

Split equality variational inequality problems for pseudomonotone mappings in Banach spaces

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Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th anniversary.

Abstract. A new algorithm for approximating solutions of the split equality variational inequality problems (SEVIP) for pseudomonotone mappings in the setting of Banach spaces is introduced. Strong convergence of the sequence generated by the proposed algorithm to a solution of the SEVIP is then derived without assuming the Lipschitz continuity of the underlying mappings and without prior knowledge of operator norms of the bounded linear operators involved. In addition, we provide several applications of our method and provide a numerical example to illustrate the convergence of the proposed algorithm. Our results improve, consolidate and complement several results reported in the literature.

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1. Introduction

Let K be a nonempty, closed and convex subset of a real Hilbert space H , and $T : C \rightarrow H$ be a nonlinear mapping. The variational inequality problem (VIP), first introduced by Stampacchia [31] and Fichera [19] in 1964, is a problem that consist of finding an element $x \in C$ such that $\langle Tx, y - x \rangle \geq 0$ for all $y \in C$. For a nonlinear mapping $T : C \rightarrow H$, we denote the solution of the VIP by $VI(C, T)$ if it is nonempty. It is known that x solves the VIP if and only if x is a fixed point of the map $P_C(I - \lambda T) : C \rightarrow C$. Variational inequality problems have been studied extensively by several authors, thanks to their relevance in various applications in areas such as mechanics, physics, engineering, convex programming and control theory. Among these studies, VIPs for continuous and pseudomonotone maps will be of particular interest to us. Let us remember that if T is continuous and pseudomonotone, then $VI(C, T)$ is closed and convex [26]. In [30, 33, 34], the authors studied algorithms for

solving uniformly continuous and weakly sequentially continuous pseudomonotone VIPs in Hilbert spaces. The distinctive feature of the algorithms constructed and analyzed in [33, 34] is mainly on the different Armijo-type line search rules used. For further reading on the VIP, particularly iterative methods for finding solutions of VIPs, the interested reader is referred to articles [2, 7, 10, 21–23, 25, 29, 38, 41], and their references.

Let K_1 and K_2 be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 , respectively. Also let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be bounded linear mappings, where H_3 is another real Hilbert space. Consider two nonlinear mappings $T : H_1 \rightarrow H_1$ and $S : H_2 \rightarrow H_2$. The split equality variational inequality problem (SEVIP) is formulated as a problem of finding:

$$(x, y) \in K_1 \times K_2 \text{ such that } (x, y) \in VI(K_1, T) \times VI(K_2, S) \text{ and } Ax = By. \quad (1.1)$$

The SEVIP is quite general and it includes as special cases, split equality zero point problem (see, [18]), common solutions of the variational inequality problem [12], common zeros of mappings [16], split equality feasibility problem [27], has been studied extensively by many authors and applied to solving many real life problems such as in modelling intensity-modulated radiation therapy treatment planning [8, 9], modelling of inverse problems arising from phase retrieval, and in sensor networks in computerised tomography and data compression [5, 17].

If, in (1.1), we consider $H_2 = H_3$, and $B = I$, the identity mapping on H_2 , the SEVIP reduces to the split variational inequality problem (SVIP) that was recently introduced by Censor *et al.* [10]. The SVIP consists of finding:

$$(x, y) \in K_1 \times K_2 \text{ such that } (x, y) \in VI(K_1, T) \times VI(K_2, S) \text{ and } y = Ax, \quad (1.2)$$

that is, the SVIP constitutes a pair of VIPs, which have to be solved so that the image $y = Ax$, under a given bounded linear operator A of the solution x of the VIP in H_1 , is a solution of another VIP in another space H_2 . In Moudafi [27], it was noted that the SVIP generalizes the split fixed point problem, split variational inequality problem, split zero point problem and split feasibility problem (see also [3, 4, 6, 13–15, 35, 40], and the references therein). Many of the results cited above were obtained in the setting of real Hilbert spaces. In [11], Censor *et al.* studied an iterative algorithm that approximates a solution of the SVIP for a monotone mapping in Hilbert spaces and proved weak convergence results of the algorithm. In [6], Byrne *et al.* constructed a scheme which approximates the solution of the SVIP for monotone type mappings in Hilbert spaces and proved weak and strong convergence results of the scheme under certain assumptions.

Motivated by the work of Censor *et al.* [11], Byrne *et al.* [6] and Thong *et al.* [33], we introduce and study a new algorithm for solving the SEVIP for uniformly continuous and weakly sequentially continuous pseudomonotone mappings in the setting of Banach spaces. Strong convergence of the proposed algorithm is proved under mild assumptions and without prior knowledge of operator norms of bounded linear mappings involved. Some applications of the main results are also provided. A numerical example is given to illustrate the convergence of the proposed algorithm. Our results improve, consolidate and complement several results in the literature.

2. Preliminaries

Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty, closed and convex subset of E . Consider the function $\phi : E \times E \rightarrow \mathbb{R}$, introduced by Alber [1], defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \text{ for } x, y \in E, \tag{2.1}$$

where $J : E \rightarrow E^*$ is the *normalized duality mapping* defined by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}, \forall x \in E.$$

It is known that if E is uniformly smooth, then J is uniformly norm to norm continuous on each bounded subset of E (see, [32]). Furthermore, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is a duality mapping from E^* into E which satisfies $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see, [32]). The *generalized projection mapping*, introduced by Alber [1], is a mapping $\Pi_C : E \rightarrow C$ that assigns an arbitrary point $x \in E$ to the minimizer, \bar{x} , of $\phi(\cdot, x)$ over C .

Lemma 2.1. [1] *Let C be a nonempty, closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$ for all $y \in C$.*

Lemma 2.2. [20] *Let E be a real smooth and uniformly convex Banach space and let (x_n) and (y_n) be two sequences in E . If either (x_n) or (y_n) is bounded and $\phi(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, then $x_n - y_n \rightarrow 0$, as $n \rightarrow \infty$.*

Lemma 2.3. [1] *Let C be a convex subset of a real smooth Banach space E . Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if $\langle z - x_0, J_E x - J_E x_0 \rangle \leq 0, \forall z \in C$.*

Consider the function $V : E \times E^* \rightarrow \mathbb{R}$, studied by Alber [1], defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \text{ for all } x \in E \text{ and } x^* \in E^*.$$

Lemma 2.4. [1] *Let E be reflexive, strictly convex and smooth Banach space with E^* as its dual. Then for all $x \in E$ and $x^*, y^* \in E^*$,*

$$V(x, x^*) + 2\langle J_E^{-1} x^* - x, y^* \rangle \leq V(x, x^* + y^*).$$

Lemma 2.5. [28] *If E is a smooth Banach space and $\{t_i\} \in (0, 1)$ with $\sum_{i=1}^N t_i = 1$, then*

$$\phi\left(z, J_E^{-1}\left(\sum_{i=1}^N t_i J_E x_i\right)\right) \leq \sum_{i=1}^N t_i \phi(z, x_i).$$

Lemma 2.6. [37] *Let (a_n) be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \beta_n)a_n + \beta_n \delta_n$, for all $n \geq 1$, where $(\beta_n) \subset (0, 1)$ and $(\delta_n) \subset \mathbb{R}$ satisfying $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Then, $\lim_{n \rightarrow \infty} a_n = 0$.*

Lemma 2.7. [24] *Let (a_n) be a sequence of real numbers such that there exists a subsequence (n_i) of (n) such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $(m_k) \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ and $\max\{a_{m_k}, a_k\} \leq a_{m_k+1}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

Lemma 2.8. [39] *Let E be a reflexive and smooth real Banach space. Then, there exists $\alpha > 0$ such that $\langle x - y, J_E x - J_E y \rangle \geq \alpha \|x - y\|^2$ for all $x, y \in E$.*

Lemma 2.9. [36] *Let E be a reflexive and smooth real Banach space. Then for each $x, y \in E$, we have $\phi(y, x) \geq \frac{1}{2} \|x - y\|^2$.*

Lemma 2.10. *Let C be a closed and convex set in a reflexive real Banach space E , h be a real-valued function on E , and K be the set $\{x \in C : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with constant $L > 0$, then*

$$\phi(x, \Pi_K x) \geq \frac{1}{2L^2} (h(x))^2, \text{ for all } x \in C. \tag{2.2}$$

