambda^2(1+x^c) + \frac{kcx^{c-1}(1+\lambda+\lambda x)}{1+x^c}\right]}{-(1-\theta)\left((1+\lambda+\lambda x)(1+\lambda)^{-1} e^{-\lambda x}(1+x^c)^{-k}\right)\log(1-\theta)},$$

for $\lambda, c, k > 0$, and $0 < \theta < 1$. The hazard rate function is given by

$$h_{LBXIIL}(x; \lambda, c, k, \theta) = \frac{\theta e^{-\lambda x} \frac{(1+x^c)^{-k}}{1+\lambda}\left[\lambda^2(1+x^c) + \frac{kcx^{c-1}(1+\lambda+\lambda x)}{1+x^c}\right]}{-(1-\theta)\left((1+\lambda+\lambda x)(1+\lambda)^{-1} e^{-\lambda x}(1+x^c)^{-k}\right)\log(1-\theta)} \times \left(1 - \frac{\log\left(1 - \theta\left((1+\lambda+\lambda x)(1+\lambda)^{-1} e^{-\lambda x}(1+x^c)^{-k}\right)\right)}{\log(1-\theta)}\right)^{-1}.$$

The pdfs of the LBXIIL distribution can take various shapes that include reverse J-shaped, symmetric and right skewed (see Fig. 5). Also, the hazard function exhibits both monotonic and non-monotonic hazard rate shapes for the selected parameter values. Table 4 and Table 5 gives the quantile values and the first 5 moments for the LBXIIL distribution, respectively.

3D plots of skewness and kurtosis for the LBXIIL distribution show that (see Figs. 6 and 7)

- When we fix the parameters c and θ, the skewness and kurtosis of LBXIIL decreases as λ and k increases.
- When we fix the parameters λ and θ, the skewness and kurtosis of LBXIIL increases as k and c increases.

8. Simulations

In this section, the performance of the maximum likelihood estimates is examined by conducting simulation studies for different sample sizes. We examine the performance of the LBXIIP distribution by conducting various simulations for different sizes (n = 30, 60, 120, 240, 480, 960, 1920) using the bbmle package in R. We simulate $N = 1000$ samples for the true parameters values given in the Table 6. The Average Bias and Root Mean Square Error (RMSE) were computed. The average bias and RMSE for the estimated parameter $\hat{\theta}$, say, are given by:

Table 5. Table of moments for LBXIIL distribution.

	(λ, c, k, θ)				
Moments	$(1.2, 3.4, 2.5, 0.5)$	$(0.9, 3, 1.5, 0.5)$	$(1.3, 2, 2, 0.6)$	$(2.1, 1.5, 1.5, 0.8)$	$(1.2, 2.5, 2.5, 0.9)$
$E(X)$	0.0516	0.1702	0.0938	0.0827	0.0932
$E(X^2)$	0.0640	0.2783	0.1427	0.1252	0.1259
$E(X^3)$	0.0826	0.5466	0.2446	0.2099	0.1824
$E(X^4)$	0.1125	1.3808	0.4901	0.3991	0.2908
$E(X^5)$	0.1649	4.6503	1.1876	0.8765	0.5279
SD	0.2477	0.4993	0.3659	0.3440	0.3424
CV	4.7999	2.9338	3.8994	4.1579	3.6738
CS	4.8019	3.3285	4.2039	4.4203	3.7083
CK	25.6100	16.9678	22.6128	23.8940	16.6770

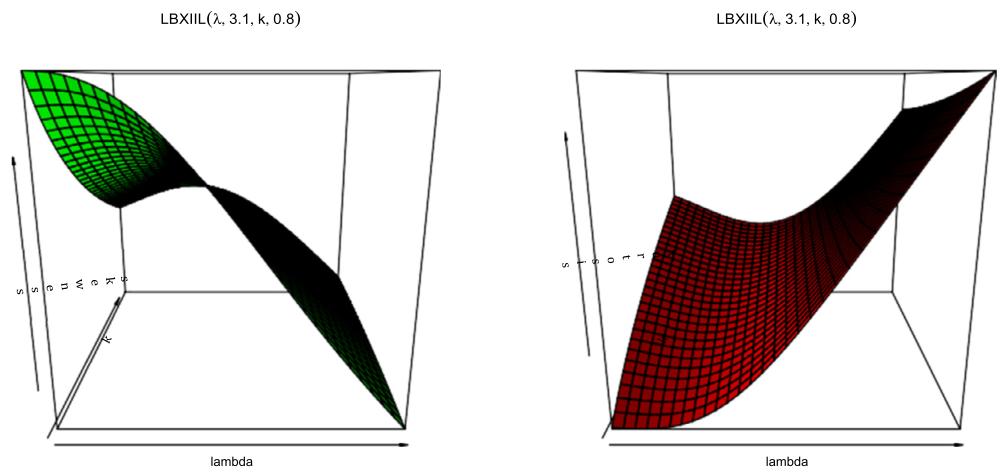

Fig. 6. 3D plots of skewness and kurtosis for LBXIIL distribution.