Proof. Clearly (2.2) holds for all $x \in K$. Hence, it suffices to show that (2.2) holds for every $x \in C \setminus K$. Let $x \in C$ but $x \notin K$. Since K is closed, there exists $y \in K$ such that $\phi(x, y) = \phi(x, \Pi_K x)$. It follows from the Lipschitz continuity of h that $|h(x) - h(y)| \leq L \|x - y\|$. Since $x \notin K$ and $y \in K$, we have $h(x) > 0$ and $h(y) \leq 0$. Thus, from Lemma 2.9, we have

$$h(x) \leq h(x) - h(y) = |h(x) - h(y)| \leq L \|x - y\| \leq L \left(2\phi(x, \Pi_K x) \right)^{\frac{1}{2}},$$

and hence the conclusion follows. □

Definition 2.11. Let $T : C \rightarrow E^*$ be a mapping. Then T is called

- (a) sequentially weakly continuous on C if for each sequence $(x_n) \subseteq C$ converging weakly to $x \in C$, the sequence (Ax_n) converges weakly to Ax ;
- (b) monotone if $\langle x - y, Tx - Ty \rangle \geq 0$ for each $x, y \in C$;
- (c) pseudomonotone on C if for all $x, y \in C$,

$$\langle y - x, Tx \rangle \geq 0 \text{ implies } \langle y - x, Ty \rangle \geq 0. \tag{2.3}$$

Remark 2.12. In [30], Shehu *et al.* asserted that using the Monte-Carlo approach, it can be shown that the map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T(x, y) = \left([x^2 + (y - 1)^2] (1 + y), -x^3 - x(y - 1)^2 \right)$$

is pseudomonotone on \mathbb{R}^2 . The correctness of this method/approach in verifying pseudomonotonicity of an operator is questionable. We claim that there could still be a pair of points, say $(x, y), (u, v) \in \mathbb{R}^2$, such that the implication (2.3) does not hold. Indeed, (2.3) fails to hold for a pair of points $(0, 1)$ and $(-1, 2)$ in \mathbb{R}^2 , as shown by simple computations below

$$\langle T(0, 1), (-1, 2) - (0, 1) \rangle = 0 \geq 0 \quad \text{and} \quad \langle T(-1, 2), (-1, 2) - (0, 1) \rangle = -4 < 0.$$

Example 2.13. Let the map $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$S(x, y) = \left([x^2 + 1 + (y - 1)^2] (1 + y), -x^3 - x [1 + (y - 1)^2] \right).$$

Claim 1: S is not monotone. Indeed, for the pair $(1, 0)$ and $(-1, -1)$, we have

$$\langle S(1, 0) - S(-1, -1), (1, 0) - (-1, -1) \rangle = -3 < 0.$$

Claim 2: S is pseudomonotone. To this end, we assume that $\langle S(x, y), (u, v) - (x, y) \rangle \geq 0$ is true for each pair $(x, y), (u, v) \in \mathbb{R}^2$. This means that

$$[x^2 + 1 + (y - 1)^2] (1 + y)(u - x) + [-x^3 - x [1 + (y - 1)^2]] (v - y) \geq 0$$

which implies that $[x^2 + 1 + (y - 1)^2] [u + uy - x - xv] \geq 0$ for all $(x, y), (u, v) \in \mathbb{R}^2$. Therefore, $u(1 + y) - x(1 + v) \geq 0$ for all $(x, y), (u, v) \in \mathbb{R}^2$. Since $u^2 + 1 + (v - 1)^2 > 0$ for any $u, v \in \mathbb{R}$, we have for any $(x, y), (u, v) \in \mathbb{R}^2$,

$$\langle S(u, v), (u, v) - (x, y) \rangle = [u^2 + 1 + (v - 1)^2] [u(1 + y) - x(1 + v)] \geq 0.$$

Lemma 2.14. [26] *Let K be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space E . Let A be a continuous pseudomonotone mapping from K into E^* . Then, $VI(K, A)$ is closed and convex, and $p \in VI(K, A)$ if and only if $\langle x - p, Ax \rangle \geq 0$, for all $x \in K$.*

3. Main results

In the sequel, we shall make use of the following assumptions:

Assumption 1:

- (A1) Let E_1 and E_2 be uniformly smooth and uniformly convex real Banach spaces with dual spaces E_1^* and E_2^* , respectively, and let E_3 be a real Banach space with dual space E_3^* .
- (A2) Let $A : E_1 \rightarrow E_3$ and $B : E_2 \rightarrow E_3$ be bounded linear mappings with adjoints $A^* : E_3^* \rightarrow E_1^*$ and $B^* : E_3^* \rightarrow E_2^*$, respectively.
- (A3) Let $C \subseteq E_1$ and $D \subseteq E_2$ be nonempty, closed and convex subsets.
- (A4) Let $T : E_1 \rightarrow E_1^*$ and $S : E_2 \rightarrow E_2^*$ be uniformly continuous pseudomonotone mappings that are sequentially weakly continuous on bounded subset of C and D , respectively.
- (A5) Let $\Gamma := \{(p, q) \in C \times D : \langle x - p, Tp \rangle \geq 0, \forall x \in C, \langle y - q, Sq \rangle \geq 0, \forall y \in D, \text{ and } Ap = Bq\} \neq \emptyset$.

Assumption 2:

- (B1) Let $\xi = \min\{\xi_1, \xi_2\}$, where ξ_1 and ξ_2 are constants given in Lemma 2.8 associated with J_{E_1} and J_{E_2} , respectively.
- (B2) Let $l \in (0, 1)$, $\mu > 0$ and $\lambda \in (0, \frac{\xi}{\mu})$.
- (B3) Let $(\alpha_n) \subset (0, e] \subset (0, 1)$, for some constant $e > 0$, be such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(B4) Let $0 < \gamma \leq \gamma_n \leq \frac{\xi \|Ax_n - By_n\|^2}{2[\|A^*J_{E_3}(Ax_n - By_n)\|^2 + \|B^*J_{E_3}(Ax_n - By_n)\|^2]}$

for $n \in \Omega = \{n \in \mathbb{N} : Ax_n - By_n \neq 0\}$, otherwise $\gamma_n = \gamma > 0$.

Now, we introduce our algorithm for the SEVIP.

Algorithm 3.1

For arbitrary $x_0, u \in C$ and $y_0, v \in D$, define an iterative algorithm by

1. **Step 1.** Compute: $u_n = \Pi_C J_{E_1}^{-1} [J_{E_1} x_n - \gamma_n A^* J_{E_3} (Ax_n - By_n)]$ and $r_1(x_n, u_n) = x_n - u_n$.
 Compute $v_n = \Pi_D J_{E_2}^{-1} [J_{E_2} y_n + \gamma_n B^* J_{E_3} (Ax_n - By_n)]$ and $s_1(y_n, v_n) = y_n - v_n$.
2. **Step 2.** Compute: $z_n = \Pi_C J_{E_1}^{-1} [J_{E_1} u_n - \lambda T u_n]$ and $r_2(u_n, z_n) = u_n - z_n$.
 Compute $w_n = \Pi_D J_{E_2}^{-1} [J_{E_2} v_n - \lambda S v_n]$ and $s_2(v_n, w_n) = v_n - w_n$.

3. **Step 3.** Compute $f_n = u_n - \tau_n r_2(u_n, z_n)$, where $\tau_n = l^{j_n}$ and j_n is the smallest non-negative integer j satisfying

$$\langle r_2(u_n, z_n), Tu_n - T(u_n - l^j r_2(u_n, z_n)) \rangle \leq \mu \|r_2(u_n, z_n)\|^2 \quad (3.1)$$

and $g_n = v_n - \kappa_n s_2(v_n, w_n)$, where $\kappa_n = l^{h_n}$ and h_n is the smallest non-negative integer h satisfying

$$\langle s_2(v_n, w_n), Sv_n - S(v_n - l^h s_2(v_n, w_n)) \rangle \leq \mu \|s_2(v_n, w_n)\|^2. \quad (3.2)$$

4. **Step 4.** Compute: $x_{n+1} = J_{E_1}^{-1}[\alpha_n J_{E_1} u + (1 - \alpha_n) J_{E_1} \Pi_{C_n} u]$, where

$$C_n := \{x \in C : \langle x - f_n, Tf_n \rangle \leq 0\}, \quad (3.3)$$

and $y_{n+1} = J_{E_2}^{-1}[\alpha_n J_{E_2} v + (1 - \alpha_n) J_{E_2} \Pi_{D_n} v]$, where

$$D_n := \{y \in D : \langle y - g_n, Sg_n \rangle \leq 0\}. \quad (3.4)$$

5. **Step 5.** Set $n := n + 1$ and go to **Step 1**.

Lemma 3.1. *Assume that Conditions (A1) - (A5) and (B1) - (B4) are satisfied. Then, the sequences (x_n) and (y_n) generated by Algorithm 3.1 are well defined.*

Proof. It is enough to show that the search rules in (3.1) and (3.2) are well defined, and the sets C_n and D_n are nonempty.

Since $l \in (0, 1)$ and T is continuous on C , it follows that

$$\langle r_2(u_n, z_n), Tu_n - T(u_n - l^j r_2(u_n, z_n)) \rangle \rightarrow 0, \text{ as } j \rightarrow \infty.$$

On the other hand, since $\|r_2(u_n, z_n)\| > 0$, there exists a non-negative integer j_n satisfying inequality (3.1). Similarly, from the continuity of the mapping S on D , there exists a non-negative integer h_n satisfying inequality (3.2).

Furthermore, since $\Gamma \neq \emptyset$, choose $(p, q) \in \Gamma$. Then by Step 3 of the algorithm, $f_n \in C$ and $g_n \in D$ for each $n \geq 0$, and hence by Lemma 2.14, $\langle p - f_n, Tf_n \rangle \leq 0$ and $\langle q - g_n, Sg_n \rangle \leq 0$ for each $n \geq 0$. Hence, $p \in C_n$ and $q \in D_n$ for each $n \geq 0$, showing that $C_n \neq \emptyset$ and $D_n \neq \emptyset$ for each $n \geq 0$. \square

Lemma 3.2. *Assume that Conditions (A1) - (A5) and (B1) - (B4) are satisfied. If (u_n) , (z_n) , (v_n) and (w_n) are sequences generated by Algorithm 3.1, then $\xi \lambda^{-1} \|r_2(u_n, z_n)\|^2 \leq \langle r_2(u_n, z_n), Tu_n \rangle$ and $\xi \lambda^{-1} \|s_2(v_n, w_n)\|^2 \leq \langle s_2(v_n, w_n), Sv_n \rangle$.*

Proof. Using Lemma 2.3 and the definition of z_n , we have

$$\langle z - z_n, J_{E_1} u_n - \lambda Tu_n - J_{E_1} z_n \rangle \leq 0, \forall z \in C.$$

In particular, for $z = u_n \in C$, we obtain $\langle u_n - z_n, J_{E_1} u_n - J_{E_1} z_n \rangle \leq \lambda \langle u_n - z_n, Tu_n \rangle$. Using Lemma 2.8, we obtain

$$\xi \|r_2(u_n, z_n)\|^2 \leq \xi_1 \|u_n - z_n\|^2 \leq \lambda \langle u_n - z_n, Tu_n \rangle.$$

The second inequality of the lemma can be proved in a similar way. \square

Lemma 3.3. *Assume that Conditions (A1) - (A5) and (B1) - (B4) are met.*

Let $(p, q) \in \Gamma$, $F_n(x) = \langle x - f_n, Tf_n \rangle$ and $G_n(y) = \langle y - g_n, Sg_n \rangle$. Then

(i) $F_n(p) \leq 0$ and $F_n(u_n) \geq \tau_n (\xi\lambda^{-1} - \mu) \|r_2(u_n, z_n)\|^2$, and

(ii) $G_n(q) \leq 0$ and $G_n(v_n) \geq \kappa_n (\xi\lambda^{-1} - \mu) \|s_2(v_n, w_n)\|^2$.

In particular, if $r_2(u_n, z_n) \neq 0$ and $s_2(v_n, w_n) \neq 0$, then $F_n(u_n) > 0$ and $G_n(v_n) > 0$, respectively.

Proof. (i) Since $(p, q) \in \Gamma$, it follows that $p \in \text{VI}(C, T)$ and $q \in \text{VI}(D, S)$.

By Lemma 2.14, $F_n(p) = \langle p - f_n, Tf_n \rangle \leq 0$ for each $n \geq 0$.

Next, we observe from Step 3 of the algorithm and the definition of F_n that

$$F_n(u_n) = \langle u_n - f_n, Tf_n \rangle = \langle \tau_n r_2(u_n, z_n), Tf_n \rangle = \tau_n \langle r_2(u_n, z_n), Tf_n \rangle.$$

But from the search rule (3.1), $\langle r_2(u_n, z_n), Tu_n - Tf_n \rangle \leq \mu \|r_2(u_n, z_n)\|^2$, which together with Lemma 3.2 imply that

$$\begin{aligned} F_n(u_n) = \tau_n \langle r_2(u_n, z_n), Tf_n \rangle &\geq \tau_n \left[\langle r_2(u_n, z_n), Tu_n \rangle - \mu \|r_2(u_n, z_n)\|^2 \right] \\ &\geq \tau_n \left[\xi\lambda^{-1} \|r_2(u_n, z_n)\|^2 - \mu \|r_2(u_n, z_n)\|^2 \right]. \end{aligned}$$

Obviously, if $r_2(u_n, z_n) \neq 0$, then from Condition (B2), we have $F_n(u_n) > 0$.

(ii) The proof is similar to the proof of part (i) above. \square

Lemma 3.4. *Assume that Conditions (A1) - (A5) and (B1) - (B4) hold.*

(a). If there exist $(u_{n_k}) \subset (u_n)$ and $(z_{n_k}) \subset (z_n)$ such that (u_{n_k}) converges weakly to $x \in E_1$ and $\tau_{n_k} \|u_{n_k} - z_{n_k}\|^2 \rightarrow 0$ as $k \rightarrow \infty$, then $x \in \text{VI}(C, T)$.

(b). If there exist $(v_{n_i}) \subset (v_n)$ and $(w_{n_i}) \subset (w_n)$ such that (v_{n_i}) converges weakly to $y \in E_2$ and $\kappa_{n_i} \|v_{n_i} - w_{n_i}\|^2 \rightarrow 0$ as $i \rightarrow \infty$, then $y \in \text{VI}(D, S)$.

Proof. (a). By considering two possible cases on τ_{n_k} , we first show that

$$\lim_{k \rightarrow \infty} \|u_{n_k} - z_{n_k}\| = 0. \quad (3.5)$$

Case I: Assume that $\liminf_{k \rightarrow \infty} \tau_{n_k} > 0$.

Then there exists a constant $\tau > 0$ such that $\tau_{n_k} \geq \tau > 0$ for all $k \in \mathbb{N}$. Then

$$\|u_{n_k} - z_{n_k}\|^2 = \tau_{n_k}^{-1} \left[\tau_{n_k} \|u_{n_k} - z_{n_k}\|^2 \right] \leq \tau^{-1} \left[\tau_{n_k} \|u_{n_k} - z_{n_k}\|^2 \right]. \quad (3.6)$$

Therefore, (3.5) follows from (3.6) and the assumption in the lemma.

Case II: Assume that $\liminf_{k \rightarrow \infty} \tau_{n_k} = 0$.

In this case, we take a subsequence (n_{k_j}) of (n_k) if necessary, we assume without loss of generality that

$$\lim_{k \rightarrow \infty} \tau_{n_k} = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|u_{n_k} - z_{n_k}\| = a > 0. \quad (3.7)$$

Let $f_k = \frac{1}{l} \tau_{n_k} z_{n_k} + (1 - \frac{1}{l} \tau_{n_k}) u_{n_k}$. Using (3.7), we get

$$\lim_{k \rightarrow \infty} \|f_k - u_{n_k}\| = \lim_{k \rightarrow \infty} l^{-1} \tau_{n_k} \|u_{n_k} - z_{n_k}\| = 0. \quad (3.8)$$

Since T is uniformly continuous on bounded subsets of C , it follows from (3.8) that $\|Tf_k - Tu_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$.