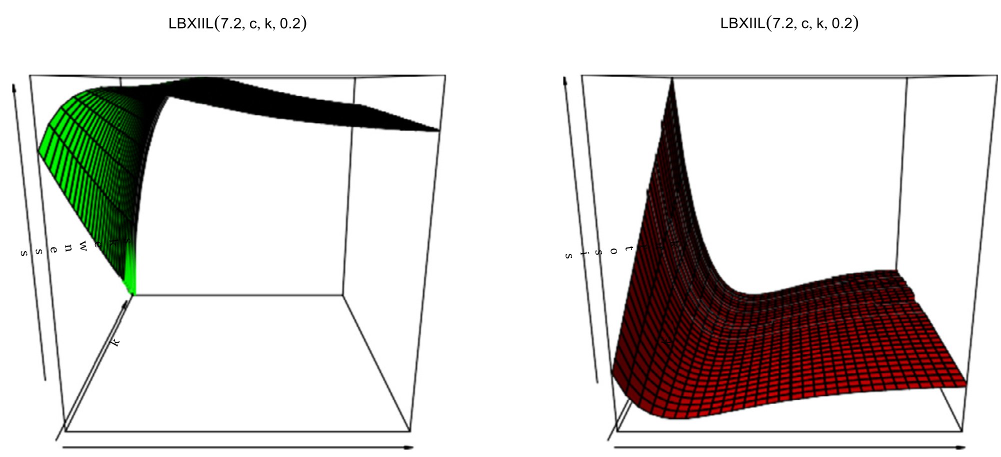

Fig. 7. 3D plots of skewness kurtosis for LBXIIL distribution.

$$ABias\left(\hat{\theta}\right) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE\left(\hat{\theta}\right) = \sqrt{\frac{\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2}{N}},$$

respectively. Table 6 list the mean MLEs of the parameters along with the respective root mean squared errors (RMSEs).

From the results we can clearly see that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter value, whereas the Average bias and the RMSEs decay towards zero as indicated in Table 6.

9. Application

A real data example of the LBXIIL distribution is presented in order to assess the flexibility of the new class of distributions. We assessed performance of the new class of distribution using several goodness-of-fit statistics that include: −2loglikelihood ($-2A$), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramer-von Mises (W^*), Andersen-Darling (A^*), Kolmogorov-Smirnov (K-S) (and its P-value), and sum of squares (SS) from probability plots.

We estimated the model parameters using the subroutine NLMIXED in SAS and the bbmle package in R. Parameter estimates (standard error in parenthesis) for the real data example is given in Table 7. We compared the new model to other competing non-nested models and the results are shown in Table 7. The non-nested models considered are the beta odd Lindley-exponential (BOL-E) and beta odd Lindley-uniform (BOL-U) by Chipepa et al. [15], the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. [16] and the odd exponentiated

Table 6. Monte Carlo simulation results: mean, average bias and RMSE.

Parameter	n	I (0.1, 0.1, 1.5, 2.5) Mean	RMSE	Average Bias	II (0.5, 0.1, 1.5, 2.5) Mean	RMSE	Average Bias
λ	30	1.7839	2.0772	1.6839	1.8684	1.8533	1.3684
	60	1.9068	2.4241	1.8068	1.9372	2.0921	1.4372
	120	1.7305	2.3683	1.6305	1.7161	1.9847	1.2161
	240	1.3286	1.8771	1.2285	1.3264	1.5754	0.8264
	480	1.0415	1.6677	0.9415	1.0405	1.2505	0.5405
	960	0.6023	0.6627	0.5023	0.7277	0.5009	0.2277
	1920	0.4703	0.4857	0.3703	0.6218	0.3605	0.1218
c	30	0.1421	0.1677	0.0421	0.1419	0.1892	0.0419
	60	0.1528	0.1518	0.0528	0.1377	0.1536	0.0377
	120	0.1659	0.1384	0.0659	0.1186	0.1137	0.0186
	240	0.1859	0.1293	0.0859	0.1250	0.0959	0.0250
	480	0.1822	0.1108	0.0822	0.1128	0.0723	0.0128
	960	0.1817	0.0988	0.0817	0.1092	0.0532	0.0092
	1920	0.1783	0.0877	0.0783	0.1068	0.0380	0.0068
k	30	1.7561	0.4016	0.2561	1.6945	0.3640	0.1945
	60	1.6988	0.2759	0.1988	1.6357	0.2306	0.1357
	120	1.6645	0.2216	0.1645	1.6037	0.1687	0.1037
	240	1.6231	0.1662	0.1231	1.5631	0.1181	0.0631
	480	1.5959	0.1295	0.0959	1.5441	0.0900	0.0441
	960	1.5693	0.0935	0.0693	1.5245	0.0634	0.0245
	1920	1.5517	0.0704	0.0517	1.5127	0.0469	0.0127
θ	30	1.7996	1.0157	−0.7004	1.9251	0.9798	−0.5749
	60	1.7025	1.0633	−0.7975	1.8452	0.9848	−0.6548
	120	1.7576	0.9995	−0.7424	1.9519	0.8722	−0.5481
	240	1.8574	0.8423	−0.6426	2.0792	0.7066	−0.4208
	480	2.0049	0.6889	−0.4950	2.2323	0.5476	−0.2677
	960	2.1716	0.4283	−0.3284	2.3644	0.3208	−0.1356
	1920	2.2452	0.3269	−0.2548	2.4265	0.2392	−0.0735