From (3.1), we have $\langle Tu_{n_k} - Tf_k, u_{n_k} - z_{n_k} \rangle > \mu \|u_{n_k} - z_{n_k}\|^2$, and it follows that $\|u_{n_k} - z_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$. This contradicts (3.7), hence the limit in (3.5) must hold. Finally, we show that $x \in \text{VI}(C, T)$.

Since C is weakly closed, we have $x \in C$. Furthermore, from the fact that J_{E_1} is uniformly continuous on bounded subsets of E_1 , we have

$$\lim_{k \rightarrow \infty} \|J_{E_1} u_{n_k} - J_{E_1} z_{n_k}\| = 0. \tag{3.9}$$

From Lemma 2.3 and $z_n \in C$, we get $\langle z - z_{n_k}, J_{E_1} u_{n_k} - \lambda Tu_{n_k} - J_{E_1} z_{n_k} \rangle \leq 0$ for all $z \in C$, which implies that

$$\langle z - z_{n_k}, J_{E_1} u_{n_k} - J_{E_1} z_{n_k} \rangle - \lambda \langle u_{n_k} - z_{n_k}, Tu_{n_k} \rangle \leq \lambda \langle z - u_{n_k}, Tu_{n_k} \rangle. \tag{3.10}$$

Taking the limit inferior as $k \rightarrow \infty$ and using (3.9), we get

$$\liminf_{k \rightarrow \infty} \langle z - u_{n_k}, Tu_{n_k} \rangle \geq 0 \quad \forall z \in C. \tag{3.11}$$

Thus the inequality in (3.11) implies that we can choose a decreasing sequence of positive real numbers (δ_k) such that (δ_k) converges to zero as $k \rightarrow \infty$, and for each δ_k there exists N_k , the smallest positive integer, such that

$$\langle z - u_{n_j}, Tu_{n_j} \rangle + \delta_k \geq 0 \quad \forall j \geq N_k \quad \text{and} \quad \forall z \in C. \tag{3.12}$$

Since (δ_k) is decreasing, the sequence (N_k) is increasing.

Note that if there exists $N > 0$ such that $Tu_{N_k} = 0$ for all $k \geq N$, then it can be shown easily that $x \in \text{VI}(C, T)$.

On the other hand, if there exists a subsequence (N_{k_i}) of (N_k) , again denoted by (N_k) , such that $Tu_{N_k} \neq 0$ for all $k \in \mathbb{N}$, then $\langle a_{N_k}, Tu_{N_k} \rangle = 1$ for each $k \in \mathbb{N}$, where

$$a_{N_k} = \frac{J_{E_1}^{-1} Tu_{N_k}}{\|Tu_{N_k}\|^2}.$$

From (3.12), we deduce that $\langle z + \delta_k a_{N_k} - u_{N_k}, Tu_{N_k} \rangle \geq 0$ for each $k \in \mathbb{N}$ and $z \in C$. Since T is pseudomonotone, it follows that

$$\langle z + \delta_k a_{N_k} - u_{N_k}, T(z + \delta_k a_{N_k}) \rangle \geq 0 \quad \forall k \in \mathbb{N} \quad \text{and} \quad \forall z \in C. \tag{3.13}$$

But by our assumption, (u_{N_k}) converges weakly to $x \in C$. Also T is sequentially weakly continuous on E_1 implies that (Tu_{N_k}) converges weakly to Tx . Moreover, we can suppose that $Tx \neq 0$ (otherwise, x is in $\text{VI}(C, T)$) and so

$$0 \leq \|Tx\| \leq \liminf_{k \rightarrow \infty} \|Tu_{N_k}\|.$$

Since $(u_{N_k}) \subset (u_{n_k})$ and (δ_k) converges to zero as $k \rightarrow \infty$, we obtain that

$$0 \leq \limsup_{k \rightarrow \infty} \|\delta_k a_{N_k}\| = \limsup_{k \rightarrow \infty} \left(\frac{\delta_k}{\|Tu_{N_k}\|} \right) \leq \frac{\limsup_{k \rightarrow \infty} \delta_k}{\liminf_{k \rightarrow \infty} \|Tu_{N_k}\|} \leq \frac{0}{\|Tx\|},$$

and hence $\|\delta_k a_{N_k}\| \rightarrow 0$ as $k \rightarrow \infty$. Therefore, taking the limit in (3.13) as $k \rightarrow \infty$, we get $\langle z - x, Tz \rangle \geq 0$ for all $z \in C$. In view of Lemma 2.14, we conclude that $x \in \text{VI}(C, T)$.

Part (b) of the Lemma can be proved in a similar way. □

Remark 3.5. If in Lemma 3.4, $T : E_1 \rightarrow E_1^*$ and $S : E_2 \rightarrow E_2^*$ are uniformly continuous and monotone mappings, then for all $z \in C$, we have from (3.10)

$$\begin{aligned} \langle z - z_{n_k}, J_{E_1} u_{n_k} - J_{E_1} z_{n_k} \rangle &+ \lambda \langle z_{n_k} - u_{n_k}, T u_{n_k} \rangle \leq \lambda \langle z - u_{n_k}, T u_{n_k} - T z \rangle \\ &+ \lambda \langle z - u_{n_k}, T z \rangle \leq \lambda \langle z - u_{n_k}, T z \rangle. \end{aligned}$$

Taking the limit as $k \rightarrow \infty$, we get $0 \leq \langle z - x, T z \rangle$. It then follows from Lemma 2.14 that $x \in \text{VI}(C, T)$. Similarly, we get $y \in \text{VI}(D, S)$.

Lemma 3.6. *Let (x_n) and (y_n) be sequences generated by Algorithm 3.1. Assume that the Conditions (A1) - (A5) and (B1) - (B4) hold. Then (x_n) and (y_n) are bounded. Hence, (u_n) , (v_n) , (z_n) and (w_n) are bounded sequences.*

Proof. Let $(p, q) \in \Gamma$. Then $p \in \text{VI}(C, T)$, $q \in \text{VI}(D, S)$ and $Ap = Bq$.

Denote $q_n = J_{E_1}^{-1} [J_{E_1} x_n - \gamma_n A^* J_{E_3} (Ax_n - By_n)]$. Then $u_n = \Pi_C q_n$ and so from Lemmas 2.1 and 2.4, and the properties of the mapping V , we obtain

$$\begin{aligned} \phi(p, u_n) &\leq \phi(p, J_{E_1}^{-1} [J_{E_1} x_n - \gamma_n A^* J_{E_3} (Ax_n - By_n)]) \\ &= V(p, J_{E_1} x_n - \gamma_n A^* J_{E_3} (Ax_n - By_n)) \\ &\leq V(p, J_{E_1} x_n) - 2 \langle q_n - p, \gamma_n A^* J_{E_3} (Ax_n - By_n) \rangle \\ &= \phi(p, x_n) - 2 \gamma_n \langle q_n - p, A^* J_{E_3} (Ax_n - By_n) \rangle \\ &= \phi(p, x_n) - 2 \gamma_n \langle Aq_n - Ap, J_{E_3} (Ax_n - By_n) \rangle. \end{aligned} \quad (3.14)$$

Furthermore, from Lemma 2.5, Lemma 2.1 and (3.14), we have

$$\begin{aligned} \phi(p, x_{n+1}) &\leq \alpha_n \phi(p, u) + (1 - \alpha_n) \phi(p, \Pi_{C_n} u_n) \\ &\leq \alpha_n \phi(p, u) + (1 - \alpha_n) \phi(p, u_n) \\ &\leq \alpha_n \phi(p, u) + (1 - \alpha_n) [\phi(p, x_n) \\ &\quad - 2 \gamma_n \langle Aq_n - Ap, J_{E_3} (Ax_n - By_n) \rangle]. \end{aligned} \quad (3.15)$$

Similarly, if we denote $t_n = J_{E_2}^{-1} [J_{E_2} y_n + \gamma_n B^* J_{E_3} (Ax_n - By_n)]$, then

$$\phi(q, v_n) \leq \phi(q, y_n) + 2 \gamma_n \langle Bt_n - Bq, J_{E_3} (Ax_n - By_n) \rangle, \quad (3.17)$$

and therefore, from Lemma 2.5, Lemma 2.1 and (3.17)

$$\begin{aligned} \phi(q, y_{n+1}) &\leq \alpha_n \phi(q, v) + (1 - \alpha_n) \phi(q, \Pi_{D_n} v_n) \\ &\leq (1 - \alpha_n) [\phi(q, y_n) + 2 \gamma_n \langle Bt_n - Bq, J_{E_3} (Ax_n - By_n) \rangle] \\ &\quad + \alpha_n \phi(q, v). \end{aligned} \quad (3.18)$$

Denote $\Upsilon = \phi(p, u) + \phi(q, v)$ and $\Theta_n = \phi(p, x_n) + \phi(q, y_n)$. Then adding (3.16) and (3.19), we get

$$\Theta_{n+1} \leq (1 - \alpha_n) [\Theta_n - 2 \gamma_n \langle Aq_n - Bt_n, J_{E_3} (Ax_n - By_n) \rangle] + \alpha_n \Upsilon. \quad (3.20)$$

Now observe that

$$\begin{aligned} -\langle Aq_n - Bt_n, J_{E_3} (Ax_n - By_n) \rangle &= -\langle Ax_n - By_n, J_{E_3} (Ax_n - By_n) \rangle \\ &\quad - \langle Aq_n - Ax_n, J_{E_3} (Ax_n - By_n) \rangle - \langle By_n - Bt_n, J_{E_3} (Ax_n - By_n) \rangle \\ &\leq \|q_n - x_n\| \|A^* J_{E_3} (Ax_n - By_n)\| - \|Ax_n - By_n\|^2 \\ &\quad + \|y_n - t_n\| \|B^* J_{E_3} (Ax_n - By_n)\|. \end{aligned} \quad (3.21)$$

From Lemma 2.8 and the definition of q_n , we obtain

$$\|q_n - x_n\| \leq \frac{1}{\xi_1} \|\gamma_n A^* J_{E_3}(Ax_n - By_n)\| \leq \frac{\gamma_n}{\xi} \|A^* J_{E_3}(Ax_n - By_n)\|. \tag{3.22}$$

Similarly, from Lemma 2.8 and the definition of t_n , we obtain

$$\|y_n - t_n\| \leq \gamma_n \xi^{-1} \|B^* J_{E_3}(Ax_n - By_n)\|. \tag{3.23}$$

Combining (3.21), (3.22) and (3.23), we obtain

$$\begin{aligned} & - 2\gamma_n \langle Aq_n - Bt_n, J_{E_3}(Ax_n - By_n) \rangle \leq -2\gamma_n \|Ax_n - By_n\|^2 \\ & + 2\gamma_n^2 \xi^{-1} [\|A^* J_{E_3}(Ax_n - By_n)\|^2 + \|B^* J_{E_3}(Ax_n - By_n)\|^2] \\ & \leq -\gamma \|Ax_n - By_n\|^2 \end{aligned} \tag{3.24}$$

for all $n \in \Omega$, where the last inequality follows from Assumption (B4). If $n \notin \Omega$, then $Ax_n - By_n = 0$, and in this case inequality (3.24) follows trivially. Finally, using (3.24) in (3.20), we obtain $\Theta_{n+1} \leq (1 - \alpha_n)\Theta_n + \alpha_n \Upsilon$. By mathematical induction, $\Theta_n \leq \max\{\Theta_0, \Upsilon\}$ for all $n \geq 0$, showing that the sequence $(\phi(p, x_n) + \phi(q, y_n))$ is bounded, which implies that $(\phi(p, x_n))$ and $(\phi(q, y_n))$ are bounded. By the properties of ϕ , we conclude that (x_n) and (y_n) are bounded. Consequently, (u_n) , (v_n) , (z_n) and (w_n) are bounded. \square

Theorem 3.7. *Suppose the Assumptions (A1) - (A5) and (B1) - (B4) hold. Then the sequence $((x_n, y_n))$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \Gamma$, where $(x^*, y^*) = \prod_{\Gamma}(u, v)$.*

Proof. Let $(x^*, y^*) \in \Gamma$ be such that $(x^*, y^*) = \prod_{\Gamma}(u, v)$. Denote

$$\Lambda_n = 2 \langle (x_n, y_n) - (x^*, y^*), (J_{E_1} u, J_{E_2} v) - (J_{E_1} x^*, J_{E_2} y^*) \rangle.$$

Then for some $M_1 > 0$, we have

$$\begin{aligned} \Lambda_{n+1} &= 2 [\langle x_n - x^*, J_{E_1} u - J_{E_1} x^* \rangle + \langle y_n - y^*, J_{E_2} v - J_{E_2} y^* \rangle] \\ &+ 2 [\langle x_{n+1} - x_n, J_{E_1} u - J_{E_1} x^* \rangle + \langle y_{n+1} - y_n, J_{E_2} v - J_{E_2} y^* \rangle] \\ &\leq \Lambda_n + M_1 [\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|]. \end{aligned} \tag{3.25}$$

Since the sequences (u_n) and (z_n) are bounded by Lemma 3.6, the sequence (f_n) is bounded. But T is uniformly continuous implies that there exists $L > 0$ such that $\|Tf_n\| \leq L$ for all $n \geq 0$. We can then deduce that for each $n \geq 0$, the mapping F_n is Lipschitz continuous with Lipschitz constant $L > 0$.

Now, from Lemma 2.1,

$$\phi(x^*, \Pi_{C_n} u_n) \leq \phi(x^*, u_n) - \phi(\Pi_{C_n} u_n, u_n). \tag{3.26}$$

Using Lemmas 2.10 and 3.3, we obtain

$$\phi(\Pi_{C_n} u_n, u_n) \geq L_1 \tau_n^2 \|r_2(u_n, z_n)\|^4 \tag{3.27}$$

for some $L_1 > 0$. Combining (3.26), (3.27) and (3.14), we get

$$\begin{aligned} \phi(x^*, \Pi_{C_n} u_n) &\leq \phi(x^*, x_n) - 2\gamma_n \langle Aq_n - Ax^*, J_{E_3}(Ax_n - By_n) \rangle \\ &- L_1 \tau_n^2 \|u_n - z_n\|^4. \end{aligned} \tag{3.28}$$

From the definition of x_n , the properties of the map V and Lemma 2.4,

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= V(x^*, \alpha_n J_{E_1} u + (1 - \alpha_n) J_{E_1} \Pi_{C_n} u_n) \\
&\leq V(x^*, \alpha_n J_{E_1} x^* + (1 - \alpha_n) J_{E_1} \Pi_{C_n} u_n) \\
&\quad + 2\alpha_n \langle x_{n+1} - x^*, J_{E_1} u - J_{E_1} x^* \rangle \\
&= \phi(x^*, J_{E_1}^{-1} [\alpha_n J_{E_1} x^* + (1 - \alpha_n) J_{E_1} \Pi_{C_n} u_n]) \\
&\quad + 2\alpha_n \langle x_{n+1} - x^*, J_{E_1} u - J_{E_1} x^* \rangle.
\end{aligned} \tag{3.29}$$

Using Lemma 2.5, (3.29) and (3.28), we obtain for some $\widehat{K}_1 > 0$,

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq (1 - \alpha_n) \phi(x^*, \Pi_{C_n} u_n) + 2\alpha_n \langle x_{n+1} - x^*, J_{E_1} u - J_{E_1} x^* \rangle \\
&\leq (1 - \alpha_n) [\phi(x^*, x_n) - 2\gamma_n \langle Aq_n - Ax^*, J_{E_3}(Ax_n - By_n) \rangle] \\
&\quad + 2\alpha_n \langle x_{n+1} - x^*, J_{E_1} u - J_{E_1} x^* \rangle - \widehat{K}_1 \tau_n^2 \|u_n - z_n\|^4.
\end{aligned} \tag{3.30}$$