Table 7. Run off data.

Model	Estimates λ	c	k	θ	Statistics $-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
LBXIIL	1.3965×10^{-01}	2.3546	1.7732	5.8112×10^{-7}	29.9	37.9	39.9	42.7	0.0169	0.1262	0.0880	0.9902
	(5.4269×10^{-01})	(3.9966×10^{-01})	(4.6575×10^{-01})	(2.0113×10^{-01})								
	a	b	λ	θ								
BOL-E	2.6963	17.1560	107.8500	0.0017	30.9	38.9	40.9	43.7	0.0298	0.2234	0.1088	0.9286
	(0.7221)	(8.2789×10^{-5})	(9.0380×10^{-5})	(4.6003×10^{-4})								
BOL-U	2.7069	10.2290	1.6768×10	5.7638×10^{-5}	30.8	38.8	40.8	43.7	0.0296	0.2216	0.1082	0.9317
	(0.7246)	(2.9318)	(1.9065×10^{-4})	(5.5463×10^{-5})								
	α	λ	a	b								
OEHL-BXII	0.1169	1.1501	12.2263	0.0967	33.8	41.8	43.8	46.7	0.0403	0.2541	0.1411	0.7017
	(0.0659)	(2.8505)	(6.9698)	(0.1129)								
	β	λ	θ	γ								
ELOLLW	0.0000	1.0687	0.9785	1.6082	33.2	41.2	43.2	46.0	0.0547	0.3922	0.1218	0.8519
	(0.2780)	(0.1615)	(0.1097)	(0.2304)								

half logistic-Burr XII (OEHL-BXII) by Aldahlan et al. [17] distributions. The pdfs of the non-nested models are given by

$$f_{ELOLLW}(x;\alpha,\beta,\gamma,\theta,\lambda) = \frac{\alpha\theta^2\gamma\lambda^\gamma x^{\gamma-1}e^{-(\lambda x)^\gamma}\left(e^{-(\lambda x)}\right)^{\alpha\theta-1}\left(1-e^{-(\lambda x)^\gamma}\right)^{\alpha-1}}{(\theta+\beta)\left(\left(1-e^{-(\lambda x)^\gamma}\right)^\alpha+e^{-\alpha(\lambda x)^\gamma}\right)^{\theta-1}} \times\left(1-\beta\log\left[\frac{e^{-(\lambda x)^\gamma}}{\left(1-e^{-(\lambda x)^\gamma}\right)^\alpha+e^{-\alpha(\lambda x)^\gamma}}\right]\right),$$

for $\alpha,\beta,\gamma,\theta,\lambda>0$,

$$f_{BOL-U}(x;a,b,\lambda,\theta) = \frac{1}{B(a,b)}\left[1-\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{a-1} \times\left[\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{b-1} \times\frac{\lambda^2}{(1+\lambda)}\frac{\theta^2}{(\theta-x)^3}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\},$$

for $a,b,\lambda,\theta>0$,

$$f_{BOL-E}(x;a,b,\lambda,\theta) = \frac{1}{B(a,b)}\left[1-\frac{\lambda+e^{-\theta x}}{(1+\lambda)e^{-\theta x}}\exp\left\{-\lambda\frac{\left(1-e^{-\theta x}\right)}{e^{-\theta x}}\right\}\right]^{a-1} \times\left[\frac{\lambda+e^{-\theta x}}{(1+\lambda)e^{-\theta x}}\exp\left\{-\lambda\frac{\left(1-e^{-\theta x}\right)}{e^{-\theta x}}\right\}\right]^{b-1} \times\frac{\lambda^2}{(1+\lambda)}\frac{\left(\theta e^{-\theta x}\right)}{e^{-3\theta x}}\exp\left\{-\lambda\frac{1-e^{-\theta x}}{e^{-\theta x}}\right\},$$