Similarly, we deduce that for each $n \geq 0$,

$$\phi(y^*, \Pi_{D_n} v_n) \leq \phi(y^*, v_n) - \phi(\Pi_{D_n} v_n, v_n), \tag{3.31}$$

and also, for some $\widehat{K}_2 > 0$, we derive

$$\begin{aligned}
\phi(y^*, y_{n+1}) &\leq (1 - \alpha_n) [\phi(y^*, y_n) + 2\gamma_n \langle Bt_n - By^*, J_{E_3}(Ax_n - By_n) \rangle] \\
&\quad + 2\alpha_n \langle y_{n+1} - y^*, J_{E_2} v - J_{E_2} y^* \rangle - \widehat{K}_2 \kappa_n^2 \|v_n - w_n\|^4.
\end{aligned} \tag{3.32}$$

Denote $\Theta_n^* = \phi(x^*, x_n) + \phi(y^*, y_n)$ and $\Upsilon^* = \phi(x^*, u) + \phi(y^*, v)$. Then combining (3.30), (3.32) and (3.24), we get

$$\Theta_{n+1}^* \leq (1 - \alpha_n) \Theta_n^* + \alpha_n \Lambda_{n+1} - L^* \left[\tau_n^2 \|u_n - z_n\|^4 + \kappa_n^2 \|v_n - w_n\|^4 \right] \tag{3.33}$$

for some $L^* > 0$. Furthermore, from (3.15), (3.26) and (3.14), we obtain

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \alpha_n \phi(x^*, u) + (1 - \alpha_n) [\phi(x^*, x_n) - \phi(\Pi_{C_n} u_n, u_n)] \\
&\quad - 2(1 - \alpha_n) \gamma_n \langle Aq_n - Ax^*, J_{E_3}(Ax_n - By_n) \rangle.
\end{aligned} \tag{3.34}$$

Similarly, from (3.18), (3.31) and (3.17), we obtain

$$\begin{aligned}
\phi(y^*, y_{n+1}) &\leq \alpha_n \phi(y^*, v) + (1 - \alpha_n) [\phi(y^*, y_n) - \phi(\Pi_{D_n} v_n, v_n)] \\
&\quad + 2(1 - \alpha_n) \gamma_n \langle Bt_n - By^*, J_{E_3}(Ax_n - By_n) \rangle.
\end{aligned} \tag{3.35}$$

Adding (3.34) and (3.35), and using (3.24), we obtain for some $M > 0$

$$\Theta_{n+1}^* \leq \Theta_n^* + \alpha_n M - \gamma \|Ax_n - By_n\|^2 - [\phi(\Pi_{C_n} u_n, u_n) + \phi(\Pi_{D_n} v_n, v_n)],$$

which implies that

$$\begin{aligned}
\phi(\Pi_{C_n} u_n, u_n) + \phi(\Pi_{D_n} v_n, v_n) + \gamma \|Ax_n - By_n\|^2 &\leq \Theta_n^* - \Theta_{n+1}^* \\
&\quad + \alpha_n M.
\end{aligned} \tag{3.36}$$

Finally, we show that the sequence (Θ_n^*) converges strongly to zero as $n \rightarrow \infty$. For this, we consider two possible cases on (Θ_n^*) .

Case I. Assume that there exists $n_0 \in \mathbb{N}$ such that the sequence of real numbers (Θ_n^*) is decreasing for all $n \geq n_0$. It then follows that (Θ_n^*) is convergent. Taking the limit in (3.36) as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \|Ax_n - By_n\| = 0. \quad (3.37)$$

From the definition of q_n , Lemma 2.1, Lemma 2.4 and (3.22), we have

$$\begin{aligned} \phi(x_n, u_n) &\leq \phi(x_n, J_{E_1}^{-1} [J_{E_1} x_n - \gamma_n A^* J_{E_3} (Ax_n - By_n)]) \\ &= V(x_n, J_{E_1} x_n - \gamma_n A^* J_{E_3} (Ax_n - By_n)) \\ &\leq V(x_n, J_{E_1} x_n) - 2\langle q_n - x_n, \gamma_n A^* J_{E_3} (Ax_n - By_n) \rangle \\ &\leq \phi(x_n, x_n) + 2\gamma_n^2 \xi^{-1} \|A\|^2 \|Ax_n - By_n\|^2. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and noticing (3.37), yield $\phi(x_n, u_n) \rightarrow 0$ as $n \rightarrow \infty$. Using Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.38)$$

Similarly, starting with the definition of t_n , we obtain

$$\lim_{n \rightarrow \infty} \|y_n - v_n\| = 0. \quad (3.39)$$

Moreover, we also obtain from (3.36) and Lemma 2.2

$$\lim_{n \rightarrow \infty} \|\Pi_{C_n} u_n - u_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\Pi_{D_n} v_n - v_n\| = 0. \quad (3.40)$$

From the definition of x_n and Lemma 2.8,

$$\begin{aligned} \|x_{n+1} - u_n\| &\leq \xi_1^{-1} \|\alpha_n J_{E_1} u + (1 - \alpha_n) J_{E_1} \Pi_{C_n} u_n - J_{E_1} u_n\| \\ &\leq \alpha_n K_1 + \xi_1^{-1} \|J_{E_1} \Pi_{C_n} u_n - J_{E_1} u_n\|, \end{aligned} \quad (3.41)$$

for some constant $K_1 > 0$. Since J_{E_1} is norm to norm uniformly continuous on bounded subsets of E_1 , we conclude from (3.40) and (3.41) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0. \quad (3.42)$$

Therefore, combining (3.38) and (3.42) yield

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.43)$$

Similarly, one can show that

$$\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0. \quad (3.44)$$

Since (x_n) and (y_n) are bounded by Lemma 3.6, we obtain from (3.33)

$$L^* \left[\tau_n^2 \|u_n - z_n\|^4 + \kappa_n^2 \|v_n - w_n\|^4 \right] \leq \Theta_n^* - \Theta_{n+1}^* + \alpha_n \Lambda_{n+1}, \quad (3.45)$$

But the convergence of (Θ_n^*) and Assumption (B3) imply that

$$\lim_{n \rightarrow \infty} \tau_n \|u_n - z_n\|^2 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \kappa_n \|v_n - w_n\|^2 = 0. \quad (3.46)$$

Now, we deduce from Lemma 3.6 that $((x_n, y_n))$ is a bounded sequence in $C \times D$. Therefore, there exists a subsequence $((x_{n_k}, y_{n_k}))$ of $((x_n, y_n))$ such that $((x_{n_k}, y_{n_k}))$ converges weakly to (x, y) in $E_1 \times E_2$ and

$$\limsup_{n \rightarrow \infty} \Lambda_n = \lim_{k \rightarrow \infty} \Lambda_{n_k}. \tag{3.47}$$

It then follows that (x_{n_k}) converges weakly to x in E_1 and (y_{n_k}) converges weakly to y in E_2 . From (3.38), (u_{n_k}) converges weakly to x in E_1 and from (3.39), (v_{n_k}) converges weakly to y in E_2 . Using (3.46) and Lemma 3.4, we conclude that $x \in \text{VI}(C, T)$ and $y \in \text{VI}(D, S)$, respectively. Moreover,

$$\|Ax - By\|^2 \leq 2\langle Ax - Ax_{n_k} + By_{n_k} - By, J_{E_3}(Ax - By) \rangle + \|Ax_{n_k} - By_{n_k}\|^2.$$

Since (x_{n_k}) converges weakly to x , it follows that (Ax_{n_k}) converges weakly to Ax . Similarly, (y_{n_k}) converges weakly to y implies that (By_{n_k}) converges weakly to By . Using (3.37), we get $Ax = By$. Consequently, $(x, y) \in \Gamma$.

From (3.25), (3.47), (3.43), (3.44) and Lemma 2.3, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \Lambda_{n+1} &\leq \limsup_{n \rightarrow \infty} \Lambda_n + M_1 \limsup_{n \rightarrow \infty} [\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|] \\ &= \lim_{k \rightarrow \infty} \Lambda_{n_k} + M_1 \lim_{k \rightarrow \infty} [\|x_{n_k+1} - x_{n_k}\| + \|y_{n_k+1} - y_{n_k}\|] \\ &= 2\langle (x, y) - (x^*, y^*), (J_{E_1}u, J_{E_2}v) - (J_{E_1}x^*, J_{E_2}y^*) \rangle \\ &\leq 0. \end{aligned} \tag{3.48}$$

Finally, from (3.33), we have $\Theta_{n+1}^* \leq (1 - \alpha_n)\Theta_n^* + \alpha_n\Lambda_{n+1}$. Therefore, from (3.48) and Lemma 2.6, we conclude that (Θ_n^*) converges to zero as $n \rightarrow \infty$. That is, $\phi(x^*, x_n) \rightarrow 0$ and $\phi(y^*, y_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence by Lemma 2.2, we have (x_n) and (y_n) converges to x^* and y^* , respectively.

Case II. Assume that there exists a subsequence $(\Theta_{n_i}^*)$ of (Θ_n^*) such that $\Theta_{n_i}^* < \Theta_{n_{i+1}}^*$ for all $i \geq 0$. Then in view of Lemma 2.7, we can define a nondecreasing sequence $(m_k) \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\Theta_{m_k}^* \leq \Theta_{m_k+1}^* \quad \text{and} \quad \Theta_k^* \leq \Theta_{m_k+1}^* \tag{3.49}$$

for all $k \in \mathbb{N}$. Following similar steps as in Case I, we derive

$$\limsup_{k \rightarrow \infty} \Lambda_{m_k+1} \leq 0. \tag{3.50}$$

From (3.33) and (3.49), we obtain $\alpha_{m_k}\Theta_{m_k+1}^* \leq \alpha_{m_k}\Lambda_{m_k+1}$, which reduces to $\Theta_{m_k+1}^* \leq \Lambda_{m_k+1}$. Taking the limit as $k \rightarrow \infty$ and using (3.50), we conclude that $\Theta_{m_k+1}^* \rightarrow 0$ as $k \rightarrow \infty$. Again from (3.49), it follows that $\Theta_k^* \rightarrow 0$ as $k \rightarrow \infty$. Therefore, $\phi(x^*, x_k) \rightarrow 0$ and $\phi(y^*, y_k) \rightarrow 0$ as $k \rightarrow \infty$. Hence by Lemma 2.2, we have $x_k \rightarrow x^*$ and $y_k \rightarrow y^*$ as $k \rightarrow \infty$. \square

If $u = 0$ and $v = 0$, then Algorithm 3.1 can be used to locate an element of the solution with the minimum norm.

Corollary 3.8. *Let the Assumptions (A1) – (A5) and (B1) – (B4) hold. Then, the sequence $((x_n, y_n))$ generated by Algorithm 3.1 with $u = 0 = v$ converges strongly to the minimum norm point $(x^*, y^*) \in \Gamma$, that is, $(x^*, y^*) = \prod_{\Gamma}(0, 0)$.*

Corollary 3.9. *Assume that $T : E_1 \rightarrow E_1^*$ and $S : E_2 \rightarrow E_2^*$ are uniformly continuous and monotone mappings. Let the Assumptions (A1) – (A3), (A5) and (B1) – (B4) be satisfied. Then the sequence $((x_n, y_n))$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \Gamma$, where $(x^*, y^*) = \prod_{\Gamma}(u, v)$.*

Proof. The mappings T and S are pseudomonotone, hence by Lemma 3.6, (x_n) and (y_n) are bounded. It then follows from (3.46) and Remark 3.5 that $x \in \text{VI}(C, T)$ and $y \in \text{VI}(D, S)$, where x and y are weak cluster points of (x_n) and (y_n) , respectively. The rest of the proof is similar to the proof of Theorem 3.7. \square

4. Applications

In this section, we apply our main result to solve the following problems: split equality zero point problem (SEZPP), common solutions of the variational inequality problem, common zeros of pseudomonotone mappings, split variational inequality problem, split zero point problem (SZPP), split equality feasibility problem (SEFP) and split feasibility problem (SFP).

4.1. Split equality zero point problem

If $C = E_1$ and $D = E_2$, then the SEVIP reduces to the SEZPP, which is to find $x \in T^{-1}(0)$ and $y \in S^{-1}(0)$ such that $Ax = By$, where $T^{-1}(0) = \{p \in E_1 : 0 = Tp\}$ and $S^{-1}(0) = \{q \in E_2 : 0 = Sq\}$. Denote the solution of this problem by $F = \{(p, q) \in E_1 \times E_2 : p \in T^{-1}(0), q \in S^{-1}(0) \text{ and } Ap = Bq\}$.

Corollary 4.1. *Assume that $F \neq \emptyset$. Let the Assumptions (A1), (A2), (A4) and (B1) – (B4) be satisfied with $C = E_1$ and $D = E_2$. Then the sequence $((x_n, y_n))$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in F$, where $(x^*, y^*) = \prod_F(u, v)$.*

4.2. Common solutions of the variational inequality problem

Let $E = E_1 = E_2 = E_3$, $A = I$ and $B = I$. In this case, the SEVIP reduces to finding common solutions of two variational inequality problems for pseudomonotone mappings. Denote $\mathcal{F} = \{(p, q) \in C \times D : \langle x - p, Tp \rangle \geq 0, \forall x \in C \text{ and } \langle y - q, Sq \rangle \geq 0, \forall y \in D \text{ such that } p = q\}$.

Corollary 4.2. *Assume that $\mathcal{F} \neq \emptyset$. Let the Assumptions (A1), (A3), (A4) and (B1) – (B4) be satisfied with $E = E_1 = E_2 = E_3$ and $A = I = B$. Then the sequence $((x_n, y_n))$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \mathcal{F}$, where $(x^*, y^*) = \prod_{\mathcal{F}}(u, v)$.*

4.3. Common zeros of pseudomonotone mappings

Let $E = E_1 = E_2 = E_3$, $A = I$ and $B = I$. If $C = E$ and $D = E$, then the SEVIP reduces to finding common zeros of pseudomonotone mappings. Denote $\mathcal{F}' = \{(p, q) \in E \times E : p \in T^{-1}(0) \text{ and } q \in S^{-1}(0) \text{ such that } p = q\}$.

Corollary 4.3. *Let the Assumptions (A1), (A4), (B1) – (B4) be satisfied with $C = D = E = E_1 = E_2 = E_3$ and $A = I = B$. If $\mathcal{F}' \neq \emptyset$, then the sequence $((x_n, y_n))$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \mathcal{F}'$, where $(x^*, y^*) = \prod_{\mathcal{F}'}(u, v)$.*

4.4. Split variational inequality problem

If $E_2 = E_3$ and $B = I$, then the SEVIP reduces to the split variational inequality problem (SVIP) which is to find $x \in VI(C, T)$ and $y \in VI(D, S)$ such that $Ax = y$. Denote $\mathcal{T} = \{(p, q) \in C \times D : p \in VI(C, T), q \in VI(D, S) \text{ and } Ap = q\}$.

Corollary 4.4. *Assume that the Assumptions (A1) – (A4) and (B1) – (B4) hold with $E_3 = E_2$ and $B = I$. If $\mathcal{T} \neq \emptyset$, then the sequence $((x_n, y_n))$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \mathcal{T}$, where $(x^*, y^*) = \prod_{\mathcal{T}}(u, v)$.*

4.5. Split zero point problem

Let $E = E_2 = E_3$ and $B = I$. If $C = E$ and $D = E$, then the SEVIP reduces to the SZPP which is to find $x \in T^{-1}(0)$ and $y \in S^{-1}(0)$ such that $Ax = y$. Denote $\mathcal{S} = \{(p, q) \in E \times E : p \in T^{-1}(0), q \in S^{-1}(0) \text{ and } Ap = q\}$.

Corollary 4.5. *Assume that the Assumptions (A1), (A2), (A4), (B1) – (B3) and (B4) with $B = I$ hold. If $\mathcal{S} \neq \emptyset$, then the sequence $((x_n, y_n))$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \mathcal{S}$, where $(x^*, y^*) = \prod_{\mathcal{S}}(u, v)$.*

Remark 4.6. (a). If $E = E_1 = E_2 = E_3$, $S = 0$, $A = 0$ and $B = 0$, then Theorem 3.7 can be used to find solutions of variational inequality problems for uniformly continuous pseudomonotone mappings that are sequentially weakly continuous on bounded subsets of E as well as for uniformly continuous monotone mappings. If in addition, we take $C = E$ and $D = E$, then Corollary 4.1 will approximate zeros of uniformly continuous pseudomonotone mappings that are sequentially weakly continuous on bounded subsets of E and also zeros of uniformly continuous monotone mappings.