for $a,b,\lambda,\theta>0$, and

$$f_{OEHLBXII}(x;\alpha,\lambda,a,b) = \frac{2\alpha\lambda abx^{a-1}\exp\left(\lambda\left[1-(1+x^a)^b\right]\right)\left(1-\exp\left(\lambda\left[1-(1+x^a)^b\right]\right)\right)^{\alpha-1}}{(1+x^a)^{-b-1}\left(1+\exp\left(\lambda\left[1-(1+x^a)^b\right]\right)\right)^{\alpha+1}},$$

for $\alpha,\lambda,a,b>0$

9.1. Run off data

The data represents runoff amounts at Jug Bridge, Maryland by Chhikara and Folks [18] as one of the data sets which were used to describe the Birnbaum–Saunders distribution. The data set is also cited

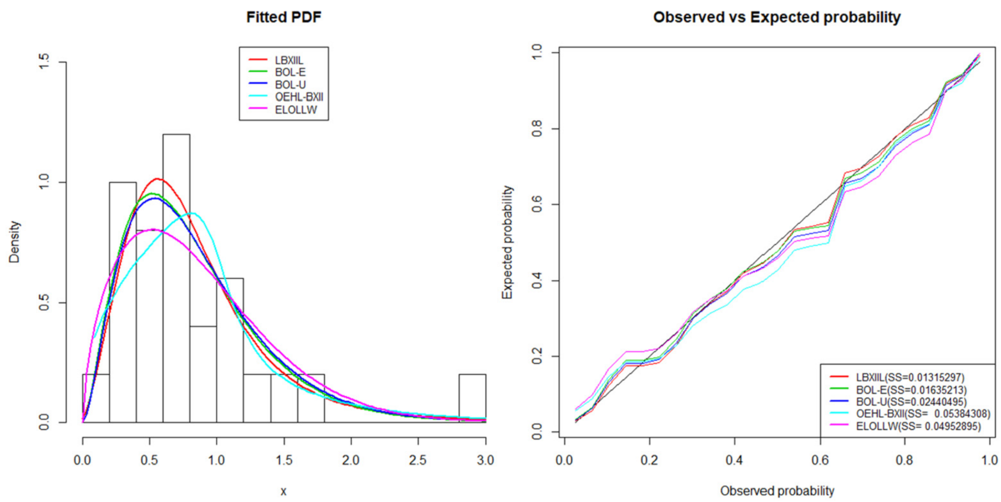

Fig. 8. Graphs for run off data.

by Gadde et al. [19]. The data are as follows; 0.17, 1.19, 0.23, 0.33, 0.39, 0.39, 0.40, 0.45, 0.52, 0.56, 0.59, 0.64, 0.66, 0.70, 0.76, 0.77, 0.78, 0.95, 0.97, 1.02, 1.12, 1.24, 1.59, 1.74, 2.92.

The approximate 95% confidence intervals for the model parameters, λ, c, k and θ are $(0.6 \pm 1.96 \times \sqrt{0.103577})$, $(0.482 \pm 1.96 \times \sqrt{1.856784})$, $(4.1 \pm 1.96 \times \sqrt{1.090633})$ and $(0.1 \pm 1.96 \times \sqrt{0.282894})$, respectively.

The values of AIC, AICC, BIC statistics are the smallest for the LBXIIL distribution when compared to the ones for the non-nested models. The values of the goodness-of-fit statistics W^*, A^*, $K-S$ and its p-value indicate that the proposed model is a better fitting model than the selected competing non-nested models on run off data.

Furthermore, results shown in Figs. 8 (a) and 8 (b), show that the new proposed model fit the run off data set better than the selected competing non-nested models. We therefore, conclude that the proposed model is indeed a useful model.

10. Concluding remarks

We have presented a new class of distributions called the Lindley-Burr XII power series (LBXIIPS) distribution and two special cases, LBXIIP and LBXIIL distributions. This generalized distribution and some of its structural properties are presented. A real data example of the model is given in order to illustrate the applicability and usefulness of the proposed class of distribution. We compared the LBXIIL distribution to several non-nested models including BOL-E, BOL-U, OEHL-BXII and ELOLLW distributions.

Declarations

Author contribution statement

B. Makubate, B. Oluyede: Conceived and designed the experiments.
M. Gabanankosi: Performed the experiments; Wrote the paper.
C. Fastel: Analyzed and interpreted the data; Wrote the paper.

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Data included in article/supplementary material/referenced in article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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