(b). In view of Corollary 3.9, and the discussion in this section, one can use the results of this section to find solutions of split equality zero point problem for uniformly continuous monotone mappings, common solutions of the variational inequality problem for uniformly continuous monotone mappings, common zeros of uniformly continuous monotone mappings, split variational inequality problems for uniformly continuous monotone mappings, split zero point problem for monotone mappings.

(c). The special cases of the above results can be obtained by taking $E_1 = H_1$, $E_2 = H_2$ and $E_3 = H_3$ to be real Hilbert spaces.

Note that if $E = H$, a real Hilbert space, then $J_E = I$, the identity mapping on H , and $\Pi_C = P_C$, the metric projection onto C . A well known example of a uniformly continuous, monotone and hence pseudomonotone map is $I - P_C$. Henceforth, $E_1 = H_1$, $E_2 = H_2$ and $E_3 = H_3$ are real Hilbert spaces, $C \subset H_1$ and $D \subset H_2$ are nonempty, closed and convex subsets. Also $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ are bounded linear mappings with adjoints A^* and B^* , respectively, $T = I - P_C$ and $S = I - P_D$. Thus, we obtain the following applications in Hilbert spaces.

4.6. Split equality feasibility problem

Replacing T with $I - P_C$ and S with $I - P_D$, then the SEZPP reduces to the SEFP which seeks to find $x \in C$ and $y \in D$ such that $Ax = By$. Denote

$$\Gamma' = \{(p, q) \in C \times D : Ax = By\}.$$

Corollary 4.7. *Assume that $\Gamma' \neq \emptyset$. Let the Assumptions (B2), (B3), (B4) be satisfied. Then the sequence $((x_n, y_n))$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \Gamma'$, where $(x^*, y^*) = P_{\Gamma'}(u, v)$.*

4.7. Split feasibility problem

Setting $T = I - P_C$, $S = I - P_D$, $H_2 = H_3$ and $B = I$, the identity mapping on H_2 , then the SEFP reduces to the SFP which seeks to find $x \in C$ such that $Ax \in D$. This problem can also be expressed as a problem of finding $x \in C$ and $y \in D$ such that $Ax = y$. Denote $\mathcal{S}'' = \{(p, q) \in C \times D : Ap = q\}$.

Corollary 4.8. *Assume that $\mathcal{S}'' \neq \emptyset$. Let the Assumptions (B2), (B3), (B4) with $B = I$ be satisfied. Then the sequence $((x_n, y_n))$ generated by Algorithm 3.1 with $B = I$ converges strongly to $(x^*, y^*) \in \mathcal{S}''$, where $(x^*, y^*) = P_{\mathcal{S}''}(u, v)$.*

Remark 4.9. If we take $u = 0$ and $v = 0$, then one can obtain elements of minimum norm for all the application areas listed in this section.

5. Numerical example

In this section, we give a numerical example to demonstrate that the sequence $(z_n) = ((x_n, y_n))$ generated by Algorithm 3.1 converges to an element $z^* = (x^*, y^*)$ in Γ for different initial values $z_0 = (x_0, y_0)$.

Example 5.1. Let $\|\cdot\|$ be the norm on \mathbb{R}^2 induced by the inner product $\langle \cdot, \cdot \rangle$. Define the map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) = \left(\frac{3}{2} + \sqrt{x^2 + y^2}\right)(x - 1, y)$. Assume that $\langle T(x, y), (u, v) - (x, y) \rangle \geq 0$ for all $(x, y), (u, v) \in \mathbb{R}^2$. Then

$$\left(\frac{3}{2} + \sqrt{x^2 + y^2}\right) \langle (x - 1, y), (u - x, v - y) \rangle \geq 0,$$

which implies that $\langle (x - 1, y), (u - x, v - y) \rangle \geq 0$ for all $(x, y), (u, v) \in \mathbb{R}^2$. Therefore,

$$\begin{aligned} \langle T(u, v), (u, v) - (x, y) \rangle &= \left(\frac{3}{2} + \sqrt{u^2 + v^2}\right) \langle (x - 1, y), (u - x, v - y) \rangle \\ &+ \left(\frac{3}{2} + \sqrt{u^2 + v^2}\right) [\langle (u - 1, v), (u - x, v - y) \rangle - \langle (x - 1, y), (u - x, v - y) \rangle] \\ &\geq \left(\frac{3}{2} + \sqrt{u^2 + v^2}\right) \|(u - x, v - y)\|^2 \geq 0, \end{aligned}$$

for all $(x, y), (u, v) \in \mathbb{R}^2$, showing that T is pseudomonotone on \mathbb{R}^2 .

Example 5.2. Let $E_1 = E_2 = E_3 = \mathbb{R}^2$ be equipped with the usual norm. Assume that $A, B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are given by $A(x, y) = (0, 3y)$ and $B(x, y) = (2x, 0)$ with adjoints $A^*(x, y) = (0, 3y)$ and $B^*(x, y) = (2x, 0)$, respectively.

Let $C = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \leq 1\}$ and $D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \leq 3\}$. Let $S, T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be pseudomonotone maps defined as in Examples 2.13 and 5.1, respectively.

Then $\langle T(1, 0), (x, y) - (1, 0) \rangle \geq 0$ for all $(x, y) \in C$, $\langle S(0, -1), (x', y') - (0, -1) \rangle \geq 0$

for all $(x', y') \in D$ and $A(1, 0) = (0, 0) = B(0, -1)$, and so $((1, 0), (0, -1)) \in \Gamma \neq \emptyset$. Also, let $\mu = 0.9$, $\lambda = 1$ and let

$$\gamma_n = \begin{cases} \frac{\|Ax_n - By_n\|^2}{8[\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2]} & \text{if } n \in \Omega \\ \frac{1}{10000} & \text{if } n \notin \Omega. \end{cases}$$

and $\alpha_n = \frac{1}{n+1}$. Thus assumptions (A1) – (A5) and (B1) – (B4) are satisfied. Using MATLAB, we get Figure 1 below which shows that for any choice of initial values the sequence generated by Algorithm 3.1 converges to a solution of the split equality variational inequality problem. The numerical example also shows that the convergence of the sequence is faster if the parameter $l \in (0, 1)$ is closer to 1 compared to when it is closer to 0.

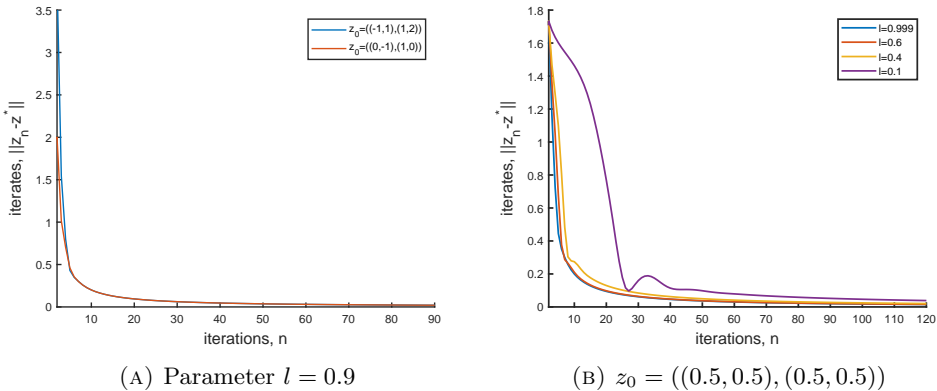


FIGURE 1. Convergence of (z_n) to z^*

6. Conclusion

In this paper, we have constructed an algorithm for solving the split equality variational inequality problem in real uniformly smooth and uniformly convex Banach space settings. We also established a strong convergence theorem under the assumption that the associated mappings are uniformly continuous, pseudomonotone and sequentially weakly continuous. The algorithm does not require prior knowledge of operator norms of A and B . We also gave some applications of our results to some problems in Banach spaces. A numerical example was also provided to demonstrate the behavior of the convergence of the proposed algorithm. Our results in this paper extend the results of Censor *et al.* [11], Byrne *et al.* [6] and Thong *et al* [33] to a more general SEVIP in uniformly smooth and uniformly convex Banach spaces more general than Hilbert spaces.

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